On a certain non-split cubic surface

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ON A CERTAIN NON-SPLIT CUBIC SURFACE

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Abstract. In this note, we establish an asymptotic formula with a power-saving error term for the number of rational points of bounded height on the singular cubic surface of $\mathbb{P}^3_{\mathbb{Q}}$

$$x_0(x_1^2 + x_2^2) = x_3^3$$

in agreement with the Manin-Peyre conjectures.

1. Introduction and results

Let $V \subset \mathbb{P}^3_{\mathbb{Q}}$ be the cubic surface defined by

$$x_0(x_1^2 + x_2^2) - x_3^3 = 0.$$

The surface V has three singular points $\xi_1 = [1 : 0 : 0 : 0]$, $\xi_2 = [0 : 1 : i : 0]$ and $\xi_3 = [0 : 1 : -i : 0]$. It is easy to see that the only three lines contained in $V_{\overline{\mathbb{Q}}} = V \times_{\mathrm{Spec}(\mathbb{Q})} \mathrm{Spec}(\overline{\mathbb{Q}})$ are

$$\ell_1 := \{x_3 = x_1 - ix_2 = 0\}, \quad \ell_2 := \{x_3 = x_1 + ix_2 = 0\},$$

and

$$\ell_3 := \{x_3 = x_0 = 0\}.$$

Clearly both ℓ_1 and ℓ_2 pass through ξ_1, which is actually the only rational point lying on these two lines.

Let $U = V \smallsetminus \{\ell_1 \cup \ell_2 \cup \ell_3\}$, and B a parameter that can approach infinity. In this note we are concerned with the behavior of the counting function

$$N_U(B) = \#\{\mathbf{x} \in U(\mathbb{Q}) : H(\mathbf{x}) \leqslant B\},$$

where H is the anticanonical height function on V defined by

$$H(\mathbf{x}) := \max\left\{|x_0|, \sqrt{x_1^2 + x_2^2}, |x_3|\right\} \tag{1.1}$$

where each $x_j \in \mathbb{Z}$ and $\gcd(x_0, x_1, x_2, x_3) = 1$. The main result of this note is the following.

Theorem 1.1. *There exists a constant $\vartheta > 0$ and a polynomial $Q \in \mathbb{R}[X]$ of degree 3 such that*

$$N_U(B) = BQ(\log B) + O(B^{1-\vartheta}). \tag{1.2}$$

The leading coefficient C of Q satisfies

$$C = \frac{7}{216}(3\pi)\left(\frac{\pi}{4}\right)^3 \tau \tag{1.3}$$

with

$$\tau = \prod_p \left(1 - \frac{1}{p}\right)^4 \left(1 - \frac{\chi(p)}{p}\right)^3 \left(1 + \frac{2 + 3\chi(p) + 2\chi^2(p)}{p} + \frac{\chi^2(p)}{p^2}\right)$$

and χ the non-principal character modulo 4. *The constant C agrees with Peyre's prediction* [29, Formule 5.1].

Remark. If follows from the arguments in [5] or [26] that, at least, any $\vartheta < \frac{1}{9}$ is acceptable in Theorem 1.1, and further improvements are possible.

The Manin-Peyre conjectures for smooth toric varieties were established by Batyrev and Tschinkel in their seminal work [1]. Since our cubic surface V is a (non-split) toric surface, the main term of the asymptotic formula (1.2) can be derived from [1]. In addition to providing a different proof of the Manin-Peyre's conjectures for V and to getting a power-saving error term of the counting function $N_U(B)$, this note also serves to complement the results in [26], in which Manin's conjecture for the cubic hypersurfaces $S_n \subset \mathbb{P}^{n+1}$ defined by the equation

$$x_0^3 = (x_1^2 + \ldots + x_n^2)x_{n+1}$$

with $n = 4k$ was established. The cubic surface V is the case for $n = 2$.

We conclude the introduction by a brief discussion of the split toric surface of $\mathbb{P}^3_{\mathbb{Q}}$ given by

$$V' : \quad x_0x_1x_2 = x_3^3.$$

The variety V' is isomorphic to V over $\mathbb{Q}(i)$ and was well studied by a number of authors. Manin's conjecture for V' is a consequence of Batyrev and Tschinkel [1]. Others include the first author [5], the first author and Swinnerton-Dyer [8], Fouvry [19], Heath-Brown and Moroz [23] and Salberger [32]. Derenthal and Janda [15] established Manin's conjecture for V' over imaginary quadratic fields of class number one and Frei [21] further generalized their work to arbitrary number fields. Of the unconditional asymptotic formulae obtained, the strongest is the one in [5], which yields the estimate

$$N_U(B) = BP(\log B) + O\big(B^{7/8}\exp(-c(\log B)^{3/5}(\log\log B)^{-1/5})\big),$$

where U is a Zariski open subset of V', and P is a polynomial of degree 6 and c is a positive constant. In [8], even the second term of the counting function $N_U(B)$ is established under the Riemann Hypothesis as well as the assumption that all the zeros of the Riemann ζ-function are simple.

2. Geometry and Peyre's constant

In [28], Peyre proposed a general conjecture about the shape of the leading constant arising in the asymptotic formula for the number of points of bounded height but only for smooth Fano varieties.

The surface V that we study in this note is singular so we can not apply directly this conjecture and [28, Définition 2.1]. To get around this, we construct explicitly in this section a minimal resolution $\pi : \widetilde{V} \to V$ of V and show that for $U = V \smallsetminus \{\ell_1 \cup \ell_2 \cup \ell_3\}$ and $\widetilde{U} = \pi^{-1}(U)$, we have $\pi_{|\widetilde{U}} : \widetilde{U} \cong U$. This implies that our counting problem on V can be seen as a counting problem on the smooth variety $\widetilde{V}$ since

$$N_U(B) = \#\{\mathbf{x} \in \widetilde{U}(\mathbb{Q}) : H \circ \pi(\mathbf{x}) \leqslant B\}$$

where $H \circ \pi$ is an anticanonical height function on $\widetilde{V}$. Indeed, by [10, Lemma 1.1] the surface V has only du Val singularities which are canonical singularities (see [25, Theorem 4.20]) and alluding to [25, 2.26, 4.3, 4.4 and 4.5], we can conclude that $\pi^* K_V = K_{\widetilde{V}}$ where K_V and $K_{\widetilde{V}}$ denote the anticanonical divisors of V and $\widetilde{V}$ respectively. However, $\widetilde{V}$ is not a Fano variety and therefore we still can not apply [28, Définition 2.1].

We nevertheless establish in this section that $\widetilde{V}$ is "almost Fano" in the sense of [29, Definition 3.1]. Alluding to the fact that the original conjecture of Peyre has been refined by Batyrev and Tscinkel [2] and Peyre [29] to this setting, we may refer to [29, Formule empirique 5.1] to interpret the constant C arising in our Theorem 1.1. According to [29, Formule empirique 5.1], the leading constant C in our Theorem 1.1 takes the form

$$C = \alpha(\widetilde{V})\beta(\widetilde{V})\tau(\widetilde{V}) \tag{2.1}$$

where $\alpha(\widetilde{V})$ is a rational number defined in terms of the cone of effective divisors, $\beta(\widetilde{V})$ a cohomological invariant and $\tau(\widetilde{V})$ a Tamagawa number. For more details, see définition 4.8 of [29].

Our main strategy to check that the constant C in Theorem 1.1 agrees with the prediction [29, Formule empirique 5.1] relies in a crucial way on the (non-split) toric structure of the surface V and on results from [2].

2.1. **Minimal resolution of V and interpretation of the power of $\log B$.** We refer the reader to the following references for details about toric varieties over arbitrary fields [27, 20, 13, 12] and especially [3, 1] and [32, End of §8].

The toric surface V is easily seen to be an equivariant compactification of the non-split torus T given by the equation $x_0(x_1^2+x_2^2)=1$. The torus T is isomorphic to $R_{\mathbb{Q}(i)/\mathbb{Q}}(\mathbb{G}_m)$ where $R_{\mathbb{Q}(i)/\mathbb{Q}}(\cdot)$ denotes the Weil restriction functor and is split by the quadratic extension $k=\mathbb{Q}(i)$. We now introduce $M=\widehat{T}_k:=\mathrm{Hom}(T,k^\times)$ the group of regular k-rational characters of T and $N=\mathrm{Hom}(M,\mathbb{Z})$. Alluding to [34, Lemma 1.3.1], we see that $M\cong N\cong\mathbb{Z}\times\mathbb{Z}$ with the Galois group $G=\mathrm{Gal}(k/\mathbb{Q})\cong\mathbb{Z}/2\mathbb{Z}$ interchanging the two factors. Let (e_1,e_2) be a $\mathbb{Z}$-basis of N. In a similar manner as in [32, Example 11.50], we denote by Δ the fan of $N_{\mathbb{R}}=N\otimes\mathbb{R}$ given by the rays ρ_1,ρ_2,ρ_2' generated by $-e_1-e_2$, $-e_1+2e_2$ and $2e_1-e_2$.

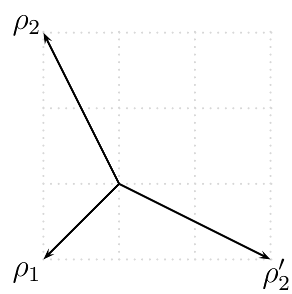

FIGURE 1. The fan Δ.

The fan Δ is G-invariant in the sense of [5, Definition 1.11] and hence defines a non-split toric surface P_Δ over $\mathbb{Q}$. Using the same arguments as in [32, Example 11.50], one easily sees that the k-variety $P_{\Delta,k}=P_\Delta\otimes_{\mathrm{Spec}(\mathbb{Q})}\mathrm{Spec}(k)$ is given by the equation $x_3^3=x_0z_1z_2$ with G exchanging z_1 and z_2. The change of variables $x_1=(z_1+z_2)/2$ and $x_2=(z_1-z_2)/(2i)$ yields that $P_{\Delta,k}$ is isomorphic to the variety of equation $x_3^3=x_0(x_1^2+x_2^2)$, all the variables being G-invariant. Hence, the surface V is a complete algebraic variety such that $V\otimes_{\mathrm{Spec}(\mathbb{Q})}\mathrm{Spec}(k)$ is isomorphic to $P_{\Delta,k}$, the isomorphism being compatible with the G-actions. Then, theorem 1.12 of [1] allows us to conclude that V is given by the G-invariant fan Δ after noting that the assumption that the fan is regular is not necessary.

The fan Δ is not complete and regular in the sense of [2, Definition 1.9] which accounts for the fact that V is singular. As in [32, Example 11.50], there exists a complete and regular refinement $\tilde{\Delta}$ of Δ given by the extra rays $\tilde{\rho}_1, \tilde{\rho}_2, \tilde{\rho}_3, \tilde{\rho}'_1, \tilde{\rho}'_2, \tilde{\rho}'_3$ generated by $-e_1$, $-e_1+e_2$, e_2, $-e_2$, e_1-e_2 and e_1.

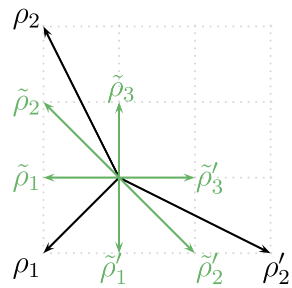

FIGURE 2. The fan $\tilde{\Delta}$.

The toric surface $\widetilde{V}$ defined over $\mathbb{Q}$ by the G-invariant fan $\tilde{\Delta}$ is then smooth by [1, Theorems 1.10 and 1.12] and, thanks to [12, 5.5.1] and [20, §2.6], comes with a proper equivariant birational morphism $\pi : \widetilde{V} \to V$ which is an isomorphism on the torus T. Here T corresponds to the open subset $U = V \smallsetminus \{\ell_1 \cup \ell_2 \cup \ell_3\}$. Now the proof of the proposition 11.2.8 of [11] yields that π is a crepant resolution and hence that it is minimal since we are in dimension 2.

We note that thanks to [13, Corollaire 3] and [1, Proposition 1.15], the minimal resolution $\widetilde{V}$ is "almost Fano" in the sense of [29, Definition 3.1].

Let us now turn to the computation of the Picard group of $\widetilde{V}$. To this end, we will exploit the exact sequence given by [1, Proposition 1.15]. With the notations of [1, Proposition 1.15], we have $M^G \cong \mathbb{Z}$ generated by $e_1^*+e_2^*$ if (e_1^*, e_2^*) is a $\mathbb{Z}$-basis of M. Moreover, a function $\varphi \in \mathrm{PL}(\tilde{\Delta})^G$ being completely determined by its integer values on ρ_1, ρ_2 and $\tilde{\rho}_1, \tilde{\rho}_2$ and $\tilde{\rho}_3$, we have $\mathrm{PL}(\tilde{\Delta})^G \cong \mathbb{Z}^5$. Finally, the $\mathbb{Z}$-module M is a permutation module and therefore $H^1(G, M)$ is trivial. Bringing all of that together yields that $\mathrm{rk}(\mathrm{Pic}(\widetilde{V})) = 5-1 = 4$, which agrees with the prediction coming from Manin's conjecture regarding the power of $\log B$ in Theorem 1.1.

2.2. **The factor α.** We will use the same method as in [4, Lemma 5] to compute the nef cone volume $\alpha(\widetilde{V})$ and we refer the reader to [4, Lemma 5] for more details and definitions.

Let T_i, T'_i, $\widetilde{T}_i$ and $\widetilde{T}'_i$ be the Zariski closures of the one dimensional tori corresponding respectively to the cones $\mathbb{R}_{\geqslant 0}\rho_i$, $\mathbb{R}_{\geqslant 0}\rho'_i$, $\mathbb{R}_{\geqslant 0}\tilde{\rho}_i$ and $\mathbb{R}_{\geqslant 0}\tilde{\rho}'_i$. We also introduce the G-invariant divisors

$$D_1 = T_1, \;\; D_2 = T_2 + T'_2, \;\; D_3 = \widetilde{T}_1 + \widetilde{T}'_1, \;\; D_4 = \widetilde{T}_2 + \widetilde{T}'_2, \;\; D_5 = \widetilde{T}_3 + \widetilde{T}'_3.$$

Using [1, Proposition 1.15], one immediately sees that $\mathrm{Pic}(\widetilde{V})$ is generated by D_1, D_2, D_3, D_4, D_5 with the relation $D_5 = 2D_1 + D_2 - D_4$ and that the divisor

$$D_1 + D_2 + D_3 + D_4 + D_5 \sim 3D_1 + 2D_2 + D_3$$

is an anticanonical divisor for $\widetilde{V}$. Following the strategy of [4, Lemma 5] and using the same notations than in [4, Lemma 5], it now follows that $C_{\mathrm{eff}}^{\vee}$ is the subset of $\mathbb{R}^4_{\geqslant 0}$ given by $2z_1 + z_2 - z_4 \geqslant 0$ and that $H_{\widetilde{V}}$ is given by the equation $3z_1 + 2z_2 + z_3 = 1$. Therefore, a straightforward computation finally yields

$$\begin{aligned}\alpha(\widetilde{V}) &= \int_0^1 (1-z_3)^2 \mathrm{d}z_3 \times \frac{1}{2}\mathrm{Vol}\left\{(z_1,z_4) \in \mathbb{R}^2_{\geqslant 0} : \begin{array}{l} 3z_1 \leqslant 1, \\ 2z_4 - z_1 \leqslant 1 \end{array}\right\} \\ &= \frac{1}{6}\int_{z_1=0}^{\frac{1}{3}} \left(\int_{z_4=0}^{\frac{1+z_1}{2}} \mathrm{d}z_4 \right) \mathrm{d}z_1 = \frac{7}{216}.\end{aligned}$$

2.3. **The factor** β**.** Let us now briefly justify that $\beta(\widetilde{V}) = 1$. We know that $\widetilde{V}$ is birational to the torus $R_{\mathbb{Q}(i)/\mathbb{Q}}(\mathbb{G}_m)$. But the open immersion $\mathbb{G}_{m,\mathbb{Q}(i)} \hookrightarrow \mathbb{A}^2_{\mathbb{Q}(i)}$ gives rise to an open immersion $R_{\mathbb{Q}(i)/\mathbb{Q}}(\mathbb{G}_m) \hookrightarrow \mathbb{A}^2_{\mathbb{Q}}$ by taking the functor $R_{\mathbb{Q}(i)/\mathbb{Q}}(\cdot)$ and by alluding to [33, Proposition 4.9]. Hence, $R_{\mathbb{Q}(i)/\mathbb{Q}}(\mathbb{G}_m)$ is rational and so is $\widetilde{V}$. Finally this implies that $\beta(\widetilde{V}) = 1$ (see [4, section 5] for details).

2.4. **The Tamagawa number.**

2.4.1. *Conjectural expression.* Let us choose $S = \{\infty, 2\}$ and note that our definition will be independent of that choice. We have from [3, Theorem 1.3.2] that $\mathrm{Pic}(\widetilde{V}_{\overline{\mathbb{Q}}})$ is the free abelian group generated by the divisors $T_1, T_2, T_2', \widetilde{T}_1, \widetilde{T}_1', \widetilde{T}_2, \widetilde{T}_2'$ defined in §2.2 with the following G-action

$$\sigma(T_2) = T_2', \quad \sigma(\widetilde{T}_i) = \widetilde{T}_i' \quad (i \in \{1,2\})$$

if σ denotes the complex conjugation. Alluding to [24, Defintion 7.1], we have the following conjectural expression

$$\tau(\widetilde{V}) := \lim_{s \to 1^+} (s-1)^4 L_S(s, \chi_{\mathrm{Pic}(\widetilde{V}_{\overline{\mathbb{Q}}})}) \omega_\infty \prod_p \lambda_p^{-1} \omega_p$$

where

$$L_S(s, \chi_{\mathrm{Pic}(\widetilde{V}_{\overline{\mathbb{Q}}})}) = \prod_{p \notin S} \det\left(\mathrm{Id} - p^{-s}\mathrm{Frob}_p \mid \mathrm{Pic}(\widetilde{V}_{\overline{\mathbb{Q}}})^{I_p}\right)^{-1}$$

with I_p the inertia group and Frob_p a representative of the Frobenius automorphism and where

$$\lambda_p = L_p(1, \chi_{\mathrm{Pic}}(\widetilde{V}_{\overline{\mathbb{Q}}})), \quad \omega_\infty = \omega_{\infty,\widetilde{V}}(\widetilde{V}(\mathbb{R})), \quad \omega_p = \omega_{p,\widetilde{V}}(\widetilde{V}(\mathbb{Q}_p))$$

for measures $\omega_{v,\widetilde{V}}$ on $\widetilde{V}(\mathbb{Q}_v)$ whose proper definitions are postponed to the next section (they are the measures $\omega_{\mathscr{K},v}$ defined in [1, §2]) and where λ_p is taken to be 1 for $p \in S$.

Let $\Re e(s) > 1$. First we notice that for all $p \notin S$, we have that I_p is trivial since p is not ramified in $\mathbb{Q}(i)$. Then the Frobenius Frob_p being trivial for all $p \equiv 1 \bmod 4$, it is easy to see that in that case

$$\det\left(\mathrm{Id} - p^{-s}\mathrm{Frob}_p \mid \mathrm{Pic}(\widetilde{V}_{\overline{\mathbb{Q}}})^{I_p}\right) = \left(1 - \frac{1}{p^s}\right)^7.$$

When $p \equiv 3 \bmod 4$, Frob_p is of order 2 with the same action than σ on $\mathrm{Pic}(\widetilde{V}_{\overline{\mathbb{Q}}})$ and hence one sees immediately that

$$\det\left(\mathrm{Id} - p^{-s}\mathrm{Frob}_p \mid \mathrm{Pic}(\widetilde{V}_{\overline{\mathbb{Q}}})^{I_p}\right) = \left(1 - \frac{1}{p^s}\right)\left(1 - \frac{1}{p^{2s}}\right)^3.$$

Bringing all of this together yields that

$$L_S(s, \chi_{\mathrm{Pic}}(\widetilde{V}_{\overline{\mathbb{Q}}})) = \prod_{p>2}\left(1 - \frac{1}{p^s}\right)^{-4}\left(1 - \frac{\chi(p)}{p^s}\right)^{-3}$$

and

$$\lim_{s\to 1}(s-1)^4 L_S(s, \chi_{\mathrm{Pic}}(\widetilde{V}_{\overline{\mathbb{Q}}})) = \frac{L(1,\chi)^2}{2^4} = \frac{1}{2^4} \times \left(\frac{\pi}{4}\right)^3$$

and therefore

$$\tau(\widetilde{V}) = \left(\frac{\pi}{4}\right)^3 \omega_\infty \times \frac{\omega_2}{2^4} \times \prod_{p>2}\left(1 - \frac{1}{p}\right)^4\left(1 - \frac{\chi(p)}{p}\right)^3 \omega_p.$$

2.4.2. *Construction of the Tamagawa measure.* Let us write here $V_{(i)}$ for the affine subset of V where $x_i \neq 0$ with coordinates $x_j^{(i)} = x_j/x_i$ for $j \neq i$. Note that $V_{(i)}$ is defined by the equation

$$f^{(i)}(x_0^{(i)}, \dots, \widehat{x_i^{(i)}}, \dots, x_3^{(i)}) = \left(\frac{x_3}{x_i}\right)^3 - \frac{x_0}{x_i}\left(\left(\frac{x_1}{x_i}\right)^2 + \left(\frac{x_2}{x_i}\right)^2\right)$$

where $(x_0^{(i)}, \dots, \widehat{x_i^{(i)}}, \dots, x_3^{(i)})$ denotes $(x_0^{(i)}, x_2^{(i)}, x_3^{(i)})$ after removing the i-th component. The same arguments as in [22, §13] go through to yield that $\omega_V \cong \mathscr{O}_V(-1)$ and that such an isomorphism is given on $V_{(i)}$ by

$$x_i^{-1} \longmapsto \frac{(-1)^{i+t}}{\partial f^{(i)}/\partial x_j^{(i)}} \mathrm{d}x_k^{(i)} \wedge \mathrm{d}x_\ell^{(i)}$$

for $k < \ell \in \{0,1,2,3\} \smallsetminus \{i\}$, $\{i,j,k,\ell\} = \{0,1,2,3\}$ and $t = k+\ell$ if $k < i < \ell$ and $t = k+\ell-1$ otherwise. Moreover, we have already seen that $\omega_{\widetilde{V}} \cong \pi^*\omega_V$. The dual sections τ_i of s_i in $\omega_V^{-1} \cong \mathscr{O}_V(1)$ define the embedding $V \hookrightarrow \mathbb{P}^3$ under consideration in this note and the morphism $\widetilde{V} \to V \hookrightarrow \mathbb{P}^3$ is given by the sections $\pi^*\tau_i$ of $H^0(\widetilde{V}, \omega_{\widetilde{V}}^{-1})$.

Consider now the subsequent Arakelov heights $(\omega_{\widetilde{V}}^{-1}, (||.||_v)_{v\in\mathrm{Val}(\mathbb{Q})})$ and $(\omega_V^{-1}, (||.||'_v)_{v\in\mathrm{Val}(\mathbb{Q})})$ defined by these global sections where for all $v \in \mathrm{Val}(\mathbb{Q})$, $x \in V(\mathbb{Q}_v)$, $y \in \widetilde{V}(\mathbb{Q}_v)$, $\tau \in \omega_{\widetilde{V}}^{-1}$ and $\sigma \in \omega_V^{-1}$ we use respectively the v-adic metrics defined by

$$||\tau||_v = \min_{\substack{0\leqslant i\leqslant 3\\ \pi^*\tau_i\neq 0}} \left\{ \left| \frac{\tau}{\pi^*\tau_i(x)} \right|_v \right\}$$

if v is finite and

$$||\tau||_\infty = \min \left\{ \min_{\substack{i\in\{0,3\}\\ \pi^*\tau_i\neq 0}} \left\{ \left| \frac{\tau}{\pi^*\tau_i(x)} \right|_\infty \right\}, \left(\left| \frac{\pi^*\tau_1(x)}{\tau} \right|_\infty^2 + \left| \frac{\pi^*\tau_2(x)}{\tau} \right|_\infty^2 \right)^{-\frac{1}{2}} \right\}$$

if v is the archimedean place and $\tau \neq 0$ and

$$||\sigma||'_v = \min_{\substack{0\leqslant i\leqslant 3\\ \tau_i\neq 0}} \left\{ \left| \frac{\sigma}{\tau_i(y)} \right|_v \right\}$$

if v is finite and

$$||\sigma||'_\infty = \min \left\{ \min_{\substack{i\in\{0,3\}\\ \tau_i\neq 0}} \left\{ \left| \frac{\sigma}{\tau_i(y)} \right|_\infty \right\}, \left(\left| \frac{\tau_1(y)}{\sigma} \right|_\infty^2 + \left| \frac{\tau_2(y)}{\sigma} \right|_\infty^2 \right)^{-\frac{1}{2}} \right\}$$

if v is the archimedean place and $\sigma \neq 0$. These heights correspond to the heights H on V and $H \circ \pi$ on $\widetilde{V}$ that we used in our counting problem. Applying the definition 2.2.1 of [29], we get a measure $\omega_{v,\widetilde{V}}$ on $\widetilde{V}(\mathbb{Q}_v)$ associated to the v-adic metric $||.||_v$ which is the measure defined in [1, §2] and used in §2.4.1 and a measure $\omega_{v,V}$ on $V(\mathbb{Q}_v)$ associated to the v-adic metric $||.||'_v$.

2.4.3. *Computation of the archimedean density.* We follow once again the strategy adopted in [22, §13]. One sees easily that $U = V_{(3)}$ and the same argument as in [22, §13] shows that

$$\omega_\infty = \omega_{\infty,\widetilde{V}}(\widetilde{V}(\mathbb{R})) = \omega_{\infty,\widetilde{V}}(\pi^{-1}(U)(\mathbb{R})).$$

Now the local coordinates $x_1^{(3)}-1$ and $x_2^{(3)}$ at the rational point $(1,1,0)$ of U give an isomorphism

$$U(\mathbb{R}) \cong W = \{(z_1,z_2)\in\mathbb{R}^2 : (z_1+1)^2+z_2^2\neq 0\}$$

and a similar computation as in [22, §13] yields

$$\begin{aligned}\omega_{\infty,\widetilde{V}}(\pi^{-1}(U)(\mathbb{R})) &= \int_{\mathbb{R}^2}\frac{\mathrm{d}z_1\mathrm{d}z_2}{\max\{1, z_1^2+z_2^2, (z_1^2+z_2^2)^{3/2}\}}\\ &= \int_{z_1^2+z_2^2\leqslant 1}\mathrm{d}z_1\mathrm{d}z_2 + \int_{z_1^2+z_2^2>1}\frac{\mathrm{d}z_1\mathrm{d}z_2}{(z_1^2+z_2^2)^{3/2}}\\ &= 3\pi\end{aligned}$$

after a polar change of coordinates. We can therefore conclude that $\omega_\infty = 3\pi$.

2.4.4. *Computation of ω_p for odd p.* Thanks to the remarks of [32, Page 187], one can construct a model $\widetilde{\mathscr{V}}$ over $\mathrm{Spec}(\mathbb{Z})$ satisfying the conditions of [29, Notation 4.5] with $S=\{\infty,2\}$. Hence, one can consider the reduction $\widetilde{\mathscr{V}}_p$ modulo p of $\widetilde{\mathscr{V}}$ for every prime number p.

The torus T has good reduction T_p for every prime $p>2$ since p is not ramified in $\mathbb{Q}(i)$ and T_p is a split torus of rank 2 if $p\equiv 1 \bmod 4$ and a non-split torus of rank 2 split by $\mathbb{F}_{p^2}$ if $p\equiv 3\bmod 4$. Hence, the reduction $\widetilde{\mathscr{V}}_p$ modulo p can be realized as the toric variety over $\mathbb{F}_p$ under the torus T_p given by the fan Δ' which is invariant under Frob_p. Since the fan stays regular and complete, we can conclude that $\widetilde{\mathscr{V}}_p$ is smooth and hence that $\widetilde{\mathscr{V}}$ has good reduction modulo $p>2$ (see [12]).

We can therefore apply [24, Corollary 6.7] to obtain for all odd p the following expression

$$\omega_p = \frac{\#\widetilde{\mathscr{V}}(\mathbb{F}_p)}{p^2}.$$

Now alluding to Weil's formula, we obtain

$$\omega_p = 1+\frac{\mathrm{Tr}(\mathrm{Frob}_p|\mathrm{Pic}(\widetilde{V}_{\overline{\mathbb{Q}}}))}{p}+\frac{1}{p^2} = 1+\frac{4+3\chi(p)}{p}+\frac{1}{p^2}$$

by using the description of the action of Frob_p on $\mathrm{Pic}(\widetilde{V}_{\overline{\mathbb{Q}}})$ given in §2.4.1.

2.4.5. *Computation of ω_2.* For $p=2$, the model $\widetilde{\mathscr{V}}$ having bad reduction, we appeal to the lemma 6.6 of [24] to compute ω_2. By [1, Proposition 2.10] and noting that the smooth assumption is not necessary, one gets

$$\omega_{2,\widetilde{V}}(\widetilde{V}(\mathbb{Q}_2)) = \omega_{2,\widetilde{V}}(\pi^{-1}(U)), \quad \omega_{2,V}(V(\mathbb{Q}_2)) = \omega_{2,V}(U).$$

Now an analogous computation as the one in §2.4.3 yields that both quantities $\omega_{2,\widetilde{V}}(\pi^{-1}(U))$ and $\omega_{2,V}(U)$ are equal to the expression

$$\int_W \frac{\mathrm{d}z_1\mathrm{d}z_2}{\max\{1,|z_1^2+z_2^2|_2,|z_1(z_1^2+z_2^2)|_2,|z_2(z_1^2+z_2^2)|_2\}}$$

with $W=\{(z_1,z_2)\in\mathbb{Q}_2^2:(z_1+1)^2+z_2^2\neq 0\}$. Therefore, ω_2 is equal to $\omega_{2,V}(V(\mathbb{Q}_2))$ and [24, Remark 6.8] implies that

$$\omega_2=\lim_{n\to+\infty}\frac{N(2^n)}{2^{3n}},$$

where

$$N(2^n):=\#\big\{\mathbf{x}\,(\mathrm{mod}\,2^n):\ x_0(x_1^2+x_2^2)\equiv x_3^3\,(\mathrm{mod}\,2^n)\big\}.$$

Let $v_2(x_1^2+x_2^2)=k$ and $v_2(x_0)=k_0$. If $k=1+2k'$ is odd and $1+2k'<n$, then the number of (x_1,x_2) satisfying $v_2(x_1^2+x_2^2)=k$ is $2^{2n-2k'-2}$. There are $2^{n-(1+2k'+k_0)/3-1}$ ways to choose x_3 and then $2^{2k'+1}$ choices for x_0. Then, in the case where $v_2(x_1^2+x_2^2)$ is odd, the number of solutions is asymptotic to

$$2^{3n}\sum_{\substack{3|1+2k'+k_0'\\ k'\geqslant 0}}2^{-2-(1+2k'+k_0')/3}\sim\frac{5}{6}2^{3n}.$$

The number of (x_1,x_2) satisfying $v_2(x_1^2+x_2^2)=2k'$ is, at least for $2k'<n$, equal to $2^{2n-2k'-1}$. There are $2^{n-(2k'+k_0)/3-1}$ ways to choose x_3 and then $2^{2k'}$ choices for x_0. Summing over $3\mid 2k'+k_0$ and $k'\geqslant 0$ we get the contribution of the case $v_2(x_1^2+x_2^2)$ even in $N(2^n)$, which is asymptotic to $\frac{7}{6}\cdot 2^{3n}$. It follows that

$$\omega_2=2=1+\frac{2+3\chi(2)+2\chi^2(2)}{2}+\frac{\chi^2(2)}{2^2}.$$

2.4.6. *Conclusion.* Bringing everything together yields the following expression for the Peyre constant

$$\alpha(\widetilde{V})\beta(\widetilde{V})\tau(\widetilde{V})=\frac{7}{216}(3\pi)\left(\frac{\pi}{4}\right)^3\tau.$$

This is in agreement with the constant C in (1.3).

3. Proof of Theorem 1.1

By symmetry, we have

$$N_U(B)=\#\left\{\mathbf{x}\in E:x_0(x_1^2+x_2^2)=x_3^3,\ \max\left\{x_0,\sqrt{x_1^2+x_2^2}\right\}\leqslant B\right\},$$

where $E := \{\mathbf{x} \in \mathbb{N} \times \mathbb{Z}^2 \times \mathbb{N} : \gcd(x_0, x_1, x_2, x_3) = 1\}$ and $\mathbb{N} = \mathbb{Z}_{\geqslant 1}$. As in [5], we parametrize $x_1^2 + x_2^2$, x_0 and x_3 by

$$x_1^2 + x_2^2 = n_1 n_2^2 n_3^3, \quad x_0 = n_1^2 n_2 n_4^3, \quad x_3 = n_1 n_2 n_3 n_4,$$

where n_1 and n_2 are squarefree and $\gcd(n_1, n_2) = 1$ which is equivalent to $\mu^2(n_1 n_2) = 1$. It follows that

$$N_U(B) = 4 \sum_{\substack{\mathbf{n} \in \mathbb{N}^4 \\ \mu^2(n_1 n_2) = 1 \\ n_1^2 n_2 n_4^3 \leqslant B \\ n_1 n_2^2 n_3^3 \leqslant B^2}} r(n_1 n_2^2 n_3^3, n_1 n_2 n_4)$$

where

$$r(n, m) := \tfrac{1}{4} \# \big\{ (x_1, x_2) \in \mathbb{Z}^2 : x_1^2 + x_2^2 = n, \ ((x_1, x_2), m) = 1 \big\}.$$

Here, we remark that our choice of height function is particularly well suited to handle the expression $r(n, m)$.

Let χ be the non-principal character modulo 4 and $r_0 := 1 * \chi$. The quantity $r(n, m)$ is a multiplicative arithmetic function in n, and we have

$$r(n, m) := \prod_p r\big(p^{v_p(n)}, p^{v_p(m)}\big).$$

We use the fact that, when $\nu \geqslant 1$,

$$r(p^\nu, p) = \begin{cases} 2 & \text{if } p \equiv 1 \pmod 4, \\ 0 & \text{if } p \equiv 3 \pmod 4, \\ 1 & \text{if } \nu = 1, \, p = 2, \\ 0 & \text{if } \nu \geqslant 2, \, p = 2. \end{cases}$$

Then, when $\nu_1+\nu_2 \leqslant 1$, the value of $r(p^{\nu_1+2\nu_2+3\nu_3}, p^{\nu_1+\nu_2+\nu_4})$ is given by

$$\begin{cases} 1 & \text{if } (\nu_1,\nu_2,\nu_3,\nu_4)=(0,0,0,\nu_4), \\ r_0(p^{3\nu_3}) & \text{if } (\nu_1,\nu_2,\nu_3,\nu_4)=(0,0,\nu_3,0), \\ 0 & \text{if } (\nu_1,\nu_2)=(0,0),\ \min\{\nu_3,\nu_4\}\geqslant 1,\ p\equiv 2,3 \,(\operatorname{mod} 4), \\ 2 & \text{if } (\nu_1,\nu_2)=(0,0),\ \min\{\nu_3,\nu_4\}\geqslant 1,\ p\equiv 1 \,(\operatorname{mod} 4), \\ 0 & \text{if } \min\{\nu_1,\nu_3\}\geqslant 1,\ p\equiv 2,3 \,(\operatorname{mod} 4), \\ 2 & \text{if } \min\{\nu_1,\nu_3\}\geqslant 1,\ p\equiv 1 \,(\operatorname{mod} 4), \\ 0 & \text{if } \nu_1=1,\ \nu_3=0,\ p\equiv 3 \,(\operatorname{mod} 4), \\ 2 & \text{if } \nu_1=1,\ \nu_3=0,\ p\equiv 1 \,(\operatorname{mod} 4), \\ 1 & \text{if } \nu_1=1,\ \nu_3=0,\ p=2, \\ 0 & \text{if } \nu_2=1,\ p\equiv 2,3 \,(\operatorname{mod} 4), \\ 2 & \text{if } \nu_2=1,\ p\equiv 1 \,(\operatorname{mod} 4). \end{cases}$$

The Dirichlet series associated to this counting problem is

$$F(s_1,s_2) := \sum_{\substack{\mathbf{n}\in\mathbb{N}^4 \\ \mu^2(n_1n_2)=1}} \frac{r(n_1n_2^2n_3^3, n_1n_2n_4)}{n_1^{2s_1+s_2}n_2^{s_1+2s_2}n_3^{3s_2}n_4^{3s_1}}, \qquad \left(\Re e(s_1), \Re e(s_2) > \frac{1}{3}\right).$$

It can be written as an Euler product of $F_p(s_1,s_2)$, where

$$F_2(s_1,s_2) = \frac{1}{1-2^{-3s_1}} + \frac{1}{2^{3s_2}-1} + \frac{1}{2^{2s_1+s_2}(1-2^{-3s_1})},$$

$$F_p(s_1,s_2) = \frac{1}{1-p^{-3s_1}} + \frac{1}{p^{6s_2}-1},$$

if $p\equiv 3 \,(\operatorname{mod} 4)$ and

$$\begin{aligned} F_p(s_1,s_2) =& \frac{1}{1-p^{-3s_1}} + \frac{4-p^{-3s_2}}{p^{3s_2}(1-p^{-3s_2})^2} \\ &+ 2\frac{p^{-3(s_1+s_2)}+p^{-(2s_1+s_2)}+p^{-(s_1+2s_2)}}{(1-p^{-3s_2})(1-p^{-3s_1})}, \end{aligned}$$

if $p\equiv 1 \,(\operatorname{mod} 4)$. For $\Re e(s)>1$, let

$$\begin{aligned} \zeta_{\mathbb{Q}(i)}(s) &:= \sum_{n\geqslant 1}\frac{r_0(n)}{n^s} = \zeta(s)L(s,\chi) \\ &= \frac{1}{1-2^{-s}} \prod_{p\equiv 3\,(\operatorname{mod} 4)} \frac{1}{1-p^{-2s}} \prod_{p\equiv 1\,(\operatorname{mod} 4)} \frac{1}{(1-p^{-s})^2}. \end{aligned}$$

Let $\mathbf{s}$ stand for the pair (s_1,s_2). Then there exists G such that

$$F(\mathbf{s}) = \zeta(3s_1)\zeta_{\mathbb{Q}(i)}(3s_2)^2\zeta_{\mathbb{Q}(i)}(s_1+2s_2)\zeta_{\mathbb{Q}(i)}(s_2+2s_1)G(\mathbf{s}).$$

The above quantity $G(\mathbf{s})$ can be written as an Euler product of $G_p(\mathbf{s})$ where

$$G_2(1/3,1/3)=2^{-3}$$

while, for $p\equiv 3\,(\mathrm{mod}\,4)$

$$G_p(\mathbf{s})=\big(1-p^{-2(s_1+2s_2)}\big)^2\big(1-p^{-2(2s_1+s_2)}\big)\big(1-p^{-3(s_1+2s_2)}\big).$$

and for $p\equiv 1\,(\mathrm{mod}\,4)$,

$$\begin{aligned}G_p(\mathbf{s})=&\big(1-p^{-3s_2}\big)^4\big(1-p^{-(s_1+2s_2)}\big)^2\big(1-p^{-(2s_1+s_2)}\big)^2\\&+\big(1-p^{-3s_1}\big)\big(4p^{-3s_2}-p^{-6s_2}\big)\\&\quad\times\big(1-p^{-3s_2}\big)^2\big(1-p^{-(s_1+2s_2)}\big)^2\big(1-p^{-(2s_1+s_2)}\big)^2\\&+2\big(p^{-3(s_1+s_2)}+p^{-(2s_1+s_2)}+p^{-(s_1+2s_2)}\big)\\&\quad\times\big(1-p^{-3s_2}\big)^3\big(1-p^{-(s_1+2s_2)}\big)^2\big(1-p^{-(2s_1+s_2)}\big)^2.\end{aligned}$$

The series F is absolutely convergent when $\Re e(s_1)>\frac{1}{3}$ and $\Re e(s_2)>\frac{1}{3}$ and the function G can be analytically continued to $\Re e(s_1)>\frac{1}{6}$ and $\Re e(s_2)>\frac{1}{6}$. Moreover, we have

$$\begin{aligned}G\Big(\frac{1}{3},\frac{1}{3}\Big)&=\frac{1}{2^3}\prod_{p\neq 2}\Big(1-\frac{1}{p}\Big)^4\Big(1-\frac{\chi(p)}{p}\Big)^3\Big(1+\frac{4+3\chi(p)}{p}+\frac{1}{p^2}\Big)\\&=\tau.\end{aligned}\tag{3.1}$$

Thus F satisfies the assumptions of Theorem 1 of [6] with $(\beta_1,\beta_2)=(1,2)$, $(\alpha_1,\alpha_2)=(\frac{1}{3},\frac{1}{3})$,

$$\ell_1(\mathbf{s})=3s_1,\quad \ell_2(\mathbf{s})=\ell_3(\mathbf{s})=3s_2,$$
$$\ell_4(\mathbf{s})=s_1+2s_2,\quad \ell_5(\mathbf{s})=2s_1+s_2.$$

It follows that there exists a constant $\vartheta>0$ and a polynomial $Q\in\mathbb{R}[X]$ of degree 3 such that

$$N_U(B)=BQ(\log B)+O(B^{1-\vartheta}).$$

Now alluding to Theorem 2 of [6] to get the leading coefficient C of Q, we obtain

$$\begin{aligned}Q(\log B)&\underset{B\to+\infty}{\sim}\frac{4L(1,\chi)^4G(\frac{1}{3},\frac{1}{3})}{B}\int_{\substack{(y_1,y_2,y_3,y_4,y_5)\in[1,+\infty[^5\\ y_1^3y_4y_5^2\leqslant B,\,y_2^3y_3^3y_4^2y_5\leqslant B^2}}\mathrm{d}\mathbf{y}\\&\underset{B\to+\infty}{\sim}4\Big(\frac{\pi}{4}\Big)^4G\Big(\frac{1}{3},\frac{1}{3}\Big)\int_{\substack{(y_3,y_4,y_5)\in[1,+\infty[^3\\ y_4y_5^2\leqslant B,\,y_3^3y_4^2y_5\leqslant B^2}}\frac{\mathrm{d}\mathbf{y}}{y_3y_4y_5}\\&\underset{B\to+\infty}{\sim}\frac{\pi^4}{2^6}G\Big(\frac{1}{3},\frac{1}{3}\Big)(\log B)^3I,\end{aligned}$$

where

$$I := \mathrm{vol}\big\{(t_3,t_4,t_5)\in\mathbb{R}_+^3 \,:\, t_4+2t_5\leqslant 1,\, 3t_3+2t_4+t_5\leqslant 2\big\}.$$

An straightforward computation immediately yields

$$I=\frac{1}{3}\int_0^{1/2}\int_0^{1-2t_5}(2-2t_4-t_5)\mathrm{d}t_4\mathrm{d}t_5=\frac{7}{72},$$

and therefore the leading coefficient C of Q is given by

$$C=\frac{7}{216}\Big(\frac{\pi}{4}\Big)^3(3\pi)G\Big(\frac{1}{3},\frac{1}{3}\Big).$$

By (3.1) we have $G(\frac{1}{3},\frac{1}{3})=\tau$, from which (1.3) follows. This completes the proof.

4. The descent argument

Our main argument in order to derive Theorem 1.1 in section 3 consists of a descent from our original variety $\tilde{V}$ onto the variety of equation

$$x_1^2+x_2^2=n_1n_2^2n_3^3.$$

Although this is not required to verify Peyre's conjecture since $\tilde{V}$ is a rational variety, it is particularly interesting to find out which torsor were used during this descent argument because $\tilde{V}$ is a non-split variety. Indeed, as mentioned in [17], versal torsors parametrizations (see [9] for precise definitions) are mostly used in the case of split varieties and the question of the right approach in the case of non-split varieties is quite natural. Using the Cox ring machinery over nonclosed fields developed in [17], all known examples of Manin's conjecture in the case of non-split varieties derived by means of a descent rely on a descent on quasi-versal torsors in the sense of [9]. For example, the descent in [18] is a descent on torsors of injective type $\mathrm{Pic}(V_{\mathbb{Q}(i)})\hookrightarrow\mathrm{Pic}(V_{\overline{\mathbb{Q}}})$ whereas it is shown in [17] that the *ad hoc* descent used in [7] is a descent on the torsor of injective type $\mathrm{Pic}(V)\hookrightarrow\mathrm{Pic}(V_{\overline{\mathbb{Q}}})$. Here, we now show in the following lemma that the descent corresponds to a torsor of a different type, which is not quasi-versal.

With the notations of §2.2, we set $\hat{T}=[D_1]\mathbb{Z}\oplus[D_3]\mathbb{Z}\oplus[D_4]\mathbb{Z}$ and $\lambda:\hat{T}\hookrightarrow\mathrm{Piv}(\tilde{V}_{\overline{\mathbb{Q}}})$ be the natural embedding.

Lemma 4.1. *Every Cox ring of injective type λ is isomorphic to the $\mathbb{Q}$-algebra*

$$R=\mathbb{Q}[x_1,x_2,\eta_1,\eta_2,\eta_3,\eta_4]/\left(x_1^2+x_2^2-\eta_2\eta_3^2\eta_4^3\right).$$

Proof. The proof is very similar to the one in [31, Proposition 2.71] and that is why we will not repeat all the details here. Since $\tilde{V}_{\overline{\mathbb{Q}}}$ is a split toric variety, we know by [32] that a Cox ring of identity type for $\tilde{V}_{\overline{\mathbb{Q}}}$ is given by

$$\overline{\mathscr{R}} = \overline{\mathbb{Q}}[t_1, t_2, t_2', \tilde{t}_1, \tilde{t}_1', \tilde{t}_2, \tilde{t}_2', \tilde{t}_3, \tilde{t}_3']$$

where $t_i = \operatorname{div}(T_i)$, $t_i' = \operatorname{div}(T_i')$, $\tilde{t}_i = \operatorname{div}(\tilde{T}_i)$ and $\tilde{t}_i' = \operatorname{div}(\tilde{T}_i')$. We then have by [31, Remark 2.51] that every Cox ring of injective type λ is isomorphic to the ring of invariant of

$$\bigoplus_{m \in \hat{T}} \overline{R}_m$$

where $\overline{R}_m$ is the vector space generated by the degree m elements of $\overline{\mathscr{R}}$. For $m \in \hat{T}$ given by $m = [a_1 D_1 + a_3 D_3 + a_4 D_4]$, we have to solve the following linear system with $e_i, e_i', \tilde{e}_i, \tilde{e}_i' \geqslant 0$ to determine $\overline{R}_m$

$$\begin{aligned}\Big[e_1 T_1 + e_2 T_2 + e_2' T_2' + \tilde{e}_1 \tilde{T}_1 + \tilde{e}_1' \tilde{T}_1' + \tilde{e}_2 \tilde{T}_2 + \tilde{e}_2' \tilde{T}_2' + \tilde{e}_3 \tilde{T}_3 + \tilde{e}_3' \tilde{T}_3'\Big]& \\ = [a_1 D_1 + a_3 D_3 + a_4 D_4]&.\end{aligned}$$

Alluding to the fan Δ' and [1, Proposition 1.15], we get that this linear system is equivalent to

$$\begin{cases} \tilde{e}_3' + \tilde{e}_1 = \tilde{e}_3 + \tilde{e}_1' \\ \tilde{e}_2' + \tilde{e}_3 - \tilde{e}_3' = \tilde{e}_2 + \tilde{e}_3' - \tilde{e}_3 \\ e_2 + \tilde{e}_3' - 2\tilde{e}_3 = e_2' + \tilde{e}_3 - 2\tilde{e}_3' = 0. \end{cases}$$

This easily yields that $\overline{R}$ is generated by

$$\eta_1 = t_1, \quad \eta_2 = \tilde{t}_1 \tilde{t}_1', \quad \eta_3 = \tilde{t}_2 \tilde{t}_2', \quad \eta_4 = t_2 t_2' \tilde{t}_3 \tilde{t}_3', \quad \eta_5 = \tilde{t}_1 \tilde{t}_2^2 t_2^3 \tilde{t}_3^2 \tilde{t}_3',$$

and $\overline{\eta}_5$ the conjugate of η_5 with the relation

$$\eta_5 \overline{\eta}_5 = \eta_2 \eta_3^2 \eta_4^3.$$

Using the Galois invariant variables

$$x_1 = \frac{\eta_5 + \overline{\eta}_5}{2}, \quad x_2 = \frac{\eta_5 - \overline{\eta}_5}{2i}$$

one finally ensures that every Cox ring of injective type λ is isomorphic to R. $\square$

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