Accuracy enhancement for finite-element simulations in curved domains; application to fluid flow

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Vitoriano Ruas[1,2*]

[1] Institut Jean Le Rond d'Alembert, CNRS UMR 7190, Sorbonne Université, Paris, France.

[2] Research grant holder of CNPq, PUC-Rio, Brazil.

e-mail: *vitoriano.ruas@upmc.fr*

Abstract

Among a few known techniques the isoparametric version of the finite element method for meshes consisting of curved triangles or tetrahedra is the one most widely employed to solve PDEs with essential conditions prescribed on curved boundaries. It allows to recover optimal approximation properties that hold for elements of order greater than one in the energy norm for polytopic domains. However, besides a geometric complexity, this technique requires the manipulation of rational functions and the use of numerical integration. We consider a simple alternative to deal with Dirichlet boundary conditions that bypasses these drawbacks, without eroding qualitative approximation properties. In the present work we first recall the main principle this technique is based upon, by taking as a model the solution of the Poisson equation with quadratic Lagrange finite elements. Then we show that it extends very naturally to viscous incompressible flow problems. Although the technique applies to any higher order velocity-pressure pairing, as an illustration a thorough study thereof is conducted in the framework of the Stokes system solved by the classical Taylor-Hood method.

Key words: Curved domain, Dirichlet, finite elements, mixed, N-simplex, optimal order, Stokes, straight-edged.

AMS Subject Classification: 65N30, 76M10.

1 Introduction

Consider the finite-element solution of second order elliptic equations posed in curved domains with Dirichlet boundary conditions. It is well known that a considerable order lowering may occur if prescribed boundary values are shifted to nodes that are not mesh vertexes of an approximating polygon or polyhedron formed by the union of straight-edged N-simplexes of a fitted mesh. Over four decades ago some techniques were designed in order to remedy such a loss of accuracy, and possibly attain the same theoretical optimal orders as in the case of a polytopic domain, assuming that the solution is sufficiently smooth. Two examples of such attempts in the framework of two-dimensional problems are the interpolated boundary condition method by Nitsche and Scott (cf. [17] and [25]), and the method introduced by Zlámal in [32] and extended by Ženíšek [30]. Among such techniques the finite element method's isoparametric version is by far the one most widely in use since the sixties (cf. [31]) in order to recover the lost optimality. One of the main reasons why it became so popular is the fact that the isoparametric technique applies to both two- and three-dimensional problems. We recall that this version of the finite element method is based on elements with curved boundary portions, aimed at better approximating a curved boundary than straight edges or plane faces. In this case the aforementioned shift of prescribed boundary values is avoided, since all nodes to which such values apply remain on the true boundary. The price to pay however is the manipulation of rational functions as both shape and trial functions defined upon the curved elements, and the resulting compulsory use of numerical integration. While on the one hand this is far from being an obstacle in most current applications such as linear problems with constant coefficients, numerical integration can be a

*Sorbonne Université, Campus de Jussieu, case 162, 4 place jussieu, Couloir 55-65, 4ème étage, 75005 Paris, France.

delicate issue in more complex situations. The technique exploited in this work allows overcoming all such issues, since it is based only on polynomial algebra upon an ordinary (i.e. a straight-edged) N-simplex. Moreover, in contrast to the simple polygonal approach no erosion of the theoretical order of a given interpolation inherent to the method occurs, especially for methods which are not of the lowest possible order. In short, our technique is aimed at ensuring a theoretical order greater or equal to two in the natural norm, without the use of curved elements and interpolating functions other than piecewise polynomials.
Actually the conception of the finite-element technique for solving boundary value problems with a smooth curvilinear boundary considered in this work is close to the interpolated (Dirichlet) boundary condition method studied in [7]. Though intuitive and known since the seventies, the latter technique has been of limited use so far. Among the reasons for this lies its difficult implementation, the lack of an extension to three-dimensional problems and restrictions on the choice of boundary nodal points to reach optimal convergence rates. In contrast our method is simple to implement in both two- and three-dimensional geometries. Moreover it is particularly handy, whenever a finite element method has normal component or normal derivative degrees of freedom as illustrated in [22]. Indeed when a method incorporates this type of degree of freedom the definition of isoparametric finite-element analogs is not always simple or clear (see e.g. [4]).
It is important to point out that efficient finite-element techniques are known since long, to optimally handle boundary conditions other than Dirichlet's, such as Neumann or Robin boundary conditions prescribed on curved boundaries. In this respect the author refers for instance to the works by Barrett and Elliott [2] and [3], besides [28] where a clear explanation on the issues brought about by Neumann conditions prescribed on curved boundaries is given.

The technique applied in this paper was introduced in [20], in connection with triangular Lagrange finite elements of any order k greater than one to solve the Poisson equation posed in a smooth curvilinear domain. In the subsequent work [21] the author addressed the case of tetrahedral Lagrange elements of arbitrary order for second order elliptic PDEs in the same class of domains. A synthesis of both papers is given in [23]. In [19] the author and co-worker used the same approach to the solution of Maxwell's equations with a Hermite finite element method. In this work we push further such studies in accordance to the following outline. In Section 2 we recall the main results on the new technique provided in [20] and [21], restricted to the case $k = 2$. Numerical examples are given in Section 3 in connection with the two-dimensional Poisson equation as well. A rigorous study of this technique is carried out in Section 4 in the framework of the finite-element solution of the equations governing incompressible viscous flows. Although such a study applies to different mixed elements or formulations of these equations such as GLS and SUPG, we confine ourselves to the case of the popular Taylor-Hood element (cf. [16]) as applied to the Stokes system in standard Galerkin formulation. In Section 5 we supply numerical examples illustrating the theoretical results of Section 4, and we conclude in Section 6 with a few comments.

2 Technique's short description

Referring to [20] and [21] for further details, here we describe our technique to solve boundary value problems with Dirichlet conditions prescribed on smooth curved boundaries, by solving a simple model problem as follows. Let Ω be a bounded N-dimensional domain for $N = 2$ or $N = 3$, and Γ be its boundary assumed to be sufficiently smooth (Γ must be at least of the C^1-class). Given a function $f \in H^1(\Omega)$ we wish to find a function $u \in H^3(\Omega)$ that solves $-\Delta u = f$ in Ω with $u = g$ on Γ assuming that $g \in H^{5/2}(\Gamma)$.

Now let $\mathcal{P} = \{\mathcal{T}_h\}_h$ be a uniformly regular family of finite element meshes consisting of straight-edged triangle or tetrahedra, according to the space dimension, satisfying the usual compatibility conditions (see e.g. [9]). Every element of $\mathcal{T}_h$ is to be viewed as a closed set. Moreover each one of these meshes is assumed to fit Ω in such a way that all the vertexes of the polygon or the polyhedron $\cup_{T \in \mathcal{T}_h} T$ lie on Γ. We denote the interior of this union set by Ω_h. Let Γ_h be the boundary of Ω_h, h_T be the diameter of $T \in \mathcal{T}_h$ and $h := \max_{T \in \mathcal{T}_h} h_T$.
We make the very reasonable assumption that every mesh triangle has no more than one edge in Γ_h and no mesh tetrahedron has either no more than one face or no more than one edge contained in Γ_h. The subset of $\mathcal{T}_h$ consisting of elements having at least one edge on Γ_h is denoted by $\mathcal{S}_h$.
Now let V_h be the finite-element space consisting of continuous functions that vanish on Γ_h, whose restriction to each $T \in \mathcal{T}_h$ belongs to P_2, where P_k is the space of polynomials of degree less than or equal to k. $\tilde{f} \in L^2(\Omega_h)$ being an extension of f to $\Omega_h \setminus \Omega$, we further set for $u, v \in H^1(\Omega_h)$:

$$\begin{cases} \bar{a}_h(u,v) := \int_{\Omega_h} \mathbf{grad}\ u \cdot \mathbf{grad}\ v \ d\mathbf{x} \\ \text{and } L_h(v) := \int_{\Omega_h} \tilde{f} v \ d\mathbf{x}. \end{cases} \quad (1)$$

To make ideas clear, and without loss of essential aspects, let us consider the case where $g \equiv 0$. If we search for $\bar{u}_h$ such that,

$$\bar{u}_h \in V_h \text{ and } \bar{a}_h(\bar{u}_h, v) = L_h(v) \; \forall v \in V_h, \tag{2}$$

it is well-known that the energy norm of $u - \bar{u}_h$ in Ω_h, that is, $\| \, u - \bar{u}_h \, \|_{\bar{e},h}$, where

$$\| \cdot \|_{\bar{e},h} := \left[\int_{\Omega_h} |\mathbf{grad}(\cdot)|^2 d\mathbf{x} \right]^{1/2}, \tag{3}$$

will be only an $O(h^{1.5})$.

In order to recover the optimal $O(h^2)$-convergence rate in the energy norm we propose the following.

Let W_h be a space consisting of functions defined in Ω_h whose restriction to every element $T \in \mathcal{T}_h$ is a polynomial of degree less than or equal to two, which are continuous at the vertexes of T and at the mid-points of the edges of T not contained in Γ_h, $\forall T \in \mathcal{T}_h$. Besides this, a function $w \in W_h$ is required to vanish at all the mesh vertexes lying on Γ_h, and at certain points of Γ belonging to the set Δ_S attached to an elements $S \in \mathcal{S}_h$ containing the underlying portion of Γ, constructed as described below $\forall S \in \mathcal{S}_h$. We consider beforehand that the expression of $w \in W_h$ in every element $S \in \mathcal{S}_h$ is extended to $\Delta_S \setminus \Omega_h$.

Referring to Figure 1, in the two-dimensional case Δ_S is the closed set delimited by Γ and the edge d of S contained in Γ_h. For every $S \in \mathcal{S}_h$, the extension of w to $\Delta_S \setminus \Omega_h$ is required to vanish at a point $P \in \Gamma$ located between two neighboring vertexes of S. To make implementation more straightforward P can be chosen to be the nearest intersection with Γ of the perpendicular to the edge d passing through its mid-point M. Henceforth Δ_S will be referred to as a skin.

Referring to Figures 2 and 3 (right), in the three-dimensional case, for every boundary edge d of an element $S \in \mathcal{S}_h$ we first denote by Π the plane bisecting the dihedral formed by the faces F and $F^{'}$ of S and another tetrahedron $S^{'} \in \mathcal{S}_h$ respectively, whose intersection is d. We generically denote by δ_d the closed subset of Π comprised between d and Γ, referred to as a plane skin hereafter, as depicted in Figure 2 for a tetrahedron S having a face F contained in Γ_h. Referring to Figure 3 (left), in the case of such a tetrahedron we denote by Δ_S the closed subset of $\Omega \cup \Omega_h$ delimited by the three plane skins δ_d, Γ and F. For the other type of tetrahedrons $S \in \mathcal{S}_h$ there will be only one such a plane skin δ_d and we set $\Delta_S = \delta_d$. Then for every $S \in \mathcal{S}_h$ the extension of w to $\Delta_S \setminus \Omega_h$ is required to vanish at a point $P \in \Gamma$ belonging to δ_d located between the end-points of d, for every edge d of S contained in Γ_h. Akin to the two-dimensional case P can be conveniently chosen to be the nearest intersection with Γ of the perpendicular to d in δ_d passing through its mid-point M.

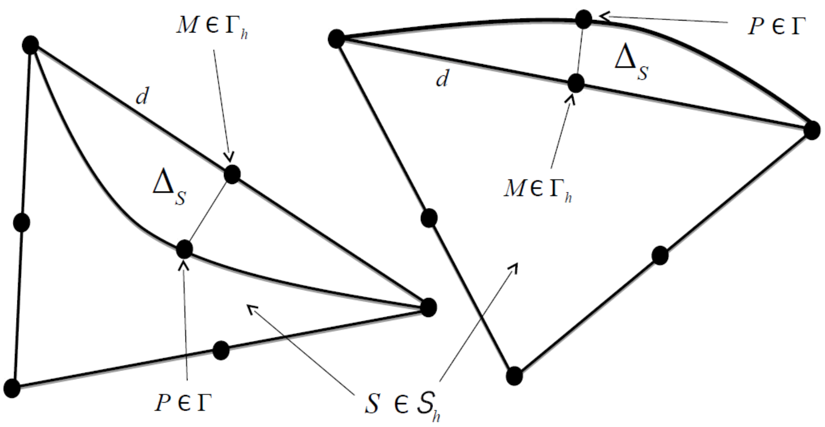

Figure 1: Set Δ_S for triangles S in $\mathcal{S}_h$ with their nodes $P \in \Gamma$ associated with $M \in \Gamma_h$

Now instead of solving (2) we search for u_h such that,

$$u_h \in W_h \text{ and } a_h(u_h, v) = L_h(v) \; \forall v \in V_h. \tag{4}$$

where

$$a_h(w, v) := \int_{\Omega_h} \mathbf{grad}_h w \cdot \mathbf{grad} \; v \; d\mathbf{x} \; \forall w \in W_h \text{ and } \forall v \in V_h, \tag{5}$$

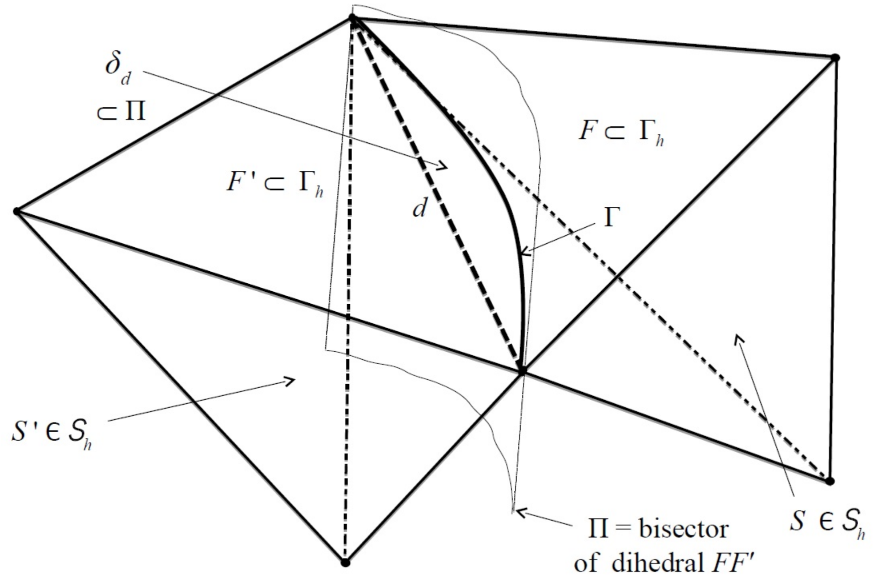

Figure 2: Plane skin δ_d for two tetrahedra $S, S^{'}$ having faces $F, F^{'} \subset \Gamma_h$ with $F \cap F^{'}$ = edge d

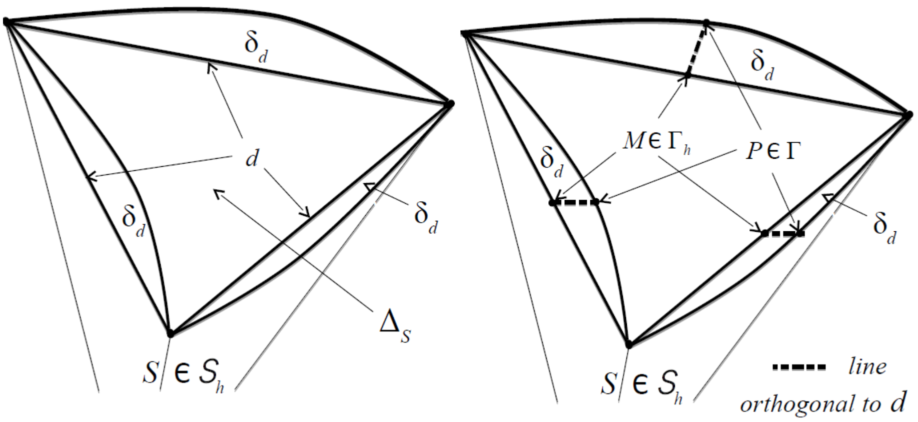

Figure 3: Set Δ_S for a tetrahedron $S \in \mathcal{S}_h$ and its nodes (right) $P \in \Gamma$ associated with $M \in \Gamma_h$

$\mathbf{grad}_h : W_h + H^1(\Omega_h) \longrightarrow [L^2(\Omega_h)]^N$ being the *broken gradient operator* defined by,

$$[\mathbf{grad}_h w]_{|T} = \mathbf{grad}[w_{|T}] \ \forall T \in \mathcal{T}_h.$$

Of course if w is continuous the forms a_h and $\bar{a}_h$ coincide. This always happens in the two-dimensional case, since $W_h \subset C^0(\overline{\Omega_h})$. However this inclusion is false in three-dimension space because functions $w \in W_h$ are not necessarily continuous on the interfaces of two tetrahedra in $\mathcal{S}_h$ (cf. [21]). For this reason in the sequel we will work with the *broken energy norm* $\| \cdot \|_{e,h}$ given by,

$$\| \cdot \|_{e,h} := \left[\int_{\Omega_h} |\mathbf{grad}_h(\cdot)|^2 d\mathbf{x} \right]^{1/2}, \tag{6}$$

and also with the *reduced broken energy norm* $\| \cdot \|_{\tilde{e},h}$ given by,

$$\| \cdot \|_{\tilde{e},h} := \left[\int_{\tilde{\Omega}_h} |\mathbf{grad}_h(\cdot)|^2 d\mathbf{x} \right]^{1/2} \text{ where } \tilde{\Omega}_h := \Omega_h \cap \Omega. \tag{7}$$

According to [20] and [21], $\| u_h - u \|_{e,h}$ (resp. $\| u_h - u \|_{\tilde{e},h}$) is an $O(h^2)$ if Ω is convex (resp. non-convex). In the remainder of this section we give a detailed proof of these results. One of the keys to the problem is the following proposition:

Proposition 2.1 *Provided h is sufficiently small, for every $w \in W_h$ there exists $v \in V_h$ such that*

$$\left\{ \begin{array}{l} \| w - v \|_{e,h} \leq C_1 h \| w \|_{e,h}, \\ a_h(w, v) \geq \| w \|_{e,h}^2 /2 \end{array} \right. \tag{8}$$

where C_1 is a strictly positive constant independent of both w and h. ■

Proof : Given $w \in W_h$ let $v \in V_h$ coincide with w at all mesh nodes, except those located on Γ which are not mesh vertexes. The latter are precisely the nodes P whose construction is illustrated in Figures 1 and 3, at which w necessarily vanishes. Clearly enough $(v - w)_{|T} \equiv 0$ for every T having no edge contained in Γ_h. Therefore all we need are estimates of $\int_S |\mathbf{grad}(v - w)_{|S}|^2 d\mathbf{x}$ for every $S \in \mathcal{S}_h$, i.e. having at least one edge $d \subset \Gamma_h$. For the sake of brevity we only consider the case of an $S \in \mathcal{S}_h$ having exactly one such an edge d; indeed the case of a tetrahedron having a face contained in Γ_h can be handled as a trivial extension of the argument that follows.
Let then $\varphi_S \in P_2$ be the Lagrange basis function that vanishes at all the five nodes of S for $N = 2$ or nine nodes for $N = 3$, which are not the mid-point of d. Denoting such a mid-point by M_S we clearly have

$$[v - w]_{|S} = w(M_S)\varphi_S. \tag{9}$$

First we refer to Figures 1 and 2 where the set Δ_S is illustrated, for $S \in \mathcal{S}_h$. Recalling that h_S is the diameter of S, by construction there exists a mesh-independent constant C_Γ such that

$$|w(M_S)| \leq C_\Gamma h_S^2 \max_{\mathbf{x} \in \Delta_S} |\mathbf{grad}\ w(\mathbf{x})|. \tag{10}$$

Since w is a polynomial in S there must exist another mesh-independent constant C_δ such that (cf [28]):

$$\max_{\mathbf{x} \in \Delta_S} |\mathbf{grad}\ w(\mathbf{x})| \leq C_\delta \max_{\mathbf{x} \in S} |\mathbf{grad}\ w(\mathbf{x})|. \tag{11}$$

Furthermore by a classical inverse inequality (cf. [9]) there exists another mesh-independent constant C_ι such that,

$$\max_{\mathbf{x} \in S} |\mathbf{grad}\ w(\mathbf{x})| \leq C_\iota h_S^{-N/2} \left[\int_S |\mathbf{grad}\ w|^2 d\mathbf{x}| \right]^{1/2}. \tag{12}$$

It follows from (9), (10), (11) and (12) that

$$\int_S |\mathbf{grad}(v - w)|^2 d\mathbf{x} \leq C_\iota^2 C_\delta^2 C_\Gamma^2 h_S^{4-N} \int_S |\mathbf{grad}\ w|^2 d\mathbf{x} \int_S |\mathbf{grad}\ \varphi_S|^2 d\mathbf{x}. \tag{13}$$

Combining (13) with the obvious estimate

$$\int_S |\mathbf{grad}\ \varphi_S|^2 d\mathbf{x} \leq C_\varphi^2 h_S^{N-2} \tag{14}$$

for a suitable mesh-independent constant C_φ, we easily obtain

$$\| v - w \|_{e,h} \leq C_1 h \| w \|_{e,h} \text{ with } C_1 = C_\varphi C_\iota C_\delta C_\Gamma. \tag{15}$$

Finally, noting that $a_h(w, v) = \| w \|_{e,h}^2 + a_h(w, v - w) \geq \| w \|_{e,h} (\| w \|_{e,h} - \| v - w \|_{e,h})$, the lower bound,

$$a_h(w, v) \geq \| w \|_{e,h}^2 /2 \tag{16}$$

trivially follows from (15) for $h = h_0$, where h_0 is the largest mesh size in the family $\mathcal{P}$ such that $1 - C_1 h_0$ is bounded below by $1/2$.

As an immediate consequence of (8) a uniform inf-sup Babuška-Brezzi condition in connection with problem (4)-(5) is satisfied. More precisely we have:

Corollary 2.2 *Provided h is sufficiently small, it holds for some $\alpha \geq 1/3$:*

$$\forall w \in W_h \neq 0, \sup_{v \in V_h \setminus \{0\}} \frac{a_h(w, v)}{\| w \|_{e,h} \| v \|_{e,h}} \geq \alpha. \ \blacksquare \tag{17}$$

Since obviously $dim(V_h) = dim(W_h)$, and both a_h and L_h are uniformly bounded independently of h, the simple fact that (17) holds implies that (4) is uniquely solvable (cf. [12]).

Next we prove error estimates for problem (4). In this aim we denote by $| \cdot |_{m,D}$ the usual semi-norm of Sobolev space $H^m(D)$ in a bounded domain $D \in \Re^N$, for $m \in \mathcal{N}$ (cf. [1]). First we have:

Theorem 2.3 *Assume that Ω is convex. Then provided h is sufficiently small, for a certain mesh-independent constant C_2 it holds:*

$$\parallel u_h - u \parallel_{e,h} \leq C_2 h^2 |u|_{3,\Omega}. \tag{18}$$

where $\| \cdot \|_{e,h}$ is the discrete H^1- semi-norm defined by (6).

Proof : From (17) we infer that

$$\parallel u_h - w \parallel_{e,h} \leq 3 \sup_{v \in V_h \setminus \{0\}} \frac{a_h(u_h - w, v)}{\parallel v \parallel_{e,h}} \; \forall w \in W_h. \tag{19}$$

Let $I_h(u) \in W_h$ be the standard interpolate of u at the mesh nodes associated with W_h. Taking in (17) $w = u_h - I_h(u)$, we add and subtract u in the first argument of a_h. Thus by straightforward calculations,

$$\parallel u_h - I_h(u) \parallel_{e,h} \leq 3 \left[\parallel u - I_h(u) \parallel_{e,h} + \sup_{v \in V_h \setminus \{0\}} \frac{a_h(u_h - u, v)}{\parallel v \parallel_{e,h}} \right]. \tag{20}$$

Noting that $a_h(u_h, v) = L_h(v)$ we come up with:

$$\parallel u_h - I_h(u) \parallel_{e,h} \leq 3 \left[\parallel u - I_h(u) \parallel_{e,h} + \sup_{v \in V_h \setminus \{0\}} \frac{|a_h(u, v) - L_h(v)|}{\parallel v \parallel_{e,h}} \right]. \tag{21}$$

Since $\Omega_h \subset \Omega$ if Ω is convex, we observe that $a_h(u,v) = \oint_{\Gamma_h} v \frac{\partial u}{\partial n_h} d\Gamma_h - \int_{\Omega_h} v \Delta u \, d\mathbf{x}$, where $\frac{\partial u}{\partial n_h}$ is the outer normal derivative of u on Γ_h. Noting that $-\Delta u = f$ in Ω_h and $v \equiv 0$ on Γ_h, we trivially obtain,

$$\parallel u_h - u \parallel_{e,h} \leq 4 \parallel u - I_h(u) \parallel_{e,h} . \tag{22}$$

Then (18) is a consequence of standard estimates of the interpolation error in Sobolev norms (cf. [7]). ■

Remark 1 It is interesting to note that although our method is non-conforming for $N = 3$, in any case V_h is a subspace of $H^1(\Omega_h)$. Therefore the variational residual $a_h(u,v) - L_h(v)$ vanishes if Ω is convex, in contrast to usual non-conforming methods. ■

Now we address the case of a non-convex Ω. Let us consider a smooth domain $\tilde{\Omega}$ close to Ω which strictly contains $\Omega \cup \Omega_h$ for all h sufficiently small. More precisely, denoting by $\tilde{\Gamma}$ the boundary of $\tilde{\Omega}$ we assume that $meas(\tilde{\Gamma}) - meas(\Gamma) \leq \varepsilon$ for ε sufficiently small. For the sake of simplicity henceforth we consider that f was extended by zero to the whole $\tilde{\Omega} \setminus \Omega$ and still denote this extension by $\tilde{f}$. In doing so the following error estimate can be proved:

Theorem 2.4 *Assume that there exists a function $\tilde{u}$ defined in $\tilde{\Omega}$, satisfying:*

- $\tilde{u}_{|\Omega} = u$;
- $\tilde{u} = 0$ *a.e. on* Γ;
- $\tilde{u} \in H^3(\tilde{\Omega})$.

Then as long as h is sufficiently small it holds:

$$\parallel u_h - u \parallel_{\tilde{e},h} \leq \tilde{C}_2 [h^2 |\tilde{u}|_{3,\tilde{\Omega}} + h^{5/2} \parallel \Delta \tilde{u} \parallel_{0,\tilde{\Omega}}], \tag{23}$$

where $\tilde{C}_2$ is a mesh-independent constant and $\| \cdot \|_{\tilde{e},h}$ denotes the standard H^1-semi-norm defined in (7).

Proof : According to (17) $\forall w \in W_h$ we have,

$$\parallel u_h - w \parallel_{e,h} \leq \frac{1}{\alpha} \sup_{v \in V_h \setminus \{0\}} \frac{|a_h(\tilde{u}, v) - F_h(v)| + |a_h(\tilde{u} - w, v)|}{\parallel v \parallel_{e,h}}. \tag{24}$$

Since $\tilde{u} \in H^3(\tilde{\Omega})$ we can apply First Green's identity to $a_h(\tilde{u}, v)$ thereby getting rid of integrals on portions of Γ. Next we note that $\Delta \tilde{u} + \tilde{f} = 0$ in every $T \in \mathcal{T}_h \setminus \mathcal{S}_h$; this is also true of elements T not belonging to the subset $\mathcal{R}_h$ of $\mathcal{S}_h$ consisting of elements R such that $R \setminus \Omega$ is not restricted to a set of vertexes of Ω_h. Finally we recall that $\Delta\tilde{u} + \tilde{f}$ vanishes identically in the set $R \cap \Omega\ \forall R \in \mathcal{R}_h$. Denoting by $\tilde{\Delta}_R$ the interior of the set $R \setminus \Omega$ for $R \in \mathcal{R}_h$ we can write:

$$|a_h(\tilde{u}, v) - F_h(v)| = \left| \sum_{R \in \mathcal{R}_h} \int_{\tilde{\Delta}_R} \Delta\tilde{u}\ v\ d\mathbf{x} \right| \leq \sum_{R \in \mathcal{R}_h} \parallel \Delta\tilde{u} \parallel_{0,\tilde{\Delta}_R} \parallel v \parallel_{0,\tilde{\Delta}_R}. \tag{25}$$

Now taking into account that $v \equiv 0$ on Γ_h and recalling the constant C_Γ defined in Proposition 8, it holds:

$$|v(\mathbf{x})| \leq C_\Gamma h_R^2 \parallel |\mathbf{grad}\ v| \parallel_{0,\infty,\tilde{\Delta}_R},\ \forall \mathbf{x} \in \tilde{\Delta}_R,$$

where $\parallel \cdot \parallel_{0,\infty,D}$ denotes the standard norm of $L^\infty(D)$, D being a bounded open set of $\Re^N$. Next, using the same arguments as in the proof of Proposition 8, we derive the estimate $\parallel |\mathbf{grad}\ v| \parallel_{0,\infty,\tilde{\Delta}_R} \leq C_I h_R^{-N/2} \parallel \mathbf{grad}\ v \parallel_{0,R}$ for a mesh-independent constant C_I. Noticing that the measure of $\tilde{\Delta}_R$ is bounded by a constant depending only on Ω times h_R^{N+1}, after straightforward calculations we obtain for a certain mesh-independent constant C_Z:

$$\parallel \Delta\tilde{u} \parallel_{0,\tilde{\Delta}_R} \parallel v \parallel_{0,\tilde{\Delta}_R} \leq C_Z h_R^{5/2} \parallel \Delta\tilde{u} \parallel_{0,\tilde{\Delta}_R} \parallel \mathbf{grad}\ v \parallel_{0,R}\ \ \forall R \in \mathcal{R}_h. \tag{26}$$

Now plugging (26) into (25) and applying the Cauchy-Schwarz inequality, we easily come up with,

$$|a_h(\tilde{u}, v) - F_h(v)| \leq C_Z h^{5/2} \parallel \Delta\tilde{u} \parallel_{0,\tilde{\Omega}} \parallel v \parallel_{e,h}. \tag{27}$$

Finally plugging (27) into (24) and taking $w = I_h(\tilde{u})$, we immediately establish the validity of error estimate (23). ■

Remark 2 There are many ways to ensure the existence of $\tilde{u}$ satisfying the assumptions of Theorem 2.4, as long as Γ is as smooth as required. For instance we refer to [27] for an interesting construction of $\tilde{u}$. ■

3 Some numerics for the Poisson equation

Let us illustrate the performance of the new technique to handle Dirichlet conditions on curved boundaries. With this aim we first solve problems (2) and (4) in case Ω is the unit disk centered at the origin, and a uniformly regular family of meshes consisting of $8n^2$ triangles for $n = 2^m$, with $m = 1, 2, \ldots$ is constructed in the way described in [21]. In these experiments we take $f(x, y) = 9r$ where $r = (x^2 + y^2)^{1/2}$, and hence the exact solution is given by $u(x, y) = 1 - r^3$. Owing to symmetry only the quarter disk corresponding to $x > 0$ and $y > 0$ is taken into account in the computations, and therefore meshes containing $2n^2$ elements are employed. For a fairer comparison we also supply results obtained for the same problem solved by the classical isoparametric technique. We denote the solution obtained with this method by $\tilde{u}_h$.

Taking $m = 2, 3, 4, 5, 6$ and observing that $h = 1/n$, in Table 1 the quantities $\parallel u_h - u \parallel_{e,h}$, $\parallel \tilde{u}_h - u \parallel_{e,h}$ and $\parallel \bar{u}_h - u \parallel_{e,h}$ for the resulting decreasing values of h are displayed. Table 1 confirms second order convergence

n	$\longrightarrow$	4	8	16	32	64
$\parallel u_h - u \parallel_{e,h}$	$\longrightarrow$	0.1329×10^{-1}	0.3343×10^{-2}	0.8381×10^{-3}	0.2097×10^{-3}	0.5245×10^{-4}
$\parallel \bar{u}_h - u \parallel_{e,h}$	$\longrightarrow$	0.5434×10^{-1}	0.1969×10^{-1}	0.7042×10^{-2}	0.2503×10^{-2}	0.8870×10^{-3}
$\parallel \tilde{u}_h - u \parallel_{e,h}$	$\longrightarrow$	0.1559×10^{-1}	0.3837×10^{-2}	0.9477×10^{-3}	0.2353×10^{-3}	0.5861×10^{-4}

Table 1: Energy errors for a test-problem in a disk solved by methods (4), (2) and isoparametric FEs

in the energy norm for the approach advocated in this paper, while the polygonal approach (2) yields only $O(h^{1.5})$ approximations in the same norm, as predicted in classical books (cf. [9]). Of course the expected second order convergence of the isoparametric solution is also observed. However the new method is a little more accurate.

Now in order to further illustrate the accuracy of method (4) in case Ω is not convex, we compare it again with method (2) by solving a problem whose exact solution is not axisymmetric. More specifically, here the domain described in polar coordinates (r, θ) is given by $\Omega := \{(r, \theta) |\ r \leq [4 + cos(4\theta)]/5\}$. Taking $g \equiv 0$

and $f = 16r^2 - 5.8r - 11.2x^2y^2/r^3$ the exact solution is the function $u = r^3 - r^4 - 1.6x^2y^2/r$. Notice that $f \in H^1(\Omega)$ and $u \in H^3(\Omega)$.
Here again symmetry allows working with the computational domain corresponding to $x > 0$ and $y > 0$. The meshes also consist of $2n^2$ elements, generated like in [22], by subdividing the radial coordinate r into n equal parts and the azimuthal coordinate $\theta \in (0, \pi/2)$ into $2n$ equal parts.

Observing that $h = 1/n$, we show in Table 2 the quantity $\parallel u_h - u \parallel_{e,h}$ for $n = 2^m$, with $m = 2, 3, 4, 5, 6$. Moreover, in order to give a better idea of how effective our method is, we also supply the errors $\parallel u_h - u \parallel_{0,h}$ and $|u_h - u|_{\infty,h}$, where $\parallel \cdot \parallel_{0,h}$ and $| \cdot |_{\infty,h}$ stand for the standard norm of $L^2(\Omega_h)$ and the maximum absolute error at the nodal points, respectively. Table 2 validates method's second order convergence in energy norm established

n	$\longrightarrow$	4	8	16	32	64
$\parallel u_h - u \parallel_{e,h}$	$\longrightarrow$	0.1566×10^{-1}	0.4235×10^{-2}	0.1089×10^{-2}	0.2751×10^{-3}	0.6910×10^{-4}
$\parallel u_h - u \parallel_{0,h}$	$\longrightarrow$	0.4875×10^{-3}	0.5859×10^{-4}	0.7315×10^{-5}	0.9195×10^{-6}	0.1156×10^{-6}
$\|u_h - u\|_{\infty,h}$	$\longrightarrow$	0.8110×10^{-3}	0.1790×10^{-3}	0.2745×10^{-4}	0.3701×10^{-5}	0.4915×10^{-6}

Table 2: Errors in different senses for a test-problem in a non-convex domain solved by method (4)

in Theorem 2.4. Even better news come from the observed convergence rates of three in the norm of $L^2(\Omega_h)$ and of a little less than three in the L^∞-semi-norm $| \cdot |_{\infty,h}$.

4 Application to the Taylor-Hood element

The classical Taylor-Hood element was introduced in [16] for the solution of the incompressible Navier-Stokes equations. It consists of continuous piecewise polynomial representations of both velocity and pressure in triangles or tetrahedra, of degree two for the former variable and of degree one for the latter. Second order convergence results for this method were established by Verfürth [29] in the case of a polygonal domain and by Boffi [5] in the case of polyhedrons. In this section we apply the method described in Section 2 in order to extend such results to the case of smooth curvilinear domains.
This study will be restricted to the linearized form of the stationary incompressible Navier-Stokes equations, which governs incompressible viscous flows at a very low Reynolds number. More specifically our theory applies to the following Stokes system in a bounded domain Ω of $\Re^N$ at least of the C^1-class, for $N = 2$ or $N = 3$:
Given a field $\mathbf{f} \in [H^1(\Omega)]^N$, and a velocity profile $\mathbf{g}$ defined on Γ assumed to belong to $[H^{5/2}(\Gamma)]^N$ and to satisfy the conservation property $\oint_\Gamma \mathbf{g} \cdot \mathbf{n} \, d\mathbf{s} = 0$, where $\mathbf{n}$ is the unit outer normal vector on Γ, we wish to determine a velocity field $\mathbf{u} \in [H^1(\Omega)]^N$ and a hydrostatic pressure $p \in L^2(\Omega)/\Re$, where A/B denotes the quotient between two vector spaces A and B, such that:

$$\left\{ \begin{array}{ll} \left. \begin{array}{l} -\Delta \mathbf{u} + \mathbf{grad}\, p = \mathbf{f} \\ div\, \mathbf{u} = 0 \end{array} \right\} & \text{in } \Omega \\ \mathbf{u} = \mathbf{g} & \text{on } \Gamma. \end{array} \right. \tag{28}$$

A suitable regularity assumption on Ω, besides those applying to $\mathbf{f}$ and $\mathbf{g}$, legitimately allows assuming in turn that $\mathbf{u} \in [H^3(\Omega)]^N$ and $p \in H^2(\Omega)$.
Although all the results to be derived hereafter apply to the inhomogeneous case, in order to avoid non essential difficulties, we further restrict the analysis conducted in this section to the case where $\mathbf{g} \equiv \vec{0}$.

Our working spaces here will be the pair $(\mathbf{V}_h, \mathbf{W}_h)$ of vector field spaces defined by $\mathbf{V}_h := [V_h]^N$ and $\mathbf{W}_h := [W_h]^N$, together with the function space $Q_h := \tilde{Q}_h \cap L^2_0(\Omega_h)$, with $\tilde{Q}_h := \{q \mid q \in C^0(\Omega_h),\ q_{|T} \in P_1,\ \forall T \in \mathcal{T}_h\}$, where $L^2_0(\Omega_h) = \{q \mid q \in L^2(\Omega_h),\ \int_{\Omega_h} q \, d\mathbf{x} = 0\}$. For the sake of simplicity, henceforth we denote by $|\mathbf{w}|_{1,h}$ the semi-norm $\parallel \mathbf{grad}_h \mathbf{w} \parallel_{0,h}$ of a field $\mathbf{w} \in \mathbf{W}_h + [H^1(\Omega_h)]^N$, where $\parallel \cdot \parallel_{0,h}$ stands for standard norm of $L^2(\Omega_h)$.
We make the same assumptions as in Section 2 on a given family $\mathcal{P}$ of meshes $\mathcal{T}_h$ of Ω into N-simplexes. In doing so we consider the extension by zero $\tilde{\mathbf{f}}$ of $\mathbf{f}$ to $\Omega_h \setminus \Omega$, if applicable, and define the broken divergence operator $div_h : \mathbf{W}_h + [H^1(\Omega_h)]^N \longrightarrow L^2(\Omega_h)$ by $[div_h \mathbf{w}]_{|T} = div\ \mathbf{w}_{|T}\ \forall T \in \mathcal{T}_h$. We further set for $\mathbf{w} \in$

$\mathbf{W}_h + [H^1(\Omega_h)]^N$, $\mathbf{v} \in [H^1(\Omega_h)]^N$ and $q \in L^2(\Omega_h)$:

$$\left\{ \begin{array}{l} c_h(\mathbf{w}, \mathbf{v}) := \int_{\Omega_h} \mathbf{grad}_h \, \mathbf{w} : \mathbf{grad} \, \mathbf{v} \, d\mathbf{x} \\ b_h(\mathbf{v}, q) := -\int_{\Omega_h} q \, div \, \mathbf{v} \, d\mathbf{x} \\ d_h(\mathbf{w}, q) := -\int_{\Omega_h} q \, div_h \mathbf{w} \, d\mathbf{x} \\ \mathbf{L}_h(\mathbf{v}) := \int_{\Omega_h} \tilde{\mathbf{f}} \cdot \mathbf{v} \, d\mathbf{x}. \end{array} \right. \tag{29}$$

Now we pose the corresponding finite-element counterpart of (28) as:

$$\left\{ \begin{array}{l} \text{Find } \mathbf{u}_h \in \mathbf{W}_h \text{ and } p_h \in Q_h \text{ such that:} \\ c_h(\mathbf{u}_h, \mathbf{v}) + b_h(\mathbf{v}, p_h) = \mathbf{L}_h(\mathbf{v}) \; \forall \mathbf{v} \in \mathbf{V}_h, \\ d_h(\mathbf{u}_h, q) = 0 \; \forall q \in Q_h. \end{array} \right. \tag{30}$$

According to the classical theory of linear variational problems (see e.g. [12]) problem (30) is well-posed thanks to the validity of the underlying Babuška-Brezzi condition, or yet the inf-sup condition (31), that is,

Proposition 4.1 *Provided h is sufficiently small, there exists a strictly positive constant A independent of h such that*

$$\inf_{(\mathbf{w},p) \in \mathbf{W}_h \times Q_h \setminus \{(\vec{0},0)\}} \; \sup_{(\mathbf{v},q) \in \mathbf{V}_h \times Q_h \setminus \{(\vec{0},0)\}} \frac{c_h(\mathbf{w}, \mathbf{v}) + b_h(\mathbf{v}, p) + d_h(\mathbf{w}, q)}{[|\mathbf{w}|_{1,h}^2 + \parallel p \parallel_{0,h}^2]^{1/2} [|\mathbf{v}|_{1,h}^2 + \parallel q \parallel_{0,h}^2]^{1/2}} \geq A. \blacksquare \tag{31}$$

<u>Proof:</u> Let the pair $(\mathbf{w}, p) \neq (\vec{0}, 0)$ be given in $\mathbf{W}_h \times Q_h$.
First we observe that, since Taylor-Hood elements are uniformly stable, the following condition holds:

$$\sup_{\mathbf{v} \in \mathbf{V}_h \setminus \{\vec{0}\}} \frac{b_h(\mathbf{v}, p)}{|\mathbf{v}|_{1,h}} \geq \beta \parallel p \parallel_{0,h} . \tag{32}$$

for a constant $\beta > 0$ independent of both p and the mesh. Actually (32) is the consequence of well-known arguments (cf. [8]), according to which there exist two mesh-independent constants C_3 and C_4 also independent of p, such that one can find $\mathbf{v}_0 \in \mathbf{V}_h$ satisfying

$$\left\{ \begin{array}{l} b_h(\mathbf{v}_0, p) \geq C_3 \parallel p \parallel_{0,h}^2 \\ |\mathbf{v}_0|_{1,h} \leq C_4 \parallel p \parallel_{0,h} . \end{array} \right. \tag{33}$$

Noticing that c_h is nothing but a_h applied to vector fields instead of functions, let $\mathbf{v}_1 \in \mathbf{V}_h$ satisfy the obvious vector analog of (8) for our given $\mathbf{w} \in \mathbf{W}_h$. For a certain parameter $\eta > 0$ we define $\mathbf{v} := \eta \mathbf{v}_0 + \mathbf{v}_1$ and take $q \equiv -p$. From the obvious vector analog of (15) we easily obtain

$$|\mathbf{v}_1|_{1,h} \leq (1 + C_1 h) |\mathbf{w}|_{1,h},$$

which together with (33) immediately yields:

$$\left\{ \begin{array}{l} c_h(\mathbf{w}, \mathbf{v}) + b_h(\mathbf{v}, p) + d_h(\mathbf{w}, q) \geq \dfrac{|\mathbf{w}|_{1,h}^2}{2} - \eta C_4 |\mathbf{w}|_{1,h} \parallel p \parallel_{0,h} + b_h(\mathbf{v}_1, p) - d_h(\mathbf{w}, p) + \eta C_3 \parallel p \parallel_{0,h}^2 \\ \text{and } |\mathbf{v}|_{1,h} + \parallel q \parallel_{0,h} \leq (1 + C_1 h) |\mathbf{w}|_{1,h} + (\eta C_4 + 1) \parallel p \parallel_{0,h} . \end{array} \right. \tag{34}$$

Next we note that

$$b_h(\mathbf{v}_1, p) - d_h(\mathbf{w}, p) = \int_{\Omega_h} p \, div_h(\mathbf{w} - \mathbf{v}_1) d\mathbf{x} \leq \sqrt{N} |\mathbf{w} - \mathbf{v}_1|_{1,h} \parallel p \parallel_{0,h} .$$

Thus using Young's inequality and recalling that $C_1 h \leq 1/2$, from (34) we obtain,

$$\left\{ \begin{array}{l} c_h(\mathbf{w}, \mathbf{v}) + b_h(\mathbf{v}, p) + d_h(\mathbf{w}, q) \geq \dfrac{|\mathbf{w}|_{1,h}^2}{4} - \sqrt{N} |\mathbf{w} - \mathbf{v}_1|_{1,h} \parallel p \parallel_{0,h} + (\eta C_3 - \eta^2 C_4^2) \parallel p \parallel_{0,h}^2 \\ \text{and } |\mathbf{v}|_{1,h} + \parallel q \parallel_{0,h} \leq \dfrac{3 |\mathbf{w}|_{1,h}}{2} + (\eta C_4 + 1) \parallel p \parallel_{0,h} . \end{array} \right. \tag{35}$$

Moreover, plugging the natural vector version of (15) into the first inequality of (35), we derive

$$
\begin{cases} c_h(\mathbf{w},\mathbf{v}) + b_h(\mathbf{v},p) + d_h(\mathbf{w},q) \geq \dfrac{|\mathbf{w}|_{1,h}^2}{4} - \sqrt{N}C_1 h|\mathbf{w}|_{1,h} \parallel p \parallel_{0,h} + (\eta C_3 - \eta^2 C_4^2) \parallel p \parallel_{0,h}^2 \\ \text{and } |\mathbf{v}|_{1,h} + \parallel q \parallel_{0,h} \leq \dfrac{3|\mathbf{w}|_{1,h}}{2} + (\eta C_4 + 1) \parallel p \parallel_{0,h} . \end{cases} \tag{36}
$$

Taking $\eta = C_3/(2C_4^2)$, setting $C_5 = \min\{1/8, [C_3/(2C_4)]^2\}$ and assuming that $\sqrt{N}C_1 h \leq 2C_5$ from (36) we come up with,

$$
\begin{cases} c_h(\mathbf{w},\mathbf{v}) + b_h(\mathbf{v},p) + d_h(\mathbf{w},q) \geq C_5(|\mathbf{w}|_{1,h}^2 + \parallel p \parallel_{0,h}^2) \\ \text{and } [|\mathbf{v}|_{1,h}^2 + \parallel q \parallel_{0,h}^2]^{1/2} \leq C_6[|\mathbf{w}|_{1,h}^2 + \parallel p \parallel_{0,h}^2]^{1/2}. \end{cases} \tag{37}
$$

with $C_6 = \{9/4 + [C_3/(2C_4) + 1]^2\}^{1/2}$.
In view of both inequalities in (37), as long as $h \leq \min[2C_5/\sqrt{N}, 1/2]/C_1$, (31) holds with $A = C_5/C_6$. ■

Now we endeavor to derive error estimates for problem (30). Essentially this task is not more complicated than the one carried out in Theorems 2.3 and 2.4. Indeed (30) can be rewritten as follows:

$$
\text{Find } U_h \in \mathcal{W}_h \text{ such that } \mathcal{A}_h(U_h, V) = \mathcal{L}_h(V)\ \forall V \in \mathcal{V}, \tag{38}
$$

where

- $U_h = (\mathbf{u}_h, p_h)$;
- $V = (\mathbf{v}, q)$;
- $\mathcal{W}_h := \mathbf{W}_h \times Q_h$;
- $\mathcal{V}_h := \mathbf{V}_h \times Q_h$;
- $\mathcal{A}_h((\mathbf{w},p),(\mathbf{v},q)) := c_h(\mathbf{w},\mathbf{v}) + b_h(\mathbf{v},p) + d_h(\mathbf{w},q)$;
- $\mathcal{L}_h(V) := \mathbf{L}_h(\mathbf{v})$.

Now we denote by $\parallel \cdot \parallel_{X,h}$ the norm over $\{\mathbf{W}_h + [H_0^1(\Omega_h)]^N\} \times L^2(\Omega_h)$, given by

$$
\parallel V \parallel_{X,h} := [|\mathbf{v}|_{1,h}^2 + \parallel q \parallel_{0,h}^2]^{1/2}. \tag{39}
$$

Then letting A play the same role as the constant $1/3$ in (17), analogously to (19) we obtain:

$$
\parallel U_h - W \parallel_{X,h} \leq \frac{1}{A} \sup_{V \in \mathcal{V}_h \setminus \{\mathcal{O}\}} \frac{\mathcal{A}_h(U_h - W, V)}{\parallel V \parallel_{X,h}}\ \forall W \in \mathcal{W}_h. \tag{40}
$$

Finally noticing that here also the variational residual $\mathcal{A}_h((\mathbf{u},p),(\mathbf{v},q)) - \mathcal{L}_h((\mathbf{v},q))$ vanishes for every $(\mathbf{v},q) \in \mathcal{V}_h$ if Ω is convex, using standard estimates for the interpolation error in Sobolev spaces, akin to Theorem 2.3, (40) leads to:

Theorem 4.2 *Provided h is small enough and Ω is convex, for a certain mesh-independent constant $\mathcal{C}$ it holds:*

$$
[|\mathbf{u} - \mathbf{u}_h|_{1,h}^2 + \parallel p - p_h \parallel_{0,h}^2]^{1/2} \leq \mathcal{C}h^2[|\mathbf{u}|_{3,\Omega} + |p|_{2,\Omega}]. \blacksquare \tag{41}
$$

The case where Ω is not convex can be treated quite in the same manner as in Section 2. The key to the problem is the existence of suitable extensions $\tilde{\mathbf{u}}$ of $\mathbf{u}$ and $\tilde{p}$ of p to the domain $\tilde{\Omega} \setminus \Omega$, where $\tilde{\Omega}$ is defined in Section 2. More precisely, we extend $\mathbf{f}$ by zero to $\tilde{\Omega} \setminus \bar{\Omega}$ and still denote by $\tilde{\mathbf{f}}$ such an extension. However, naturally enough, more technicalities come into play here.
To begin with we need the following preliminary result:

Lemma 4.3 *Let ϕ be a function in $H^1(\tilde{\Omega})$ that vanishes on Γ. There exists a mesh-independent constant C_X such that*

$$
\parallel \phi \parallel_{0,\tilde{\Delta}_R} \leq C_X h_R^2 |\phi|_{1,\tilde{\Delta}_R}\ \forall R \in \mathcal{R}_h. \tag{42}
$$

Proof : We refer to [15] for the terminology and some properties of diffeomorphisms used in this proof.
Let us cover the whole non-convex region of Γ by a set of M overlapping local maps, say, ω_i, $i = 1, \ldots, M$. Owing to our regularity assumptions, there exists a C^1-diffeomorphism $\mathcal{F}_i$ that transforms ω_i into a set $\hat{\omega}_i$ such that $\hat{\Gamma}_i := \mathcal{F}_i(\Gamma \cap \omega_i)$ is a line segment for $N = 2$ or a plane bounded set for $N = 3$. Without loss of generality we assume that the measure of $\hat{\Gamma}_i$ is not zero, and moreover that we can assign each $R \in \mathcal{R}_h$ to a certain local map ω_i, in such a way that $R \subset \omega_i$.
We generically denote by $\hat{\mathbf{x}} = (\hat{\mathbf{t}}, \hat{n})$ the local Cartesian coordinate system of $\Re^N$ with coordinates $\hat{\mathbf{t}} = \hat{t}$ for $N = 2$ or $\hat{\mathbf{t}} = (\hat{t}_1, \hat{t}_2)$ for $N = 3$ along or upon $\hat{\Gamma}_i$, and by $\hat{n}$ the coordinate along the axis orthogonal to $\hat{\Gamma}_i$ oriented from this manifold outwards the image of $\omega_i \cap \Omega$ under $\mathcal{F}_i$. Let $R \in \mathcal{R}_h$ and $\hat{\Delta}_R$ be the transformation of $\tilde{\Delta}_R$ under $\mathcal{F}_i$ for the appropriate i. We denote by $\hat{\phi}$ the transformation of ϕ under $\mathcal{F}_i$ defined in $\hat{\omega}_i$. Since $\hat{\phi} = 0$ on $\hat{\Gamma}_i$ we can write

$$\hat{\phi}(\hat{\mathbf{t}}, \hat{n}) = \int_{\nu=0}^{\nu=\hat{n}} \left[\frac{\partial \hat{\phi}}{\partial \hat{n}}\right] (\hat{\mathbf{t}}, \nu) \, d\nu.$$

Hence, we obtain successively,

$$\int_{\hat{\Delta}_R} |\hat{\phi}|^2 \, d\hat{\mathbf{x}} \leq \int_{\hat{\Delta}_R} \left| \int_{\nu=0}^{\nu=\hat{n}} \left[\frac{\partial \hat{\phi}}{\partial \hat{n}}\right] (\hat{\mathbf{t}}, \nu) \, d\nu \right|^2 \, d\hat{\mathbf{t}} d\hat{n},$$

$$\int_{\hat{\Delta}_R} |\hat{\phi}|^2 \, d\hat{\mathbf{x}} \leq \int_{\hat{\Delta}_R} l(\hat{\mathbf{t}}) \left\{ \int_{\nu=0}^{\nu=l(\hat{\mathbf{t}})} \left| \left[\frac{\partial \hat{\phi}}{\partial \hat{n}}\right] (\hat{\mathbf{t}}, \nu) \right|^2 \, d\nu \right\} d\hat{\mathbf{t}} d\hat{n},$$

where $l(\hat{\mathbf{t}})$ is the width of $\hat{\Delta}_R$ measured in the direction normal to $\hat{\Gamma}_i$ from point $(\hat{\mathbf{t}}, 0)$. Then denoting by $\hat{l}$ the maximum of $l(\hat{\mathbf{t}})$ over $(\hat{\mathbf{t}}, 0) \in \hat{\Gamma}_i \cap \hat{\Delta}_R$, we trivially obtain,

$$\int_{\hat{\Delta}_R} |\hat{\phi}|^2 \, d\hat{\mathbf{x}} \leq \hat{l}^2 \int_{\hat{\Delta}_R} \left| \left[\frac{\partial \hat{\phi}}{\partial \hat{n}}\right] (\hat{\mathbf{t}}, \hat{n}) \right|^2 \, d\hat{n} d\hat{\mathbf{t}},$$

and further,

$$\int_{\hat{\Delta}_R} |\hat{\phi}|^2 \, d\hat{\mathbf{x}} \leq \hat{C}_i h_R^2 \int_{\hat{\Delta}_R} |\widehat{\mathbf{grad}} \, \hat{\phi}|^2 \, d\hat{\mathbf{x}}$$

where $\widehat{\mathbf{grad}}(\cdot)$ represents the gradient operator of a function defined in $\hat{\omega}_i$, and the constant $\hat{C}_i$ depends only on Ω and $\hat{\omega}_i$.
Next we make straightforward changes of variables in the above integrals, thereby transforming them into integrals in $\tilde{\Delta}_R$, and observe that $\widehat{\mathbf{grad}} \, \hat{\phi} = F_i^{-1} \mathbf{grad} \, \phi$ where F_i is the Jacobian matrix of $\mathcal{F}_i$. From a basic property of diffeomorphisms the spectral norm of F_i can be uniformly bounded above by a constant independent of the mesh, as much as the Jacobian of both $\mathcal{F}_i$ and $\mathcal{F}_i^{-1}$. Finally taking the extrema over i of those constants and of $\hat{C}_i$ in the required senses, the result follows. ■

Now we have

Theorem 4.4 *Assume that there exists $\tilde{\mathbf{u}}$ and $\tilde{p}$ satisfying the following conditions:*

- $\tilde{\mathbf{u}}_{|\Omega} = \mathbf{u}$ *and* $\tilde{p}_{|\Omega} = p$
- $\tilde{\mathbf{u}} = \overrightarrow{0}$ *a.e. on* Γ*;*
- $\tilde{\mathbf{u}} \in [H^3(\tilde{\Omega})]^N$ *and* $\tilde{p} \in H^2(\tilde{\Omega})$.

Then, as long as h is small enough, for a certain mesh-independent constant $\tilde{\mathcal{C}}$ it holds:

$$[|\mathbf{u} - \mathbf{u}_h|_{1,\tilde{\Omega}_h}^2 + \parallel p - p_h \parallel_{\tilde{\Omega}_h}^2]^{1/2} \leq \tilde{\mathcal{C}}\{h^2[|\tilde{\mathbf{u}}|_{3,\tilde{\Omega}} + |\tilde{p}|_{2,\tilde{\Omega}}] + h^{5/2}[|\tilde{\mathbf{u}}|_{2,\tilde{\Omega}} + |\tilde{p}|_{1,\tilde{\Omega}}]\}, \quad (43)$$

where $\tilde{\Omega}_h := \Omega_h \cap \Omega$.

Proof : The proof of this theorem is based on the same arguments as the proof of Theorem 2.4. Therefore we skip some details.
First we set $\tilde{U} := (\tilde{\mathbf{u}}, \tilde{p})$. For every $W = (\mathbf{w}, r) \in \mathcal{W}_h$ we have:

$$\parallel U_h - W \parallel_{X,h} \leq \frac{1}{A} \sup_{V=(\mathbf{v},q) \in \mathcal{V}_h \neq \mathcal{O}} \frac{|\mathcal{A}_h(\tilde{U}, V) - \mathcal{L}_h(V)| + |\mathcal{A}_h(W, V) - \mathcal{A}_h(\tilde{U}, V)|}{\parallel V \parallel_{X,h}}. \quad (44)$$

The second term in the numerator of (44) can be handled in a standard manner by means of classical interpolation theory. This yields for a mesh-independent constant C_J:

$$\inf_{W \in \mathcal{W}_h} \frac{|\mathcal{A}_h(W,V) - \mathcal{A}_h(\tilde{U},V)|}{\| V \|_{X,h}} \leq C_J h^2 [|\tilde{\mathbf{u}}|_{3,\tilde{\Omega}} + |\tilde{p}|_{2,\tilde{\Omega}}]. \tag{45}$$

Next, thanks to the fact that $\tilde{\mathbf{u}} \in [H^3(\tilde{\Omega})]^N$ and $\tilde{p} \in H^2(\tilde{\Omega})$ we can write,

$$|\mathcal{A}_h(\tilde{U},V) - \mathcal{L}_h(V)| \leq \sum_{R \in \mathcal{R}_h} \left\{ \int_{\tilde{\Delta}_R} [|\Delta \tilde{\mathbf{u}}| + |\mathbf{grad}\ \tilde{p}|] |\mathbf{v}|\ d\mathbf{x} + \int_{\tilde{\Delta}_R} |q|\ |div\ \tilde{\mathbf{u}}|\ d\mathbf{x} \right\}. \tag{46}$$

Similarly to Theorem 2.4 the summation of the first integral on the right hand side of (46) can be bounded above as follows, for a suitable mesh-independent constant C_Y:

$$\sum_{R \in \mathcal{R}_h} \int_{\tilde{\Delta}_R} [|\Delta \tilde{\mathbf{u}}| + |\mathbf{grad}\ \tilde{p}|] |\mathbf{v}|\ d\mathbf{x} \leq C_Y h^{5/2} [\| \Delta \tilde{\mathbf{u}} \|_{0,\tilde{\Omega}} + |p|_{1,\tilde{\Omega}}] |\mathbf{v}|_{1,h}. \tag{47}$$

On the other hand we have,

$$\int_{\tilde{\Delta}_R} |q|\ |div\ \tilde{\mathbf{u}}|\ d\mathbf{x} \leq \| q \|_{0,\tilde{\Delta}_R} \| div\ \tilde{\mathbf{u}} \|_{0,\tilde{\Delta}_R}. \tag{48}$$

Now, since $div\ \tilde{\mathbf{u}}$ vanishes on Γ, using Lemma 4.3, it holds for a certain mesh-independent constant C_X:

$$\| div\ \tilde{\mathbf{u}} \|_{0,\tilde{\Delta}_R} \leq C_X h_R^2 |div\ \tilde{\mathbf{u}}|_{1,\tilde{\Delta}_R} \tag{49}$$

Moreover using the fact that $meas(\tilde{\Delta}_R) \leq C_Q h_R^{N+1}$ for a mesh-independent constant C_Q, together with the inverse inequality $\| q \|_{0,\infty,R} \leq C_I h_R^{-N/2} \| q \|_{0,R}$, from (48) and (49) we derive,

$$\int_{\tilde{\Delta}_R} |q|\ |div\ \tilde{u}|\ d\mathbf{x} \leq C_I C_Q C_X h_R^{5/2} |div\ \tilde{\mathbf{u}}|_{1,\tilde{\Delta}_R} \| q \|_{0,R}. \tag{50}$$

This trivially yields

$$\sum_{R \in \mathcal{R}_h} \int_{\tilde{\Delta}_R} |q|\ |div\ \tilde{\mathbf{u}}|\ d\mathbf{x} \leq \sqrt{N} C_I C_Q C_X h^{5/2} \| q \|_{0,h} |\tilde{\mathbf{u}}|_{2,\tilde{\Omega}}. \tag{51}$$

Finally combining (44), (45), (46), (47) and (51) we come up with (43). ■

As pointed out in Remark 2, the construction of a pair $(\tilde{\mathbf{u}}, \tilde{p})$ satisfying the assumptions of Theorem 4.4 can be performed in different manners. In this respect we refer for instance to [27].

5 Numerical validation for confined rotating flows

One of the most remarkable applications of the method studied in the previous section is the simulation of confined rotating flows. Indeed in this case a viscous fluid adhere to the curved wall of the flow region, and thus handling the underlying Dirichlet boundary condition with a method of order higher than one requires the use of an accurate technique. In this section we present results obtained with ours, for two test-problems governed by the Stokes system.

In the tables of this section the acronym OCR stands for *observed convergence rate*.

5.1 Test-problem with a manufactured solution

First we apply the Taylor-Hood method combined with our technique to solve (28) with a manufactured solution corresponding to the following data: Ω is the unit disk (centered at the origin), $\mathbf{f} = (8,8)(x-y)$, and $\mathbf{g} \equiv \vec{0}$. Prescribing $p(\sqrt{2}/2, \sqrt{2}/2) = 0$, the exact solution has polynomial expressions, namely $\mathbf{u} = (y,-x)(1-x^2-y^2)$ and $p = x^2 - y^2$.
We use meshes constructed like in the first test-problem of Section 3, but here the computational domain is the whole disk. More specifically now we compute with $(2n \times 2n)$-meshes containing $8n^2$ triangles, each mesh being symmetric with respect to the axes $x = 0$ and $y = 0$, for $n = 2^m$ with $m = 2, 3, 4, 5$. We recall that $h = 1/n$.

In order to discard any particularity inherent to the problem being solved, we compared the numerical solution with the one obtained by the simple polygonal approach.
We display in Table 3 the velocity and pressure errors in the norms $|\cdot|_{1,h}$ and $\|\cdot\|_{0,h}$ for both approaches. The notations $\bar{\mathbf{u}}_h$ and $\bar{p}_h$ are employed to represent the velocity and pressure obtained by the simple polygonal approach. These results completely validate the analysis carried out in the previous sections for the case of a convex domain (cf. Theorem 4.2). It is no surprise that the polygonal approach does erode the order of the velocity approximation. It is interesting to note however that, at least in this test-case, this simple approach does not affect the pressure approximation.

$2n$	$\longrightarrow$	8	16	32	64	OCR
$\|\mathbf{u}_h - \mathbf{u}\|_{1,h}$	$\longrightarrow$	0.3585×10^{-1}	0.8833×10^{-2}	0.2157×10^{-2}	0.5299×10^{-3}	$O(h^2)$
$\|\bar{\mathbf{u}}_h - \mathbf{u}\|_{1,h}$	$\longrightarrow$	0.7959×10^{-1}	0.2746×10^{-1}	0.9588×10^{-2}	0.3370×10^{-2}	$O(h^{1.5})$
$\Vert p_h - p \Vert_{0,h}$	$\longrightarrow$	0.4266×10^{-1}	0.1047×10^{-1}	0.2567×10^{-2}	0.6332×10^{-3}	$O(h^2)$
$\Vert \bar{p}_h - p \Vert_{0,h}$	$\longrightarrow$	0.4264×10^{-1}	0.1047×10^{-1}	0.2567×10^{-2}	0.6332×10^{-3}	$O(h^2)$

Table 3: Errors for a test-(flow) problem in a disk solved by method (30) and the polygonal approach

5.2 Pseudo circular Couette flow

In order to check our method's performance in the case of a non-convex flow domain we used it to solve the problem described as follows.
Circular Couette flow of an incompressible viscous fluid with density ρ in a region comprised between two concentric cylinders, where the inner one of radius r_i rotates at an angular velocity ω and the outer one with radius r_e is kept fixed, is governed by the stationary Navier-Stokes equations with a zero body-force right hand side. As long as the Reynolds number is sufficiently low, the flow is laminar and the solution to the problem is given by $\mathbf{u} = (sin\theta, -cos\theta)u_\theta(r)$ where $u_\theta(r) = \omega r_i^2(r_e^2 - r^2)/[r(re^2 - ri^2)]$ and $p(r) = \rho\omega^2 r_i^4/(r_e^2 - r_i^2)^2[r^2/2 - r_e^4/(2r^2) - 2r_e^2 log(r)] + c$, c being a constant. If we enforce zero pressure on the outer wall, then c takes the value $2r_e^2 log(r_e)\rho\omega^2 r_i^4/(r_e^2 - r_i^2)^2$.
Although there is no particular difficulty to solve the Navier-Stokes equations with our method, in order to focus on our essentially validating goal, we apply it to a modified problem, in which the exact inertia term $\rho[\mathbf{grad}\ \mathbf{u}]\mathbf{u}$ with a minus sign is input as right hand side datum $\mathbf{f}$. Of course the pair $(\mathbf{u}, p)$ is still the solution to the resulting Stokes system (28) in the annulus Ω with inner radius r_i and outer radius r_e. The datum $\mathbf{g}$ in turn equals $\vec{0}$ for $r = r_e$, while its value for $r = r_i$ conforms to the given azimuthal velocity $r_i\omega$ and a zero radial velocity.
Taking $r_e = 1$, $r_i = 0.5$, $\omega = 1$ and $\rho = 1$, we proceeded to the numerical solution of thus defined (pseudo) circular Couette flow problem with the Taylor-Hood method combined with our technique to approximate the boundary conditions. In order to avoid non physical boundary conditions, computations were carried out for the whole annulus. With this aim we used again $(2n \times 2n)$ symmetric meshes, for $n = 2^m$, with $m = 3, 4, 5, 6$, constructed in the way described in the previous subsection, except for the fact that now the elements inside the disk with radius r_i were disregarded. This yields meshes consisting of $6n^2$ triangles, with $h = 1/n$.
We display in Table 4 the velocity errors measured in the norms $|\cdot|_{1,h}$ and $\|\cdot\|_{0,h}$, together with the pressure errors measured in the $\|\cdot\|_{0,h}$-norm. It is interesting to note that the latter are decreasing at a rate faster than the $O(h^2)$ observed in the test-problem of the previous sub-section. This seems to be due to the fact that in circular Couette flow the inertia term with a minus sign is nothing but the pressure gradient. On the other hand the velocity errors in the H^1-semi-norm are in perfect agreement with the theoretical predictions. The velocity errors in the L^2-norm in turn seem to decrease like an $O(h^3)$, which is optimal.

6 Extensions to other mixed elements and final comments

In the four previous sections we focused on the application of the technique introduced in [20] and [21] to solve boundary value problems in smooth curved domains with Dirichlet boundary conditions, in the particular case of quadratic Lagrange interpolation in N-simplexes. More specifically we considered the solution of the Poisson

$2n$	$\longrightarrow$	16	32	64	128	OCR
$\vert \mathbf{u}_h - \mathbf{u} \vert_{1,h}$	$\longrightarrow$	$0.1592\times10^{+0}$	0.4261×10^{-1}	0.1090×10^{-1}	0.2741×10^{-2}	$O(h^2)$
$\parallel \mathbf{u}_h - \mathbf{u} \parallel_{0,h}$	$\longrightarrow$	0.3833×10^{-2}	0.5339×10^{-3}	0.6923×10^{-4}	0.8744×10^{-5}	$O(h^3)$
$\parallel p_h - p \parallel_{0,h}$	$\longrightarrow$	$0.1209\times10^{+0}$	0.2095×10^{-1}	0.3952×10^{-2}	0.6638×10^{-3}	$O(h^{\approx 2.5})$

Table 4: Errors for the pseudo circular Couette flow problem solved by method (30)

equation as a basis for the solution of incompressible viscous flow problems by the popular Taylor-Hood element. However this second order mixed finite element was chosen here only for illustrative purposes. As a matter of fact our technique basically applies to most known reliable mixed methods of order greater than one, to solve this kind of problems, as long as velocity degrees of freedom must be prescribed at boundary points different from vertexes. Let us be more specific about some of these methods.

1. If we stick to second-order methods based on the standard Galerkin formulation (such as Taylor-Hood elements), the convergence results that apply to the Crouzeix-Raviart method on triangles [11] for the polygonal case extend to method's obvious modification using our technique. Notice however that the application of this technique to the Crouzeix-Raviart method's extension to tetrahedra considered in [18] must be the object of a specific study. This is because it employs certain mean values along element edges as velocity degrees of freedom, instead of nodal values.

2. Methods using a piecewise quadratic representation of the velocity combined with the Petrov-Galerkin formulation due to Franca & Hughes [14] or the one of Douglas & Wang [13] can be combined with our technique quite in the same manner as Taylor-Hood elements. The final (second-order) qualitative results remain unchanged.

3. Any third-order method in the natural norms using a cubic velocity representation can also be optimally handled in association with our technique. This is true of Taylor-Hood element's cubic extension using the standard Galerkin formulation considered by Boffi [6], and also of the method in the Petrov-Galerkin formulation mentioned in the previous item. In both cases the analysis is based on the arguments developed in [20] and [21] for cubic Lagrange finite elements.

4. Methods of order $k \geq 4$ in the natural norm, though of limited interest, can also be combined with our technique. More particularly this is the case of the generalized Taylor-Hood pairing consisting of the continuous $P_K - P_{k-1}$ velocity-pressure representation considered in [6], or yet the *continuous* P_k - *discontinuous* P_{k-1} velocity-pressure method studied by Scott & Vogelius [26]. However here optimal convergence results hold under the condition that a numerical quadrature formula with a compatible order and without integration points in the interior of boundary edges or faces be employed to compute the right hand side term. We refer to [20] and [21] for more details about such a restriction, which also applies to isoparametric elements (cf. [10]).

In conclusion the author emphasizes that the scope in Computational Engineering of the approach adopted in this work to handle Dirichlet conditions prescribed on curved boundaries is much wider than the one of classical techniques such as isoparametric finite elements. This was shown in [19] and [22] in the framework of Maxwell's equations of Electromagnetism and deformations of elastic membranes in mixed formulation, respectively. Moreover, even in cases where the use of classical techniques is consolidated, our approach is at least as reliable and competitive in terms of accuracy.
We would also like to point out that, as far as we can see, our technique has only two drawbacks: first of all it is necessary to solve a non symmetric problem, even when the original problem is symmetric. Moreover for each boundary element a small matrix has to be inverted in order to determine the local basis functions. However none of both issues are a real problem nowadays, taking into account the state-of-the-art of Computational Linear Algebra.
A final remark on the choice of nodal points on Γ different from vertexes is in order. As one can easily infer from the analysis carried out throughout the paper, the construction of these nodes advocated in Section 2 is not compulsory at all. Actually, referring to Figures 1 and 3, any other choice in $\Delta_S \cap \Gamma$ for $S \in \mathcal{S}_h$ will do. However

intuitively we can say that these nodes should not be too close to the boundary vertexes of element S, since this may lead to a worse conditioning of the resulting linear system.

Remark 3 *Besides those considered in [22], applications to Solid Mechanics of the technique studied in this paper can be found in [24].* ■

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