Some uniform estimates for scalar curvature type equations

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SOME UNIFORM ESTIMATES FOR SCALAR CURVATURE TYPE EQUATIONS

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We consider the prescribed scalar curvature equation on an open set Ω of $\mathbb{R}^n$, $-\Delta u = V u^{(n+2)/(n-2)} + u^{n/(n-2)}$ with $V \in C^{1,\alpha}$ ($0 < \alpha \leq 1$), and we prove the inequality $\sup_K u \times \inf_\Omega u \leq c$ where K is a compact set of Ω.

In dimension 4, we have an idea on the supremum of the solution of the prescribed scalar curvature if we control the infimum. For this case we suppose the scalar curvature $C^{1,\alpha}$ ($0 < \alpha \leq 1$).

1. Introduction and main result

In our work, we denote by $\Delta = \nabla^i \nabla_i$ the Laplace–Beltrami operator in dimension $n \geq 2$.

Without loss of generality, we suppose $\Omega = B = B_2(0)$ the ball of radius 2 centered at 0 of $\mathbb{R}^n$.

Here, we study some a priori estimates of type $\sup \times \inf$ for a perturbed prescribed scalar curvature equation in all dimensions $n \geq 4$.

We have a counterexample to the sharp $\sup \times \inf$ inequality for the prescribed scalar curvature [Chen and Lin 1997, Proposition 4.3]. In our work the perturbation by a subcritical term is a sufficient condition to obtain such an inequality.

The $\sup \times \inf$ inequality is characteristic of those equations as the usual Harnack inequalities are for harmonic functions.

Note that the prescribed scalar curvature equation was studied a lot. We can find — see, for example, [Aubin 1998; Bahoura 2004; Brezis and Merle 1991; Brezis et al. 1993; Chen and Lin 1997; 1998; Li 1993; 1995; 1996; 1999; Li and Shafrir 1994; Li and Zhang 2004; Li and Zhu 1999; Shafrir 1992] — many results about uniform estimates in dimensions $n = 2$ and $n \geq 3$.

In dimension 2, the corresponding equation is

$$(E_0) \qquad -\Delta u = V e^u.$$

Note that Shafrir [1992] obtained an inequality of type $\sup u + C \inf u < c$ with only an L^∞ assumption on V.

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To obtain exactly the estimate $\sup u + \inf u < c$, Brezis, Li and Shafrir [Brezis et al. 1993] assumed that the prescribed scalar curvature V is Lipschitz continuous. Later, Chen and Lin [1998] proved that, if V is uniformly Hölder continuous, we can obtain a $\sup + \inf$ inequality.

In dimension $n \geq 3$, the prescribed curvature equation on general manifold M is

$$(E_0') \qquad -\Delta u + R_g u = V u^{(n+2)/(n-2)}.$$

When $M = \mathbb{S}_n$, Li [1993; 1995; 1996] proved a priori estimates for the solutions of the previous equation. He used the notion of simple isolated points and some flatness conditions on V.

If we suppose $n = 3, 4$, we can find in [Li and Zhang 2004; Li and Zhu 1999] a uniform bound for the energy and a $\sup \times \inf$ inequality. Note that Li and Zhu [1999] proved the compactness of the solutions to the Yamabe problem using the positive mass theorem.

In [Bahoura 2004], we can see (on a bounded domain of $\mathbb{R}^4$) that we have a uniform estimate for the solutions of (E_0') ($n = 4$ and Euclidean case) by assuming that those solutions are bounded below by a positive constant; in this case we have assumed that the prescribed scalar curvature V is only Lipschitz.

Here we extend some result of [Bahoura 2004] to equations with nonlinear terms or with minimal condition on the prescribed scalar curvature.

For the Euclidean case, Chen and Lin [1997] got some a priori estimates for general equations

$$(E_0'') \qquad -\Delta u = V u^{(n+2)/(n-2)} + g(u)$$

with some assumption on g and the Li-flatness conditions on V.

Here, we give some a priori estimates with some minimal conditions on the prescribed curvature, for perturbed scalar curvature equation, in all dimensions $n \geq 4$.

In our work, we use the *blow-up* analysis, the *moving-plane* method and a flatness condition (of order 1) for the prescribed scalar curvature. Note that the flatness condition which we use is also obtained by a *moving-plane* argument of Chen and Lin [1997]. The method of moving plane was developed in particular by Gidas, Ni and Nirenberg [Gidas et al. 1979] and Serrin [1971].

First, consider the equation

$$(E_1) \qquad -\Delta u = V u^{(n+2)/(n-2)} + u^{n/(n-2)}$$

with $0 < a \leq V(x) \leq b$ and $\|V\|_{C^{1,\alpha}} \leq A$, $0 < \alpha \leq 1$.

We have:

Theorem 1. *For all $a, b, A, \alpha > 0$ $(0 < \alpha \leq 1)$, and all compact sets K of Ω of dimension $n \geq 4$, there is a positive constant $c = c(a, b, A, \alpha, K, \Omega, n)$ such that*

$$\sup_K u \times \inf_\Omega u \leq c$$

for all solutions u of (E_1) relative to V.

Now, if we suppose $V \in C^1(\Omega)$ and $V \geq a > 0$, we have:

Theorem 2. *For all $a > 0$, V and all compact K of Ω of dimension $n \geq 4$, there is a positive constant $c = c(a, V, K, \Omega, n)$ such that*

$$\sup_K u \times \inf_\Omega u \leq c$$

for all solutions u of (E_1) relative to V.

Now, we suppose $n = 4$, and we consider the equation (prescribed scalar curvature equation)

$$(E_2) \qquad -\Delta u = V u^3 \quad \text{on } \Omega \subset \mathbb{R}^4$$

with $0 < a \leq V(x) \leq b$ and $\|V\|_{C^{1,\alpha}} \leq A$, $0 < \alpha \leq 1$. We have:

Theorem 3. *For all $a, b, m, A, \alpha > 0$ $(0 < \alpha \leq 1)$ and all compact K of Ω, there is a positive constant $c = c(a, b, m, A, \alpha, K, \Omega)$ such that*

$$\sup_K u \leq c \quad \textit{if} \quad \min_\Omega u \geq m$$

for all solutions u of (E_2) relative to V.

If we suppose $n = 4$ and $V \in C^1(\Omega)$ and $V \geq a > 0$ on Ω, we have:

Theorem 4. *For all $a, m > 0$, $V \in C^1(\Omega)$ and all compact $K \in \Omega$, there is a positive constant $c = c(a, m, V, K, \Omega)$ such that*

$$\sup_K u \leq c \quad \textit{if} \quad \min_\Omega u \geq m$$

for all solutions u of (E_2) relative to V.

2. Proofs of the theorems

Proof of Theorems 1 and 2.

Proof of Theorem 1. Without loss of generality, we suppose $\Omega = B_1$ the unit ball of $\mathbb{R}^n$. We want to prove an a priori estimate around 0.

Let (u_i) and (V_i) be sequences of functions on Ω such that

$$-\Delta u_i = V_i {u_i}^{(n+2)/(n-2)} + u_i^{n/(n-2)}, \quad u_i > 0,$$

with $0 < a \leq V_i(x) \leq b$ and $\|V_i\|_{C^{1,\alpha}} \leq A$.

We argue by contradiction, and we suppose that the sup $\times$ inf is not bounded.

We have that for all c, $R>0$ there exists $u_{c,R}$ a solution of (E_1) such that

$$(H) \qquad R^{n-2}\sup_{B(0,R)} u_{c,R}\times \inf_M u_{c,R}\geq c.$$

Proposition (blow-up analysis). *There is a sequence of points $(y_i)_i$, $y_i\to 0$, and two sequences of positive real numbers $(l_i)_i$ and $(L_i)_i$ (see below), $l_i\to 0$ and $L_i\to+\infty$, such that, if we set $v_i(y)=u_i(y+y_i)/u_i(y_i)$, we have*

$$0<v_i(y)\leq \beta_i\leq 2^{(n-2)/2},$$

$$\beta_i\to 1,$$

$$v_i(y)\to\left(\frac{1}{1+|y|^2}\right)^{(n-2)/2} \text{ uniformly on all compact sets of } \mathbb{R}^n,$$

$$l_i^{(n-2)/2}u_i(y_i)\times\inf_{B_1}u_i\to+\infty,$$

Proof. We use the hypothesis (H); we take two sequences $R_i>0$, $R_i\to 0$, and $c_i\to+\infty$ such that

$$R_i{}^{(n-2)}\sup_{B(0,R_i)} u_i\times\inf_{B_1}u_i\geq c_i\to+\infty.$$

Let $x_i\in B(x_0,R_i)$ be a point such that $\sup_{B(0,R_i)}u_i=u_i(x_i)$ and $s_i(x)=(R_i-|x-x_i|)^{(n-2)/2}u_i(x)$, $x\in B(x_i,R_i)$. Then $x_i\to 0$.

We have

$$\max_{B(x_i,R_i)} s_i(x)=s_i(y_i)\geq s_i(x_i)=R_i{}^{(n-2)/2}u_i(x_i)\geq\sqrt{c_i}\to+\infty.$$

We set

$$l_i=R_i-|y_i-x_i|,\qquad \bar{u}_i(y)=u_i(y_i+y),\qquad v_i(z)=\frac{u_i[y_i+(z/[u_i(y_i)]^{2/(n-2)})]}{u_i(y_i)}.$$

Clearly, we have $y_i\to x_0$.

We take

$$L_i=\frac{l_i}{(c_i)^{1/2(n-2)}}[u_i(y_i)]^{2/(n-2)}=\frac{[s_i(y_i)]^{2/(n-2)}}{c_i^{1/2(n-2)}}\geq\frac{c_i^{1/(n-2)}}{c_i^{1/2(n-2)}}=c_i^{1/2(n-2)}\to+\infty.$$

If $|z|\leq L_i$, then $y=[y_i+z/[u_i(y_i)]^{2/(n-2)}]\in B(y_i,\delta_i l_i)$ with $\delta_i=1/(c_i)^{1/2(n-2)}$ and $|y-y_i|<R_i-|y_i-x_i|$; thus, $|y-x_i|<R_i$ and $s_i(y)\leq s_i(y_i)$. We can write

$$u_i(y)(R_i-|y-y_i|)^{(n-2)/2}\leq u_i(y_i)(l_i)^{(n-2)/2}.$$

But $|y-y_i| \le \delta_i l_i$, $R_i > l_i$ and $R_i - |y-x_i| \ge R_i - |x_i-y_i| - \delta_i l_i > l_i - \delta_i l_i = l_i(1-\delta_i)$. We obtain

$$0 < v_i(z) = \frac{u_i(y)}{u_i(y_i)} \le \left[\frac{l_i}{l_i(1-\delta_i)}\right]^{(n-2)/2} \le 2^{(n-2)/2}.$$

We set $\beta_i = (1/(1-\delta_i))^{(n-2)/2}$; clearly, we have $\beta_i \to 1$.

The function v_i satisfies

$$-\Delta v_i = \widetilde{V}_i v_i{}^{(n+2)/(n-2)} + \frac{v_i^{n/(n-2)}}{[u_i(y_i)]^{2/(n-2)}},$$

where $\widetilde{V}_i(y) = V_i[y + y/[u_i(y_i)]^{2/(n-2)}]$. Without loss of generality, we can suppose that $\widetilde{V}_i \to V(0) = n(n-2)$.

We use the elliptic estimates of the Ascoli and Ladyzhenskaya theorems to have the uniform convergence of (v_i) to v on a compact set of $\mathbb{R}^n$. The function v satisfies

$$-\Delta v = n(n-2)v^{N-1}, \qquad v(0) = 1, \qquad 0 \le v \le 1 \le 2^{(n-2)/2}, \qquad N = \frac{2n}{n-2}.$$

By the maximum principle, we have $v > 0$ on $\mathbb{R}^n$. If we use the result of Caffarelli, Gidas and Spruck [Caffarelli et al. 1989], we obtain $v(y) = (1/(1+|y|^2))^{(n-2)/2}$. We have the same properties as in [Bahoura 2004]. $\square$

Remark. When we use the convergence on compact sets of the sequence (v_i), we can take an increasing sequence of compact sets and we see that we can obtain a sequence (ϵ_i) such that $\epsilon_i \to 0$ and after we choose $(\widetilde{R}_i)$ such that $\widetilde{R}_i \to +\infty$ and finally

$$\widetilde{R}_i^{n-2} \|v_i - v\|_{B(0,\widetilde{R}_i)} \le \epsilon_i.$$

We can say that we are in the case of [Chen and Lin 1997, step 1 of the proof of Theorem 1.2].

Fundamental point (a consequence of the blow-up). According to the work of Chen and Lin [1997, step 2 of the proof of Theorem 1.3], in the blow-up point, the prescribed scalar curvature V is such that

$$(P_0) \qquad \lim_{i\to+\infty} |\nabla V_i(y_i)| = 0.$$

Polar coordinates (moving-plane method). Now we must use the same method as in [Bahoura 2004, Theorem 1]. We will use the moving-plane method.

We must prove [Bahoura 2004, Lemma 2].

We set $t \in]-\infty, -\log 2]$ and $\theta \in \mathbb{S}_{n-1}$:

$$w_i(t,\theta) = e^{(n-2)t/2} u_i(y_i + e^t\theta) \quad \text{and} \quad \overline{V}_i(t,\theta) = V_i(y_i + e^t\theta).$$

We consider the operator $L = \partial_{tt} + \Delta_\sigma - (n-2)^2/4$, with Δ_σ the Laplace–Beltrami operator on $\mathbb{S}_{n-1}$.

The function w_i satisfies

$$-Lw_i = \overline{V}_i {w_i}^{N-1} + e^t \times {w_i}^{n/(n-2)}, \quad N = \frac{2n}{n-2}.$$

Remark. Here w_i is a solution to the previous equation with a perturbed term which contains e^t. The term e^t is fundamental in the computations; it corrects the variation of V_i.

For $\lambda \leq 0$, we set

$$t^\lambda = 2\lambda - t, \qquad w_i^\lambda(t, \theta) = w_i(t^\lambda, \theta), \qquad \overline{V}_i^\lambda(t, \theta) = \overline{V}_i(t^\lambda, \theta).$$

First, like in [Bahoura 2004], we have the following lemma.

Lemma 5. *Let A_λ be the property*

$$A_\lambda = \{\lambda \leq 0 \mid \textit{there exists } (t_\lambda, \theta_\lambda) \in \,]\lambda, t_i] \times \mathbb{S}_{n-1},\ {w_i}^\lambda(t_\lambda, \theta_\lambda) - w_i(t_\lambda, \theta_\lambda) \geq 0\}.$$

Then there is $\nu \leq 0$ such that, for $\lambda \leq \nu$, A_λ is not true.

Remark. Here we choose $t_i = \log \sqrt{l_i}$, where l_i is chosen as in the proposition.

Like in proof of the Theorem 1 of [Bahoura 2004], we want to prove the following lemma.

Lemma 6. *For $\lambda \leq 0$ we have*

$${w_i}^\lambda - w_i \leq 0 \quad \Longrightarrow \quad -L({w_i}^\lambda - w_i) \leq 0$$

on $]\lambda, t_i] \times \mathbb{S}_{n-1}$.

Like in [Bahoura 2004], we have:

A useful point. Let $\xi_i = \sup\{\lambda \leq \bar{\lambda}_i = 2 + \log \eta_i \mid {w_i}^\lambda - w_i < 0 \text{ on }]\lambda, t_i] \times \mathbb{S}_{n-1}\}$. The real ξ_i exists.

First,

$$w_i(2\xi_i - t, \theta) = w_i[(\xi_i - t + \xi_i - \log \eta_i - 2) + (\log \eta_i + 2)].$$

Proof of Lemma 6. In fact, for each i we have $\lambda = \xi_i \leq \log \eta_i + 2$, where $\eta_i = [u_i(y_i)]^{(-2)/(n-2)}$.

Note that

$$w_i(2\xi_i - t, \theta) = w_i[(\xi_i - t + \xi_i - \log \eta_i - 2) + (\log \eta_i + 2)];$$

if we use the definition of w_i, then for $\xi_i \le t$,

$$w_i(2\xi_i - t, \theta) = e^{[(n-2)(\xi_i - t + \xi_i - \log\eta_i - 2)]/2} e^{n-2} v_i[\theta e^2 e^{(\xi_i - t)+(\xi_i - \log\eta_i - 2)}] \le 2^{(n-2)/2} e^{n-2} = \bar{c}.$$

We know that

$$-L(w_i^{\xi_i} - w_i) = [\overline{V}_i^{\xi_i}(w_i^{\xi_i})^{(n+2)/(n-2)} - \overline{V}_i {w_i}^{(n+2)/(n-2)}] + [e^{t^{\xi_i}}(w_i^{\xi_i})^{n/(n-2)} - e^t {w_i}^{n/(n-2)}].$$

We denote by Z_1 and Z_2 the terms

$$Z_1 = (\overline{V}_i^{\xi_i} - \overline{V}_i)(w_i^{\xi_i})^{(n+2)/(n-2)} + \overline{V}_i[(w_i^{\xi_i})^{(n+2)/(n-2)} - {w_i}^{(n+2)/(n-2)}]$$

and

$$Z_2 = e^{t^{\xi_i}}[(w_i^{\xi_i})^{n/(n-2)} - {w_i}^{n/(n-2)}] + {w_i}^{n/(n-2)}(e^{t^{\xi_i}} - e^t).$$

Like in the proof of Theorem 1 of [Bahoura 2004], we have

$$ {w_i}^{\xi_i} \le w_i \quad \text{and} \quad w_i^{\xi_i}(t,\theta) \le \bar{c} \quad \text{for all } (t,\theta) \in [\xi_i, -\log 2] \times \mathbb{S}_{n-1},$$

where $\bar{c}$ is a positive constant independent of i and $w_i^{\xi_i}$ for $\xi_i \le \log\eta_i + 2$.

The (P_0) *hypothesis*. Now we use (P_0) (this hypothesis is the same hypothesis as in the first part of the paper: $|\nabla V_i(y_i)| \to 0$). We write

$$|\nabla V_i(y_i + e^t\theta) - \nabla V_i(y_i)| \le A e^{\alpha t},$$

Thus,

$$\left| V_i(y_i + e^{t^{\xi_i}}\theta) - V_i(y_i + e^t\theta) - \langle \nabla V_i(y_i) \mid \theta\rangle (e^{t^{\xi_i}} - e^t) \right| \le \frac{A}{1+\alpha}[e^{(1+\alpha)t^{\xi_i}} - e^{(1+\alpha)t}].$$

Then

$$|V_i^{\xi_i} - V_i| \le |o(1)|(e^t - e^{t^{\xi_i}}).$$

Thus, $Z_1 \le |o(1)|(w_i^{\xi_i})^{(n+2)/(n-2)}(e^t - e^{t^{\xi_i}})$ and $Z_2 \le (w_i^{\xi_i})^{n/(n-2)} \times (e^{t^{\xi_i}} - e^t)$. Then

$$-L(w_i^{\xi_i} - w_i) \le (w_i^{\xi_i})^{n/(n-2)}[(|o(1)| {w_i^{\xi_i}}^{2/(n-2)} - 1)(e^t - e^{t^{\xi_i}})] \le 0.$$

The lemma is proved. □

We set

$$\xi_i = \sup\{\mu \le \log\eta_i + 2 \mid w_i^{\mu}(t,\theta) - w_i(t,\theta) \le 0 \text{ for all } (t,\theta) \in [\mu, t_i] \times \mathbb{S}_{n-1}\},$$

with t_0 small enough.

Like in the proof of Theorem 1 of [Bahoura 2004], the maximum principle implies

$$\min_{\theta\in\mathbb{S}_{n-1}} w_i(t_i,\theta) \le \max_{\theta\in\mathbb{S}_{n-1}} w_i(2\xi_i - t_i).$$

But

$$w_i(t_i,\theta) = e^{t_i} u_i(y_i + e^{t_i}\theta) \ge e^{t_i} \min u_i \quad \text{and} \quad w_i(2\xi_i - t_i) \le \frac{c_0}{u_i(y_i)};$$

thus,

$$l_i{}^{(n-2)/2} u_i(y_i) \times \min u_i \le c.$$

The proposition is contradicted. □

Proof of Theorem 2. The proof of Theorem 2 is similar to the proof of Theorem 1. Only the "fundamental point" changes.

According to the work of Chen and Lin [1997, step 2 of the proof of Theorem 1.1], in the blow-up point, the prescribed scalar curvature V is such that

$$\nabla V(0) = 0.$$

The function ∇V is continuous on $B_r(0)$ (with r small enough), so it is uniformly continuous and we write (because $y_i \to 0$)

$$|\nabla V(y_i + y) - \nabla V(y_i)| \le \epsilon \quad \text{for } |y| \le \delta \ll r \text{ for all } i.$$

Thus,

$$|V^{\xi_i} - V| \le o(1)(e^t - e^{t^{\xi_i}}).$$

We see that we have the same computations as in the "polar coordinates" in the proof of Theorem 1. □

Proof of Theorems 3 and 4. Here, only the "polar coordinates" change; the proposition of the first theorem stays true. First, we have:

Fundamental point (a consequence of the blow-up). According to the work of Chen and Lin [1997, step 2 of the proof of Theorem 1.3], in the blow-up point, the prescribed scalar curvature V is such that:

Case 1 (Theorem 3). $\lim_{i\to+\infty} |\nabla V_i(y_i)| = 0.$

We write

$$|\nabla V_i(y_i + e^t\theta) - \nabla V_i(y_i)| \le A e^{\alpha t}.$$

Thus,

$$|V_i^{\xi_i} - V_i| \le |o(1)|(e^t - e^{t^{\xi_i}}).$$

Case 2 (Theorem 4). $\nabla V(0) = 0.$

The function ∇V is continuous on $B_r(0)$ (for r small enough), so it is uniformly continuous and we write (because $y_i \to 0$)

$$|\nabla V(y_i + y) - \nabla V(y_i)| \le \epsilon \quad \text{for } |y| \le \delta \ll r \text{ for all } i.$$

Thus,

$$|V^{\xi_i} - V| \le o(1)(e^t - e^{t^{\xi_i}}),$$

Conclusion of the proofs of Theorems 3 and 4. Finally, we can note that we are in the case of Theorem 2 of [Bahoura 2004]. We have the same computations if we consider the function

$$\overline{w}_i(t,\theta) = w_i(t,\theta) - \frac{m}{2}e^t.$$

We set $L = \partial_{tt} + \Delta_\sigma - 1$, where Δ_σ is the Laplace–Beltrami operator on $\mathbb{S}_3$, and $\overline{V}_i(t,\theta) = V_i(y_i + e^t\theta)$.

Like in [Bahoura 2004], we want to prove the following lemma.

Lemma 7. $$\overline{w}_i^{\xi_i} - \overline{w}_i \le 0 \quad \Longrightarrow \quad -L(\overline{w}_i^{\xi_i} - \overline{w}_i) \le 0.$$

Proof of Lemma 7. We have

$$-L(\overline{w}_i^{\xi_i} - \overline{w}_i) = \overline{V}_i^{\xi_i}(w_i^{\xi_i})^3 - \overline{V}_i w_i^3.$$

Then

$$-L(\overline{w}_i^{\xi_i} - \overline{w}_i) = (\overline{V}_i^{\xi_i} - \overline{V}_i)(w_i^{\xi_i})^3 + [(w_i^{\xi_i})^3 - w_i^3]\overline{V}_i.$$

For $t \in [\xi_i, t_i]$ and $\theta \in \mathbb{S}_3$,

$$|\overline{V}_i^{\xi_i}(t,\theta) - \overline{V}_i(t,\theta)| = |V_i(y_i + e^{2\xi_i - t}\theta) - V_i(y_i + e^t\theta)| \le |o(1)|(e^t - e^{2\xi_i - t}).$$

The real $t_i = \log\sqrt{l_i} \to -\infty$, where l_i is chosen as in the proposition of Theorem 1.

But if $\overline{w}_i^{\xi_i} - \overline{w}_i \le 0$, we obtain

$$w_i^{\xi_i} - w_i \le \frac{m}{2}(e^{2\xi_i - t} - e^t) < 0.$$

Using the fact that $0 < w_i^{\xi_i} < w_i$, we have

$$(w_i^{\xi_i})^3 - w_i^3 = (w_i^{\xi_i} - w_i)[(w_i^{\xi_i})^2 + w_i^{\xi_i} w_i + (w_i)^2] \le 3(w_i^{\xi_i} - w_i) \times (w_i^{\xi_i})^2.$$

Thus, we have for $t \in [\xi_i, t_i]$ and $\theta \in \mathbb{S}_3$

$$(w_i^{\xi_i})^3 - {w_i}^3 \le 3\frac{m}{2}(w_i^{\xi_i})^2(e^{2\xi_i - t} - e^t).$$

We can write

$$(**) \qquad -L(\overline{w}_i^{\xi_i} - \overline{w}_i) \le (w_i^{\xi_i})^2\big(\tfrac{3}{2}m\overline{V}_i - |o(1)|{w_i}^{\xi_i}\big)(e^{2\xi_i - t} - e^t).$$

We know that, for $t \le \log(l_i) - \log 2 + \log \eta_i$, we have

$$w_i(t,\theta) = e^t \times \frac{u_i(y_i + e^t\theta/u_i(y_i))}{u_i(y_i)} \le 2e^t.$$

We find

$$w_i^{\xi_i}(t,\theta) \le 2e^2\sqrt{\frac{8}{a}},$$

because $\xi_i - \log \eta_i \le 2 + \frac{1}{2}\log(8/V(0))$ and $\xi_i \le t \le t_i$.

Finally, $(**)$ is negative and the lemma is proved. □

Now, if we use the Hopf maximum principle, we obtain

$$\min_{\theta\in\mathbb{S}^3} \bar{w}_i(t_i,\theta) \le \max_{\theta\in\mathbb{S}^3} \bar{w}_i(2\xi_i - t_i,\theta),$$

which implies that

$$l_i u_i(y_i) \le c.$$

It is a contradiction. □

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