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FIXED POINTS FOR NON-EXPANSIVE SET-VALUED MAPPINGS

JEAN SAINT RAYMOND

Let E be a Banach space and $F : E \rightrightarrows E$ be a 1-Lipschitz set-valued mapping with closed convex non-empty values. We study the set of fixed points $\text{Fix}(F) = \{x \in E : x \in F(x)\}$ and provide in any space E with $\dim(E) \geq 2$ an example of such a mapping F such that $\text{Fix}(F)$ is not connected.

1. Introduction

In this paper we are concerned with set-valued mappings from a Banach space E into itself having closed convex values. We will consider only mappings F which are 1-Lipschitz for the Hausdorff distance d_H on the set $\mathcal{F}(E)$ of non-empty closed subsets of E . Recall that, for A and B in $\mathcal{F}(E)$:

$$d_H(A, B) = \max\left(\sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A)\right)$$

so $d(z, F(y)) \leq \|x - y\|$ for all $x, y \in E$ and $z \in F(x)$.

For such a mapping F we will be essentially interested in the set $\text{Fix}(F) = \{x \in E : x \in F(x)\}$ of fixed points of F which is clearly closed in E . Of course it can happen that $\text{Fix}(F) = \emptyset$, for example if $F(x) = \{x + a\}$ where a is a fixed non-zero vector in E .

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The case of multivalued contraction mappings, i.e. the case where F is q -Lipschitz for a $q < 1$ was extensively studied for long (see [2], [1], [3], [4]) and many properties of structure or conservation for the set $\text{Fix}(F)$ of fixed points were shown. For example:

- i. $\text{Fix}(F) \neq \emptyset$ (even for non convex-valued mappings). ([2], [1])
- ii. $\text{Fix}(F)$ is an absolute retract, in particular it is path-connected. ([3])
- iii. $\text{Fix}(F)$ is not a singleton if all values $F(x)$ have several points ([4] for the case where $q < \frac{1}{2}$ or E is a Hilbert space, [5] for the general case)
- iv. $\text{Fix}(F)$ is bounded if so are all values $F(x)$ for $x \in E$ (or even for only one $x \in E$). ([5])
- v. $\text{Fix}(F)$ is compact if so are all values $F(x)$ for $x \in E$.([5])

We shall show in this paper that most of these results disappear when the Lipschitz constant q of F (which is < 1 if F is a contraction mapping) is only assumed to be ≤ 1 and $\dim(E) \geq 2$.

In the sequel we will call *quasi-contraction* any 1-Lipschitz set-valued mapping from E to E with (non-empty) closed convex values.

Clearly properties (iv) and (v) become false already in the trivial example where $E = \mathbb{R}$ and $F(x) = \{x\}$ since $F(x)$ is then always single-valued, and a fortiori compact and bounded though $\text{Fix}(F) = \mathbb{R}$ is unbounded. We provide in section 4 an example of quasi-contraction in a Hilbert space for which property (iii) does not hold. Concerning property (ii) and namely the connectedness of the set of fixed points of a quasi-contraction, the main part of this paper consists in proving that it does not hold in general.

After studying in section 2 the very simple case where E has dimension 1, we look in section 3 at the case where $F(x)$ is single-valued. It turns out that if E is finite-dimensional we can prove that $\text{Fix}(F)$ is connected but that this is no more true for infinite-dimensional spaces.

The remainder of the paper is devoted to show that in every normed space of dimension at least 2 one can construct a quasi-contraction having a non-connected set of fixed points. In section 5 we provide such a construction for the 2-dimensional euclidean space, and generalize it to every 2-dimensional smooth normed space in section 7. The general case is dealt in sections 6 and 7.

For any two points a and b in a normed space E we will denote by $[a, b] \subset E$ the segment with endpoints a and b , it is the set $\{(1-t)a + tb : t \in [0, 1]\}$. The following simple lemma will be of constant use throughout the paper.

Lemma 1.1. *Let E be a normed space, a, b, a', b' be points of E . Then*

$$d_H([a, b], [a', b']) \leq \max(\|a - a'\|, \|b - b'\|).$$

Proof. If $w \in [a, b]$ we have $w = ta + (1 - t)b$ for some $t \in [0, 1]$ hence

$$\begin{aligned} d(w, [a', b']) &\leq d(ta + (1 - t)b, ta' + (1 - t)b') \\ &= \|t((a - a') + (1 - t)(b - b'))\| \\ &\leq t\|a - a'\| + (1 - t)\|b - b'\| \\ &\leq \max(\|a - a'\|, \|b - b'\|) \end{aligned}$$

whence it follows that $\sup_{w \in [a, b]} d(w, [a', b']) \leq \max(\|a - a'\|, \|b - b'\|)$, hence that $d_H([a, b], [a', b']) \leq \max(\|a - a'\|, \|b - b'\|)$. □

2. The case of dimension 1

Proposition 2.1. *Let F be a quasi-contraction from \mathbb{R} to \mathbb{R} (the values of F are closed intervals). Then $\text{Fix}(F)$ is either empty or a closed interval. In particular $\text{Fix}(F)$ is connected.*

Proof. Since F is a quasi-contraction, it is easy to see that there are two mappings a and b from \mathbb{R} to $\bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ such that $F(x) = \mathbb{R} \cap [a(x), b(x)]$ where $a(x) = -\infty$ for all x or $a(x) > -\infty$ for all x (and $b(x) = +\infty$ for all x or $b(x) < +\infty$ for all x). If $-\infty < a(x) \leq a(y)$ we have $d(a(x), F(y)) = |a(y) - a(x)|$ hence $d_H(F(x), F(y)) \geq |a(y) - a(x)|$ and similarly $d_H(F(x), F(y)) \geq |b(y) - b(x)|$ if $b(x) < +\infty$. So by Lemma 1.1

$$d_H(F(x), F(y)) = \max(|a(x) - a(y)|, |b(x) - b(y)|)$$

whence it follows that both a and b are 1-Lipschitz or constantly infinite.

Suppose towards a contradiction that there exist $x_0, x'_0 \in \text{Fix}(F)$, $x'_0 < x < x_0$ and $x \notin \text{Fix}(F)$. Then we have $b(x) = b(x_0) = +\infty$ or $b(x_0) \geq x_0$, hence

$$b(x) \geq b(x_0) - |x - x_0| \geq x_0 - (x_0 - x) = x$$

Thus since $x \notin \text{Fix}(F)$ we necessarily get $a(x) > x$, so $a(x) > -\infty$, and since $x'_0 < x$,

$$a(x'_0) \geq a(x) - |x - x'_0| > x - (x - x'_0) = x'_0$$

hence $x'_0 \notin F(x_0)$, a contradiction. This shows that $\text{Fix}(F)$ is an interval. □

3. The case of functions

In this section we consider single-valued quasi-contractions F , which we identify with 1-Lipschitz functions f by $F(x) = \{f(x)\}$. More generally we will study the case where H is a closed convex subset of the Banach space E and $f : H \rightarrow H$ is 1-Lipschitz.

Proposition 3.1. *Let E be a strictly convex normed space, $H \subset E$ be closed and convex, $f : H \rightarrow H$ be 1-Lipschitz. Then $\text{Fix}(F)$ is convex, possibly empty.*

Proof. Clearly, if $H = E$, $a \in E$ is not zero and f is the translation $x \mapsto x + a$, f is an isometry and $\text{Fix}(f) = \emptyset$.

If $f : H \rightarrow H$ is 1-Lipschitz and u, v are two distinct points of $\text{Fix}(f)$ then for all $t \in]0, 1[$ the points $x_t = tu + (1 - t)v$ and $y_t = f(x_t)$ satisfy

$$\begin{aligned} \|y_t - u\| &= \|f(x_t) - f(u)\| \leq \|x_t - u\| = (1 - t)\|u - v\| \\ \|y_t - v\| &= \|f(x_t) - f(v)\| \leq \|x_t - v\| = t\|u - v\| \end{aligned}$$

thus $\|u - v\| \leq \|y_t - u\| + \|y_t - v\| \leq \|u - v\|$, whence $y_t \in [u, v]$ because E is strictly convex, and $y_t - v = s(u - v)$ for some $s \in [0, 1]$. Then since $s\|u - v\| = \|y_t - v\| = t\|u - v\|$, we conclude that $s = t$ and $y_t = x_t$, hence that $x_t \in \text{Fix}(F)$. □

If E is not strictly convex, the previous result does not hold any more. For example if $E = \mathbb{R}^2$ equipped with the norm $u = (x, y) \mapsto \|u\|_\infty = \max(|x|, |y|)$, the function $f : (x, y) \mapsto (x, \sin x)$ is 1-Lipschitz: indeed

$$\|f(x, y) - f(x', y')\| = \max(|x - x'|, |\sin x - \sin x'|) = |x - x'| \leq \|(x, y) - (x', y')\|$$

and $\text{Fix}(f) = \{(x, \sin x) : x \in \mathbb{R}\}$ is connected, but not convex. So far it is unclear, even in a finite-dimensional space, whether a 1-Lipschitz mapping could have a non-connected set of fixed points. Nevertheless we shall see later on, in Theorems 3.3, 3.4 and Corollary 7.7 what really happens.

Lemma 3.2. *Let H be a non-empty convex compact subset of a finite-dimensional space E and $f : H \rightarrow H$ be a 1-Lipschitz function. Then the set $\text{Fix}(f)$ is compact connected and non-empty.*

Proof. It follows readily from Brouwer's theorem that $\text{Fix}(f)$ is non-empty. And it is closed in H hence compact. For the connectedness we proceed by induction on the dimension of E , or more precisely on the dimension $\delta(H)$ of the linear subspace A_H generated by $H - H$. If $\delta(H) = 1$ then $A_H \approx \mathbb{R}$ is strictly convex and it follows from Proposition 3.1 that $\text{Fix}(f)$ is convex, hence connected.

Assume that the statement of the lemma holds for all compact convex K such that $\delta(K) < n$, that H is a compact convex subset of E such that $\delta(H) = n$ and that a_0 and a_1 are two distinct fixed points of the 1-Lipschitz function $f : H \rightarrow H$. By translation invariance we can and do assume that $0 \in H$ and H spans E . Choose by Hahn-Banach's theorem a linear functional $\xi \in E^*$ of norm 1 such that $\langle \xi, a_1 - a_0 \rangle = \|a_1 - a_0\|$.

For all $t \in [0, 1]$ consider the set

$$H_t = \{x \in H : \|x - a_0\| \leq t\|a_1 - a_0\| \text{ and } \|x - a_1\| \leq (1 - t)\|a_1 - a_0\|\}$$

which is convex and compact. Then $f(H_t) \subset H_t$: indeed if $x \in H_t, y = f(x) \in H$ and

$$\begin{cases} \|y - a_0\| = \|f(x) - f(a_0)\| \leq \|x - a_0\| \leq t\|a_1 - a_0\| \\ \|y - a_1\| = \|f(x) - f(a_1)\| \leq \|x - a_1\| \leq (1 - t)\|a_1 - a_0\|. \end{cases}$$

Moreover for $x \in H_t$,

$$\begin{cases} \langle \xi, x - a_0 \rangle \leq \|\xi\| \cdot \|x - a_0\| \leq t\|a_1 - a_0\| \\ \langle \xi, a_1 - x \rangle \leq \|\xi\| \cdot \|x - a_1\| \leq (1 - t)\|a_1 - a_0\| \end{cases}$$

hence

$$\begin{aligned} 0 &= \langle \xi, a_1 - a_0 \rangle - \|a_1 - a_0\| = \langle \xi, x - a_0 \rangle + \langle \xi, a_1 - x \rangle - \|a_1 - a_0\| \\ &= (\langle \xi, x - a_0 \rangle - t\|a_1 - a_0\|) + (\langle \xi, a_1 - x \rangle - (1 - t)\|a_1 - a_0\|) \end{aligned}$$

and

$$0 \leq t\|a_1 - a_0\| - \langle \xi, x - a_0 \rangle = (\langle \xi, a_1 - x \rangle - (1 - t)\|a_1 - a_0\|) \leq 0$$

from what we deduce that

$$\langle \xi, x - a_0 \rangle = t\|a_1 - a_0\| \text{ and } \langle \xi, a_1 - x \rangle = (1 - t)\|a_1 - a_0\|,$$

hence that $\langle \xi, x \rangle = \theta := \langle \xi, a_0 \rangle + t\|a_1 - a_0\|$. Thus this shows that H_t is included in the affine hyperplane $V_\theta = \{x \in E : \langle \xi, x \rangle = \theta\}$ for which $\delta(V_\theta) < \dim(E) = n$. It follows then from the induction hypothesis that $H_t \cap \text{Fix}(f) = \text{Fix}(f|_{H_t})$ is compact connected and non-empty.

Assume that $\text{Fix}(f)$ is not connected. So it would exist two disjoint compact subsets A_0 and A_1 of H such that $\text{Fix}(f) \subset A_0 \cup A_1$ and two points a_0 and a_1 with $a_i \in A_i$. For $t \in [0, 1]$, let H_t be the set introduced above which corresponds to the points a_0, a_1 . Then, from what precedes, for all $t \in [0, 1]$, $\text{Fix}(f|_{H_t}) \subset A_0 \cup A_1$ ($i \in \{0, 1\}$) and, by connectedness of $\text{Fix}(f|_{H_t})$, $\text{Fix}(f|_{H_t}) \subset A_i$ for some i . Then the sets $T_i = \{t \in [0, 1] : \text{Fix}(f|_{H_t}) \subset A_i\}$ form a partition of $[0, 1]$. Moreover,

note that $0 \in T_0$ and $1 \in T_1$. Thus, if we prove that T_0 and T_1 are also closed, we will get a contradiction which will complete the proof of the connectedness of $\text{Fix}(f)$.

Let (t_n) be a sequence in T_0 which converges to t^* ; there exists for all n a point $x_n \in A_0 \cap \text{Fix}(f_{H_n})$. Since $H \cap B(a_0, \|a_1 - a_0\|)$ is compact and $H_{t_n} \subset H \cap B(a_0, \|a_1 - a_0\|)$, up to passing to a subsequence we can assume that (x_n) converges to some point $x^* \in A_0$. We have

$$\|x^* - a_0\| = \lim_n \|x_n - a_0\| \leq \limsup_n t_n \|a_1 - a_0\| = t^* \|a_1 - a_0\|$$

and similarly $\|x^* - a_1\| \leq (1 - t^*) \|a_1 - a_0\|$.

Thus $x^* \in H_{t^*}$. Moreover $\|f(x^*) - x^*\| = \lim \|f(x_n) - x_n\| = 0$. It follows that $x^* \in \text{Fix}(f_{H_{t^*}}) \cap A_0$, hence that $\text{Fix}(f_{H_{t^*}}) \cap A_0 \neq \emptyset$ and that $t^* \in T_0$. By the same argument one can show that T_1 is closed. This completes the proof of the connectedness of $\text{Fix}(f)$, hence this of Lemma 3.2. \square

Theorem 3.3. *Let E be a normed finite-dimensional space, $H \subset E$ be a closed convex subset and $f : H \rightarrow H$ be a 1-Lipschitz function. Then the set of fixed points of f is connected.*

Proof. It is enough to consider the case where $\text{Fix}(f)$ is non-empty. Then let $a \in \text{Fix}(f)$. For all integer $n \geq 1$ the set $H_n = \{x \in H : \|x - a\| \leq n\}$ is compact convex non-empty and stable under f . So it follows from Lemma 3.2 that $\text{Fix}(f|_{H_n}) = \text{Fix}(f) \cap H_n$ is connected and contains a . Then $\bigcup_n \text{Fix}(f_{H_n}) = \text{Fix}(f)$ is connected. \square

We now show that for infinite-dimensional spaces E there is no particular topological property of the sets $\text{Fix}(f)$ for 1-Lipschitz functions $f : E \rightarrow E$. Indeed :

Theorem 3.4. *Let X be a complete metric space. Then there exist a Banach space E and a 1-Lipschitz function $f : E \rightarrow E$ such that $\text{Fix}(f)$ is isometric to X . Moreover if X is separable the space E can be chosen separable.*

Proof. Remark first that since $\text{Fix}(f)$ is closed in E hence complete, the completeness of X is a necessary condition.

It is well-known that any metric space X can be isometrically embedded into a Banach space. For example if $a \in X$ and D is a dense subset of X the function $\psi : x \mapsto \left(d(x, y) - d(a, y) \right)_{y \in D}$ is an isometry from X to a subset of the space ℓ_D^∞ . And if X is separable, the space $\overline{\text{span}}(\psi(X))$ is a separable Banach space.

Recall that c_0 denotes the Banach space of all real sequences converging to 0 equipped with the norm $\|u\| = \sup_n |x_n|$ and denote $\underline{0}$ the null

sequence in c_0 . Choose an isometric embedding $j : X \rightarrow W$ for some Banach space W and define $H = j(X) \subset W$ and $E = W \times c_0$ equipped with the norm $(w, u) \mapsto \max(\|w\|, \|u\|)$.

For $w \in W$ and $u \in c_0$ define $f(w, u) = (w, v)$ where $v = (v_n) \in c_0$ is defined by

$$v_n = \begin{cases} d(w, H) & \text{if } n = 0 \\ u_{n-1} & \text{if } n > 0 \end{cases}$$

Claim 3.5. The function f is 1-Lipschitz and even an isometry.

Proof. We have

$$\begin{aligned} \|f(w, u) - f(w', u')\| &= \max(\|w - w'\|, |d(w, H) - d(w', H)|, \sup_{n \geq 1} |u_{n-1} - u'_{n-1}|) \\ &= \max(\|w - w'\|, |d(w, H) - d(w', H)|, \|u - u'\|) \\ &= \max(\|w - w'\|, \|u - u'\|) = \|(w, u) - (w', u')\| \end{aligned}$$

since $|d(w, H) - d(w', H)| \leq \|w - w'\|$. ◇

Claim 3.6. $\text{Fix}(f) = H \times \{\underline{0}\}$.

Proof. It is clear that if $w \in H$ then $f(w, \underline{0}) = (w, \underline{0})$.

Conversely, if $u = (x_n)$ and (w, u) is a fixed point of f we have $x_0 = d(w, H)$ and $x_n = x_{n-1}$ for all $n \geq 1$. Thus u is the constant sequence with value $d(w, H)$, which does not belong to c_0 if $d(w, H) \neq 0$. So $w \in H$ and $u = \underline{0}$. ◇

It follows from previous claim that the function $x \mapsto (j(x), \underline{0})$ is an isometry from X onto $\text{Fix}(f)$. □

4. Uniqueness of fixed points

We provide in this section an example of a quasi-contraction F on a Hilbert space H such that $F(x)$ is a singleton for no $x \in H$ but $\text{Fix}(F)$ is a singleton. And this shows that Property (iii) in the Introduction does not hold in general for quasi-contractions.

Theorem 4.1. *There exists a quasi-contraction F on a Hilbert space H such that $\text{diam}(F(x)) = 1$ for all $x \in H$ but $\text{Fix}(F)$ is a singleton.*

Proof. Let H be the Hilbert space ℓ^2 , $S : H \rightarrow H$ be the isometric mapping defined by $x = (x_n)_{n \geq 0} \mapsto y = (y_n)_{n \geq 0}$ where $y_0 = 0$ and $y_n = x_{n-1}$ for $n > 0$. Let $u = (u_n) \in H$ be the unit vector such that $u_0 = 1$ and $u_n = 0$ for $n > 0$, and $\underline{0}$ be the null vector of H .

For $x \in H$ define $F(x)$ as the segment $[S(x), S(x) + u]$ whose diameter is 1. We claim that F is a quasi-contraction. Indeed by Lemma 1.1:

$$\begin{aligned} d_H(F(y), F(x)) &\leq \max(\|S(y) - S(x)\|, \|(S(y) + u) - (S(x) + u)\|) \\ &= \|S(y) - S(x)\| = \|S(y - x)\| = \|y - x\|. \end{aligned}$$

If $x^* = (x_n^*)$ is a fixed point of F there exists some $t \in [0, 1]$ such that

$$x^* = (1 - t)S(x^*) + t(S(x^*) + u) = S(x^*) + tu$$

so $x_0^* = tu_0 = t$ and $x_n^* = x_{n-1}^*$ for $n > 0$. This implies that the sequence $(x_n) \in \ell^2$ has to be constant with the value t , which is possible only with $t = 0$, and $x^* = \underline{0}$. And this shows that $\text{Fix}(F) = \{\underline{0}\}$ is a singleton. \square

5. The 2-dimensional euclidean space

The aim of this section is to construct a quasi-contraction on the 2-dimensional euclidean space \mathbb{R}^2 whose set of fixed points is not connected. It follows from Proposition 2.1 that such a construction cannot be achieved in a 1-dimensional space, and from Proposition 3.1 that it is impossible with a single-valued quasi-contraction.

Consider the points $x_0 = (-1, 0)$ and $x_1 = (1, 0)$ of the euclidean space \mathbb{R}^2 and the symmetry $S : (u, v) \mapsto (-u, v)$ of \mathbb{R}^2 exchanging x_0 and x_1 . We want to define two 1-Lipschitz mappings α and β from \mathbb{R}^2 to itself such that $\alpha(x_0) = x_0$ and $S \circ \beta(z) = \alpha \circ S(z)$ for all z . In particular this implies $\beta(x_1) = S \circ \alpha(x_0) = S(x_0) = x_1$. We define then the set-valued mapping F by $F(z) = [\alpha(z), \beta(z)]$, which is clearly convex and closed.

Lemma 5.1. *If the mapping α is 1-Lipschitz then F is a quasi-contraction.*

Proof. Since S is an isometry it is clear that $\beta = S \circ \alpha \circ S$ is 1-Lipschitz too. Then by Lemma 1.1, if z and z' are in \mathbb{R}^2

$$d_H(F(z), F(z')) \leq \max(\|\alpha(z) - \alpha(z')\|, \|\beta(z) - \beta(z')\|) \leq \|z - z'\|,$$

the wanted inequality. \square

Fix $\varepsilon \in]0, 1]$ and define the function $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ by

$$\varphi(t) = \begin{cases} \varepsilon & \text{if } t \leq 0 \\ t + \varepsilon e^{-t} & \text{if } t \geq 0 \end{cases}$$

Claim 5.2. The function φ is 1-Lipschitz and has no fixed point on \mathbb{R} .

Proof. It is immediately checked that φ is continuous, derivable on $\mathbb{R} \setminus \{0\}$, that $\varphi'(t) = 0$ for $t < 0$ and $\varphi'(t) = 1 - \varepsilon e^{-t} \geq 0$ if $t > 0$. Since $|\varphi'(t)| \leq 1$ for $t \neq 0$, the function φ is 1-Lipschitz. It is non-decreasing with values in $[\varepsilon, +\infty[$ and if t^* were a fixed point of φ we would have $t^* = \varphi(t^*) \geq \varepsilon > 0$ and $t^* = \varphi(t^*) = t^* + \varepsilon e^{-t^*}$ hence $e^{-t^*} = 0$, that is impossible. \diamond

Define now the function α on $X = \{x_0, x_1\} \cup (\{0\} \times \mathbb{R})$ by

$$\begin{cases} \alpha(x_0) = x_0 \\ \alpha(x_1) = (0, \varepsilon) \\ \alpha(0, v) = \left(-\frac{1}{2}, \varphi(v)\right) \end{cases}$$

Lemma 5.3. *It is possible to choose $\varepsilon \in]0, 1]$ such that this function α be 1-Lipschitz on X .*

Proof. For $v \in \mathbb{R}$ denote $y_v = (0, v) \in X$. We have to prove that for some convenient $\varepsilon > 0$:

- i. $\|\alpha(x_0) - \alpha(x_1)\|^2 \leq \|x_0 - x_1\|^2 = 4$,
- ii. $\forall v \in \mathbb{R}, \|\alpha(x_0) - \alpha(y_v)\|^2 \leq \|x_0 - y_v\|^2$,
- iii. $\forall v \in \mathbb{R}, \|\alpha(x_1) - \alpha(y_v)\|^2 \leq \|x_1 - y_v\|^2$,
- iv. $\forall v, w \in \mathbb{R}, \|\alpha(y_v) - \alpha(y_w)\| \leq \|y_v - y_w\|$.

For (i) we must have $1 + \varepsilon^2 \leq 4$, that is true since $\varepsilon \leq 1 < \sqrt{3}$.

For (ii) we must have

$$\left(-1 + \frac{1}{2}\right)^2 + \varphi(v)^2 \leq 1 + v^2$$

it is $\frac{1}{4} + \varphi(v)^2 \leq 1 + v^2$. And since $\varphi(v)^2 = \varepsilon^2$ if $v \leq 0$ and if $v \geq 0$:

$$\begin{aligned} \varphi(v)^2 &= (v + \varepsilon e^{-v})^2 = v^2 + \varepsilon^2 e^{-2v} + 2\varepsilon v e^{-v} \leq v^2 + \varepsilon^2 + 2\varepsilon \sup_{t \geq 0} t e^{-t} \\ &= v^2 + \varepsilon^2 + 2e^{-1} \varepsilon \leq v^2 + \varepsilon^2 + \varepsilon \end{aligned}$$

we must have $\frac{1}{4} + v^2 + \varepsilon^2 + \varepsilon \leq 1 + v^2$, that holds as soon as $\varepsilon^2 + \varepsilon \leq \frac{3}{4}$, hence whenever $0 < \varepsilon \leq \frac{1}{2}$.

For (iii) we must have $(\frac{1}{2})^2 + (\varphi(v) - \varepsilon)^2 \leq 1 + v^2$. And since $\varphi(v) \geq \varepsilon$ we have $(\varphi(v) - \varepsilon)^2 \leq \varphi(v)^2$. We have seen that if ε is chosen in $]0, \frac{1}{2}]$ then for all v : $\frac{1}{4} + \varphi(v)^2 \leq 1 + v^2$, so a fortiori $\frac{1}{4} + (\varphi(v) - \varepsilon)^2 \leq 1 + v^2$.

Finally for (iv), we have to show that

$$\|\alpha(y_v) - \alpha(y_w)\| = |\varphi(v) - \varphi(w)| \leq \|y_v - y_w\| = |v - w|$$

but this follows immediately from Claim 5.2.

Taking $\varepsilon = \frac{1}{2}$ completes the proof of Lemma 5.3. \square

Using Kirszbraun-Valentine's Theorem, we can extend the function α into a 1-Lipschitz function (still denoted by α) from \mathbb{R}^2 to \mathbb{R}^2 , and then define $\beta = S \circ \alpha \circ S$, which is 1-Lipschitz too.

Theorem 5.4. *The set-valued mapping $F : z \mapsto [\alpha(z), \beta(z)]$ is 1-Lipschitz, but the set $\text{Fix}(F)$ of its fixed points is not connected.*

Proof. That F be 1-Lipschitz follows from Lemma 5.1. Since $x_0 = \alpha(x_0) \in F(x_0)$ we have $x_0 \in \text{Fix}(F)$ and since $x_1 = \beta(x_1) \in F(x_1)$ we have $x_1 \in \text{Fix}(F)$. Hence $\{x_0, x_1\} \subset \text{Fix}(F)$.

We now show that $(\{0\} \times \mathbb{R}) \cap \text{Fix}(F) = \emptyset$. Indeed if there were some $y_v = (0, v)$ in $\text{Fix}(F)$ we should have $y_v \in \text{conv}(\alpha(y_v), \beta(y_v))$. Since $\alpha(y_v) = (-\frac{1}{2}, \varphi(v))$ and $\beta(y_v) = (\frac{1}{2}, \varphi(v))$ we would get

$$(0, v) = y_v \in \text{conv}(\alpha(y_v), \beta(y_v)) = [-\frac{1}{2}, \frac{1}{2}] \times \{\varphi(v)\}$$

hence $\varphi(v) = v$, in contradiction with Claim 5.2.

It follows that the two disjoint open subsets $W_0 = \{(x, y) \in \text{Fix}(F) : x < 0\}$ and $W_1 = \{(x, y) \in \text{Fix}(F) : x > 0\}$ of $\text{Fix}(F)$ are both non-empty and cover $\text{Fix}(F)$. Thus $\text{Fix}(F)$ is not connected. \square

6. The non-smooth case

It is also possible to give a simple example in any normed space E whose dual space E^* is not strictly convex (in particular if the norm of E itself is not smooth) of a quasi-contraction whose set of fixed points is not connected.

If E^* is not strictly convex there are two non-zero vectors u and v of E^* such that $\|u\| = \|u + v\| = \|u - v\| = 1$. Define then the real function h on E by

$$h(x) = \langle u, x \rangle + \sin^2(\langle v, x \rangle)$$

Lemma 6.1. *The function h is 1-Lipschitz.*

Proof. In fact h is of class \mathcal{C}^1 and its differential at x is $h'(x) = u + \sin(2\langle v, x \rangle) \cdot v$. The convex function $v : t \mapsto \|u + tv\|$ satisfies $v(-1) = v(1) = 1$ hence $v(t) \leq 1$ for $t \in [-1, 1]$. It follows that $\|h'(x)\| \leq 1$ for all x hence that h is 1-Lipschitz. \square

Lemma 6.2. *The set-valued mapping $P : \mathbb{R} \rightrightarrows E$ defined by $P(t) = \{y \in E : \langle u, y \rangle \geq t\}$ is 1-Lipschitz and takes closed convex non-empty values.*

Proof. It is clear that $P(x)$ is convex closed and non-empty. Notice that if $t \leq t'$ then we have $P(t') \subset P(t)$, so $d_H(P(t), P(t')) = \sup_{y \in P(t')} d(y, P(t))$. If $y \in P(t)$ and $\varepsilon > 0$ we can find some $z \in E$ with $\|z\| \leq 1 + \varepsilon$ and $\langle u, z \rangle = 1$.

Then $y' = y + (t' - t)z$ satisfies $\langle u, y' \rangle = \langle u, y \rangle + (t' - t) \geq t'$, hence $y' \in P(t')$ and $\|y - y'\| \leq (1 + \varepsilon)(t' - t)$. So $d(y, P(t')) \leq t' - t$ and P is 1-Lipschitz. \square

Theorem 6.3. *Let E be a normed space. Assume that the norm on E^* is not strictly convex. Then there exists a quasi-contraction $F : E \rightrightarrows E$ with closed convex values such that $\text{Fix}(F)$ is not connected.*

Proof. Take h and P as in previous Lemma, and define $F = P \circ h$ which is clearly 1-Lipschitz since so are P and h . If $x \in \text{Fix}(F)$ we must have

$$\langle u, x \rangle \geq h(x) = \langle u, x \rangle + \sin^2(\langle v, x \rangle)$$

hence $\sin(\langle v, x \rangle) = 0$, that implies $\langle v, x \rangle = k\pi$ for some integer $k \in \mathbb{Z}$. If $a \in E$ satisfies $\langle v, a \rangle = 1$ (and such points exist since $v \neq 0$) we get

$$\text{Fix}(F) = \bigcup_{k \in \mathbb{Z}} (k \cdot a + \ker v)$$

which is the discrete union of a countable family of pairwise disjoint closed hyperplanes, hence it cannot be connected. \square

7. The smooth case

We now want to extend Theorem 5.4 to every normed space E of dimension 2. It follows from Theorem 6.3 that one can assume the norm of E is smooth. Recall that a basis (e_1, e_2, \dots, e_n) of a finite-dimensional normed space E is called an *Auerbach basis* of E if $\|e_j\| = 1$ for all $j = 1, 2, \dots, n$ and moreover $\|e_j^*\| = 1$ for all $j = 1, 2, \dots, n$ where $(e_1^*, e_2^*, \dots, e_n^*)$ is the dual basis of E^* (what means $\langle e_j^*, e_k \rangle = \delta_j^k$).

Lemma 7.1. *If E is a 2-dimensional normed space with smooth norm, there exists an Auerbach basis (e_1, e_2) of E such that $\|e_2 + te_1\| > 1$ for all real $t \neq 0$.*

Proof. Let B be the unit ball of E . It is a well-known fact that if the determinant function $\Delta : (u, v) \mapsto u \wedge v$ attains at (x, y) its supremum on $B \times B$ then (x, y) is an Auerbach basis. It can be easily seen that the converse is not true: the canonical basis (e_1, e_2) of ℓ_2^∞ satisfies $\Delta(e_1, e_2) = 1$ though $e_1 + e_2$ and $e_2 - e_1$ have norm 1 and $\Delta(e_1 + e_2, e_2 - e_1) = 2$.

Assume that (e_1, e_2) is such an “extremal” Auerbach basis. If there is some $t \neq 0$ such that $e_2 + te_1 \in B$ then we have $e_1 \wedge e_2 > 0$ and for all $s > 0$

$$\left(\left(1 - \frac{s}{2}\right)e_1 - \frac{s}{t}e_2 \right) \wedge (e_2 + te_1) = \left(1 - \frac{s}{2} + s\right)e_1 \wedge e_2 = \left(1 + \frac{s}{2}\right)e_1 \wedge e_2 > e_1 \wedge e_2$$

what shows that $z_s = \left(1 - \frac{s}{2}\right)e_1 - \frac{s}{t}e_2 \notin B$: indeed if not the basis

$$(z_s, e'_2) = (z_s, e_2 + te_1)$$

would satisfy $\Delta(z_s, e'_2) > \Delta(e_1, e_2)$ with $(z_s, e'_2) \in B \times B$.

For $s < 0$ we have $\|z_s\| \geq \langle e_1^*, z_s \rangle = 1 - \frac{s}{2} > 1$. It follows that $\|z_s\| \geq 1$ for all $s \in \mathbb{R}$. Denote $u^* = e_1^* - \frac{t}{2}e_2^*$. If $u \in \{v : \langle u^*, v \rangle = 1\}$ we have $\langle u^*, u - e_1 \rangle = 0$ so $u = z_s$ for some $s \in \mathbb{R}$, hence $\|u\| \geq 1$. This shows that $\|u^*\| \leq 1$. Then $\|e_1^*\| = 1$, $\|u^*\| \leq 1$ and

$$1 \geq \|\lambda u^* + (1 - \lambda)e_1^*\| = \|e_1^* - \lambda \frac{t}{2}e_2^*\| \geq \langle e_1^* - \lambda \frac{t}{2}e_2^*, e_1 \rangle = 1$$

for $\lambda \in [0, 1]$, what shows that the norm of E^* is not strictly convex, in contradiction with the hypothesis of smoothness of E . □

Lemma 7.2. *Let $f : \mathbb{R}^+ \rightarrow \mathbb{R}$ be a continuous positive function such that $f(0) \leq 1$. Then there exists a convex non-increasing positive and 1-Lipschitz function $\varphi : \mathbb{R}^+ \rightarrow \mathbb{R}$ satisfying $\varphi(x) \leq f(x)$ for all $x \geq 0$.*

Proof. For all $\alpha > 0$ set $\tilde{f}(\alpha) = \inf_{0 \leq t \leq 2\alpha} f(t)$ which is positive by compactness of $[0, 2\alpha]$. And the affine decreasing function

$$f_\alpha : x \mapsto \tilde{f}(2\alpha)\left(1 - \frac{x}{2\alpha}\right)$$

satisfies $f_\alpha(x) \leq 0 < f(x)$ for $x \geq 2\alpha$, $f_\alpha(x) \leq \tilde{f}(2\alpha) \leq f(x)$ for $0 \leq x \leq 2\alpha$ and $f_\alpha(\alpha) = \frac{1}{2}\tilde{f}(2\alpha) > 0$. It follows that $\varphi : x \mapsto \sup_{\alpha \geq 1/2} f_\alpha(x)$ is convex, non-increasing, everywhere positive on $[\frac{1}{2}, +\infty[$ hence a fortiori on \mathbb{R}^+ , and that $\varphi \leq f$.

Finally since the function f_α is $\frac{\tilde{f}(2\alpha)}{2\alpha}$ -Lipschitz the function φ is λ -Lipschitz

for $\lambda = \sup_{\alpha \geq 1/2} \frac{\tilde{f}(2\alpha)}{2\alpha} = \tilde{f}(1) \leq f(0) \leq 1$. □

Lemma 7.3. *If the basis (e_1, e_2) of E is as in Lemma 7.1 there exists a positive convex non-decreasing 1-Lipschitz function $\varphi : \mathbb{R}^+ \rightarrow \mathbb{R}$ such that for all $x \in \mathbb{R}^+$ the inequality*

$$|x| + \varphi(|x|) \leq \|e_1 + xe_2\|$$

holds true.

Proof. Consider the function $f_+ : x \mapsto \|e_1 + xe_2\| - x$ on \mathbb{R}^+ . Since (e_1, e_2) is an Auerbach basis we have $\|e_1 + xe_2\| \geq \|xe_2\| = x$, hence $f_+(x) \geq 0$. And if we had $f_+(x) = 0$ for some $x \in \mathbb{R}^+$ we would have $x = \|e_1 + xe_2\| \geq 1$ hence $1 = \|e_2 + \frac{1}{x}e_1\| > \|e_2\| = 1$ since by hypothesis $\|e_2 + se_1\| > 1$ for all $s \neq 0$. It follows that f_+ is positive. And in the same way one sees that the function $f_- : x \mapsto \|e_1 - xe_2\| - x$ is positive on \mathbb{R}^+ . Moreover $f = \min(f_+, f_-)$ satisfies $f(0) = 1$.

Applying then Lemma 7.2 to f we get a positive convex non-decreasing 1-Lipschitz function $\varphi : \mathbb{R}^+ \rightarrow \mathbb{R}$ such that $|x| + \varphi(|x|) \leq \|e_1 + xe_2\|$ for all $x \in \mathbb{R}$. □

Still assuming the basis (e_1, e_2) of E satisfies the condition of Lemma 7.1, we define the closed convex set H by

$$H = \{x \in E : \langle e_1^*, x \rangle \in [-1, 1] \text{ and } \langle e_2^*, x \rangle \geq 0\}$$

Claim 7.4. There exists a 1-Lipschitz retraction p from E to H .

Proof. The function $p_1 : (x, y) \mapsto (x, \max(y, 0))$ is 1-Lipschitz: indeed if u and v belong to E , $p_1(u) - p_1(v)$ is a convex combination of the vectors $u - v$ and $\langle e_1^*, u - v \rangle e_1$ which have both a norm at most $\|u - v\|$. Thus $\|p_1(u) - p_1(v)\| \leq \|u - v\|$. Moreover p_1 is the identity mapping on H and $p_1(E) \subset \mathbb{R} \times \mathbb{R}^+$.

In the same way the mapping $p_2 : (x, y) \mapsto (\max(-1, \min(1, x)), y)$ is the identity on H and is 1-Lipschitz since when u and v belong to E , $p_2(u) - p_2(v)$ is a convex combination of the vectors $u - v$ and $\langle e_2^*, u - v \rangle e_2$ which have both a norm at most $\|u - v\|$. Moreover $p_2(\mathbb{R} \times \mathbb{R}^+) \subset H$.

Then $p = p_2 \circ p_1$ is the identity on H , is 1-Lipschitz and satisfies $p(E) \subset H$, so is a 1-Lipschitz retraction on H . ◇

Theorem 7.5. *If E is a 2-dimensional normed space with smooth norm, there exists on E a quasi-contraction F such that $\text{Fix}(F)$ is not connected.*

Proof. Choose the basis (e_1, e_2) , the function φ and the set H as above. We will define two 1-Lipschitz functions α and β from H to H and set $F(x) = [\alpha(x), \beta(x)]$ which will be a quasi-contraction by Lemma 1.1.

In order to ensure $\text{Fix}(F)$ is not connected we want to have $\alpha(-e_1 + te_2) = -e_1 + te_2$, $\beta(e_1 + te_2) = e_1 + te_2$, and $te_2 \notin \text{Fix}(F)$ for all $t \geq 0$, so $\{-1, 1\} \times \mathbb{R}^+ \subset \text{Fix}(F)$ and $(\{0\} \times \mathbb{R}^+) \cap \text{Fix}(F) = \emptyset$.

Let φ be as in Lemma 7.3 and define the function $\alpha : H_0 = \{-1, 0\} \times \mathbb{R}^+ \rightarrow E$ by :

$$\alpha(u) = -e_1 + \lambda(u)e_2$$

where $\lambda : H_0 \rightarrow \mathbb{R}^+$ is defined by $\lambda(-e_1 + te_2) = t$ and $\lambda(te_2) = t + \varphi(t)$. In particular $\alpha(H_0) \subset H$, $\alpha(-e_1) = -e_1$ and $\alpha(ye_2) = (-1, y + \varphi(y))$.

Claim 7.6. The function λ is 1-Lipschitz from H_0 to \mathbb{R}^+ .

Proof. Denote $a_t = (-1, t)$ and $c_t = (0, t)$. We have to prove the following inequalities, for s and $t \geq 0$:

$$i. \quad |\lambda(a_s) - \lambda(a_t)| \leq \|a_s - a_t\|,$$

$$ii. \quad |\lambda(c_s) - \lambda(c_t)| \leq \|c_s - c_t\|,$$

$$iii. \quad |\lambda(a_s) - \lambda(c_t)| \leq \|a_s - c_t\|.$$

For (i) we have $\|\lambda(a_s) - \lambda(a_t)\| = |s - t| = \|a_s - a_t\|$.

For (ii) we have $\|c_s - c_t\| = \|(s - t)e_2\| = |s - t|$ and

$$\|\lambda(c_s) - \lambda(c_t)\| = |(t + \varphi(t)) - (s + \varphi(s))| = |s + \varphi(s) - t - \varphi(t)|$$

Without loss of generality we can assume $s \leq t$; so we have $\varphi(t) \leq \varphi(s)$ and $s + \varphi(s) \leq t + \varphi(t)$ since $s \mapsto s + \varphi(s)$ is non-decreasing, so

$$|s + \varphi(s) - t - \varphi(t)| = t - s - (\varphi(s) - \varphi(t)) \leq t - s = |s - t| = \|c_s - c_t\|$$

Finally for (iii) we have

$$\begin{aligned} |\lambda(a_s) - \lambda(c_t)| &= |s - (t + \varphi(t))| = |s - \varphi(t) - t| \\ &\leq |s - t| + \varphi(t) \leq |s - t| + \varphi(|s - t|) \end{aligned}$$

since $t \geq |s - t|$ and by Lemma 7.3 applied to $x = t - s$:

$$\|a_s - c_t\| = \|-e_1 + se_2 - te_2\| = \|e_1 + (t - s)e_2\| \geq |s - t| + \varphi(|s - t|)$$

hence $|\lambda(a_s) - \lambda(c_t)| \leq |s - t| + \varphi(|s - t|) \leq \|a_s - c_t\|$. \diamond

It is well-known that any 1-Lipschitz function λ defined on a subset H_0 of the metric space H can be extended into a 1-Lipschitz function $\tilde{\lambda}$ on H by the formula

$$\tilde{\lambda}(x) = \inf_{y \in H_0} (\lambda(y) + d(x, y))$$

which yields a non-negative function $\tilde{\lambda}$ if $\lambda \geq 0$. Define then the function α on H by $\alpha(u) = -e_1 + \tilde{\lambda}(u)e_2$. We clearly have for u and v in H :

$$\|a(u) - a(v)\| = \|(\tilde{\lambda}(u) - \tilde{\lambda}(v))e_2\| = |\tilde{\lambda}(u) - \tilde{\lambda}(v)| \leq \|u - v\|$$

Replacing $-e_1$ by e_1 we can define in the same way a 1-Lipschitz function $\beta : H \rightarrow H$ such that $\beta(e_1 + te_2) = e_1 + te_2$ and $\beta(te_2) = e_1 + (t + \varphi(t))e_2$. Then the set-valued mapping F defined on H by $F(u) = [\alpha(u), \beta(u)]$ takes closed convex values. It is 1-Lipschitz because by Lemma 1.1:

$$d_H(F(u), F(v)) \leq \max(\|\alpha(u) - \alpha(v)\|, \|\beta(u) - \beta(v)\|) \leq \|u - v\|.$$

By definition we have $\alpha(-1, t) = (-1, t)$ and $\beta(1, t) = (1, t)$; so all points of $\{-1, 1\} \times \mathbb{R}^+$ are fixed points for F .

Conversely for $t \geq 0$ we have $F(0, t) = [-1, 1] \times \{t + \varphi(t)\}$ and this shows that $(0, t) \notin \text{Fix}(F)$ since $\varphi(t) \neq 0$, hence that the two non-empty open subsets $W_0 = \{u = (x, y) \in \text{Fix}(F) : x < 0\}$ and $W_1 = \{u = (x, y) \in \text{Fix}(F) : x > 0\}$ of $\text{Fix}(F)$ form a partition of $\text{Fix}(F)$. Thus $\text{Fix}(F)$ is not connected. And since $G = F \circ p$ satisfies $\text{Fix}(G) = \text{Fix}(F)$, we have just constructed a quasi-contraction on E whose set of fixed points is not connected. And this completes the proof of Theorem 7.5. □

Corollary 7.7. *If the normed space E has dimension at least 2, there exists on E a quasi-contraction G such that $\text{Fix}(G)$ is not connected.*

Proof. Notice that following Theorem 2.1 the condition “ $\dim(E) \geq 2$ ” is necessary and that following Theorem 3.3 such a quasi-contraction cannot be single-valued if E is finite-dimensional.

Take a closed linear subspace E_0 of codimension 2 and denote π the canonical projection onto the quotient space E/E_0 . Recall that the norm on E/E_0 is given by $\|y\| = \inf\{\|x\| : x \in \pi^{-1}(y)\}$.

It follows from Theorems 7.5 and 6.3 that there exists on the 2-dimensional space E/E_0 a quasi-contraction F with closed convex values such that $\text{Fix}(F)$ is not connected. Define then for $x \in E : G(x) = \pi^{-1}(F(\pi(x)))$ which is clearly a non-empty closed convex subset of E . And

$$x \in \text{Fix}(G) \iff x \in G(x) \iff \pi(x) \in F(\pi(x)) \iff \pi(x) \in \text{Fix}(F)$$

so $\text{Fix}(G) = \pi^{-1}(\text{Fix}(F))$, and $\pi(\text{Fix}(G)) = \text{Fix}(F)$ since π is onto.

If $\text{Fix}(G)$ were connected so would be $\text{Fix}(F) = \pi(\text{Fix}(G))$. Thus $\text{Fix}(G)$ is not connected. It remains to show that G is 1-Lipschitz. And this follows

immediately from the facts that F is 1-Lipschitz and that the mapping $T \mapsto \pi^{-1}(T)$ is 1-Lipschitz from $\mathcal{F}(E/E_0)$ to $\mathcal{F}(E)$. Indeed :

$$\begin{aligned} d(x, \pi^{-1}(T)) &= \inf_{t \in T} \inf_{y \in t} \|x - y\| = \inf_{t \in T} \inf_{u \in E_0, y \in t} \|(x - u) - y\| \\ &= \inf_{t \in T} \|\pi(x) - t\| = d(\pi(x), T) \end{aligned}$$

whence $d_H(\pi^{-1}(S), \pi^{-1}(T)) = d_H(S, T)$. □

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