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FIXED POINTS
FOR NON-EXPANSIVE SET-VALUED MAPPINGS

JEAN SAINT RAYMOND

Let $E$ be a Banach space and $F : E \rightrightarrows E$ be a 1-Lipschitz set-valued mapping with closed convex non-empty values. We study the set of fixed points $\text{Fix}(F) = \{x \in E : x \in F(x)\}$ and provide in any space $E$ with $\dim(E) \geq 2$ an example of such a mapping $F$ such that $\text{Fix}(F)$ is not connected.

1. Introduction

In this paper we are concerned with set-valued mappings from a Banach space $E$ into itself having closed convex values. We will consider only mappings $F$ which are 1-Lipschitz for the Hausdorff distance $d_H$ on the set $\mathcal{F}(E)$ of non-empty closed subsets of $E$. Recall that, for $A$ and $B$ in $\mathcal{F}(E)$:

$$d_H(A, B) = \max \left( \sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A) \right)$$

so $d(z, F(y)) \leq \|x - y\|$ for all $x, y \in E$ and $z \in F(x)$.

For such a mapping $F$ we will be essentially interested in the set $\text{Fix}(F) = \{x \in E : x \in F(x)\}$ of fixed points of $F$ which is clearly closed in $E$. Of course it can happen that $\text{Fix}(F) = \emptyset$, for example if $F(x) = \{x + a\}$ where $a$ is a fixed non-zero vector in $E$.

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The case of multivalued contraction mappings, i.e. the case where $F$ is $q$-Lipschitz for a $q < 1$ was extensively studied for long (see [2], [1], [3], [4]) and many properties of structure or conservation for the set $\text{Fix}(F)$ of fixed points were shown. For example:

i. $\text{Fix}(F) \neq \emptyset$ (even for non convex-valued mappings). ([2], [1])

ii. $\text{Fix}(F)$ is an absolute retract, in particular it is path-connected. ([3])

iii. $\text{Fix}(F)$ is not a singleton if all values $F(x)$ have several points ([4] for the case where $q < \frac{1}{2}$ or $E$ is a Hilbert space, [5] for the general case)

iv. $\text{Fix}(F)$ is bounded if so are all values $F(x)$ for $x \in E$ (or even for only one $x \in E$). ([5])

v. $\text{Fix}(F)$ is compact if so are all values $F(x)$ for $x \in E$.([5])

We shall show in this paper that most of these results disappear when the Lipschitz constant $q$ of $F$ (which is $< 1$ if $F$ is a contraction mapping) is only assumed to be $\leq 1$ and $\dim(E) \geq 2$.

In the sequel we will call quasi-contraction any 1-Lipschitz set-valued mapping from $E$ to $E$ with (non-empty) closed convex values.

Clearly properties (iv) and (v) become false already in the trivial example where $E = \mathbb{R}$ and $F(x) = \{x\}$ since $F(x)$ is then always single-valued, and a fortiori compact and bounded though $\text{Fix}(F) = \mathbb{R}$ is unbounded. We provide in section 4 an example of quasi-contraction in a Hilbert space for which property (iii) does not hold. Concerning property (ii) and namely the connectedness of the set of fixed points of a quasi-contraction, the main part of this paper consists in proving that it does not hold in general.

After studying in section 2 the very simple case where $E$ has dimension 1, we look in section 3 at the case where $F(x)$ is single-valued. It turns out that if $E$ is finite-dimensional we can prove that $\text{Fix}(F)$ is connected but that this is no more true for infinite-dimensional spaces.

The remainder of the paper is devoted to show that in every normed space of dimension at least 2 one can construct a quasi-contraction having a non-connected set of fixed points. In section 5 we provide such a construction for the 2-dimensional euclidean space, and generalize it to every 2-dimensional smooth normed space in section 7. The general case is dealt in sections 6 and 7.

For any two points $a$ and $b$ in a normed space $E$ we will denote by $[a,b] \subset E$ the segment with endpoints $a$ and $b$, it is the set $\{(1-t)a+tb : t \in [0,1]\}$. The following simple lemma will be of constant use throughout the paper.
Lemma 1.1. Let $E$ be a normed space, $a$, $b$, $a'$, $b'$ be points of $E$. Then

$$d_H([a,b],[a',b']) \leq \max(\|a-a'\|, \|b-b'\|).$$

Proof. If $w \in [a,b]$ we have $w = ta + (1-t)b$ for some $t \in [0,1]$ hence

$$d(w,[a',b']) \leq d(ta + (1-t)b,ta' + (1-t)b')$$

$$= \|t((a-a') + (1-t)(b-b'))\|$$

$$\leq t\|a-a'\| + (1-t)\|b-b'\|$$

$$\leq \max(\|a-a'\|, \|b-b'\|)$$

whence it follows that $\sup_{w \in [a,b]} d(w,[a',b']) \leq \max(\|a-a'\|, \|b-b'\|)$, hence that $d_H([a,b],[a',b']) \leq \max(\|a-a'\|, \|b-b'\|)$. \qed

2. The case of dimension 1

Proposition 2.1. Let $F$ be a quasi-contraction from $\mathbb{R}$ to $\mathbb{R}$ (the values of $F$ are closed intervals). Then Fix($F$) is either empty or a closed interval. In particular Fix($F$) is connected.

Proof. Since $F$ is a quasi-contraction, it is easy to see that there are two mappings $a$ and $b$ from $\mathbb{R}$ to $\mathbb{R} \cup \{-\infty, +\infty\}$ such that $F(x) = \mathbb{R} \cap [a(x), b(x)]$ where $a(x) = -\infty$ for all $x$ or $a(x) > -\infty$ for all $x$ and $b(x) = +\infty$ for all $x$ or $b(x) < +\infty$ for all $x$. If $-\infty < a(x) \leq a(y)$ we have $d(a(x), F(y)) = |a(y) - a(x)|$ hence $d_H(F(x), F(y)) \geq |a(y) - a(x)|$ and similarly $d_H(F(x), F(y)) \geq |b(y) - b(x)|$ if $b(x) < +\infty$. So by Lemma 1.1

$$d_H(F(x), F(y)) = \max(\|a(x) - a(y)\|, \|b(x) - b(y)\|)$$

whence it follows that both $a$ and $b$ are 1-Lipschitz or constantly infinite.

Suppose towards a contradiction that there exist $x_0, x'_0 \in \text{Fix}(F)$, $x'_0 < x < x_0$ and $x \notin \text{Fix}(F)$. Then we have $b(x) = b(x_0) = +\infty$ or $b(x_0) \geq x_0$, hence

$$b(x) \geq b(x_0) - |x-x_0| \geq x_0 - (x_0 - x) = x$$

Thus since $x \notin \text{Fix}(F)$ we necessarily get $a(x) > x$, so $a(x) > -\infty$, and since $x'_0 < x$,

$$a(x'_0) \geq a(x) - |x-x'_0| > x - (x-x'_0) = x'_0$$

hence $x'_0 \notin F(x_0)$, a contradiction. This shows that Fix($F$) is an interval. \qed
3. The case of functions

In this section we consider single-valued quasi-contractions \( F \), which we identify with 1-Lipschitz functions \( f \) by \( F(x) = \{ f(x) \} \). More generally we will study the case where \( H \) is a closed convex subset of the Banach space \( E \) and \( f : H \to H \) is 1-Lipschitz.

**Proposition 3.1.** Let \( E \) be a strictly convex normed space, \( H \subseteq E \) be closed and convex, \( f : H \to H \) be 1-Lipschitz. Then \( \text{Fix}(F) \) is convex, possibly empty.

**Proof.** Clearly, if \( H = E \), \( a \in E \) is not zero and \( f \) is the translation \( x \mapsto x + a \), \( f \) is an isometry and \( \text{Fix}(f) = \emptyset \).

If \( f : H \to H \) is 1-Lipschitz and \( u, v \) are two distinct points of \( \text{Fix}(f) \) then for all \( t \in ]0, 1[ \) the points \( x_t = tu + (1 - t)v \) and \( y_t = f(x_t) \) satisfy

\[
\|y_t - u\| = \|f(x_t) - f(u)\| \leq \|x_t - u\| = (1 - t)\|u - v\|
\]

\[
\|y_t - v\| = \|f(x_t) - f(v)\| \leq \|x_t - v\| = t\|u - v\|
\]

thus \( \|u - v\| \leq \|y_t - u\| + \|y_t - v\| \leq \|u - v\| \), whence \( y_t \in [u, v] \) because \( E \) is strictly convex, and \( y_t - v = s(u - v) \) for some \( s \in [0, 1] \). Then since \( s\|u - v\| = \|y_t - v\| = t\|u - v\| \), we conclude that \( s = t \) and \( y_t = x_t \), hence that \( x_t \in \text{Fix}(F) \).

If \( E \) is not strictly convex, the previous result does not hold any more. For example if \( E = \mathbb{R}^2 \) equipped with the norm \( u = (x, y) \mapsto \|u\|_\infty = \max(|x|, |y|) \), the function \( f : (x, y) \mapsto (x, \sin x) \) is 1-Lipschitz: indeed

\[
\|f(x, y) - f(x', y')\| = \max(|x - x'|, |\sin x - \sin x'|) = |x - x'| \leq \|(x, y) - (x', y')\|
\]

and \( \text{Fix}(f) = \{(x, \sin x) : x \in \mathbb{R}\} \) is connected, but not convex. So far it is unclear, even in a finite-dimensional space, whether a 1-Lipschitz mapping could have a non-connected set of fixed points. Nevertheless we shall see later on, in Theorems 3.3, 3.4 and Corollary 7.7 what really happens.

**Lemma 3.2.** Let \( H \) be a non-empty convex compact subset of a finite-dimensional space \( E \) and \( f : H \to H \) be a 1-Lipschitz function. Then the set \( \text{Fix}(f) \) is compact connected and non-empty.

**Proof.** It follows readily from Brouwer’s theorem that \( \text{Fix}(f) \) is non-empty. And it is closed in \( H \) hence compact. For the connectedness we proceed by induction on the dimension of \( E \), or more precisely on the dimension \( \delta(H) \) of the linear subspace \( A_H \) generated by \( H - H \). If \( \delta(H) = 1 \) then \( A_H \approx \mathbb{R} \) is strictly convex and it follows from Proposition 3.1 that \( \text{Fix}(f) \) is convex, hence connected.
Assume that the statement of the lemma holds for all compact convex $K$ such that $\delta(K) < n$, that $H$ is a compact convex subset of $E$ such that $\delta(H) = n$ and that $a_0$ and $a_1$ are two distinct fixed points of the 1-Lipschitz function $f : H \to H$. By translation invariance we can and do assume that $0 \in H$ and $H$ spans $E$. Choose by Hahn-Banach’s theorem a linear functional $\xi \in E^*$ of norm 1 such that $\langle \xi, a_1 - a_0 \rangle = \|a_1 - a_0\|$. 

For all $t \in [0, 1]$ consider the set 

$$
H_t = \{ x \in H : \|x - a_0\| \leq t\|a_1 - a_0\| \text{ and } \|x - a_1\| \leq (1-t)\|a_1 - a_0\| \}
$$

which is convex and compact. Then $f(H_t) \subset H_t$: indeed if $x \in H_t$, $y = f(x) \in H$ and

$$
\begin{align*}
\|y - a_0\| &= \|f(x) - f(a_0)\| \leq \|x - a_0\| \leq t\|a_1 - a_0\| \\
\|y - a_1\| &= \|f(x) - f(a_1)\| \leq \|x - a_1\| \leq (1-t)\|a_1 - a_0\| .
\end{align*}
$$

Moreover for $x \in H_t$, 

$$
\begin{align*}
\langle \xi, x - a_0 \rangle &\leq \|\xi\| \cdot \|x - a_0\| \leq t\|a_1 - a_0\| \\
\langle \xi, a_1 - x \rangle &\leq \|\xi\| \cdot \|x - a_1\| \leq (1-t)\|a_1 - a_0\|
\end{align*}
$$

hence 

$$
0 = \langle \xi, a_1 - a_0 \rangle - \|a_1 - a_0\| = \langle \xi, x - a_0 \rangle + \langle \xi, a_1 - x \rangle - \|a_1 - a_0\| \\
= (\langle \xi, x - a_0 \rangle - t\|a_1 - a_0\|) + (\langle \xi, a_1 - x \rangle - (1-t)\|a_1 - a_0\|)
$$

and 

$$
0 \leq t\|a_1 - a_0\| - \langle \xi, x - a_0 \rangle = (\langle \xi, a_1 - x \rangle - (1-t)\|a_1 - a_0\|) \leq 0
$$

from what we deduce that 

$$
\langle \xi, x - a_0 \rangle = t\|a_1 - a_0\| \text{ and } \langle \xi, a_1 - x \rangle = (1-t)\|a_1 - a_0\|,
$$

hence that $\langle \xi, x \rangle = \theta := \langle \xi, a_0 \rangle + t\|a_1 - a_0\|$. Thus this shows that $H_t$ is included in the affine hyperplane $V_\theta = \{ x \in E : \langle \xi, x \rangle = \theta \}$ for which $\delta(V_\theta) < \dim(E) = n$. It follows then from the induction hypothesis that $H_t \cap \Fix(f) = \Fix(f|_{H_t})$ is compact connected and non-empty. 

Assume that $\Fix(f)$ is not connected. So it would exist two disjoint compact subsets $A_0$ and $A_1$ of $H$ such that $\Fix(f) \subset A_0 \cup A_1$ and two points $a_0$ and $a_1$ with $a_i \in A_i$. For $t \in [0, 1]$, let $H_t$ be the set introduced above which corresponds to the points $a_0, a_1$. Then, from what precedes, for all $t \in [0, 1]$, $\Fix(f|_{H_t}) \subset A_0 \cup A_1$ ($i \in \{0, 1\}$) and, by connectedness of $\Fix(f|_{H_t})$, $\Fix(f|_{H_t}) \subset A_i$ for some $i$. Then the sets $T_i = \{ t \in [0, 1] : \Fix(f|_{H_t}) \subset A_i \}$ form a partition of $[0, 1]$. Moreover,
note that 0 ∈ T₀ and 1 ∈ T₁. Thus, if we prove that T₀ and T₁ are also closed, we
will get a contradiction which will complete the proof of the connectedness of
Fix(f).

Let (tn) be a sequence in T₀ which converges to t*; there exists for all n
a point xn ∈ A₀ ∩ Fix(fHₙ). Since H ∩ B(a₀, ‖a₁ − a₀‖) is compact and Hₙ ⊂
H ∩ B(a₀, ‖a₁ − a₀‖), up to passing to a subsequence we can assume that (xn)
converges to some point x* ∈ A₀. We have

||x* − a₀|| = limₙ ‖xₙ − a₀‖ ≤ lim supₙ |a₁ − a₀| = t*‖a₁ − a₀‖

and similarly ‖x* − a₁‖ ≤ (1 − t)‖a₁ − a₀‖.

Thus x* ∈ H₁. Moreover ‖f(x*) − x*‖ = lim ‖f(xₙ) − xₙ‖ = 0. It follows
that x* ∈ Fix(fHₙ) ∩ A₀, hence that Fix(fHₙ) ∩ A₀ ≠ ∅ and that t* ∈ T₀. By the
same argument one can show that T₁ is closed. This completes the proof of the
connectedness of Fix(f), hence this of Lemma 3.2.

Theorem 3.3. Let E be a normed finite-dimensional space, H ⊂ E be a closed
convex subset and f : H → H be a 1-Lipschitz function. Then the set of fixed
points of f is connected.

Proof. It is enough to consider the case where Fix(f) is non-empty. Then let
a ∈ Fix(f). For all integer n ≥ 1 the set Hₙ = {x ∈ H : ‖x − a‖ ≤ n} is com-
 pact convex non-empty and stable under f. So it follows from Lemma 3.2 that
Fix(fHₙ) = Fix(f) ∩ Hₙ is connected and contains a. Then ∪ₙ Fix(fHₙ) = Fix(f)
is connected.

We now show that for infinite-dimensional spaces E there is no particular
topological property of the sets Fix(f) for 1-Lipschitz functions f : E → E.
Indeed :

Theorem 3.4. Let X be a complete metric space. Then there exist a Banach
space E and a 1-Lipschitz function f : E → E such that Fix(f) is isometric to
X. Moreover is X is separable the space E can be chosen separable.

Proof. Remark first that since Fix(f) is closed in E hence complete, the com-
pleteness of X is a necessary condition.

It is well-known that any metric space X can be isometrically embedded into
a Banach space. For example if a ∈ X and D is a dense subset of X the function
ψ : x ↦ (d(x, y) − d(a, y)) y∈D is an isometry from X to a subset of the space
ℓ∞. And if X is separable, the space span(ψ(X)) is a separable Banach space.

Recall that c₀ denotes the Banach space of all real sequences converging to 0
equipped with the norm : u = (xₙ) ↦ ‖u‖ = supₙ |xₙ| and denote 0 the null
sequence in $c_0$. Choose an isometric embedding $j : X \to W$ for some Banach space $W$ and define $H = j(X) \subset W$ and $E = W \times c_0$ equipped with the norm 
$(w,u) \mapsto \max(\|w\|, \|u\|)$.

For $w \in W$ and $u \in c_0$ define $f(w,u) = (w,v)$ where $v = (v_n) \in c_0$ is defined by

$$
{v_n} = \begin{cases} 
    d(w,H) & \text{if } n = 0 \\
    u_{n-1} & \text{if } n > 0 
\end{cases}
$$

Claim 3.5. The function $f$ is 1-Lipschitz and even an isometry.

Proof. We have

$$
\|f(w,u) - f(w',u')\| = \max(\|w-w'\|, |d(w,H) - d(w',H)|, \sup_{n \geq 1} |u_{n-1} - u'_{n-1}|)
$$

$$
= \max(\|w-w'\|, |d(w,H) - d(w',H)|, \|u-u'\|)
$$

$$
= \max(\|w-w'\|, \|u-u'\|) = \| (w,u) - (w',u') \|
$$

since $|d(w,H) - d(w',H)| \leq \|w-w'\|$.

Claim 3.6. Fix$(f)$ = $H \times \{0\}$.

Proof. It is clear that if $w \in H$ then $f(w,0) = (w,0)$.

Conversely, if $u = (x_n)$ and $(w,u)$ is a fixed point of $f$ we have $x_0 = d(w,H)$ and $x_n = x_{n-1}$ for all $n \geq 1$. Thus $u$ is the constant sequence with value $d(w,H)$, which does not belong to $c_0$ if $d(w,H) \neq 0$. So $w \in H$ and $u = 0$.

It follows from previous claim that the function $x \mapsto (j(x),0)$ is an isometry from $X$ onto Fix$(f)$.

4. Uniqueness of fixed points

We provide in this section an example of a quasi-contraction $F$ on a Hilbert space $H$ such that $F(x)$ is a singleton for no $x \in H$ but Fix$(F)$ is a singleton. And this shows that Property $(iii)$ in the Introduction does not hold in general for quasi-contractions.

Theorem 4.1. There exists a quasi-contraction $F$ on a Hilbert space $H$ such that diam$(F(x)) = 1$ for all $x \in H$ but Fix$(F)$ is a singleton.

Proof. Let $H$ be the Hilbert space $\ell^2$, $S : H \to H$ be the isometric mapping defined by $x = (x_n)_{n \geq 0} \mapsto y = (y_n)_{n \geq 0}$ where $y_0 = 0$ and $y_n = x_{n-1}$ for $n > 0$. Let $u = (u_n) \in H$ be the unit vector such that $u_0 = 1$ and $u_n = 0$ for $n > 0$, and $0$ be the null vector of $H$. 


For $x \in H$ define $F(x)$ as the segment $[S(x), S(x) + u]$ whose diameter is 1. We claim that $F$ is a quasi-contraction. Indeed by Lemma 1.1:

$$d_H(F(y), F(x)) \leq \max \left( \|S(y) - S(x)\|, \|S(y) + u - (S(x) + u)\| \right)$$

$$= \|S(y) - S(x)\| = \|S(y - x)\| = \|y - x\|.$$ 

If $x^* = (x_n^*)$ is a fixed point of $F$ there exists some $t \in [0, 1]$ such that

$$x^* = (1 - t)S(x^*) + t(S(x^*) + u) = S(x^*) + tu$$

so $x_0^* = tu_0 = t$ and $x_n^* = x_{n-1}^*$ for $n > 0$. This implies that the sequence $(x_n) \in l^2$ has to be constant with the value $t$, which is possible only with $t = 0$, and $x^* = 0$. And this shows that $\text{Fix}(F) = \{0\}$ is a singleton.

5. The 2-dimensional euclidean space

The aim of this section is to construct a quasi-contraction on the 2-dimensional euclidean space $\mathbb{R}^2$ whose set of fixed points is not connected. It follows from Proposition 2.1 that such a construction cannot be achieved in a 1-dimensional space, and from Proposition 3.1 that it is impossible with a single-valued quasi-contraction.

Consider the points $x_0 = (-1, 0)$ and $x_1 = (1, 0)$ of the euclidean space $\mathbb{R}^2$ and the symmetry $S : (u, v) \mapsto (-u, v)$ of $\mathbb{R}^2$ exchanging $x_0$ and $x_1$. We want to define two 1-Lipschitz mappings $\alpha$ and $\beta$ from $\mathbb{R}^2$ to itself such that $\alpha(x_0) = x_0$ and $S \circ \beta(z) = \alpha \circ S(z)$ for all $z$. In particular this implies $\beta(x_1) = S \circ \alpha(x_0) = S(x_0) = x_1$. We define then the set-valued mapping $F$ by $F(z) = [\alpha(z), \beta(z)]$, which is clearly convex and closed.

**Lemma 5.1.** If the mapping $\alpha$ is 1-Lipschitz then $F$ is a quasi-contraction.

**Proof.** Since $S$ is an isometry it is clear that $\beta = S \circ \alpha \circ S$ is 1-Lipschitz too. Then by Lemma 1.1, if $z$ and $z'$ are in $\mathbb{R}^2$

$$d_H(F(z), F(z')) \leq \max(\|\alpha(z) - \alpha(z')\|, \|\beta(z) - \beta(z')\|) \leq \|z - z'\|,$$

the wanted inequality. □

Fix $\varepsilon \in ]0, 1]$ and define the function $\varphi : \mathbb{R} \to \mathbb{R}$ by

$$\varphi(t) = \begin{cases} 
\varepsilon & \text{if } t \leq 0 \\
\varepsilon e^{-t} & \text{if } t \geq 0
\end{cases}$$

**Claim 5.2.** The function $\varphi$ is 1-Lipschitz and has no fixed point on $\mathbb{R}$.
Proof. It is immediately checked that $\varphi$ is continuous, derivable on $\mathbb{R} \setminus \{0\}$, that $\varphi'(t) = 0$ for $t < 0$ and $\varphi'(t) = 1 - \varepsilon e^{-t} \geq 0$ if $t > 0$. Since $|\varphi'(t)| \leq 1$ for $t \neq 0$, the function $\varphi$ is 1-Lipschitz. It is non-decreasing with values in $[\varepsilon, +\infty]$ and if $t^*$ were a fixed point of $\varphi$ we would have $t^* = \varphi(t^*) \geq \varepsilon > 0$ and $t^* = \varphi(t^*) = t^* + \varepsilon e^{-t^*}$ hence $e^{-t^*} = 0$, that is impossible. $\diamond$

Define now the function $\alpha$ on $X = \{x_0, x_1\} \cup (\{0\} \times \mathbb{R})$ by

\[
\begin{align*}
\alpha(x_0) &= x_0 \\
\alpha(x_1) &= (0, \varepsilon) \\
\alpha(0, v) &= (-\frac{1}{2}, \varphi(v))
\end{align*}
\]

Lemma 5.3. It is possible to choose $\varepsilon \in ]0, 1]$ such that this function $\alpha$ be 1-Lipschitz on $X$.

Proof. For $v \in \mathbb{R}$ denote $y_v = (0, v) \in X$. We have to prove that for some convenient $\varepsilon > 0$:

i. $\|\alpha(x_0) - \alpha(x_1)\|^2 \leq \|x_0 - x_1\|^2 = 4,$

ii. $\forall v \in \mathbb{R}, \|\alpha(x_0) - \alpha(y_v)\|^2 \leq \|x_0 - y_v\|^2,$

iii. $\forall v \in \mathbb{R}, \|\alpha(x_1) - \alpha(y_v)\|^2 \leq \|x_1 - y_v\|^2,$

iv. $\forall v, w \in \mathbb{R}, \|\alpha(y_v) - \alpha(y_w)\| \leq \|y_v - y_w\|.$

For (i) we must have $1 + \varepsilon^2 \leq 4$, that is true since $\varepsilon \leq 1 < \sqrt{3}$.

For (ii) we must have

$$(-1 + \frac{1}{2})^2 + \varphi(v)^2 \leq 1 + v^2$$

it is $\frac{1}{4} + \varphi(v)^2 \leq 1 + v^2$. And since $\varphi(v)^2 = \varepsilon^2$ if $v \leq 0$ and if $v \geq 0$:

$$\varphi(v)^2 = (v + \varepsilon e^{-v})^2 = v^2 + \varepsilon^2 e^{-2v} + 2\varepsilon ve^{-v} \leq v^2 + \varepsilon^2 + 2\varepsilon \sup_{t \geq 0} t e^{-t}$$

$$= v^2 + \varepsilon^2 + 2\varepsilon^{-1} e \leq v^2 + \varepsilon^2 + \varepsilon$$

we must have $\frac{1}{4} + v^2 + \varepsilon^2 + \varepsilon \leq 1 + v^2$, that holds as soon as $\varepsilon^2 + \varepsilon \leq \frac{3}{4}$, hence whenever $0 < \varepsilon \leq \frac{1}{2}$. 

For (iii) we must have \( \left( \frac{1}{2} \right)^2 + (\varphi(v) - \varepsilon)^2 \leq 1 + v^2 \). And since \( \varphi(v) \geq \varepsilon \) we have \( (\varphi(v) - \varepsilon)^2 \leq \varphi(v)^2 \). We have seen that if \( \varepsilon \) is chosen in \([0, \frac{1}{2}]\) then for all \( v: \frac{1}{4} + \varphi(v)^2 \leq 1 + v^2 \), so a fortiori \( \frac{1}{4} + (\varphi(v) - \varepsilon)^2 \leq 1 + v^2 \).

Finally for (iv), we have to show that

\[
\|\alpha(y_v) - \alpha(y_w)\| = |\varphi(v) - \varphi(w)| \leq \|y_v - y_w\| = |v - w|
\]

but this follows immediately from Claim 5.2.

Taking \( \varepsilon = \frac{1}{2} \) completes the proof of Lemma 5.3. \( \square \)

Using Kirszbraun-Valentine’s Theorem, we can extend the function \( \alpha \) into a 1-Lipschitz function (still denoted by \( \alpha \)) from \( \mathbb{R}^2 \) to \( \mathbb{R}^2 \), and then define \( \beta = S \circ \alpha \circ S \), which is 1-Lipschitz too.

**Theorem 5.4.** The set-valued mapping \( F : z \mapsto [\alpha(z), \beta(z)] \) is 1-Lipschitz, but the set \( \text{Fix}(F) \) of its fixed points is not connected.

**Proof.** That \( F \) be 1-Lipschitz follows from Lemma 5.1. Since \( x_0 = \alpha(x_0) \in F(x_0) \) we have \( x_0 \in \text{Fix}(F) \) and since \( x_1 = \beta(x_1) \in F(x_1) \) we have \( x_1 \in \text{Fix}(F) \). Hence \( \{x_0, x_1\} \subset \text{Fix}(F) \).

We now show that \( (\{0\} \times \mathbb{R}) \cap \text{Fix}(F) = \emptyset \). Indeed if there were some \( y_v = (0, v) \) in \( \text{Fix}(F) \) we should have \( y_v \in \text{conv}(\alpha(y_v), \beta(y_v)) \). Since \( \alpha(y_v) = \left(-\frac{1}{2}, \varphi(v)\right) \) and \( \beta(y_v) = \left(\frac{1}{2}, \varphi(v)\right) \) we would get

\[
(0, v) = y_v \in \text{conv}(\alpha(y_v), \beta(y_v)) = \left[-\frac{1}{2}, \frac{1}{2}\right] \times \{\varphi(v)\}
\]

hence \( \varphi(v) = v \), in contradiction with Claim 5.2.

It follows that the two disjoint open subsets \( W_0 = \{(x, y) \in \text{Fix}(F) : x < 0\} \) and \( W_1 = \{(x, y) \in \text{Fix}(F) : x > 0\} \) of \( \text{Fix}(F) \) are both non-empty and cover \( \text{Fix}(F) \). Thus \( \text{Fix}(F) \) is not connected. \( \square \)

### 6. The non-smooth case

It is also possible to give a simple example in any normed space \( E \) whose dual space \( E^* \) is not strictly convex (in particular if the norm of \( E \) itself is not smooth) of a quasi-contraction whose set of fixed points is not connected.

It \( E^* \) is not strictly convex there are two non-zero vectors \( u \) and \( v \) of \( E^* \) such that \( \|u\| = \|u + v\| = \|u - v\| = 1 \). Define then the real function \( h \) on \( E \) by

\[
h(x) = \langle u, x \rangle + \sin^2(\langle v, x \rangle)
\]
**Lemma 6.1.** The function $h$ is 1-Lipschitz.

*Proof.* In fact $h$ is of class $\mathcal{C}^1$ and its differential at $x$ is $h'(x) = u + \sin(2\langle v, x \rangle)v$. The convex function $v : t \mapsto \|u + t v\|$ satisfies $v(-1) = v(1) = 1$ hence $v(t) \leq 1$ for $t \in [-1, 1]$. It follows that $\|h'(x)\| \leq 1$ for all $x$ hence that $h$ is 1-Lipschitz.

**Lemma 6.2.** The set-valued mapping $P : \mathbb{R} \nrightarrow E$ defined by $P(t) = \{ y \in E : \langle u, y \rangle \geq t \}$ is 1-Lipschitz and takes closed convex non-empty values.

*Proof.* It is clear that $P(x)$ is convex closed and non-empty. Notice that if $t \leq t'$ then we have $P(t') \subset P(t)$, so $d_H(P(t), P(t')) = \sup_{y \in P(t)} d(y, P(t'))$. If $y \in P(t)$ and $\varepsilon > 0$ we can find some $z \in E$ with $\|z\| \leq 1 + \varepsilon$ and $\langle u, z \rangle = 1$.

Then $y' = y + (t' - t)z$ satisfies $\langle u, y' \rangle = \langle u, y \rangle + (t' - t) \geq t'$, hence $y' \in P(t')$ and $\|y - y'\| \leq (1 + \varepsilon)(t' - t)$. So $d(y, P(t')) \leq t' - t$ and $P$ is 1-Lipschitz.

**Theorem 6.3.** Let $E$ be a normed space. Assume that the norm on $E^*$ is not strictly convex. Then there exists a quasi-contraction $F : E \nrightarrow E$ with closed convex values such that $\text{Fix}(F)$ is not connected.

*Proof.* Take $h$ and $P$ as in previous Lemma, and define $F = P \circ h$ which is clearly 1-Lipschitz since so are $P$ and $h$. If $x \in \text{Fix}(F)$ we must have

$$\langle u, x \rangle \geq h(x) = \langle u, x \rangle + \sin^2(\langle v, x \rangle)$$

hence $\sin(\langle v, x \rangle) = 0$, that implies $\langle v, x \rangle = k\pi$ for some integer $k \in \mathbb{Z}$. If $a \in E$ satisfies $\langle v, a \rangle = 1$ (and such points exist since $v \neq 0$) we get

$$\text{Fix}(F) = \bigcup_{k \in \mathbb{Z}} (k.a + \ker v)$$

which is the discrete union of a countable family of pairwise disjoint closed hyperplanes, hence it cannot be connected.

**7. The smooth case**

We now want to extend Theorem 5.4 to every normed space $E$ of dimension 2. It follows from Theorem 6.3 that one can assume the norm of $E$ is smooth. Recall that a basis $(e_1, e_2, \ldots, e_n)$ of a finite-dimensional normed space $E$ is called an Auerbach basis of $E$ if $\|e_j\| = 1$ for all $j = 1, 2, \ldots, n$ and moreover $\|e_j^*\| = 1$ for all $j = 1, 2, \ldots, n$ where $(e_1^*, e_2^*, \ldots, e_n^*)$ is the dual basis of $E^*$ (what means $\langle e_j^*, e_k \rangle = \delta_{jk}^k$).

**Lemma 7.1.** If $E$ is a 2-dimensional normed space with smooth norm, there exists an Auerbach basis $(e_1, e_2)$ of $E$ such that $\|e_2 + te_1\| > 1$ for all real $t \neq 0$. 
Proof. Let $B$ be the unit ball of $E$. It is a well-known fact that if the determinant function $\Delta : (u, v) \mapsto u \wedge v$ attains at $(x, y)$ its supremum on $B \times B$ then $(x, y)$ is an Auerbach basis. It can be easily seen that the converse is not true: the canonical basis $(e_1, e_2)$ of $\ell^2$ satisfies $\Delta(e_1, e_2) = 1$ though $e_1 + e_2$ and $e_2 - e_1$ have norm 1 and $\Delta(e_1 + e_2, e_2 - e_1) = 2$.

Assume that $(e_1, e_2)$ is such an “extremal” Auerbach basis. If there is some $t \neq 0$ such that $e_2 + te_1 \in B$ then we have $e_1 \wedge e_2 > 0$ and for all $s > 0$

$((1 - \frac{s}{2})e_1 - \frac{s}{t}e_2) \wedge (e_2 + te_1) = (1 - \frac{s}{2} + s)e_1 \wedge e_2 = (1 + \frac{s}{2})e_1 \wedge e_2 > e_1 \wedge e_2$

what shows that $e_2 = (1 - \frac{s}{2})e_1 - \frac{s}{t}e_2 \not\in B$: indeed if not the basis

$(z_s, e_2') = (z_s, e_2 + te_1)$

would satisfy $\Delta(z_s, e_2') > \Delta(e_1, e_2)$ with $(z_s, e_2') \in B \times B$.

For $s < 0$ we have $\|z_s\| \geq \langle e_1^*, z_s \rangle = 1 - \frac{s}{2} > 1$. It follows that $\|z_s\| \geq 1$ for all $s \in \mathbb{R}$.

Denote $u^* = e_1^* - \frac{t}{2}e_2^*$. If $u \in \{v : \langle u^*, v \rangle = 1\}$ we have $\langle u^*, u - e_1 \rangle = 0$ so $u = z_s$ for some $s \in \mathbb{R}$, hence $\|u\| \geq 1$. This shows that $\|u^*\| \leq 1$. Then $\|e_1^*\| = 1$, $\|u^*\| \leq 1$ and

$1 \geq \|\lambda u^* + (1 - \lambda)e_1^*\| = \|e_1^* - \lambda \frac{t}{2}e_2^*\| \geq \langle e_1^* - \lambda \frac{t}{2}e_2^*, e_1 \rangle = 1$

for $\lambda \in [0, 1]$, which shows that the norm of $E^*$ is not strictly convex, in contradiction with the hypothesis of smoothness of $E$. \hfill \square

Lemma 7.2. Let $f : \mathbb{R}^+ \to \mathbb{R}$ be a continuous positive function such that $f(0) \leq 1$. Then there exists a convex non-increasing positive and 1-Lipschitz function $\varphi : \mathbb{R}^+ \to \mathbb{R}$ satisfying $\varphi(x) \leq f(x)$ for all $x \geq 0$.

Proof. For all $\alpha > 0$ set $\bar{f}(\alpha) = \inf_{0 \leq t \leq 2\alpha} f(t)$ which is positive by compactness of $[0, 2\alpha]$. And the affine decreasing function

$f_\alpha : x \mapsto \bar{f}(2\alpha)(1 - \frac{x}{2\alpha})$

satisfies $f_\alpha(x) \leq 0 < f(x)$ for $x \geq 2\alpha$, $f_\alpha(x) \leq \bar{f}(2\alpha) \leq f(x)$ for $0 \leq x \leq 2\alpha$ and $f_\alpha(\alpha) = \frac{1}{2} \bar{f}(2\alpha) > 0$. It follows that $\varphi : x \mapsto \sup_{\alpha \geq 1/2} f_\alpha(x)$ is convex, non-increasing, everywhere positive on $[\frac{1}{2}, +\infty[$ hence a fortiori on $\mathbb{R}^+$, and that $\varphi \leq f$.

Finally since the function $f_\alpha$ is $\frac{\bar{f}(2\alpha)}{2\alpha}$-Lipschitz the function $\varphi$ is $\lambda$-Lipschitz for $\lambda = \sup_{\alpha \geq 1/2} \frac{\bar{f}(2\alpha)}{2\alpha} = \bar{f}(1) \leq f(0) \leq 1$. \hfill \square
Lemma 7.3. If the basis \((e_1,e_2)\) of \(E\) is as in Lemma 7.1 there exists a positive convex non-decreasing 1-Lipschitz function \(\varphi : \mathbb{R}^+ \to \mathbb{R}\) such that for all \(x \in \mathbb{R}^+\) the inequality

\[ |x| + \varphi(|x|) \leq \|e_1 + xe_2\| \]

holds true.

Proof. Consider the function \(f_+ : x \mapsto \|e_1 + xe_2\| - x\) on \(\mathbb{R}^+\). Since \((e_1,e_2)\) is an Auerbach basis we have \(\|e_1 + xe_2\| \geq \|xe_2\| = x\), hence \(f_+(x) \geq 0\). And if we had \(f_+(x) = 0\) for some \(x \in \mathbb{R}^+\) we would have \(x = \|e_1 + xe_2\| \geq 1\) hence 1 = \(\|e_2 + \frac{1}{x}e_1\| > \|e_2\| = 1\) since by hypothesis \(\|e_2 + se_1\| > 1\) for all \(s \neq 0\). It follows that \(f_+\) is positive. And in the same way one sees that the function \(f_- : x \mapsto \|e_1 - xe_2\| - x\) is positive on \(\mathbb{R}^+\). Moreover \(f = \min(f_+, f_-)\) satisfies \(f(0) = 1\).

Applying then Lemma 7.2 to \(f\) we get a positive convex non-decreasing 1-Lipschitz function \(\varphi : \mathbb{R}^+ \to \mathbb{R}\) such that \(|x| + \varphi(|x|) \leq \|e_1 + xe_2\|\) for all \(x \in \mathbb{R}\). \qed

Still assuming the basis \((e_1,e_2)\) of \(E\) satisfies the condition of Lemma 7.1, we define the closed convex set \(H\) by

\[ H = \{x \in E : \langle e_1^*, x \rangle \in [-1,1] \text{ and } \langle e_2^*, x \rangle \geq 0\} \]

Claim 7.4. There exists a 1-Lipschitz retraction \(p\) from \(E\) to \(H\).

Proof. The function \(p_1 : (x,y) \mapsto (x, \max(y,0))\) is 1-Lipschitz: indeed if \(u\) and \(v\) belong to \(E\), \(p_1(u) - p_1(v)\) is a convex combination of the vectors \(u - v\) and \(\langle e_1^*, u - v \rangle e_1\) which have both a norm at most \(\|u - v\|\). Thus \(\|p_1(u) - p_1(v)\| \leq \|u - v\|\). Moreover \(p_1\) is the identity mapping on \(H\) and \(p_1(E) \subset \mathbb{R} \times \mathbb{R}^+\).

In the same way the mapping \(p_2 : (x,y) \mapsto (\max(-1, \min(1,x)), y)\) is the identity on \(H\) and is 1-Lipschitz since when \(u\) and \(v\) belong to \(E\), \(p_2(u) - p_2(v)\) is a convex combination of the vectors \(u - v\) and \(\langle e_2^*, u - v \rangle e_2\) which have both a norm at most \(\|u - v\|\). Moreover \(p_2(\mathbb{R} \times \mathbb{R}^+) \subset H\).

Then \(p = p_2 \circ p_1\) is the identity on \(H\), is 1-Lipschitz and satisfies \(p(E) \subset H\), so is a 1-Lipschitz retraction on \(H\). \qed

Theorem 7.5. If \(E\) is a 2-dimensional normed space with smooth norm, there exists on \(E\) a quasi-contraction \(F\) such that \(\text{Fix}(F)\) is not connected.

Proof. Choose the basis \((e_1,e_2)\), the function \(\varphi\) and the set \(H\) as above. We will define two 1-Lipschitz functions \(\alpha\) and \(\beta\) from \(H\) to \(H\) and set \(F(x) = [\alpha(x), \beta(x)]\) which will be a quasi-contraction by Lemma 1.1.
In order to ensure $\text{Fix}(F)$ is not connected we want to have $\alpha(-e_1 + te_2) = -e_1 + te_2$, $\beta(e_1 + te_2) = e_1 + te_2$, and $te_2 \notin \text{Fix}(F)$ for all $t \geq 0$, so $\{-1, 1\} \times \mathbb{R}^+ \subset \text{Fix}(F)$ and $(\{0\} \times \mathbb{R}^+) \cap \text{Fix}(F) = \emptyset$.

Let $\phi$ be as in Lemma 7.3 and define the function $\alpha : H_0 = \{-1, 0\} \times \mathbb{R}^+ \rightarrow E$ by:

$$\alpha(u) = -e_1 + \lambda(u)e_2$$

where $\lambda : H_0 \rightarrow \mathbb{R}^+$ is defined by $\lambda(-e_1 + te_2) = t$ and $\lambda(te_2) = t + \phi(t)$. In particular $\alpha(H_0) \subset H$, $\alpha(-e_1) = -e_1$ and $\alpha(ye_2) = (-1, y + \phi(y))$.

**Claim 7.6.** The function $\lambda$ is 1-Lipschitz from $H_0$ to $\mathbb{R}^+$.

**Proof.** Denote $a_t = (-1, t)$ and $c_t = (0, t)$. We have to prove the following inequalities, for $s$ and $t \geq 0$:

1. $|\lambda(a_s) - \lambda(a_t)| \leq \|a_s - a_t\|$,
2. $|\lambda(c_s) - \lambda(c_t)| \leq \|c_s - c_t\|$,
3. $|\lambda(a_s) - \lambda(c_t)| \leq \|a_s - c_t\|$.

For (i) we have $\|\lambda(a_s) - \lambda(a_t)\| = |s - t| = \|a_s - a_t\|$.

For (ii) we have $\|c_s - c_t\| = \|(s - t)e_2\| = |s - t|$ and

$$\|\lambda(c_s) - \lambda(c_t)\| = |(t + \phi(t)) - (s + \phi(s))| = |s + \phi(s) - t - \phi(t)|$$

Without loss of generality we can assume $s \leq t$; so we have $\phi(t) \leq \phi(s)$ and $s + \phi(s) \leq t + \phi(t)$ since $s \mapsto s + \phi(s)$ is non-decreasing, so

$$|s + \phi(s) - t - \phi(t)| = t - s - (\phi(s) - \phi(t)) \leq t - s = |s - t| = \|c_s - c_t\|$$

Finally for (iii) we have

$$|\lambda(a_s) - \lambda(c_t)| = |s - (t + \phi(t))| = |s - \phi(t) - t| 
\leq |s - t| + \phi(t) \leq |s - t| + \phi(|s - t|)$$

since $t \geq |s - t|$ and by Lemma 7.3 applied to $x = t - s$ :

$$\|a_s - c_t\| = \|-e_1 + se_2 - te_2\| = \|e_1 + (t - s)e_2\| \geq |s - t| + \phi(|s - t|)$$

hence $|\lambda(a_s) - \lambda(c_t)| \leq |s - t| + \phi(|s - t|) \leq \|a_s - c_t\|$.

It is well-known that any 1-Lipschitz function $\lambda$ defined on a subset $H_0$ of the metric space $H$ can be extended into a 1-Lipschitz function $\tilde{\lambda}$ on $H$ by the formula

$$\tilde{\lambda}(x) = \inf_{y \in H_0} \left( \lambda(y) + d(x, y) \right)$$
which yields a non-negative function \( \tilde{\lambda} \) if \( \lambda \geq 0 \). Define then the function \( \alpha \) on \( H \) by \( \alpha(u) = -e_1 + \tilde{\lambda}(u)e_2 \). We clearly have for \( u \) and \( v \) in \( H \):

\[
\|a(u) - a(v)\| = \| (\tilde{\lambda}(u) - \tilde{\lambda}(v)) e_2 \| = |\tilde{\lambda}(u) - \tilde{\lambda}(v)| \leq \|u - v\|
\]

Replacing \(-e_1\) by \(e_1\) we can define in the same way a 1-Lipschitz function \( \beta : H \to H \) such that \( \beta(e_1 + te_2) = e_1 + te_2 \) and \( \beta(te_2) = e_1 + (t + \varphi(t))e_2 \). Then the set-valued mapping \( F \) defined on \( H \) by \( F(u) = [\alpha(u), \beta(u)] \) takes closed convex values. It is 1-Lipschitz because by Lemma 1.1:

\[
d_H(F(u), F(v)) \leq \max(\|\alpha(u) - \alpha(v)\|, \|\beta(u) - \beta(v)\|) \leq \|u - v\|.
\]

By definition we have \( \alpha(-1,t) = (-1,t) \) and \( \beta(1,t) = (1,t) \); so all points of \( \{-1,1\} \times \mathbb{R}^+ \) are fixed points for \( F \).

Conversely for \( t \geq 0 \) we have \( F(0,t) = [-1,1] \times \{t + \varphi(t)\} \) and this shows that \( (0,t) \notin \text{Fix}(F) \) since \( \varphi(t) \neq 0 \), hence that the two non-empty open subsets \( W_0 = \{u = (x,y) \in \text{Fix}(F) : x < 0\} \) and \( W_1 = \{u = (x,y) \in \text{Fix}(F) : x > 0\} \) of \( \text{Fix}(F) \) form a partition of \( \text{Fix}(F) \). Thus \( \text{Fix}(F) \) is not connected. And since \( G = F \circ p \) satisfies \( \text{Fix}(G) = \text{Fix}(F) \), we have just constructed a quasi-contraction on \( E \) whose set of fixed points is not connected. And this completes the proof of Theorem 7.5. \( \square \)

**Corollary 7.7.** If the normed space \( E \) has dimension at least 2, there exists on \( E \) a quasi-contraction \( G \) such that \( \text{Fix}(G) \) is not connected.

**Proof.** Notice that following Theorem 2.1 the condition “\( \dim(E) \geq 2 \)” is necessary and that following Theorem 3.3 such a quasi-contraction cannot be single-valued if \( E \) is finite-dimensional.

Take a closed linear subspace \( E_0 \) of codimension 2 and denote \( \pi \) the canonical projection onto the quotient space \( E/E_0 \). Recall that the norm on \( E/E_0 \) is given by \( \|y\| = \inf\{\|x\| : x \in \pi^{-1}(y)\} \).

It follows from Theorems 7.5 and 6.3 that there exists on the 2-dimensional space \( E/E_0 \) a quasi-contraction \( F \) with closed convex values such that \( \text{Fix}(F) \) is not connected. Define then for \( x \in E : G(x) = \pi^{-1}(F(\pi(x))) \) which is clearly a non-empty closed convex subset of \( E \). And

\[
x \in \text{Fix}(G) \iff x \in G(x) \iff \pi(x) \in F(\pi(x)) \iff \pi(x) \in \text{Fix}(F)
\]

so \( \text{Fix}(G) = \pi^{-1}(\text{Fix}(F)) \), and \( \pi(\text{Fix}(G)) = \text{Fix}(F) \) since \( \pi \) is onto.

If \( \text{Fix}(G) \) were connected so would be \( \text{Fix}(F) = \pi(\text{Fix}(G)) \). Thus \( \text{Fix}(G) \) is not connected. It remains to show that \( G \) is 1-Lipschitz. And this follows
immediately from the facts that $F$ is 1-Lipschitz and that the mapping $T \mapsto \pi^{-1}(T)$ is 1-Lipschitz from $\mathcal{F}(E/E_0)$ to $\mathcal{F}(E)$. Indeed:

\[
d(x, \pi^{-1}(T)) = \inf_{t \in T} \inf_{y \in t} \|x - y\| = \inf_{t \in T} \inf_{u \in E_0, y \in t} \|(x - u) - y\| \\
= \inf_{t \in T} \|\pi(x) - t\| = d(\pi(x), T)
\]

whence $d_H(\pi^{-1}(S), \pi^{-1}(T)) = d_H(S, T)$. □

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