Unlikely intersections in semi-abelian surfaces

Daniel Bertrand, Harry Schmidt

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D. Bertrand and H. Schmidt *

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Abstract. - We consider a family, depending on a parameter, of multiplicative extensions of an elliptic curve with complex multiplications. They form a 3-dimensional variety G which admits a dense set of special curves, known as Ribet curves, which strictly contains the torsion curves. We show that an irreducible curve W in G meets this set Zariski-densely only if W lies in a fiber of the family or is a translate of a Ribet curve by a multiplicative section. We further deduce from this result a proof of the Zilber-Pink conjecture (over number fields) for the mixed Shimura variety attached to the threefold G, when the parameter space is the universal one.

Contents

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* Authors' addresses, AMS Classification, Acknowledgements : see end of the paper.

1 Introduction

1.1 Statement of the results and plan of the proofs

Let $E_0/\mathbb{Q}^{alg}$ be an elliptic curve with complex multiplications. On any extension G_0 of E_0 by $\mathbb{G}_m$ defined over $\mathbb{Q}^{alg}$, there exists a particular subgroup Γ_0 of $G_0(\mathbb{Q}^{alg})$, whose elements are called Ribet points. We refer to §1.2 below for their precise definition, but point out right now that Γ_0 contains the torsion subgroup G_0^{tor} of $G_0(\mathbb{Q}^{alg})$. In fact $\Gamma_0 = G_0^{tor}$ if the extension G_0 is isosplit, while Γ_0 has rank 1 otherwise.

Let further $X/\mathbb{Q}^{alg}$ be a smooth irreducible algebraic curve and let G/X be an X-extension of $E_{0/X}$ by $\mathbb{G}_{m/X}$. Let q be the section of $\hat{E}_{0/X} \to X$ representing the isomorphism class of the extension G/X. We identify q with its image in $E_0(X)$ under the standard polarization $\hat{E}_0 \simeq E_0$, and write $G \simeq G_q$. Given a section s of G/X, we denote by $p = \pi \circ s \in E_0(X)$ its composition with the projection $\pi : G \to E_0 \times X$.

Let $\delta \neq 0$ be a purely imaginary complex multiplication of E_0, and let $\xi \in X(\mathbb{Q}^{alg})$. A first property of Ribet points is that if $s(\xi)$ is a Ribet point of its fiber $G_\xi \simeq G_{q(\xi)}$, then its projection $p(\xi)$ to E_0 and the point $\delta q(\xi)$ are linearly dependent over $\mathbb{Z}$. Usually, this condition alone will be satisfied by infinitely many ξ's. But asking that $s(\xi)$ be a Ribet point in the fiber of $G_\xi \to E_0$ above $p(\xi)$ brings a second condition, unlikely to be satisfied infinitely often. And indeed, we prove in this paper :

Theorem 1. *Let $G \simeq G_q$ be a non constant (hence non isosplit) extension of $E_{0/X}$ by $\mathbb{G}_{m/X}$, and let s be a section of $G \to X$, all defined over $\mathbb{Q}^{alg}$. Assume that the set*

$$\Xi = \Xi_s := \{\xi \in X(\mathbb{Q}^{alg}),\ s(\xi) \text{ is a Ribet point of its fiber } G_\xi \simeq G_{q(\xi)}\}$$

is infinite. Then, the sections p and q are linearly dependent over $End(E_0)$.

Referring again to §1.2 for the definition of the Ribet sections of G/X (which in view of the hypothesis on G, also form a group Γ of rank 1, containing the torsion sections), we deduce the following (actually equivalent) version of Theorem 1 :

Theorem 2. *Assume that the hypotheses of Theorem 1 on the extension G, the section s and the set Ξ are satisfied. Then, there exists a non constant or trivial section s' in $\mathbb{G}_m(X)$ such that $s - s'$ is a Ribet section of G/X.*

The conclusion of Theorem 2 is best possible. Indeed, let s' be such a section in $\mathbb{G}_m(X)$ and let s'' be a Ribet section. Then, $s''(\xi)$ is a Ribet point of G_ξ for any $\xi \in X$, while $s'(\xi)$ lies in $\mathbb{G}_m^{tor}$ infinitely often. The set Ξ_s attached to $s = s' + s''$ is therefore infinite.

As a corollary to Theorem 1, we consider the case when the curve $X = \hat{E}_0 \simeq Ext(E_0, \mathbb{G}_m)$ is the parameter space of the universal extension $\mathcal{P}_0$ of E_0 by $\mathbb{G}_m$. This extension, which identifies with the Poincaré bi-extension of $E_0 \times \hat{E}_0$ by $\mathbb{G}_m$, is naturally endowed with the structure of a mixed Shimura variety, for which we prove :

Theorem 3. *Let $W/\mathbb{Q}^{alg}$ be an irreducible algebraic curve in $\mathcal{P}_0$. Assume that W contains infinitely many points lying on special curves of the mixed Shimura variety $\mathcal{P}_0$. Then, W is contained in a special surface of $\mathcal{P}_0$.*

Combined with Gao's work on the André-Oort conjecture, this readily implies the following conclusion, which answers a question of J. Pila.

Theorem 4. *The mixed Shimura variety $\mathcal{P}_0$ satisfies the Zilber-Pink conjecture over number fields.*

See §5 below for the statement of this conjecture, and for the deduction of Theorems 3 and 4 from Theorem 1.

The proof of Theorem 1 will distinguish three cases. In the first one, we establish the following weaker version, where the conclusion is replaced by a "weakly special" one. Denote by $E_0(\mathbb{Q}^{alg}) \subset E_0(X)$ the group of constant sections of $E_{0/X}$.

Theorem 1.w. *Same hypotheses as in Theorem 1. Then, the sections p and q are linearly dependent over $End(E_0)$ modulo $E_0(\mathbb{Q}^{alg})$.*

The proof of Theorem 1.w (see §2) follows the *o*-minimal strategy of Pila-Zannier and Masser-Zannier, starting with the observation that if its conclusion does not hold, then the points ξ of Ξ have bounded height.

In the remaining cases, we suppose that p and q are linearly dependent over $End(E_0)$ modulo $E_0(\mathbb{Q}^{alg})$. In the second one (see §3), we assume that they are linearly dependent over $\mathbb{Z}$ modulo $E_0(\mathbb{Q}^{alg})$, but that p is not (i.e. p is not constant). Here again, we use the *o*-minimal strategy, but a new argument is required to check bounded height.

In the last case (see §4), we reduce a weaky special relation over $End(E_0)$ to one over $\mathbb{Z}$, and therefore to a constant section p. We finally show that p must be torsion, thanks to a duality argument which turns the problem into a special case of the Mordell-Lang theorem (recalled in §1.3.(v) below) for a constant semi-abelian variety attached not to q, but to p.

1.2 Ribet sections and points

Let $\mathcal{X}/\mathbb{Q}^{alg}$ be a smooth irreducible variety, let A be an abelian scheme over $\mathcal{X}$, let $q \in \hat{A}(\mathcal{X})$ be a section of the dual abelian scheme $\hat{A}/\mathcal{X} \simeq Ext_{\mathcal{X}}(A, \mathbb{G}_m)$, and let $G = G_q$ be the corresponding $\mathcal{X}$-extension of A by $\mathbb{G}_{m/\mathcal{X}}$, obtained by removing its zero section from the line bundle defined by q. We point out that G_q is an isosplit extension (i.e. isogenous to the product $\mathbb{G}_m \times A$) if and only if q is a torsion section. When $A/\mathcal{X}$ is a constant group scheme, G_q is a constant group scheme if and only if q is a constant section (for instance a torsion one).

Let $\mathcal{P}$ be the Poincaré bi-extension of $A \times_{\mathcal{X}} \hat{A}$ by $\mathbb{G}_m$. For any $\varphi \in Hom_{\mathcal{X}}(\hat{A}, A)$, with transpose $\hat{\varphi}$, there is a canonical isomorphism $\sigma_{\varphi,q} : \mathcal{P}((\varphi - \hat{\varphi})(q), q) \simeq \mathbb{G}_{m/\mathcal{X}}$ of $\mathbb{G}_m$-torsors over $\mathcal{X}$ (see [9] Prop. 6.3, whose description of $\sigma_{\varphi,q}$ works over an arbitrary base scheme; [6], Prop. 3.1). We define the *basic Ribet section* associated to φ as the section $s_{\varphi,q} = \sigma^*_{\varphi,q}(1_{\mathcal{X}})$ of the semi-abelian scheme $G = G_q = (id_A, q)^*\mathcal{P} = \mathcal{P}_{|A\times q}$ over $\mathcal{X}$. We say "point" instead of "section" if $\mathcal{X}$ is a point, and drop the index q when the context is clear.

The Ribet section $s_\varphi \in G(\mathcal{X})$ depends additively on φ, and in fact only on $\varphi - \hat{\varphi}$ (cf. [15], Prop. 4.2; [6], Formula 3.1.2). Its projection under $\pi : G \to A$ is the section

$$p_\varphi := \pi \circ q_\varphi = (\varphi - \hat{\varphi}) \circ q \in A(\mathcal{X}).$$

So, when φ varies, the basic Ribet sections form a finitely generated subgroup of $G(\mathcal{X})$, of rank r_q at most equal to the rank of the $\mathbb{Z}$-module $\mathcal{E} = \{\varphi - \hat{\varphi}, \varphi \in Hom_{\mathcal{X}}(\hat{A}, A)\}$, and equal to it when q is sufficiently general. On the other hand, $r_q = 0$ if q is a torsion section. Indeed, although their dependence in q is *not linear*, the Ribet sections s_φ satisfy the following "lifting property" (for $(i) \Rightarrow (ii)$, see [4], §1, [7], Thm. 3.(i) in the case of points, and [6], Prop. 3.3 in general) :

Lemma 1. *Let $\varphi \in Hom_{\mathcal{X}}(\hat{A}, A)$, let $q \in \hat{A}(\mathcal{X})$ and consider the conditions*
(i) q is a torsion section
(ii) s_φ is a torsion section
(iii) p_φ is a torsion section.
Then, $(i) \Rightarrow (ii) \Rightarrow (iii)$, and if $\varphi - \hat{\varphi}$ is an isogeny, the three conditions are equivalent.

More generally, let s be a local section of $G \to \mathcal{X}$ (for the étale topology). We say that s is a *Ribet section* of $G/\mathcal{X}$ if there exists a positive integer n satisfying : $n.s = s_\varphi$ for some φ, with multiplication by n in the sense of the group scheme $G/\mathcal{X}$. The projection p of s to A satisfies : $np = (\varphi - \hat{\varphi}) \circ q$. All (local) torsion sections of $G/\mathcal{X}$ now appear as such Ribet sections, and Lemma 1 extends to this more general setting. Viewed as points above the generic point η of $\mathcal{X}$, with $K = \mathbb{Q}^{alg}(\mathcal{X}_\eta)$, the Ribet sections form a subgroup Γ of the group $G_\eta(K^{alg})$, of same rank r_q as above.

The construction of Ribet sections commutes with any base change. For instance, given a basic Ribet section $s_{\varphi,q}$ of $G/\mathcal{X}$, and a point ξ in $\mathcal{X}(\mathbb{Q}^{alg})$, $s_{\varphi,q}(\xi) = s_{\varphi_\xi,q(\xi)}$ is the basic Ribet point of the fiber G_ξ attached to the specialization φ_ξ of φ at ξ. Conversely, let s^ξ be a Ribet point of $G_\xi(\mathbb{Q}^{alg})$. By definition, there exist $n_\xi \in \mathbb{Z}_{>0}$ and $\varphi_\xi \in Hom(\hat{A}_\xi, A_\xi)$ such that $n_\xi s^\xi = s_{\varphi_\xi,q(\xi)}$. *Assume* further *that φ_ξ extends to an element* $\varphi \in Hom(\hat{A}, A)$ (which occurs automatically if $A/\mathcal{X}$ is a constant abelian scheme as in §1.1). Then, $s_{\varphi_\xi,q(\xi)} = s_{\varphi,q}(\xi)$, and there exists a local section s of $G/\mathcal{X}$ such that $n_\xi.s = s_\varphi$, whose image in G contains s^ξ. So, the Ribet point s^ξ extends locally to a Ribet section of $G/\mathcal{X}$.

Let us now return to the situation of §1.1, where $A = E_0 \times \mathcal{X}$, for a CM elliptic curve E_0, and $\mathcal{X}$ is either the curve X or a point ξ on X. Then, the $\mathbb{Z}$-module $\mathcal{E}$ above identifies with

$$\mathcal{E} = \{\varphi - \overline{\varphi}, \varphi \in End(E_0)\} = \mathbb{Z}\delta,$$

where $\delta = \alpha - \overline{\alpha} \neq 0$ is a purely imaginary quadratic number, which will be fixed from now on. Consequently, for any $q \in E_0(X)$, the group of basic Ribet sections of $G = G_q$ is cyclic, generated by the section

$$s^R := s_{\alpha,q} \in G(X)\ , \quad \text{with } p^R := \pi \circ s^R = \delta q \in E_0(X).$$

Viewed at the generic point η of X, *the Ribet sections of G/X then form the divisible hull Γ of the group $\mathbb{Z}.s^R(\eta)$ in $G_\eta(K^{alg})$.* Furthermore, for any $\xi \in X(\mathbb{Q}^{alg})$, the value $s^R(\xi) = s_{\alpha,q(\xi)}$ of s^R at ξ generates the group of basic Ribet sections of $G_\xi = G_{q(\xi)}$, and the Ribet points of G_ξ form the divisible hull

$$\Gamma_\xi = \{s^\xi \in G_\xi(\mathbb{Q}^{alg}), \exists (n,m) \in \mathbb{Z}^2, n \neq 0, ns^\xi = ms^R(\xi)\} \supset G_\xi^{tor}$$

of $\mathbb{Z}.s^R(\xi)$ in $G_\xi(\mathbb{Q}^{alg})$.

Under the assumptions of §1.1, the section q is not constant, hence not torsion, while δ is an isogeny, so s^R is not torsion by Lemma 1, and the rank r_q of Γ is equal to 1. On the other hand, by Lemma 1 (now at the level of points), given a point $\xi \in X(\mathbb{Q}^{alg})$,

$$q(\xi) \in E_0^{tor} \Leftrightarrow s^R(\xi) \in G_\xi^{tor} \Leftrightarrow \Gamma_\xi = G_\xi^{tor},$$

and this occurs for *infinitely many* ξ's since q is not constant (cf. [4], Thm. 1). Otherwise, Γ_ξ has rank 1, but for $s(\xi) \in \Gamma_\xi$, we still have : $s(\xi) \in G_\xi^{tor} \Leftrightarrow p(\xi) \in E_0^{tor}$.

In view of these descriptions of the groups Γ and Γ_ξ, our work can be interpreted as a particular case of the study of unlikely intersections within an isogeny class (cf. [10]), or of a relative version of the Mordell-Lang problem (compare with §1.3.(v) below).

1.3 The context

We here put the results of §1.1 in perspective with other statements of unlikely intersections. Two sets

$$\Xi^{tor} \subset \Xi \subset \Xi^{\ell d}$$

related to the section $s \in G(X)$ naturally appear in the process.

(i) Theorem 1 gives a positive answer to the "Question 2" raised in [5], §5, while a positive answer to its "Question 1" was recently obtained by Barroero [1]. However, the applications to Pink's conjecture given in [5] require clarification, because of their ambiguous use of Hecke orbits. We by-pass this problem for the mixed Shimura variety $\mathcal{P}_0$ studied in §5, by describing all its possible special curves. Theorem 3 will then follow from Theorem 1, along the method of [5].

(ii) Contrary to the convention of [7], the torsion points are here viewed as particular cases of Ribet points. Therefore, Theorem 2 implies the restriction to the case of our semi-abelian scheme G/X of the main theorem of [7], which concerns the subset

$$\Xi^{tor} = \Xi_s^{tor} := \{\xi \in X(\mathbb{Q}^{alg}),\ s(\xi) \text{ is a torsion point of its fiber } G_\xi\ \}$$

of Ξ, and asserts :

Lemma 2. *Let G/X and s be as in Theorem 1, and assume moreover that the subset Ξ^{tor} of Ξ is infinite. Then s is a Ribet section or a torsion translate of a non constant section in $\mathbb{G}_m(X)$.*

For $\xi \in \Xi^{tor}$, $p(\xi)$ too is torsion, so (by the Manin-Mumford theorem [12] for the image of (p, s') in $E_0 \times \mathbb{G}_m$), the conclusion of Theorem 2 can be sharpened to the same statement.

Let $\Xi^{tor}_{s^R}$ be the set attached to the Ribet section s^R, defined similarly as Ξ^{tor}_s. We pointed out at the end of §1.2 that $\Xi^{tor}_{s^R}$ is infinite. Therefore, Lemma 2 too is best possible.

(iii) In relation with the two sections s, s^R of G/X, consider the set

$$\Xi^{\ell d} = \Xi^{\ell d}_{s,s^R} := \{\xi \in X(\mathbb{Q}^{alg}),\ s(\xi) \text{ and } s^R(\xi) \text{ are linearly dependent over } \mathbb{Z}\}\ .$$

For ξ in this set, either $s(\xi)$ lies in the divisible hull Γ_ξ of $\mathbb{Z}.s^R(\xi)$, or $s^R(\xi)$ is a torsion point. So $\Xi^{\ell d}$ is the (non necessarily disjoint) union of Ξ and $\Xi^{tor}_{s^R}$ and in particular, is always infinite. More generally, given two sections s, s' in $G(X)$, the similarly defined set $\Xi^{\ell d}_{s,s'}$ will be infinite as soon as the group generated by s and s' in $G(X)$ contains a non-torsion Ribet section. So, in contrast with the case of abelian schemes (see [21], [2]), the subgroup schemes of $G \times_X G$ do not suffice to control the finiteness of $\Xi^{\ell d}_{s,s'}$: as in [6], the special subvarieties of the corresponding mixed Shimura variety should also be taken into account.

(iv) Consider the curve $W = s(X)$ in G and define a Ribet curve as the image in G of a Ribet section. Theorem 2 then says that W is the translate of a Ribet curve by a section in $\mathbb{G}_m(X)$. Since any curve W in G dominating X can be viewed as the image of a section after a base extension, while any Ribet point of a fiber G_ξ locally extends to a Ribet section, this justifies the last but one sentence of the abstract.

(v) Assume that contrary to the hypothesis of Theorem 1 , $G = G_0 \times X$ for some constant semi-abelian surface $G_0/\mathbb{Q}^{alg}$, and that s is not constant. Then, the projection W_0 of $W = s(X)$ to G_0 is a curve, which contains infinitely many points of the group Γ_0 of Ribet points of G_0. Since Γ_0 has finite rank (at most 1), the solution by Vojta and McQuillan [20] of the *Mordell-Lang* conjecture for semi-abelian varieties implies that s factors through a translate by a Ribet point of a strict connected algebraic subgroup of G_0. If the section q, here constant, is not torsion, the only such one is $\mathbb{G}_m$. So the conclusions of Theorems 1 and 2 still hold true in this case.

(vi) Same as in (v), but assume furthermore that q is a torsion section, say the trivial one, so $G_0 \simeq \mathbb{G}_m \times E_0$. Then, $s = (s', p)$ for some section $s' \in \mathbb{G}_m(X)$, while the group Γ_0 of Ribet points of G_0 coincides with G_0^{tor}. By Manin-Mumford, $\Xi = \Xi^{tor}$ is then infinite if and only if s' is a torsion section, or p is a torsion section.

(vii) In this paper, we do not touch on the question of replacing $\mathbb{Q}^{alg}$ by $\mathbb{C}$, or of applying Theorem 2 to generalized Pell equations as in [21], [2]. Nor do we study how effective our results can be made. Note that Lemma 2 above is made effective in the ongoing work [17].

Due to the use of Pfaffian methods, in particular [18] and [16], the bounds for the counting problem in [17] are uniform and effective.

We take opportunity of these comments to show that

Theorem 1 $\Leftrightarrow$ *Theorem 2* : Theorem 2 clearly implies Theorem 1. Indeed, the sections s and $s'' = s - s'$ have the same projection p to E_0. Since s'' is a Ribet section, p and δq are linearly dependent over $\mathbb{Z}$, so p and q are linearly dependent over $End(E_0)$.

Conversely, assume that the hypotheses and the conclusion of Theorem 1 hold true, and let $np - \rho\, q = 0$ be a non-trivial relation with $n \in \mathbb{Z}, \rho \in End(E_0)$ not both 0 (equivalently, $n \neq 0$ since q is not a torsion section). Without loss of generality, we can assume that Ξ^{tor} is finite, otherwise Lemma 2 readily implies the conclusion of Theorem 2. For any $\xi \in \Xi$, $\delta q(\xi)$ and the the projection $p(\xi)$ of the Ribet point $s(\xi)$ are linearly dependent over $\mathbb{Z}$, so there exist $n_\xi, m_\xi \in \mathbb{Z}$, not both zero, such that $n_\xi p(\xi) - m_\xi \delta q(\xi) = 0$, while the generic relation implies : $np(\xi) - \rho\, q(\xi) = 0$. If these two relations are linearly independent over $End(E_0)$, then $q(\xi)$, hence $s^R(\xi)$, hence $s(\xi)$, are torsion points and ξ lies in Ξ^{tor}. So, for infinitely many, hence at least one, ξ, these two relations must be linearly dependent over $End(E_0)$, and in fact over $\mathbb{Z}$, since n does not vanish. This implies that ρ is a rational multiple of δ, and by their very construction, this in turn implies the existence of a Ribet section s'' projecting to p. So, $s' = s - s''$ factors through $\mathbb{G}_m$. Finally, if s' is a constant section, it must be a torsion one since $s'(\xi)$ is a Ribet point of G_ξ projecting to 0 for one (any) $\xi \in \Xi$. In this case, s itself is a Ribet section, and otherwise s' is not constant, so the conclusion of Theorem 2 holds in all cases. □

2 Proof of Theorem 1.w

Recall the hypotheses of Theorem 1.w, as well as the notation $s^R, \Gamma_\xi, ...$ of §1.2. So, $q \in E_0(X)$ is not constant, s is a section of $G = G_q \to X$ projecting to the section $\pi \circ s = p \in E_0(X)$, and the set $\Xi = \{\xi \in X(\mathbb{Q}^{alg}), s(\xi) \in \Gamma_\xi\}$, concretely described as

$$\Xi = \{\xi \in X(\mathbb{Q}^{alg}), \exists (n,m) \in \mathbb{Z}^2, n \neq 0, ns(\xi) - ms^R(\xi) = 0\}.$$

is infinite. *We assume that the sections p and q are linearly independent over $End(E_0)$ modulo $E_0(\mathbb{Q}^{alg})$*, and search for a contradiction.

We fix a number field k over which X and G, hence the sections q and s^R, as well as the section s, hence p, and the isogeny δ, are defined. We recall that the basic Ribet section s^R projects to E_0 on the section $p^R = \delta q$.

2.1 The o-minimal strategy

The proof of Theorem 1.w will be done in 5 steps. The 3rd one is developed in §2.2. By a "constant" c, γ, we mean a positive real number which depends only on the data X, E_0, q, s and the number field k. The constants C may depend on further data introduced in the proof.

We point out that any finite set of points can without loss of generality be withdrawn from the curve X. To ease a technical point in the 3rd step, we will for instance require that the sections p, q and $p+q \in E_0(X)$ never vanish on X. The complement is a finite set since q is not constant, p can be assumed to be so (constant p's are treated by a direct method in §4.2), and if $p+q$ is constant, we can make it non constant by replacing s by $2s$, so p by $2p$, without modifying the content of the theorems.

2.1.1. Bounded heights of points

Let h denote a height on $X(\mathbb{Q}^{alg})$ attached to a divisor of degree 1 on the completed curve. Consider the set

$$\Xi_{p,\delta q}^{\mathbb{Z}\ell d} = \{\xi \in X(\mathbb{Q}^{alg}), p(\xi) \text{ and } \delta q(\xi) \text{ are linearly dependent over } \mathbb{Z}\}.$$

Since the projection $p(\xi) = \pi \circ s(\xi)$ of a Ribet point $s(\xi)$ lies in the divisible hull of the group $\mathbb{Z}.\delta q(\xi)$ in $E_0(\mathbb{Q}^{alg})$, this set contains Ξ.

Lemma 3. *Let $p, q \in E_0(X)$ be linearly independent over $End(E_0)$ modulo $E_0(\mathbb{Q}^{alg})$. There exists a constant c_0 such that $h(\xi) \le c_0$ for any $\xi \in \Xi_{p,\delta q}^{\mathbb{Z}\ell d}$, and in particular, for any $\xi \in \Xi$.*

Proof. - In view of the hypothesis on p, q, bounded height on $\Xi_{p,\delta q}^{\mathbb{Z}\ell d}$ follows directly from [25], Theorem 4 (and one can even replace $\mathbb{Z}$ by $End(E_0)$ in the definition of $\Xi_{p,\delta q}^{\mathbb{Z}\ell d}$). Alternatively, one can appeal to Silverman's specialization theorem [24]. □

To get the desired contradiction, it remains to show that the degrees

$$d_\xi = [k(\xi) : \mathbb{Q}]$$

too are bounded from above on the set Ξ.

2.1.2. Heights of relations bounded by degrees

Lemma 4. *There exist two constants c, γ such that for any point $\xi \in \Xi$, there exist two integers $n \neq 0, m$ with $|n|, |m| \le c d_\xi^\gamma$ such that $ns(\xi) - ms^R(\xi) = 0$.*

Proof. - By [7], Corollary of §3.1, there exists a constant c' such that if $s(\xi)$ is a torsion point of G_ξ, its order n is bounded from above by $c' d_\xi^4$, so $(n, 0)$ satisfies the required condition. We can therefore assume that the Ribet point $s(\xi)$, hence $q(\xi)$ by Lemma 1, is not a torsion point. For $\xi \in \Xi$, there exist $a, b \in \mathbb{Z}$, not both 0, such that $ap(\xi) - b\delta q(\xi) = 0$, and since $q(\xi) \notin E_0^{tor}$, any such relation will automatically imply $a \neq 0$. The points $p(\xi), \delta q(\xi)$ are defined over $k(\xi)$, and have heights $\le c_0$. By works of Masser and David (see for instance Lemma 6.1 of [1]), there then exists such a relation with $max(|a|, |b|) \le c_1 d_\xi^{\gamma_1}$ for some constants c_1, γ_1.

By our running hypothesis that $q(\xi)$ is not torsion, the set of such relations (trivial one included) is a free $\mathbb{Z}$ module of rank 1, and its generator (a_0, b_0) satisfies the above bound.

Consider now the non-torsion Ribet point $s(\xi)$ (so, $s^R(\xi)$ too is non-torsion), and let $(n_0 \neq 0, m_0) \in \mathbb{Z}^2$ be a generator of the group of relations $ns(\xi) - ms^R(\xi) = 0$, which is again free of rank 1. Projecting to E_0, we then have $n_0 p(\xi) - m_0 \delta q(\xi) = 0$. So, there exists $d \in \mathbb{N}$ such that $(n_0, m_0) = d.(a_0, b_0)$, and $a_0 s(\xi) - b s_0^R(\xi)$ is a torsion point of $G_{q(\xi)}$, of exact order d since (n_0, m_0) is minimal. Since it projects to 0 on E_0, it is actually a d-th root of unity ζ_d . Now, both $s(\xi)$ and $s^R(\xi)$ are defined over $k(\xi)$ (since s and s^R are global sections of $G \to X$), so ζ_d too lies in $k(\xi)$. Since ζ_d has order d, this implies that $d \leq c_2 d_\xi^{\gamma_2}$, say with $\gamma_2 = 2$.

In conclusion, for any $\xi \in \Xi$, there is a linear relation $ns(\xi) - ms^R(\xi) = 0$, with $(n, m) \in \mathbb{Z}^2, n \neq 0$ and $max(|n|, |m|) \leq c d_\xi^\gamma$ for some constants c and $\gamma = \gamma_1 + \gamma_2$. □

2.1.3. Counting relations of bounded height

In this step and the next one, we extend the scalars from $\mathbb{Q}^{alg}$ to $\mathbb{C}$, but still write $X, K = \mathbb{C}(X)$, etc, instead of $X_{\mathbb{C}}, K \otimes \mathbb{C}, \ldots$. We sometimes indicate by the exponent ^{an} the analytic object attached to an algebraic one over $\mathbb{C}$.

We now follow the usual procedure of studying the lifts to a universal covering of the relations considered in Lemma 4, and bounding their number via (generalizations of) the Pila-Wilkie theorem for a relevant o-minimal structure. There are several ways to implement this method. For instance, we can

(A) choose a fundamental domain $\mathcal{F}$ for the uniformization map unif : $\tilde{G} \simeq \mathbb{C} \rtimes (\mathbb{C} \times \tilde{X}) \to G^{an}$, and count the relations in $\tilde{G}$ when the transcendence degree over $\mathbb{C}$ of the field of definition of $(\mathrm{unif}_{|\mathcal{F}})^{-1} \circ s$ is large enough. Here, $\mathcal{F}$ is unbounded, but by work of Peterzil and Starchenko, a convenient choice allows to work in the o-minimal structure $\mathbb{R}_{an,\exp}$;

(B) or fix a simply connected domain $D \subset X^{an}$, consider the exponential morphism $\exp_G$, restricted over D, and count the relations in $(LieG)/D \simeq (\mathbb{C} \rtimes \mathbb{C}) \times D$ when the transcendence degree over $\mathbb{C}(X)$ of the field of definition of $\exp_G^{-1}(s_{|D})$ is sufficiently large. Here, D can be compact, and it suffices to work in the o-minimal structure $\mathbb{R}_{an}$.

An advantage of (A) is its impact on effectivity, as alluded to in Comment (vii) of §1.3 (see also Remark 3 of §4.3). But as in [7], §3.3, we here follow the more elementary approach (B), taking advantage of the computation of transcendence degrees already established in this paper.

So, let (D, ξ_0) be a pointed set in X^{an}, homeomorphic to a closed disk. The group scheme G/X defines an analytic family G^{an} of Lie groups over the Riemann surface X^{an}. Similarly, its relative Lie algebra $(LieG)/X$ defines an analytic vector bundle $LieG^{an}$ over X^{an}, of rank 2. We denote by Π_G the $\mathbb{Z}$-local system of periods of G^{an}/X^{an}; it is the kernel of the exponential exact sequence of analytic sheaves over X^{an} :

$$0 \longrightarrow \Pi_G \longrightarrow LieG^{an} \overset{\exp_G}{\longrightarrow} G^{an} \longrightarrow 0 \ ,.$$

For any U_0 in $Lie(G_{\xi_0}(\mathbb{C}))$ such that $\exp_{G_{\xi_0}}(U_0) = s(\xi_0) \in G_{\xi_0}(\mathbb{C})$, there exists a unique analytic section U of $Lie(G^{an})/D$ (meaning : over a neighbourhood of D), such that

$$U(\xi_0) = U_0 \ \text{ and } \ \forall \xi \in D, \exp_{G_\xi^{an}}(U(\xi)) = s(\xi).$$

Since D is fixed, we will just write $U = \log_G(s)$, although only its class modulo Π_G is well defined. Similarly, let $U^R = \log_G(s^R)$ for the Ribet section s^R. By the same process for $E_{0/X}$ (and the tacit assumption that the logarithms at ξ_0 are chosen in a compatible way), the projection $p = \pi \circ s \in E_0(X)$ admits as logarithm $\log_{E_0}(p) := u = d\pi(U)$; we also set $v = \log_{E_0}(q)$, so $d\pi(U^R) := u^R = \delta v$.

We will use the explicit expressions given in [7] for U, U^R and Π_G. These hold on any simply connected domain of X^{an} where u, v and $u + v$ do not assume period values. This is ensured by the hypothesis, made at the beginning of §2.1, that p, q and $p + q$ vanish nowhere on X.

Let $K = \mathbb{C}(X)$ be the field of rational functions of X. Since $LieG$ is a vector bundle over X, it makes sense to speak of the field of definition $K(U)$ of U over K. Similarly, let $F_G = K(\Pi_G)$ be the field of definition of Π_G. Notice that the field $F_G(U)$ now depends only on the section s. Moreover, for the Ribet section s^R, we have :

Lemma 5. *The field of definition $F^R = K(U^R)$ of any logarithm U^R of s^R coincides with the field of periods F_G of G.*

Proof. - The explicit expressions of Π_G and U^R given in [7], §A.1, show that both fields coincide with the field $K(v, \zeta(v))$, where ζ denotes the Weierstrass zeta function of the elliptic curve E_0. □

For any real number $T \geq 1$, set $\mathbb{Z}[T] = \{n \in \mathbb{Z}, |n| \leq T\}$, and consider the subset

$$\Xi[T] := \{\xi \in X(\mathbb{Q}^{alg}), \exists (n, m) \in (\mathbb{Z}[T])^2, n \neq 0, ns(\xi) - ms^R(\xi) = 0\}$$

of $\Xi = \Xi_s$. We then have :

Proposition 1. *Let D be a closed disk in X^{an}. For any $\epsilon > 0$, there exists a real number C_ϵ, depending only on X, E_0, q, s, D and ϵ, such that*

- (a) *either, for any $T \geq 1$, there are at most $C_\epsilon T^\epsilon$ points in $D \cap \Xi[T]$;*
- (b) *or the field $F_G(U)$ has transcendence degree at most 1 over the field F_G.*

The proof of Proposition 1 is given in §2.2 below, as a corollary of Habegger-Pila's "semi-rational" count [13], Corollary 7.2.

2.1.4. Logarithmic Ax

Assume that Conclusion (b) of Proposition 1 holds. Since $u = d\pi(U)$, the field $F_G(U)$ has transcendence degree at most 1 over $F_G(u)$, and :

- (b1) either u is algebraic over $F_G = K(v, \zeta(v))$, in which case we know by the Ax-Schanuel theorem on the universal vectorial extension of the elliptic curve E_0 (see for instance [7], §6, Case (SC3)) that p and q are linearly dependent over $End(E_0)$ modulo constants;

- (b2) or $U = \log_G(s)$ is algebraic over $F_G(u)$, hence over $K(u, \zeta(u), v, \zeta(v))$, in which case we know by [7], Lemma 5.1, that s is a translate of a Ribet section by a constant one, i.e. one in $\mathbb{G}_m(\mathbb{C})$ since G is not isosplit. Then, $p = \pi \circ s$ and q are linearly dependent over $End(E_0)$.

In both cases, we get a contradiction to our hypothesis that p and q are linearly independent over $End(E_0)$ modulo $E_0(\mathbb{Q}^{alg})$. So, Conclusion (a) must hold.

2.1.5. Conclusion

It follows from Lemma 3 and a compactness argument (see [21], Lemma 8.2 and the paragraph after (9.2)) that there exists a finite set of closed disks D_i in X^{an} and a constant c' such that the following holds : for any $\xi \in \Xi$, a positive proportion $\frac{1}{c'} d_\xi$ of the conjugates of ξ over k lie in one of the D_i's, say D_1. Now, all these conjugates are still in Ξ, since $\sigma(s^R(\xi)) = s^R(\sigma\xi)$ is a Ribet point of $G_{q(\sigma\xi)}$ for $\sigma \in Gal(\mathbb{Q}^{alg}/k)$. Actually, by Lemma 4, all the conjugates of ξ over k lie in $\Xi[T]$ with $T = c d_\xi^\gamma$. Choosing $\epsilon = 1/2\gamma$, we deduce from Conclusion (a) that $D_1 \cap \Xi$ has at most $c'' d_\xi^{1/2}$ (and at least $\frac{1}{c'} d_\xi$) elements. Therefore, d_ξ is bounded from above on Ξ, and this concludes the proof of Theorem 1.w.

2.2 The semi-rational count

The proof of Proposition 1 uses Betti coordinates and maps, defined as follows. We recall that $D \subset X^{an}$ is homeomorphic to a closed complex disk.

The sections of the local system Π_G over D form a $\mathbb{Z}$-module $\Pi_G(D) \subset LieG^{an}(D)$ of rank 3, with a basis $\{\varpi_0, \varpi_1, \varpi_2\}$ such that ϖ_0 generates $\Pi_{\mathbb{G}_m}(D)$, and ϖ_1, ϖ_2 project to a basis ω_1, ω_2 of $\Pi_{E_0}(D)$. Then, any logarithm $U := \log_G(s)$ of a section s of G/X over the disk D can uniquely be written as

$$U = b_0\varpi_0 + b_1\varpi_1 + b_2\varpi_2,$$

where b_0, b_1, b_2 are real analytic functions on D, with values in $\mathbb{C}$ for b_0, and in $\mathbb{R}$ for b_1 and b_2. We call (b_0, b_1, b_2) the Betti coordinates of U, and define the Betti map attached to U as

$$U_B = (b_0; b_1, b_2) : D \to \mathbb{C} \times \mathbb{R}^2,$$

Similarly, we write $U_B^R = (b_0^R; b_1^R, b_2^R)$ for the Betti map attached to $U^R = \log_G(s^R)$, and denote by $\mathcal{S}$ the image of the disk D under the map

$$\mathcal{U}_B := (U_B, U_B^R) : D \twoheadrightarrow \mathcal{S} \subset \mathbb{R}^4 \times \mathbb{R}^4 = \mathbb{R}^8.$$

We will work in the o-minimal structure $\mathbb{R}_{an}$ of globally subanalytic sets.

Lemma 6. *$\mathcal{S} = \mathcal{U}_B(D)$ is a compact 2-dimensional set, definable in the structure $\mathbb{R}_{an}$.*

Proof. - By definition (or by inspection of the formulae in [7]), the maps U_B and U_B^R extend to real analytic maps on a neighbourhood of the compact disk D. Therefore, $\mathcal{S} = \mathcal{U}_B(D)$ is a compact definable set. Furthermore, the Betti map $\pi \circ U_B^R := u_B^R = (b_1^R, b_2^R)$ attached to $u^R = \log_{E_0}(p^R)$ is an immersion (since $p^R = \delta q \in E_0(X)$ is not a constant section), so $\mathcal{S}$ is indeed a real surface. □

With this notation in mind, a point ξ of D lies in $D \cap \Xi$ if and only if

$$\exists(\nu \neq 0, \mu) \in \mathbb{Z}^2, \exists(\beta_0, \beta_1, \beta_2) \in \mathbb{Z}^3, \nu U(\xi) - \mu U^R(\xi) = \beta_0 \varpi_0(\xi) + \beta_1 \varpi_1(\xi) + \beta_2 \varpi_2(\xi),$$

or alternatively, in terms of the Betti maps :

$$\exists(\nu \neq 0, \mu) \in \mathbb{Z}^2, \exists(\beta_0, \beta_1, \beta_2) \in \mathbb{Z}^3, \nu U_B(\xi) - \mu U_B^R(\xi) = (\beta_0; \beta_1, \beta_2) \in \mathbb{Z} \times \mathbb{Z}^2 \subset \mathbb{C} \times \mathbb{R}^2.$$

Remark that

- *if $|\nu|, |\mu|$ are bounded by some number T, then* $|\beta_0|, |\beta_1|, |\beta_2| \leq C_1 T$ for some constant C_1, since D is compact;

- *given any* real *numbers* $\nu \neq 0, \mu, \beta_0, \beta_1, \beta_2$, there are only *finitely many ξ's in D such that* $\nu U_B(\xi) - \mu U_B^R(\xi) = (\beta_0; \beta_1, \beta_2)$. Otherwise, $\nu u - \mu \delta v$ would be constant on D, contradicting the Ax-Schanuel theorem invoked in §2.1.4.(b1).

We can now describe the definable set $\mathcal{Z}$ to which Habegger-Pila's semi-rational count [13] will be applied. On the one hand, we have the affine space $\mathbb{R}^5$ with real coordinates $(\nu, \mu, \beta_0, \beta_1, \beta_2)$; we will indicate by the index $_*$ the complement of the hyperplane $\nu = 0$. On the other hand, we have the affine space $\mathbb{C} \times \mathbb{R}^2 = \mathbb{R}^4$ and its square $\mathbb{R}^8$, which is the target space of the map $\mathcal{U}_B$. We consider the incidence variety $\mathcal{Z}$ in $\mathbb{R}^5 \times \mathbb{R}^8$, with projections π_1 to $\mathbb{R}^5_* \subset \mathbb{R}^5$ and π_2 to $\mathcal{S} = \mathcal{U}_B(D) \subset \mathbb{R}^8$:

$$\mathcal{Z} = \{\big((\nu, \mu, \beta_0, \beta_1, \beta_2); \big(w := (w_0; w_1, w_2), w^R := (w_0^R; w_1^R, w_2^R)\big)\big) \in \mathbb{R}^5 \times \mathcal{S} \subset \mathbb{R}^5 \times \mathbb{R}^8,$$

$$\text{such that } \nu \neq 0 \text{ and } \nu.w - \mu.w^R = (\beta_0; \beta_1, \beta_2) \in \mathbb{R} \times \mathbb{R}^2 \subset \mathbb{C} \times \mathbb{R}^2 = \mathbb{R}^4\}$$

By Lemma 6, $\mathcal{Z}$ is a definable subset of $\mathbb{R}^{13}$. Furthermore, $\mathcal{U}_B(D \cap \Xi) = \pi_2(\pi_1^{-1}(\mathbb{Z}^5_*))$.

Let $\epsilon \in \mathbb{R}_{>0}$. Given $T \geq 1$, let $\mathcal{Z}[T]$ be the subset $\pi_1^{-1}((\mathbb{Z}[T])^5_*)$ formed by those elements of $\mathcal{Z}$ whose projection to $\mathbb{R}^5_*$ have integer coordinates of height $\leq T$. By [13], Corollary 7.2 (with no $\mathbb{R}^\ell$), there is a constant C'_ϵ such that :

- (a') either $\pi_2(\mathcal{Z}[T]) \subset \mathcal{U}_B(D \cap \Xi[T]) \subset \mathcal{S}$ has less than $C'_\epsilon T^\epsilon$ elements. Recalling the two remarks above, we then deduce from an o-minimal uniformity argument (or from a zero estimate as in [7], Prop. 3.3) that for some constant C_ϵ, there are at most $C_\epsilon T^\epsilon$ points $\xi \in D \cap \Xi$ for which $\nu U(\xi) - \mu U^R(\xi) \in \Pi_{G_\xi}$ for some $(\nu \neq 0, \mu) \in (\mathbb{Z}[T])^2$. This is Conclusion (a) of Proposition 3;

- (b') or there is a definable connected curve $\mathcal{C} \subset \mathcal{Z}$ such that $\pi_1(\mathcal{C}) \subset \mathbb{R}^5_*$ is semi-algebraic and $\pi_2(\mathcal{C}) \subset \mathcal{S}$ has (real) dimension 1. Let $\mathcal{T} \subset D \subset X(\mathbb{C})$ be the inverse image of $\pi_2(\mathcal{C})$ under the map $\mathcal{U}_B$. We can view $\mathcal{C}$ as parametrized by the curve $\mathcal{T}$.

The coordinates $\mu, \nu, \beta_0, \beta_1, \beta_2; w_0, w_1, w_2, w_0^R, w_1^R, w_2^R$ on $\mathbb{R}^5 \times \mathbb{R}^8$, restricted to $\mathcal{C}$, then become functions of the (real) variable $\gamma \in \mathcal{T}$. Since $\pi_1(\mathcal{C})$ is semi-algebraic, the functions $\mu(\gamma), \nu(\gamma), \beta_0(\gamma), \beta_1(\gamma), \beta_2(\gamma)$ generate a field of transcendence degree 1 (or 0, if constant) over $\mathbb{C}$. In view of the incidence relations, *whose ν-component does not vanish* by definition, the restrictions to $\mathcal{T}$ of the functions $w_0 = b_0, w_1 = b_1, w_2 = b_2$ generate a field of transcendence degre ≤ 1 over the field generated by the restrictions to $\mathcal{T}$ of the functions $w_0^R = b_0^R, w_1^R = b_1^R, w_2 = b_2^R$. Recalling that $U = b_0\varpi_0 + b_1\varpi_1 + b_2\varpi_2$, and similarly with U^R, we deduce that $U_{|\mathcal{T}}$ generate a field of transcendence degree ≤ 1 over the field generated by $U^R_{|\mathcal{T}}$ and the $\varpi_{i|\mathcal{T}}$'s. By complex analyticity, the corresponding algebraic relation extends to D, so U generates a field of transcendence degree ≤ 1 over the field $F^R.F_G$ generated over $\mathbb{C}(X)$ by U^R and the ϖ_i's. In view of Lemma 5, this is Conclusion (b), and the proof of Proposition 1 is completed. □

3 The weakly special case over $\mathbb{Z}$

From now on, we assume that *the sections p and q* are *linearly dependent over $End(E_0)$ modulo the subgroup $E_0(\mathbb{Q}^{alg})$* of constant sections of $E_0(X)$, and look for a proof of Theorem 1. Since its statement is invariant under multiplication of s by a positive integer, and since q is not constant, we can assume without loss of generality that the generic relation they satisfy takes the form

$$p = \rho q + p_0, \text{ with } \rho \in End(E_0), p_0 \in E_0(\mathbb{Q}^{alg}), p_0 \notin E_0^{tor}(\mathbb{Q}^{alg})$$

(if p_0 is torsion, the conclusion of Theorem 1 is trivially satisfied). In such a case, the initial Step 2.1.1 of the previous proof simply does not hold : contrary to the situation of Lemma 5, the set

$$\Xi^{\mathbb{Z}\ell d}_{p,\delta q} = \{\xi \in X(\mathbb{Q}^{alg}), p(\xi) \text{ and } \delta q(\xi) \text{ are linearly dependent over } \mathbb{Z}\}$$

may well have unbounded height.

In this section, we show that if

$$\rho = r \in \mathbb{Z}, r \neq 0,$$

bounded height for $\Xi^{\mathbb{Z}\ell d}_{p,\delta q}$, hence for its subset Ξ, can still be recovered, thanks to Silverman's theorem and basic orthogonality properties of Néron-Tate pairings. Theorem 1 then follows by reproducing most of the previous proof.

3.1 Bounded height

Let again h denote the height on $X(\mathbb{Q}^{alg})$ attached to a divisor of degree 1.

Proposition 2. *Let $p, q \in E_0(X), p_0 \in E_0(\mathbb{Q}^{alg})$, q not constant, and assume that there exists a non-zero integer r such that $p = rq + p_0$. Then, there exists a constant c_0' such that $h(\xi) \leq c_0'$ for any $\xi \in \Xi^{\mathbb{Z}\ell d}_{p,\delta q}$, hence for any $\xi \in \Xi$.*

Proof. - This follows from an elementary computation, using the fact that for any $\rho \in End(E_0)$, the Néron-Tate height of $\rho q(\xi)$ is $\rho\overline{\rho}$ times that of $q(\xi)$. The following argument is based solely on orthogonality properties. Assume for a contradiction that there exists a sequence $\xi_n, n \in \mathbb{N}$, of points of $\Xi^{\mathbb{Z}\ell d}_{p,\delta q}$ whose heights $h(\xi_n)$ tend to infinity. Denote by $\langle \, , \rangle_{geo}$ the (geometric) Néron-Tate pairing on $E_0(K^{alg}) \times E_0(K^{alg})$, where $K = \mathbb{Q}^{alg}(X)$, and by $\langle \, , \rangle_{ari}$ the (arithmetic) Néron-Tate pairing on $E_0(\mathbb{Q}^{alg}) \times E_0(\mathbb{Q}^{alg})$.

Recall that for both pairings, the adjoint of $\rho \in End(E_0)$ is its complex conjugate. In particular, $\delta q(\xi) = -\overline{\delta} q(\xi)$ is orthogonal to $q(\xi)$, so $\langle p(\xi_n) \, , q(\xi_n)\rangle_{ari} = 0$ for all n. By Silverman ([24], or see [19], p. 306), we deduce that

$$\langle p \, , q\rangle_{geo} = \lim_{n\to\infty} \frac{\langle p(\xi_n) \, , q(\xi_n)\rangle_{ari}}{h(\xi_n)} = 0.$$

Now, $p = rq + p_0$, and the constant part $E_0(\mathbb{Q}^{alg})$ is orthogonal to the full space $E_0(K^{alg})$ for the geometric pairing. So

$$\langle p \, , q\rangle_{geo} = \langle rq \, , q\rangle_{geo} + \langle p_0 \, , q\rangle_{geo} = r\langle q \, , q\rangle_{geo} \text{ with } r \neq 0.$$

Therefore, the section q has vanishing Néron-Tate height, hence must be constant, contrary to our hypothesis. □

3.2 Algebraic (in)dependence

Assuming that $p = rq + p_0$ as above, we now follow the proof of §2.1. All its steps go through, except that Conclusion (b) of Proposition 1 is now automatically satisfied. Indeed, we have $u = rv + u_0$, where $u_0 \in LieE_0(\mathbb{C})$ is a conveniently chosen elliptic logarithem of p_0, so $K(u)$ lies in the field $K(v) \subset F_G$, and automatically, $U = \log_G(s)$ generates a field of transcendence degree at most 1 over F_G.

To overcome this difficulty, we will now deduce from the generic relation $p = rq + p_0$ that Conclusion (b) can here be replaced by the more precise

- (b$^\sharp$) or the field $F_G(U)$ is algebraic over the field $F_G(u) = F_G$

(which is actually Conclusion (b2) of §2.1.4).

To check this, we use the same incidence variety $\mathcal{Z}$ as in §2.2, and follow Alternative (b') of the discussion. Notice that any relation $\nu U(\xi) - \mu U^R(\xi) = \beta_0\varpi_0(\xi) + \beta_1\varpi_1(\xi) + \beta_2\varpi_2(\xi)$, projected to $LieE_0$, yields $\nu u(\xi) - \mu u^R(\xi) = \beta_1\omega_1 + \beta_2\omega_1$ hence since $u^R = \delta v$:

$$(\nu\, r - \mu\delta)v(\xi) = \beta_1\omega_1 + \beta_2\omega_2 - \nu u_0.$$

Restricting this relation to the real curve $\mathcal{T} \subset \mathcal{D}$, and recalling that $\nu \neq 0, r \neq 0$ and $\delta \notin \mathbb{R}$, we deduce that if Alternative (b') holds, then the field generated over $\mathbb{C}$ by the restriction of the function v to $\mathcal{T}$ lies in the field generated over $\mathbb{C}$ by the restriction to $\mathcal{T}$ of the real functions μ, ν and the β_i's, $i = 1, 2$. Since the latter field has transcendence degree at most 1 over $\mathbb{C}$, while v is not constant, the two fields have the same algebraic closure, in which u lies. The full incidence relation then implies that U is algebraic over the field $F^R.F_G(u) = F_G$. This is Conclusion (b$^\sharp$).

So, $\log_G(s)$ is algebraic over F_G. As explained in case (b2) of §2.1.4, Lemma 5.1 of [7] then implies that p and q are linearly dependent over $End(E_0)$ and Theorem 1 is established in this "$\rho = r \in \mathbb{Z}, r \neq 0$-weakly special" case. □

4 End of proof of Theorem 1

4.1 From weakly special to constant

In this subsection, we assume that the projection $p \in E_0(X)$ of $s \in G(X)$ and the section $q \in E_0(X)$ are linked by a generic relation of arbitrary shape :

$$p = \rho q + p_0, \text{ with } \rho \in End(E_0), p_0 \in E_0(\mathbb{Q}^{alg}).$$

We will deduce from the previous section that either p and q are linearly dependent over $End(E_0)$ (as predicted by Theorem 1), or we may assume that $\rho = 0$, i.e. p itself is a constant section.

Replacing s by $2s$ if necessary, we can write $\rho = r + r'\delta \in \mathbb{Z} \oplus \mathbb{Z}\delta \subset End(E_0)$, and consider the basic Ribet section $s_{r'\alpha} = r's^R$ of $G = G_q$ over X. Its projection to $E_0(X)$ is the section $r'p^R = r'\delta q$. Therefore, the section $s' := s - s_{r'\alpha}$ of G/X projects to

$$\pi(s') := p' = p - r'\delta q = rq + p_0.$$

Moreover, for any $\xi \in X(\mathbb{Q}^{alg})$, $s_{r'\alpha}(\xi) = s_{r'\alpha, q(\xi)}$ is by definition a Ribet point of $G_{q(\xi)}$. Consequently, the set $\Xi := \Xi_s$ of points of $X(\mathbb{Q}^{alg})$ where $s(\xi)$ is a Ribet point coincides with the set $\Xi_{s'}$ similarly attached to s', which is therefore infinite. Since $r \in \mathbb{Z}$, we deduce from the result of §3 that either p' and q, hence p and q, are linearly dependent over $End(E_0)$, or that $r = 0$.

Assume now that $r = 0$, so the generic relation reads : $p = r'\delta q + p_0$, and consider again the section $s' = s - r's^R$, which projects to $p' = p_0$. The corresponding set $\Xi_{s'}$ is still infinite. Therefore, we have reduced the proof of Theorem 1 to the case where $\rho = 0$, i.e. where the projection p of s is a constant section p_0. We must then show that p_0 is necessarily a torsion point.

4.2 The constant case

The word constant here refers not to the semi-abelian scheme G/X, which we still assume to be non constant ($q \notin E_0(\mathbb{Q}^{alg})$), but to the section $\pi \circ s := p = p_0 \in E_0(\mathbb{Q}^{alg})$. However, the duality properties of the Poincaré bi-extension $\mathcal{P}_0$ of $E_0 \times \hat{E}_0$ by $\mathbb{G}_m$ enable us to permute the roles of q and p, thereby translating the problem into one on the constant semi-abelian variety $G'_{p_0} = \mathcal{P}_{0|p_0 \times \hat{E}_0} \in Ext(\hat{E}_0, \mathbb{G}_m)$ parametrized by the point p_0 of (the bidual of) E_0 . We must then prove that p_0 is torsion, i.e. that G'_{p_0} is isosplit.

Assume for a contradiction that p_0 is not torsion. Then for each ξ in the set Ξ, there is a relation $np_0 - m\delta q(\xi) = 0$ with $nm \neq 0$, so $q(\xi)$ lies in the divisible hull of $\mathbb{Z}.\delta p_0$, and is not torsion either. Consider the constant semi-abelian surface $G'_{p_0} \in Ext(\hat{E}_0, \mathbb{G}_m)$. By duality, we can view s as a section $\check{s} \in G'_{p_0}(X)$, and $s(\xi)$ as a point $\check{s}(\xi)$ on G'_{p_0} projecting to $q(\xi)$ in $\hat{E}_0$. Furthermore, $\check{s}(\xi)$ is a non torsion Ribet point of G'_{p_0} if and only if $s(\xi)$ is a non torsion Ribet point of $G_{q(\xi)}$: in the setting of §1.2, this is clear when $\varphi - \hat{\varphi}$ is an isomorphism, and it remains true in general via an isogeny. (In fact, it is proven in [6], Remark 5.4.1, that the 1-motive attached to $s_{\varphi,q}$ is isogenous to its Cartier dual as soon as $\varphi - \hat{\varphi}$ is an isogeny.)

Therefore, the image $\check{s}(X)$ of $\check{s}$ is an irreducible curve in G'_{p_0} which contains infinitely many points of the group Γ'_0 formed by all the Ribet points of G'_{p_0}. Since this group has finite rank (at most 1), McQuillan's Mordell-Lang theorem [20] , as recalled in §1.3.(v), can be applied to G'_{p_0}. We derive that $\check{s}$ factors through a translate by a Ribet point of a strict connected algebraic subgroup of G'_{p_0}. Since p_0 is not torsion, the only such one is $\mathbb{G}_m$, so $q(X)$ reduces to a point of $\hat{E}_0$. This contradicts our assumption that q is not constant, and concludes the proof of Theorem 1. □

4.3 Further comments

We here list properties of Ribet points and sections which although not used in the proof, may be relevant to further studies of unlikely intersections.

Remark 1 (in relation with Proposition 2) : attached to the divisor at infinity D_ξ of the standard compactification of $G_{q(\xi)}$, there is a canonical "relative height" $\hat{h}_{D_\xi}$, which vanishes on the Ribet points of $G_{q(\xi)}$ (cf. [3], §3). Is there a Zimmer-like comparison of $\hat{h}_{D_\xi}$ with a Weil height h_{D_ξ}, of the type $\hat{h}_{D_\xi} - h_{D_\xi} = O\big((\hat{h}(q(\xi)))^{1/2}\big)$, or even just $o\big(\hat{h}(q(\xi))\big)$, where $\hat{h}$ is the Néron-Tate height on $\hat{E}_0(\mathbb{Q}^{alg})$? Bounded height on Ξ would then follow in all cases, "weakly special" or not. See [9], Thm 5.5, for an Arakelov approach to this problem.

Remark 2 (on the Betti maps) : let $\xi \in \Xi$. By [3], Thm. 4, the Ribet point $s(\xi)$ lies in the maximal compact subgroup of its fiber G^{an}_ξ. So its logarithm $U(\xi)$ lies in $\Pi_{G_\xi} \otimes \mathbb{R}$, and its Betti coordinate $b_0(\xi)$ is a real number. Similarly, the Betti coordinate b^R_0 of the Betti map U^R_B attached to $U^R = \log_G(s^R)$ is actually real-valued. But a priori, not the Betti coordinate b_0 of U. It would be interesting to characterize the sections $s \in G(X)$ whose images meet the union of the maximal compact subgroups of all the fibers infinitely often.

Remark 3 (about effectivity) : as suggested in §2.1.3.(A) (see also §1.3.(vii)), making the "constants" of the text effective in terms of the initial datas X, E_0, q, s, requires a global version of Proposition 1. One should here start with the uniformization map Unif $: \tilde{\mathcal{P}}_0 \simeq \mathbb{C} \rtimes (\mathbb{C} \times \mathbb{C}) \to \mathcal{P}^{an}_0$ of the Poincaré bi-extension itself, thereby reflecting the symmetric roles played by p and q in the construction of Ribet sections. As far as the dependence in s is concerned, a first aim would be to show that these constants are uniformly bounded in terms of the degree of the curve $W = s(X)$ in a projective embedding of G. We point out that this aim has indeed been reached in various versions of the Mordell-Lang problem itself: see [14] for a differential algebraic approach (inspired by work of Buium, and recently sharpened in [8]), and [23], Theorem 2.4 for the general case.

5 The Zilber-Pink conjecture for $\mathcal{P}_0$

Pink's generalization of the conjectures on unlikely intersections proposed by Bombieri, Masser, Zannier and by Zilber asserts :

Conjecture. *([22], Conjecture 1.3). Let $\mathcal{S}/\mathbb{C}$ be a mixed Shimura variety, and let W be an irreducible algebraic subvariety of $\mathcal{S}$, of dimension d. Assume that the intersection of W with the union of all the special subvarieties of $\mathcal{S}$ of codimension $> d$ is Zariski dense in W. Then, W is contained in a special subvariety of $\mathcal{S}$ of positive codimension.*

As in the text, let again $E_0/\mathbb{Q}^{alg}$ be an elliptic curve with complex multiplications, with dual $\hat{E}_0 \simeq Ext(E_0, \mathbb{G}_m)$, and let $\mathcal{P}_0/\mathbb{Q}^{alg}$ be the Poincaré bi-extension of $E_0 \times \hat{E}_0$ by $\mathbb{G}_m$. This is a $\mathbb{G}_m$-torsor over $E_0 \times \hat{E}_0$, which admits two families of group laws. Namely, for any $q \in \hat{E}_0$, the restriction of $\mathcal{P}_0$ above $E_0 \times \{q\}$ is the semi-abelian variety attached to q, viewed as a point in $Ext(E_0, \mathbb{G}_m)$, while for any $p \in E_0$, the restriction of $\mathcal{P}_0$ above $\{p\} \times \hat{E}_0$ is the semi-abelian variety attached to p, viewed by biduality as a point in $Ext(\hat{E}_0, \mathbb{G}_m) \simeq E_0$. The important point in this section is that $\mathcal{P}_0$ admits a canonical structure of a mixed Shimura variety, which is described in detail in [6]. However, only a minimal knowledge of MSV theory will be needed to prove Theorem 3 of the introduction.

Before proving this theorem, we note (as pointed out by J. Pila) that it completely establishes Theorem 4, i.e. Pink's conjecture for the MSV $\mathcal{S} = \mathcal{P}_0$ when the variety W is defined over $\mathbb{Q}^{alg}$. Indeed, if the dimension d of W is 0 or 3, there is nothing to prove. If $d = 2$, then the special subvarieties of $\mathcal{P}_0$ of codimension $> d$ are its special points, and the statement reduces to the André-Oort conjecture, which follows in this case from [11], Theorem 13.6. So, only the case $d = 1$, i.e. Theorem 3, needs to be treated.

Through the first familiy of group laws above, the projection $\varpi : \mathcal{P}_0 \to \hat{E}_0$ turns $\mathcal{P}_0$ into the universal extension $\mathcal{G}$ of E_0 by $\mathbb{G}_m$, over the moduli space $\hat{E}_0$. For any integer n, we will denote by $[n]_{\mathcal{G}}$ the morphism of multiplication by n of the group scheme $\mathcal{G}/\hat{E}_0$. Its Ribet sections are well-defined, and we call their images *Ribet curves of $\mathcal{P}_0$, in the sense of $\mathcal{G}/\hat{E}_0$*. Similarly, the projection $\varpi' : \mathcal{P}_0 \to E_0$ turns $\mathcal{P}_0$ into a group scheme $\mathcal{G}'/E_0$, with morphisms $[n]_{\mathcal{G}'}$ and *Ribet curves of $\mathcal{P}_0$, in the sense of $\mathcal{G}'/E_0$*. Furthermore, $[n]_{\mathcal{G}}$ and $[n]_{\mathcal{G}'}$ induce the same morphism $[n]$ on the fiber $\mathbb{G}_m$ of (ϖ, ϖ') above $(0, 0)$. With these definitions in mind, we have the following explicit necessary conditions for an irreducible curve to be special in $\mathcal{P}_0$. It follows from [6], §5 (see also [4], §2) that they are also sufficient, but we will not need this sharper result.

Proposition 3. *Let C be a special curve of the MSV $\mathcal{P}_0$. Then,*

i) if $\varpi : C \to \hat{E}_0$ is dominant, C is a Ribet curve in the sense of $\mathcal{G}/\hat{E}_0$;

ii) if $\varpi' : C \to E_0$ is dominant, C is a Ribet curve in the sense of $\mathcal{G}'/E_0$;

iii) if $(\varpi', \varpi)(C)$ is a point (p_0, q_0) of $E_0 \times \hat{E}_0$, this point is a torsion point, and C is the fiber of $\mathcal{P}_0$ above (p_0, q_0).

Notice that most special curves C satisfy both (i) and (ii), and are therefore Ribet curves in both senses. This reflects the self-duality of non torsion Ribet sections, already encountered in §4.2. As for (iii), it occurs if neither (i) nor (ii) are satisfied.

Proof. - We will use the following facts, for which we refer to [22], [10].

(F1) : a point P of $\mathcal{P}_0$ is special (if and) only if $(p,q) = (\varpi', \varpi)(P)$ is torsion in $E_0 \times \hat{E}_0$ and P is torsion in the (isosplit) extension $\mathcal{G}_q$ (equivalently, in the isosplit $\mathcal{G}'_p$).

(F2) a special curve of $\mathcal{P}_0$ contains a Zariski-dense set of special points, hence by F1 a Zariski-dense set of torsion points of the various fibers of $\mathcal{G}/\hat{E}_0$ (or of $\mathcal{G}'/E_0$).

(F3) : the image of a special subvariety under a Shimura morphism (such as $\varpi', \varpi, [n]_{\mathcal{G}}, [n]_{\mathcal{G}'}$) is a special subvariety.

Let then $C \subset \mathcal{P}_0 = \mathcal{G}$ be a special curve, dominating $\hat{E}_0$ as in (i). By base extension along the finite cover $\varpi : X := C \to \hat{E}_0$, we can view the diagonal map $X \to C_X$ as a section s of the group scheme $G = \mathcal{G}_X := \mathcal{G} \times_{\hat{E}_0} X$ over X. We can now apply Lemma 2 of §1.3 (relative Manin-Mumford) to $s \in G(X)$: by Facts F1 and F2, the set Ξ_s^{tor} is infinite and we infer that s is a Ribet section of G/X, or factors through a torsion translate of $\mathbb{G}_{m/X} = \mathbb{G}_m \times X$. In the first case, the image $C \subset \mathcal{G}$ of $s(X) \subset C_X \subset \mathcal{G}_X$ is a Ribet curve of $\mathcal{P}_0$ in the sense of $\mathcal{G}/\hat{E}_0$, as was to be shown.

In the second case, a multiple $C' := [n]_{\mathcal{G}}(C)$ of C lies in the fiber $\mathbb{G}_m \times \hat{E}_0$ of $\mathcal{P}_0$ above $p = 0$, and is still a special curve of $\mathcal{P}_0$ by F3. So, by F2, C' contains infinitely many special points of $\mathcal{P}_0$ lying in $\mathbb{G}_m \times \hat{E}_0$. But by F1, these special points are contained in (in fact, fill up) the torsion of the group $\mathbb{G}_m \times \hat{E}_0$. We can now apply the standard Manin-Mumford theorem [12] to $C' \cap (\mathbb{G}_m \times \hat{E}_0)^{tor}$, and deduce that C' is a torsion translate of $\mathbb{G}_m \times \{0\}$ or of $\{1\} \times \hat{E}_0$. The first conclusion cannot occur since C' too dominates $\hat{E}_0$. So, a multiple $[m]C' = [mn]_{\mathcal{G}}(C)$ of C is the image of the unit section of $\mathcal{G}/\hat{E}_0$. Therefore, C is in all cases a Ribet curve of $\mathcal{P}_0$ in the sense of $\mathcal{G}/\hat{E}_0$.

The same proof applies to (ii), while (iii) easily follows from F1 (or from F3, in view of [10]). This concludes the proof of Proposition 3. □

We can now turn to the proof of Theorem 1. We will need the following complement to Fact F3.

(F4) : under a Shimura morphism, the irreducible components of the inverse image of a special subvariety are special subvarieties.

So, let $W/\mathbb{Q}^{alg}$ be an irreducible algebraic curve in $\mathcal{P}_0$, which contains infinitely many points lying on special curves of $\mathcal{P}_0$. We must show that W is contained in a special surface of $\mathcal{P}_0$. We deduce from Proposition 3 that

(a) : W contains infinitely many points lying on Ribet curves in the sense of $\mathcal{G}/\hat{E}_0$, and if not,

(b) : ditto in the sense of $\mathcal{G}'/E_0$, and if not,

(c) : ditto with the fibers of $\mathcal{P}_0$ above the torsion points of $E_0 \times \hat{E}_0$.

Assume first that $\varpi : W \to \hat{E}_0$ is dominant, and that we are in Case (a). Base changing along $\varpi : X = W \to \hat{E}_0$ as above, we may view the diagonal map $X \to W_X \subset \mathcal{G}_X = G$ as a section $s \in G(X)$, to which Theorem 1 (or the relative Mordell-Lang Theorem 2) of §1.1 applies. By (a), the set Ξ_s is infinite, and we infer that the sections p and q attached to s are linearly dependent over $End(E_0)$. So, $(\varpi', \varpi)(W)$ lies in a torsion translate of an elliptic curve $B \subset E_0 \times \hat{E}_0$ passing through 0. By [10], these are special curves of the MSV $E_0 \times \hat{E}_0$. Therefore, by F4, W lies in a special surface of $\mathcal{P}_0$. Vice versa, the same conclusion holds if $\varpi' : W \to E_0$ is dominant and we are in Case (b).

Secondly, assume that W still dominates $\hat{E}_0$, but that we are in Case (b). As just pointed out, we can then assume that W does not dominate E_0, and so, projects to a point $p \in E_0$ under ϖ'. If p is not torsion, W lies in the non isosplit extension $\mathcal{G}'_p = \varpi'^{-1}(p)$ (which is then not a special surface of $\mathcal{P}_0$). Now, the Ribet curves in the sense of $\mathcal{G}'/E_0$ meet $\mathcal{G}'_p$ at Ribet points of $\mathcal{G}'_p$, so by (b), W contains infinitely many Ribet points of $\mathcal{G}'_p$. We deduce from the standard Mordell-Lang theorem [20] that W lies in a translate of $\mathbb{G}_m$ by a Ribet point. But then, W cannot dominate $\hat{E}_0$. So, p is a torsion point, and W lies in $\varpi'^{-1}(p)$, which is a special surface of $\mathcal{P}_0$ by F4. Vice versa, the same conclusion holds if $\varpi' : W \to E_0$ is dominant and we are in Case (a).

Thirdly, assume that W dominates $\hat{E}_0$ or E_0, and that we are in Case (c). Then, the projection W' of W in $E_0 \times \hat{E}_0$ is a curve which contains infinitely many torsion points of $E_0 \times \hat{E}_0$. By Manin-Mumford, we deduce that W' lies in a torsion translate of an elliptic curve $B \subset E_0 \times \hat{E}_0$ passing through 0. So, W lies in a special surface of $\mathcal{P}_0$.

It remains to study the case when W projects to a point (p, q) of $E_0 \times \hat{E}_0$ under (ϖ', ϖ). Then, the only special curve of type (c) which meets W is the closure of W itself, so in Case (c), (p, q) is a torsion point, and W lies in (many) a special surface of $\mathcal{P}_0$. Assume finally that we are in Case (a), or in Case (b). Then, W contains a Ribet point of $\mathcal{G}_q$, or of $\mathcal{G}'_p$, projecting to $p \in E_0$, or to $q \in \hat{E}_0$. In both cases, we deduce that the points p and q are linearly dependent over $End(E_0)$. So, the projection to $E_0 \times \hat{E}_0$ of W lies in a torsion translate of an elliptic curve B passing through 0, and W lies in a special surface of $\mathcal{P}_0$. This concludes the proof of Theorem 3. □

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Authors' addresses :

D.B.: daniel.bertrand@imj-prg.fr
Sorbonne Université, IMJ-PRG, Case 247, 75 252 Paris Cedex 05, France.

H.S.: harry.schmidt@manchester.ac.uk
School of Mathematics, The University of Manchester, Manchester M13 9PL, UK.

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