Fast voltage dynamics of voltage-conductance models for neural networks

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Abstract. We present the conductance limit of the voltage-conductance model with random firing voltage when conductance dynamics are slower than the voltage dynamics. The result of the limiting procedure is a transport/Fokker-Planck equation for conductance variable with a non-linear drift which depends on the total firing rate. We analyze the asymptotic behavior of the limit equation under two possible rescalings which relate the voltage scale, the conductance scale and the firing rate. We provide the sufficient framework in which the limiting procedure can be rigorously justified. Moreover, we also suggest a sufficient condition on the parameters and firing distribution in the limiting conductance equation under which we are able to obtain a unique stationary state and its asymptotic stability. Finally, we provide several numerical illustrations supporting the analytic results.

1. Introduction

Dynamics of neurons is characterized by the membrane potential and the synaptic or ionic channels conductance. When considering an assembly of neurons, a mean-field equation describes the neural network, called the voltage-conductance model which has been introduced in the neuroscience literature, see [7, 8, 19], and studied mathematically in [6, 17, 18]. This model enters a larger class of equations that are used to describe neural nets as the time-elapse model [4, 15, 10, 16] and the Integrate and Fire model [1, 5, 9, 13] (and the references therein). Mathematically, these models are focused on a single variable, either time from the last firing or voltage itself, assuming equilibrium for the others.

The voltage-conductance model describes the dynamics of a probability density function $p(t, v, g)$, the probability of finding neuron with membrane potential v and conductance g at time t. Here we consider that the domain of voltage variable is $v \in (V_R, V_E)$, where V_R and V_E denote the rest potential and excitatory reversal potential. The conductance variable g varies over nonnegative real number, i.e., $g \in [0, \infty)$. The prototype of voltage-conductance model reads as

$$\partial_t p + \partial_v \big[(g_L(V_L - v) + g(V_E - v))p \big] + \partial_g \left[\frac{g_{in}(t) - g}{\sigma_E} p \right] - \frac{a(t)}{\sigma_E} \partial^2_{gg} p = 0.$$

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Here, $g_L > 0$ and V_L, with $V_R < V_L < V_E$, stand for leak conductance and leak potential respectively. For the Integrate-and-Fire model, a firing potential V_F is introduced which determines the boundary conditions in v. Due to this difficulty, the original voltage-conductance model was simplified in [18] assuming a distribution of V_F which leads to the equation

$$\begin{aligned} &\partial_t p + \partial_v[(g_L(V_L - v) + g(V_E - v))p] + \partial_g[(G_{eq}(v, b\mathcal{N}(t)) - g)p] \\ &\qquad\qquad - a\partial^2_{gg} p + \phi_F(v)p = 0, \quad t \geq 0, \quad V_R < v < V_E, \quad g \geq 0, \end{aligned} \tag{1.1}$$

with the no-flux boundary conditions at $g = 0$ and $v = V_E$, and entering flux at $v = V_R$ (because $V_R < V_L < V_E$),

$$\begin{cases} (G_{eq}(v, b\mathcal{N}) - g)p(t, v, g) - a\partial_g p(t, v, g) = 0, \quad \text{for } g = 0, \\ [g_L(V_L - V_R) + g(V_E - V_R)]\, p(t, V_R, g) = N(t, g), \quad p(t, V_E, g) = 0. \end{cases} \tag{1.2}$$

The individual firing distribution of neurons with potential v is given by $\phi_F(v)$, which is generally a non-negative increasing function. The function $N(t, g)$ is the network firing rate of neurons with conductance g and $\mathcal{N}(t)$ denotes the total firing rate of neurons, which are defined as

$$N(t, g) := \int_{V_R}^{V_E} \phi_F(v) p \, dv, \qquad \mathcal{N}(t) := \int_0^\infty N(t, g)\, dg. \tag{1.3}$$

Moreover, $G_{eq}(v, \cdot) \geq 0$ is the conductance equilibrium at voltage v when there is no noise in conductance variable. Finally, the positive constants a and b denote synaptic noise and synaptic strength of network coupling respectively. The flux in v-direction vanishes at a single point which appears often in the present analysis, therefore we introduce the notation

$$0 \leq V_L < V_*(g) := \frac{g_L V_L + g V_E}{g + g_L} < V_E. \tag{1.4}$$

We complete the equation with an initial data p^0 satisfying

$$p^0(v, g) \geq 0, \quad \int_{V_R}^{V_E} \int_0^\infty p^0(v, g)\, dv\, dg = 1,$$

$$\int_0^\infty \int_{V_R}^{V_E} g^k p^0(v, g)\, dv\, dg < \infty, \qquad \forall k \geq 0. \tag{1.5}$$

Note that the no-flux boundary conditions, together with sufficiently fast decay of p at $g = \infty$ implies the conservation of total number of neurons:

$$\int_{V_R}^{V_E} \int_0^\infty p(t, v, g)\, dv\, dg = 1.$$

In the present paper, we are interested in limiting procedure when voltage dynamics is fast compared to the conductance dynamics. More precisely, we first consider the scaled equation of (1.1) given by

$$
\left\{
\begin{aligned}
&\partial_t p_\varepsilon + \frac{1}{\varepsilon}\partial_v\big[\big(g_L(V_L - v) + g(V_E - v)\big)p_\varepsilon\big] + \partial_g\big[\big(G_{eq}(v, b\mathcal{N}_\varepsilon(t)) - g\big)p_\varepsilon\big] \\
&\qquad\qquad\qquad -a\partial^2_{gg}p_\varepsilon + \phi_F(v)p_\varepsilon = 0, \qquad t \geq 0,\ V_R < v < V_E,\ g \geq 0, \\
&\frac{1}{\varepsilon}\big(g_L(V_L - V_R) + g(V_E - V_R)\big)p_\varepsilon(t, V_R, g) = N_\varepsilon(t, g) := \int_{V_R}^{V_E} \phi_F(v)p_\varepsilon \, dv.
\end{aligned}
\right.
\tag{1.6}
$$

Among other possible scales, we also mention the rescaled version of (1.1) with fast firing regime

$$
\left\{
\begin{aligned}
&\partial_t p_\varepsilon + \frac{1}{\varepsilon}\partial_v\big[\big(g_L(V_L - v) + g(V_E - v)\big)p_\varepsilon\big] + \partial_g\big[\big(G_{eq}(v, b\mathcal{N}_\varepsilon(t)) - g\big)p_\varepsilon\big] \\
&\qquad\qquad\qquad -a\partial^2_{gg}p_\varepsilon + \frac{1}{\varepsilon}\phi_F(v)p_\varepsilon = 0, \\
&\big(g_L(V_L - V_R) + g(V_E - V_R)\big)p_\varepsilon(t, V_R, g) = N_\varepsilon(t, g) := \int_{V_R}^{V_E} \phi_F(v)p_\varepsilon \, dv.
\end{aligned}
\right.
\tag{1.7}
$$

From now on, to simplify our settings, we only consider the case when the conductance equilibrium $G = G_{eq}$ is bounded above, depends only on the total firing rate $\mathcal{N}$ but not on the voltage variable v. Under this assumption, the heuristic limiting equation is obtained by integration in v and should be written as

$$
\partial_t n + \partial_g\big[\big(G(b\mathcal{N}(t)) - g\big)n\big] - a\partial^2_{gg}n = 0, \qquad t \geq 0,\ g \geq 0,
\tag{1.8}
$$

together with boundary condition at $g = 0$. We will specify the boundary conditions for the case $a = 0$ and $a > 0$ respectively in the next sections as well as the definition of $\mathcal{N}(t)$.

Notice that, in [18], the authors considered the fast conductance limit, i.e., following scaled version of equation (1.1) in which fast relaxation of the conductance variable g was considered:

$$
\begin{aligned}
\partial_t p_\varepsilon + \partial_v\big[\big(g_L(V_L - v) + g(V_E - v)\big)p_\varepsilon\big] + \frac{1}{\varepsilon}\partial_g\big[\big(G_{eq}(v, b\mathcal{N}_\varepsilon(t)) - g\big)p_\varepsilon\big], \\
- \frac{a}{\varepsilon}\partial^2_{gg}p_\varepsilon + \phi_F(v)p_\varepsilon = 0,
\end{aligned}
$$

and they derived the following Integrate-and-Fire type equation by letting the scaling parameter $\varepsilon \to 0$:

$$
\left\{
\begin{aligned}
&\partial_t n + \partial_v\big[\mathcal{G}(v, b\mathcal{N}(t))\big(V(b\mathcal{N}(t)) - v\big)n\big] + \phi_F(v)n = 0, \quad t \geq 0, \quad V_R < v < V_E, \\
&\mathcal{N}(t) := \int_{V_R}^{V_E} \phi_F(v)n(t, v)\, dv, \quad \mathcal{G}(V_R, b\mathcal{N}(t))\big(\mathcal{V}(b\mathcal{N}(t)) - V_R\big)n(t, V_R) = \mathcal{N}(t), \\
&n(t, V_E) = 0.
\end{aligned}
\right.
\tag{1.9}
$$

A major issue in neural networks is the appearance of synchronisation. This was observed in the voltage-conductance model by [6], in the Integrate-and-Fire model by [2, 3, 13, 20] and is mathematically related to blow-up in such equation [5, 12]. Synchronisation also occurs for the equation (1.9). A major issue is therefore to know if this phenomena also persists for one of the versions of equation (1.8) derived from the two proposed rescalings.

Thus, the goal of this paper is twofold. Firstly, we rigorously show the limiting procedure from (1.6) to (1.8); we derive a uniform-in-ε estimate for the moments of probability distribution p_ε as well as the firing rate $\mathcal{N}_\varepsilon$, from which the weak convergence of p_ε can be obtained. Secondly, we study the asymptotic behaviors of limiting equation (1.8), considering the cases $a = 0$ and $a > 0$ separately. The asymptotic distributions are the Dirac mass and Gaussian distribution for each case respectively.

The paper is organized as follows. In Section 2, we show the slow-fast limit from the voltage-conductance equation (1.6) to the conductance equation. Section 3 contains several properties and asymptotic behavior of the limiting conductance equation (1.8). In Section 4, we consider the other scaling (1.7) where the firing rate ϕ_F has also a fast dynamics. In Section 5, we provide several numerical tests for the models, supporting our analytical results in the previous sections. Finally, Section 6 is devoted to concluding remarks and discussion.

2. From voltage conductance model to conductance model

To begin with, from equation (1.6), we derive the limiting equation (1.8) at the formal level, and then we provide the rigorous convergence result. We start with the case without a conductance noise, i.e., $a = 0$ and apply a similar methodology to the case of $a > 0$.

2.1. **Hyperbolic equation.** When the synaptic noise a is neglected, the scaled equation (1.6) becomes

$$\partial_t p_\varepsilon + \frac{1}{\varepsilon}\partial_v\big[\big(g_L(V_L - v) + g(V_E - v)\big)p_\varepsilon\big] + \partial_g\big[\big(G(b\mathcal{N}_\varepsilon(t)) - g\big)p_\varepsilon\big] + \phi_F(v)p_\varepsilon = 0 \tag{2.1}$$

and the corresponding no-flux boundary condition becomes

$$\begin{cases} \frac{1}{\varepsilon}\big(g_L(V_L - V_R) + g(V_E - V_R)\big)p_\varepsilon(t, V_R, g) = N_\varepsilon(t, g) := \displaystyle\int_{V_R}^{V_E} \phi_F(v)p_\varepsilon dv, \\ p_\varepsilon(t, v, 0) = 0. \end{cases} \tag{2.2}$$

To derive the conductance-only equation, we consider a v-marginal of p_ε defined as

$$n_\varepsilon(t, g) := \int_{V_R}^{V_E} p_\varepsilon(t, v, g)\, dv.$$

Next, we integrate (2.1) with respect to v-variable and utilizing the boundary condition (2.2) to obtain

$$\partial_t n_\varepsilon + \partial_g\big[\big(G(b\mathcal{N}_\varepsilon(t)) - g\big)n_\varepsilon\big] = 0,$$

where the total firing rate $\mathcal{N}_\varepsilon$ is still

$$\mathcal{N}_\varepsilon(t) := \int_0^\infty \int_{V_R}^{V_E} \phi_F(v)p_\varepsilon(t, v, g)\, dv\, dg.$$

In order to close the equation, we need to express the total firing rate $\mathcal{N}_\varepsilon$ in terms of n_ε. However, due to the scaling, the formal limit density $p(t, v, g)$ should satisfy

$$\partial_v\big[\big(g_L(V_L - v) + g(V_E - v)\big)p\big] = 0.$$

This implies that there exists some $C = C(t, g)$ such that

$$\big(g_L(V_L - v) + g(V_E - v)\big)p = C(t, g).$$

However, since $p_\varepsilon(t, V_E, g) = 0$ for all $\varepsilon > 0$, we have $C(g) = 0$ and this implies that the limit density p should be concentrated on $v = V_*(g)$ (using the notation (1.4)), i.e.,

$$p(t, v, g) = n(t, g)\delta_{V_*(g)}(v). \tag{2.3}$$

Therefore, at the formal level, the limiting conductance equation should be

$$\partial_t n + \partial_g\big[\big(G(b\mathcal{N}(t)) - g\big)n\big] = 0, \qquad t \geq 0,\ g \geq 0,$$

subjected to the zero-flux boundary condition $n(t,0) = 0$ and

$$\mathcal{N}(t) = \int_0^\infty \int_{V_R}^{V_E} \phi_F(v) n(t,g) \delta_{V_*(g)}(v)\, dv\, dg = \int_0^\infty \phi_F\left(V_*(g)\right) n(t,g)\, dg.$$

2.2. **Diffusive equation.** Similar to the hyperbolic equation, the limit probability density $p(t,v,g)$ should be concentrated according to (2.3) and hence, the reduced equation becomes,

$$\partial_t n + \partial_g\big[\big(G(b\mathcal{N}(t)) - g\big)n\big] - a\partial_{gg}^2 n = 0, \quad \mathcal{N}(t) = \int_0^\infty \phi_F\left(V_*(g)\right) n(t,g)\, dg.$$

However, in the diffusive case, the zero-flux boundary condition is given as

$$G(b\mathcal{N}(t))n(t,0) = a\partial_g n(t,0).$$

Of course, this boundary condition can include the case when $a = 0$. However, the analysis, and more precisely the proof of compactness of $\mathcal{N}_\varepsilon(t)$, is much harder when $a > 0$. Therefore, we separate the two cases when $a = 0$ and $a > 0$. In the next section, we prove the slow-fast limit for the hyperbolic case first. The diffusive case is discussed later.

2.3. **Uniform-in-ε estimates on moments and firing rate, $a = 0$.** In order to rigorously derive the limit equation, we follow the strategy in [18]. The first argument we need is the uniform-in-ε boundedness of moments of p so as to ensure that there is no mass loss at infinity in g.

Lemma 2.1. *Let $\{p_\varepsilon\}_{\varepsilon>0}$ be a family of solution to (2.1), (2.2) with initial data $\{p_\varepsilon^0\}_{\varepsilon>0}$ with finite g-moments, i.e., (1.5) holds uniformly in ε. Then, for all $k \geq 0$, there exists $C(k)$ independent of ε satisfying*

$$M_{g,\varepsilon}^{(k)}(t) := \int_0^\infty \int_{V_R}^{V_E} g^k p_\varepsilon(t,v,g)\, dv\, dg \leq C(k), \qquad \forall t \geq 0.$$

Proof. Since the case when $k = 0$ is nothing but the conservation of total mass, we only consider the case when $k \geq 1$. We multiply (2.1) by g^k, for $k \geq 1$, and use integration by parts to find, thanks to the boundary condition at V_R in (2.2),

$$\frac{dM_{g,\varepsilon}^{(k)}}{dt} - \int_0^\infty g^k N(t,g) dg - k\int_0^\infty \int_{V_R}^{V_E} g^{k-1}(G(b\mathcal{N}_\varepsilon(t)) - g)p_\varepsilon\, dv\, dg$$
$$+ \int_0^\infty \int_{V_R}^{V_E} g^k \phi_F(v) p_\varepsilon(t,v,g)\, dv\, dg = 0.$$

Since G is bounded, we have

$$\frac{dM_{g,\varepsilon}^{(k)}}{dt} + kM_{g,\varepsilon}^{(k)} \leq CkM_{g,\varepsilon}^{(k-1)}.$$

However, for any $R>0$,

$$
\begin{aligned}
M_{g,\varepsilon}^{(k-1)}(t) &= \int_{V_R}^{V_F}\int_0^R g^{k-1}p_\varepsilon(t,v,g)\,dv\,dg + \int_{V_R}^{V_F}\int_R^\infty g^{k-1}p_\varepsilon(t,v,g)\,dv\,dg \\
&\le R^{k-1} + \frac{1}{R}\int_{V_R}^{V_F}\int_0^\infty g^k p_\varepsilon(t,v,g)\,dv\,dg = R^{k-1} + \frac{1}{R}M_{g,\varepsilon}^{(k)}(t).
\end{aligned}
$$

Therefore,

$$
\frac{dM_{g,\varepsilon}^{(k)}}{dt} + kM_{g,\varepsilon}^{(k)} \le CM_{g,\varepsilon}^{(k-1)} \le CkR^{k-1} + \frac{Ck}{R}M_{g,\varepsilon}^{(k)}.
$$

By taking R sufficiently large, we have

$$
\frac{dM_{g,\varepsilon}^{(k)}}{dt} + M_{g,\varepsilon}^{(k)} \le CR^{k-1}, \quad \text{i.e.,} \quad M_{g,\varepsilon}^{(k)}(t) \le M_{g,\varepsilon}^{(k)}(0)e^{-t} + CR^{k-1}\left(1-e^{-t}\right)
$$

which implies the uniform boundedness of any k-th g-moment. □

Next, we study the uniform-in-ε boundedness of total firing rate $\mathcal{N}_\varepsilon(t)$.

Lemma 2.2 (Compactness of $\mathcal{N}_\varepsilon(t)$, $a=0$)**.** *Let $\{p_\varepsilon\}_{\varepsilon>0}$ be a family of solution to (2.1) with initial data $\{p_\varepsilon^0\}_{\varepsilon>0}$ with finite g-moments, i.e., (1.5) holds uniformly in ε for $0<\varepsilon<1$. Then, we have the uniform bound*

$$
\|\mathcal{N}_\varepsilon\|_{L^\infty(0,\infty)} \le \|\phi_F\|_{L^\infty}
$$

Moreover, after initial time layer τ of order $\sqrt{\varepsilon}$, we also have the Lipschitz bound

$$
\left\|\frac{d\mathcal{N}_\varepsilon}{dt}\right\|_{L^\infty(\tau,\infty)} \le C(\tau).
$$

Proof. The uniform boundedness of $\mathcal{N}_\varepsilon$ follows immediately from its definition (1.3).

Next, we estimate the derivative of $\mathcal{N}_\varepsilon$ as

$$
\begin{aligned}
\frac{d\mathcal{N}_\varepsilon}{dt} &= \int_0^\infty\int_{V_R}^{V_E}\phi_F(v)\Big(-\frac{1}{\varepsilon}\partial_v\big[\big(g_L(V_L-v)+g(V_E-v)\big)p_\varepsilon\big] \\
&\qquad\qquad -\partial_g\big[\big(G(b\mathcal{N}_\varepsilon(t))-g\big)p_\varepsilon\big]-\phi_F(v)p_\varepsilon\Big)\,dv\,dg \\
&= \int_0^\infty\left(\frac{1}{\varepsilon}\phi_F(V_R)\big(g_L(V_L-V_R)+g(V_E-V_R)\big)p_\varepsilon(t,V_R,g)\right)dg \\
&\quad + \frac{1}{\varepsilon}\int_0^\infty\int_{V_R}^{V_E}\phi_F'(v)\big(g_L(V_L-v)+g(V_E-v)\big)p_\varepsilon\,dv\,dg - \int_0^\infty\int_{V_R}^{V_E}\phi_F(v)^2p_\varepsilon\,dv\,dg.
\end{aligned}
$$

Using the boundary condition, this yields

$$
\begin{aligned}
\frac{d\mathcal{N}_\varepsilon}{dt} &= \int_0^\infty\int_{V_R}^{V_E}\big(\phi_F(V_R)-\phi_F(v)\big)\,\phi_F(v)\,p_\varepsilon\,dv\,dg \\
&\quad + \frac{1}{\varepsilon}\int_0^\infty\int_{V_R}^{V_E}\phi_F'(v)\big(g_L(V_L-v)+g(V_E-v)\big)p_\varepsilon\,dv\,dg.
\end{aligned}
\tag{2.4}
$$

In order to show that $N_\varepsilon(t)$ is uniformly Lipschitz continuous in ε, we need to control the second term. To do so, we define

$$
Q_\varepsilon(t) := \int_0^\infty\int_{V_R}^{V_E}\big|g_L(V_L-v)+g(V_E-v)\big|p_\varepsilon\,dv\,dg.
$$

To estimate Q_ε, we multiply (2.1) by $|g_L(V_L-v)+g(V_E-v)|$ and use integration by parts to obtain

$$\begin{aligned}\frac{d}{dt}Q_\varepsilon(t) \le \frac{1}{\varepsilon}\int_0^\infty & \big|g_L(V_L-V_R)+g(V_E-V_R)\big|\big(g_L(V_L-V_R)+g(V_E-V_R)\big)p_\varepsilon(V_R,g)\,dg\\ &-\frac{1}{\varepsilon}\int_0^\infty\int_{V_R}^{V_E}(g_L+g)\,\big|g_L(V_L-v)+g(V_E-v)\big|p_\varepsilon\,dv\,dg\\ &+\int_0^\infty\int_{V_R}^{V_E}\partial_g\big|g_L(V_L-v)+g(V_E-v)\big|\big(G(b\mathcal{N}_\varepsilon(t))-g\big)p_\varepsilon\,dv\,dg\\ &-\int_0^\infty\int_{V_R}^{V_E}\big|g_L(V_L-v)+g(V_E-v)\big|\phi_F(v)p_\varepsilon\,dv\,dg.\end{aligned}$$

Here, we have used the relation

$$\partial_v\big[|g_L(V_L-v)+g(V_E-v)|\big]\big(g_L(V_L-v)+g(V_E-v)\big) = -(g+g_L)\,\big|g_L(V_L-v)+g(V_E-v)\big|.$$

Using the boundary condition, this implies

$$\begin{aligned}\frac{d}{dt}Q_\varepsilon(t)+\frac{1}{\varepsilon}\int_0^\infty\int_{V_R}^{V_E}&(g_L+g)\big|g_L(V_L-v)+g(V_E-v)\big|p_\varepsilon\,dv\,dg\\ &\le\int_0^\infty\Big(\big|g_L(V_L-V_R)+g(V_E-V_R)\big|-\big|g_L(V_L-v)+g(V_E-v)\big|\Big)N(t,g)\,dg\\ &\quad+\int_0^\infty\int_{V_R}^{V_E}(V_E-v)\big|G(b\mathcal{N}_\varepsilon(t))-g\big|p_\varepsilon\,dv\,dg.\end{aligned}$$

Since the first g-moment of p_ε is uniformly bounded by Lemma 2.1, we have

$$\frac{d}{dt}Q_\varepsilon(t)+\frac{g_L}{\varepsilon}Q_\varepsilon(t)\le C, \tag{2.5}$$

which, together with the Grönwall inequality, implies

$$Q_\varepsilon(t)\le Q_\varepsilon(0)e^{-\frac{g_L}{\varepsilon}t}+\frac{C\varepsilon}{g_L}.$$

Therefore, for $\tau=O(\sqrt{\varepsilon})$, we have

$$Q_\varepsilon(\tau)\le Q_\varepsilon(0)e^{-\frac{C}{\sqrt{\varepsilon}}}+\frac{C\varepsilon}{g_L}\le Ce^{-\frac{C}{\sqrt{\varepsilon}}}+\frac{C\varepsilon}{g_L}\le C\varepsilon$$

for sufficiently small ε. Therefore, we conclude that for $\varepsilon\ll 1$, using again (2.5),

$$Q_\varepsilon(t)\le Q_\varepsilon(\tau)e^{-\frac{g_L}{\varepsilon}(t-\tau)}+\frac{C\varepsilon}{g_L}\le C\varepsilon,\quad \text{for}\quad t\ge\tau.$$

Thus, we can control the quantity Q_ε after initial layer, and therefore, we can substitute this estimate to (2.4) to obtain

$$\begin{aligned}\frac{d\mathcal{N}_\varepsilon}{dt} &= \int_0^\infty\int_{V_R}^{V_E}\big(\phi_F(V_R)-\phi_F(v)\big)\big(\phi_F(v)\big)p_\varepsilon\,dv\,dg\\ &\quad+\frac{1}{\varepsilon}\int_0^\infty\int_{V_R}^{V_E}\phi_F'(v)\big(g_L(V_L-v)+g(V_E-v)\big)p_\varepsilon\,dv\,dg\\ &\le 2\|\phi_F\|_{L^\infty}^2+\frac{\|\phi_F'\|_{L^\infty}}{\varepsilon}Q_\varepsilon(t)\le C,\quad\text{for}\quad t\ge\tau.\end{aligned}$$

□

Remark 2.3. *One can get rid of the initial layer region if we consider a well-prepared initial data. More precisely, if we consider the initial data p^0_ε with $Q_\varepsilon(0) < C\varepsilon$, then, it automatically follows from (2.5) that $Q_\varepsilon(t) \le C\varepsilon$, for $t \ge 0$.*

2.4. **Uniform-in-ε estimates on moments and firing rate, $a > 0$.** Next, we prove the compactness of the total firing rate when there is a synaptic noise. The major difficulty lies in estimating the Lipschitz norm of $\mathcal{N}_\varepsilon$ and this requires more step than in the case $a = 0$.

Again, we start with an estimate for moments.

Lemma 2.4. *Let $\{p_\varepsilon\}_{\varepsilon>0}$ be a family of solution to (1.6) with initial data $\{p^0_\varepsilon\}_{\varepsilon>0}$ with finite g-moments, i.e., (1.5) holds uniformly in ε. Then, for all $k \ge 0$, there exists $C(k)$ independent of ε satisfying*

$$M^{(k)}_{g,\varepsilon}(t) := \int_0^\infty \int_{V_R}^{V_E} g^k p_\varepsilon(t, v, g)\, dv\, dg \le C(k), \qquad \forall t \ge 0.$$

Proof. Again, we only consider the case when $k \ge 1$. After exactly same procedure with Lemma 2.1 with $k \ge 2$, we obtain

$$\frac{dM^{(k)}_{g,\varepsilon}}{dt} + kM^{(k)}_{g,\varepsilon} \le C\left(M^{(k-1)}_{g,\varepsilon} + M^{(k-2)}_{g,\varepsilon}\right).$$

However, for any $R > 0$, we have,

$$M^{(k-1)}_{g,\varepsilon} \le R^{k-1} + \frac{1}{R}M^{(k)}_{g,\varepsilon}, \quad M^{(k-2)}_{g,\varepsilon} \le R^{k-2} + \frac{1}{R^2}M^{(k)}_{g,\varepsilon},$$

and by taking R sufficiently large, it holds

$$\frac{dM^{(k)}_{g,\varepsilon}}{dt} + M^{(k)}_{g,\varepsilon} \le C(R^{k-1} + R^{k-2}), \quad k \ge 2,$$

which implies a uniform-in-ε bound of $M^{(k)}_{g,\varepsilon}$ for $k \ge 2$. The case of $k = 1$ immediately follows by using the Cauchy-Schwarz inequality, together with mass conservation. □

As before, the last and main step is to estimate the total firing rate.

Lemma 2.5 (Compactness of $\mathcal{N}_\varepsilon(t)$, $a > 0$)**.** *Let $\{p_\varepsilon\}_{\varepsilon>0}$ be a family of solution to (1.6) with initial data $\{p^0_\varepsilon\}_{\varepsilon>0}$ satisfying (1.5) uniformly in ε for $0 < \varepsilon < 1$ and*

$$\phi'_F(V_L) = \phi''_F(V_L) = 0. \tag{2.6}$$

Then, we have the estimate

$$\|\mathcal{N}_\varepsilon\|_{L^\infty(0,\infty)} \le \|\phi_F\|_{L^\infty}.$$

Moreover, after initial time layer τ of order $\sqrt{\varepsilon}$, we also have the Lipschitz bound

$$\left\|\frac{d\mathcal{N}_\varepsilon}{dt}\right\|_{L^\infty(\tau,\infty)} \le C(\tau, T).$$

Proof. Since the estimate of $\mathcal{N}_\varepsilon$ is identical to previous case, we focus on estimate of $\frac{d\mathcal{N}_\varepsilon}{dt}$. Considering boundary condition, the equality (2.4) stills holds, and we need to estimate the term

$$R_\varepsilon(t) = \int_0^\infty \int_{V_R}^{V_E} \phi'_F(v)\big(g_L(V_L - v) + g(V_E - v)\big)p_\varepsilon\, dv\, dg = O(\varepsilon)$$

The previous strategy based on the quantity $Q_\varepsilon(t)$ is not compatible with the diffusion in g. Therefore we introduce another approach and set

$$R^1_\varepsilon(t) = \int_0^\infty \int_{V_R}^{V_E} \big(\phi'_F(v) - \phi'_F(V_*(g))\big)\big(g_L(V_L - v) + g(V_E - v)\big)p_\varepsilon \, dv \, dg,$$

$$R^2_\varepsilon(t) = \int_0^\infty \int_{V_R}^{V_E} \phi'_F(V_*(g))\big(g_L(V_L - v) + g(V_E - v)\big)p_\varepsilon \, dv \, dg.$$

We are going to prove that both terms are of order ε for $t \geq \tau$.

Step 1. The term $R^1_\varepsilon(t)$. This term can be treated as $Q_\varepsilon(t)$ in the case $a = 0$ because, here, the multiplier is smooth. To begin with, notice that, because of linearity

$$g_L(V_L - v) + g(V_E - v) = -(g_L + g)(v - V^*(g)),$$

we can write using that ϕ'_F is Lipschitz

$$|R^1_\varepsilon(t)| \leq C\overline{R}^1_\varepsilon(t), \qquad \overline{R}^1_\varepsilon(t) := \int_0^\infty \int_{V_R}^{V_E} (g_L + g)\big(v - V_*(g)\big)^2 p_\varepsilon \, dv \, dg,$$

It remains to prove that $\overline{R}^1_\varepsilon(t)$ is of order ε. We estimate $\overline{R}^1_\varepsilon$ by using equation (1.6) and integration by parts as

$$\begin{aligned} \frac{d\overline{R}^1_\varepsilon(t)}{dt} =& -\frac{2}{\varepsilon}\int_0^\infty \int_{V_R}^{V_E} (g_L + g)^2\big(v - V_*(g)\big)^2 p_\varepsilon \, dv \, dg \\ &+ \int_0^\infty \int_{V_R}^{V_E} \partial_g\big[(g_L + g)(v - V_*(g))\big]\big[(G - g)p_\varepsilon - a\partial_g p_\varepsilon\big] \, dv \, dg \\ &+ O(1) \end{aligned}$$

where the terms $O(1)$ are computed exactly as in the case $a = 0$. Only the control of the term with the derivative in g is new. We write, after one more integration by parts ,

$$\begin{aligned} \frac{d\overline{R}^1_\varepsilon(t)}{dt} =& -\frac{2}{\varepsilon}\int_0^\infty \int_{V_R}^{V_E} (g_L + g)^2\big(v - V_*(g)\big)^2 p_\varepsilon \, dv \, dg \\ &+ a\int_{V_R}^{V_E} \partial_g\big[(g_L + g)(v - V_*(g))\big] \; p_\varepsilon \, dv \,\bigg|_{g=0} \\ &+ O(1) \end{aligned}$$

Using the formula (1.4), it remains to notice that

$$\partial_g\big[(g_L + g)(v - V_*(g))\big] = v - V_E \leq 0.$$

This allows to conclude, as in Lemma 2.2, that $|\overline{R}^1_\varepsilon(t)| \leq C\varepsilon$ for $t \geq \tau$ and conclude the estimate on the term R^1_ε.

Step 2. The term $R^2_\varepsilon(t)$. For this term, we define

$$S(t, g) := \int_{V_R}^{V_E} \big(g_L(V_L - v) + g(V_E - v)\big)p_\varepsilon \, dv, \qquad R^2_\varepsilon(t) = \int_0^\infty \phi'_F(V_*(g))S(t, g) \, dg.$$

By using (1.6), one can compute the equations for $S(t,g)$ as

$$
\begin{aligned}
\partial_t S(t,g) + \frac{g_L+g}{\varepsilon} S = (g+g_L) \int_{V_R}^{V_E} (v-V_R)\phi_F(v) p_\varepsilon \, dv \\
- \int_{V_R}^{V_E} \big(g_L(V_L-v)+g(V_E-v)\big)\partial_g \big[\big(G(b\mathcal{N}_\varepsilon(t))-g\big)p_\varepsilon - a\partial_g p_\varepsilon\big]\, dv.
\end{aligned}
$$

We multiply $e^{\frac{g_L+g}{\varepsilon}t}$ on the both sides and integrate in time to derive

$$
\begin{aligned}
S(t,g) = S(0,g)e^{-\frac{g_L+g}{\varepsilon}t} + \int_0^t e^{\frac{g_L+g}{\varepsilon}(s-t)}(g+g_L) \int_{V_R}^{V_E} (v-V_R)\phi_F(v)p_\varepsilon(s,v,g)dvds \\
- \int_0^t e^{\frac{g_L+g}{\varepsilon}(s-t)} \int_{V_R}^{V_E} \big(g_L(V_L-v)+g(V_E-v)\big)\partial_g\big[\big(G(b\mathcal{N}_\varepsilon(t))-g\big)p_\varepsilon - a\partial_g p_\varepsilon\big]\, dv\, ds
\end{aligned}
$$

Therefore, we can recover that

$$
R_\varepsilon^2(t) = \int_0^\infty \phi_F'(V_*(g))S(0,g)e^{-\frac{g_L+g}{\varepsilon}t}dg + \mathrm{I} + \mathrm{II},
$$

where

$$
\mathrm{I} = \int_0^\infty \phi_F'(V_*(g)) \int_0^t e^{\frac{g_L+g}{\varepsilon}(s-t)}(g+g_L) \int_{V_R}^{V_E} (v-V_R)\phi_F(v)p_\varepsilon(s,v,g)dv\,ds\,dg
$$

which, using the moment bounds in Lemma 2.4, is estimated as

$$
\begin{aligned}
\big|\mathrm{I}\big| &\le \|\phi_F'\|_\infty \|\phi_F\|_\infty (V_E-V_R) \int_0^\infty \int_0^t e^{-\frac{g_L}{\varepsilon}(t-s)}(g+g_L) \int_{V_R}^{V_E} p_\varepsilon(s,v,g)\, dv\, ds\, dg \\
&\le \|\phi_F'\|_\infty \|\phi_F\|_\infty (V_E-V_R) \int_0^t e^{-\frac{g_L}{\varepsilon}(t-s)}ds \sup_{0\le s\le t} \int_0^\infty \int_{V_R}^{V_E} (g+g_L)p_\varepsilon(s,v,g)\, dv\, dg \\
&\le C\varepsilon.
\end{aligned}
$$

And the other term is written, thanks to the zero flux boundary condition at $g=0$,

$$
\begin{aligned}
\mathrm{II} &= -\int_0^\infty \phi_F'(V_*(g)) \int_0^t e^{\frac{g_L+g}{\varepsilon}(s-t)} \\
&\qquad \times \int_{V_R}^{V_E} \big(g_L(V_L-v)+g(V_E-v)\big)\partial_g\big[\big(G(b\mathcal{N}_\varepsilon(t))-g\big)p_\varepsilon - a\partial_g p_\varepsilon\big]\, dv\, ds\, dg \\
&= \int_0^\infty \int_{V_R}^{V_E} \int_0^t \partial_g\Big[\phi_F'(V_*(g))e^{\frac{g_L+g}{\varepsilon}(s-t)}\big(g_L(V_L-v)+g(V_E-v)\big)\Big] \\
&\qquad \times \big[\big(G(b\mathcal{N}_\varepsilon(t))-g\big)p_\varepsilon - a\partial_g p_\varepsilon\big]\, dv\, ds\, dg \\
&= O(\varepsilon) - a\int_0^\infty \int_{V_R}^{V_E} \int_0^t \partial_g\Big[\phi_F'(V_*(g))e^{\frac{g_L+g}{\varepsilon}(s-t)}\big(g_L(V_L-v)+g(V_E-v)\big)\Big]\partial_g p_\varepsilon\, dv\, ds\, dg \\
&= O(\varepsilon) + a\int_0^\infty \int_{V_R}^{V_E} \int_0^t \partial_{gg}^2\Big[\phi_F'(V_*(g))e^{\frac{g_L+g}{\varepsilon}(s-t)}\big(g_L(V_L-v)+g(V_E-v)\big)\Big]p_\varepsilon\, dv\, ds\, dg
\end{aligned}
$$

and the boundary term at $g=0$ vanishes thanks to the assumption (2.6).

Using again Lemma 2.4, this provides a control

$$
\big|R_\varepsilon^2(t)\big| \le Ce^{-\frac{g_L t}{\varepsilon}} + C\varepsilon.
$$

The remaining procedure is identical to that of Lemma 2.2. □

2.5. **Weak convergence.** We have gathered the material to present our first main theorem, which holds both for the hyperbolic and diffusive regimes (noiseless or noisy models)

Theorem 2.6. *Let $\{p_\varepsilon\}_{\varepsilon>0}$ be a family of solution to (2.1) with initial data $\{p_\varepsilon^0\}_{\varepsilon>0}$ with finite g-moments, i.e., (1.5) holds uniformly in ε. Then, as $\varepsilon \to 0$,*

$$p_\varepsilon(t,v,g) \rightharpoonup n(t,g)\delta_{V_*(g)}(v), \quad t \geq \tau, \quad \tau = O(\sqrt{\varepsilon})$$

(weakly in the sense of probability measures), up to a subsequence. the probability density $n(t,g)$ satisfies

$$\partial_t n + \partial_g((G(b\mathcal{N}(t)) - g)n) - a\partial_{gg}^2 n = 0, \qquad t \geq 0,\ g \geq 0, \tag{2.7}$$

$$n^0(g) := \int_{V_R}^{V_E} p^0(v,g)\,dv, \qquad \mathcal{N}(t) := \int_0^\infty \phi_F\big(V_*(g)\big)n(t,g)\,dg, \tag{2.8}$$

with the no-flux boundary condition

$$G(b\mathcal{N}(t))n(t,0) = a\partial_g n(t,0).$$

Proof. Since $\|\mathcal{N}\|_{L^\infty(0,\infty)}$ and $\left\|\frac{d\mathcal{N}}{dt}\right\|_{L^\infty(\tau,\infty)}$ are uniformly bounded, by the Arzela-Ascoli theorem, for any $T > 0$, there exists a Lipschitz function $\mathcal{N} : (\tau, T) \to [0,\infty)$ such that

$$\lim_{\varepsilon\to 0} \|\mathcal{N}_\varepsilon - \mathcal{N}\|_{L^\infty(\tau,T)} = 0.$$

Moreover, since $\{p_\varepsilon\}_{\varepsilon\geq 0}$ is a family of probability measures which moments are uniformly bounded, there exists a probability measure p such that, in the weak sense of bounded measures, and up to extraction of a subsequence

$$p_\varepsilon \rightharpoonup p, \qquad gp_\varepsilon \rightharpoonup gp.$$

Then, after applying the arguments used in deriving the formal limit equation in Section 2.1 and Section 2.2, we conclude that the limit probability measure p is of the form

$$p(t,v,g) = n(t,g)\delta_{V_*(g)}(v), \quad t \geq \tau,$$

with n satisfying (2.7)–(2.8). □

3. Asymptotic stability of conductance equation

Since the conductance equation (2.7) is established, we now look for the behavior of solutions. We first establish the existence of a stationary solution, and then we study its asymptotic stability. Again, we first study the hyperbolic equation.

3.1. **Hyperbolic equation.** The limiting equation for the hyperbolic model is

$$\partial_t n + \partial_g\big[\big(G(b\mathcal{N}(t)) - g\big)n\big] = 0, \quad \mathcal{N}(t) := \int_0^\infty \phi_F\left(V_*(g)\right)n(t,g)\,dg \tag{3.1}$$

with no-flux boundary condition $n(t,0) = 0$. From now on, we consider the linear case when $G(b\mathcal{N}(t)) = b\mathcal{N}(t)$ for analytic simplicity. Then, a stationary solution n_∞ should satisfy

$$\partial_g\big[\big(b\mathcal{N} - g\big)n_\infty\big] = 0, \quad \mathcal{N} = \int_0^\infty \phi_F\left(V_*(g)\right)n_\infty(g)\,dg.$$

Therefore, considering the boundary condition, the stationary solution n_∞ has to be

$$n_\infty(g) = \delta_{b\mathcal{N}}(g).$$

We again substitute this form of stationary solution to the equation for $\mathcal{N}$ which yields

$$\mathcal{N} = \phi_F(V_*(b\mathcal{N})) = \phi_F\left(\frac{g_L V_L + b\mathcal{N} V_E}{b\mathcal{N} + g_L}\right).$$

Therefore, finding a stationary solution is equivalent to finding a fixed point of the map

$$\Phi : \mathcal{N} \mapsto \phi_F\left(\frac{g_L V_L + b\mathcal{N} V_E}{b\mathcal{N} + g_L}\right) =: \tilde{\phi}_F(b\mathcal{N}).$$

Here, we defined the auxiliary function $\tilde{\phi}$ for notational convenience.

Proposition 3.1. *Suppose that b is sufficiently small so that the following relation is satisfied:*

$$b < \frac{g_L}{(V_E - V_L)\|\phi_F'\|_{L^\infty}}. \tag{3.2}$$

Then, the map Φ has the unique fixed point $\mathcal{N}_\infty$. Thus, there is a unique stationary state of (3.1).

Proof. First of all, let us briefly study the property of $\tilde{\phi}_F$. Since the function ϕ_F is increasing and the map (using the notation (1.4))

$$g \mapsto V_*(g) = V_E - \frac{g_L(V_E - V_L)}{g + g_L}$$

is increasing with respect to g, so is Φ, the composition of two increasing maps when $b > 0$. Since $\Phi(0) = \tilde{\phi}_F(0) = \phi_F(V_L) > 0$, Φ has a unique fixed point $\mathcal{N}_\infty$ if $\Phi' < 1$. However, this condition is equivalent to

$$\frac{d\Phi}{d\mathcal{N}} = \frac{d}{d\mathcal{N}}\tilde{\phi}_F(b\mathcal{N}) = b\tilde{\phi}_F'(b\mathcal{N}) = b\phi_F'\left(\frac{g_L V_L + b\mathcal{N} V_E}{b\mathcal{N} + g_L}\right)\frac{g_L(V_E - V_L)}{(b\mathcal{N} + g_L)^2} < 1.$$

Indeed, assuming the condition (3.2) on b, we have

$$\frac{d\Phi}{d\mathcal{N}} = b\phi_F'\left(\frac{g_L V_L + b\mathcal{N} V_E}{b\mathcal{N} + g_L}\right)\frac{g_L(V_E - V_L)}{(b\mathcal{N} + g_L)^2} < \left(\frac{g_L}{b\mathcal{N} + g_L}\right)^2 < 1$$

and we conclude that Φ has a unique fixed point if (3.2) holds. □

From now on, we assume (3.2) so that we have the unique fixed point $\mathcal{N}_\infty$. Our next goal is to study the long term convergence of solution $n(t, g)$ toward the stationary solution $n_\infty(g) = \delta_{b\mathcal{N}_\infty}(g)$. In order to show the convergence, we consider the Lyapunov functional

$$\mathcal{L}(t) := \int_0^\infty (g - b\mathcal{N}_\infty)^2 n \, dg.$$

Note that this Lyapunov functional can be understood as a measure of how far the solution n is from the stationary solution $\delta_{b\mathcal{N}_\infty}(g)$. The following theorem shows that the condition (3.2) which guarantees the existence and uniqueness of fixed point $\mathcal{N}_\infty$ is also sufficient condition to exponential decay of $\mathcal{L}$.

Theorem 3.2. *Let $n = n(t, g)$ satisfy (3.1) and assume the smallness condition (3.2) on b. Let $\mathcal{N}_\infty$ be the unique fixed point of Φ obtained in Proposition 3.1. Then, the Lyapunov functional $\mathcal{L}$ decays exponentially towards 0, namely*

$$\mathcal{L}(t) \le \mathcal{L}(0) e^{-2(1-b\|\tilde{\phi}'_F\|_{L^\infty})t}.$$

Let us mention that the method based on Monge-Kantorowich distance (see [14]) allows to prove that solutions will converge to a steady state, even when it is not unique. However, the norm is weaker and should be adapted which is beyond the scope of the present analysis.

Proof. We directly take a derivative to the definition of $\mathcal{L}$ to obtain

$$\begin{aligned}\frac{d\mathcal{L}}{dt} &= \int_0^\infty (g - b\mathcal{N}_\infty)^2 \big[-\partial_g \big((b\mathcal{N}(t) - g) n \big) \big] \, dg = 2 \int_0^\infty (g - b\mathcal{N}_\infty)(b\mathcal{N}(t) - g) n \, dg \\ &= -2 \int_0^\infty (b\mathcal{N}_\infty - g)^2 n \, dg + 2b(\mathcal{N}(t) - \mathcal{N}_\infty) \int_0^\infty (g - b\mathcal{N}_\infty) n \, dg \\ &\le -2\mathcal{L} + 2b|\mathcal{N}(t) - \mathcal{N}_\infty| \sqrt{\mathcal{L}}\end{aligned} \tag{3.3}$$

where we have used the Cauchy-Schwarz inequality together with a mass conservation law

$$\int_0^\infty |g - b\mathcal{N}_\infty| n \, dg \le \left(\int_0^\infty n \, dg \right)^{1/2} \left(\int_0^\infty |g - b\mathcal{N}_\infty|^2 n \, dg \right)^{1/2} = \sqrt{\mathcal{L}}.$$

On the other hand, since $\mathcal{N}_\infty$ is the unique fixed point of Φ, it satisfies $\mathcal{N}_\infty = \tilde{\phi}_F(b\mathcal{N}_\infty)$. Therefore, we have

$$\begin{aligned}|\mathcal{N}(t) - \mathcal{N}_\infty| &= \left| \int_0^\infty \Big(\tilde{\phi}_F(g) - \tilde{\phi}_F(b\mathcal{N}_\infty) \Big) n \, dg \right| \le \int_0^\infty \Big| \tilde{\phi}_F(g) - \tilde{\phi}_F(b\mathcal{N}_\infty) \Big| n \, dg \\ &\le \int_0^\infty \|\tilde{\phi}'_F\|_{L^\infty} |g - b\mathcal{N}_\infty| n \, dg \le \|\tilde{\phi}'_F\|_{L^\infty} \sqrt{\mathcal{L}}.\end{aligned} \tag{3.4}$$

Now, we substitute (3.4) into (3.3) to obtain

$$\frac{d\mathcal{L}}{dt} \le -2(1 - b\|\tilde{\phi}'_F\|_{L^\infty})\mathcal{L}.$$

Therefore, if we have $b\|\tilde{\phi}'_F\|_{L^\infty} < 1$, then we have a desired exponential decay of $\mathcal{L}$. Note that this condition is exactly same condition with (3.2). $\square$

3.2. **Diffusive equation.** When considering the noisy conductance equation, the limit equation is given by

$$\partial_t n + \partial_g \big[(b\mathcal{N}(t) - g) n \big] - a\partial^2_{gg} n = 0 \tag{3.5}$$

with the zero-flux boundary condition

$$b\mathcal{N}(t) n(t, 0) = a\partial_g n(t, 0).$$

As in the hyperbolic case, we first search for a stationary solution n_∞ which should satisfy

$$\partial_g \big[(b\mathcal{N}_\infty - g) n_\infty \big] = a\partial^2_{gg} n_\infty.$$

Thanks to the boundary condition, we have

$$(b\mathcal{N}_\infty - g) n_\infty = a\partial_g n_\infty \quad \text{or equivalently} \quad n_\infty(g) = \frac{1}{Z_{\mathcal{N}_\infty}} \exp\left(-\frac{(g - b\mathcal{N}_\infty)^2}{2a} \right)$$

where $\mathcal{N}_\infty$ is given by the implicit equation and normalizing constant

$$\mathcal{N}_\infty = \int_0^\infty \tilde{\phi}_F(g) n_\infty(g)\, dg, \qquad Z_{\mathcal{N}_\infty} := \int_0^\infty \exp\left(-\frac{(g-b\mathcal{N}_\infty)^2}{2a}\right) dg.$$

Based on our experience in hyperbolic case, finding a stationary state is equivalent to finding a fixed point of the following map:

$$\Psi : \mathcal{N} \mapsto \int_0^\infty \tilde{\phi}_F(g) \frac{1}{Z_{\mathcal{N}}} \exp\left(-\frac{(g-b\mathcal{N})^2}{2a}\right) dg.$$

The following proposition provides a sufficient condition so that the map Ψ has a unique fixed point.

Proposition 3.3. *Suppose that b satisfies condition (3.2) in Proposition 3.1. Then, the map Ψ has a unique fixed point $\mathcal{N}_\infty$.*

Proof. First of all, we note that

$$\Psi(0) = \frac{1}{Z_0} \int_0^\infty \tilde{\phi}_F(g) \exp\left(-\frac{g^2}{2a}\right) dg > 0.$$

Therefore, if we have $\Psi'(\mathcal{N}) < 1$, then we have the existence and uniqueness of fixed point. Hence we first investigate $\Psi'(\mathcal{N})$. In fact, we have

$$\begin{aligned} \frac{d}{d\mathcal{N}} Z_{\mathcal{N}} &= \frac{b}{a} \int_0^\infty (g-b\mathcal{N}) \exp\left(-\frac{(g-b\mathcal{N})^2}{2a}\right) dg = \frac{b}{a} \int_{-b\mathcal{N}}^\infty g \exp\left(-\frac{g^2}{2a}\right) dg \\ &= \frac{b}{a} \left[-a \exp\left(-\frac{g^2}{2a}\right)\right]_{b\mathcal{N}}^\infty = b \exp\left(-\frac{b^2\mathcal{N}^2}{2a}\right). \end{aligned} \tag{3.6}$$

We take a derivative to Ψ to obtain

$$\begin{aligned} \frac{d\Psi}{d\mathcal{N}} &= -\frac{\frac{d}{d\mathcal{N}}}{Z_{\mathcal{N}}^2} \int_0^\infty \tilde{\phi}_F(g) \exp\left(-\frac{(g-b\mathcal{N})^2}{2a}\right) dg \\ &\quad + \frac{1}{Z_{\mathcal{N}}} \int_0^\infty \tilde{\phi}_F(g) \frac{b}{a}(g-b\mathcal{N}) \exp\left(-\frac{(g-b\mathcal{N})^2}{2a}\right) dg. \end{aligned} \tag{3.7}$$

However, the integration by parts yields

$$\begin{aligned} &\frac{1}{a} \int_0^\infty \tilde{\phi}_F(g)(g-b\mathcal{N}) \exp\left(-\frac{(g-b\mathcal{N})^2}{2a}\right) dg = -\int_0^\infty \tilde{\phi}_F(g) \partial_g \left(\exp\left(-\frac{(g-b\mathcal{N})^2}{2a}\right)\right) dg \\ &= \tilde{\phi}_F(0) \exp\left(-\frac{b^2\mathcal{N}^2}{2a}\right) + \int_0^\infty \tilde{\phi}'_F(g) \exp\left(-\frac{(g-b\mathcal{N})^2}{2a}\right) dg, \end{aligned}$$

which, combined to (3.7), gives

$$\begin{aligned}
\frac{d\Psi}{d\mathcal{N}} &= -\frac{\frac{d}{d\mathcal{N}}Z_{\mathcal{N}}}{Z_{\mathcal{N}}^2}\int_0^\infty \tilde{\phi}_F(g)\exp\left(-\frac{(g-b\mathcal{N})^2}{2a}\right)dg + \frac{b}{Z_{\mathcal{N}}}\tilde{\phi}_F(0)\exp\left(-\frac{b^2\mathcal{N}^2}{2a}\right)\\
&\quad + \frac{b}{Z_{\mathcal{N}}}\int_0^\infty \tilde{\phi}_F'(g)\exp\left(-\frac{(g-b\mathcal{N})^2}{2a}\right)dg\\
&= \frac{b}{Z_{\mathcal{N}}}\int_0^\infty \tilde{\phi}_F'(g)\exp\left(-\frac{(g-b\mathcal{N})^2}{2a}\right)dg\\
&\quad - \frac{\frac{d}{d\mathcal{N}}Z_{\mathcal{N}}}{Z_{\mathcal{N}}^2}\int_0^\infty \left(\tilde{\phi}_F(g)-\tilde{\phi}_F(0)\right)\exp\left(-\frac{(g-b\mathcal{N})^2}{2a}\right)dg\\
&= \frac{1}{Z_{\mathcal{N}}}\int_0^\infty \left(b\tilde{\phi}_F'(g) - \frac{\frac{d}{d\mathcal{N}}Z_{\mathcal{N}}}{Z_{\mathcal{N}}}(\tilde{\phi}_F(g)-\tilde{\phi}_F(0))\right)\exp\left(-\frac{(g-b\mathcal{N})^2}{2a}\right)dg.
\end{aligned}$$

Therefore, if the network connectivity b satisfies

$$b\tilde{\phi}_F'(g) - \frac{\frac{d}{d\mathcal{N}}Z_{\mathcal{N}}}{Z_{\mathcal{N}}}(\tilde{\phi}_F(g)-\tilde{\phi}_F(0)) < 1 \tag{3.8}$$

for any g and $\mathcal{N}$, we have

$$\frac{d\Psi}{d\mathcal{N}} < \frac{1}{Z_{\mathcal{N}}}\int_0^\infty \exp\left(-\frac{(g-b\mathcal{N})^2}{2a}\right)dg = 1,$$

which implies the existence and uniqueness of fixed point. However, the condition (3.8) is easily satisfied when

$$b\|\tilde{\phi}_F'\|_{L^\infty} < 1$$

since $\tilde{\phi}_F(g) \geq \tilde{\phi}_F(0)$ and $\frac{d}{d\mathcal{N}}Z_{\mathcal{N}} > 0$ and therefore,

$$b\tilde{\phi}_F'(g) - \frac{\frac{d}{d\mathcal{N}}Z_{\mathcal{N}}}{Z_{\mathcal{N}}}\left(\tilde{\phi}_F(g)-\tilde{\phi}_F(0)\right) \leq b\|\tilde{\phi}_F'\|_{L^\infty} < 1.$$

□

Remark 3.4. *We investigate the condition* (3.8) *more precisely as follows.*
1. We can estimate $Z_{\mathcal{N}}$ also as follows:

$$\begin{aligned}
Z_{\mathcal{N}} &:= \int_{-b\mathcal{N}}^\infty \exp\left(-\frac{g^2}{2a}\right)dg = \sqrt{\frac{a\pi}{2}} + \int_0^{b\mathcal{N}}\exp\left(-\frac{g^2}{2a}\right) = \sqrt{\frac{a\pi}{2}} + \sqrt{2a}\int_0^{\frac{b\mathcal{N}}{\sqrt{2a}}}\exp(-x^2)\,dx\\
&= \sqrt{\frac{a\pi}{2}}\left(1+\operatorname{erf}\left(\frac{b\mathcal{N}}{\sqrt{2a}}\right)\right) = \sqrt{\frac{a\pi}{2}}\left(2-\operatorname{erfc}\left(\frac{b\mathcal{N}}{\sqrt{2a}}\right)\right)
\end{aligned}$$

However, the complementary error function $\operatorname{erfc}(x)$ *can be estimated as in* [11], $\operatorname{erfc}(x) \leq e^{-x^2}$, *which implies*

$$\sqrt{\frac{a\pi}{2}}\left(2-\exp\left(-\frac{b^2\mathcal{N}^2}{2a}\right)\right) \leq Z_{\mathcal{N}} \leq \sqrt{2a\pi}.$$

2. The condition (3.8) *can be rewritten as*

$$b\tilde{\phi}_F'(g) < 1 + \frac{\frac{d}{d\mathcal{N}}Z_{\mathcal{N}}}{Z_{\mathcal{N}}}\left(\tilde{\phi}_F(g)-\tilde{\phi}_F(0)\right), \tag{3.9}$$

for any g and $\mathcal{N}$. On the other hand, according to (3.6) and estimate on $Z_{\mathcal{N}}$ in (1),

$$\frac{\frac{d}{d\mathcal{N}}Z_{\mathcal{N}}}{Z_{\mathcal{N}}} \leq \frac{b\exp\left(-\frac{b^2\mathcal{N}^2}{2a}\right)}{\sqrt{\frac{a\pi}{2}}\left(2-\exp\left(-\frac{b^2\mathcal{N}^2}{2a}\right)\right)} \to 0, \quad \text{as} \quad \mathcal{N}\to\infty.$$

Since (3.9) should be satisfied as $\mathcal{N}\to\infty$, it becomes

$$b\tilde{\phi}'_F(g) < 1$$

for all $g \geq 0$. Therefore, the condition $b\|\tilde{\phi}'_F\|_{L^\infty} < 1$ is indeed equivalent to (3.8).

We now study the asymptotic behavior of (3.5). Basically, we use a relative entropy method. To this end, we define

$$h(t,g) := \frac{n(t,g)}{n_\infty(g)}.$$

Lemma 3.5. *For any convex function S, we have*

$$\frac{d}{dt}\int_0^\infty n_\infty S(h(g,t))\,dg = -a\int_0^\infty n_\infty S''(h)(\partial_g h)^2\,dg + b(\mathcal{N}_\infty - \mathcal{N})\int_0^\infty n_\infty \partial_g(S(h)-S'(h)h)\,dg$$

Proof. Since $n = hn_\infty$, we have

$$\partial_g n = n_\infty\partial_g h + h\partial_g n_\infty, \quad \text{and} \quad \partial^2_{gg}n = n_\infty\partial^2_g h + 2\partial_g n_\infty\partial_g h + \partial^2_{gg}n_\infty h.$$

Therefore,

$$\begin{aligned}
\partial_t h &= \frac{1}{n_\infty}(-\partial_g((b\mathcal{N}(t)-g)n) + a\partial^2_{gg}n) = \frac{1}{n_\infty}(n + (g-b\mathcal{N}(t))\partial_g n + a\partial^2_{gg}n)\\
&= \frac{n}{n_\infty} + (g-b\mathcal{N})\left(\partial_g h + \frac{\partial_g n_\infty}{n_\infty}h\right) + a\partial^2_{gg}h + 2a\frac{\partial_g n_\infty}{n_\infty}\partial_g h + a\frac{\partial^2_{gg}n_\infty}{n_\infty}h\\
&= \left(g - b\mathcal{N} + 2a\frac{\partial_g n_\infty}{n_\infty}\right)\partial_g h + a\partial^2_{gg}h + \frac{n}{n_\infty^2}\left(n_\infty + (g-b\mathcal{N})\partial_g n_\infty + a\partial^2_{gg}n_\infty\right)\\
&= \left(g - b\mathcal{N} + 2a\frac{\partial_g n_\infty}{n_\infty}\right)\partial_g h + a\partial^2_{gg}h + b(\mathcal{N}_\infty - \mathcal{N})\frac{n}{n_\infty^2}\partial_g n_\infty
\end{aligned}$$

where we used the equation for n_∞:

$$n_\infty + (g-b\mathcal{N}_\infty)\partial_g n_\infty + a\partial^2_{gg}n_\infty = -\partial_g((b\mathcal{N}_\infty - g)n_\infty) + a\partial^2_{gg}n_\infty = 0.$$

Therefore,

$$\begin{aligned}
\partial_t S(h) = S'(h)\partial_t h &= S'(h)\left(\left(g - b\mathcal{N} + 2a\frac{\partial_g n_\infty}{n_\infty}\right)\partial_g h + a\partial^2_{gg}h + b(\mathcal{N}_\infty - \mathcal{N})\frac{n}{n_\infty^2}\partial_g n_\infty\right)\\
&= \left(g - b\mathcal{N} + 2a\frac{\partial_g n_\infty}{n_\infty}\right)\partial_g S(h) + a\left(\partial^2_{gg}S(h) - S''(h)(\partial_g h)^2\right) + b(\mathcal{N}_\infty - \mathcal{N})\frac{n}{n_\infty^2}S'(h)\partial_g n_\infty
\end{aligned}$$

and, consequently,

$$\begin{aligned}
\partial_t(n_\infty S(h)) &= (g-b\mathcal{N})n_\infty\partial_g S(h) - a\partial^2_{gg}n_\infty S(h) + a\partial^2_{gg}(n_\infty S(h)) - an_\infty S''(h)(\partial_g h)^2\\
&\quad + b(\mathcal{N}_\infty - \mathcal{N})hS'(h)\partial_g n_\infty\\
&= \partial_g((g-b\mathcal{N}_\infty)n_\infty S(h)) + a\partial^2_{gg}(n_\infty S(h)) - an_\infty S''(h)(\partial_g h)^2\\
&\quad + b(\mathcal{N}_\infty - \mathcal{N})(S'(h)hn'_\infty + n_\infty\partial_g S(h)).
\end{aligned}$$

Hence, we have

(3.10)
$$
\begin{aligned}
\frac{d}{dt}\int_0^\infty n_\infty S(h)\,dg &= b\mathcal{N}_\infty n_\infty(0)S(h(0)) - a\partial_g(n_\infty S(h))|_{g=0} - a\int_0^\infty n_\infty S''(h)(\partial_g h)^2\,dg \\
&\quad + b(\mathcal{N}_\infty - \mathcal{N})\int_0^\infty S'(h)hn'_\infty + n_\infty \partial_g S(h)\,dg \\
&= -an_\infty\partial_g S(h)|_{g=0} - a\int_0^\infty n_\infty S''(h)(\partial_g h)^2\,dg \\
&\quad + b(\mathcal{N}_\infty - \mathcal{N})\int_0^\infty n'_\infty(S'(h)h - S(h))\,dg - b(\mathcal{N}_\infty - \mathcal{N})n_\infty(0)S(h(0))
\end{aligned}
$$

However, since we have

$$
\partial_g S(h) = S'(h)\frac{(\partial_g n)n_\infty - n(\partial_g n_\infty)}{n_\infty^2}
$$

and boundary conditions

$$
b\mathcal{N}(t)n(0) = a\partial_g n(0), \quad b\mathcal{N}_\infty n_\infty(0) = a\partial_g n_\infty(0),
$$

we obtain

$$
-an_\infty\partial_g S(h)|_{g=0} = b(\mathcal{N}_\infty - \mathcal{N})n(0)S'(h(0)). \tag{3.11}
$$

Therefore, we substitute (3.11) into (3.10) to derive

$$
\begin{aligned}
&\frac{d}{dt}\int_0^\infty n_\infty S(h)\,dg \\
&= -a\int_0^\infty n_\infty S''(h)(\partial_g h)^2\,dg \\
&\quad + b(\mathcal{N}_\infty - \mathcal{N})\left(n(0)S'(h(0)) - n_\infty(0)S(h(0)) + \int_0^\infty n'_\infty(S'(h)h - S(h))\,dg\right) \\
&= -a\int_0^\infty n_\infty S''(h)(\partial_g h)^2\,dg + b(\mathcal{N}_\infty - \mathcal{N})\int_0^\infty n_\infty\partial_g(S(h) - S'(h)h)\,dg.
\end{aligned}
$$

□

Now, we take a specific form of convex function $S(x) = (x-1)^2$ and we define a relative entropy Lyapunov functional $\mathcal{E}$ as

$$
\mathcal{E}(t) := \int_0^\infty n_\infty S(h)\,dg = \int_0^\infty n_\infty(h-1)^2\,dg,
$$

and its dissipation $\mathcal{D}$ as

$$
\mathcal{D}(t) := \frac{1}{2}\int_0^\infty n_\infty S''(h)(\partial_g h)^2\,dg = \int_0^\infty n_\infty(\partial_g h)^2\,dg.
$$

Then the entropy equality in Lemma 3.5 becomes

$$
\frac{d\mathcal{E}}{dt} = -2a\mathcal{D} + b(\mathcal{N}_\infty - \mathcal{N})\int_0^\infty n_\infty\partial_g(1-h^2)\,dg. \tag{3.12}
$$

In order to estimate further, we use the following Poincaré-type inequality [5, 9]: there exists a positive constant γ such that for any function h satisfying $\int_0^\infty n_\infty(g)h(g)\,dg = 1$,

$$\gamma \int_0^\infty n_\infty(g)(h(g)-1)^2\,dg \le \int_0^\infty n_\infty(g)(\partial_g h)^2(g)\,dg,$$

or in our notation, $\gamma\mathcal{E} \le \mathcal{D}$. Below, we provide a sufficient condition on parameters and initial data which guarantee the exponential decay of relative entropy $\mathcal{E}$.

Theorem 3.6. *Suppose $n = n(t,g)$ satisfies (3.5). Moreover, we assume the following conditions on parameters*

$$b\|\tilde{\phi}'_F\|_{L^\infty} < 1, \quad b\left(1 + \frac{2\|\phi_F\|^2_{L^\infty}}{\gamma}\right) < 2a$$

and initial data

$$2b\mathcal{E}(0) < \left(2a - b\left(1 + \frac{2\|\phi_F\|^2_{L^\infty}}{\gamma}\right)\right).$$

Let $\mathcal{N}_\infty$ be the unique fixed point of Ψ obtained in Proposition 3.3. Then, the relative entropy functional $\mathcal{E}$ decays exponentially towards 0: there exists a positive constant $\mu > 0$ such that

$$\mathcal{E}(t) \le \mathcal{E}(0)e^{-\mu t}.$$

Proof. From equation (3.12), we have

$$\begin{aligned}
\frac{d\mathcal{E}}{dt} &= -2a\mathcal{D} + 2b(\mathcal{N}-\mathcal{N}_\infty)\int_0^\infty n_\infty((h-1)\partial_g h + \partial_g h)\,dg \\
&= -2a\mathcal{D} + 2b(\mathcal{N}-\mathcal{N}_\infty)\int_0^\infty n_\infty((h-1)\partial_g h)\,dg + 2b(\mathcal{N}-\mathcal{N}_\infty)\int_0^\infty n_\infty\partial_g h\,dg \\
&\le -2a\mathcal{D} + b\left((\mathcal{N}-\mathcal{N}_\infty)^2 + \left(\int_0^\infty n_\infty((h-1)\partial_g h)\,dg\right)^2\right) \qquad (3.13)\\
&\quad + b\left((\mathcal{N}-\mathcal{N}_\infty)^2 + \left(\int_0^\infty n_\infty\partial_g h\,dg\right)^2\right) \\
&\le -2a\mathcal{D} + b((\mathcal{N}-\mathcal{N}_\infty)^2 + \mathcal{E}\mathcal{D}) + b((\mathcal{N}-\mathcal{N}_\infty)^2 + \mathcal{D}) \\
&= -\mathcal{D}(2a - b - b\mathcal{E}) + 2b(\mathcal{N}-\mathcal{N}_\infty)^2
\end{aligned}$$

However, from the definition of $\mathcal{N}$,

$$\begin{aligned}
(\mathcal{N}-\mathcal{N}_\infty)^2 &= \left(\int_0^\infty \tilde{\phi}_F(g)(n-n_\infty)\,dg\right)^2 \le \left(\int_0^\infty \tilde{\phi}_F(g)n_\infty|h-1|\,dg\right)^2 \\
&\le \|\phi_F\|^2_{L^\infty}\left(\int_0^\infty n_\infty\,dg\right)\left(\int_0^\infty n_\infty(h-1)^2\,dg\right) = \|\phi_F\|^2_{L^\infty}\mathcal{E}
\end{aligned}$$

Therefore, we substitute the estimate of $(\mathcal{N}-\mathcal{N}_\infty)^2$ in (3.13) to obtain

$$\frac{d\mathcal{E}}{dt} \le -\mathcal{D}(2a - b - b\mathcal{E}) + 2b\|\phi_F\|^2_{L^\infty}\mathcal{E}.$$

Now, by the assumption on initial data, $2a - b - b\mathcal{E}(0) > 0$ and hence, by the Poincaré-type inequality, we have,

$$\frac{d\mathcal{E}}{dt}\Big|_{t=0} \le -\gamma\mathcal{E}\left(2a - b - b\mathcal{E}(0) - \frac{2b\|\phi_F\|^2_{L^\infty}}{\gamma}\right) \le 0.$$

Therefore, $\mathcal{E}$ decays in times and hence

$$2b\mathcal{E}(t) < \left(2a - b\left(1 + \frac{2\|\phi_F\|^2_{L^\infty}}{\gamma}\right)\right).$$

This implies the following exponential decay of $\mathcal{E}$ with decay rate $\mu = \frac{\gamma}{2}\left(2a - b - \frac{2b\|\phi_F\|^2_{L^\infty}}{\gamma}\right)$:

$$\frac{d\mathcal{E}}{dt} \leq -\gamma\mathcal{E}\left(2a - b - b\mathcal{E} - \frac{2b\|\phi_F\|^2_{L^\infty}}{\gamma}\right) \leq -\frac{\gamma}{2}\left(2a - b - \frac{2b\|\phi_F\|^2_{L^\infty}}{\gamma}\right)\mathcal{E}.$$

□

4. Slow-fast limit with fast firing regime

We now consider the case when both the voltage dynamics and the firing rate have fast dynamics. A scaling of interest, because it leads to a different analysis than in the previous section, is when both terms have the same strength. More precisely, we consider the scaled equation, where the factor $\frac{1}{\varepsilon}$ also appears in front of firing rate ϕ_F,

$$\begin{aligned} \partial_t p_\varepsilon + \frac{1}{\varepsilon}\partial_v\big[\big(g_L(V_L - v) + g(V_E - v)\big)p_\varepsilon\big] + \partial_g\big[\big(G_{eq}(v, b\mathcal{N}_\varepsilon(t)) - g\big)p_\varepsilon\big] \\ -a\partial^2_{gg}p_\varepsilon + \frac{1}{\varepsilon}\phi_F(v)p_\varepsilon = 0, \qquad t \geq 0,\ v \in (V_L, V_E),\ g \geq 0, \end{aligned} \tag{4.1}$$

with the corresponding no-flux boundary conditions

$$\begin{cases} \big(g_L(V_L - V_R) + g(V_E - V_R)\big)p_\varepsilon(t, V_R, g) = N_\varepsilon(t, g) := \displaystyle\int_{V_R}^{V_E} \phi_F(v)p_\varepsilon\, dv, \\ p_\varepsilon(t, V_E, g) = 0, \qquad \big(G_{eq}(v, b\mathcal{N}_\varepsilon(t)) - g\big)p_\varepsilon - a\partial_g p_\varepsilon(t, v, 0) = 0, \\ \mathcal{N}_\varepsilon(t) := \displaystyle\int_0^\infty N_\varepsilon(t, g)dg = \int_0^\infty \int_{V_R}^{V_E} \phi_F(v)p_\varepsilon\, dv\, dg. \end{cases} \tag{4.2}$$

We first identify the limiting density as $\varepsilon \to 0$. Formally, the probability density function $p = \lim_{\varepsilon\to 0} p_\varepsilon$ should satisfy

$$\partial_v\big[\big(g_L(V_L - v) + g(V_E - v)\big)p\big] + \phi_F(v)p = 0,$$

which is equivalent to

$$\partial_v(\log p) = \frac{g_L + g - \phi_F(v)}{g_L(V_L - v) + g(V_E - v)}. \tag{4.3}$$

When $V_R \leq v < V_*(g)$ (see (1.4)), we integrate (4.3) from V_R to v to obtain

$$p(t, v, g) = p(t, V_R, g)\exp\left(\int_{V_R}^v \frac{g_L + g - \phi_F(w)}{g_L(V_L - w) + g(V_E - w)}\, dw\right), \qquad V_R \leq v < V_*(g).$$

For $V_*(g) < v \leq V_E$, we integrate (4.3) from v to V_E to get

$$p(t, v, g) = p(t, V_E, g)\exp\left(-\int_v^{V_E} \frac{g_L + g - \phi_F(w)}{g_L(V_L - w) + g(V_E - w)}\, dw\right) = 0.$$

In summary, this case is very close to that studied in [18], and the limit density p is more complex, given as

$$p(t,v,g) = \begin{cases} p(t,V_R,g)\exp\left(\displaystyle\int_{V_R}^{v} \frac{g_L+g-\phi_F(w)}{g_L(V_L-w)+g(V_E-w)}\,dw\right) & V_R \le v < V_*(g), \\ 0 & V_*(g) < v < V_E. \end{cases} \tag{4.4}$$

For the special case when ϕ_F is constant, i.e., $\phi_F \equiv \phi$, we can explicitly calculate the limiting distribution p. First of all, the exponent in (4.4) can be calculated as

$$\begin{aligned}\int_{V_R}^{v} \frac{g_L+g-\phi}{g_L(V_L-w)+g(V_E-w)}\,dw &= -\frac{(g_L+g-\phi)}{g_L+g}\Big[\log((g_LV_L+gV_E)-(g_L+g)w)\Big]_{V_R}^{v} \\ &= \frac{\phi-g_L-g}{g_L+g}\log\left(\frac{g_LV_L+gV_E-(g_L+g)v}{g_LV_L+gV_E-(g_L+g)V_R}\right).\end{aligned}$$

Thus, the density p is given by

$$\begin{aligned}p(t,v,g) =& p(t,V_R,g)\exp\left(\frac{\phi-g_L-g}{g_L+g}\log\left(\frac{g_LV_L+gV_E-(g_L+g)v}{g_LV_L+gV_E-(g_L+g)V_R}\right)\right) \\ =& \, p(t,V_R,g)\left(\frac{g_LV_L+gV_E-(g_L+g)v}{g_LV_L+gV_E-(g_L+g)V_R}\right)^{\frac{\phi-g_L-g}{g_L+g}},\end{aligned}$$

which behaves like $x^{\frac{\phi-g_L-g}{g_L+g}}$ near $x=0$ as $v \to V_*(g)$. Since $\frac{\phi-g_L-g}{g_L+g} > -1$, the distribution is integrable with respect to v, although it blows-up when $g > \phi - g_L$.

For the sake of completeness, we mention the a priori bounds for the g-moments

Lemma 4.1. *Let $\{p_\varepsilon\}_{\varepsilon>0}$ be a family of solution to (4.1), (4.2) with initial data $\{p_\varepsilon^0\}_{\varepsilon>0}$ with finite g-moments, i.e., (1.5) holds uniformly in ε. Then, for all $k \ge 0$, there exists $C(k)$ independent of ε satisfying*

$$M_{g,\varepsilon}^{(k)}(t) := \int_0^\infty \int_{V_R}^{V_E} g^k p_\varepsilon(t,v,g)\,dv\,dg \le C(k), \qquad \forall t \ge 0.$$

We omit the proof which is similar to those of Lemma 2.1 for $a=0$ or Lemma 2.4 for $a>0$.

Even though the convergence theory seems much harder than with the previous scalings, the behavior as $\varepsilon \to 0$ is tested numerically in Subsection 5.1 and the formal asymptotic is confirmed numerically.

5. Numerical simulation

We now illustrate our theoretical results with numerical simulations. First, we test the convergence of solution to (1.6) and (1.7) as $\varepsilon \to 0$. In particular, we observe that, for (1.6), the solution converges to the Dirac delta distribution concentrated on the 1-dimensional manifold $\{(v,g)\ :\ v = V_*(g)\}$ and that, for (1.7), we obtain that the formal limit distribution for fast firing regime has algebraic order near the critical voltage $v = V_*(g)$. Secondly, we move on to the numerical simulation for limit equations (3.1) and (3.5), for both $a=0$ and $a>0$. We provide the dynamics of $\mathcal{N}$ as well as the dynamics of $n(t,g)$ and the numerical results indicate that the asymptotic convergence analysis to a stationary state on the limit model is also observable in the numerical simulation. Moreover, with specific choice of ϕ, we can numerically observe the emergence of a bifurcation behavior with respect to the parameter b.

5.1. **Convergence of solution as $\varepsilon \to 0$.** We first simulate equation (1.6) and (1.7) for various values of ε and observe the behavior of the solutions. We take a set of parameters as follows:

$$g_L = 3, \quad g_{Max} = 20, \quad V_R = 0, \quad V_L = 0.1, \quad V_E = 1, \quad a = 1, \quad b = 1,$$

and the conductance equilibrium function G and firing function $\phi_F(v)$ as

$$G(v, b\mathcal{N}) = b\mathcal{N}, \quad \phi_F(v) = 1 + v.$$

The initial data is chosen as a Gaussian distribution given as

$$p_0(v,g) := \frac{1}{Z}\exp\left(-30(v-0.5)^2 - 0.5(g-5)^2\right),$$

where Z is a normalization constant making p_0 as a probability density. The numerical solutions are calculated for $\varepsilon = 0.1,\ 0.15,\ 0.2,\ 0.3,\ 0.5$ and 1 until $t = 0.05$. The results are shown in Figure 1. Here the red line is a reference line for the 1-dimensional manifold $\left\{(v,g)\ :\ v = V_*(g) := \frac{g_L V_L + g V_E}{g_L + g}\right\}$ which carries the limiting Dirac distribution as $\varepsilon \to 0$. As we proved it, the solutions to (1.6) indeed converges to Dirac delta distribution in v variable as $\varepsilon \to 0$.

Next, we discuss the dynamics of the total firing rate $\mathcal{N}_\varepsilon(t)$ for different values of ε and we compare their dynamics, still for equation (1.6). The dynamics of $\mathcal{N}_\varepsilon$ of the equation (1.6) for the same set of ε as before is depicted in Figure 2. As we expect, the upper bound for $\mathcal{N}_\varepsilon$ is around 1.65, which is always less than the $\|\phi_F\|_{L^\infty} = 2$. Moreover, the time derivative of $\mathcal{N}_\varepsilon(t)$ is also bounded after the initial layer $0 \le t \le \tau$, where τ decreases as ε decays.

As a final numerical test of this subsection for (1.6), we compare the case of the slow firing scaling (1.6) and the fast firing scaling (1.7). In Figure 3, we plot the section of the distributions at $g = 5$, where $\varepsilon = 0.1$ and $t = 0.05$. As we derived in Section 4, the limit distribution $p(t,v,g) := \lim_{\varepsilon\to 0} p_\varepsilon(t,v,g)$ for fast firing regime has algebraic order near the critical voltage $v = V_*(g)$, while the limit distribution for slow firing regime is just a delta distribution.

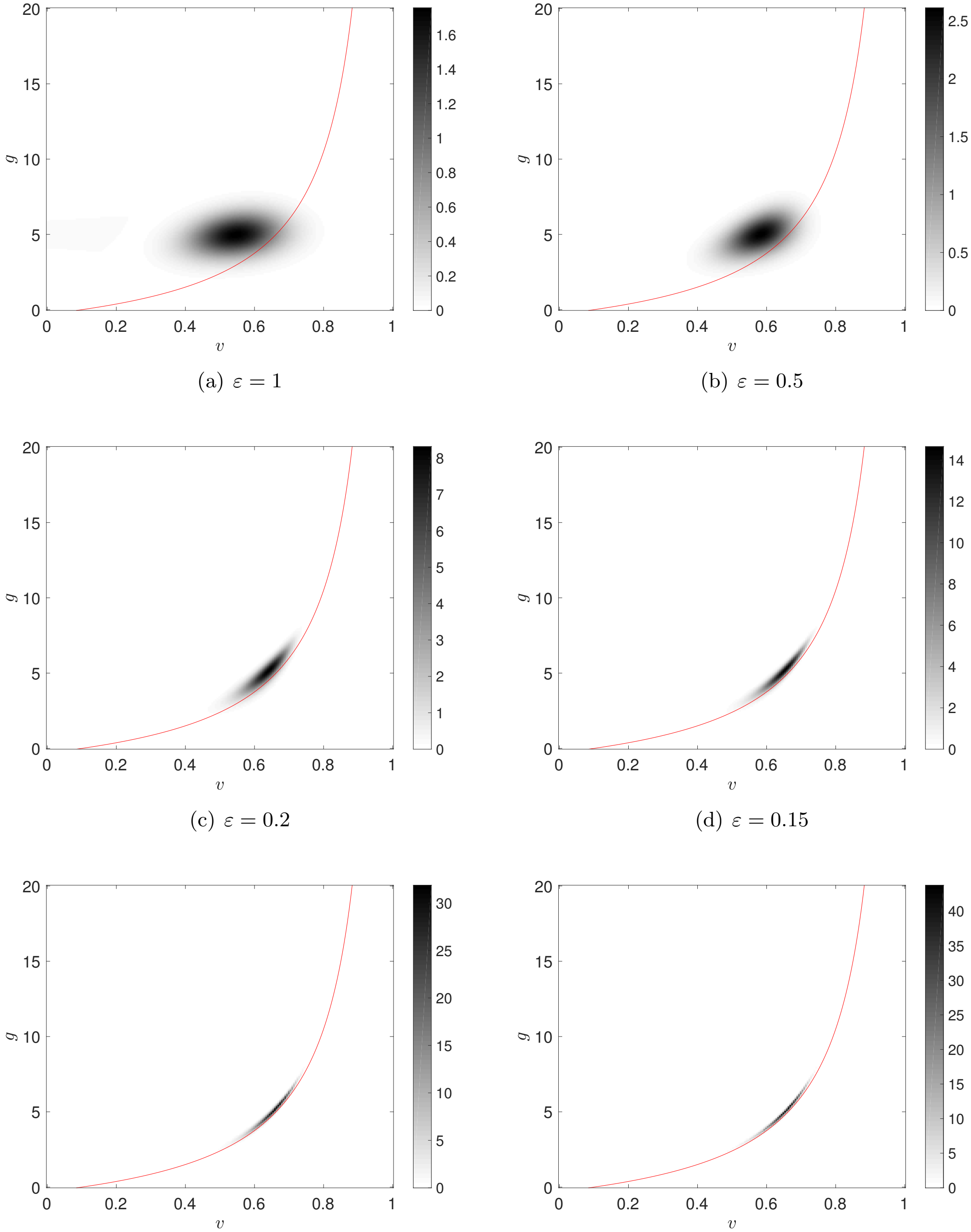
5.2. **Long term behaviors of the limit equations (3.1) and (3.5).** We now move to the numerical simulations for asymptotic behaviors of the limit equations (3.1) and (3.5). We first consider the equation (3.1) which is a hyperbolic case, i.e., $a = 0$. For numerical simulation, we use $b = 1$ and the initial data

$$n_0(g) = \frac{1}{Z}\exp\left(-0.2(g-10)^2\right).$$

Figure 4 shows the graph of $\phi_F\left(\frac{g_L V_L + b\mathcal{N} V_E}{b\mathcal{N} + g_L}\right)$ with respect to $\mathcal{N}$. The intersection gives the stationary state. Since

$$b = 1 < \frac{10}{3} = \frac{3}{(1-0.1)1} = \frac{g_L}{(V_E - V_L)\|\phi_F'\|_{L^\infty}},$$

the condition (3.2) is satisfied and we expect that a fixed point of map Φ is unique. As we can find in Figure 4, the map Φ has a unique fixed point $\mathcal{N}_\infty \approx 1.384$. Moreover, Figure 5 and Figure 6 show that the total firing rate $\mathcal{N}(t)$ converges to unique fixed point $\mathcal{N}_\infty$



(a) $\varepsilon = 1$

(b) $\varepsilon = 0.5$

(c) $\varepsilon = 0.2$

(d) $\varepsilon = 0.15$

(e) $\varepsilon = 0.1$

(f) $\varepsilon = 0.05$

FIGURE 1. Numerical results for (1.6) at $t = 0.05$ with various values of ε illustrating the convergence to a Dirac distribution.

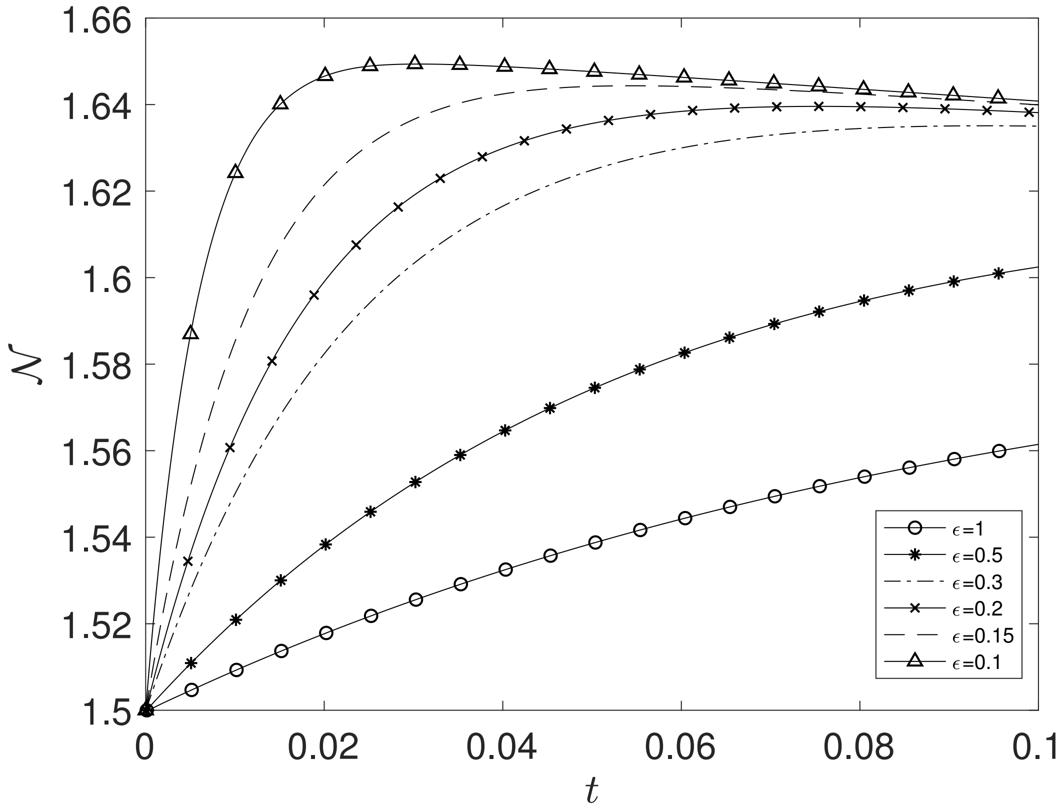


FIGURE 2. The dynamics of $\mathcal{N}(t)$ for various ε in equation (1.6)

exhibited in Figure 4 and the distribution $n(t,g)$ converges to the Dirac delta distribution $n_\infty(g) = \delta_{b\mathcal{N}_\infty}(g)$.

For the case of diffusive equation (3.5), we choose the diffusion coefficient $a = 1$ and the same initial data as in the case of hyperbolic case. Here, we chose parameter sets which may not satisfy the condition in Theorem 3.6, since we don't know the exact value of the Poincaré constant γ. However, even in this case, we are able to observe that the total firing rate $\mathcal{N}$ converges to equilibrium $\mathcal{N}_\infty$ (Figure 7) and the distribution $n(t,g)$ converges to the Maxwellian equilibrium $n_\infty(g)$ (Figure 8).

5.3. **Bifurcating behavior of equilibrium.** As a last numerical simulation, we present the behavior of equation (3.1) when ϕ_F is not smooth and there are more than one steady state, i.e., a fixed point of ϕ_F. As an example, we consider the following choice of ϕ_F:

$$\phi_F(x) = \begin{cases} 0, & 0 < x < 0.3, \\ 1, & 0.3 < x < 0.8, \\ 2, & x > 0.8. \end{cases}$$

We observe the firing rate dynamics for several values of b in Figure 9. The result of the numerical simulation implies that the equilibrium of the firing rate $\mathcal{N}(t)$ suddenly changed

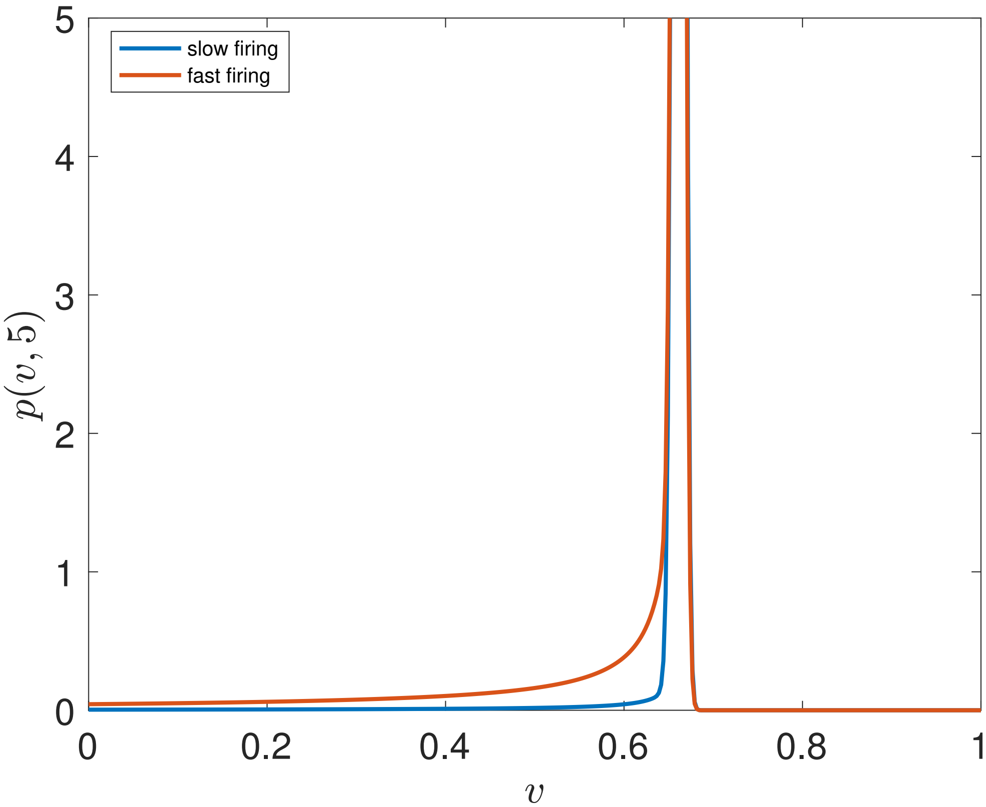


FIGURE 3. Comparison of the section of distributions $p_\varepsilon(v,5)$ with $\varepsilon = 0.1$ for slow and fast firing regimes.

from 1 to 2 at $b \approx 7.633$. Since there is no significant difference in the graph of ϕ_F between the case of $b = 7.632$ and $b = 7.636$ (Figure 10), this bifurcation is not due to the structure of ϕ_F which does not vary qualitatively.

6. Conclusion

For a voltage-conductance nonlinear model (1.6) describing neural assemblies coupled through the total activity (firing rate) $\mathcal{N}$ of the network. We have established rigorously the conductance limit using rescalings of the equation. To do so, we obtain the uniform boundedness of moments of the solution of (1.6), together with the $W^{1,\infty}$ boundedness of the total firing rate after the initial layer.

We have also established the asymptotic stability analysis of the limit conductance equation (1.8) for both cases $a = 0$ (hyperbolic model when noise is neglected) and $a > 0$ (cases including Gaussian noise). For this stability analysis, we provide a sufficient condition for the existence of a unique stationary solution to (1.8) and provide the asymptotic convergence of the solution toward this unique stationary solution. The method uses Lyapunov functionals measuring the distance of the solution from the stationary solution.

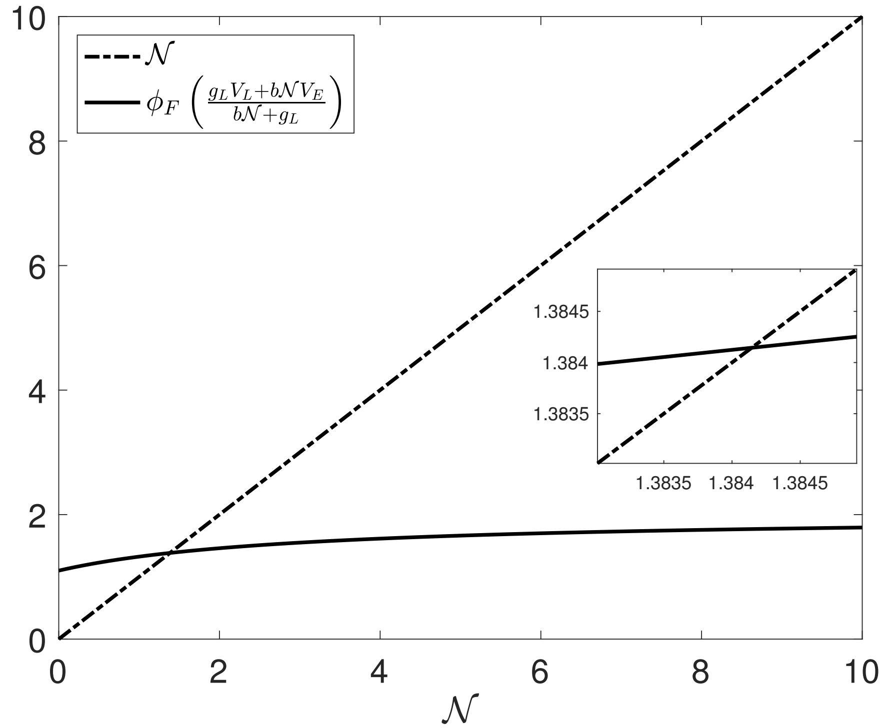


FIGURE 4. The graph of $\mathcal{N}$ and $\phi_F\left(\frac{g_L V_L + b\mathcal{N} V_E}{b\mathcal{N}+g_L}\right)$.

All the results in the paper are supported by numerical simulations. In particular, we have focused on showing the stability of the solution to the limit equation. No oscillations or synchronisation effect occurs in such a simple conductance model, neither in the voltage-conductance with the regime $\varepsilon \ll 1$, which does not contradict the periodic regime exhibited in [6], neither that in the voltage only model in [18].

However, there are still several remaining questions. A first issue would be to make more precise the convergence to the Dirac limiting solution and establish convergence rates. Secondly, the compactness in the case of fast firing, equation (1.7) is left open. Thirdly, as pointed out in Section 5.3, if there exist multiple fixed points of ϕ_F, it is not proved that the solution of (3.1) will converge to Dirac mass on one of the fixed points, let alone specifying the fixed point on which the solution will concentrated. A hint towards this goal is a result using Monge-Kantorovich distance in [14]. Furthermore, all the numerical solutions converge to its equilibrium and it does not show any oscillatory or periodic behavior. Thus, it will be also interesting if one can understand more physically based models where the conductance-only limit also shows periodic phenomena.

References

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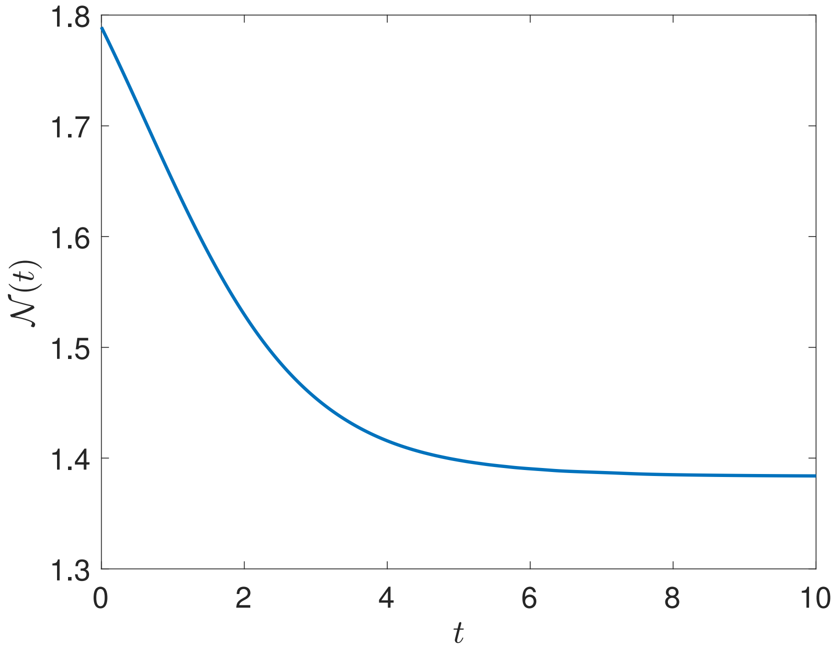


FIGURE 5. The dynamics of $\mathcal{N}$ for $a = 0$.

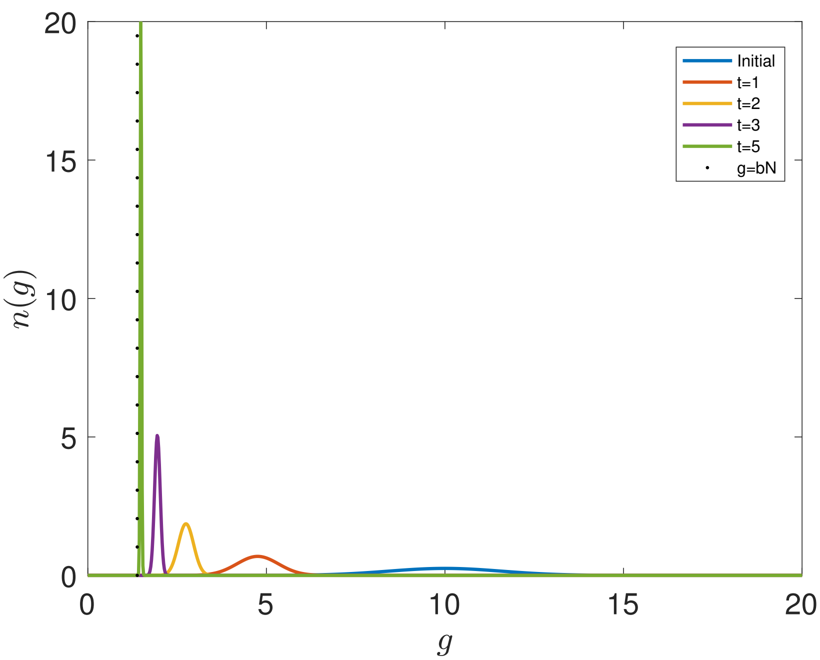


FIGURE 6. The graph of $n(t, g)$ when $a = 0$.

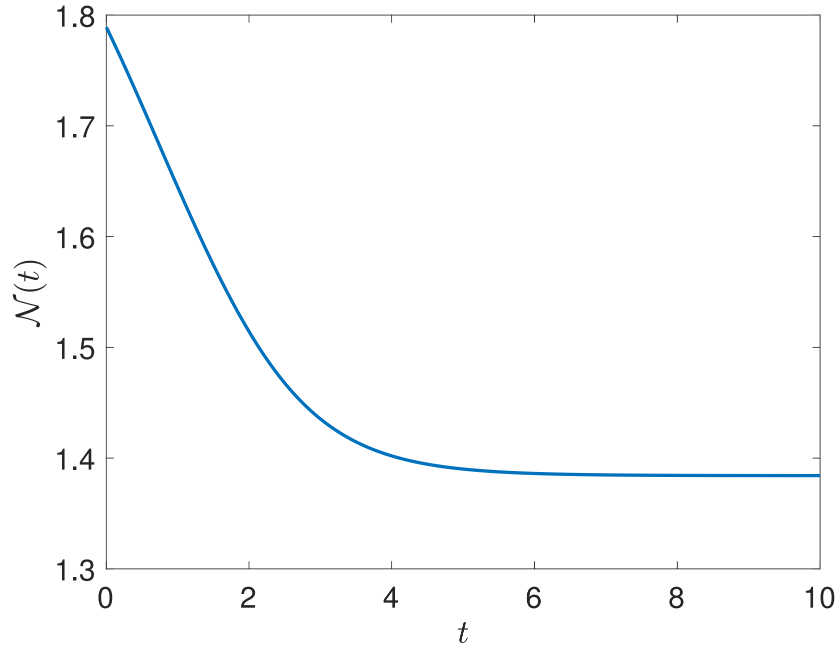


FIGURE 7. The dynamics of $\mathcal{N}$ for $a = 1$.

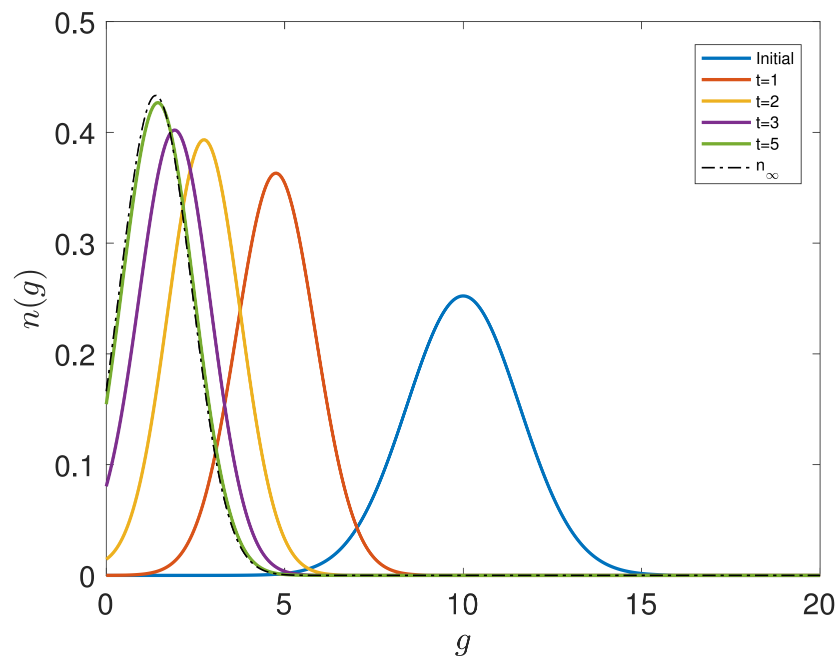


FIGURE 8. The dynamics of $n(t, g)$ when $a = 1$.

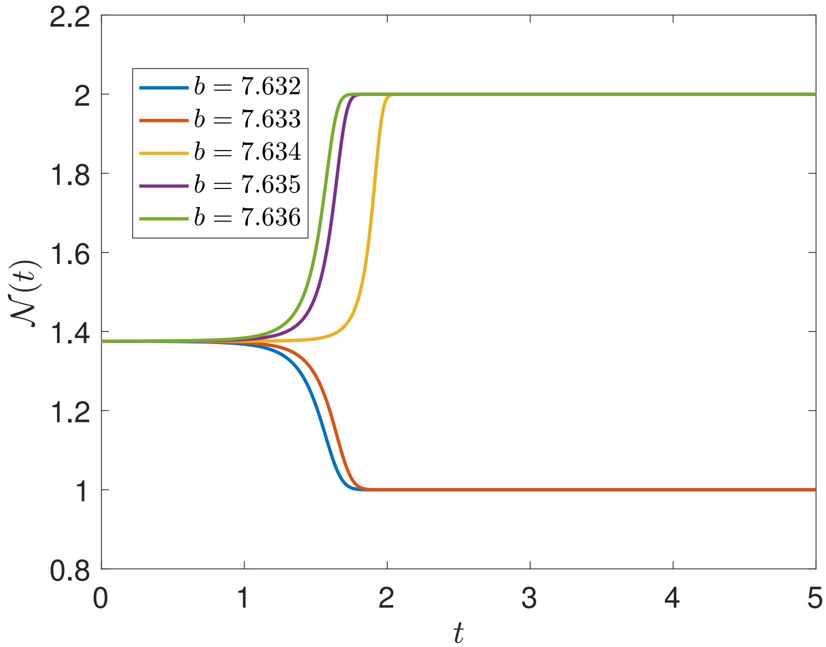
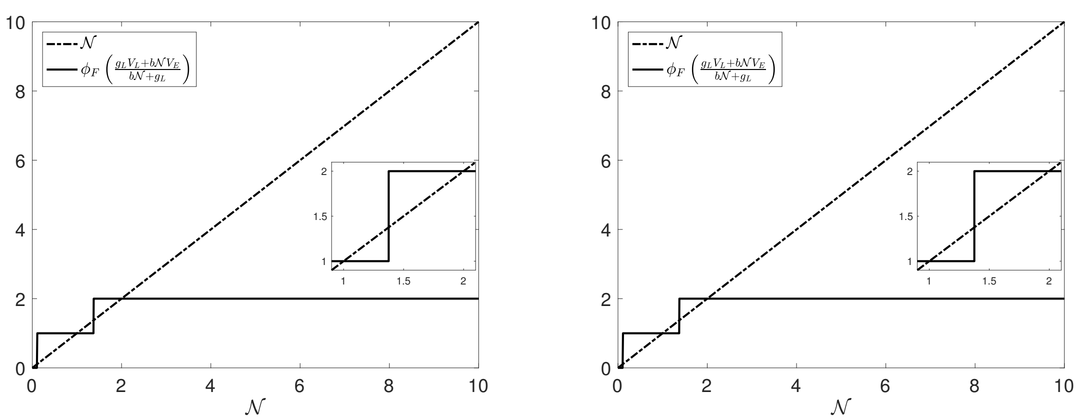


FIGURE 9. Bifurcating behavior of equilibrium



(a) The graph of $\mathcal{N}$ and ϕ_F when $b = 7.632$

(b) The graph of $\mathcal{N}$ and ϕ_F when $b = 7.636$

FIGURE 10. Graph of $\phi_F(\mathcal{N})$ for different values of b.

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