FIXED POINTS FOR NON-EXPANSIVE SET-VALUED MAPPINGS
Jean Saint Raymond

To cite this version:

HAL Id: hal-02409175
https://hal.sorbonne-universite.fr/hal-02409175
Submitted on 13 Dec 2019

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L’archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d’enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.
FIXED POINTS
FOR NON-EXPANSIVE SET-VALUED MAPPINGS

JEAN SAINT RAYMOND

Let $E$ be a Banach space and $F : E \rightrightarrows E$ be a 1-Lipschitz set-valued mapping with closed convex non-empty values. We study the set of fixed points $\text{Fix}(F) = \{ x \in E : x \in F(x) \}$ and provide in any space $E$ with $\dim(E) \geq 2$ an example of such a mapping $F$ such that $\text{Fix}(F)$ is not connected.

1. Introduction

In this paper we are concerned with set-valued mappings from a Banach space $E$ into itself having closed convex values. We will consider only mappings $F$ which are 1-Lipschitz for the Hausdorff distance $d_H$ on the set $\mathcal{F}(E)$ of non-empty closed subsets of $E$. Recall that, for $A$ and $B$ in $\mathcal{F}(E)$:

$$d_H(A, B) = \max \left( \sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A) \right)$$

so $d(z, F(y)) \leq \|x - y\|$ for all $x, y \in E$ and $z \in F(x)$.

For such a mapping $F$ we will be essentially interested in the set $\text{Fix}(F) = \{ x \in E : x \in F(x) \}$ of fixed points of $F$ which is clearly closed in $E$. Of course it can happen that $\text{Fix}(F) = \emptyset$, for example if $F(x) = \{ x + a \}$ where $a$ is a fixed non-zero vector in $E$.

Received on June 10, 2019

AMS 2010 Subject Classification: 49J53, 55M20, 54H25

Keywords: non-expansive mappings, fixed points, set-valued mappings
The case of multivalued contraction mappings, i.e. the case where $F$ is $q$-Lipschitz for a $q < 1$ was extensively studied for long (see [2], [1], [3], [4]) and many properties of structure or conservation for the set $\text{Fix}(F)$ of fixed points were shown. For example:

\begin{enumerate}
  \item $\text{Fix}(F) \neq \emptyset$ (even for non convex-valued mappings). ([2], [1])
  \item $\text{Fix}(F)$ is an absolute retract, in particular it is path-connected. ([3])
  \item $\text{Fix}(F)$ is not a singleton if all values $F(x)$ have several points ([4] for the case where $q < \frac{1}{2}$ or $E$ is a Hilbert space, [5] for the general case)
  \item $\text{Fix}(F)$ is bounded if so are all values $F(x)$ for $x \in E$ (or even for only one $x \in E$). ([5])
  \item $\text{Fix}(F)$ is compact if so are all values $F(x)$ for $x \in E$.([5])
\end{enumerate}

We shall show in this paper that most of these results disappear when the Lipschitz constant $q$ of $F$ (which is $< 1$ if $F$ is a contraction mapping) is only assumed to be $\leq 1$ and $\dim(E) \geq 2$.

In the sequel we will call \textit{quasi-contraction} any 1-Lipschitz set-valued mapping from $E$ to $E$ with (non-empty) closed convex values.

Clearly properties (iv) and (v) become false already in the trivial example where $E = \mathbb{R}$ and $F(x) = \{x\}$ since $F(x)$ is then always single-valued, and a fortiori compact and bounded though $\text{Fix}(F) = \mathbb{R}$ is unbounded. We provide in section 4 an example of quasi-contraction in a Hilbert space for which property (iii) does not hold. Concerning property (ii) and namely the connectedness of the set of fixed points of a quasi-contraction, the main part of this paper consists in proving that it does not hold in general.

After studying in section 2 the very simple case where $E$ has dimension 1, we look in section 3 at the case where $F(x)$ is single-valued. It turns out that if $E$ is finite-dimensional we can prove that $\text{Fix}(F)$ is connected but that this is no more true for infinite-dimensional spaces.

The remainder of the paper is devoted to show that in every normed space of dimension at least 2 one can construct a quasi-contraction having a non-connected set of fixed points. In section 5 we provide such a construction for the 2-dimensional euclidean space, and generalize it to every 2-dimensional smooth normed space in section 7. The general case is dealt in sections 6 and 7.

For any two points $a$ and $b$ in a normed space $E$ we will denote by $[a,b] \subset E$ the segment with endpoints $a$ and $b$, it is the set $\{(1-t)a+tb : t \in [0,1]\}$. The following simple lemma will be of constant use throughout the paper.
Lemma 1.1. Let \( E \) be a normed space, \( a, b, a', b' \) be points of \( E \). Then
\[
d_H([a, b], [a', b']) \leq \max(\|a - a', \|b - b'\|) .
\]

Proof. If \( w \in [a, b] \) we have \( w = ta + (1 - t)b \) for some \( t \in [0, 1] \) hence
\[
d(w, [a', b']) \leq d(ta + (1 - t)b, ta' + (1 - t)b') \\
= \|t((a - a') + (1 - t)(b - b'))\| \\
\leq t\|a - a'\| + (1 - t)\|b - b'\| \\
\leq \max(\|a - a'\|, \|b - b'\|)
\]
whence it follows that \( \sup_{w \in [a, b]} d(w, [a', b']) \leq \max(\|a - a'\|, \|b - b'\|) \), hence that
\[
d_H([a, b], [a', b']) \leq \max(\|a - a'\|, \|b - b'\|).
\]

2. The case of dimension 1

Proposition 2.1. Let \( F \) be a quasi-contraction from \( \mathbb{R} \) to \( \mathbb{R} \) (the values of \( F \) are closed intervals). Then \( \text{Fix}(F) \) is either empty or a closed interval. In particular \( \text{Fix}(F) \) is connected.

Proof. Since \( F \) is a quasi-contraction, it is easy to see that there are two mappings \( a \) and \( b \) from \( \mathbb{R} \) to \( \widehat{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\} \) such that \( F(x) = \mathbb{R} \cap [a(x), b(x)] \) where \( a(x) = -\infty \) for all \( x \) or \( a(x) > -\infty \) for all \( x \) (and \( b(x) = +\infty \) for all \( x \) or \( b(x) < +\infty \) for all \( x \)). If \( -\infty < a(x) \leq a(y) \) we have \( d(a(x), F(y)) = |a(y) - a(x)| \) hence \( d_H(F(x), F(y)) \geq |a(y) - a(x)| \) and similarly \( d_H(F(x), F(y)) \geq |b(y) - b(x)| \) if \( b(x) < +\infty \). So by Lemma 1.1
\[
d_H(F(x), F(y)) = \max(\|a(x) - a(y)\|, \|b(x) - b(y)\|)
\]
whence it follows that both \( a \) and \( b \) are 1-Lipschitz or constantly infinite.

Suppose towards a contradiction that there exist \( x_0, x'_0 \in \text{Fix}(F), x'_0 < x < x_0 \) and \( x \notin \text{Fix}(F) \). Then we have \( b(x) = b(x_0) = +\infty \) or \( b(x_0) \geq x_0 \), hence
\[
b(x) \geq b(x_0) - |x - x_0| \geq x_0 - (x_0 - x) = x
\]
Thus since \( x \notin \text{Fix}(F) \) we necessarily get \( a(x) > x \), so \( a(x) > -\infty \), and since \( x'_0 < x \),
\[
a(x'_0) \geq a(x) - |x - x'_0| > x - (x - x'_0) = x'_0
\]
hence \( x'_0 \notin F(x_0) \), a contradiction. This shows that \( \text{Fix}(F) \) is an interval. \( \square \)
3. The case of functions

In this section we consider single-valued quasi-contractions \( F \), which we identify with 1-Lipschitz functions \( f \) by \( F(x) = \{ f(x) \} \). More generally we will study the case where \( H \) is a closed convex subset of the Banach space \( E \) and \( f : H \to H \) is 1-Lipschitz.

**Proposition 3.1.** Let \( E \) be a strictly convex normed space, \( H \subset E \) be closed and convex, \( f : H \to H \) be 1-Lipschitz. Then \( \text{Fix}(F) \) is convex, possibly empty.

**Proof.** Clearly, if \( H = E \), \( a \in E \) is not zero and \( f \) is the translation \( x \mapsto x + a \), \( f \) is an isometry and \( \text{Fix}(f) = \emptyset \).

If \( f : H \to H \) is 1-Lipschitz and \( u, v \) are two distinct points of \( \text{Fix}(f) \) then for all \( t \in ]0, 1[ \) the points \( x_t = tu + (1 - t)v \) and \( y_t = f(x_t) \) satisfy

\[
\| y_t - u \| = \| f(x_t) - f(u) \| \leq \| x_t - u \| = (1 - t)\| u - v \|
\]

\[
\| y_t - v \| = \| f(x_t) - f(v) \| \leq \| x_t - v \| = t\| u - v \|
\]

thus \( \| u - v \| \leq \| y_t - u \| + \| y_t - v \| \leq \| u - v \| \), whence \( y_t \in [u, v] \) because \( E \) is strictly convex, and \( y_t - v = s(u - v) \) for some \( s \in [0, 1] \). Then since \( s\| u - v \| = \| y_t - v \| = t\| u - v \| \), we conclude that \( s = t \) and \( y_t = x_t \), hence that \( x_t \in \text{Fix}(F) \). \( \square \)

If \( E \) is not strictly convex, the previous result does not hold any more. For example if \( E = \mathbb{R}^2 \) equipped with the norm \( u = (x, y) \mapsto \| u \|_\infty = \max(|x|, |y|) \), the function \( f : (x, y) \mapsto (x, \sin x) \) is 1-Lipschitz: indeed

\[
\| f(x, y) - f(x', y') \| = \max(|x - x'|, | \sin x - \sin x'|) = |x - x'| \leq \|(x, y) - (x', y')\|
\]

and \( \text{Fix}(f) = \{(x, \sin x) : x \in \mathbb{R}\} \) is connected, but not convex. So far it is unclear, even in a finite-dimensional space, whether a 1-Lipschitz mapping could have a non-connected set of fixed points. Nevertheless we shall see later on, in Theorems 3.3, 3.4 and Corollary 7.7 what really happens.

**Lemma 3.2.** Let \( H \) be a non-empty convex compact subset of a finite-dimensional space \( E \) and \( f : H \to H \) be a 1-Lipschitz function. Then the set \( \text{Fix}(f) \) is compact connected and non-empty.

**Proof.** It follows readily from Brouwer’s theorem that \( \text{Fix}(f) \) is non-empty. And it is closed in \( H \) hence compact. For the connectedness we proceed by induction on the dimension of \( E \), or more precisely on the dimension \( \delta(H) \) of the linear subspace \( A_H \) generated by \( H - H \). If \( \delta(H) = 1 \) then \( A_H \approx \mathbb{R} \) is strictly convex and it follows from Proposition 3.1 that \( \text{Fix}(f) \) is convex, hence connected.
Assume that the statement of the lemma holds for all compact convex $K$ such that $\delta(K) < n$, that $H$ is a compact convex subset of $E$ such that $\delta(H) = n$ and that $a_0$ and $a_1$ are two distinct fixed points of the 1-Lipschitz function $f : H \rightarrow H$. By translation invariance we can and do assume that $0 \in a$ that $\xi \in E^*$ of norm 1 such that $\langle \xi, a_1 - a_0 \rangle = \|a_1 - a_0\|$. 

For all $t \in [0, 1]$ consider the set

$$H_t = \{x \in H : \|x - a_0\| \leq t\|a_1 - a_0\| \text{ and } \|x - a_1\| \leq (1 - t)\|a_1 - a_0\|\}$$

which is convex and compact. Then $f(H_t) \subset H_t$: indeed if $x \in H_t$, $y = f(x) \in H$ and

$$\begin{cases} \|y - a_0\| = \|f(x) - f(a_0)\| \leq \|x - a_0\| \leq t\|a_1 - a_0\| \\ \|y - a_1\| = \|f(x) - f(a_1)\| \leq \|x - a_1\| \leq (1 - t)\|a_1 - a_0\|. \end{cases}$$

Moreover for $x \in H_t$,

$$\begin{cases} \langle \xi, x - a_0 \rangle \leq \|\xi\| \|x - a_0\| \leq t\|a_1 - a_0\| \\ \langle \xi, a_1 - x \rangle \leq \|\xi\| \|x - a_1\| \leq (1 - t)\|a_1 - a_0\| \end{cases}$$

hence

$$0 = \langle \xi, a_1 - a_0 \rangle - \|a_1 - a_0\| = \langle \xi, x - a_0 \rangle + \langle \xi, a_1 - x \rangle - \|a_1 - a_0\|$$

$$= (\langle \xi, x - a_0 \rangle - t\|a_1 - a_0\|) + (\langle \xi, a_1 - x \rangle - (1 - t)\|a_1 - a_0\|)$$

and

$$0 \leq t\|a_1 - a_0\| - \langle \xi, x - a_0 \rangle = (\langle \xi, a_1 - x \rangle - (1 - t)\|a_1 - a_0\|) \leq 0$$

from what we deduce that

$$\langle \xi, x - a_0 \rangle = t\|a_1 - a_0\| \text{ and } \langle \xi, a_1 - x \rangle = (1 - t)\|a_1 - a_0\|,$$

hence that $\langle \xi, x \rangle = \theta := \langle \xi, a_0 \rangle + t\|a_1 - a_0\|$. Thus this shows that $H_t$ is included in the affine hyperplane $V_\theta = \{x \in E : \langle \xi, x \rangle = \theta\}$ for which $\delta(V_\theta) < \dim(E) = n$. It follows then from the induction hypothesis that $H_t \cap \Fix(f) = \Fix(f|_{H_t})$ is compact connected and non-empty.

Assume that $\Fix(f)$ is not connected. So it would exist two disjoint compact subsets $A_0$ and $A_1$ of $H$ such that $\Fix(f) \subset A_0 \cup A_1$ and two points $a_0$ and $a_1$ with $a_i \in A_i$. For $t \in [0, 1]$, let $H_t$ be the set introduced above which corresponds to the points $a_0$, $a_1$. Then, from what precedes, for all $t \in [0, 1]$, $\Fix(f|_{H_t}) \subset A_0 \cup A_1$ ($i \in \{0, 1\}$) and, by connectedness of $\Fix(f|_{H_t})$, $\Fix(f|_{H_t}) \subset A_i$ for some $i$. Then the sets $T_i = \{t \in [0, 1] : \Fix(f|_{H_t}) \subset A_i\}$ form a partition of $[0, 1]$. Moreover,
note that $0 \in T_0$ and 1 \in T_1$. Thus, if we prove that $T_0$ and $T_1$ are also closed, we will get a contradiction which will complete the proof of the connectedness of Fix($f$).

Let $(t_n)$ be a sequence in $T_0$ which converges to $t^*$; there exists for all $n$ a point $x_n \in A_0 \cap \text{Fix}(f_{H_n})$. Since $H \cap B(a_0, \|a_1 - a_0\|)$ is compact and $H_{t_n} \subset H \cap B(a_0, \|a_1 - a_0\|)$, up to passing to a subsequence we can assume that $(x_n)$ converges to some point $x^* \in A_0$. We have

$$\|x^* - a_0\| = \lim_{n} \|x_n - a_0\| \leq \lim_{n} \sup t_n \|a_1 - a_0\| = t^* \|a_1 - a_0\|$$

and similarly $\|x^* - a_1\| \leq (1 - t) \|a_1 - a_0\|$.

Thus $x^* \in H_{t^*}$. Moreover $\|f(x^*) - x^*\| = \lim \|f(x_n) - x_n\| = 0$. It follows that $x^* \in \text{Fix}(f_{H_{t^*}}) \cap A_0$, hence that $\text{Fix}(f_{H_{t^*}}) \cap A_0 \neq \emptyset$ and that $t^* \in T_0$. By the same argument one can show that $T_1$ is closed. This completes the proof of the connectedness of Fix($f$), hence this of Lemma 3.2.

**Theorem 3.3.** Let $E$ be a normed finite-dimensional space, $H \subset E$ be a closed convex subset and $f : H \rightarrow H$ be a 1-Lipschitz function. Then the set of fixed points of $f$ is connected.

*Proof.* It is enough to consider the case where Fix($f$) is non-empty. Then let $a \in \text{Fix}(f)$. For all integer $n \geq 1$ the set $H_n = \{x \in H : \|x - a\| \leq n\}$ is compact convex non-empty and stable under $f$. So it follows from Lemma 3.2 that $\text{Fix}(f_{H_n}) = \text{Fix}(f) \cap H_n$ is connected and contains $a$. Then $\bigcup_n \text{Fix}(f_{H_n}) = \text{Fix}(f)$ is connected. \hfill \qedsymbol

We now show that for infinite-dimensional spaces $E$ there is no particular topological property of the sets Fix($f$) for 1-Lipschitz functions $f : E \rightarrow E$. Indeed:

**Theorem 3.4.** Let $X$ be a complete metric space. Then there exist a Banach space $E$ and a 1-Lipschitz function $f : E \rightarrow E$ such that Fix($f$) is isometric to $X$. Moreover is $X$ is separable the space $E$ can be chosen separable.

*Proof.* Remark first that since Fix($f$) is closed in $E$ hence complete, the completeness of $X$ is a necessary condition.

It is well-known that any metric space $X$ can be isometrically embedded into a Banach space. For example if $a \in X$ and $D$ is a dense subset of $X$ the function $\psi : x \mapsto \left( d(x,y) - d(a,y) \right)_{y \in D}$ is an isometry from $X$ to a subset of the space $\ell^\infty_D$. And if $X$ is separable, the space $\overline{\text{span}}(\psi(X))$ is a separable Banach space.

Recall that $c_0$ denotes the Banach space of all real sequences converging to 0 equipped with the norm $\|u\| = \sup_n |x_n|$ and denote 0 the null
sequence in $c_0$. Choose an isometric embedding $j : X \to W$ for some Banach space $W$ and define $H = j(X) \subset W$ and $E = W \times c_0$ equipped with the norm $(w,u) \mapsto \max(\|w\|,\|u\|)$.

For $w \in W$ and $u \in c_0$ define $f(w,u) = (w,v)$ where $v = (v_n) \in c_0$ is defined by

$$v_n = \begin{cases} d(w,H) & \text{if } n = 0 \\ u_{n-1} & \text{if } n > 0 \end{cases}$$

**Claim 3.5.** The function $f$ is 1-Lipschitz and even an isometry.

**Proof.** We have

$$
\|f(w,u) - f(w',u')\| = \max(\|w-w'\|, |d(w,H) - d(w',H)|, \sup_{n \geq 1} |u_{n-1} - u'_{n-1}|)
$$

$$= \max(\|w-w'\|, |d(w,H) - d(w',H)|, \|u-u'\|)
$$

$$= \max(\|w-w'\|, \|u-u'\|) = \|f(w,u) - f(w',u')\|$$

since $|d(w,H) - d(w',H)| \leq \|w-w'\|$.

\*\*\*

**Claim 3.6.** $\text{Fix}(f) = H \times \{0\}$.

**Proof.** It is clear that if $w \in H$ then $f(w,0) = (w,0)$.

Conversely, if $u = (x_n)$ and $(w,u)$ is a fixed point of $f$ we have $x_0 = d(w,H)$ and $x_n = x_{n-1}$ for all $n \geq 1$. Thus $u$ is the constant sequence with value $d(w,H)$, which does not belong to $c_0$ if $d(w,H) \neq 0$. So $w \in H$ and $u = 0$.

It follows from previous claim that the function $x \mapsto (j(x),0)$ is an isometry from $X$ onto $\text{Fix}(f)$.

\*\*\*

4. **Uniqueness of fixed points**

We provide in this section an example of a quasi-contraction $F$ on a Hilbert space $H$ such that $F(x)$ is a singleton for no $x \in H$ but $\text{Fix}(F)$ is a singleton.

And this shows that Property (iii) in the Introduction does not hold in general for quasi-contractions.

**Theorem 4.1.** There exists a quasi-contraction $F$ on a Hilbert space $H$ such that $\text{diam}(F(x)) = 1$ for all $x \in H$ but $\text{Fix}(F)$ is a singleton.

**Proof.** Let $H$ be the Hilbert space $\ell^2$, $S : H \to H$ be the isometric mapping defined by $x = (x_n)_{n \geq 0} \mapsto y = (y_n)_{n \geq 0}$ where $y_0 = 0$ and $y_n = x_{n-1}$ for $n > 0$. Let $u = (u_n) \in H$ be the unit vector such that $u_0 = 1$ and $u_n = 0$ for $n > 0$, and $0$ be the null vector of $H$. 
For $x \in H$ define $F(x)$ as the segment $[S(x), S(x) + u]$ whose diameter is 1. We claim that $F$ is a quasi-contraction. Indeed by Lemma 1.1:

$$d_H(F(y), F(x)) \leq \max(\|S(y) - S(x)\|, \|(S(y) + u) - (S(x) + u)\|)$$

$$= \|S(y) - S(x)\| = \|S(y - x)\| = \|y - x\|.$$ 

If $x^* = (x^*_n)$ is a fixed point of $F$ there exists some $t \in [0, 1]$ such that

$$x^* = (1 - t)S(x^*) + t(S(x^*) + u) = S(x^*) + tu$$

so $x_0^* = tu_0 = t$ and $x_n^* = x_{n-1}^*$ for $n > 0$. This implies that the sequence $(x_n) \in l^2$ has to be constant with the value $t$, which is possible only with $t = 0$, and $x^* = 0$. And this shows that $\text{Fix}(F) = \{0\}$ is a singleton. 

5. The 2-dimensional euclidean space

The aim of this section is to construct a quasi-contraction on the 2-dimensional euclidean space $\mathbb{R}^2$ whose set of fixed points is not connected. It follows from Proposition 2.1 that such a construction cannot be achieved in a 1-dimensional space, and from Proposition 3.1 that it is impossible with a single-valued quasi-contraction.

Consider the points $x_0 = (-1, 0)$ and $x_1 = (1, 0)$ of the euclidean space $\mathbb{R}^2$ and the symmetry $S: (u, v) \mapsto (-u, v)$ of $\mathbb{R}^2$ exchanging $x_0$ and $x_1$. We want to define two 1-Lipschitz mappings $\alpha$ and $\beta$ from $\mathbb{R}^2$ to itself such that $\alpha(x_0) = x_0$ and $S \circ \beta(z) = \alpha \circ S(z)$ for all $z$. In particular this implies $\beta(x_1) = S \circ \alpha(x_0) = S(x_0) = x_1$. We define then the set-valued mapping $F$ by $F(z) = [\alpha(z), \beta(z)]$, which is clearly convex and closed.

**Lemma 5.1.** If the mapping $\alpha$ is 1-Lipschitz then $F$ is a quasi-contraction.

**Proof.** Since $S$ is an isometry it is clear that $\beta = S \circ \alpha \circ S$ is 1-Lipschitz too. Then by Lemma 1.1, if $z$ and $z'$ are in $\mathbb{R}^2$

$$d_H(F(z), F(z')) \leq \max(\|\alpha(z) - \alpha(z')\|, \|\beta(z) - \beta(z')\|) \leq \|z - z'\|,$$

the wanted inequality. 

Fix $\epsilon \in ]0, 1]$ and define the function $\varphi: \mathbb{R} \to \mathbb{R}$ by

$$\varphi(t) = \begin{cases} 
\epsilon & \text{if } t \leq 0 \\
t + \epsilon e^{-t} & \text{if } t \geq 0
\end{cases}$$

**Claim 5.2.** The function $\varphi$ is 1-Lipschitz and has no fixed point on $\mathbb{R}$.
Proof. It is immediately checked that $\phi$ is continuous, derivable on $\mathbb{R} \setminus \{0\}$, that $\phi'(t) = 0$ for $t < 0$ and $\phi''(t) = 1 - \varepsilon e^{-t} \geq 0$ if $t > 0$. Since $|\phi'(t)| \leq 1$ for $t \neq 0$, the function $\phi$ is 1-Lipschitz. It is non-decreasing with values in $[\varepsilon, +\infty[$ and if $t^*$ were a fixed point of $\phi$ we would have $t^* = \phi(t^*) = \varepsilon > 0$ and $t^* = \phi(t^*) = t^* + \varepsilon e^{-t^*}$ hence $e^{-t^*} = 0$, that is impossible. $\diamond$

Define now the function $\alpha$ on $X = \{x_0, x_1\} \cup (\{0\} \times \mathbb{R})$ by

\[
\left\{
\begin{array}{l}
\alpha(x_0) = x_0 \\
\alpha(x_1) = (0, \varepsilon) \\
\alpha(0, v) = \left(-\frac{1}{2}, \phi(v)\right)
\end{array}
\right.
\]

**Lemma 5.3.** It is possible to choose $\varepsilon \in [0, 1]$ such that this function $\alpha$ be 1-Lipschitz on $X$.

Proof. For $v \in \mathbb{R}$ denote $y_v = (0, v) \in X$. We have to prove that for some convenient $\varepsilon > 0$:

i. $\|\alpha(x_0) - \alpha(x_1)\|^2 \leq \|x_0 - x_1\|^2 = 4$,

ii. $\forall v \in \mathbb{R}, \|\alpha(x_0) - \alpha(y_v)\|^2 \leq \|x_0 - y_v\|^2$,

iii. $\forall v \in \mathbb{R}, \|\alpha(x_1) - \alpha(y_v)\|^2 \leq \|x_1 - y_v\|^2$,

iv. $\forall v, w \in \mathbb{R}, \|\alpha(y_v) - \alpha(y_w)\| \leq \|y_v - y_w\|$.

For (i) we must have $1 + \varepsilon^2 \leq 4$, that is true since $\varepsilon \leq 1 < \sqrt{3}$.

For (ii) we must have

\[
(-1 + \frac{1}{2})^2 + \phi(v)^2 \leq 1 + v^2
\]

it is $\frac{1}{4} + \phi(v)^2 \leq 1 + v^2$. And since $\phi(v)^2 = \varepsilon^2$ if $v \leq 0$ and if $v \geq 0$:

\[
\phi(v)^2 = (v + \varepsilon e^{-v})^2 = v^2 + \varepsilon^2 e^{-2v} + 2\varepsilon v e^{-v} \leq v^2 + \varepsilon^2 + 2\varepsilon \sup_{t \geq 0} t e^{-t}
\]

\[
= v^2 + \varepsilon^2 + 2e^{-1} \varepsilon \leq v^2 + \varepsilon^2 + \varepsilon
\]

we must have $\frac{1}{4} + v^2 + \varepsilon^2 + \varepsilon \leq 1 + v^2$, that holds as soon as $\varepsilon^2 + \varepsilon \leq \frac{3}{4}$, hence whenever $0 < \varepsilon \leq \frac{1}{2}$. 


For (iii) we must have \((\frac{1}{2})^2 + (\varphi(v) - \varepsilon)^2 \leq 1 + v^2\). And since \(\varphi(v) \geq \varepsilon\) we have \((\varphi(v) - \varepsilon)^2 \leq \varphi(v)^2\). We have seen that if \(\varepsilon\) is chosen in \([0, \frac{1}{2}]\) then for all \(v: \frac{1}{4} + \varphi(v)^2 \leq 1 + v^2\), so a fortiori \(\frac{1}{4} + (\varphi(v) - \varepsilon)^2 \leq 1 + v^2\).

Finally for (iv), we have to show that
\[
\|\alpha(y_v) - \alpha(y_w)\| = |\varphi(v) - \varphi(w)| \leq \|y_v - y_w\| = |v - w|
\]
but this follows immediately from Claim 5.2.

Taking \(\varepsilon = \frac{1}{2}\) completes the proof of Lemma 5.3. \(\square\)

Using Kirszbraun-Valentine’s Theorem, we can extend the function \(\alpha\) into a 1-Lipschitz function (still denoted by \(\alpha\)) from \(\mathbb{R}^2\) to \(\mathbb{R}^2\), and then define \(\beta = S \circ \alpha \circ S\), which is 1-Lipschitz too.

**Theorem 5.4.** The set-valued mapping \(F: z \mapsto [\alpha(z), \beta(z)]\) is 1-Lipschitz, but the set Fix\((F)\) of its fixed points is not connected.

**Proof.** That \(F\) be 1-Lipschitz follows from Lemma 5.1. Since \(x_0 = \alpha(x_0) \in F(x_0)\) we have \(x_0 \in \text{Fix}(F)\) and since \(x_1 = \beta(x_1) \in F(x_1)\) we have \(x_1 \in \text{Fix}(F)\). Hence \(\{x_0, x_1\} \subseteq \text{Fix}(F)\).

We now show that \((\{0\} \times \mathbb{R}) \cap \text{Fix}(F) = \emptyset\). Indeed if there were some \(y_v = (0, v)\) in \(\text{Fix}(F)\) we should have \(y_v \in \text{conv}(\alpha(y_v), \beta(y_v))\). Since \(\alpha(y_v) = (-\frac{1}{2}, \varphi(v))\) and \(\beta(y_v) = (\frac{1}{2}, \varphi(v))\) we would get
\[
(0, v) = y_v \in \text{conv}(\alpha(y_v), \beta(y_v)) = [-\frac{1}{2}, \frac{1}{2}] \times \{\varphi(v)\}
\]
hence \(\varphi(v) = v\), in contradiction with Claim 5.2.

It follows that the two disjoint open subsets \(W_0 = \{(x, y) \in \text{Fix}(F) : x < 0\}\) and \(W_1 = \{(x, y) \in \text{Fix}(F) : x > 0\}\) of \(\text{Fix}(F)\) are both non-empty and cover \(\text{Fix}(F)\). Thus \(\text{Fix}(F)\) is not connected. \(\square\)

### 6. The non-smooth case

It is also possible to give a simple example in any normed space \(E\) whose dual space \(E^*\) is not strictly convex (in particular if the norm of \(E\) itself is not smooth) of a quasi-contraction whose set of fixed points is not connected.

It \(E^*\) is not strictly convex there are two non-zero vectors \(u\) and \(v\) of \(E^*\) such that \(\|u\| = \|u + v\| = \|u - v\| = 1\). Define then the real function \(h\) on \(E\) by
\[
h(x) = \langle u, x \rangle + \sin^2(\langle v, x \rangle)
\]
Lemma 6.1. The function $h$ is 1-Lipschitz.

Proof. In fact $h$ is of class $\mathcal{C}^1$ and its differential at $x$ is $h'(x) = u + \sin(2\langle v, x \rangle)v$. The convex function $v : t \mapsto \|u + tv\|$ satisfies $v(-1) = v(1) = 1$ hence $v(t) \leq 1$ for $t \in [-1, 1]$. It follows that $\|h'(x)\| \leq 1$ for all $x$ hence that $h$ is 1-Lipschitz. \qed

Lemma 6.2. The set-valued mapping $P : \mathbb{R} \rightrightarrows E$ defined by $P(t) = \{y \in E : \langle u, y \rangle \geq t\}$ is 1-Lipschitz and takes closed convex non-empty values.

Proof. It is clear that $P(x)$ is convex closed and non-empty. Notice that if $t \leq t'$ then we have $P(t') \subset P(t)$, so $d_H(P(t), P(t')) = \sup_{y \in P(t)} d(y, P(t'))$. If $y \in P(t)$ and $\varepsilon > 0$ we can find some $z \in E$ with $\|z\| \leq 1 + \varepsilon$ and $\langle u, z \rangle = 1$.

Then $y' = y + (t' - t)z$ satisfies $\langle u, y' \rangle = \langle u, y \rangle + (t' - t) \geq t'$, hence $y' \in P(t')$ and $\|y - y'\| \leq (1 + \varepsilon)(t' - t)$. So $d(y, P(t')) \leq t' - t$ and $P$ is 1-Lipschitz. \qed

Theorem 6.3. Let $E$ be a normed space. Assume that the norm on $E^*$ is not strictly convex. Then there exists a quasi-contraction $F : E \rightrightarrows E$ with closed convex values such that $\text{Fix}(F)$ is not connected.

Proof. Take $h$ and $P$ as in previous Lemma, and define $F = P \circ h$ which is clearly 1-Lipschitz since so are $P$ and $h$. If $x \in \text{Fix}(F)$ we must have

$$\langle u, x \rangle \geq h(x) = \langle u, x \rangle + \sin^2(\langle v, x \rangle)$$

hence $\sin(\langle v, x \rangle) = 0$, that implies $\langle v, x \rangle = k\pi$ for some integer $k \in \mathbb{Z}$. If $a \in E$ satisfies $\langle v, a \rangle = 1$ (and such points exist since $v \neq 0$) we get

$$\text{Fix}(F) = \bigcup_{k \in \mathbb{Z}} \{k \cdot a + \ker v\}$$

which is the discrete union of a countable family of pairwise disjoint closed hyperplanes, hence it cannot be connected. \qed

7. The smooth case

We now want to extend Theorem 5.4 to every normed space $E$ of dimension 2. It follows from Theorem 6.3 that one can assume the norm of $E$ is smooth. Recall that a basis $(e_1, e_2, \ldots, e_n)$ of a finite-dimensional normed space $E$ is called an Auerbach basis of $E$ if $\|e_j\| = 1$ for all $j = 1, 2, \ldots, n$ and moreover $\|e_j^*\| = 1$ for all $j = 1, 2, \ldots, n$ where $(e_1^*, e_2^*, \ldots, e_n^*)$ is the dual basis of $E^*$ (what means $\langle e_j^*, e_k \rangle = \delta^k_j$).

Lemma 7.1. If $E$ is a 2-dimensional normed space with smooth norm, there exists an Auerbach basis $(e_1, e_2)$ of $E$ such that $\|e_2 + te_1\| > 1$ for all real $t \neq 0$. 
Proof. Let $B$ be the unit ball of $E$. It is a well-known fact that if the determinant function $\Delta : (u,v) \mapsto u \wedge v$ attains at $(x,y)$ its supremum on $B \times B$ then $(x,y)$ is an Auerbach basis. It can be easily seen that the converse is not true: the canonical basis $(e_1,e_2)$ of $\ell^2_2$ satisfies $\Delta(e_1,e_2) = 1$ though $e_1 + e_2$ and $e_2 - e_1$ have norm $1$ and $\Delta(e_1 + e_2, e_2 - e_1) = 2$.

Assume that $(e_1,e_2)$ is such an “extremal” Auerbach basis. If there is some $t \neq 0$ such that $e_2 + te_1 \in B$ then we have $e_1 \wedge e_2 > 0$ and for all $s > 0$

$$((1 - \frac{s}{2})e_1 - \frac{s}{t}e_2) \wedge (e_2 + te_1) = (1 - \frac{s}{2} + s) e_1 \wedge e_2 = (1 + \frac{s}{2})e_1 \wedge e_2 > e_1 \wedge e_2$$

what shows that $z_s = (1 - \frac{s}{2})e_1 - \frac{s}{t}e_2 \notin B$: indeed if not the basis

$$(z_s,e'_2) = (z_s,e_2 + te_1)$$

would satisfy $\Delta(z_s,e'_2) > \Delta(e_1,e_2)$ with $(z_s,e'_2) \in B \times B$.

For $s < 0$ we have $\|z_s\| \geq \langle e^*_1,z_s \rangle = 1 - \frac{s}{2} > 1$. It follows that $\|z_s\| \geq 1$ for all $s \in \mathbb{R}$. Denote $u^* = e^*_1 - \frac{t}{2} e^*_2$. If $u \in \{v : \langle u^*,v \rangle = 1\}$ we have $\langle u^*,u - e_1 \rangle = 0$ so $u = z_s$ for some $s \in \mathbb{R}$, hence $\|u\| \geq 1$. This shows that $\|u^*\| \leq 1$. Then $\|e^*_1\| = 1$, $\|u^*\| \leq 1$ and

$$1 \geq \|\lambda u^* + (1 - \lambda)e^*_1\| = \|e^*_1 - \lambda \frac{t}{2} e^*_2\| \geq \langle e^*_1 - \lambda \frac{t}{2} e^*_2, e_1 \rangle = 1$$

for $\lambda \in [0,1]$, what shows that the norm of $E^*$ is not strictly convex, in contradiction with the hypothesis of smoothness of $E$. \qed

Lemma 7.2. Let $f : \mathbb{R}^+ \to \mathbb{R}$ be a continuous positive function such that $f(0) \leq 1$. Then there exists a convex non-increasing positive and $1$-Lipschitz function $\varphi : \mathbb{R}^+ \to \mathbb{R}$ satisfying $\varphi(x) \leq f(x)$ for all $x \geq 0$.

Proof. For all $\alpha > 0$ set $\tilde{f}(\alpha) = \inf_{0 \leq t \leq 2\alpha} f(t)$ which is positive by compactness of $[0,2\alpha]$. And the affine decreasing function

$$f_\alpha : x \mapsto \tilde{f}(2\alpha)(1 - \frac{x}{2\alpha})$$

satisfies $f_\alpha(x) \leq 0 < f(x)$ for $x \geq 2\alpha$, $f_\alpha(x) \leq \tilde{f}(2\alpha) \leq f(x)$ for $0 \leq x \leq 2\alpha$ and $f_\alpha(\alpha) = \frac{1}{2} \tilde{f}(2\alpha) > 0$. It follows that $\varphi : x \mapsto \sup_{\alpha \geq 1/2} f_\alpha(x)$ is convex, non-increasing, everywhere positive on $[\frac{1}{2},+\infty]$ hence a fortiori on $\mathbb{R}^+$, and that $\varphi \leq f$.

Finally since the function $f_\alpha$ is $\tilde{f}(2\alpha) -$Lipschitz the function $\varphi$ is $\lambda$-Lipschitz for $\lambda = \sup_{\alpha \geq 1/2} \frac{\tilde{f}(2\alpha)}{2\alpha} = \tilde{f}(1) \leq f(0) \leq 1$. \qed
Lemma 7.3. If the basis \((e_1, e_2)\) of \(E\) is as in Lemma 7.1 there exists a positive convex non-decreasing 1-Lipschitz function \(\varphi : \mathbb{R}^+ \to \mathbb{R}\) such that for all \(x \in \mathbb{R}^+\) the inequality

\[ |x| + \varphi(|x|) \leq \|e_1 + xe_2\| \]

holds true.

Proof. Consider the function \(f_+ : x \mapsto \|e_1 + xe_2\| - x\) on \(\mathbb{R}^+\). Since \((e_1, e_2)\) is an Auerbach basis we have \(\|e_1 + xe_2\| \geq \|xe_2\| = x\), hence \(f_+(x) \geq 0\). And if we had \(f_+(x) = 0\) for some \(x \in \mathbb{R}^+\) we would have \(x = \|e_1 + xe_2\| \geq 1\) hence \(1 = \|e_2 + xe_1\| > \|e_2\| = 1\) since by hypothesis \(\|e_2 + se_1\| > 1\) for all \(s \neq 0\). It follows that \(f_+\) is positive. And in the same way one sees that the function \(f_- : x \mapsto \|e_1 - xe_2\| - x\) is positive on \(\mathbb{R}^+\). Moreover \(f = \min(f_+, f_-)\) satisfies \(f(0) = 1\).

Applying then Lemma 7.2 to \(f\) we get a positive convex non-decreasing 1-Lipschitz function \(\varphi : \mathbb{R}^+ \to \mathbb{R}\) such that \(|x| + \varphi(|x|) \leq \|e_1 + xe_2\|\) for all \(x \in \mathbb{R}\). \(\square\)

Still assuming the basis \((e_1, e_2)\) of \(E\) satisfies the condition of Lemma 7.1, we define the closed convex set \(H\) by

\[ H = \{ x \in E : \langle e_1^*, x \rangle \in [-1, 1] \text{ and } \langle e_2^*, x \rangle \geq 0 \} \]

Claim 7.4. There exists a 1-Lipschitz retraction \(p\) from \(E\) to \(H\).

Proof. The function \(p_1 : (x, y) \mapsto (x, \max(y, 0))\) is 1-Lipschitz: indeed if \(u\) and \(v\) belong to \(E\), \(p_1(u) - p_1(v)\) is a convex combination of the vectors \(u - v\) and \(\langle e_1^*, u - v \rangle e_1\) which have both a norm at most \(\|u - v\|\). Thus \(\|p_1(u) - p_1(v)\| \leq \|u - v\|\). Moreover \(p_1\) is the identity mapping on \(H\) and \(p_1(E) \subset \mathbb{R} \times \mathbb{R}^+\).

In the same way the mapping \(p_2 : (x, y) \mapsto (\max(-1, \min(1, x)), y)\) is the identity on \(H\) and is 1-Lipschitz since when \(u\) and \(v\) belong to \(E\), \(p_2(u) - p_2(v)\) is a convex combination of the vectors \(u - v\) and \(\langle e_2^*, u - v \rangle e_2\) which have both a norm at most \(\|u - v\|\). Moreover \(p_2(\mathbb{R} \times \mathbb{R}^+) \subset H\).

Then \(p = p_2 \circ p_1\) is the identity on \(H\), is 1-Lipschitz and satisfies \(p(E) \subset H\), so is a 1-Lipschitz retraction on \(H\). \(\diamond\)

Theorem 7.5. If \(E\) is a 2-dimensional normed space with smooth norm, there exists on \(E\) a quasi-contraction \(F\) such that \(\text{Fix}(F)\) is not connected.

Proof. Choose the basis \((e_1, e_2)\), the function \(\varphi\) and the set \(H\) as above. We will define two 1-Lipschitz functions \(\alpha\) and \(\beta\) from \(H\) to \(H\) and set \(F(x) = [\alpha(x), \beta(x)]\) which will be a quasi-contraction by Lemma 1.1.
In order to ensure \( \text{Fix}(F) \) is not connected we want to have \( \alpha(-e_1 + te_2) = -e_1 + te_2, \beta(e_1 + te_2) = e_1 + te_2, \) and \( te_2 \notin \text{Fix}(F) \) for all \( t \geq 0, \) so \( \{-1, 1\} \times \mathbb{R}^+ \subset \text{Fix}(F) \) and \( \{0\} \times \mathbb{R}^+ \cap \text{Fix}(F) = \emptyset. \)

Let \( \varphi \) be as in Lemma 7.3 and define the function \( \alpha : H_0 = \{-1, 0\} \times \mathbb{R}^+ \to E \) by :

\[
\alpha(u) = -e_1 + \lambda(u)e_2
\]

where \( \lambda : H_0 \to \mathbb{R}^+ \) is defined by \( \lambda(-e_1 + te_2) = t \) and \( \lambda(te_2) = t + \varphi(t). \) In particular \( \alpha(H_0) \subset H, \) \( \alpha(-e_1) = -e_1 \) and \( \alpha(ye_2) = (-1, y + \varphi(y)). \)

**Claim 7.6.** The function \( \lambda \) is 1-Lipschitz from \( H_0 \) to \( \mathbb{R}^+. \)

**Proof.** Denote \( a_t = (-1, t) \) and \( c_t = (0, t). \) We have to prove the following inequalities, for \( s \) and \( t \geq 0: \)

i. \( |\lambda(a_s) - \lambda(a_t)| \leq \|a_s - a_t\|, \)

ii. \( |\lambda(c_s) - \lambda(c_t)| \leq \|c_s - c_t\|, \)

iii. \( |\lambda(a_s) - \lambda(c_t)| \leq \|a_s - c_t\|. \)

For (i) we have \( \|\lambda(a_s) - \lambda(a_t)\| = |s - t| = \|a_s - a_t\|. \)

For (ii) we have \( \|c_s - c_t\| = \|(s - t)e_2\| \) and

\[
|\lambda(c_s) - \lambda(c_t)| = |(t + \varphi(t)) - (s + \varphi(s))| = |s + \varphi(s) - t - \varphi(t)|
\]

Without loss of generality we can assume \( s \leq t; \) so we have \( \varphi(t) \leq \varphi(s) \) and \( s + \varphi(s) \leq t + \varphi(t) \) since \( s \mapsto s + \varphi(s) \) is non-decreasing, so

\[
|s + \varphi(s) - t - \varphi(t)| = t - s - (\varphi(s) - \varphi(t)) \leq t - s = |s - t| = \|c_s - c_t\|
\]

Finally for (iii) we have

\[
|\lambda(a_s) - \lambda(c_t)| = |s - (t + \varphi(t))| = |s - \varphi(t) - t| \\
\leq |s - t| + \varphi(t) \leq |s - t| + \varphi(|s - t|)
\]

since \( t \geq |s - t| \) and by Lemma 7.3 applied to \( x = t - s: \)

\[
\|a_s - c_t\| = \|-e_1 + se_2 - te_2\| \geq \|e_1 + (t - s)e_2\| \geq |s - t| + \varphi(|s - t|)
\]

hence \( |\lambda(a_s) - \lambda(c_t)| \leq |s - t| + \varphi(|s - t|) \leq \|a_s - c_t\|. \)

It is well-known that any 1-Lipschitz function \( \lambda \) defined on a subset \( H_0 \) of the metric space \( H \) can be extended into a 1-Lipschitz function \( \check{\lambda} \) on \( H \) by the formula

\[
\check{\lambda}(x) = \inf_{y \in H_0} \left( \lambda(y) + d(x, y) \right)
\]
which yields a non-negative function $\tilde{\lambda}$ if $\lambda \geq 0$. Define then the function $\alpha$ on $H$ by $\alpha(u) = -e_1 + \tilde{\lambda}(u)e_2$. We clearly have for $u$ and $v$ in $H$:

$$\|a(u) - a(v)\| = \| (\tilde{\lambda}(u) - \tilde{\lambda}(v)) e_2 \| = |\tilde{\lambda}(u) - \tilde{\lambda}(v)| \leq \|u - v\|$$

Replacing $-e_1$ by $e_1$ we can define in the same way a 1-Lipschitz function $\beta : H \to H$ such that $\beta(e_1 + te_2) = e_1 + te_2$ and $\beta(te_2) = e_1 + (t + \varphi(t)) e_2$. Then the set-valued mapping $F$ defined on $H$ by $F(u) = [\alpha(u), \beta(u)]$ takes closed convex values. It is 1-Lipschitz because by Lemma 1.1:

$$d_H(F(u), F(v)) \leq \max(\|\alpha(u) - \alpha(v)\|, \|\beta(u) - \beta(v)\|) \leq \|u - v\|.$$ 

By definition we have $\alpha(-1, t) = (-1, t)$ and $\beta(1, t) = (1, t)$; so all points of $\{-1, 1\} \times \mathbb{R}^+$ are fixed points for $F$.

Conversely for $t \geq 0$ we have $F(0, t) = [-1, 1] \times \{t + \varphi(t)\}$ and this shows that $(0, t) \notin \text{Fix}(F)$ since $\varphi(t) \neq 0$, hence that the two non-empty open subsets $W_0 = \{u = (x, y) \in \text{Fix}(F) : x < 0\}$ and $W_1 = \{u = (x, y) \in \text{Fix}(F) : x > 0\}$ of $\text{Fix}(F)$ form a partition of $\text{Fix}(F)$. Thus $\text{Fix}(F)$ is not connected. And since $G = F \circ p$ satisfies $\text{Fix}(G) = \text{Fix}(F)$, we have just constructed a quasi-contraction on $E$ whose set of fixed points is not connected. And this completes the proof of Theorem 7.5.

\[ \square \]

**Corollary 7.7.** If the normed space $E$ has dimension at least 2, there exists on $E$ a quasi-contraction $G$ such that $\text{Fix}(G)$ is not connected.

**Proof.** Notice that following Theorem 2.1 the condition “dim$(E) \geq 2$” is necessary and that following Theorem 3.3 such a quasi-contraction cannot be single-valued if $E$ is finite-dimensional.

Take a closed linear subspace $E_0$ of codimension 2 and denote $\pi$ the canonical projection onto the quotient space $E/E_0$. Recall that the norm on $E/E_0$ is given by $\|y\| = \inf\{\|x\| : x \in \pi^{-1}(y)\}$.

It follows from Theorems 7.5 and 6.3 that there exists on the 2-dimensional space $E/E_0$ a quasi-contraction $F$ with closed convex values such that $\text{Fix}(F)$ is not connected. Define then for $x \in E : G(x) = \pi^{-1}(F(\pi(x)))$ which is clearly a non-empty closed convex subset of $E$. And

$$x \in \text{Fix}(G) \iff x \in G(x) \iff \pi(x) \in F(\pi(x)) \iff \pi(x) \in \text{Fix}(F)$$

so $\text{Fix}(G) = \pi^{-1}(\text{Fix}(F))$, and $\pi(\text{Fix}(G)) = \text{Fix}(F)$ since $\pi$ is onto.

If $\text{Fix}(G)$ were connected so would be $\text{Fix}(F) = \pi(\text{Fix}(G))$. Thus $\text{Fix}(G)$ is not connected. It remains to show that $G$ is 1-Lipschitz. And this follows
immediately from the facts that $F$ is 1-Lipschitz and that the mapping $T \mapsto \pi^{-1}(T)$ is 1-Lipschitz from $\mathcal{F}(E/E_0)$ to $\mathcal{F}(E)$. Indeed:

$$d(x, \pi^{-1}(T)) = \inf_{t \in T} \inf_{y \in t} ||x - y|| = \inf_{t \in T} \inf_{u \in E_0, y \in t} ||(x - u) - y||$$

$$= \inf_{t \in T} ||\pi(x) - t|| = d(\pi(x), T)$$

whence $d_H(\pi^{-1}(S), \pi^{-1}(T)) = d_H(S, T)$.

REFERENCES


