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# ON A CERTAIN NON-SPLIT CUBIC SURFACE

R. DE LA BRETÈCHE, K. DESTAGNOL, J. LIU, J. WU & Y. ZHAO

ABSTRACT. In this note, we establish an asymptotic formula with a power-saving error term for the number of rational points of bounded height on the singular cubic surface of  $\mathbb{P}_{\mathbb{Q}}^3$

$$x_0(x_1^2 + x_2^2) = x_3^3$$

in agreement with the Manin-Peyre conjectures.

## 1. INTRODUCTION AND RESULTS

Let  $V \subset \mathbb{P}_{\mathbb{Q}}^3$  be the cubic surface defined by

$$x_0(x_1^2 + x_2^2) - x_3^3 = 0.$$

The surface  $V$  has three singular points  $\xi_1 = [1 : 0 : 0 : 0]$ ,  $\xi_2 = [0 : 1 : i : 0]$  and  $\xi_3 = [0 : 1 : -i : 0]$ . It is easy to see that the only three lines contained in  $V_{\overline{\mathbb{Q}}} = V \times_{\text{Spec}(\mathbb{Q})} \text{Spec}(\overline{\mathbb{Q}})$  are

$$\ell_1 := \{x_3 = x_1 - ix_2 = 0\}, \quad \ell_2 := \{x_3 = x_1 + ix_2 = 0\},$$

and

$$\ell_3 := \{x_3 = x_0 = 0\}.$$

Clearly both  $\ell_1$  and  $\ell_2$  pass through  $\xi_1$ , which is actually the only rational point lying on these two lines.

Let  $U = V \setminus \{\ell_1 \cup \ell_2 \cup \ell_3\}$ , and  $B$  a parameter that can approach infinity. In this note we are concerned with the behavior of the counting function

$$N_U(B) = \#\{\mathbf{x} \in U(\mathbb{Q}) : H(\mathbf{x}) \leq B\},$$

where  $H$  is the anticanonical height function on  $V$  defined by

$$H(\mathbf{x}) := \max \left\{ |x_0|, \sqrt{x_1^2 + x_2^2}, |x_3| \right\} \quad (1.1)$$

where each  $x_j \in \mathbb{Z}$  and  $\gcd(x_0, x_1, x_2, x_3) = 1$ . The main result of this note is the following.

**Theorem 1.1.** *There exists a constant  $\vartheta > 0$  and a polynomial  $Q \in \mathbb{R}[X]$  of degree 3 such that*

$$N_U(B) = BQ(\log B) + O(B^{1-\vartheta}). \quad (1.2)$$

The leading coefficient  $C$  of  $Q$  satisfies

$$C = \frac{7}{216}(3\pi)\left(\frac{\pi}{4}\right)^3 \tau \quad (1.3)$$

with

$$\tau = \prod_p \left(1 - \frac{1}{p}\right)^4 \left(1 - \frac{\chi(p)}{p}\right)^3 \left(1 + \frac{2 + 3\chi(p) + 2\chi^2(p)}{p} + \frac{\chi^2(p)}{p^2}\right)$$

and  $\chi$  the non-principal character modulo 4. The constant  $C$  agrees with Peyre's prediction [29, Formule 5.1].

**Remark.** It follows from the arguments in [5] or [26] that, at least, any  $\vartheta < \frac{1}{9}$  is acceptable in Theorem 1.1, and further improvements are possible.

The Manin-Peyre conjectures for smooth toric varieties were established by Batyrev and Tschinkel in their seminal work [1]. Since our cubic surface  $V$  is a (non-split) toric surface, the main term of the asymptotic formula (1.2) can be derived from [1]. In addition to providing a different proof of the Manin-Peyre's conjectures for  $V$  and to getting a power-saving error term of the counting function  $N_U(B)$ , this note also serves to complement the results in [26], in which Manin's conjecture for the cubic hypersurfaces  $S_n \subset \mathbb{P}^{n+1}$  defined by the equation

$$x_0^3 = (x_1^2 + \dots + x_n^2)x_{n+1}$$

with  $n = 4k$  was established. The cubic surface  $V$  is the case for  $n = 2$ .

We conclude the introduction by a brief discussion of the split toric surface of  $\mathbb{P}_{\mathbb{Q}}^3$  given by

$$V' : x_0x_1x_2 = x_3^3.$$

The variety  $V'$  is isomorphic to  $V$  over  $\mathbb{Q}(i)$  and was well studied by a number of authors. Manin's conjecture for  $V'$  is a consequence of Batyrev and Tschinkel [1]. Others include the first author [5], the first author and Swinnerton-Dyer [8], Fouvry [19], Heath-Brown and Moroz [23] and Salberger [32]. Derenthal and Janda [15] established Manin's conjecture for  $V'$  over imaginary quadratic fields of class number one and Frei [21] further generalized their work to arbitrary number fields. Of the unconditional asymptotic formulae obtained, the strongest is the one in [5], which yields the estimate

$$N_U(B) = BP(\log B) + O(B^{7/8} \exp(-c(\log B)^{3/5}(\log \log B)^{-1/5})),$$

where  $U$  is a Zariski open subset of  $V'$ , and  $P$  is a polynomial of degree 6 and  $c$  is a positive constant. In [8], even the second term of the counting

function  $N_U(B)$  is established under the Riemann Hypothesis as well as the assumption that all the zeros of the Riemann  $\zeta$ -function are simple.

## 2. GEOMETRY AND PEYRE'S CONSTANT

In [28], Peyre proposed a general conjecture about the shape of the leading constant arising in the asymptotic formula for the number of points of bounded height but only for smooth Fano varieties.

The surface  $V$  that we study in this note is singular so we can not apply directly this conjecture and [28, Définition 2.1]. To get around this, we construct explicitly in this section a minimal resolution  $\pi : \tilde{V} \rightarrow V$  of  $V$  and show that for  $U = V \setminus \{\ell_1 \cup \ell_2 \cup \ell_3\}$  and  $\tilde{U} = \pi^{-1}(U)$ , we have  $\pi|_{\tilde{U}} : \tilde{U} \cong U$ . This implies that our counting problem on  $V$  can be seen as a counting problem on the smooth variety  $\tilde{V}$  since

$$N_U(B) = \#\{\mathbf{x} \in \tilde{U}(\mathbb{Q}) : H \circ \pi(\mathbf{x}) \leq B\}$$

where  $H \circ \pi$  is an anticanonical height function on  $\tilde{V}$ . Indeed, by [10, Lemma 1.1] the surface  $V$  has only du Val singularities which are canonical singularities (see [25, Theorem 4.20]) and alluding to [25, 2.26, 4.3, 4.4 and 4.5], we can conclude that  $\pi^*K_V = K_{\tilde{V}}$  where  $K_V$  and  $K_{\tilde{V}}$  denote the anticanonical divisors of  $V$  and  $\tilde{V}$  respectively. However,  $\tilde{V}$  is not a Fano variety and therefore we still can not apply [28, Définition 2.1].

We nevertheless establish in this section that  $\tilde{V}$  is “almost Fano” in the sense of [29, Definition 3.1]. Alluding to the fact that the original conjecture of Peyre has been refined by Batyrev and Tscinkel [2] and Peyre [29] to this setting, we may refer to [29, Formule empirique 5.1] to interpret the constant  $C$  arising in our Theorem 1.1. According to [29, Formule empirique 5.1], the leading constant  $C$  in our Theorem 1.1 takes the form

$$C = \alpha(\tilde{V})\beta(\tilde{V})\tau(\tilde{V}) \tag{2.1}$$

where  $\alpha(\tilde{V})$  is a rational number defined in terms of the cone of effective divisors,  $\beta(\tilde{V})$  a cohomological invariant and  $\tau(\tilde{V})$  a Tamagawa number. For more details, see définition 4.8 of [29].

Our main strategy to check that the constant  $C$  in Theorem 1.1 agrees with the prediction [29, Formule empirique 5.1] relies in a crucial way on the (non-split) toric structure of the surface  $V$  and on results from [2].

**2.1. Minimal resolution of  $V$  and interpretation of the power of  $\log B$ .** We refer the reader to the following references for details about toric varieties over arbitrary fields [27, 20, 13, 12] and especially [3, 1] and [32, End of §8].

The toric surface  $V$  is easily seen to be an equivariant compactification of the non-split torus  $T$  given by the equation  $x_0(x_1^2 + x_2^2) = 1$ . The torus  $T$  is isomorphic to  $R_{\mathbb{Q}(i)/\mathbb{Q}}(\mathbb{G}_m)$  where  $R_{\mathbb{Q}(i)/\mathbb{Q}}(\cdot)$  denotes the Weil restriction functor and is split by the quadratic extension  $k = \mathbb{Q}(i)$ . We now introduce  $M = \widehat{T}_k := \text{Hom}(T, k^\times)$  the group of regular  $k$ -rational characters of  $T$  and  $N = \text{Hom}(M, \mathbb{Z})$ . Alluding to [34, Lemma 1.3.1], we see that  $M \cong N \cong \mathbb{Z} \times \mathbb{Z}$  with the Galois group  $G = \text{Gal}(k/\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z}$  interchanging the two factors. Let  $(e_1, e_2)$  be a  $\mathbb{Z}$ -basis of  $N$ . In a similar manner as in [32, Example 11.50], we denote by  $\Delta$  the fan of  $N_{\mathbb{R}} = N \otimes \mathbb{R}$  given by the rays  $\rho_1, \rho_2, \rho'_2$  generated by  $-e_1 - e_2, -e_1 + 2e_2$  and  $2e_1 - e_2$ .

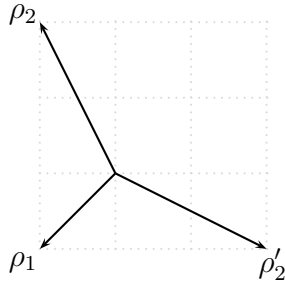


FIGURE 1. The fan  $\Delta$ .

The fan  $\Delta$  is  $G$ -invariant in the sense of [5, Definition 1.11] and hence defines a non-split toric surface  $P_\Delta$  over  $\mathbb{Q}$ . Using the same arguments as in [32, Example 11.50], one easily sees that the  $k$ -variety  $P_{\Delta,k} = P_\Delta \otimes_{\text{Spec}(\mathbb{Q})} \text{Spec}(k)$  is given by the equation  $x_3^3 = x_0 z_1 z_2$  with  $G$  exchanging  $z_1$  and  $z_2$ . The change of variables  $x_1 = (z_1 + z_2)/2$  and  $x_2 = (z_1 - z_2)/(2i)$  yields that  $P_{\Delta,k}$  is isomorphic to the variety of equation  $x_3^3 = x_0(x_1^2 + x_2^2)$ , all the variables being  $G$ -invariant. Hence, the surface  $V$  is a complete algebraic variety such that  $V \otimes_{\text{Spec}(\mathbb{Q})} \text{Spec}(k)$  is isomorphic to  $P_{\Delta,k}$ , the isomorphism being compatible with the  $G$ -actions. Then, theorem 1.12 of [1] allows us to conclude that  $V$  is given by the  $G$ -invariant fan  $\Delta$  after noting that the assumption that the fan is regular is not necessary.

The fan  $\Delta$  is not complete and regular in the sense of [2, Definition 1.9] which accounts for the fact that  $V$  is singular. As in [32, Example 11.50], there exists a complete and regular refinement  $\tilde{\Delta}$  of  $\Delta$  given by

the extra rays  $\tilde{\rho}_1, \tilde{\rho}_2, \tilde{\rho}_3, \tilde{\rho}'_1, \tilde{\rho}'_2, \tilde{\rho}'_3$  generated by  $-e_1, -e_1 + e_2, e_2, -e_2, e_1 - e_2$  and  $e_1$ .

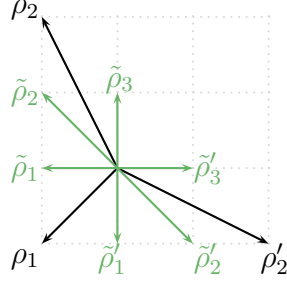


FIGURE 2. The fan  $\tilde{\Delta}$ .

The toric surface  $\tilde{V}$  defined over  $\mathbb{Q}$  by the  $G$ -invariant fan  $\tilde{\Delta}$  is then smooth by [1, Theorems 1.10 and 1.12] and, thanks to [12, 5.5.1] and [20, §2.6], comes with a proper equivariant birational morphism  $\pi : \tilde{V} \rightarrow V$  which is an isomorphism on the torus  $T$ . Here  $T$  corresponds to the open subset  $U = V \setminus \{\ell_1 \cup \ell_2 \cup \ell_3\}$ . Now the proof of the proposition 11.2.8 of [11] yields that  $\pi$  is a crepant resolution and hence that it is minimal since we are in dimension 2.

We note that thanks to [13, Corollaire 3] and [1, Proposition 1.15], the minimal resolution  $\tilde{V}$  is “almost Fano” in the sense of [29, Definition 3.1].

Let us now turn to the computation of the Picard group of  $\tilde{V}$ . To this end, we will exploit the exact sequence given by [1, Proposition 1.15]. With the notations of [1, Proposition 1.15], we have  $M^G \cong \mathbb{Z}$  generated by  $e_1^* + e_2^*$  if  $(e_1^*, e_2^*)$  is a  $\mathbb{Z}$ -basis of  $M$ . Moreover, a function  $\varphi \in \text{PL}(\tilde{\Delta})^G$  being completely determined by its integer values on  $\rho_1, \rho_2$  and  $\tilde{\rho}_1, \tilde{\rho}_2$  and  $\tilde{\rho}_3$ , we have  $\text{PL}(\tilde{\Delta})^G \cong \mathbb{Z}^5$ . Finally, the  $\mathbb{Z}$ -module  $M$  is a permutation module and therefore  $H^1(G, M)$  is trivial. Bringing all of that together yields that  $\text{rk}(\text{Pic}(\tilde{V})) = 5 - 1 = 4$ , which agrees with the prediction coming from Manin’s conjecture regarding the power of  $\log B$  in Theorem 1.1.

**2.2. The factor  $\alpha$ .** We will use the same method as in [4, Lemma 5] to compute the nef cone volume  $\alpha(\tilde{V})$  and we refer the reader to [4, Lemma 5] for more details and definitions.

Let  $T_i, T'_i, \tilde{T}_i$  and  $\tilde{T}'_i$  be the Zariski closures of the one dimensional tori corresponding respectively to the cones  $\mathbb{R}_{\geq 0}\rho_i, \mathbb{R}_{\geq 0}\rho'_i, \mathbb{R}_{\geq 0}\tilde{\rho}_i$  and  $\mathbb{R}_{\geq 0}\tilde{\rho}'_i$ . We also introduce the  $G$ -invariant divisors

$$D_1 = T_1, \quad D_2 = T_2 + T'_2, \quad D_3 = \tilde{T}_1 + \tilde{T}'_1, \quad D_4 = \tilde{T}_2 + \tilde{T}'_2, \quad D_5 = \tilde{T}_3 + \tilde{T}'_3.$$

Using [1, Proposition 1.15], one immediately sees that  $\text{Pic}(\tilde{V})$  is generated by  $D_1, D_2, D_3, D_4, D_5$  with the relation  $D_5 = 2D_1 + D_2 - D_4$  and that the divisor

$$D_1 + D_2 + D_3 + D_4 + D_5 \sim 3D_1 + 2D_2 + D_3$$

is an anticanonical divisor for  $\tilde{V}$ . Following the strategy of [4, Lemma 5] and using the same notations than in [4, Lemma 5], it now follows that  $C_{\text{eff}}^V$  is the subset of  $\mathbb{R}_{\geq 0}^4$  given by  $2z_1 + z_2 - z_4 \geq 0$  and that  $H_{\tilde{V}}$  is given by the equation  $3z_1 + 2z_2 + z_3 = 1$ . Therefore, a straightforward computation finally yields

$$\begin{aligned} \alpha(\tilde{V}) &= \int_0^1 (1 - z_3)^2 dz_3 \times \frac{1}{2} \text{Vol} \left\{ (z_1, z_4) \in \mathbb{R}_{\geq 0}^2 : \begin{array}{l} 3z_1 \leq 1, \\ 2z_4 - z_1 \leq 1 \end{array} \right\} \\ &= \frac{1}{6} \int_{z_1=0}^{\frac{1}{3}} \left( \int_{z_4=0}^{\frac{1+z_1}{2}} dz_4 \right) dz_1 = \frac{7}{216}. \end{aligned}$$

**2.3. The factor  $\beta$ .** Let us now briefly justify that  $\beta(\tilde{V}) = 1$ . We know that  $\tilde{V}$  is birational to the torus  $R_{\mathbb{Q}(i)/\mathbb{Q}}(\mathbb{G}_m)$ . But the open immersion  $\mathbb{G}_{m, \mathbb{Q}(i)} \hookrightarrow \mathbb{A}_{\mathbb{Q}(i)}^2$  gives rise to an open immersion  $R_{\mathbb{Q}(i)/\mathbb{Q}}(\mathbb{G}_m) \hookrightarrow \mathbb{A}_{\mathbb{Q}}^2$  by taking the functor  $R_{\mathbb{Q}(i)/\mathbb{Q}}(\cdot)$  and by alluding to [33, Proposition 4.9]. Hence,  $R_{\mathbb{Q}(i)/\mathbb{Q}}(\mathbb{G}_m)$  is rational and so is  $\tilde{V}$ . Finally this implies that  $\beta(\tilde{V}) = 1$  (see [4, section 5] for details).

#### 2.4. The Tamagawa number.

**2.4.1. Conjectural expression.** Let us choose  $S = \{\infty, 2\}$  and note that our definition will be independent of that choice. We have from [3, Theorem 1.3.2] that  $\text{Pic}(\tilde{V}_{\overline{\mathbb{Q}}})$  is the free abelian group generated by the divisors  $T_1, T_2, T_2', \tilde{T}_1, \tilde{T}_1', \tilde{T}_2, \tilde{T}_2'$  defined in §2.2 with the following  $G$ -action

$$\sigma(T_2) = T_2', \quad \sigma(\tilde{T}_i) = \tilde{T}_i' \quad (i \in \{1, 2\})$$

if  $\sigma$  denotes the complex conjugation. Alluding to [24, Definition 7.1], we have the following conjectural expression

$$\tau(\tilde{V}) := \lim_{s \rightarrow 1^+} (s-1)^4 L_S(s, \chi_{\text{Pic}(\tilde{V}_{\overline{\mathbb{Q}}})}) \omega_{\infty} \prod_p \lambda_p^{-1} \omega_p$$

where

$$L_S(s, \chi_{\text{Pic}(\tilde{V}_{\overline{\mathbb{Q}}})}) = \prod_{p \notin S} \det \left( \text{Id} - p^{-s} \text{Frob}_p \mid \text{Pic}(\tilde{V}_{\overline{\mathbb{Q}}})^{I_p} \right)^{-1}$$

with  $I_p$  the inertia group and  $\text{Frob}_p$  a representative of the Frobenius automorphism and where

$$\lambda_p = L_p(1, \chi_{\text{Pic}(\tilde{V}_{\mathbb{Q}})}), \quad \omega_\infty = \omega_{\infty, \tilde{V}}(\tilde{V}(\mathbb{R})), \quad \omega_p = \omega_{p, \tilde{V}}(\tilde{V}(\mathbb{Q}_p))$$

for measures  $\omega_{v, \tilde{V}}$  on  $\tilde{V}(\mathbb{Q}_v)$  whose proper definitions are postponed to the next section (they are the measures  $\omega_{\mathcal{X}, v}$  defined in [1, §2]) and where  $\lambda_p$  is taken to be 1 for  $p \in S$ .

Let  $\Re(s) > 1$ . First we notice that for all  $p \notin S$ , we have that  $I_p$  is trivial since  $p$  is not ramified in  $\mathbb{Q}(i)$ . Then the Frobenius  $\text{Frob}_p$  being trivial for all  $p \equiv 1 \pmod{4}$ , it is easy to see that in that case

$$\det \left( \text{Id} - p^{-s} \text{Frob}_p \mid \text{Pic}(\tilde{V}_{\mathbb{Q}})^{I_p} \right) = \left( 1 - \frac{1}{p^s} \right)^7.$$

When  $p \equiv 3 \pmod{4}$ ,  $\text{Frob}_p$  is of order 2 with the same action than  $\sigma$  on  $\text{Pic}(\tilde{V}_{\mathbb{Q}})$  and hence one sees immediately that

$$\det \left( \text{Id} - p^{-s} \text{Frob}_p \mid \text{Pic}(\tilde{V}_{\mathbb{Q}})^{I_p} \right) = \left( 1 - \frac{1}{p^s} \right) \left( 1 - \frac{1}{p^{2s}} \right)^3.$$

Bringing all of this together yields that

$$L_S(s, \chi_{\text{Pic}(\tilde{V}_{\mathbb{Q}})}) = \prod_{p>2} \left( 1 - \frac{1}{p^s} \right)^{-4} \left( 1 - \frac{\chi(p)}{p^s} \right)^{-3}$$

and

$$\lim_{s \rightarrow 1} (s-1)^4 L_S(s, \chi_{\text{Pic}(\tilde{V}_{\mathbb{Q}})}) = \frac{L(1, \chi)^2}{2^4} = \frac{1}{2^4} \times \left( \frac{\pi}{4} \right)^3$$

and therefore

$$\tau(\tilde{V}) = \left( \frac{\pi}{4} \right)^3 \omega_\infty \times \frac{\omega_2}{2^4} \times \prod_{p>2} \left( 1 - \frac{1}{p} \right)^4 \left( 1 - \frac{\chi(p)}{p} \right)^3 \omega_p.$$

**2.4.2. Construction of the Tamagawa measure.** Let us write here  $V_{(i)}$  for the affine subset of  $V$  where  $x_i \neq 0$  with coordinates  $x_j^{(i)} = x_j/x_i$  for  $j \neq i$ . Note that  $V_{(i)}$  is defined by the equation

$$f^{(i)}(x_0^{(i)}, \dots, \widehat{x_i^{(i)}}, \dots, x_3^{(i)}) = \left( \frac{x_3}{x_i} \right)^3 - \frac{x_0}{x_i} \left( \left( \frac{x_1}{x_i} \right)^2 + \left( \frac{x_2}{x_i} \right)^2 \right)$$

where  $(x_0^{(i)}, \dots, \widehat{x_i^{(i)}}, \dots, x_3^{(i)})$  denotes  $(x_0^{(i)}, x_2^{(i)}, x_3^{(i)})$  after removing the  $i$ -th component. The same arguments as in [22, §13] go through to



yield that  $\omega_V \cong \mathcal{O}_V(-1)$  and that such an isomorphism is given on  $V_{(i)}$  by

$$x_i^{-1} \mapsto \frac{(-1)^{i+t}}{\partial f^{(i)}/\partial x_j^{(i)}} dx_k^{(i)} \wedge dx_\ell^{(i)}$$

for  $k < \ell \in \{0, 1, 2, 3\} \setminus \{i\}$ ,  $\{i, j, k, \ell\} = \{0, 1, 2, 3\}$  and  $t = k + \ell$  if  $k < i < \ell$  and  $t = k + \ell - 1$  otherwise. Moreover, we have already seen that  $\omega_{\tilde{V}} \cong \pi^*\omega_V$ . The dual sections  $\tau_i$  of  $s_i$  in  $\omega_V^{-1} \cong \mathcal{O}_V(1)$  define the embedding  $V \hookrightarrow \mathbb{P}^3$  under consideration in this note and the morphism  $\tilde{V} \rightarrow V \hookrightarrow \mathbb{P}^3$  is given by the sections  $\pi^*\tau_i$  of  $H^0(\tilde{V}, \omega_{\tilde{V}}^{-1})$ .

Consider now the subsequent Arakelov heights  $(\omega_{\tilde{V}}^{-1}, (\|\cdot\|_v)_{v \in \text{Val}(\mathbb{Q})})$  and  $(\omega_V^{-1}, (\|\cdot\|'_v)_{v \in \text{Val}(\mathbb{Q})})$  defined by these global sections where for all  $v \in \text{Val}(\mathbb{Q})$ ,  $x \in V(\mathbb{Q}_v)$ ,  $y \in \tilde{V}(\mathbb{Q}_v)$ ,  $\tau \in \omega_{\tilde{V}}^{-1}$  and  $\sigma \in \omega_V^{-1}$  we use respectively the  $v$ -adic metrics defined by

$$\|\tau\|_v = \min_{\substack{0 \leq i \leq 3 \\ \pi^*\tau_i \neq 0}} \left\{ \left| \frac{\tau}{\pi^*\tau_i(x)} \right|_v \right\}$$

if  $v$  is finite and

$$\|\tau\|_\infty = \min \left\{ \min_{\substack{i \in \{0, 3\} \\ \pi^*\tau_i \neq 0}} \left\{ \left| \frac{\tau}{\pi^*\tau_i(x)} \right|_\infty \right\}, \left( \left| \frac{\pi^*\tau_1(x)}{\tau} \right|_\infty^2 + \left| \frac{\pi^*\tau_2(x)}{\tau} \right|_\infty^2 \right)^{-\frac{1}{2}} \right\}$$

if  $v$  is the archimedean place and  $\tau \neq 0$  and

$$\|\sigma\|'_v = \min_{\substack{0 \leq i \leq 3 \\ \tau_i \neq 0}} \left\{ \left| \frac{\sigma}{\tau_i(y)} \right|_v \right\}$$

if  $v$  is finite and

$$\|\sigma\|'_\infty = \min \left\{ \min_{\substack{i \in \{0, 3\} \\ \tau_i \neq 0}} \left\{ \left| \frac{\sigma}{\tau_i(y)} \right|_\infty \right\}, \left( \left| \frac{\tau_1(y)}{\sigma} \right|_\infty^2 + \left| \frac{\tau_2(y)}{\sigma} \right|_\infty^2 \right)^{-\frac{1}{2}} \right\}$$

if  $v$  is the archimedean place and  $\sigma \neq 0$ . These heights correspond to the heights  $H$  on  $V$  and  $H \circ \pi$  on  $\tilde{V}$  that we used in our counting problem. Applying the definition 2.2.1 of [29], we get a measure  $\omega_{v, \tilde{V}}$  on  $\tilde{V}(\mathbb{Q}_v)$  associated to the  $v$ -adic metric  $\|\cdot\|_v$  which is the measure defined in [1, §2] and used in §2.4.1 and a measure  $\omega_{v, V}$  on  $V(\mathbb{Q}_v)$  associated to the  $v$ -adic metric  $\|\cdot\|'_v$ .

**2.4.3. Computation of the archimedean density.** We follow once again the strategy adopted in [22, §13]. One sees easily that  $U = V_{(3)}$  and the same argument as in [22, §13] shows that

$$\omega_\infty = \omega_{\infty, \tilde{V}}(\tilde{V}(\mathbb{R})) = \omega_{\infty, \tilde{V}}(\pi^{-1}(U)(\mathbb{R})).$$

Now the local coordinates  $x_1^{(3)} - 1$  and  $x_2^{(3)}$  at the rational point  $(1, 1, 0)$  of  $U$  give an isomorphism

$$U(\mathbb{R}) \cong W = \{(z_1, z_2) \in \mathbb{R}^2 : (z_1 + 1)^2 + z_2^2 \neq 0\}$$

and a similar computation as in [22, §13] yields

$$\begin{aligned} \omega_{\infty, \tilde{V}}(\pi^{-1}(U)(\mathbb{R})) &= \int_{\mathbb{R}^2} \frac{dz_1 dz_2}{\max\{1, z_1^2 + z_2^2, (z_1^2 + z_2^2)^{3/2}\}} \\ &= \int_{z_1^2 + z_2^2 \leq 1} dz_1 dz_2 + \int_{z_1^2 + z_2^2 > 1} \frac{dz_1 dz_2}{(z_1^2 + z_2^2)^{3/2}} \\ &= 3\pi \end{aligned}$$

after a polar change of coordinates. We can therefore conclude that  $\omega_{\infty} = 3\pi$ .

2.4.4. *Computation of  $\omega_p$  for odd  $p$ .* Thanks to the remarks of [32, Page 187], one can construct a model  $\tilde{\mathcal{V}}$  over  $\text{Spec}(\mathbb{Z})$  satisfying the conditions of [29, Notation 4.5] with  $S = \{\infty, 2\}$ . Hence, one can consider the reduction  $\tilde{\mathcal{V}}_p$  modulo  $p$  of  $\tilde{\mathcal{V}}$  for every prime number  $p$ .

The torus  $T$  has good reduction  $T_p$  for every prime  $p > 2$  since  $p$  is not ramified in  $\mathbb{Q}(i)$  and  $T_p$  is a split torus of rank 2 if  $p \equiv 1 \pmod{4}$  and a non-split torus of rank 2 split by  $\mathbb{F}_{p^2}$  if  $p \equiv 3 \pmod{4}$ . Hence, the reduction  $\tilde{\mathcal{V}}_p$  modulo  $p$  can be realized as the toric variety over  $\mathbb{F}_p$  under the torus  $T_p$  given by the fan  $\Delta'$  which is invariant under  $\text{Frob}_p$ . Since the fan stays regular and complete, we can conclude that  $\tilde{\mathcal{V}}_p$  is smooth and hence that  $\tilde{\mathcal{V}}$  has good reduction modulo  $p > 2$  (see [12]).

We can therefore apply [24, Corollary 6.7] to obtain for all odd  $p$  the following expression

$$\omega_p = \frac{\#\tilde{\mathcal{V}}(\mathbb{F}_p)}{p^2}.$$

Now alluding to Weil's formula, we obtain

$$\omega_p = 1 + \frac{\text{Tr}(\text{Frob}_p | \text{Pic}(\tilde{V}_{\overline{\mathbb{Q}}}))}{p} + \frac{1}{p^2} = 1 + \frac{4 + 3\chi(p)}{p} + \frac{1}{p^2}$$

by using the description of the action of  $\text{Frob}_p$  on  $\text{Pic}(\tilde{V}_{\overline{\mathbb{Q}}})$  given in §2.4.1.

2.4.5. *Computation of  $\omega_2$ .* For  $p = 2$ , the model  $\tilde{\mathcal{V}}$  having bad reduction, we appeal to the lemma 6.6 of [24] to compute  $\omega_2$ . By [1, Proposition 2.10] and noting that the smooth assumption is not necessary, one gets

$$\omega_{2, \tilde{V}}(\tilde{V}(\mathbb{Q}_2)) = \omega_{2, \tilde{V}}(\pi^{-1}(U)), \quad \omega_{2, V}(V(\mathbb{Q}_2)) = \omega_{2, V}(U).$$

Now an analogous computation as the one in §2.4.3 yields that both quantities  $\omega_{2,\tilde{V}}(\pi^{-1}(U))$  and  $\omega_{2,V}(U)$  are equal to the expression

$$\int_W \frac{dz_1 dz_2}{\max\{1, |z_1^2 + z_2^2|_2, |z_1(z_1^2 + z_2^2)|_2, |z_2(z_1^2 + z_2^2)|_2\}}$$

with  $W = \{(z_1, z_2) \in \mathbb{Q}_2^2 : (z_1 + 1)^2 + z_2^2 \neq 0\}$ . Therefore,  $\omega_2$  is equal to  $\omega_{2,V}(V(\mathbb{Q}_2))$  and [24, Remark 6.8] implies that

$$\omega_2 = \lim_{n \rightarrow +\infty} \frac{N(2^n)}{2^{3n}},$$

where

$$N(2^n) := \#\{\mathbf{x} \pmod{2^n} : x_0(x_1^2 + x_2^2) \equiv x_3^3 \pmod{2^n}\}.$$

Let  $v_2(x_1^2 + x_2^2) = k$  and  $v_2(x_0) = k_0$ . If  $k = 1 + 2k'$  is odd and  $1 + 2k' < n$ , then the number of  $(x_1, x_2)$  satisfying  $v_2(x_1^2 + x_2^2) = k$  is  $2^{2n-2k'-2}$ . There are  $2^{n-(1+2k'+k_0)/3-1}$  ways to choose  $x_3$  and then  $2^{2k'+1}$  choices for  $x_0$ . Then, in the case where  $v_2(x_1^2 + x_2^2)$  is odd, the number of solutions is asymptotic to

$$2^{3n} \sum_{\substack{3|1+2k'+k_0 \\ k' \geq 0}} 2^{-2-(1+2k'+k_0)/3} \sim \frac{5}{6} 2^{3n}.$$

The number of  $(x_1, x_2)$  satisfying  $v_2(x_1^2 + x_2^2) = 2k'$  is, at least for  $2k' < n$ , equal to  $2^{2n-2k'-1}$ . There are  $2^{n-(2k'+k_0)/3-1}$  ways to choose  $x_3$  and then  $2^{2k'}$  choices for  $x_0$ . Summing over  $3 | 2k' + k_0$  and  $k' \geq 0$  we get the contribution of the case  $v_2(x_1^2 + x_2^2)$  even in  $N(2^n)$ , which is asymptotic to  $\frac{7}{6} \cdot 2^{3n}$ . It follows that

$$\omega_2 = 2 = 1 + \frac{2 + 3\chi(2) + 2\chi^2(2)}{2} + \frac{\chi^2(2)}{2^2}.$$

2.4.6. *Conclusion.* Bringing everything together yields the following expression for the Peyre constant

$$\alpha(\tilde{V})\beta(\tilde{V})\tau(\tilde{V}) = \frac{7}{216}(3\pi)\left(\frac{\pi}{4}\right)^3 \tau.$$

This is in agreement with the constant  $C$  in (1.3).

### 3. PROOF OF THEOREM 1.1

By symmetry, we have

$$N_U(B) = \#\left\{\mathbf{x} \in E : x_0(x_1^2 + x_2^2) = x_3^3, \max\left\{x_0, \sqrt{x_1^2 + x_2^2}\right\} \leq B\right\},$$

where  $E := \{\mathbf{x} \in \mathbb{N} \times \mathbb{Z}^2 \times \mathbb{N} : \gcd(x_0, x_1, x_2, x_3) = 1\}$  and  $\mathbb{N} = \mathbb{Z}_{\geq 1}$ . As in [5], we parametrize  $x_1^2 + x_2^2$ ,  $x_0$  and  $x_3$  by

$$x_1^2 + x_2^2 = n_1 n_2^2 n_3^3, \quad x_0 = n_1^2 n_2 n_4^3, \quad x_3 = n_1 n_2 n_3 n_4,$$

where  $n_1$  and  $n_2$  are squarefree and  $\gcd(n_1, n_2) = 1$  which is equivalent to  $\mu^2(n_1 n_2) = 1$ . It follows that

$$N_U(B) = 4 \sum_{\substack{\mathbf{n} \in \mathbb{N}^4 \\ \mu^2(n_1 n_2) = 1 \\ n_1^2 n_2 n_4^3 \leq B \\ n_1 n_2^2 n_3^3 \leq B^2}} r(n_1 n_2^2 n_3^3, n_1 n_2 n_4)$$

where

$$r(n, m) := \frac{1}{4} \# \{(x_1, x_2) \in \mathbb{Z}^2 : x_1^2 + x_2^2 = n, ((x_1, x_2), m) = 1\}.$$

Here, we remark that our choice of height function is particularly well suited to handle the expression  $r(n, m)$ .

Let  $\chi$  be the non-principal character modulo 4 and  $r_0 := 1 * \chi$ . The quantity  $r(n, m)$  is a multiplicative arithmetic function in  $n$ , and we have

$$r(n, m) := \prod_p r(p^{v_p(n)}, p^{v_p(m)}).$$

We use the fact that, when  $\nu \geq 1$ ,

$$r(p^\nu, p) = \begin{cases} 2 & \text{if } p \equiv 1 \pmod{4}, \\ 0 & \text{if } p \equiv 3 \pmod{4}, \\ 1 & \text{if } \nu = 1, p = 2, \\ 0 & \text{if } \nu \geq 2, p = 2. \end{cases}$$

Then, when  $\nu_1 + \nu_2 \leq 1$ , the value of  $r(p^{\nu_1+2\nu_2+3\nu_3}, p^{\nu_1+\nu_2+\nu_4})$  is given by

$$\left\{ \begin{array}{ll} 1 & \text{if } (\nu_1, \nu_2, \nu_3, \nu_4) = (0, 0, 0, \nu_4), \\ r_0(p^{3\nu_3}) & \text{if } (\nu_1, \nu_2, \nu_3, \nu_4) = (0, 0, \nu_3, 0), \\ 0 & \text{if } (\nu_1, \nu_2) = (0, 0), \min\{\nu_3, \nu_4\} \geq 1, p \equiv 2, 3 \pmod{4}, \\ 2 & \text{if } (\nu_1, \nu_2) = (0, 0), \min\{\nu_3, \nu_4\} \geq 1, p \equiv 1 \pmod{4}, \\ 0 & \text{if } \min\{\nu_1, \nu_3\} \geq 1, p \equiv 2, 3 \pmod{4}, \\ 2 & \text{if } \min\{\nu_1, \nu_3\} \geq 1, p \equiv 1 \pmod{4}, \\ 0 & \text{if } \nu_1 = 1, \nu_3 = 0, p \equiv 3 \pmod{4}, \\ 2 & \text{if } \nu_1 = 1, \nu_3 = 0, p \equiv 1 \pmod{4}, \\ 1 & \text{if } \nu_1 = 1, \nu_3 = 0, p = 2, \\ 0 & \text{if } \nu_2 = 1, p \equiv 2, 3 \pmod{4}, \\ 2 & \text{if } \nu_2 = 1, p \equiv 1 \pmod{4}. \end{array} \right.$$

The Dirichlet series associated to this counting problem is

$$F(s_1, s_2) := \sum_{\substack{\mathbf{n} \in \mathbb{N}^4 \\ \mu^2(n_1 n_2) = 1}} \frac{r(n_1 n_2^2 n_3^3, n_1 n_2 n_4)}{n_1^{2s_1+s_2} n_2^{s_1+2s_2} n_3^{3s_2} n_4^{3s_1}}, \quad \left( \Re(s_1), \Re(s_2) > \frac{1}{3} \right).$$

It can be written as an Euler product of  $F_p(s_1, s_2)$ , where

$$F_2(s_1, s_2) = \frac{1}{1 - 2^{-3s_1}} + \frac{1}{2^{3s_2} - 1} + \frac{1}{2^{2s_1+s_2}(1 - 2^{-3s_1})},$$

$$F_p(s_1, s_2) = \frac{1}{1 - p^{-3s_1}} + \frac{1}{p^{6s_2} - 1},$$

if  $p \equiv 3 \pmod{4}$  and

$$F_p(s_1, s_2) = \frac{1}{1 - p^{-3s_1}} + \frac{4 - p^{-3s_2}}{p^{3s_2}(1 - p^{-3s_2})^2} + 2 \frac{p^{-3(s_1+s_2)} + p^{-(2s_1+s_2)} + p^{-(s_1+2s_2)}}{(1 - p^{-3s_2})(1 - p^{-3s_1})},$$

if  $p \equiv 1 \pmod{4}$ . For  $\Re(s) > 1$ , let

$$\begin{aligned} \zeta_{\mathbb{Q}(i)}(s) &:= \sum_{n \geq 1} \frac{r_0(n)}{n^s} = \zeta(s) L(s, \chi) \\ &= \frac{1}{1 - 2^{-s}} \prod_{p \equiv 3 \pmod{4}} \frac{1}{1 - p^{-2s}} \prod_{p \equiv 1 \pmod{4}} \frac{1}{(1 - p^{-s})^2}. \end{aligned}$$

Let  $\mathbf{s}$  stand for the pair  $(s_1, s_2)$ . Then there exists  $G$  such that

$$F(\mathbf{s}) = \zeta(3s_1) \zeta_{\mathbb{Q}(i)}(3s_2)^2 \zeta_{\mathbb{Q}(i)}(s_1 + 2s_2) \zeta_{\mathbb{Q}(i)}(s_2 + 2s_1) G(\mathbf{s}).$$

The above quantity  $G(\mathbf{s})$  can be written as an Euler product of  $G_p(\mathbf{s})$  where

$$G_2(1/3, 1/3) = 2^{-3}$$

while, for  $p \equiv 3 \pmod{4}$

$$G_p(\mathbf{s}) = (1 - p^{-2(s_1+2s_2)})^2 (1 - p^{-2(2s_1+s_2)}) (1 - p^{-3(s_1+2s_2)}).$$

and for  $p \equiv 1 \pmod{4}$ ,

$$\begin{aligned} G_p(\mathbf{s}) &= (1 - p^{-3s_2})^4 (1 - p^{-(s_1+2s_2)})^2 (1 - p^{-(2s_1+s_2)})^2 \\ &\quad + (1 - p^{-3s_1}) (4p^{-3s_2} - p^{-6s_2}) \\ &\quad \times (1 - p^{-3s_2})^2 (1 - p^{-(s_1+2s_2)})^2 (1 - p^{-(2s_1+s_2)})^2 \\ &\quad + 2(p^{-3(s_1+s_2)} + p^{-(2s_1+s_2)} + p^{-(s_1+2s_2)}) \\ &\quad \times (1 - p^{-3s_2})^3 (1 - p^{-(s_1+2s_2)})^2 (1 - p^{-(2s_1+s_2)})^2. \end{aligned}$$

The series  $F$  is absolutely convergent when  $\Re(s_1) > \frac{1}{3}$  and  $\Re(s_2) > \frac{1}{3}$  and the function  $G$  can be analytically continued to  $\Re(s_1) > \frac{1}{6}$  and  $\Re(s_2) > \frac{1}{6}$ . Moreover, we have

$$\begin{aligned} G\left(\frac{1}{3}, \frac{1}{3}\right) &= \frac{1}{2^3} \prod_{p \neq 2} \left(1 - \frac{1}{p}\right)^4 \left(1 - \frac{\chi(p)}{p}\right)^3 \left(1 + \frac{4 + 3\chi(p)}{p} + \frac{1}{p^2}\right) \\ &= \tau. \end{aligned} \tag{3.1}$$

Thus  $F$  satisfies the assumptions of Theorem 1 of [6] with  $(\beta_1, \beta_2) = (1, 2)$ ,  $(\alpha_1, \alpha_2) = (\frac{1}{3}, \frac{1}{3})$ ,

$$\begin{aligned} \ell_1(\mathbf{s}) &= 3s_1, \quad \ell_2(\mathbf{s}) = \ell_3(\mathbf{s}) = 3s_2, \\ \ell_4(\mathbf{s}) &= s_1 + 2s_2, \quad \ell_5(\mathbf{s}) = 2s_1 + s_2. \end{aligned}$$

It follows that there exists a constant  $\vartheta > 0$  and a polynomial  $Q \in \mathbb{R}[X]$  of degree 3 such that

$$N_U(B) = BQ(\log B) + O(B^{1-\vartheta}).$$

Now alluding to Theorem 2 of [6] to get the leading coefficient  $C$  of  $Q$ , we obtain

$$\begin{aligned} Q(\log B) &\underset{B \rightarrow +\infty}{\sim} \frac{4L(1, \chi)^4 G\left(\frac{1}{3}, \frac{1}{3}\right)}{B} \int_{\substack{(y_1, y_2, y_3, y_4, y_5) \in [1, +\infty[^5 \\ y_1^3 y_4 y_5^2 \leq B, y_2^3 y_3^3 y_4^2 y_5 \leq B^2}} \mathbf{dy} \\ &\underset{B \rightarrow +\infty}{\sim} 4 \left(\frac{\pi}{4}\right)^4 G\left(\frac{1}{3}, \frac{1}{3}\right) \int_{\substack{(y_3, y_4, y_5) \in [1, +\infty[^3 \\ y_4 y_5^2 \leq B, y_3^3 y_4^2 y_5 \leq B^2}} \frac{\mathbf{dy}}{y_3 y_4 y_5} \\ &\underset{B \rightarrow +\infty}{\sim} \frac{\pi^4}{2^6} G\left(\frac{1}{3}, \frac{1}{3}\right) (\log B)^3 I, \end{aligned}$$

where

$$I := \text{vol}\{(t_3, t_4, t_5) \in \mathbb{R}_+^3 : t_4 + 2t_5 \leq 1, 3t_3 + 2t_4 + t_5 \leq 2\}.$$

An straightforward computation immediately yields

$$I = \frac{1}{3} \int_0^{1/2} \int_0^{1-2t_5} (2 - 2t_4 - t_5) dt_4 dt_5 = \frac{7}{72},$$

and therefore the leading coefficient  $C$  of  $Q$  is given by

$$C = \frac{7}{216} \left(\frac{\pi}{4}\right)^3 (3\pi) G\left(\frac{1}{3}, \frac{1}{3}\right).$$

By (3.1) we have  $G(\frac{1}{3}, \frac{1}{3}) = \tau$ , from which (1.3) follows. This completes the proof.

#### 4. THE DESCENT ARGUMENT

Our main argument in order to derive Theorem 1.1 in section 3 consists of a descent from our original variety  $\tilde{V}$  onto the variety of equation

$$x_1^2 + x_2^2 = n_1 n_2^2 n_3^3.$$

Although this is not required to verify Peyre's conjecture since  $\tilde{V}$  is a rational variety, it is particularly interesting to find out which torsor were used during this descent argument because  $\tilde{V}$  is a non-split variety. Indeed, as mentioned in [17], versal torsors parametrizations (see [9] for precise definitions) are mostly used in the case of split varieties and the question of the right approach in the case of non-split varieties is quite natural. Using the Cox ring machinery over nonclosed fields developed in [17], all known examples of Manin's conjecture in the case of non-split varieties derived by means of a descent rely on a descent on quasi-versal torsors in the sense of [9]. For example, the descent in [18] is a descent on torsors of injective type  $\text{Pic}(V_{\mathbb{Q}(i)}) \hookrightarrow \text{Pic}(V_{\overline{\mathbb{Q}}})$  whereas it is shown in [17] that the *ad hoc* descent used in [7] is a descent on the torsor of injective type  $\text{Pic}(V) \hookrightarrow \text{Pic}(V_{\overline{\mathbb{Q}}})$ . Here, we now show in the following lemma that the descent corresponds to a torsor of a different type, which is not quasi-versal.

With the notations of §2.2, we set  $\hat{T} = [D_1]\mathbb{Z} \oplus [D_3]\mathbb{Z} \oplus [D_4]\mathbb{Z}$  and  $\lambda : \hat{T} \hookrightarrow \text{Piv}(\tilde{V}_{\overline{\mathbb{Q}}})$  be the natural embedding.

**Lemma 4.1.** *Every Cox ring of injective type  $\lambda$  is isomorphic to the  $\mathbb{Q}$ -algebra*

$$R = \mathbb{Q}[x_1, x_2, \eta_1, \eta_2, \eta_3, \eta_4] / (x_1^2 + x_2^2 - \eta_2 \eta_3^2 \eta_4^3).$$

*Proof.* The proof is very similar to the one in [31, Proposition 2.71] and that is why we will not repeat all the details here. Since  $\tilde{V}_{\mathbb{Q}}$  is a split toric variety, we know by [32] that a Cox ring of identity type for  $\tilde{V}_{\mathbb{Q}}$  is given by

$$\overline{\mathcal{R}} = \overline{\mathbb{Q}}[t_1, t_2, t_2', \tilde{t}_1, \tilde{t}_1', \tilde{t}_2, \tilde{t}_2', \tilde{t}_3, \tilde{t}_3']$$

where  $t_i = \text{div}(T_i)$ ,  $t_i' = \text{div}(T_i')$ ,  $\tilde{t}_i = \text{div}(\tilde{T}_i)$  and  $\tilde{t}_i' = \text{div}(\tilde{T}_i')$ . We then have by [31, Remark 2.51] that every Cox ring of injective type  $\lambda$  is isomorphic to the ring of invariant of

$$\bigoplus_{m \in \hat{T}} \overline{R}_m$$

where  $\overline{R}_m$  is the vector space generated by the degree  $m$  elements of  $\overline{\mathcal{R}}$ . For  $m \in \hat{T}$  given by  $m = [a_1 D_1 + a_3 D_3 + a_4 D_4]$ , we have to solve the following linear system with  $e_i, e_i', \tilde{e}_i, \tilde{e}_i' \geq 0$  to determine  $\overline{R}_m$

$$\begin{aligned} \left[ e_1 T_1 + e_2 T_2 + e_2' T_2' + \tilde{e}_1 \tilde{T}_1 + \tilde{e}_1' \tilde{T}_1' + \tilde{e}_2 \tilde{T}_2 + \tilde{e}_2' \tilde{T}_2' + \tilde{e}_3 \tilde{T}_3 + \tilde{e}_3' \tilde{T}_3' \right] \\ = [a_1 D_1 + a_3 D_3 + a_4 D_4]. \end{aligned}$$

Alluding to the fan  $\Delta'$  and [1, Proposition 1.15], we get that this linear system is equivalent to

$$\begin{cases} \tilde{e}_3' + \tilde{e}_1 = \tilde{e}_3 + \tilde{e}_1' \\ \tilde{e}_2' + \tilde{e}_3 - \tilde{e}_3' = \tilde{e}_2 + \tilde{e}_3' - \tilde{e}_3 \\ e_2 + \tilde{e}_3' - 2\tilde{e}_3 = e_2' + \tilde{e}_3 - 2\tilde{e}_3' = 0. \end{cases}$$

This easily yields that  $\overline{R}$  is generated by

$$\eta_1 = t_1, \quad \eta_2 = \tilde{t}_1 \tilde{t}_1', \quad \eta_3 = \tilde{t}_2 \tilde{t}_2', \quad \eta_4 = t_2 t_2' \tilde{t}_3 \tilde{t}_3', \quad \eta_5 = \tilde{t}_1 \tilde{t}_2^2 \tilde{t}_3^3 \tilde{t}_3',$$

and  $\overline{\eta}_5$  the conjugate of  $\eta_5$  with the relation

$$\eta_5 \overline{\eta}_5 = \eta_2 \eta_3^2 \eta_4^3.$$

Using the Galois invariant variables

$$x_1 = \frac{\eta_5 + \overline{\eta}_5}{2}, \quad x_2 = \frac{\eta_5 - \overline{\eta}_5}{2i}$$

one finally ensures that every Cox ring of injective type  $\lambda$  is isomorphic to  $R$ .  $\square$

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