



HAL
open science

Cumulants of multiinformation density in the case of a multivariate normal distribution

Guillaume Marrelec, Alain Giron

► **To cite this version:**

Guillaume Marrelec, Alain Giron. Cumulants of multiinformation density in the case of a multivariate normal distribution. *Statistics and Probability Letters*, 2020, 156, pp.108587. 10.1016/j.spl.2019.108587. hal-02374446

HAL Id: hal-02374446

<https://hal.sorbonne-universite.fr/hal-02374446>

Submitted on 21 Nov 2019

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Cumulants of multiinformation density in the case of a multivariate normal distribution

Guillaume Marrelec^{1,2} and Alain Giron^{1,2}

¹Laboratoire d'imagerie biomédicale (LIB), Sorbonne Université, CNRS, INSERM, F-75006, Paris, France. email: `firstname.lastname@inserm.fr`.

²Centre de recherches et d'études en sciences des interactions (CRÉSI), Center for Interaction Science (CIS), F-75006, Paris, France

November 13, 2019

Abstract

We consider a generalization of information density to a partitioning into $N \geq 2$ subvectors. We calculate its cumulant-generating function and its cumulants, showing that these quantities are only a function of all the regression coefficients associated with the partitioning.

Keywords: dependence; information; cumulant-generating function; cumulants; mutual information; multiinformation

1 Introduction

Let \mathbf{X} be a multivariate normal random variable with distribution $f(\mathbf{x})$, and $(\mathbf{X}_1, \mathbf{X}_2)$ a partitioning of \mathbf{X} into 2 subvectors with corresponding marginals $f_1(\mathbf{x}_1)$ and $f_2(\mathbf{x}_2)$. The information density relative to \mathbf{X} and the partitioning $(\mathbf{X}_1, \mathbf{X}_2)$ is the random variable defined as (Polyanskiy and Wu, 2017, §17.1)

$$i_d(\mathbf{X}; \mathbf{X}_1, \mathbf{X}_2) = \ln \frac{f(\mathbf{X}_1, \mathbf{X}_2)}{f_1(\mathbf{X}_1) f_2(\mathbf{X}_2)}. \quad (1)$$

One of the key features of information density is that its expectation yields mutual information (Kullback, 1968, Chap. 1, §2; Polyanskiy and Wu, 2017, §17.1). In the present paper, we consider a partitioning of \mathbf{X} into $N \geq 2$ subvectors $(\mathbf{X}_1, \dots, \mathbf{X}_N)$ with corresponding marginals f_1, \dots, f_N and define *multiinformation density* as

$$i_d(\mathbf{X}; \mathbf{X}_1, \dots, \mathbf{X}_N) = \ln \left[\frac{f(\mathbf{X})}{\prod_{n=1}^N f_n(\mathbf{X}_n)} \right].$$

The mean of this quantity

$$I(\mathbf{X}; \mathbf{X}_1, \dots, \mathbf{X}_N) = \int f(\mathbf{x}) \ln \left[\frac{f(\mathbf{x})}{\prod_{n=1}^N f_n(\mathbf{x}_n)} \right] d\mathbf{x}$$

is itself a generalization of mutual information known under different names: total correlation (Watanabe, 1960), multivariate constraint (Garner, 1962), δ (Joe, 1989), or multiinformation (Studený and Vejnarová, 1998). Our interest in i_d is driven by its close connection with mutual independence. Indeed, when the \mathbf{X}_n 's are mutually independent, multiinformation is classically equal to 0, but we also have $i_d \equiv 0$ and $\text{Var}(i_d) = 0$ (see Appendix A), yielding other statistical markers of independence. By contrast, dependence between the \mathbf{X}_n 's is a multivariate phenomenon that multiinformation, as a one-dimensional measure, can only partially quantify. We expect i_d to give a more detailed characterization of dependence, e.g., through its moments or cumulants.

We here focus on multivariate normal distributions. The family of multivariate normal distribution with given mean $\boldsymbol{\mu}$ can be parameterized by either a covariance matrix or a concentration/precision (i.e., inverse covariance) matrix. Either parameterization shows multivariate distributions according to a certain perspective and emphasizes different features (e.g., Markov properties for the concentration matrix). In this context, we wished to investigate the existence of a natural way of parameterizing dependencies, i.e., a parameter that would emphasize the dependence properties of the distribution.

The core of the present paper is the following theorem:

Theorem 1 *Let \mathbf{X} be a d -dimensional variable following a multivariate normal distribution with mean $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$. Partition \mathbf{X} into N subvectors $(\mathbf{X}_1, \dots, \mathbf{X}_N)$, and set i_d the corresponding multiinformation density. Then the cumulant-generating function of i_d is given by*

$$\ln \mathbb{E}(e^{ti_d}) = tI(\mathbf{X}_1; \dots; \mathbf{X}_N) - \frac{1}{2} \ln |\mathbf{I}_d - t\boldsymbol{\Gamma}|, \quad (2)$$

where

$$\boldsymbol{\Gamma} = \boldsymbol{\Sigma} \text{diag}(\boldsymbol{\Sigma}_{11}, \dots, \boldsymbol{\Sigma}_{NN})^{-1} - \mathbf{I}_d$$

is the block matrix whose diagonal blocks are equal to $\mathbf{0}$ and where each off-diagonal block (m, n) is the matrix of regression coefficients of \mathbf{X}_m on \mathbf{x}_n

$$\boldsymbol{\Gamma}_{m|n} = \boldsymbol{\Sigma}_{mn} \boldsymbol{\Sigma}_{nn}^{-1}, \quad m \neq n. \quad (3)$$

The cumulants of i_d are given by

$$\kappa_1(i_d) = I(\mathbf{X}_1; \dots, \mathbf{X}_N) \quad \text{and} \quad \kappa_l(i_d) = \frac{(l-1)!}{2} \text{tr}(\boldsymbol{\Gamma}^l), \quad l \geq 2.$$

This theorem is proved in Section 2. In Section 3, we investigate some consequences of this result. Section 4 is devoted to the discussion.

2 Proof of theorem

2.1 Cumulant-generating function

We partition $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ in accordance with the partitioning of \mathbf{X} , so that $\boldsymbol{\mu}_n$ is the expectation of \mathbf{X}_n and $\boldsymbol{\Sigma}_{mn}$ the matrix of covariances between \mathbf{X}_m and \mathbf{X}_n . Multiinformation between the \mathbf{X}_n 's yields

$$I(\mathbf{X}_1; \dots; \mathbf{X}_N) = \frac{1}{2} \ln \frac{\prod_{n=1}^N |\boldsymbol{\Sigma}_{nn}|}{|\boldsymbol{\Sigma}|}.$$

From there, we can express multiinformation density as

$$i_d = I(\mathbf{X}_1; \dots; \mathbf{X}_N) + j_d, \quad (4)$$

where j_d is defined as

$$j_d = \frac{1}{2} (\mathbf{X} - \boldsymbol{\mu})^t \boldsymbol{\Phi} (\mathbf{X} - \boldsymbol{\mu}). \quad (5)$$

and $\boldsymbol{\Phi}$ as

$$\boldsymbol{\Phi} = \text{diag}(\boldsymbol{\Sigma}_{11}, \dots, \boldsymbol{\Sigma}_{NN})^{-1} - \boldsymbol{\Sigma}^{-1}.$$

Here, $\text{diag}(\boldsymbol{\Sigma}_{11}, \dots, \boldsymbol{\Sigma}_{NN})$ stands for the block-diagonal matrix with diagonal blocks equal to the $\boldsymbol{\Sigma}_{nn}$'s. The moment-generating function of j_d yields

$$\mathbb{E}(e^{tj_d}) = \int (2\pi)^{-\frac{d}{2}} |\boldsymbol{\Sigma}|^{-\frac{1}{2}} e^{-\frac{1}{2}(\mathbf{x} - \boldsymbol{\mu})^t (\boldsymbol{\Sigma}^{-1} - t\boldsymbol{\Phi})(\mathbf{x} - \boldsymbol{\mu})} d\mathbf{x}.$$

Since it can be shown that $\boldsymbol{\Sigma}^{-1} - t\boldsymbol{\Phi}$ is positive definite at least in a neighborhood of $t = 0$ (see Appendix B), the integrand is proportional to a multivariate normal distribution with mean $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}^{-1} - t\boldsymbol{\Phi}$. Integration with respect to \mathbf{x} therefore yields

$$\mathbb{E}(e^{tj_d}) = \frac{|\boldsymbol{\Sigma}^{-1}|^{\frac{1}{2}}}{|\boldsymbol{\Sigma}^{-1} - t\boldsymbol{\Phi}|^{\frac{1}{2}}} = |\mathbf{I}_d - t\boldsymbol{\Gamma}|^{-\frac{1}{2}}$$

and

$$\ln \mathbb{E}(e^{tj_d}) = -\frac{1}{2} \ln |\mathbf{I}_d - t\boldsymbol{\Gamma}|, \quad (6)$$

where \mathbf{I}_d is the d -by- d unit matrix and $\boldsymbol{\Gamma} = \boldsymbol{\Sigma}\boldsymbol{\Phi}$ the block matrix whose diagonal blocks are equal to $\mathbf{0}$ and where each nondiagonal block (m, n) is the matrix of regression coefficients of \mathbf{X}_m on \mathbf{x}_n given by (Anderson, 2003, Definition 2.5.1)

$$\boldsymbol{\Gamma}_{m|n} = \boldsymbol{\Sigma}_{mn}\boldsymbol{\Sigma}_{nn}^{-1}, \quad m \neq n. \quad (7)$$

2.2 Cumulants

The cumulants of i_d can be calculated in closed form from those of j_d and Equation (4) by noting that the first cumulant, $\kappa_1(i_d)$, is shift-equivariant, while the others, $\kappa_i(i_d)$ for $i \geq 2$, are shift invariant (Kendall, 1945, §3.13). This leads to

$$\begin{cases} \kappa_1(i_d) &= I(\mathbf{X}_1; \dots; \mathbf{X}_N) + \kappa_1(j_d) \\ \kappa_l(i_d) &= \kappa_l(j_d), \end{cases} \quad l \geq 2. \quad (8)$$

Now, the cumulants of j_d can be easily computed as follows. Using the fact that $|\mathbf{A}| = e^{\text{tr}[\ln(\mathbf{A})]}$ (Higham, 2007), which, for a positive definite matrix, can be expressed as $\ln |\mathbf{A}| = \text{tr}[\ln(\mathbf{A})]$, we have from Equation (6)

$$\ln \mathbb{E} (e^{tj_d}) = -\frac{1}{2} \text{tr} [\ln(\mathbf{I}_d - t\mathbf{\Gamma})].$$

For t sufficiently small, we can perform a Taylor expansion of the log function around \mathbf{I}_d (Abramowitz and Stegun, 1972, Eq. 4.1.24), leading to

$$\begin{aligned} \ln \mathbb{E} (e^{tj_d}) &= \frac{1}{2} \text{tr} \left[\sum_{l=1}^{\infty} \frac{(t\mathbf{\Gamma})^l}{l} \right] \\ &= \sum_{l=1}^{\infty} \frac{t^l}{2l} \text{tr}(\mathbf{\Gamma}^l). \end{aligned}$$

Identification with the decomposition of the same function in terms of cumulants (Kendall, 1945, §3.12)

$$\ln \mathbb{E} (e^{tj_d}) = \sum_{l=1}^{\infty} \kappa_l \frac{t^l}{l!},$$

yields for the cumulants of j_d

$$\kappa_l(j_d) = \frac{(l-1)!}{2} \text{tr}(\mathbf{\Gamma}^l). \quad (9)$$

The same result could have been reached by using the fact that j_d is a quadratic function of a multidimensional normal variate \mathbf{x} , as evidenced in Equation (5), together with the expression of the cumulants of such functions (Magnus, 1986, Lemma 2).

The cumulants of i_d therefore yield

$$\kappa_1(i_d) = I(\mathbf{X}_1; \dots, \mathbf{X}_N),$$

as expected, since the first cumulant is also the mean (Kendall, 1945, §3.14), and, for $l \geq 2$,

$$\kappa_l(i_d) = \frac{(l-1)!}{2} \text{tr}(\mathbf{\Gamma}^l).$$

In particular, the variance, which is equal to the second cumulant (Kendall, 1945, §3.14) is given by

$$\text{Var}(i_d) = \kappa_2(i_d) = \sum_{1 \leq m < n \leq N} \text{tr}(\boldsymbol{\Sigma}_{mn} \boldsymbol{\Sigma}_{mm}^{-1} \boldsymbol{\Sigma}_{mn} \boldsymbol{\Sigma}_{nn}^{-1}). \quad (10)$$

In the even more particular case where all subvectors are 1-dimensional, we have

$$\text{Var}(i_d) = \sum_{1 \leq m < n \leq N} \rho_{mn}^2,$$

where ρ_{mn} is the usual correlation coefficient between X_m and X_n .

3 Consequences

We here investigate some consequences of the previous results: the particular case of partitioning into two subvectors, the irrelevance of the variances, and a graphical interpretation.

3.1 Partitioning into two subvectors

In the particular case where $N = 2$, multiinformation boils down to mutual information. The various powers of $\boldsymbol{\Gamma}$ can easily be calculated, yielding

$$\boldsymbol{\Gamma}^l = \begin{pmatrix} \mathbf{0} & \boldsymbol{\chi}_l \\ \boldsymbol{\Upsilon}_l & \mathbf{0} \end{pmatrix} \quad \text{for } l \text{ odd}$$

and

$$\boldsymbol{\Gamma}^l = \begin{pmatrix} \boldsymbol{\chi}_l & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Upsilon}_l \end{pmatrix} \quad \text{for } l \text{ even,}$$

where we set

$$\begin{cases} \boldsymbol{\chi}_l &= \underbrace{\boldsymbol{\Gamma}_{1|2} \boldsymbol{\Gamma}_{2|1} \dots}_{l \text{ factors}} \\ \boldsymbol{\Upsilon}_l &= \boldsymbol{\Gamma}_{2|1} \boldsymbol{\Gamma}_{1|2} \dots, \end{cases}$$

with the relationship that $\boldsymbol{\chi}_l = \boldsymbol{\Gamma}_{1|2} \boldsymbol{\Upsilon}_{l-1}$ and $\boldsymbol{\Upsilon}_l = \boldsymbol{\Gamma}_{2|1} \boldsymbol{\chi}_{l-1}$. For l odd, the trace of $\boldsymbol{\Gamma}^l$ is equal to 0; for l even, it is equal to $\text{tr}(\boldsymbol{\chi}_l) + \text{tr}(\boldsymbol{\Upsilon}_l)$, which can alternatively be expressed as

$$2 \text{tr}(\boldsymbol{\chi}_l) = 2 \text{tr} \left[(\boldsymbol{\Gamma}_{1|2} \boldsymbol{\Gamma}_{2|1})^{\frac{l}{2}} \right]$$

or

$$2 \text{tr}(\boldsymbol{\Upsilon}_l) = 2 \text{tr} \left[(\boldsymbol{\Gamma}_{2|1} \boldsymbol{\Gamma}_{1|2})^{\frac{l}{2}} \right].$$

In particular, the variance of i_d is equal to

$$\text{Var}(i_d) = \kappa_2(i_d) = \text{tr}(\boldsymbol{\Sigma}_{12} \boldsymbol{\Sigma}_{22}^{-1} \boldsymbol{\Sigma}_{21} \boldsymbol{\Sigma}_{11}^{-1}). \quad (11)$$

This quantity, which is a particular case of Equation (10), was introduced by Jupp and Mardia (1980) as an extension of the classical correlation coefficient in the case of multidimensional variates, with application to directed data (Mardia and Jupp, 2000, §11.2). It is the sum of the squared canonical correlation coefficients between \mathbf{X}_1 and \mathbf{X}_2 (Anderson, 2003, Chap. 12; Jupp and Mardia, 1980).

If we furthermore assume that X_1 is a 1-dimensional vector, the cumulant-generating function yields

$$\ln E(e^{ti_d}) = -\frac{t}{2} \ln(1 - \bar{R}^2) - \ln(1 - t^2 \bar{R}^2),$$

where \bar{R} is the multiple correlation coefficient (Anderson, 2003, § 2.5.2)

$$\bar{R}^2 = \frac{\Sigma_{12} \Sigma_{22}^{-1} \Sigma_{12}^t}{\sigma_1^2}.$$

The cumulants are equal to 0 for l odd and to $\kappa_l(i_d) = (l-1)! (\bar{R}^2)^{\frac{l}{2}}$ for l even. In particular, the variance reads $\text{Var}(i_d) = \bar{R}^2$.

Finally, if both X_1 and X_2 are assumed to be 1-dimensional vectors with correlation coefficient ρ , the cumulant-generating function reads

$$\ln E(e^{ti_d}) = -\frac{t}{2} \ln(1 - \rho^2) - \ln(1 - t^2 \rho^2),$$

with cumulants equal to 0 for l odd and $\kappa_l(i_d) = (l-1)! |\rho|^l$ for l even. In particular, the variance yields $\text{Var}(i_d) = \rho^2$.

3.2 Irrelevance of variances

Mutual information and multiinformation are both quantities that do not depend on the variance coefficients Σ_{kk} 's. This result can be generalized to the cumulant-generating function of i_d , and hence, its distribution, moments and cumulants. Indeed, let

$$\Sigma = \Delta \mathbf{R} \Delta$$

be the decomposition of the covariance matrix Σ such that $\Delta = (\Delta_{kl})$ is a diagonal matrix with $\Delta_{kk} = \sqrt{\Sigma_{kk}}$ and $\mathbf{R} = (R_{kl})$ is the correlation matrix with $R_{kl} = \Sigma_{kl} / \sqrt{\Sigma_{kk} \Sigma_{ll}}$. Block multiplication of $\Delta \mathbf{R} \Delta$ shows that we have $\Sigma_{mn} = \Delta_{mm} \mathbf{R}_{mn} \Delta_{nn}$ for any block Σ_{mn} of Σ . Using the fact that $(\mathbf{A}\mathbf{B})^{-1} = \mathbf{B}^{-1} \mathbf{A}^{-1}$ for any two invertible square matrices \mathbf{A} and \mathbf{B} , we have

$$\Gamma_{m|n} = \Delta_{mm} \mathbf{R}_{mn} \mathbf{R}_{nn}^{-1} \Delta_{nn}^{-1} = \Delta_{mm} \tilde{\Gamma}_{m|n} \Delta_{nn}^{-1},$$

where $\tilde{\Gamma}_{m|n}$ is the matrix of regression coefficients obtained by application of Equation (7) to the correlation matrix \mathbf{R} instead of the covariance matrix Σ . This result shows that $\Gamma = (\Gamma_{m|n})$ can be factorized into

$$\Gamma = \Delta \tilde{\Gamma} \Delta^{-1},$$

where we set $\tilde{\mathbf{\Gamma}} = (\tilde{\mathbf{\Gamma}}_{m|n})$. This takes us to

$$\begin{aligned} |\mathbf{I}_d - t\mathbf{\Gamma}| &= |\mathbf{\Delta}(\mathbf{I}_d - t\tilde{\mathbf{\Gamma}})\mathbf{\Delta}^{-1}| \\ &= |\mathbf{I}_d - t\tilde{\mathbf{\Gamma}}|. \end{aligned}$$

As a conclusion, we have that the cumulant-generating function of a multivariate distribution with covariance matrix $\mathbf{\Sigma}$ is the same as the cumulant-generating function of a multivariate distribution with covariance matrix \mathbf{R} , where \mathbf{R} is the correlation matrix associated with $\mathbf{\Sigma}$. This is a translation of the fact that i_d does not depend on the variance coefficients.

3.3 Graphical interpretation

While simplification of the expression for the cumulants through the explicit calculation of $\mathbf{\Gamma}^l$ is challenging in the general case, one can resort to a graphical interpretation of this matrix. Note first that the block (n, m) of $\mathbf{\Gamma}^l$ is given by

$$(\mathbf{\Gamma}^l)_{mn} = \sum_{q_1, \dots, q_{l-1}} \underbrace{\mathbf{\Gamma}_{mq_1} \dots \mathbf{\Gamma}_{q_{l-1}n}}_{l \text{ terms}}$$

and $\text{tr}(\mathbf{\Gamma}^l)$ by

$$\text{tr}(\mathbf{\Gamma}^l) = \sum_{n=1}^N \text{tr} [(\mathbf{\Gamma}^l)_{nn}] = \sum_{n=1}^N \text{tr} \left(\sum_{q_1, \dots, q_{l-1}} \mathbf{\Gamma}_{nq_{l-1}} \dots \mathbf{\Gamma}_{q_1 n} \right).$$

Consider then the directed and fully connected graph with N nodes $\{1, \dots, N\}$, an arrow from any m to any $n \neq m$ (no self-connections), and corresponding (potentially matrix) weight $\mathbf{\Gamma}_{n|m}$. In this graph, a directed loop is a directed path that begins and starts at the same node. It is a k -loop if the directed path is composed of exactly k arrows. For any node n and integer k , let $\mathcal{L}_k^{\rightarrow}(n)$ be the set of all directed k -loops starting and ending at node n and $\mathcal{L}_k^{\rightarrow}$ the set of all k -loops. For any directed path $p = (q_1 \rightarrow \dots \rightarrow q_k)$, define $\tau(p)$ as the trace of the product of the weights along p

$$\tau(p) = \text{tr} (\mathbf{\Gamma}_{q_k|q_{k-1}} \dots \mathbf{\Gamma}_{q_2|q_1}).$$

With these notations, $\text{tr}(\mathbf{\Gamma}^l)$ can be interpreted as the sum of the values taken by τ along all directed l -loops starting and ending at every node of the graph

$$\text{tr}(\mathbf{\Gamma}^l) = \sum_{n=1}^N \sum_{p \in \mathcal{L}_l^{\rightarrow}(n)} \tau(p).$$

It can also be seen as the sum of the values taken by τ along all directed l -loops

$$\text{tr}(\mathbf{\Gamma}^l) = \sum_{p \in \mathcal{L}_l^{\rightarrow}} \tau(p).$$

See Figure 1 for an illustration of this interpretation. Note that the fact that 1-loops do not exist is interpreted as $\text{tr}(\mathbf{\Gamma}) = 0$.

This graphical interpretation is in particular compatible with a partitioning into two subvectors. In that case, the corresponding directed graph only has two nodes and two arrows, and $\mathcal{L}_l^\rightarrow = \emptyset$ for l odd, which is in agreement with the fact that all cumulants of odd order are equal to zero.

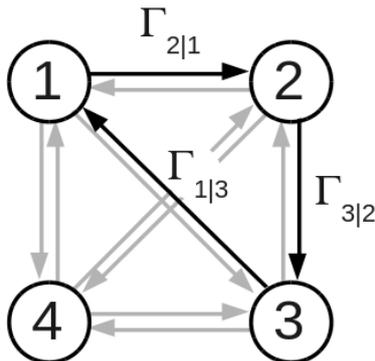


Figure 1: **Graphical interpretation of $\text{tr}(\mathbf{\Gamma}^l)$.** We consider the case $N = 4$ and $l = 3$. The directed 3-loop $lp = (1 \rightarrow 2 \rightarrow 3 \rightarrow 1)$ is represented with dark arrows. The value of τ on this loop is equal to $\tau(l) = \text{tr}(\mathbf{\Gamma}_{1|3}\mathbf{\Gamma}_{3|2}\mathbf{\Gamma}_{2|1})$. $\text{tr}(\mathbf{\Gamma}^3)$ is obtained by summing $\tau(p)$ over all 3-loops.

4 Discussion

In the present manuscript, we introduced multiinformation density, a random variable that generalizes information density and whose expectation defines multiinformation. We focused on the case of a multivariate normal distribution and derived a closed form for its cumulant-generating function as well as its cumulants. We showed that the cumulant-generating function does not depend on the values taken by the variance coefficients of the covariance matrix, and that the computations required have a simple graphical interpretation. We also considered the special case of a partitioning into two subvectors, showing the relationship between our results and existing quantities.

Interestingly, the results show that the cumulant-generating function of multiinformation density is a function of a specific quantity, namely the block matrix $\mathbf{\Gamma}$ composed of all matrices of regression coefficients corresponding to the partitioning of \mathbf{X} into $(\mathbf{X}_1, \dots, \mathbf{X}_N)$ as defined in Equation (7). This entails that the probability distribution of multiinformation density as well as all its moments are fully defined by $\mathbf{\Gamma}$. In particular, it can be shown that mutual information and multiinformation themselves, as expectations,

are functions of $\mathbf{\Gamma}$ only, namely (see Appendix C)

$$I(\mathbf{X}_1; \dots; \mathbf{X}_N) = -\frac{1}{2} \ln |\mathbf{I}_d + \mathbf{\Gamma}|.$$

Going back to our question of knowing whether the multivariate normal distribution family could be parameterized in a natural way with emphasis on its dependence properties, it can be argued that $\mathbf{\Gamma}$ is a good candidate to this aim.

Generalization The expression of the cumulant-generating function of multiinformation in the case of multivariate normal distributions is quite simple. This simplicity is mostly a consequence of the stability of the multivariate normal family to most operations performed here: (i) the product of the marginals is also a multivariate normal distribution (this is strongly related to the fact that independence and uncorrelatedness are equivalent for multivariate normal distributions); (ii) the ratio of the joint distribution to its marginals takes a simple form that is again closely related to the multivariate normal family; and (iii) the exponentiation of t times the log of this ratio still has the form of a multivariate normal distribution. It would be of interest to determine such a simplicity would still hold in more general settings such as more general distribution families and more general functions of the ratio $f_{\mathbf{X}}(\mathbf{X})/\prod f_{\mathbf{X}_i}(\mathbf{X}_i)$.

Regarding the type of family considered, a first step could be to consider the family of multivariate t distributions (Kotz and Nadarajah, 2004). In this case, $f_{\mathbf{X}}(\mathbf{x})$ and $f_{\mathbf{Y}}(\mathbf{y})$ are both multivariate t distributions (Kotz and Nadarajah, 2004, §1.10) but $f_{\mathbf{X}}(\mathbf{x}) f_{\mathbf{Y}}(\mathbf{y})$ itself is not a multivariate t distribution, and the cumulant-generating function of the log of the resulting ratio does not have a simple closed form. Another, further step would be to consider elliptically contoured distributions (Anderson, 2003, §2.7), but again no simplification seems to occur.

Another potential generalization would be obtained by replacing the log function of multiinformation density with a quantity of the form

$$g \left[\frac{f_{\mathbf{X}}(\mathbf{X})}{\prod f_{\mathbf{X}_i}(\mathbf{X}_i)} \right],$$

in the spirit of f -divergences (Csiszár, 1963, 1967; Ali and Silvey, 1966; Vajda, 1972). However, by replacing $g(t) = \ln(t)$ with another, more general form, we lose the key property that the log is the inverse of the exponentiation taken to compute the cumulant-generating function, which also simplifies calculations.

Future work Investigating the level of dependence between variables is still a thorny issue. Such an issue is usually tackled by investigating the

properties of mutual information or multiinformation. We believe that considering multiinformation density instead of multiinformation (i.e., the random variable instead of its expectation) could contribute to a better characterization of dependence. For instance, the variance of i_d , given in Equation (10), could be considered, in addition to or instead of multiinformation, to quantify the presence or absence of dependence. An important point to advance in this direction would be to provide estimators for the quantities obtained here. In the case of two variables, an estimator of this quantity, expressed in Equation (11), is readily available (Jupp and Mardia, 1980). Its generalization to more than two variables could yield a new tool to investigate mutual independence.

Another point of interest is the investigation of the behavior of multiinformation density in the case of large dimension. For instance, the asymptotic normality of this measure could be of interest. A quick derivation (see Appendix D) shows that, in the simple case of a homogeneous correlation matrix and a partitioning into 1-dimensional vectors, i_d is not asymptotically normal. We hope that tackling this issue in the more general case will help better understand dependences within systems composed of many variables.

A Mutual independence, i_d and multiinformation

We here show the equivalence between the following

1. The \mathbf{X}_n 's are mutually independent;
2. $i_d \equiv 0$;
3. $I(\mathbf{X}_1; \dots; \mathbf{X}_N) = 0$;
4. $\text{Var}(i_d) = 0$.

The equivalence between (1) and (2) is straightforward by definition of i_d . If $i_d \equiv 0$, then all its moments of i_d are equal to 0, including its expectation (multiinformation) and its variance, leading to $I(\mathbf{X}_1; \dots; \mathbf{X}_N) = 0$ and $\text{Var}(i_d) = 0$.

Since multiinformation is a Kullback-Leibler divergence, $I(\mathbf{X}_1; \dots; \mathbf{X}_N) = 0$, entails that we have

$$f(\mathbf{x}) = \prod_{n=1}^N f_n(\mathbf{x}_n)$$

for all \mathbf{x} , i.e., the \mathbf{X}_n 's are mutually independent.

Finally, if $\text{Var}(i_d) = 0$, then i_d is a constant, that is,

$$f(\mathbf{X}) = k \prod_{n=1}^N f_n(\mathbf{x}_n).$$

The fact that f and the f_n 's are distributions, and therefore must norm to 1, entails that we must have $k = 1$, that is, $i_d \equiv 0$.

B Positive-definiteness of $\Sigma^{-1} - t\Phi$

We need to show that $\Sigma^{-1} - t\Phi$ is a symmetric positive definite matrix in a neighborhood of $t = 0$. This matrix can be expressed as

$$\Sigma^{-1} - t\Phi = (1+t)\Sigma^{-1} - t \operatorname{diag}(\Sigma_{11}, \dots, \Sigma_{NN})^{-1}.$$

As a difference of two symmetric matrices, it is also symmetric. Furthermore, since the two matrices in the right-hand side of the equation are positive definite they are diagonalizable in the same basis, i.e., there exists a nonsingular matrix \mathbf{F} such that (Anderson, 2003, Theorem A.2.2)

$$\mathbf{F}^t \Sigma^{-1} \mathbf{F} = \begin{pmatrix} \lambda_1^2 & & \\ & \ddots & \\ & & \lambda_d^2 \end{pmatrix}$$

and

$$\mathbf{F}^t \operatorname{diag}(\Sigma_{11}, \dots, \Sigma_{NN})^{-1} \mathbf{F} = \mathbf{I}.$$

Since Σ^{-1} is positive definite, we furthermore have $\lambda_i^2 > 0$. $\Sigma^{-1} - t\Phi$ is therefore diagonalizable as well, with eigenvalues given by $(1+t)\lambda_i^2 - t = (\lambda_i^2 - 1)t + \lambda_i^2$, which is positive in a neighborhood of $t = 0$. $\Sigma^{-1} - t\Phi$ is therefore positive definite in a neighborhood of $t = 0$.

C Alternative expression of multiinformation

For a decomposition of a multidimensional normal variable into several subvectors, multiinformation reads

$$I(\mathbf{X}_1; \dots; \mathbf{X}_N) = \frac{1}{2} \ln \frac{\prod_{n=1}^N |\Sigma_{nn}|}{|\Sigma|}.$$

By comparison, we calculate

$$\begin{aligned} \mathbf{I}_d + \mathbf{\Gamma} &= \mathbf{I}_d + \Sigma \Phi \\ &= \mathbf{I}_d + \Sigma [\operatorname{diag}(\Sigma_{11}, \dots, \Sigma_{NN})^{-1} - \Sigma^{-1}] \\ &= \Sigma \operatorname{diag}(\Sigma_{11}, \dots, \Sigma_{NN})^{-1}, \end{aligned}$$

leading to

$$\begin{aligned} |\mathbf{I}_d + \mathbf{\Gamma}| &= |\Sigma \operatorname{diag}(\Sigma_{11}, \dots, \Sigma_{NN})^{-1}| \\ &= \frac{|\Sigma|}{\prod_{n=1}^N |\Sigma_{nn}|}, \end{aligned}$$

and, finally,

$$-\frac{1}{2} \ln |\mathbf{I}_d + \mathbf{\Gamma}| = \frac{1}{2} \ln \frac{\prod_{n=1}^N |\boldsymbol{\Sigma}_{nn}|}{|\boldsymbol{\Sigma}|}.$$

D Checking asymptotic normality

Let the correlation matrix \mathbf{R}_d be a d -by- d homogeneous matrix with parameter ρ , i.e., a matrix with 1s on the diagonal and all off-diagonal elements equal to ρ . Such a matrix has two eigenvalues: $1 + (d-1)\rho$ with multiplicity 1 (associated with the vector composed only of 1s) and $1 - \rho$ with multiplicity $d - 1$ (associated with the subspace of vectors with a zero mean). Such a matrix is positive definite for

$$-\frac{1}{d-1} \leq \rho < 1.$$

The expectation of i_d is given by

$$-\frac{1}{2} \{(d-1) \ln(1-\rho) + \ln[1 + (d-1)\rho]\}$$

To compute the higher cumulants of i_d , let \mathbf{U}_d the d -by- d matrix with all elements equal to 1. Using the fact that $\mathbf{\Gamma} = \rho(\mathbf{U}_d - \mathbf{I}_d)$ together with $\mathbf{U}_d^l = d^{l-1}\mathbf{U}_d$ for $l \geq 2$, we obtain

$$\begin{aligned} \mathbf{\Gamma}^l &= \rho^l \left[(-1)^l \mathbf{I}_d + \frac{(d-1)^l - (-1)^l}{d} \mathbf{U}_d \right] \\ \text{tr}(\mathbf{\Gamma}^l) &= \rho^l \left[(-1)^l d + (d-1)^l - (-1)^l \right] \\ \kappa_l(i_d) &= \frac{(l-1)!}{2} \rho^l \left[(-1)^l d + (d-1)^l - (-1)^l \right]. \end{aligned}$$

In particular, we have $\text{Var}(i_d) = \rho^2 d(d-1)/2$. For large d , we have $\kappa_l(i_d) \sim \frac{(l-1)!}{2} \rho^l d^l$ for $l \geq 2$ and, in particular, $\text{Var}(i_d) \sim \rho^2 d^2/2$. To investigate the asymptotic normality of i_d , we classically consider $u = [i_d - \text{E}(i_d)]/\sqrt{\text{Var}(i_d)}$. Using the fact that the cumulant of order l is homogeneous of degree l , we obtain $\kappa_l(u) \sim 2^{l/2-1}(l-1)! = \text{cte}$. If u were asymptotically normal, $\kappa_l(u)$ for $l \geq 3$ would tend to 0 as $d \rightarrow \infty$, which is not the case. As a consequence, u is not asymptotically normal.

References

Abramowitz, M., Stegun, I.A. (Eds.), 1972. Handbook of Mathematical Functions. Number 55 in Applied Math., National Bureau of Standards.

- Ali, S., Silvey, S.D., 1966. A general class of coefficients of divergence of one distribution from another. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)* 28, 131–142.
- Anderson, T.W., 2003. *An Introduction to Multivariate Statistical Analysis*. Wiley Series in Probability and Mathematical Statistics. 3rd ed., John Wiley and Sons, New York.
- Csiszár, I., 1963. Eine informationstheoretische Ungleichung und ihre Anwendung auf den Beweis der Ergodizität von Markoffschen Ketten. *A Magyar Tudományos Akadémia Matematikai és Fizikai Tudományok Osztályának Közleményei* 8, 85–108.
- Csiszár, I., 1967. Information-type measures of difference of probability distributions and indirect observation. *Studia Scientiarum Mathematicarum Hungarica* 2, 229–318.
- Garner, W.R., 1962. *Uncertainty and Structure as Psychological Concepts*. John Wiley & Sons, New York.
- Higham, N.J., 2007. Functions of matrices, in: Hogben, L. (Ed.), *Handbook of Linear Algebra*. Chapman & Hall/CRC Press, Boca Raton. *Discrete Mathematics and its Applications*. chapter 11.
- Joe, H., 1989. Relative entropy measures of multivariate dependence. *Journal of the American Statistical Association* 84, 157–164.
- Jupp, P.E., Mardia, K.V., 1980. A general correlation coefficient for directional data and related regression problems. *Biometrika* 67, 163–173.
- Kendall, M.G., 1945. *The Advanced Theory of Statistics*. volume 1. 2nd ed., Charles Griffin & Co. Ltd., London.
- Kotz, S., Nadarajah, S., 2004. *Multivariate t Distributions and their Applications*. Cambridge University Press, Cambridge, UK.
- Kullback, S., 1968. *Information Theory and Statistics*. Dover, Mineola, NY.
- Magnus, J., 1986. The exact moments of a ratio of quadratic forms. *Annales d'économie et de statistique* 4, 95–109.
- Mardia, K.V., Jupp, P.E., 2000. *Directional Statistics*. Wiley Series in Probability and Statistics, Wiley, Chichester.
- Polyanskiy, Y., Wu, Y., 2017. *Lecture notes on information theory*. <http://www.stat.yale.edu/~yw562/ln.html>.

- Studený, M., Vejnarová, J., 1998. The multiinformation function as a tool for measuring stochastic dependence, in: Jordan, M.I. (Ed.), Proceedings of the NATO Advanced Study Institute on Learning in Graphical Models, pp. 261–298.
- Vajda, I., 1972. On the f -divergence and singularity of probability measures. *Periodica Mathematica Hungarica* 2, 223–234.
- Watanabe, S., 1960. Information theoretical analysis of multivariate correlation. *IBM Journal of Research and Development* 4, 66–82.