ON TOPOLOGICAL GENERICITY OF THE MODE-LOCKING PHENOMENON

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ABSTRACT. We study circle homeomorphisms extensions over a strictly ergodic homeomorphism. Under a very mild restriction, we show that the fibered rotation number is locally constant on an open and dense subset of all circle homeomorphisms extensions homotopic to the trivial extension. In the complement of this set, we find a dense subset in which every map is conjugate to a direct product. Our result provides a generalisation of Avila-Bochi-Damanik's result on $SL(2,\mathbb{R})-$cocycles, and Jäger-Wang-Zhou's result on quasi-periodically forced maps, to a broader setting.

1. INTRODUCTION

The study of circle homeomorphisms is a classical subject in dynamical systems. For each orientation-preserving homeomorphism $f : \mathbb{T} \to \mathbb{T}$ with a lift $F : \mathbb{R} \to \mathbb{R}$, the limit $\rho(F) = \lim_{n\to\infty}(F^n(x) - x)/n$ exists and is independent of x. The *rotation number* of f is defined as $\rho(f) = \rho(F) \mod 1$. It was already known to Poincaré that the dynamic of f is largely determined by $\rho(f)$. The function $f' \mapsto \rho(f')$ is locally constant at f, or in other terms f is *mode-locked*, if and only if there exists a non-empty open interval $I \subsetneq \mathbb{T}$ such that $f^p(\bar{I}) \subset I$ for some $p \in \mathbb{Z}$. In particular, such behaviour occurs only when $\rho(f)$ is rational.

Inspired by a question of Herman in [15], Bjerklöv and Jäger found in [8] a precise analogy to the above one-dimensional result for *quasi-periodically forced maps* on $\mathbb{T}^2$ defined as follows.

Definition 1. A *quasi-periodically forced map* on $\mathbb{T}^2$ (*qpf-map* for short) is a homeomorphism $f : \mathbb{T}^2 \to \mathbb{T}^2$, homotopic to the identity, with skew-product structure $f(\theta, x) = (\theta + \omega, f_\theta(x))$, where $\omega \in (\mathbb{R} \setminus \mathbb{Q})/\mathbb{Z}$ is called the *frequency*.

Through out this paper, we denote $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. For a qpf-map f, we say that a continuous map $F : \mathbb{T} \times \mathbb{R} \to \mathbb{T} \times \mathbb{R}$ is a *lift* of f if $F(\theta, x) = (\theta + \omega, F_\theta(x))$ such that $F_\theta(x) \mod \mathbb{Z} = f_\theta(x \mod \mathbb{Z})$ and $F_\theta(x + 1) = F_\theta(x) + 1$ for any $(\theta, x) \in \mathbb{T} \times \mathbb{R}$. Similar to the case of circle homeomorphisms, the limit $\rho(F) = \lim_{n\to\infty}((F^n)_\theta(x) - x)/n$ exists and is independent of the choice of (θ, x). Here we set $(F^n)_\theta = F_{\theta+(n-1)\omega} \cdots F_\theta$ for all integer $n \geq 1$. The *fibered rotation number* of f is defined as $\rho(f) := \rho(F) \mod 1$, and is independent of the choice of F.

For any qpf-map f with a lift F, for any $t \in \mathbb{R}$ we set $F_t(\theta, x) = (\theta + \omega, F_\theta(x) + t)$. A qpf-map f is said to be *mode-locked* if $\varepsilon \mapsto \rho(F_\varepsilon)$ is constant on a neighborhood of $\varepsilon = 0$. This is equivalent to say that $g \mapsto \rho(g)$ is locally constant at $g = f$. In [8], the authors showed that f is mode-locked if and only if there exists a closed annulus, bounded by continuous curves, which is mapped into its own interior

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by some iterate of f. Moreover, they showed that whenever $\omega, \rho(f)$ and 1 are rationally independent, the map $\varepsilon \mapsto \rho(F_\varepsilon)$ is strictly monotonically increasing at $\varepsilon = 0$. In particular, in the latter case, f is not mode-locked.

There is a closely related line of research focused on the structure of quasi-periodically forced maps on $\mathbb{T}^2$. In [17], the authors showed that any qpf-map f of bounded mean motion such that $\omega, \rho(f), 1$ are rationally independent, is semi-conjugate to the irrational torus translation $(\theta, x) \mapsto (\theta + \omega, x + \rho(f))$ via a fibre-respecting semi-conjugacy (see [17, Theorem 3.1] and [8, Theorem 2.3] for the formal statement). Such result is crucial for the proof of strict monotonicity in [8, Lemma 3.2]. Jäger [16] later generalised [17, Theorem 3.1] to any minimal totally irrational pseudo-rotations on $\mathbb{T}^2$ with bounded mean motion.

Aside from the deterministic results mentioned above, it is also natural to study the generic picture of quasi-periodically forced maps. A natural question is:

Is a generic quasi-periodically forced map on $\mathbb{T}^2$ mode-locked ?

Besides its intrinsic interest, the above question also has roots in the study of differentiable dynamical systems and Schrödinger operators. We will elaborate this point in Subsection 1.1.

Depending on one's interpretation of the notion of genericity, and also on the regularity of the maps, the answer to the above question may vary. So far, this question has been studied by different means. In [18], Jäger and Wang used a multi-scale argument originated from the classical works of Benedicks-Carleson [7] and Young [23] to construct a C^1 family of quasi-periodically forced *diffeomorphisms*, among which mode-locked parameters have small measure. In [22], Jäger, Wang and Zhou showed that for a topologically generic frequency ω, the set of mode-locked qpf-maps with frequency ω is residual. We refer to Subsection 1.1 for further details on the genericity condition. For a recent result on the local density of mode-locking in a related setting, we mention [20].

In this paper, we prove the topological genericity of mode-locking property for a much more general class of maps which we call *dynamically forced maps*. This class includes circle homeomorphism extensions over any strictly ergodic homeomorphism of a compact manifold. As a special case, we are able to prove the topological genericity of mode-locking for qpf-maps with any given irrational frequency. Moreover, we show that among the set of qpf-maps which are not mode-locked, a dense subset corresponds to maps which are topologically linearizable.

Let X be a compact metric space, and let $g : X \to X$ be a uniquely ergodic homeomorphism with μ as the unique $g-$invariant measure.

Definition 2 ($g-$forced maps). Let $g : X \to X$ be given as above. We say that a homeomorphism $f : X \times \mathbb{T} \to X \times \mathbb{T}$ is a *$g-$forced map* if it is of the form

$$f(\theta, y) = (g(\theta), f_\theta(y))$$

and admits a *lift*, i.e., a homeomorphism $F : X \times \mathbb{R} \to X \times \mathbb{R}$ satisfying $F_\theta(x) \mod \mathbb{Z} = f_\theta(x \mod \mathbb{Z})$ and $F_\theta(x+1) = F_\theta(x) + 1$ for any $(\theta, x) \in X \times \mathbb{R}$. We denote the set of $g-$forced maps by $\mathcal{F}_g$. The space $\mathcal{F}_g$ is a complete metric space under the C^0 distance d_{C^0}.

By definition, for a $g-$forced map f, f_θ is orientation-preserving for each $\theta \in X$. Similar to the case of quasi-periodically forced maps on $\mathbb{T}^2$, for any $g-$forced map

f with a lift F, for any integer $n \geq 1$, we use the notation

$$(F^n)_\theta \stackrel{\text{def}}{=} F_{g^{n-1}(\theta)} \cdots F_\theta.$$

We define $(f^n)_\theta$ in a similar way. The limit $\rho(F) = \lim_{n\to\infty}((F^n)_\theta(y) - y)/n$ exists and is independent of the choice of (θ, y). It is easy to see that a lift of f is unique up to the addition of a function in $C^0(X, \mathbb{Z})$ to the $\mathbb{R}-$coordinate. Set $D = \{\int \varphi d\mu \mid \varphi \in C^0(X, \mathbb{Z})\}$. Then the *fibered rotation number* of f, defined as $\rho(f) = \rho(F) \mod D$, is independent of the lift F. For any $t \in \mathbb{R}$, we set $f_t(\theta, y) = (g(\theta), f_\theta(y) + t)$ and $F_t(\theta, y) = (g(\theta), F_\theta(y) + t)$. It is clear that F_t is a lift of f_t for all $t \in \mathbb{R}$.

Based on the local behaviour of ρ, we can give a crude classification for $g-$forced maps as follows. We say that a function $\varphi \in C^0(\mathbb{R}, \mathbb{R})$ is strictly increasing at $t = t_0$ if $\varphi(t_0 - \delta) < \varphi(t_0) < \varphi(t_0 + \delta)$ for any $\delta > 0$.

Definition 3. Given a $g-$forced map f with a lift F. Then f is said to be

- *mode-locked* if the function $\varepsilon \mapsto \rho(F_\varepsilon)$ is constant on an open neighborhood of 0;
- *semi-locked* if it is not mode-locked and the function $\varepsilon \mapsto \rho(F_\varepsilon)$ is constant on either $(-\varepsilon', 0]$ or $[0, \varepsilon')$ for some $\varepsilon' > 0$;
- *unlocked* if the function $\varepsilon \mapsto \rho(F_\varepsilon)$ is strictly increasing at 0.

We denote the set of mode-locked (resp. semi-locked, unlocked) $g-$forced maps by $\mathcal{ML}_g$ (resp. $\mathcal{SL}_g$,$\mathcal{UL}_g$). Often we omit the subscript and write $\mathcal{ML}$, $\mathcal{SL}$ and $\mathcal{UL}$ instead. Every $g-$forced map belongs to exactly one of these three subsets.

Our first result shows that: under a very mild condition on g, the set of $g-$forced maps which are topologically conjugate to $g \times R_\alpha$ for some $\alpha \in \mathbb{T}$, where R_α is the circle rotation $x \mapsto x + \alpha$, is dense in the complement of $\mathcal{ML}$.

In the following, we let X be a compact metric space, and let $g : X \to X$ be a strictly ergodic (i.e. uniquely ergodic and minimal) homeomorphism with a non-periodic factor of finite dimension. That is, there is a homeomorphism $\bar{g} : Y \to Y$, where Y is an infinite compact subset of some Euclidean space $\mathbb{R}^d$, and there is a onto continuous map $h : X \to Y$ such that $hg = \bar{g}h$. In particular, g can be any strictly ergodic homeomorphism of a compact manifold of positive dimension.

THEOREM 1. *For (X, g) as above, the following is true. For any $g-$forced map f that is not mode-locked, for any lift F of f, for any $\varepsilon > 0$, there exists a $g-$forced map f' with $d_{C^0}(f', f) < \varepsilon$, and a homeomorphism $h : X \times \mathbb{T} \to X \times \mathbb{T}$ of the form $h(\theta, x) = (\theta, h_\theta(x))$ with a lift $H : X \times \mathbb{R} \to X \times \mathbb{R}$ (that is, $H(\theta, x) \mod \mathbb{Z} = h_\theta(x \mod \mathbb{Z})$ and $H(\theta, x + 1) = H(\theta, x) + 1$ for any $(\theta, x) \in X \times \mathbb{R}$), such that*

$$f'h(x, y) = h(g(x), y + \rho(F)), \quad \forall (x, y) \in X \times \mathbb{T}.$$

Theorem 1 can be seen as a nonlinear version of the reducibility result [4, Theorem 5] for $\mathrm{SL}(2, \mathbb{R})-$cocycles [1]. For $\mathrm{SL}(2, \mathbb{R})-$cocycles with high regularity over torus translations, KAM method and Renormalization (see [13, 6, 5]) are effective tools for studying reducibility. While in our case, we use a topological method. As an easy but interesting consequence of Theorem 1, we have the following.

THEOREM 2. *For (X, g) as in Theorem 1, mode-locked $g-$forced maps form an open and dense subset of the space of $g-$forced maps.*

[1]Indeed, it is proved in [4, Appendix C] that a $\mathrm{SL}(2, \mathbb{R})-$cocycle is mode-locked if and only if it is *uniformly hyperbolic*.

We prove Theorem 2 using Theorem 1 as an intermediate step. This strategy is inspired by [4] in their study of Schrödinger operators. In contrast to the result in [4], the density of mode-locking for dynamically forced maps is true regardless of the range of the Schwartzman asymptotic cycle $G(g)$ (for its definition, see [21] or [4, Section 1.1]), while a $\mathrm{SL}(2,\mathbb{R})-$cocycle f over base map g can be mode-locked only if $\rho(f) \in G(g) \mod \mathbb{Z}$. This is due to the fact that for dynamically forced maps, one can perform perturbations that only act locally within the fibers, a convenient feature that is not shared by $\mathrm{SL}(2,\mathbb{R})-$cocycles.

We note that, even in the case where $X = \mathbb{T}$ and g is given by an irrational rotation $g(\theta) = \theta + \omega$, Theorem 2 improves the result in [22]. Indeed, in [22] the authors need to require the frequency ω to satisfy a topologically generic condition. In particular, it was unknown in [22] that mode-locking could be dense for any Diophantine frequency ω (see the remark below [22, Corollary 1.5]). Moreover, our result covers very general base maps. Thus our Theorem 2 can be viewed as a clear strengthening of the main result in [22]. Also, we can deduce from Theorem 2 a generalisation of [22, Theorem 1.6].

COROLLARY A. *Let (X, g) be as in Theorem 1, and let $\mathcal{P}$ denote the set of continuous maps from $\mathbb{T}$ to $\mathcal{F}_g$ endowed with the uniform distance. Then for a topologically generic $\hat{f} \in \mathcal{P}$, the function $\tau \mapsto \rho(\hat{f}(\tau))$ is locally constant on an open and dense subset of $\mathbb{T}$.*

One can adapt the proof of Corollary A so as to consider maps in $\mathcal{P}$ satisfying the twist condition in [22, Theorem 1.6].

1.1. **Background and further perspective.** A prominent example of qpf-map is the Arnold circle map,

$$f_{\alpha,\beta,\tau} : \mathbb{T}^2 \to \mathbb{T}^2, \quad (\theta, x) \mapsto (\theta + \omega, x + \tau + \frac{\alpha}{2\pi}\sin(2\pi x) + \beta g(\theta) \mod 1),$$

with parameters $\alpha \in [0,1], \tau, \beta \in \mathbb{R}$ and a continuous forcing function $g : \mathbb{T} \to \mathbb{R}$. It was introduced in [12] as a simple model of an oscillator forced at two incommensurate frequencies. Mode-locking was observed numerically on open regions in the $(\alpha, \tau)-$parameter space, known as the Arnold tongues.

Another well-known class of qpf-maps is the so-called (generalised) quasi-periodic Harper map,

$$s_E : \mathbb{T} \times \overline{\mathbb{R}} \to \mathbb{T} \times \overline{\mathbb{R}}, \quad (\theta, x) \mapsto (\theta + \omega, V(\theta) - E - \frac{1}{x})$$

where $V : \mathbb{T} \to \mathbb{R}$ is a continuous function and $E \in \mathbb{R}$. Here we use the identification $\overline{\mathbb{R}} \simeq \mathbb{T}$ to simplify the notations. This class of map arises naturally in the study of $1D$ discrete Schrödinger operators $(H_\theta u)_n = u_{n+1} + u_{n-1} + V(\theta + n\omega)u_n$. It is by-now well-understood that the spectrum of H_θ equals to the set of parameter E such that, as a projective cocycle, s_E is not uniformly hyperbolic. More refined links relating the dynamics of s_E with the spectral property of H_θ have been found, for instance in [1]. We refer the readers to [11, 19] for related topics as there is a vast literature dedicated to Schrödinger operators.

Just as qpf-maps are special cases of our dynamically forced maps, many results on quasi-periodic Harper maps were generalised to the general $\mathrm{SL}(2,\mathbb{R})-$cocycles, motivated by the study of Schrödinger operators with general potential functions and the techniques therein. In [4], Avila, Bochi and Damanik showed that the spectrum of a $1D$ Schrödinger operator with a C^0 generic potential generated by a map

g as in Theorem 1 has open gaps at all the labels, which extends their previous result [3] on Cantor spectra. One may ask whether mode-locking could be generic in much higher regularities. To this end, Chulaevsky and Sinai in [10] suggested that in contrast to the circle rotations, for translations on the two-dimensional torus the spectrum can be an interval for generic large smooth potentials (see also [14, Introduction]), in which case the mode-locked parameters for the Schrödinger cocycles would not be dense. In the analytic category, Goldstein, Schlag and Voda [14, Theorem A, Remark 1.2(b)] recently proved that for a multi-dimensional shift on torus with Diophantine frequency, the spectrum is an interval for almost every large trigonometric polynomial potential. This hints the failure of the genericity of mode-locking in higher regularity and higher dimension (see also [18]). Nevertheless, by combining the method of the present paper and ideas from Bochi's work [9] on Lyapunov exponents, we can show that mode-locking for multi-frequency forced maps remains generic under some stronger topology, such as the one induced by the norm $d_{C^{0,\infty}}(f,g) = \sup_{\theta\in X} d_{C^\infty}(f_\theta, g_\theta)$. We will treat this in a separate note.

1.2. **Idea of the proof.** Our idea is originated from [3, 4]. Given f, a $g-$forced map which is not mode-locked, we will perturbe f into a direct product, modulo conjugation. This is rest on the basic observation that for any unlocked map, a small perturbation can promote linear displacement for the iterates of any given point.

A key observation in the proof of Theorem 1 is that the mean motion, which is sublinear in time, can be cancelled out by a small perturbation. The global perturbation is divided into local perturbations in Section 4 at finitely many stages, and is organised using the dynamical stratification in Section 3. Theorem 2 is an easy consequence of Theorem 1. Theorem 1 and 2 are proved in Section 5.

In the quasi-periodic setting, some of the above arguments can be simplied. For instance in the case where the base dynamic is an irrational rotation of the circle, the dynamical stratification in Section 3 is well-known. Moreover, since the Schwarzman cocycle is trivial in this case, there is an alternative approach which relies on the control of *fiberwise Lyapunov exponent* that can be used to show genericity in certain stronger topology. However, even with these simplifications in mind, the C^1 density of mode-locking remains challenging.

Notation. Given a subset $M \subset X$, we denote by $int(M)$ the interior of M, and denote by $\overline{M}$ the closure of M in X. We denote by $B(M,r)$ the $r-$open neighbourhood of M in X for any $r > 0$.

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2. PRELIMINARY

2.1. **Basic properties of dynamically forced maps.** In this subsection, we only require g to be a uniquely ergodic homeomorphism of a compact topological space X. We start by giving the following basic relations between $\mathcal{ML}, \mathcal{SL}$ and $\mathcal{UL}$.

LEMMA 1. *For any $g-$forced map $f \in \mathcal{SL}$, any $\varepsilon > 0$, there exists $f' \in \mathcal{ML}$ and $f'' \in \mathcal{UL}$ such that $d_{C^0}(f, f'), d_{C^0}(f, f'') < \varepsilon$.*

Proof. Let F be a lift of f. Without loss of generality, we assume that there exists $\varepsilon_0 \in (0, \varepsilon)$ such that $\rho(F_{\varepsilon'}) = \rho(F)$ for all $\varepsilon' \in (-\varepsilon_0, 0)$; and $\rho(F_{\varepsilon'}) > \rho(F)$ for all $\varepsilon' \in (0, \varepsilon_0)$.

By choosing $f' = f_{\varepsilon'}$ for some $\varepsilon' \in (-\varepsilon_0, 0)$ sufficiently close to 0, we can ensure that $d_{C^0}(f', f) < \varepsilon$. It is clear that $f' \in \mathcal{ML}$ by Definition 3. On the other hand, for each $\varepsilon' \in (0, \varepsilon_0)$ such that $f_{\varepsilon'} \in \mathcal{ML} \bigcup \mathcal{SL} = \mathcal{UL}^c$, there exists an open interval $J \subset (0, \varepsilon_0)$ so that $\rho(F_{\varepsilon'}) = \rho(F_{\varepsilon''}), \forall \varepsilon'' \in J$. This immediately implies that the set $A := \{\rho(F_{\varepsilon'}) \mid \varepsilon' \in (0, \varepsilon_0) \text{ and } f_{\varepsilon'} \notin \mathcal{UL}\}$ is countable. Moreover, by our hypothesis on F, $\rho(F) \notin A$. Thus there exists arbitrarily small $\varepsilon'' \in (0, \varepsilon_0)$ with $f_{\varepsilon''} \in \mathcal{UL}$. This concludes the proof. □

LEMMA 2. *A $g-$forced map f with a lift F is in $\mathcal{ML}$ if and only if there exists $\varepsilon > 0$ such that for any F' which is a lift of a $g-$forced map and satisfies $d_{C^0}(F, F') < \varepsilon$, we have $\rho(F) = \rho(F')$.*

Proof. We just need to show the "only if" part, for the other direction is obvious. By Definition 3, there exists $\delta > 0$ such that $\rho(F_{-\delta}) = \rho(F) = \rho(F_\delta)$. Take $\varepsilon \in (0, \delta)$. Then $(F_{-\delta})_\theta(y) < F'_\theta(y) < (F_\delta)_\theta(y)$ for any F' in the lemma, and any $(\theta, y) \in X \times \mathbb{R}$. By monotonicity, we have $\rho(F_{-\delta}) \le \rho(F') \le \rho(F_\delta)$. This ends the proof. □

COROLLARY B. *The set $\mathcal{ML}$ is open. Moreover, for any homeomorphism $h : X \times \mathbb{T} \to X \times \mathbb{T}$ of the form $h(\theta, x) = (\theta, h_\theta(x))$ with a lift $H : X \times \mathbb{R} \to X \times \mathbb{R}$ (see Theorem 1), we have $f \in \mathcal{ML}$ if and only if $hfh^{-1} \in \mathcal{ML}$.*

Proof. We claim that for any $f \in \mathcal{ML}$ with a lift F, for any $f' \in \mathcal{F}_g$ with $d_{C^0}(f, f') < 1/4$, there exists a lift of F' of f' such that $d_{C^0}(F, F') = d_{C^0}(f, f')$. Indeed, we take an arbitrary lift F'' of f'. For any $\theta \in X$, F''_θ is a lift of f'_θ. Thus there is a function $\phi : X \to \mathbb{Z}$ such that $d_{C^0}(F''_\theta + \phi(\theta), F_\theta) = d_{C^0}(f'_\theta, f_\theta)$ for any $\theta \in X$. By continuity and $d_{C^0}(f, f') < 1/4$, ϕ is continuous. It suffices to take $F'(\theta, x) = (g(\theta), F''_\theta(x) + \phi(\theta))$.

By Lemma 2, the above claim immediately implies that $\mathcal{ML}$ is C^0 open. For the second statement, we note that for any lift F of f, HFH^{-1} is a lift of hfh^{-1}. Then we obtain $hfh^{-1} \in \mathcal{ML}$ by Lemma 2 and Definition 3. □

Proof of Corollary A. Take an arbitrary $\varepsilon > 0$. By Theorem 2, for any $\hat{f} \in \mathcal{P}$, any $\delta > 0$, one can find $f' \in \mathcal{ML}$ such that $d_{C^0}(f', \hat{f}(0)) < \delta$.

Let $\delta > 0$ be a small constant to be determined, and let F' be as above. For $s \in [-1, 1]$ we define $g^s \in C^0(\mathbb{T}, \mathbb{T})$ by

$$g^s(t) = |s|\hat{f}(0)(t) + (1 - |s|)f'(t), \quad \forall t \in \mathbb{T}.$$

The addition in the above expression is given by the natural group structure of $\mathbb{T}$. It is straightforward to verify that $g^s \in \mathrm{Homeo}_+(\mathbb{T})$ for all $s \in [-1, 1]$.

Let $\sigma > 0$ be a small constant to be determined. We define

$$\hat{f}'(s) = \begin{cases} \hat{f}(s), & \forall s \notin (-2\sigma, 2\sigma), \\ \hat{f}(2(s - \sigma)), & \forall s \in (\sigma, 2\sigma], \\ \hat{f}(2(s + \sigma)), & \forall s \in [-2\sigma, -\sigma), \\ g^{\sigma^{-1}s}, & \forall s \in [-\sigma, \sigma]. \end{cases}$$

By definition, $\hat{f}' \in \mathcal{P}$. By letting σ, δ be sufficiently small depending on $\hat{f}, \varepsilon$, we can ensure that $\hat{f}'$ is ε-close to $\hat{f}$ in the uniform topology. By Corollary B, $\hat{f}'(\tau) \in \mathcal{ML}$ for every τ in a neighbourhood of 0.

We may apply a similar perturbation for $\hat{f}$ near an arbitrary $\tau_0 \in \mathbb{T}$ instead of 0. It is clear that $\mathcal{P}$ is a complete metric space under the uniform distance. We conclude the proof by taking τ_0 over a dense subset of $\mathbb{T}$ and by a Baire category argument. □

Definition 4. For any $g-$forced map f with a lift F, for any integer $n > 0$ we set

$$\underline{M}(F,n) = \inf_{(\theta,y)\in X\times\mathbb{R}} ((F^n)_\theta(y) - y) \text{ and } \overline{M}(F,n) = \sup_{(\theta,y)\in X\times\mathbb{R}} ((F^n)_\theta(y) - y).$$

We collect some general properties of $g-$forced maps. The following is an immediate consequence of the unique ergodicity of g. We omit the proof.

LEMMA 3. *Given a $g-$forced map f with a lift F. For any $\kappa_0 > 0$, there exists $N_0 = N_0(F, \kappa_0) > 0$ such that for any $n > N_0$, we have $[\underline{M}(F,n), \overline{M}(F,n)] \subset n\rho(F) + (-n\kappa_0, n\kappa_0)$.*

We also have the following.

LEMMA 4. *Let f be a $g-$forced map with lift F. If for some constant $\delta > 0$, we have $\rho(F_{-\delta}) < \rho(F) < \rho(F_\delta)$, then there exist $\kappa_1 = \kappa_1(F,\delta) > 0$ and an integer $N_1 = N_1(F,\delta) > 0$ such that for any $n > N_1$, we have*

$$\overline{M}(F_{-\delta}, n) < \underline{M}(F,n) - n\kappa_1 < \overline{M}(F,n) + n\kappa_1 < \underline{M}(F_\delta, n).$$

Proof. The first and the third inequality are immediate consequences of Lemma 3 and our hypotheses. The second inequality is obvious. □

2.2. **An inverse function theorem.** The following lemma is simple but convenient for constructing perturbations that depend continuously on parameters.

LEMMA 5. *Let $\varepsilon > 0$ and Y be a metric space. Let $\Sigma \subset Y \times \mathbb{R}$ be an open set so that $\pi_Y(\Sigma) = Y$, and let $\varphi : (-\varepsilon, \varepsilon) \times Y \to \mathbb{R}$ be a continuous function such that*

(1) for any $x \in Y$, the function $t \mapsto \varphi(t,x)$ is monotonically increasing, and

$$\{z \in \mathbb{R} \mid (x,z) \in \Sigma\} \subset \varphi((-\varepsilon,\varepsilon) \times \{x\}),$$

(2) for any $(t_0, x) \in (-\varepsilon, \varepsilon) \times Y$ such that $(x, \varphi(t_0, x)) \in \Sigma$, the function $t \mapsto \varphi(t,x)$ is strictly increasing at $t = t_0$.

Then there exists a continuous map $s : \Sigma \to (-\varepsilon, \varepsilon)$ such that

$$\varphi(s(x,z),x) = z, \quad \forall (x,z) \in \Sigma.$$

Proof. By (1), for any $(x,z) \in \Sigma$, there exists $t \in (-\varepsilon, \varepsilon)$ such that $\varphi(t,x) = z$. By (1),(2), we see that such t is unique. Thus there exists a unique function $s : \Sigma \to (-\varepsilon, \varepsilon)$ such that $\varphi(s(x,z),x) = z$ for any $(x,z) \in \Sigma$. It remains to show that s is continuous.

Given an arbitrary $(x,z) \in \Sigma$, let $t_0 = s(x,z)$. Then by $\varphi(t_0, x) = z$ and (2), we see that the function $t \mapsto \varphi(t,x)$ is strictly increasing at $t = t_0$. Thus for any $\delta \in (0, \varepsilon - t_0)$, there exists $\kappa > 0$ such that $\varphi(t_0 + \delta, x) > z + 2\kappa$. Since φ is continuous, there exists $\sigma > 0$ such that for any $x' \in B(x,\sigma)$, $\varphi(t_0 + \delta, x') > z + \kappa$.

Then for any $(x', z') \in \Sigma \cap (B(x, \sigma) \times (z - \kappa, z + \kappa))$, we have $s(x', z') < t_0 + \delta$. Indeed, if $s(x', z') \geq t_0 + \delta$, then we would have a contradiction since

$$z' = \varphi(s(x', z'), x') \geq \varphi(t_0 + \delta, x') > z + \kappa > z'.$$

By a similar argument, we can see that for any $\delta \in (0, t_0 + \varepsilon)$, there exist $\sigma', \kappa' > 0$ such that $s(x', z') > t_0 - \delta$ for any $(x', z') \in \Sigma \cap (B(x, \sigma') \times (z - \kappa', z + \kappa'))$. This shows that s is continuous and thus concludes the proof. □

3. STRATIFICATION

Let $g : X \to X$ be given as in Theorem 1. That is, X is a compact metric space, and g is a strictly ergodic homeomorphism with a non-periodic factor of finite dimension. As in [4], for any integers $n, D, d > 0$, a compact subset $K \subset X$ is said

(1) $n-$*good* if $g^k(K)$ for $0 \leq k \leq n - 1$ are disjoint subsets,
(2) $D-$*spanning* if the union of $g^k(K)$ for $0 \leq k \leq D - 1$ covers X,
(3) $d-$*mild* if for any $x \in X$, $\{g^k(x) \mid k \in \mathbb{Z}\}$ enters ∂K at most d times.

The following is contained in [4, Proposition 5.1].

LEMMA 6. *There exists an integer $d > 0$ such that for every integer $n > 0$, there exist an integer $D > 0$ and a compact subset $K \subset X$ that is $n-$good, $D-$spanning and $d-$mild.*

Let $K \subset X$ be a $n-$good, $D-$spanning and $d-$mild compact subset. For each $x \in X$, we set

$$\ell^+(x) = \min\{j > 0 \mid g^j(x) \in int(K)\}, \quad \ell^-(x) = \min\{j \geq 0 \mid g^{-j}(x) \in int(K)\},$$
$$\ell(x) = \min\{j > 0 \mid g^j(x) \in K\},$$
$$T(x) = \{j \in \mathbb{Z} \mid -\ell^-(x) < j < \ell^+(x)\}, \quad T_B(x) = \{j \in T(x) \mid g^j(x) \in \partial K\},$$
$$N(x) = \#T_B(x), \quad K^i = \{x \in K \mid N(x) \geq d - i\}, \forall -1 \leq i \leq d.$$

Let $Z^i = K^i \setminus K^{i-1} = \{x \in K \mid N(x) = d - i\}$ for each $0 \leq i \leq d$.

Lemma 6 and the notations introduced above are minor modifications of those in the proof of [4, Lemma 4.1]. We will use them in Section 5 to construct a global perturbation from its local counterparts. In [4], it is important to give a good upper bound for the constant D appearing in Lemma 6, while in our case, we do not need such bound. In the following, we collect some facts from [4].

LEMMA 7. *We have*

(1) For any $x \in K$, $\ell(x) \leq \ell^+(x)$ and $n \leq \ell(x) \leq D$,
(2) T and T_B are upper-semicontinuous,
(3) T and T_B, and hence also ℓ, are locally constant on Z^i,
(4) K^i is closed for all $-1 \leq i \leq d$ and $\emptyset = K^{-1} \subset K^0 \subset \cdots \subset K^d = K$,
(5) For any $x \in K^i$, any $0 \leq m < \ell^+(x)$ such that $g^m(x) \in K$, we have $g^m(x) \in K^i$.

Proof. Items (1),(2) are immediate consequences of the definition. Item (4) follows from (2) and the fact that K is $d-$mild. Item (5) is true since for any such m, we have $\{g^j(x) \mid j \in T(x)\} = \{g^j(g^m(x)) \mid j \in T(g^m(x))\}$, and as a consequence, $N(x) = N(g^m(x))$.

Item (3) is essentially contained in the proof of the claim in [4, Proof of Lemma 4.1]. For the convenience of the readers, we recall the proof here. Fix $x \in Z^i$. Then we have $g^{\ell^+(x)}(x) \in int(K)$. Let $y \in Z^i$ be close to x. Then we have $g^{\ell^+(x)}(y) \in int(K)$ and hence $\ell^+(y) \leq \ell^+(x)$. Similarly, we have $\ell^-(y) \leq \ell^-(x)$. If $j \in T(x)$

then either (i) $g^j(x) \in X \setminus K$, or (ii) $g^j(x) \in \partial K$. In case (i), $g^j(y) \in X \setminus K$ since y is close to x and K is closed. Thus we have $T_B(y) \subset T_B(x)$. By $y \in Z^i$, we have equality $T_B(y) = T_B(x)$. Thus in case (ii), $g^j(y) \in \partial K$. Consequently, we also have $T(y) = T(x)$. □

In the rest of this paper, we denote by $\widetilde{Homeo}$ the set of 1−periodic orientation preserving homeomorphisms of $\mathbb{R}$. In other words, $\widetilde{Homeo}$ is the set of maps which are lifts of maps in the group $Homeo_+(\mathbb{T})$ of orientation preserving homeomorphisms of $\mathbb{T}$. Given any integer $k \geq 1$ and $g_1, \cdots, g_k \in \widetilde{Homeo}$, we denote by $\prod_{i=1}^k g_i$ the map $g_k \cdots g_1$. We set $\prod_{i=1}^0 g_i = \mathrm{Id}$.

4. Perturbation lemmata

In this section, we fix a constant $\delta \in (0, \frac{1}{4})$ and a $g-$forced map f with a lift F, satisfying

$$\rho(F_{-\delta}) < \rho(F) < \rho(F_\delta). \tag{4.1}$$

We stress that f is not necessarily unlocked.

4.1. **Cancellation of the mean motion.**

For any $\kappa > 0$ and any integer $N > 0$, let

$$\Gamma_N(F, \kappa) := \{(\theta, y, z) \in X \times \mathbb{R} \times \mathbb{R} \mid |z - (F^N)_\theta(y)| < N\kappa\}.$$

LEMMA 8. *Let $\kappa_1 = \kappa_1(F, \delta) > 0$ and $N_1 = N_1(F, \delta) > 0$ be given by Lemma 4. Then for any $g-$forced map $\check{f}$ with a lift $\check{F}$ such that $d_{C^0}(\check{F}, F) < \delta$, for any integer $N \geq N_1$, there exists a continuous map $\Phi_N^{\check{F}} : \Gamma_N(F, \kappa_1) \to \widetilde{Homeo}^N$ such that the following is true. For any $(\theta, y, z) \in \Gamma_N(F, \kappa_1)$, let $\Phi_N^{\check{F}}(\theta, y, z) = (G_0, \cdots, G_{N-1})$, then*

1. *$d_{C^0}(G_i, \check{F}_{g^i(\theta)}) < 2\delta$ for any $0 \leq i \leq N-1$,*
2. *$G_{N-1} \cdots G_0(y) = z$,*
3. *if $z = (\check{F}^N)_\theta(y)$, then $G_i = \check{F}_{g^i(\theta)}$ for any $0 \leq i \leq N-1$.*

Proof. It is clear that for any $\theta \in X, y \in \mathbb{R}$, the function $\delta' \mapsto (\check{F}_{\delta'}^N)_\theta(y)$ is strictly increasing. By $d_{C^0}(\check{F}, F) < \delta$, we have $(\check{F}_{2\delta})_\theta(y) \geq (F_\delta)_\theta(y)$ for any $\theta \in X$ and $y \in \mathbb{R}$. Then by Lemma 4,

$$(\check{F}_{2\delta}^N)_\theta(y) \geq (F_\delta^N)_\theta(y) > (F^N)_\theta(y) + N\kappa_1.$$

Similarly, we have $(\check{F}_{-2\delta}^N)_\theta(y) < (F^N)_\theta(y) - N\kappa_1$. Then we can verify (1),(2) in Lemma 5 for $(2\delta, X \times \mathbb{R}, \Gamma_N(F, \kappa_1))$ in place of (ε, Y, Σ), and for function $(\delta', \theta, y) \mapsto (\check{F}_{\delta'})_\theta^N(y)$ in place of φ. By Lemma 5 there exists a continuous function $s : \Gamma_N(F, \kappa_1) \to (-2\delta, 2\delta)$ such that $\Phi_N^{\check{F}}$ defined by

$$\Phi_N^{\check{F}}(\theta, y, z) := ((\check{F}_{s(\theta,y,z)})_{g^i(\theta)})_{i=0}^{N-1}$$

satisfies our lemma. □

4.2. **Local perturbation.** For any $g-$forced map f' with a lift denoted by F', for any integer $N > 0$, we define

$$\begin{aligned}\Omega_N(F') &:= \{(\theta, \underline{y}, y, \overline{y}, z) \in X \times \mathbb{R}^4 \mid \underline{y} < y < \overline{y} \leq \underline{y}+1, z \in ((F'^N)_\theta(\underline{y}), (F'^N)_\theta(\overline{y}))\},\\ \overline{\Omega} &:= \{(\theta, \underline{y}, y, \overline{y}) \in X \times \mathbb{R}^3 \mid \underline{y} < y < \overline{y} \leq \underline{y}+1\}.\end{aligned}$$

LEMMA 9. *There exist $\kappa_2 = \kappa_2(F,\delta) \in (0,\delta)$ and $N_2 = N_2(F,\delta) > 0$ such that the following is true. For any $g-$forced map $\check{f}$ with a lift $\check{F}$ such that $d_{C^0}(\check{F}, F) < \kappa_2$, for any integer $N \geq N_2$, there exists a continuous map $\Psi^{\check{F}}_{0,N} : \Omega_N(\check{F}) \to \widetilde{Homeo}^N$ with the following properties: for any $(\theta, \underline{y}, y, \overline{y}, z) \in \Omega_N(\check{F})$, let $\Psi^{\check{F}}_{0,N}(\theta, \underline{y}, y, \overline{y}, z) = (G_0, \cdots, G_{N-1})$, then*

1. *$d_{C^0}(G_i, \check{F}_{g^i(\theta)}) < 2\delta$ for any $0 \leq i \leq N-1$,*
2. *$G_{N-1} \cdots G_0(y) = z$,*
3. *for any $1 \leq i \leq N$, for any $w \in [\overline{y}, \underline{y}+1]$, we have $G_{i-1} \cdots G_0(w) = (\check{F}^i)_\theta(w)$,*
4. *if $z = (\check{F}^N)_\theta(y)$, then $G_i = \check{F}_{g^i(\theta)}$ for any $0 \leq i \leq N-1$.*

Proof. Let $\kappa_1 = \kappa_1(F,\delta), N_1 = N_1(F,\delta)$ be given by Lemma 4, and let $N_2 = \max(10\kappa_1^{-1}, N_1)$. We let $\kappa_2 = \kappa_2(F,\delta) \in (0,\delta)$ be sufficiently small such that for any $n > N_2$, for any f' with a lift F' such that $d_{C^0}(F', F) < \kappa_2$, we have

$$A_n := \sup_{\theta \in X, y \in \mathbb{R}} |(F'^n)_\theta(y) - (F^n)_\theta(y)| < \frac{1}{2}\kappa_1 n. \tag{4.2}$$

The existence of κ_2 is guaranteed by the compactness of X, and the formula $\lfloor A_{n+m} \rfloor \leq \lfloor A_m \rfloor + \lfloor A_n \rfloor + 2$ for any integers $n, m \geq 1$.

For each $1 \leq i \leq N$, we set

$$\underline{y}_i := (\check{F}^i)_\theta(\underline{y}) \quad \text{and} \quad \overline{y}_i := (\check{F}^i)_\theta(\overline{y}).$$

It is clear that $\underline{y}_i < \overline{y}_i$ for every $1 \leq i \leq N$, and $\underline{y}_N < z < \overline{y}_N$ for any $(\theta, \underline{y}, y, \overline{y}, z) \in \Omega_N(\check{F})$.

Let $\Delta_2 := \{(y,z) \in \mathbb{R}^2 \mid y < z \leq y+1\}$. We define a continuous map $\phi_\delta : (-1,1) \times \Delta_2 \to \widetilde{Homeo}$ as follows. For every $t \in (-1,1)$, every $(y,z) \in \Delta_2$, and every $k \in \mathbb{Z}$, we require that the restriction of $\phi_\delta(t,y,z)$ to $[y+k, z+k]$ is a homeomorphism of $[y+k, z+k]$ fixing the boundary points. More precisely, we have the following two cases to consider:

• Case $z - y \leq 2\delta$: the graph of $\phi_\delta(t,y,z)$ contains the points

$$(y+n, y+n), (\frac{(1+t)y + (1-t)z}{2} + n, \frac{(1-t)y + (1+t)z}{2} + n), (z+n, z+n), \quad n \in \mathbb{Z}$$

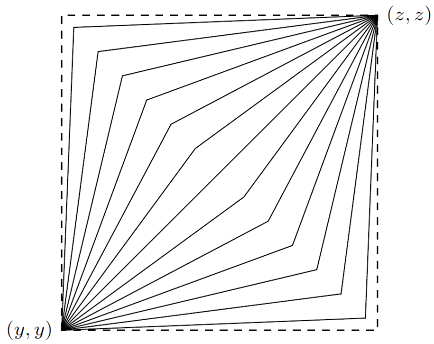
and interpolates affinely between them.

• Case $z - y > 2\delta$: the graph of $\phi_\delta(t,y,z)$ contains the points

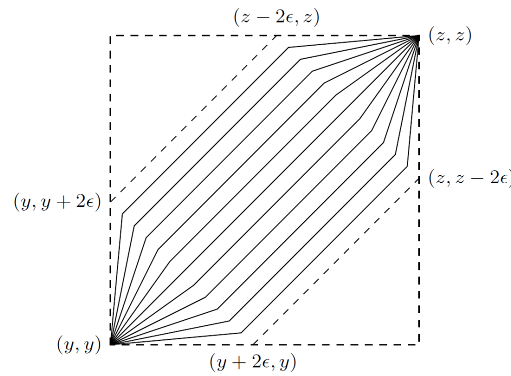
$$\begin{gathered}(y+n, y+n), (y + (1-t)\epsilon + n, y + (1+t)\epsilon + n),\\ (z - (1+t)\epsilon + n, z - (1-t)\epsilon + n), (z+n, z+n), \quad n \in \mathbb{Z}\end{gathered}$$

and interpolates affinely between them.

The figures below show the graphs of the functions $\phi_\epsilon(t,y,z)(\cdot)$ on the interval $[y,z]$, for various values of $t \in (-1,1)$:



(I) $z - y \leq 2\delta$.



(II) $z - y > 2\delta$.

Notice that by the above definition, $\phi_\delta(t, y, z)$ is identity on $\mathbb{Z} + y$ and $\mathbb{Z} + z$. Thus by letting $\phi_\delta(t, y, z)(x) = x$ for $x \in \mathbb{R} \setminus (\mathbb{Z} + [y, z])$, the map ϕ_δ is valued in $\widetilde{Homeo}$. By straightforward computations, we see that $\phi_\delta(t, y, z)(x)$ is continuous in $t \in (-1, 1)$, $(y, z) \in \Delta_2$ and $x \in (y, z)$. It is clear that the map ϕ_δ satisfies the following properties:

($a1$) for any $t \in (-1, 1), (y, z) \in \Delta_2, d_{C^0}(\phi_\delta(t, y, z), \mathrm{Id}) < 2\delta$,
($a2$) the function $t \mapsto \phi(t, y, z)(x)$ is *strictly* increasing for any $x \in (y, z)$,
($a3$) for any $(y, z) \in \Delta_2$, for any $x \in (y, z)$, we have $\lim_{t \to 1} \phi_\delta(t, y, z)(x) \in \{x + 2\delta, z\}$ and $\lim_{t \to -1} \phi_\delta(t, y, z)(x) = \{x - 2\delta, y\}$,
($a4$) for any $t \in (-1, 1)$, any $x \in \mathbb{Z} + [z, y + 1]$, we have $\phi_\delta(t, y, z)(x) = x$,
($a5$) for any $(y, z) \in \Delta_2$, we have $\phi_\delta(0, y, z) = \mathrm{Id}$.

Let $N \geq N_2$ be chosen as above. Define a continuous map $\varphi : (-1, 1) \times \overline{\Omega} \to \mathbb{R}$ by

$$\varphi(t, \theta, \underline{y}, y, \overline{y}) = \prod_{i=1}^{N} [\phi_\delta(t, \underline{y}_i, \overline{y}_i) \check{F}_{g^{i-1}(\theta)}](y).$$

We have the following,
Claim: we have $\lim_{t \to 1} \varphi(t, \theta, \underline{y}, y, \overline{y}) = (\check{F}^N)_\theta(\overline{y})$.

Proof. Let $y'_0 := y$, and for each $1 \leq k \leq N$, let

$$y'_k := \lim_{t \to 1} \prod_{i=1}^{k} [\phi_\delta(t, \underline{y}_i, \overline{y}_i) \check{F}_{g^{i-1}(\theta)}](y).$$

It is clear that $y'_k > \underline{y}_k$ for every $1 \leq k \leq N$. Assume that for every $1 \leq k \leq N$, we have $y'_k < \overline{y}_k$. Then by property ($a3$) above, for every $1 \leq k \leq N$ we would have

$$y'_k = \lim_{t \to 1} \phi_\delta(t, \underline{y}_k, \overline{y}_k) \check{F}_{g^{k-1}(\theta)}(y'_{k-1}) = \check{F}_{g^{k-1}(\theta)}(y'_{k-1}) + 2\delta = (\check{F}_{2\delta})_{g^{k-1}(\theta)}(y'_{k-1}).$$

Then by (4.2), $d_{C^0}(\check{F},F)<\kappa_2<\delta$ and $N>N_2=\max(10\kappa_1^{-1},N_1)$, we obtain

$$\lim_{t\to 1}\prod_{i=1}^{N}[\phi_\delta(t,\underline{y}_i,\overline{y}_i)\check{F}_{g^{i-1}(\theta)}](y)=(\check{F}^N_{2\delta})_\theta(y)\geq(F^N_\delta)_\theta(y)$$
$$>\quad (F^N)_\theta(y)+N\kappa_1>(F^N)_\theta(\overline{y})+\frac{1}{2}N\kappa_1>(\check{F}^N)_\theta(\overline{y}).$$

But this contradicts $y<\overline{y}$ and $(a4)$. Thus there exists $1\leq k\leq N$ such that $y'_k=\overline{y}_k$. Then by $(a3)$,$(a4)$ and $(a5)$, we have

$$(\check{F}^N)_\theta(\overline{y})\geq\lim_{t\to 1}\varphi(t,\theta,\underline{y},y,\overline{y})\geq\prod_{i=k+1}^{N}\check{F}_{g^{i-1}(\theta)}(y'_k)=\overline{y}_N=(\check{F}^N)_\theta(\overline{y}).$$

This concludes the proof of the claim. □

Similar to the above claim, we also have $\lim_{t\to -1}\varphi(t,\theta,\underline{y},y,\overline{y})=\check{F}^N_\theta(\underline{y})$. We thereby verify condition (1) in Lemma 5 for map φ, and for $(1,\overline{\Omega},\Omega_N(\check{F}))$ in place of (ε,Y,Σ). Moreover, by $(a2)$ and $(a4)$, we can directly verify that the function $t\mapsto\varphi(t,\theta,\underline{y},y,\overline{y})$ is strictly increasing for any $(\theta,\underline{y},y,\overline{y})\in\overline{\Omega}$. This verify condition (2) in Lemma 5. Then by Lemma 5, we obtain a continuous function $s:\Omega_N(\check{F})\to(-1,1)$ such that for any $(\theta,\underline{y},y,\overline{y},z)\in\Omega_N(\check{F})$, we have

$$\varphi(s(\theta,\underline{y},y,\overline{y},z),\theta,\underline{y},y,\overline{y})=z.$$

We define $\Psi^{\check{F}}_{0,N}(\theta,\underline{y},y,\overline{y},z)=(G_0,\cdots,G_{N-1})$ where for each $0\leq i\leq N-1$

$$G_i=\phi_\delta(s(\theta,\underline{y},y,\overline{y},z),\underline{y}_{i+1},\overline{y}_{i+1})\check{F}_{g^i(\theta)}.$$

We can see that (1)-(4) follows from $(a1)$-$(a5)$ and Lemma 5. □

4.3. **Concatenation.** In this section, we briefly denote $\rho(F)$ by ρ.

PROPOSITION 1. *There exist $\kappa_3=\kappa_3(F,\delta)\in(0,\delta)$, $N_3=N_3(F,\delta)>0$ such that for any $g-$forced map $\check{f}$ with a lift $\check{F}$ satisfying $d_{C^0}(\check{F},F)<\kappa_3$, for any integer $N\geq N_3$, there exists a continuous function $\Psi^{\check{F}}_N:X\to\widetilde{Homeo}^N$ such that the following is true: for any $\theta\in X$, let $\Psi^{\check{F}}_N(\theta)=(G_0,\cdots,G_{N-1})$. Then*

(1) $d_{C^0}(G_i,\check{F}_{g^i(\theta)})<2\delta$ for any $0\leq i\leq N-1$,
(2) $G_{N-1}\cdots G_0(y)=y+N\rho$ for any $y\in\mathbb{R}$,
(3) if $(\check{F}^N)_\theta(y)=y+N\rho$ for any $y\in\mathbb{R}$, then $G_i=\check{F}_{g^i(\theta)}$ for any $0\leq i\leq N-1$.

Proof. We let $N_0=N_0(F,\frac{1}{3}\kappa_1)$ be given by Lemma 3; let $\kappa_1=\kappa_1(F,\delta),N_1=N_1(F,\delta)$ be given by Lemma 4; let $\kappa_2=\kappa_2(F,\delta)$, $N_2=N_2(F,\delta)$ be given by Lemma 9. We let $\kappa_3=\kappa_3(F,\delta)\in(0,\min(\kappa_2,\delta))$ be a sufficiently small constant depending only on F and δ, such that for any $n>N_0$, for any $g-$forced map f' with a lift F' satisfying $d_{C^0}(F',F)<\kappa_3$, we have

$$B_n:=\sup_{\theta\in X,y\in\mathbb{R}}|(F'^n)_\theta(y)-y-n\rho|<\frac{2}{3}n\kappa_1. \tag{4.3}$$

The existence of κ_3 is guaranteed by the compactness of X and the formula $B_{p+q}\leq B_p+B_q$ for any integers $p,q\geq 1$.

We let

(4.4) $N_2' = N_0 + N_2, \quad p = \lceil 10\delta^{-1} \rceil \quad \text{and} \quad N_3 = N_1 + 10(p-1)N_2' + 1.$

For each $1 \le i \le p$, we set $y_i := \frac{i-1}{p}$. Take an arbitrary $N > N_3$, set $z_i := y_i + N\rho$ for each $1 \le i \le p$. Given $\theta \in X$, we inductively define $G_0, \cdots, G_{N-1}$ as follows.

We let $w_1 = (\check{F}^{(p-1)N_2'+1})^{-1}_{g^{N-(p-1)N_2'-1}(\theta)}(z_1)$. By $N_2' > N_0$, (4.3) and the hypothesis that $d_{C^0}(\check{F}, F) < \kappa_3$, we have

$$|w_1 + ((p-1)N_2'+1)\rho - z_1| < \frac{2}{3}((p-1)N_2'+1)\kappa_1. \tag{4.5}$$

By (4.4), $N - (p-1)N_2' - 1 > N_0$. Then by Lemma 3, we have

(4.6)
$$|(F^{N-(p-1)N_2'-1})_\theta(y_1) - y_1 - (N-(p-1)N_2'-1)\rho| < \frac{1}{3}(N-(p-1)N_2'-1)\kappa_1.$$

By (4.4), (4.5), (4.6) and $z_1 = y_1 + N\rho$, we have

$$\begin{aligned} & |w_1 - (F^{N-(p-1)N_2'-1})_\theta(y_1)| \\ < \; & \frac{1}{3}(N-(p-1)N_2'-1)\kappa_1 + \frac{2}{3}((p-1)N_2'+1)\kappa_1 < \kappa_1(N-(p-1)N_2'-1). \end{aligned}$$

In particular, we have $(\theta, y_1, w_1) \in \Gamma_{N-(p-1)N_2'-1}(F, \kappa_1)$. By (4.4), we have $N - (p-1)N_2' - 1 > N_1$. We also have $d_{C^0}(\check{F}, F) < \delta$. Then we can apply Lemma 8 to define

$$(G_0, \cdots, G_{N-(p-1)N_2'-2}) := \Phi^{\check{F}}_{N-(p-1)N_2'-1}(\theta, y_1, w_1).$$

By Lemma 8, we have $G_{N-(p-1)N_2'-2} \cdots G_0(y_1) = w_1$.

Assume that for some $k \in \{2, \cdots, p\}$, $G_0, \cdots, G_{N-(p-k+1)N_2'-2}$ are given so that

(4.7)
$$(\check{F}^{(p-k+1)N_2'+1})_{g^{N-(p-k+1)N_2'-1}(\theta)} G_{N-(p-k+1)N_2'-2} \cdots G_0(y_l) = z_l, \quad \forall 1 \le l \le k-1.$$

This is the case when $k = 2$ by our construction above. Let

$$w_k = (\check{F}^{(p-k)N_2'+1})^{-1}_{g^{N-(p-k)N_2'-1}(\theta)}(z_k). \tag{4.8}$$

We let

$$\begin{aligned} \underline{y} &= G_{N-(p-k+1)N_2'-2} \cdots G_0(y_{k-1}), & \overline{y} &= G_{N-(p-k+1)N_2'-2} \cdots G_0(y_1+1), \\ y' &= G_{N-(p-k+1)N_2'-2} \cdots G_0(y_k), & \theta' &= g^{N-(p-k+1)N_2'-1}(\theta). \end{aligned}$$

By $y_{k-1} < y_k < y_1 + 1$, we have $\underline{y} < y' < \overline{y}$. Moreover, by (4.7) and (4.8) we have

$$\begin{aligned} & (\check{F}^{N_2'})_{\theta'}(\underline{y}) = (\check{F}^{(p-k)N_2'+1})^{-1}_{g^{N-(p-k)N_2'-1}(\theta)}(z_{k-1}) \\ < \; & w_k = (\check{F}^{(p-k)N_2'+1})^{-1}_{g^{N-(p-k)N_2'-1}(\theta)}(z_k) \\ < \; & (\check{F}^{N_2'})_{\theta'}(\overline{y}) = (\check{F}^{(p-k)N_2'+1})^{-1}_{g^{N-(p-k)N_2'-1}(\theta)}(z_1+1). \end{aligned}$$

Then by $d_{C^0}(\check{F}, F) < \kappa_3 < \kappa_2$ and $N_2' > N_2$, we can apply Lemma 9 to define

$$(G_{N-(p-k+1)N_2'-1}, \cdots, G_{N-(p-k)N_2'-2}) := \Psi^{\check{F}}_{0,N_2'}(\theta', \underline{y}, y', \overline{y}, w_k).$$

By Lemma 9, we have

$$G_{N-(p-k)N_2'-2} \cdots G_{N-(p-k+1)N_2'-1}(y') \quad = \quad w_k.$$

Moreover, by (2),(3) in Lemma 9 and (4.8), we verify (4.7) for $k+1$ in place of k. This recovers the induction hypothesis for $k+1$. We complete the definition of G_i for all $0 \le i \le N-2$ when $k = p+1$. Then we have

$$\check{F}_{g^{N-1}(\theta)} G_{N-2} \cdots G_0(y_i) = z_i, \quad \forall i = 1, \cdots, p. \tag{4.9}$$

Define

$$H = (\check{F}_{g^{N-1}(\theta)} G_{N-2} \cdots G_0)^{-1} + N\rho \quad \text{and} \quad G_{N-1} = H\check{F}_{g^{N-1}(\theta)}.$$

Then by (4.9), we have $H(z_i) = z_i$ for any $1 \le i \le p$. Thus $d_{C^0}(H, \mathrm{Id}) \le \frac{1}{p} < \delta$. Define $\Psi^{\check{F}}_N(\theta) = (G_0, \cdots, G_{N-1})$. It is direct to see (1)-(3) by our construction. □

5. Proof of the main results

Proof of Theorem 1: Recall that F is a lift of f. In the course of the proof, we abbreviate $\rho(F)$ as ρ. Let integer $d > 0$ be given by Lemma 6. We inductively define positive constants $0 < \varepsilon_{-1} < \varepsilon_0 < \cdots < \varepsilon_d$ by the following formula:

$$\varepsilon_d = \frac{\varepsilon}{4(d+2)} \quad \text{and} \quad \varepsilon_{d-k} = \frac{1}{2(d+2)}\kappa_3(F, \varepsilon_{d-k+1}) < \varepsilon_{d-k+1}, \forall 1 \le k \le d+1.$$

Then we have

$$2(\varepsilon_{-1} + \cdots + \varepsilon_d) < \varepsilon, \quad 2(\varepsilon_{-1} + \cdots + \varepsilon_k) < \kappa_3(F, \varepsilon_{k+1}), \forall -1 \le k < d. \tag{5.1}$$

We define

$$n_0 = \max_{-1 \le i \le d} N_3(F, \varepsilon_i) + 1.$$

By Lemma 6, we can choose K, a compact subset of X, that is n_0-good, $D-$spanning and $d-$mild for some $D > 0$.

Let $\{K^i\}_{i=-1}^d, \{Z^i\}_{i=0}^d$ be defined using K as in Section 3. We will define a sequence of $g-$forced map $f^{(i)}$ for every $-1 \le i \le d$ by induction.

We set $f^{(-1)} := f$ and set $F^{(-1)} := F$. Assume that we have defined $f^{(k)}$ with a lift $F^{(k)}$ for some $-1 \le k \le d-1$ such that

(H1) $d_{C^0}(F^{(k)}, F) < 2(\varepsilon_{-1} + \cdots + \varepsilon_k)$, and
(H2) for any $\theta \in K^k$, we have $[(F^{(k)})^{\ell(\theta)}]_\theta(y) = y + \ell(\theta)\rho$ for every $y \in \mathbb{R}$.

These properties hold for $k = -1$. For each $-1 \le j \le d$, we set

$$W^j = \bigcup_{\theta \in K^j} \bigcup_{0 \le j < \ell(\theta)} \{g^i(\theta)\}. \tag{5.2}$$

By Lemma 7, we have $\ell(\theta) \le D$ for every $\theta \in K$, and $W^d = X$. We also have the following.

LEMMA 10. *Given an integer $0 \le j \le d$, let $\{\theta_n\}_{n \ge 0}$ be a sequence of points in K^j converging to θ', and let $\{\ell_n \in [0, \ell(\theta_n))\}_{n \ge 0}$ be a sequence of integers converging to ℓ'. Then after passing to a subsequence, we have exactly one of the following possibilities:*

(1) $\theta' \in Z^j$ *and* $0 \le \ell' < \ell(\theta')$,

or (2) $\theta' \in K^{j-1}$, *and there exist a unique* $\theta'' \in K^{j-1}$ *and a unique* $0 \le \ell'' < \ell(\theta'')$ *such that* $g^{\ell'}(\theta') = g^{\ell''}(\theta'') \in W^{j-1}$.

In particular, W^j *is closed.*

Proof. By our hypothesis, the limit of the sequence $g^{\ell_n}(\theta_n)$ is $g^{\ell'}(\theta')$. By Lemma 7(4), we have $\theta' \in K^j$.

We first assume that $\theta' \in Z^j$. In this case, by Lemma 7(3), we have $\ell(\theta_n) = \ell(\theta')$ for all sufficiently large n. Then for all sufficiently large n, we have $\ell' = \ell_n < \ell(\theta_n) = \ell(\theta')$. In particular, $g^{\ell'}(\theta') \in W_j$.

Now assume that $\theta' \in K^{j-1}$. We claim that $g^i(\theta') \notin int(K)$ for any $1 \le i \le \ell'$. Indeed, otherwise there would exist $1 \le i \le \ell'$ such that $g^i(\theta_n) \in int(K)$ for all sufficiently large n. But this is a contradiction, since $\ell^+(\theta_n) \ge \ell(\theta_n) > \ell_n = \ell'$ for sufficiently large n. Thus our claim is true. In particular, our claim implies that $\ell^+(\theta') > \ell'$. Let k be the largest integer in $\{0, \cdots, \ell'\}$ such that $g^k(\theta') \in K$ (such k exists since $\theta' \in K$). Then by Lemma 7(5), we have $g^k(\theta') \in K^{j-1}$ and $0 \le \ell' - k < \ell(g^k(\theta'))$. We let $\theta'' = g^k(\theta')$ and $\ell'' = \ell' - k$. It is direct to see that (θ'', ℓ'') is the unique pair which fulfills (2).This finishes the proof. □

In the following, we will define a homeomorphism $\tilde{F} : W^{k+1} \times \mathbb{R} \to g(W^{k+1}) \times \mathbb{R}$ of the form $\tilde{F}(\theta, y) = (g(\theta), \tilde{F}_\theta(y))$ such that $\tilde{F}_\theta \in \widetilde{Homeo}$ for every $\theta \in W^{k+1}$, and $\tilde{F}_{\theta'} = F^{(k)}_{\theta'}$ for every $\theta' \in W^k$.

We define

$$\tilde{F}_{\theta'} = F^{(k)}_{\theta'}, \quad \forall \theta' \in W^k. \tag{5.3}$$

By hypothesis (H1) and (5.1), $d_{C^0}(F^{(k)}, F) < \kappa_3(F, \varepsilon_{k+1})$. By the definitions of n_0 and K, we have $\ell(\theta) > N_3(F, \varepsilon_{k+1})$ for any $\theta \in Z^{k+1}$. By Proposition 1, we define

$$(\tilde{F}_\theta, \cdots, \tilde{F}_{g^{\ell(\theta)-1}(\theta)}) := \Psi^{F^{(k)}}_{\ell(\theta)}(\theta), \quad \forall \theta \in Z^{k+1}. \tag{5.4}$$

By Proposition 1, we have

$$d_{C^0}(\tilde{F}_{g^i(\theta)}, F^{(k)}_{g^i(\theta)}) \;<\; 2\varepsilon_{k+1}, \quad \forall \theta \in Z^{k+1}, 0 \le i < \ell(\theta), \tag{5.5}$$

$$(\tilde{F}^{\ell(\theta)})_\theta(y) \;=\; y + \ell(\theta)\rho, \quad \forall \theta \in Z^{k+1}, y \in \mathbb{R}. \tag{5.6}$$

We have the following.

LEMMA 11. *The map* $\tilde{F}$ *is continuous.*

Proof. It is enough to show that for any $\{\theta_n\}, \{\ell_n\}, \theta', \ell'$ in Lemma 10 with $j = k+1$, we have

$$\tilde{F}_{g^{\ell_n}(\theta_n)} \to \tilde{F}_{g^{\ell'}(\theta')} \quad \text{in } \widetilde{Homeo} \text{ as } \quad n \to \infty. \tag{5.7}$$

We first assume that conclusion (1) in Lemma 10 is true, that is, $\theta' \in Z^{k+1}$. Then (5.7) follows immediately from Lemma 7(3) and the continuity of $\Psi^{F^{(k)}}_{\ell(\theta')}$.

Now assume that conclusion (2) in Lemma 10 is true. In particular, $\theta' \in K^k$ and $g^{\ell'}(\theta') \in W^k$. It is enough to prove (5.7) for two cases: (1) $\theta_n \in K^k$ for all n; or (2) $\theta_n \in Z^{k+1}$ for all n.

In the first case, we have $g^{\ell_n}(\theta_n) \in W^k$ for all n. By Lemma 10, we have $g^{\ell'}(\theta') \in W^k$. Then (5.7) follows from (5.3) and the fact that $F^{(k)}$ is continuous. Now assume that the second case is true, namely, $\theta_n \in Z^{k+1}$ for all n. By Lemma 7(1), $\ell(\theta_n) \leq D$ for all n. After passing to a subsequence, we can assume that there exists an integer $\ell_0 \leq D$ such that $\ell(\theta_n) = \ell_0$ for all n. By (5.4), we have

$$\tilde{F}_{g^{\ell'}(\theta_n)} = \text{ the } \ell'\text{−th coordinate of } \Psi^{F^{(k)}}_{\ell_0}(\theta_n)\ .$$

Then by the continuity of $\Psi^{F^{(k)}}_{\ell_0}$ we have

$$\tilde{F}_{g^{\ell_n}(\theta_n)} \to \text{ the } \ell'\text{−th coordinate of } \Psi^{F^{(k)}}_{\ell_0}(\theta') \quad \text{as } n \to \infty.$$

It is then enough to show that the $\ell'-$th coordinate of $\Psi^{F^{(k)}}_{\ell_0}(\theta')$ is $F^{(k)}_{g^{\ell'}(\theta')}$. By Proposition 1(3), it is enough to verify that $[(F^{(k)})^{\ell_0}]_{\theta'}(y) = y + \ell_0\rho$ for every $y \in \mathbb{R}$. This last statement follows from (H2) and Lemma 7(5). Indeed, by Lemma 7(5) we can express $[(F^{(k)})^{\ell_0}]_{\theta'}$ as a composition of maps of the form $[(F^{(k)})^{\ell(\theta'')}]_{\theta''}$ where $\theta'' \in K^k$; by (H2), $[(F^{(k)})^{\ell(\theta'')}]_{\theta''}(y) = y + \ell(\theta'')\rho$ for every $\theta'' \in K^k$ and $y \in \mathbb{R}$. □

We need the following lemma, which is proved in the Appendix.

PROPOSITION 2 (Tietze's Extension Theorem for $\widetilde{Homeo}$). *Let $F : X \times \mathbb{R} \to X \times \mathbb{R}$ be a lift of a $g-$forced map, and let M be a compact subset of X. Let $G : M \times \mathbb{R} \to g(M) \times \mathbb{R}$ be a continuous map of the form $G(\theta, y) = (g(\theta), G_\theta(y))$ such that $G_\theta \in \widetilde{Homeo}$ for every $\theta \in M$, and $d_{C^0}(F|_{M\times\mathbb{R}}, G) < c$. Then there exists G', a lift of a $g-$forced map, such that $G'|_{M\times\mathbb{R}} = G|_{M\times\mathbb{R}}$ and $d_{C^0}(F, G') < c$.*

By (5.5) and Proposition 2, we can choose $F^{(k+1)} : X \times \mathbb{R} \to X \times \mathbb{R}$, a lift of a $g-$forced map $f^{(k+1)}$, so that: $F^{(k+1)}_\theta = \tilde{F}_\theta$ for any $\theta \in W^{k+1}$; and $d_{C^0}(F^{(k+1)}, F^{(k)}) < 2\varepsilon_{k+1}$. It is straightforward to verify (H1),(H2) for $k+1$. By induction, (H1) and (H2) hold for $k = d$

We let $f' = f^{(d)}$ and $F' = F^{(d)}$. By (5.1) and (H1), $d_{C^0}(f', f) < \varepsilon$. For every $\alpha \in \mathbb{R}/\mathbb{Z}$, let T_α denote the translation $y \mapsto y + \alpha$ on $\mathbb{R}$. There exists a map $H : X \times \mathbb{R} \to X \times \mathbb{R}$ of the form $H(\theta, y) = (\theta, H_\theta(y))$ and

$$H_{g^k(\theta)} = (F'^k)_\theta T_{-k\rho}, \quad \forall \theta \in K, 0 \leq k < \ell(\theta). \tag{5.8}$$

In particular, we have

$$H_\theta = \text{Id}, \quad \forall \theta \in K. \tag{5.9}$$

By definition, it is clear that for each $\theta \in X$, $H_\theta \in \widetilde{Homeo}$. By (H2) for $k = d$, we have

$$(F'^{\ell(\theta)})_\theta T_{-\ell(\theta)\rho} = \text{Id}, \quad \forall \theta \in K. \tag{5.10}$$

We can verify that H is a homeomorphism by induction. Here we only give an outline since the proof strongly resembles that of Lemma 11. We inductively show that the restriction of H to $W^k \times \mathbb{R}$ is continuous for $k = -1, \cdots, d$ (W^k is given by (5.2)). To pass from stage k to stage $k+1$, it is enough to show that

$$H_{g^{\ell_n}(\theta_n)} \to H_{g^{\ell'}(\theta')} \quad \text{as} \quad n \to \infty \tag{5.11}$$

for any $\{\theta_n\}$,$\{\ell_n\}$, θ', ℓ' given in Lemma 10 for $j = k+1$. We again divide the proof into two cases, corresponding to $\theta' \in Z^{k+1}$ and $\theta' \in K^k$. In the first case, (5.11) follows from the continuity of F' and Lemma 7(3). In the second case, (5.11) follows from the induction hypothesis if $\theta_n \in K^k$ for all n. Otherwise we can assume that $\theta_n \in Z^{k+1}$ for all n. Then as in the proof of Lemma 11, we verify (5.11) using Lemma 7(5), the continuity of F' and (5.10).

Let h be the factor of H on $X \times \mathbb{T}$. We now verify that for all $(x,y) \in X \times \mathbb{T}$

$$f'h(x,y) = h(g(x), y+\rho). \tag{5.12}$$

Take an arbitrary $x \in X$. There exist $\theta \in K$ and an integer $0 \le k < \ell(\theta)$ such that $x = g^k(\theta)$. By (5.8), we have

$$F'_x H_x = F'_{g^k(\theta)}(F')^k_\theta T_{-k\rho} = (F')^{k+1}_\theta T_{-(k+1)\rho} T_\rho.$$

If we have $k < \ell(\theta) - 1$, then by (5.8) we have

$$(F')^{k+1}_\theta T_{-(k+1)\rho} T_\rho = H_{g^{k+1}(\theta)} T_\rho = H_{g(x)} T_\rho.$$

If $k = \ell(\theta) - 1$, then by (5.10) and (5.9) we have

$$(F')^{k+1}_\theta T_{-(k+1)\rho} T_\rho = T_\rho = H_{g(x)} T_\rho.$$

This verifies (5.12). □

Proof of Theorem 2: By Corollary B, we only need to show the density of $\mathcal{ML}$. Given an arbitrary $g-$forced map f that is not mode-locked. Let F be a lift of f. By Theorem 1, for any $\varepsilon > 0$, there exists a $g-$forced map f', and a homeomorphism $h : X \times \mathbb{T} \to X \times \mathbb{T}$ of the form $h(\theta, x) = (\theta, h_\theta(x))$ with a lift $H : X \times \mathbb{R} \to X \times \mathbb{R}$, such that $d_{C^0}(f, f') < \frac{1}{2}\varepsilon$ and $f' = hRh^{-1}$, where R is defined as

$$R(x,y) = (g(x), y + \rho(F)), \quad \forall (x,y) \in X \times \mathbb{T}.$$

We let Q be a mode-locked circle homeomorphism that is sufficiently close to rotation $y \mapsto y + \rho(F)$, such that $f'' := h(g \times Q)h^{-1}$ satisfies that $d_{C^0}(f'', f') < \frac{1}{2}\varepsilon$. Then $d_{C^0}(f, f'') < \varepsilon$. We can verify that $g \times Q \in \mathcal{ML}$ by definition. Thus by Corollary B, $f'' \in \mathcal{ML}$. This concludes the proof. □

6. APPENDIX

Proof of Proposition 2: Let $\zeta : \mathbb{R}^2_{\ge 0} \to \mathbb{R}_{\ge 0}$ be a continuous function such that: $\zeta(r, r') > 0$ for any $r' \in (0, 2r)$; and $\zeta(r, r') = 0$ for any $r' \in [2r, \infty)$.

We first show the following weaker version of Proposition 2.

LEMMA 12. *Proposition 2 is true assuming in addition that:*

1. $\overline{int(M)} = M$; *and*
2. *for any* $\theta \in X \setminus M$, *any* $\theta' \in M$ *with* $d(\theta, \theta') < 2d(\theta, M)$, *we have* $d_{C^0}(G_{\theta'}, F_\theta) < c$.

Proof. Recall that μ is an $f-$invariant measure. Since f is minimal, μ is of full support. We set $G'(\theta, y) = (g(\theta), G'_\theta(y))$ with $G'_\theta(y)$ defined as follows. For any $\theta \in M$, we set $G'_\theta = G_\theta$; and for any $\theta \in X \setminus M$, we set

$$G'_\theta(y) := \left(\int_M G_{\theta'}(y)\zeta(d(\theta, M), d(\theta, \theta'))d\mu(\theta')\right) / \left(\int_M \zeta(d(\theta, M), d(\theta, \theta'))d\mu(\theta')\right).$$

The above definition makes sense since by (1) and the definition of ζ, for any $\theta \in X \setminus M$, there exists $\theta' \in int(M)$ with $d(\theta,\theta') < 2d(\theta,M)$, and as a consequence $\zeta(d(\theta,M), d(\theta,\theta'')) > 0$ for all $\theta'' \in M$ sufficiently close to θ'. It is direct to verify that $G'_\theta \in \widetilde{Homeo}$ for every $\theta \in X$, and $\theta \mapsto G'_\theta$ is continuous over X. Thus G' is a homeomorphism. By (2), we have that $d_{C^0}(F, G') < c$. This ends the proof. $\square$

Given an integer $n \geq 2$, we use Tietze's Extension Theorem to define functions $\varphi_{n,i} \in C^0(X,\mathbb{R})$,$i = 0, \cdots, n-1$, which extend functions $(\theta \mapsto G(\theta, i/n)) \in C^0(M,\mathbb{R})$,$i = 0, \cdots, n-1$ respectively. We set $\varphi_{n,n} = \varphi_{n,0} + 1$. For each $n \geq 2$, we define G_n, a lift of a $g-$forced map, as follows. We set $G_n(\theta, y) = (g(\theta), (G_n)_\theta(y))$, where for every $\theta \in X$, every $i \in \{0, \cdots, n-1\}$, every $y \in (i/n, (i+1)/n]$, we set

$$(G_n)_\theta(y+k) := ((ny-i)\varphi_{n,i}(\theta) + (i+1-ny)\varphi_{n,i+1}(\theta)) + k, \forall k \in \mathbb{Z}.$$

In another words, G_n is obtained by piecewise affine interpolations using functions $\{\varphi_{n,i}\}_{i=0}^n$. Hence $(G_n)_\theta \in \widetilde{Homeo}$ for every $\theta \in X$. Let $\{U_n\}_{n\geq 2}$ be a sequence of open neighbourhoods of M such that $\cap_{n\geq 2} U_n = M$. Moreover, we assume that $U_n \neq X$ and $\overline{U_{n+1}} \subset U_n$ for every n. By Urysohn's lemma, for each $n \geq 2$, there exists $\psi_n \in C^0(X, [0,1])$ such that $\psi_n|_{X\setminus U_n} \equiv 1$ and $\psi_n|_{\overline{U_{n+1}}} \equiv 0$. We set $\psi_1 \equiv 0$.

We define a map G'' as follows. For every $\theta \in X$, $y \in \mathbb{R}$, we set $G''(\theta, y) = (g(\theta), G''_\theta(y))$ where $G''_\theta = G_\theta$ if $\theta \in M$; and

$$G''_\theta(y) = \frac{\int \sum_{n\geq 1}(\psi_{n+1} - \psi_n)(\theta')\zeta(d(\theta,M), d(\theta,\theta'))(G_n)_{\theta'}(y)d\mu(\theta')}{\int \sum_{n\geq 1}(\psi_{n+1} - \psi_n)(\theta')\zeta(d(\theta,M), d(\theta,\theta'))d\mu(\theta')}$$

if $\theta \in X \setminus M$. It is direct to see that $G''_\theta \in \widetilde{Homeo}$ for any $\theta \in X$. We claim that: G'' is continuous. Indeed, for each $\theta' \in X$, there exist one or two integers $n \geq 1$ such that $(\psi_{n+1} - \psi_n)(\theta') \neq 0$. Thus G'' is continuous on $X \setminus M$. Moreover, for any $n > 0$, there exists $\tau > 0$ such that inequalities $0 < d(\theta, M) < \tau$ implies that any θ' with $\zeta(d(\theta,M), d(\theta,\theta')) \neq 0$ satisfies that $\theta' \in B(\theta, 2\tau) \subset B(M, 3\tau) \subset U_{n+1}$; for any such θ', we have $(\psi_{m+1} - \psi_m)(\theta') = 0$ for any $m < n$. Then we can verifies our claim by noting there exists an integer $n > 0$ such that $d_{C^0}(G_m|_{U_n\times\mathbb{R}}, F|_{U_n\times\mathbb{R}}) < c$ for every $m \geq n$. Thus G'' is a lift of a $g-$forced map whose restriction to $M \times \mathbb{R}$ equals G.

Finally, we let $U' \subset U$ be two small open neighborhoods of M with $\overline{U'} \subset U$. We set $C = \overline{U'} \cup (X \setminus U)$. We define $G''' : C \times \mathbb{R} \to g(C) \times \mathbb{R}$ by $G'''|_{\overline{U'}\times\mathbb{R}} = G''|_{\overline{U'}\times\mathbb{R}}$ and $G'''|_{(X\setminus U)\times\mathbb{R}} = F|_{(X\setminus U)\times\mathbb{R}}$. It is clear that $\overline{int(C)} = C$. By our hypothesis that $d_{C^0}(G|_{M\times\mathbb{R}}, F|_{M\times\mathbb{R}}) < c$, and by letting U be sufficiently small, we can ensure that condition Lemma 12(2) holds for (G''', F, C) in place of (G, F, M). We obtain G' as the extension given by Lemma 12 for (G''', F, C) in place of (G, F, M). $\square$

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Institut de Mathématique de Jussieu & KTH Royal Institute of Technology
Email address: zzzhangzhiyuan@gmail.com