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Samy Skander Bahoura

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SOME UNIFORM ESTIMATES FOR SCALAR CURVATURE TYPE EQUATIONS

SAMY SKANDER BAHOURA

We consider the prescribed scalar curvature equation on an open set Ω of \mathbb{R}^n , $-\Delta u = Vu^{(n+2)/(n-2)} + u^{n/(n-2)}$ with $V \in C^{1,\alpha}$ ($0 < \alpha \leq 1$), and we prove the inequality $\sup_K u \times \inf_\Omega u \leq c$ where K is a compact set of Ω .

In dimension 4, we have an idea on the supremum of the solution of the prescribed scalar curvature if we control the infimum. For this case we suppose the scalar curvature $C^{1,\alpha}$ ($0 < \alpha \leq 1$).

1. Introduction and main result

In our work, we denote by $\Delta = \nabla^i \nabla_i$ the Laplace–Beltrami operator in dimension $n \geq 2$.

Without loss of generality, we suppose $\Omega = B = B_2(0)$ the ball of radius 2 centered at 0 of \mathbb{R}^n .

Here, we study some a priori estimates of type $\sup \times \inf$ for a perturbed prescribed scalar curvature equation in all dimensions $n \geq 4$.

We have a counterexample to the sharp $\sup \times \inf$ inequality for the prescribed scalar curvature [Chen and Lin 1997, Proposition 4.3]. In our work the perturbation by a subcritical term is a sufficient condition to obtain such an inequality.

The $\sup \times \inf$ inequality is characteristic of those equations as the usual Harnack inequalities are for harmonic functions.

Note that the prescribed scalar curvature equation was studied a lot. We can find—see, for example, [Aubin 1998; Bahoura 2004; Brezis and Merle 1991; Brezis et al. 1993; Chen and Lin 1997; 1998; Li 1993; 1995; 1996; 1999; Li and Shafrir 1994; Li and Zhang 2004; Li and Zhu 1999; Shafrir 1992]—many results about uniform estimates in dimensions $n = 2$ and $n \geq 3$.

In dimension 2, the corresponding equation is

$$(E_0) \quad -\Delta u = Ve^u.$$

Note that Shafrir [1992] obtained an inequality of type $\sup u + C \inf u < c$ with only an L^∞ assumption on V .

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To obtain exactly the estimate $\sup u + \inf u < c$, Brezis, Li and Shafrir [Brezis et al. 1993] assumed that the prescribed scalar curvature V is Lipschitz continuous. Later, Chen and Lin [1998] proved that, if V is uniformly Hölder continuous, we can obtain a sup + inf inequality.

In dimension $n \geq 3$, the prescribed curvature equation on general manifold M is

$$(E'_0) \quad -\Delta u + R_g u = V u^{(n+2)/(n-2)}.$$

When $M = \mathbb{S}_n$, Li [1993; 1995; 1996] proved a priori estimates for the solutions of the previous equation. He used the notion of simple isolated points and some flatness conditions on V .

If we suppose $n = 3, 4$, we can find in [Li and Zhang 2004; Li and Zhu 1999] a uniform bound for the energy and a sup \times inf inequality. Note that Li and Zhu [1999] proved the compactness of the solutions to the Yamabe problem using the positive mass theorem.

In [Bahoura 2004], we can see (on a bounded domain of \mathbb{R}^4) that we have a uniform estimate for the solutions of (E'_0) ($n = 4$ and Euclidean case) by assuming that those solutions are bounded below by a positive constant; in this case we have assumed that the prescribed scalar curvature V is only Lipschitz.

Here we extend some result of [Bahoura 2004] to equations with nonlinear terms or with minimal condition on the prescribed scalar curvature.

For the Euclidean case, Chen and Lin [1997] got some a priori estimates for general equations

$$(E''_0) \quad -\Delta u = V u^{(n+2)/(n-2)} + g(u)$$

with some assumption on g and the Li-flatness conditions on V .

Here, we give some a priori estimates with some minimal conditions on the prescribed curvature, for perturbed scalar curvature equation, in all dimensions $n \geq 4$.

In our work, we use the *blow-up* analysis, the *moving-plane* method and a flatness condition (of order 1) for the prescribed scalar curvature. Note that the flatness condition which we use is also obtained by a *moving-plane* argument of Chen and Lin [1997]. The method of moving plane was developed in particular by Gidas, Ni and Nirenberg [Gidas et al. 1979] and Serrin [1971].

First, consider the equation

$$(E_1) \quad -\Delta u = V u^{(n+2)/(n-2)} + u^{n/(n-2)}$$

with $0 < a \leq V(x) \leq b$ and $\|V\|_{C^{1,\alpha}} \leq A$, $0 < \alpha \leq 1$.

We have:

Theorem 1. For all $a, b, A, \alpha > 0$ ($0 < \alpha \leq 1$), and all compact sets K of Ω of dimension $n \geq 4$, there is a positive constant $c = c(a, b, A, \alpha, K, \Omega, n)$ such that

$$\sup_K u \times \inf_{\Omega} u \leq c$$

for all solutions u of (E_1) relative to V .

Now, if we suppose $V \in C^1(\Omega)$ and $V \geq a > 0$, we have:

Theorem 2. For all $a > 0, V$ and all compact K of Ω of dimension $n \geq 4$, there is a positive constant $c = c(a, V, K, \Omega, n)$ such that

$$\sup_K u \times \inf_{\Omega} u \leq c$$

for all solutions u of (E_1) relative to V .

Now, we suppose $n = 4$, and we consider the equation (prescribed scalar curvature equation)

$$(E_2) \quad -\Delta u = Vu^3 \quad \text{on } \Omega \subset \mathbb{R}^4$$

with $0 < a \leq V(x) \leq b$ and $\|V\|_{C^{1,\alpha}} \leq A$, $0 < \alpha \leq 1$. We have:

Theorem 3. For all $a, b, m, A, \alpha > 0$ ($0 < \alpha \leq 1$) and all compact K of Ω , there is a positive constant $c = c(a, b, m, A, \alpha, K, \Omega)$ such that

$$\sup_K u \leq c \quad \text{if} \quad \min_{\Omega} u \geq m$$

for all solutions u of (E_2) relative to V .

If we suppose $n = 4$ and $V \in C^1(\Omega)$ and $V \geq a > 0$ on Ω , we have:

Theorem 4. For all $a, m > 0, V \in C^1(\Omega)$ and all compact $K \in \Omega$, there is a positive constant $c = c(a, m, V, K, \Omega)$ such that

$$\sup_K u \leq c \quad \text{if} \quad \min_{\Omega} u \geq m$$

for all solutions u of (E_2) relative to V .

2. Proofs of the theorems

Proof of Theorems 1 and 2.

Proof of Theorem 1. Without loss of generality, we suppose $\Omega = B_1$ the unit ball of \mathbb{R}^n . We want to prove an a priori estimate around 0.

Let (u_i) and (V_i) be sequences of functions on Ω such that

$$-\Delta u_i = V_i u_i^{(n+2)/(n-2)} + u_i^{n/(n-2)}, \quad u_i > 0,$$

with $0 < a \leq V_i(x) \leq b$ and $\|V_i\|_{C^{1,\alpha}} \leq A$.

We argue by contradiction, and we suppose that the $\sup \times \inf$ is not bounded. We have that for all $c, R > 0$ there exists $u_{c,R}$ a solution of (E_1) such that

$$(H) \quad R^{n-2} \sup_{B(0,R)} u_{c,R} \times \inf_M u_{c,R} \geq c.$$

Proposition (blow-up analysis). *There is a sequence of points $(y_i)_i$, $y_i \rightarrow 0$, and two sequences of positive real numbers $(l_i)_i$ and $(L_i)_i$ (see below), $l_i \rightarrow 0$ and $L_i \rightarrow +\infty$, such that, if we set $v_i(y) = u_i(y + y_i)/u_i(y_i)$, we have*

$$0 < v_i(y) \leq \beta_i \leq 2^{(n-2)/2},$$

$$\beta_i \rightarrow 1,$$

$$v_i(y) \rightarrow \left(\frac{1}{1 + |y|^2} \right)^{(n-2)/2} \quad \text{uniformly on all compact sets of } \mathbb{R}^n,$$

$$l_i^{(n-2)/2} u_i(y_i) \times \inf_{B_1} u_i \rightarrow +\infty,$$

Proof. We use the hypothesis (H) ; we take two sequences $R_i > 0$, $R_i \rightarrow 0$, and $c_i \rightarrow +\infty$ such that

$$R_i^{(n-2)} \sup_{B(0,R_i)} u_i \times \inf_{B_1} u_i \geq c_i \rightarrow +\infty.$$

Let $x_i \in B(x_0, R_i)$ be a point such that $\sup_{B(0,R_i)} u_i = u_i(x_i)$ and $s_i(x) = (R_i - |x - x_i|)^{(n-2)/2} u_i(x)$, $x \in B(x_i, R_i)$. Then $x_i \rightarrow 0$.

We have

$$\max_{B(x_i, R_i)} s_i(x) = s_i(y_i) \geq s_i(x_i) = R_i^{(n-2)/2} u_i(x_i) \geq \sqrt{c_i} \rightarrow +\infty.$$

We set

$$l_i = R_i - |y_i - x_i|, \quad \bar{u}_i(y) = u_i(y_i + y), \quad v_i(z) = \frac{u_i[y_i + (z/[u_i(y_i)]^{2/(n-2)})]}{u_i(y_i)}.$$

Clearly, we have $y_i \rightarrow x_0$.

We take

$$L_i = \frac{l_i}{(c_i)^{1/2(n-2)}} [u_i(y_i)]^{2/(n-2)} = \frac{[s_i(y_i)]^{2/(n-2)}}{c_i^{1/2(n-2)}} \geq \frac{c_i^{1/(n-2)}}{c_i^{1/2(n-2)}} = c_i^{1/2(n-2)} \rightarrow +\infty.$$

If $|z| \leq L_i$, then $y = [y_i + z/[u_i(y_i)]^{2/(n-2)}] \in B(y_i, \delta_i l_i)$ with $\delta_i = 1/(c_i)^{1/2(n-2)}$ and $|y - y_i| < R_i - |y_i - x_i|$; thus, $|y - x_i| < R_i$ and $s_i(y) \leq s_i(y_i)$. We can write

$$u_i(y)(R_i - |y - y_i|)^{(n-2)/2} \leq u_i(y_i)(l_i)^{(n-2)/2}.$$

But $|y - y_i| \leq \delta_i l_i$, $R_i > l_i$ and $R_i - |y - x_i| \geq R_i - |x_i - y_i| - \delta_i l_i > l_i - \delta_i l_i = l_i(1 - \delta_i)$. We obtain

$$0 < v_i(z) = \frac{u_i(y)}{u_i(y_i)} \leq \left[\frac{l_i}{l_i(1 - \delta_i)} \right]^{(n-2)/2} \leq 2^{(n-2)/2}.$$

We set $\beta_i = (1/(1 - \delta_i))^{(n-2)/2}$; clearly, we have $\beta_i \rightarrow 1$.

The function v_i satisfies

$$-\Delta v_i = \tilde{V}_i v_i^{(n+2)/(n-2)} + \frac{v_i^{n/(n-2)}}{[u_i(y_i)]^{2/(n-2)}},$$

where $\tilde{V}_i(y) = V_i[y + y/[u_i(y_i)]^{2/(n-2)}]$. Without loss of generality, we can suppose that $\tilde{V}_i \rightarrow V(0) = n(n-2)$.

We use the elliptic estimates of the Ascoli and Ladyzhenskaya theorems to have the uniform convergence of (v_i) to v on a compact set of \mathbb{R}^n . The function v satisfies

$$-\Delta v = n(n-2)v^{N-1}, \quad v(0) = 1, \quad 0 \leq v \leq 1 \leq 2^{(n-2)/2}, \quad N = \frac{2n}{n-2}.$$

By the maximum principle, we have $v > 0$ on \mathbb{R}^n . If we use the result of Caffarelli, Gidas and Spruck [Caffarelli et al. 1989], we obtain $v(y) = (1/(1 + |y|^2))^{(n-2)/2}$. We have the same properties as in [Bahoura 2004]. \square

Remark. When we use the convergence on compact sets of the sequence (v_i) , we can take an increasing sequence of compact sets and we see that we can obtain a sequence (ϵ_i) such that $\epsilon_i \rightarrow 0$ and after we choose (\tilde{R}_i) such that $\tilde{R}_i \rightarrow +\infty$ and finally

$$\tilde{R}_i^{n-2} \|v_i - v\|_{B(0, \tilde{R}_i)} \leq \epsilon_i.$$

We can say that we are in the case of [Chen and Lin 1997, step 1 of the proof of Theorem 1.2].

Fundamental point (a consequence of the blow-up). According to the work of Chen and Lin [1997, step 2 of the proof of Theorem 1.3], in the blow-up point, the prescribed scalar curvature V is such that

$$(P_0) \quad \lim_{i \rightarrow +\infty} |\nabla V_i(y_i)| = 0.$$

Polar coordinates (moving-plane method). Now we must use the same method as in [Bahoura 2004, Theorem 1]. We will use the moving-plane method.

We must prove [Bahoura 2004, Lemma 2].

We set $t \in]-\infty, -\log 2]$ and $\theta \in \mathbb{S}_{n-1}$:

$$w_i(t, \theta) = e^{(n-2)t/2} u_i(y_i + e^t \theta) \quad \text{and} \quad \bar{V}_i(t, \theta) = V_i(y_i + e^t \theta).$$

We consider the operator $L = \partial_{tt} + \Delta_\sigma - (n-2)^2/4$, with Δ_σ the Laplace–Beltrami operator on \mathbb{S}_{n-1} .

The function w_i satisfies

$$-Lw_i = \bar{V}_i w_i^{N-1} + e^t \times w_i^{n/(n-2)}, \quad N = \frac{2n}{n-2}.$$

Remark. Here w_i is a solution to the previous equation with a perturbed term which contains e^t . The term e^t is fundamental in the computations; it corrects the variation of V_i .

For $\lambda \leq 0$, we set

$$t^\lambda = 2\lambda - t, \quad w_i^\lambda(t, \theta) = w_i(t^\lambda, \theta), \quad \bar{V}_i^\lambda(t, \theta) = \bar{V}_i(t^\lambda, \theta).$$

First, like in [Bahoura 2004], we have the following lemma.

Lemma 5. *Let A_λ be the property*

$$A_\lambda = \{\lambda \leq 0 \mid \text{there exists } (t_\lambda, \theta_\lambda) \in]\lambda, t_i] \times \mathbb{S}_{n-1}, w_i^\lambda(t_\lambda, \theta_\lambda) - w_i(t_\lambda, \theta_\lambda) \geq 0\}.$$

Then there is $v \leq 0$ such that, for $\lambda \leq v$, A_λ is not true.

Remark. Here we choose $t_i = \log \sqrt{l_i}$, where l_i is chosen as in the proposition.

Like in proof of the Theorem 1 of [Bahoura 2004], we want to prove the following lemma.

Lemma 6. *For $\lambda \leq 0$ we have*

$$w_i^\lambda - w_i \leq 0 \quad \implies \quad -L(w_i^\lambda - w_i) \leq 0$$

on $] \lambda, t_i] \times \mathbb{S}_{n-1}$.

Like in [Bahoura 2004], we have:

A useful point. Let $\xi_i = \sup\{\lambda \leq \bar{\lambda}_i = 2 + \log \eta_i \mid w_i^\lambda - w_i < 0 \text{ on }]\lambda, t_i] \times \mathbb{S}_{n-1}\}$. The real ξ_i exists.

First,

$$w_i(2\xi_i - t, \theta) = w_i[(\xi_i - t + \xi_i - \log \eta_i - 2) + (\log \eta_i + 2)].$$

Proof of Lemma 6. In fact, for each i we have $\lambda = \xi_i \leq \log \eta_i + 2$, where $\eta_i = [u_i(y_i)]^{(-2)/(n-2)}$.

Note that

$$w_i(2\xi_i - t, \theta) = w_i[(\xi_i - t + \xi_i - \log \eta_i - 2) + (\log \eta_i + 2)];$$

if we use the definition of w_i , then for $\xi_i \leq t$,

$$w_i(2\xi_i - t, \theta) = e^{[(n-2)(\xi_i - t + \xi_i - \log \eta_i - 2)]/2} e^{n-2} v_i [\theta e^2 e^{(\xi_i - t) + (\xi_i - \log \eta_i - 2)}] \leq 2^{(n-2)/2} e^{n-2} = \bar{c}.$$

We know that

$$-L(w_i^{\xi_i} - w_i) = [\bar{V}_i^{\xi_i} (w_i^{\xi_i})^{(n+2)/(n-2)} - \bar{V}_i w_i^{(n+2)/(n-2)}] + [e^{t\xi_i} (w_i^{\xi_i})^{n/(n-2)} - e^t w_i^{n/(n-2)}].$$

We denote by Z_1 and Z_2 the terms

$$Z_1 = (\bar{V}_i^{\xi_i} - \bar{V}_i) (w_i^{\xi_i})^{(n+2)/(n-2)} + \bar{V}_i [(w_i^{\xi_i})^{(n+2)/(n-2)} - w_i^{(n+2)/(n-2)}]$$

and

$$Z_2 = e^{t\xi_i} [(w_i^{\xi_i})^{n/(n-2)} - w_i^{n/(n-2)}] + w_i^{n/(n-2)} (e^{t\xi_i} - e^t).$$

Like in the proof of Theorem 1 of [Bahoura 2004], we have

$$w_i^{\xi_i} \leq w_i \quad \text{and} \quad w_i^{\xi_i}(t, \theta) \leq \bar{c} \quad \text{for all } (t, \theta) \in [\xi_i, -\log 2] \times \mathbb{S}_{n-1},$$

where \bar{c} is a positive constant independent of i and $w_i^{\xi_i}$ for $\xi_i \leq \log \eta_i + 2$.

The (P₀) hypothesis. Now we use (P₀) (this hypothesis is the same hypothesis as in the first part of the paper: $|\nabla V_i(y_i)| \rightarrow 0$). We write

$$|\nabla V_i(y_i + e^t \theta) - \nabla V_i(y_i)| \leq A e^{\alpha t},$$

Thus,

$$|V_i(y_i + e^{t\xi_i} \theta) - V_i(y_i + e^t \theta) - \langle \nabla V_i(y_i) | \theta \rangle (e^{t\xi_i} - e^t)| \leq \frac{A}{1+\alpha} [e^{(1+\alpha)t\xi_i} - e^{(1+\alpha)t}].$$

Then

$$|V_i^{\xi_i} - V_i| \leq |o(1)| (e^t - e^{t\xi_i}).$$

Thus, $Z_1 \leq |o(1)| (w_i^{\xi_i})^{(n+2)/(n-2)} (e^t - e^{t\xi_i})$ and $Z_2 \leq (w_i^{\xi_i})^{n/(n-2)} \times (e^{t\xi_i} - e^t)$.

Then

$$-L(w_i^{\xi_i} - w_i) \leq (w_i^{\xi_i})^{n/(n-2)} [(|o(1)| w_i^{\xi_i 2/(n-2)} - 1) (e^t - e^{t\xi_i})] \leq 0.$$

The lemma is proved. □

We set

$$\xi_i = \sup\{\mu \leq \log \eta_i + 2 \mid w_i^\mu(t, \theta) - w_i(t, \theta) \leq 0 \text{ for all } (t, \theta) \in [\mu, t_i] \times \mathbb{S}_{n-1}\},$$

with t_0 small enough.

Like in the proof of Theorem 1 of [Bahoura 2004], the maximum principle implies

$$\min_{\theta \in \mathbb{S}_{n-1}} w_i(t_i, \theta) \leq \max_{\theta \in \mathbb{S}_{n-1}} w_i(2\xi_i - t_i).$$

But

$$w_i(t_i, \theta) = e^{t_i} u_i(y_i + e^{t_i} \theta) \geq e^{t_i} \min u_i \quad \text{and} \quad w_i(2\xi_i - t_i) \leq \frac{c_0}{u_i(y_i)};$$

thus,

$$l_i^{(n-2)/2} u_i(y_i) \times \min u_i \leq c.$$

The proposition is contradicted. \square

Proof of Theorem 2. The proof of Theorem 2 is similar to the proof of Theorem 1. Only the “fundamental point” changes.

According to the work of Chen and Lin [1997, step 2 of the proof of Theorem 1.1], in the blow-up point, the prescribed scalar curvature V is such that

$$\nabla V(0) = 0.$$

The function ∇V is continuous on $B_r(0)$ (with r small enough), so it is uniformly continuous and we write (because $y_i \rightarrow 0$)

$$|\nabla V(y_i + y) - \nabla V(y_i)| \leq \epsilon \quad \text{for } |y| \leq \delta \ll r \text{ for all } i.$$

Thus,

$$|V^{\xi_i} - V| \leq o(1)(e^t - e^{t^{\xi_i}}).$$

We see that we have the same computations as in the “polar coordinates” in the proof of Theorem 1. \square

Proof of Theorems 3 and 4. Here, only the “polar coordinates” change; the proposition of the first theorem stays true. First, we have:

Fundamental point (a consequence of the blow-up). According to the work of Chen and Lin [1997, step 2 of the proof of Theorem 1.3], in the blow-up point, the prescribed scalar curvature V is such that:

$$\text{Case 1 (Theorem 3).} \quad \lim_{i \rightarrow +\infty} |\nabla V_i(y_i)| = 0.$$

We write

$$|\nabla V_i(y_i + e^t \theta) - \nabla V_i(y_i)| \leq A e^{\alpha t}.$$

Thus,

$$|V_i^{\xi_i} - V_i| \leq |o(1)|(e^t - e^{t^{\xi_i}}).$$

$$\text{Case 2 (Theorem 4).} \quad \nabla V(0) = 0.$$

The function ∇V is continuous on $B_r(0)$ (for r small enough), so it is uniformly continuous and we write (because $y_i \rightarrow 0$)

$$|\nabla V(y_i + y) - \nabla V(y_i)| \leq \epsilon \quad \text{for } |y| \leq \delta \ll r \text{ for all } i.$$

Thus,

$$|V^{\xi_i} - V| \leq o(1)(e^t - e^{t^{\xi_i}}),$$

Conclusion of the proofs of Theorems 3 and 4. Finally, we can note that we are in the case of Theorem 2 of [Bahoura 2004]. We have the same computations if we consider the function

$$\bar{w}_i(t, \theta) = w_i(t, \theta) - \frac{m}{2}e^t.$$

We set $L = \partial_{tt} + \Delta_\sigma - 1$, where Δ_σ is the Laplace–Beltrami operator on \mathbb{S}_3 , and $\bar{V}_i(t, \theta) = V_i(y_i + e^t\theta)$.

Like in [Bahoura 2004], we want to prove the following lemma.

Lemma 7. $\bar{w}_i^{\xi_i} - \bar{w}_i \leq 0 \implies -L(\bar{w}_i^{\xi_i} - \bar{w}_i) \leq 0.$

Proof of Lemma 7. We have

$$-L(\bar{w}_i^{\xi_i} - \bar{w}_i) = \bar{V}_i^{\xi_i}(w_i^{\xi_i})^3 - \bar{V}_i w_i^3.$$

Then

$$-L(\bar{w}_i^{\xi_i} - \bar{w}_i) = (\bar{V}_i^{\xi_i} - \bar{V}_i)(w_i^{\xi_i})^3 + [(w_i^{\xi_i})^3 - w_i^3]\bar{V}_i.$$

For $t \in [\xi_i, t_i]$ and $\theta \in \mathbb{S}_3$,

$$|\bar{V}_i^{\xi_i}(t, \theta) - \bar{V}_i(t, \theta)| = |V_i(y_i + e^{2\xi_i-t}\theta) - V_i(y_i + e^t\theta)| \leq |o(1)|(e^t - e^{2\xi_i-t}).$$

The real $t_i = \log \sqrt{l_i} \rightarrow -\infty$, where l_i is chosen as in the proposition of Theorem 1.

But if $\bar{w}_i^{\xi_i} - \bar{w}_i \leq 0$, we obtain

$$w_i^{\xi_i} - w_i \leq \frac{m}{2}(e^{2\xi_i-t} - e^t) < 0.$$

Using the fact that $0 < w_i^{\xi_i} < w_i$, we have

$$(w_i^{\xi_i})^3 - w_i^3 = (w_i^{\xi_i} - w_i)[(w_i^{\xi_i})^2 + w_i^{\xi_i}w_i + (w_i)^2] \leq 3(w_i^{\xi_i} - w_i) \times (w_i^{\xi_i})^2.$$

Thus, we have for $t \in [\xi_i, t_i]$ and $\theta \in \mathbb{S}_3$

$$(w_i^{\xi_i})^3 - w_i^3 \leq 3\frac{m}{2}(w_i^{\xi_i})^2(e^{2\xi_i-t} - e^t).$$

We can write

$$(**) \quad -L(\bar{w}_i^{\xi_i} - \bar{w}_i) \leq (w_i^{\xi_i})^2\left(\frac{3}{2}m\bar{V}_i - |o(1)|w_i^{\xi_i}\right)(e^{2\xi_i-t} - e^t).$$

We know that, for $t \leq \log(l_i) - \log 2 + \log \eta_i$, we

have
$$w_i(t, \theta) = e^t \times \frac{u_i(y_i + e^t \theta / u_i(y_i))}{u_i(y_i)} \leq 2e^t.$$

We find

$$w_i^{\xi_i}(t, \theta) \leq 2e^2 \sqrt{\frac{8}{a}},$$

because $\xi_i - \log \eta_i \leq 2 + \frac{1}{2} \log(8/V(0))$ and $\xi_i \leq t \leq t_i$.

Finally, (**) is negative and the lemma is proved. □

Now, if we use the Hopf maximum principle, we obtain

$$\min_{\theta \in \mathbb{S}^3} \bar{w}_i(t_i, \theta) \leq \max_{\theta \in \mathbb{S}^3} \bar{w}_i(2\xi_i - t_i, \theta),$$

which implies that

$$l_i u_i(y_i) \leq c.$$

It is a contradiction. □

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SAMY SKANDER BAHOURA
EQUIPE D’ANALYSE COMPLEXE ET GÉOMÉTRIE
UNIVERSITE PIERRE ET MARIE CURIE
PARIS
FRANCE
samybahoura@gmail.com