Null controllability of semi-linear fourth order parabolic equations

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Abstract

In this paper, we consider a semi-linear fourth order parabolic equation in a bounded smooth domain Ω with homogeneous Dirichlet and Neumann boundary conditions. The main result of this paper is the null controllability and the exact controllability to the trajectories at any time $T > 0$ for the associated control system with a control function acting at the interior.

Résumé

Dans ce papier, on considère une équation parabolique semi-linéaire de quatrième ordre dans un domaine borné régulier Ω avec des conditions aux limites de type Dirichlet et Neumann homogènes. Le résultat principal de ce papier concerne la contrôlabilité à zéro et la contrôlabilité exacte pour tout $T > 0$ du système de contrôle associé avec un contrôle agissant à l'interieur.

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1 Introduction

In the present paper, we consider $\Omega \subset \mathbb{R}^N$ with $(N \geq 2)$ a bounded connected open set whose boundary $\partial\Omega$ is regular enough. Let $\omega \subset \Omega$ be a (small) nonempty open subset and let $T > 0$. We will use the notation $Q = (0,T) \times \Omega$ and $\Sigma = (0,T) \times \partial\Omega$ and we will denote by $\vec{n}(x)$ the outward unit normal vector to Ω at the point $x \in \partial\Omega$. On the other hand, we will denote by C_0 a generic positive constant which may depend on Ω and ω but not on T.

We consider parabolic systems of the form

$$\begin{cases} \partial_t y + \Delta^2 y + f_1(y, \nabla y, \nabla^2 y)\mathbb{1}_{\{f_2 \equiv 0\}} + f_2(y, \nabla y, \nabla^2 y, \nabla^3 y) = \chi_\omega v & \text{in } Q \ , \\ y = \dfrac{\partial y}{\partial \vec{n}} = 0 & \text{on } \Sigma \ , \\ y(0, \cdot) = y_0(\cdot) & \text{in } \Omega \ , \end{cases} \tag{1}$$

where χ_ω is the characteristic function of ω, y_0 and v are given in appropriate spaces and

$$f_1 : \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^{N^2} \to \mathbb{R}$$

and

$$f_2 : \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^{N^2} \times \mathbb{R}^{N^3} \to \mathbb{R}.$$

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Here, we introduced f_1 to show that the conditions we impose may be less restrictive when there is no $\nabla^3 y$ (i.e $f_2 \equiv 0$). Before presenting our result, we will cite some physical motivations which are related to the system under view.

In [25], the authors studied the epitaxial growth of nanoscale thin films, which is modeled by the following system :

$$\begin{cases} \partial_t u + \Delta^2 u - \nabla.\left(f(\nabla u)\right) = g & \text{in } \tilde{Q} \ , \\ \dfrac{\partial u}{\partial \vec{n}} = \dfrac{\partial \triangle u}{\partial \vec{n}} = 0 & \text{on } \tilde{\Sigma} \ , \\ u(0,\cdot) = u_0(\cdot) & \text{in } \tilde{\Omega} \ , \end{cases} \tag{2}$$

where $\tilde{\Omega} = (0, L)^2$, $\tilde{Q} = (0, T) \times \tilde{\Omega}$, $\tilde{\Sigma} = (0, T) \times \partial\tilde{\Omega}$, $u_0 \in L^2(\tilde{\Omega})$, $f \in C^1(\mathbb{R}^N, \mathbb{R}^N)$ and $g \in L^2((0, T) \times \tilde{\Omega})$. In this context, u is the scaled film height, the term $\Delta^2 u$ represents the capillarity-driven surface diffusion and g denotes the deposition flux, while $\nabla \cdot (f(\nabla u))$ describes the upward hopping of atoms.

Furthermore, in [18] the authors studied the following system :

$$\begin{cases} \partial_t u + \nabla.\left(|\nabla \triangle u|^{p(x)-2} \nabla \triangle u\right) = f(x, u) & \text{in } Q \ , \\ u = \triangle u = 0 & \text{on } \Sigma \ , \\ u(0,\cdot) = u_0(\cdot) & \text{in } \Omega \ , \end{cases} \tag{3}$$

where p and f are specific functions and u_0 is an initial data. The previous model may describe some properties of medical magnetic resonance images in space and time. When the nonlinear source $f(x, u)$ is equal to $\eta(x, t)$, then the functions $u(x, t)$ and $\eta(x, t)$, respectively, represent the pixel intensity value of a digital image and a random noise. On the other hand, the author in [26] studied a fourth order parabolic system similar to (1) that models the long range effect of insects dispersal. Moreover, the authors in [10] were interested by a fourth order parabolic system where the solution describes the height of a viscous droplet spreading on a plain. For more details about this subject, see for instance [28], [12], [3], [4], [31], [33].

Let us now suppose that f_1 and f_2 are a locally Lipschitz-continuous functions and

$$f_1(0_{\mathbb{R}}, 0_{\mathbb{R}^N}, 0_{\mathbb{R}^{N^2}}) = f_2(0_{\mathbb{R}}, 0_{\mathbb{R}^N}, 0_{\mathbb{R}^{N^2}}, 0_{\mathbb{R}^{N^3}}) = 0. \tag{4}$$

Observe that, under the hypothesis above, we can write

$$f_1(s, p, q) = g_1(s, p, q)s + G_1(s, p, q) \cdot p + E_1(s, p, q) : q \ , \ \forall (s, p, q) \in \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^{N^2}, \tag{5}$$

where

$$g_1(s, p, q) = \int_0^1 \frac{\partial}{\partial s} f_1(\lambda\, s, \lambda\, p, \lambda\, q) d\lambda,$$

$$G_{1,i}(s, p, q) = \int_0^1 \frac{\partial}{\partial p_i} f_1(\lambda\, s, \lambda\, p, \lambda\, q) d\lambda,$$

for $1 \le i \le N$ and

$$E_{1,jk}(s, p, q) = \int_0^1 \frac{\partial}{\partial q_{jk}} f_1(\lambda\, s, \lambda\, p, \lambda\, q) d\lambda,$$

for $1 \le j, k \le N$. Let us notice that $g_1 \in L^\infty_{loc}(\mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^{N^2})$, $G_1 \in L^\infty_{loc}(\mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^{N^2})^N$ and $E_1 \in L^\infty_{loc}(\mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^{N^2})^{N^2}$. On the other hand, we can write also f_2 in the same way as before and we will have

$$\begin{gathered} f_2(s, p, q, r) = g_2(s, p, q, r)s + G_2(s, p, q, r) \cdot p + E_2(s, p, q, r) : q + M_2(s, p, q, r) \therefore r \ , \\ \forall (s, p, q, r) \in \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^{N^2} \times \mathbb{R}^{N^3}, \end{gathered} \tag{6}$$

where

$$g_2(s,p,q,r) = \int_0^1 \frac{\partial}{\partial s} f_2(\lambda s, \lambda p, \lambda q, \lambda r) d\lambda,$$

$$G_{2,i}(s,p,q,r) = \int_0^1 \frac{\partial}{\partial p_i} f_2(\lambda s, \lambda p, \lambda q, \lambda r) d\lambda,$$

for $1 \le i \le N$

$$E_{2,jk}(s,p,q,r) = \int_0^1 \frac{\partial}{\partial q_{jk}} f_2(\lambda s, \lambda p, \lambda q, \lambda r) d\lambda,$$

for $1 \le j,k \le N$ and

$$M_{2,jkl}(s,p,q,r) = \int_0^1 \frac{\partial}{\partial r_{jkl}} f_2(\lambda s, \lambda p, \lambda q, \lambda r) d\lambda,$$

for $1 \le j,k,l \le N$. Let us notice that $g_2 \in L^\infty_{loc}(\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}^{N^2}\times\mathbb{R}^{N^3})$, $G_2 \in L^\infty_{loc}(\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}^{N^2}\times\mathbb{R}^{N^3})^N$, $E_2 \in L^\infty_{loc}(\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}^{N^2}\times\mathbb{R}^{N^3})^{N^2}$ and $M_2 \in L^\infty_{loc}(\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}^{N^2}\times\mathbb{R}^{N^3})^{N^3}$.

We will assume also the following conditions on g_1, G_1, E_1, g_2, G_2, E_2 and M_2 :

$$\begin{aligned} \lim_{|s|,|p|,|q|\to\infty} \frac{|g_1(s,p,q)|}{\log^3(1+|s|+|p|+|q|)} &= 0, \\ \lim_{|s|,|p|,|q|\to\infty} \frac{|G_1(s,p,q)|}{\log^2(1+|s|+|p|+|q|)} &= 0, \\ \lim_{|s|,|p|,|q|\to\infty} \frac{|E_1(s,p,q)|}{\log(1+|s|+|p|+|q|)} &= 0 \end{aligned} \tag{7}$$

and we suppose that there exists $\gamma > 1$ such that

$$\begin{aligned} \lim_{|s|,|p|,|q|,|r|\to\infty} \frac{|g_2(s,p,q,r)|}{\log^{3/\gamma}(1+|s|+|p|+|q|+|r|)} &= 0, \\ \lim_{|s|,|p|,|q|,|r|\to\infty} \frac{|G_2(s,p,q,r)|}{\log^{2/\gamma}(1+|s|+|p|+|q|+|r|)} &= 0, \\ \lim_{|s|,|p|,|q|,|r|\to\infty} \frac{|E_2(s,p,q,r)|}{\log^{1/\gamma}(1+|s|+|p|+|q|+|r|)} &= 0, \\ \lim_{|s|,|p|,|q|,|r|\to\infty} \frac{e^{\frac{\gamma}{\gamma-1}|M_2(s,p,q,r)|^{1/2}}}{\log(1+|s|+|p|+|q|+|r|)} &= 0. \end{aligned} \tag{8}$$

Our first goal is to establish the null controllability for system (1). Let us start by giving the definition of null controllability for (1).

Definition 1.1. *It is said that (1) is null controllable at time $T > 0$ if for each $y_0 \in W^{2,\infty}(\Omega) \cap H_0^2(\Omega)$, there exists $v \in L^\infty((0,T)\times\omega)$ such that the corresponding initial problem (1) admits a solution $y \in C^0([0,T];L^2(\Omega))$ satisfying*

$$y(T,\cdot) = 0 \ \text{ in } \Omega.$$

Before we continue, let us present the following remark that concerns the regularity of y_0 :

Remark 1.2. *In this paper we can assume that $y_0 \in W^{4-4/p,p}(\Omega) \cap H_0^2(\Omega)$ for any $2 \le p < +\infty$. In fact, it suffices to take $v = 0$ in (1) for $t \in [0,\varepsilon]$ and to work in the time interval $[\varepsilon, T]$, looking at $y(\varepsilon,\cdot)$ as the initial state. In order to prove that $y(\varepsilon,\cdot)$ belongs to $W^{4-4/p,p}(\Omega)$, one may perform a fixed-point argument based on Lemma 3.3.*

Now let us present the main result of this paper.

Theorem 1.3. *Assume that f verifies (4) and (7). Then (1) is null controllable at any time $T > 0$.*

In the next theorem, we prove the existence of a class of functions $f = f_1 + f_2$ such that the system (1) is not null controllable for all $T > 0$.

Theorem 1.4. *Assume that $\theta > 4$. Then there exists functions $f = f(s)$ with $f(0) = 0$ and*

$$f(s) \sim |s| \log^{\theta}(1 + |s|) \quad \textit{when} \quad s \to +\infty$$

such that the system (1) with $f_1 + f_2 = f$ is not null controllable at any time $T > 0$.

We give the proof by using the same ideas in [15] and [16]. See for instance [21] and [22] for examples of systems that fail to be null controllable.

Proof. Let us consider the following function :

$$f(s) = \int_0^{|s|} \log^{\theta}(1 + z)dz \quad \forall s \in \mathbb{R}.$$

At first, let us prove the existence of a non-negative function $\rho \in \mathcal{D}(\Omega)$ such that

$$\rho = 0 \;\text{ in } \omega, \quad \int_{\Omega} \rho\, dx = 1 \quad \text{and} \quad \rho f^*(\frac{2\Delta^2 \rho}{\rho}) \in L^1(\Omega)$$

where f^* is the convex conjugate of f.

Let us notice that, from the definition of the convex conjugate, we have

$$f^*(s) = s(f')^{-1}(s) - f((f')^{-1}(s)).$$

Then, from the definition of f, we deduce that

$$f^*(s) = s(e^{s^{1/\theta}} - 1) - \int_0^{e^{s^{1/\theta}}-1} \log^{\theta}(1 + z)dz$$

and

$$f^*(s) \sim s^{1-1/\theta} e^{s^{1/\theta}} \quad \text{when} \quad s \to +\infty. \tag{9}$$

In fact, we used the Hopital's rule to deduce the last behavior when $s \to +\infty$. For the existence of the function ρ, without loss of generality, let us choose $r > 0$ such that $B(0, r) \subset \Omega \setminus \omega$ and let us set $\rho(x) = e^{-(r-|x|)^{-m}} \mathbb{1}_{B(0,r)}$ for $m > \dfrac{4}{\theta - 4}$. To prove that $\rho f^*(\dfrac{2\Delta^2 \rho}{\rho}) \in L^1(\Omega)$, the difficulties exist only at $\partial B(0, r)$ (i.e. $|x| \to r^-$). By doing several computations, we can deduce that

$$\frac{\Delta^2 \rho(x)}{\rho(x)} \sim (r - |x|)^{-4(m+1)} \quad \text{when} \quad |x| \to r^-.$$

Combining the last result with (9), we have

$$f^*(\frac{2\Delta^2 \rho(x)}{\rho(x)}) \sim (r - |x|)^{-4(m+1)(1-1/\theta)} e^{(r-|x|)^{-4(m+1)/\theta}} \quad \text{when} \quad |x| \to r^-.$$

Then, we deduce that

$$\rho(x) f^*(\frac{2\Delta^2 \rho(x)}{\rho(x)}) \sim (r - |x|)^{-4(m+1)(1-1/\theta)} \frac{e^{(r-|x|)^{-4(m+1)/\theta}}}{e^{(r-|x|)^{-m}}} \quad \text{when} \quad |x| \to r^-.$$

It is easy to see that $\rho f^*(\frac{2\Delta^2\rho}{\rho}) \in L^1(\Omega)$ if and only if $m > \frac{4(m+1)}{\theta}$ (i.e. $m > \frac{4}{\theta - 4}$).

To complete the proof, by multiplying (1)$_1$ for $f_1 + f_2 = f$ by ρ and integrating by parts, we have

$$\frac{d}{dt}\Big(- \int_\Omega \rho\, y\, dx \Big) = \int_\Omega \Delta^2 \rho\, y\, dx + \int_\Omega \rho\, f(y) dx. \tag{10}$$

From Young's inequality, we deduce

$$\begin{aligned} \Big| \int_\Omega \Delta^2 \rho\, y\, dx \Big| &\leq \int_\Omega \rho \frac{|\Delta^2 \rho|}{\rho} |y|\, dx \\ &\leq \frac{1}{2} \int_\Omega \rho\, f^* \Big(\frac{2|\Delta^2\rho|}{\rho} \Big) dx + \frac{1}{2} \int_\Omega \rho\, f(|y|)\, dx \end{aligned} \tag{11}$$

where f^* is the convex conjugate of f. Combining the last inequality with (10), we deduce

$$\frac{d}{dt}\Big(- \int_\Omega \rho\, y\, dx \Big) \geq -\frac{1}{2} \int_\Omega \rho\, f^* \Big(\frac{2|\Delta^2\rho|}{\rho} \Big) dx + \frac{1}{2} \int_\Omega \rho\, f(|y|)\, dx. \tag{12}$$

From the convexity of f and the fact that f is increasing, we deduce

$$\frac{d}{dt}\Big(- \int_\Omega \rho\, y\, dx \Big) \geq -\frac{1}{2} \int_\Omega \rho\, f^* \Big(\frac{2|\Delta^2\rho|}{\rho} \Big) dx + \frac{1}{2} f\Big(- \int_\Omega \rho\, y\, dx \Big). \tag{13}$$

Here, from what we did previously, we have $\rho\, f^* \Big(\frac{2|\Delta^2\rho|}{\rho} \Big) \in L^1(\Omega)$. Now, let us denote $z(t) = - \int_\Omega \rho\, y\, dx$, $z_0 = \int_\Omega \rho\, y_0\, dx$ and $k = \frac{1}{2} \int_\Omega \rho\, f^* \Big(\frac{2|\Delta^2\rho|}{\rho} \Big) dx$. We find the following system :

$$\begin{cases} z'(t) \geq -k + \frac{1}{2} f(z(t)), \\ z(0) = z_0. \end{cases} \tag{14}$$

The main idea is to show that z blows up at a finite time. So, let $y_0 \in L^2(\Omega)$ be such that

$$z_0 = - \int_\Omega \rho\, y_0\, dx > 0 \quad \text{and} \quad f(z_0) > 2k.$$

Let us notice that z is not decreasing. Moreover, let us introduce the following function :

$$G(z_0, s) = \int_{z_0}^{s} \frac{1}{-k + \frac{1}{2} f(\sigma)}\, d\sigma \ \ \forall s \geq z_0.$$

By differentiating $G(z_0, z(t))$, we have

$$\frac{d}{dt} G(z_0, z(t)) = \frac{z'(t)}{-k + \frac{1}{2} f(z(t)} \geq 1 \ \ \forall t \in [0, T^*). \tag{15}$$

From the definition of f, by applying the Hopital's rule, we deduce

$$f(\sigma) \sim \sigma \log^\theta (1 + \sigma) \quad \text{when} \quad |x| \to +\infty.$$

Then we easily deduce that $G(z_0, +\infty) < +\infty$. Integrating (15) in $(0, t)$, we have

$$t \leq G(z_0, z(t)) < +\infty \ \ \forall t \in [0, T^*).$$

Then we deduce easily that

$$T^* \leq \int_{z_0}^{\infty} \frac{1}{-k + \frac{1}{2} f(\sigma)}\, d\sigma < +\infty.$$

This completes the proof. In fact, we proved that, whatever $T > 0$, if we take z_0 sufficiently large such that $G(z_0, +\infty) < T$, then the blow up time of z and the blow up time of y in $L^1(\Omega)$ is smaller than T, which means that y is not globally defined in $[0, T]$. □ □

We can also extend Theorem 1.3 to more general functions f_1 and f_2:

Remark 1.5. *Let us notice that Theorem 1.3 holds true even if we replace f_1 (resp. f_2) by $\tilde{f}_1$ (resp. $\tilde{f}_2$) of the form :*

$$\tilde{f}_1(t,x,s,p,q) = \tilde{g}_1(t,x,s,p,q)s + \tilde{G}_1(t,x,s,p,q)\cdot p + \tilde{E}_1(t,x,s,p,q) : q, \tag{16}$$

$$\tilde{f}_2(t,x,s,p,q,r) = \tilde{g}_2(t,x,s,p,q,r)s + \tilde{G}_2(t,x,s,p,q,r)\cdot p + \tilde{E}_2(t,x,s,p,q,r) : q + \tilde{M}_2(t,x,s,p,q,r) \therefore r, \tag{17}$$

for all $(t,x,s,p,q,r) \in Q \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^{N^2} \times \mathbb{R}^{N^3}$ and where $\tilde{f}_1$ and $\tilde{f}_2$ satisfy :

1. *$\tilde{f}_1(\cdot,\cdot,s,p,q)$, $\tilde{f}_2(\cdot,\cdot,s,p,q,r) \in L^\infty(Q)$, $\forall\,(s,p,q,r) \in \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^{N^2} \times \mathbb{R}^{N^3}$.*
2. *$\tilde{f}_1(t,x,\cdot,\cdot,\cdot)$ and $\tilde{f}_2(t,x,\cdot,\cdot,\cdot,\cdot)$ are locally Lipschitz-continuous for (t,x) almost everywhere in Q, where the Lipschitz constant is independent of (t,x) for all (t,x) in any bounded set of $\mathbb{R} \times \mathbb{R}^N$.*
3. *Uniformly in $(t,x) \in Q$, we have for some $\gamma > 1$*

$$\begin{aligned} \lim_{|s|,|p|,|q|\to\infty} \frac{|\tilde{g}_1(t,x,s,p,q)|}{\log^3(1+|s|+|p|+|q|)} &= 0, \\ \lim_{|s|,|p|,|q|\to\infty} \frac{|\tilde{G}_1(t,x,s,p,q)|}{\log^2(1+|s|+|p|+|q|)} &= 0, \\ \lim_{|s|,|p|,|q|\to\infty} \frac{|\tilde{E}_1(t,x,s,p,q)|}{\log(1+|s|+|p|+|q|)} &= 0 \end{aligned} \tag{18}$$

and

$$\begin{aligned} \lim_{|s|,|p|,|q|,|r|\to\infty} \frac{|\tilde{g}_2(s,p,q,r)|}{\log^{3/\gamma}(1+|s|+|p|+|q|+|r|)} &= 0, \\ \lim_{|s|,|p|,|q|,|r|\to\infty} \frac{|\tilde{G}_2(s,p,q,r)|}{\log^{2/\gamma}(1+|s|+|p|+|q|+|r|)} &= 0, \\ \lim_{|s|,|p|,|q|,|r|\to\infty} \frac{|\tilde{E}_2(s,p,q,r)|}{\log^{1/\gamma}(1+|s|+|p|+|q|+|r|)} &= 0, \\ \lim_{|s|,|p|,|q|,|r|\to\infty} \frac{e^{\frac{\gamma}{\gamma-1}|\tilde{M}_2(s,p,q,r)|^{1/2}}}{\log(1+|s|+|p|+|q|+|r|)} &= 0. \end{aligned} \tag{19}$$

Let us now give the definition of the exact controllability to the trajectories for (1).

Definition 1.6. *Let $\bar{y}_0 \in W^{2,\infty}(\Omega) \cap H_0^2(\Omega)$ and let $\bar{y} \in C^0([0,T]; W^{2,\infty}(\Omega))$ fulfill the following system :*

$$\begin{cases} \partial_t \bar{y} + \Delta^2 \bar{y} + f_1(\bar{y},\nabla\bar{y},\nabla^2\bar{y})\mathbb{1}_{\{f_2\equiv 0\}} + f_2(\bar{y},\nabla\bar{y},\nabla^2\bar{y},\nabla^3\bar{y}) = 0 & \text{in } Q\ , \\ \bar{y} = \dfrac{\partial \bar{y}}{\partial \vec{n}} = 0 & \text{on } \Sigma\ , \\ \bar{y}(0,\cdot) = \bar{y}_0(\cdot) & \text{in } \Omega\ . \end{cases} \tag{20}$$

It is said that (1) is exactly controllable to the trajectory $\bar{y}$ at time $T > 0$, if there exists $v \in L^\infty((0,T)\times\omega)$ such that the corresponding initial problem (1) admits a solution $y \in C^0([0,T]; L^2(\Omega))$ satisfying

$$y(T,\cdot) = \bar{y}(T,\cdot) \ \text{ in } \Omega.$$

Let us now present the second main result of this paper, which is the exact controllability to the trajectories of (1) :

Theorem 1.7. *Assume that f_1 (resp. f_2) verifies (4) and (18) (resp. (19)) and the following conditions :*

$$\begin{aligned}
&\lim_{|s|,|p|,|q|\to\infty}\left|\int_0^1 \frac{\partial}{\partial s} f_1(s^*+\lambda\, s, p^*+\lambda\, p, q^*+\lambda\, q)d\lambda\right|\frac{1}{\log^3(1+|s|+|p|+|q|)}=0,\\
&\lim_{|s|,|p|,|q|\to\infty}\left|\int_0^1 \frac{\partial}{\partial p_i} f_1(s^*+\lambda\, s, p^*+\lambda\, p, q^*+\lambda\, q)d\lambda\right|\frac{1}{\log^2(1+|s|+|p|+|q|)}=0,\\
&\lim_{|s|,|p|,|q|\to\infty}\left|\int_0^1 \frac{\partial}{\partial q_{jk}} f_1(s^*+\lambda\, s, p^*+\lambda\, p, q^*+\lambda\, q)d\lambda\right|\frac{1}{\log(1+|s|+|p|+|q|)}=0,
\end{aligned}\tag{21}$$

for any $1\le i,j,k\le N$ and for all (s^*,p^*,q^*) in any bounded set of $\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}^{N^2}$ and for some $\gamma>1$

$$\begin{aligned}
&\lim_{|s|,|p|,|q|,|r|\to\infty}\left|\int_0^1 \frac{\partial}{\partial s} f_2(s^*+\lambda\, s, p^*+\lambda\, p, q^*+\lambda\, q, r^*+\lambda\, r)d\lambda\right|\frac{1}{\log^{3/\gamma}(1+|s|+|p|+|q|+|r|)}=0,\\
&\lim_{|s|,|p|,|q|,|r|\to\infty}\left|\int_0^1 \frac{\partial}{\partial p_i} f_2(s^*+\lambda\, s, p^*+\lambda\, p, q^*+\lambda\, q, r^*+\lambda\, r)d\lambda\right|\frac{1}{\log^{2/\gamma}(1+|s|+|p|+|q|+|r|)}=0,\\
&\lim_{|s|,|p|,|q|,|r|\to\infty}\left|\int_0^1 \frac{\partial}{\partial q_{jk}} f_2(s^*+\lambda\, s, p^*+\lambda\, p, q^*+\lambda\, q, r^*+\lambda\, r)d\lambda\right|\frac{1}{\log^{1/\gamma}(1+|s|+|p|+|q|+|r|)}=0,\\
&\lim_{|s|,|p|,|q|,|r|\to\infty}\exp\left(\frac{\gamma}{\gamma-1}\left|\int_0^1 \frac{\partial}{\partial r_{lmn}} f_2(s^*+\lambda\, s, p^*+\lambda\, p, q^*+\lambda\, q, r^*+\lambda\, r)d\lambda\right|^{1/2}\right)\frac{1}{\log(1+|s|+|p|+|q|+|r|)}=0,
\end{aligned}\tag{22}$$

for any $1\le i,j,k,l,m,n\le N$ and for all (s^*,p^*,q^*,r^*) in any bounded set of $\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}^{N^2}\times\mathbb{R}^{N^3}$. Then (1) is exactly controllable to the trajectories at any time $T>0$.

Proof. Let us denote $w_0=y_0-\bar{y}_0$ and $w=y-\bar{y}$. Then w verifies the following system :

$$\begin{cases}
\partial_t w+\Delta^2 w+\tilde{f}(\cdot,\cdot,w,\nabla w,\nabla^2 w)\mathbb{1}_{\{\tilde{f}_2\equiv 0\}}+\tilde{f}_2(\cdot,\cdot,w,\nabla w,\nabla^2 w,\nabla^3 w)=\chi_\omega v & \text{in } Q\ ,\\
w=\dfrac{\partial w}{\partial \vec{n}}=0 & \text{on } \Sigma\ ,\\
w(0,\cdot)=w_0(\cdot) & \text{in } \Omega\ ,
\end{cases}$$

where $\tilde{f}_1$ and $\tilde{f}_2$ satisfy (16) and (17) with

$$\begin{aligned}
\tilde{g}_1(t,x,s,p,q)&=\int_0^1 \frac{\partial}{\partial s} f_1(\bar{y}(t,x)+\lambda\, s,\nabla\,\bar{y}(t,x)+\lambda\, p,\nabla^2\,\bar{y}(t,x)+\lambda\, q)d\lambda,\\
\tilde{G}_{1,i}(t,x,s,p,q)&=\int_0^1 \frac{\partial}{\partial p_i} f_1(\bar{y}(t,x)+\lambda\, s,\nabla\,\bar{y}(t,x)+\lambda\, p,\nabla^2\,\bar{y}(t,x)+\lambda\, q)d\lambda\\
\tilde{E}_{1,jk}(t,x,s,p,q)&=\int_0^1 \frac{\partial}{\partial q_{jk}} f_1(\bar{y}(t,x)+\lambda\, s,\nabla\,\bar{y}(t,x)+\lambda\, p,\nabla^2\,\bar{y}(t,x)+\lambda\, q)d\lambda
\end{aligned}$$

for $1 \leq i,j,k \leq N$ and

$$\tilde{g}_2(t,x,s,p,q,r) = \int_0^1 \frac{\partial}{\partial s} f_2(\bar{y}(t,x) + \lambda\, s, \nabla\, \bar{y}(t,x) + \lambda\, p, \nabla^2\, \bar{y}(t,x) + \lambda\, q, \nabla^3\, \bar{y}(t,x) + \lambda\, r) d\lambda,$$

$$\tilde{G}_{2,i}(t,x,s,p,q) = \int_0^1 \frac{\partial}{\partial p_i} f_2(\bar{y}(t,x) + \lambda\, s, \nabla\, \bar{y}(t,x) + \lambda\, p, \nabla^2\, \bar{y}(t,x) + \lambda\, q, \nabla^3 \bar{y}(t,x) + \lambda\, r) d\lambda$$

$$\tilde{E}_{2,jk}(t,x,s,p,q) = \int_0^1 \frac{\partial}{\partial q_{jk}} f_2(\bar{y}(t,x) + \lambda\, s, \nabla\, \bar{y}(t,x) + \lambda\, p, \nabla^2\, \bar{y}(t,x) + \lambda\, q, \nabla^3 \bar{y}(t,x) + \lambda\, r) d\lambda$$

$$\tilde{M}_{2,lmn}(t,x,s,p,q) = \int_0^1 \frac{\partial}{\partial r_{lmn}} f_2(\bar{y}(t,x) + \lambda\, s, \nabla\, \bar{y}(t,x) + \lambda\, p, \nabla^2\, \bar{y}(t,x) + \lambda\, q, \nabla^3 \bar{y}(t,x) + \lambda\, r) d\lambda$$

for $1 \leq i,j,k,l,m,n \leq N$. Then, from Remark 1.5 we deduce our desired result.

□

Let us now present some results concerning the null controllability and the exact controllability to the trajectories of fourth order parabolic equations.

• We start by the linear case. The author in [19], using the same ideas as in [20], proved the boundary null controllability of a linear fourth order parabolic equation in dimension 1 where the control acts on the whole boundary. Using the ideas of [27], the author in [32] proved the null controllability of a linear fourth order parabolic equation in higher dimension. Moreover, the author in [6] proved the null controllability of the linear Kuramoto-Sivashinsky equation by using a moments method and spectral analysis. On the other hand, by using the ideas in [16], the authors in [5], [7], [8] and [34] proved some Carleman estimates for fourth order parabolic operators in dimension 1 which led to null controllability results. In higher dimension, a Carleman estimate for a fourth order parabolic equation has recently been proved in [17]. That result implies the null controllability with L^2-controls of the following linear system :

$$\begin{cases} \partial_t y + \Delta^2 y + a_0 y + \nabla \cdot (B_0\, y) + \displaystyle\sum_{ij=1}^{N} \partial_{ij}(D_{ij}\, y) + \Delta(a_1\, y) = \chi_\omega v & \text{in } Q\ , \\ y = \dfrac{\partial\, y}{\partial \vec{n}} = 0 & \text{on } \Sigma\ , \\ y(0,\cdot) = y_0(\cdot) & \text{in } \Omega\ , \end{cases} \tag{23}$$

where $a_0,\ a_1 \in L^\infty(Q;\mathbb{R})$, $B_0 \in L^\infty(Q;\mathbb{R}^N)$, $D \in L^\infty(Q;\mathbb{R}^{N^2})$ and $y_0 \in L^2(\Omega)$.

• Let us now cite some works concerning the null and the exact controllability to the trajectories of semi-linear parabolic equations. In [11], by using a Carleman estimate and Kakutani's fix point theorem, the authors proved some controllability properties for the following system :

$$\begin{cases} \partial_t y - \Delta\, y + f(y, \nabla y) = \chi_\omega v & \text{in } Q\ , \\ y = 0 & \text{on } \Sigma\ , \\ y(0,\cdot) = y_0(\cdot) & \text{in } \Omega\ , \end{cases}$$

where f is a locally Lipschitz-continuous function satisfying some properties similar to (4) and (7) and y_0 is given in an appropriate space. Before that, the authors in [15] and [1] proved some controllability properties for the above system, where the nonlinear term is assumed to be of the form $f(y)$ and $div(f(y))$ respectively. Furthermore, using the ideas of [23] and [24], the authors proved in [14] the exact controllability to the trajectories of the following system :

$$\begin{cases} \partial_t y - \Delta\, y + F(y, \nabla y) = \chi_\omega v & \text{in } Q\ , \\ \dfrac{\partial\, y}{\partial \vec{n}} + f(y) = 0 & \text{on } \Sigma\ , \\ y(0,\cdot) = y_0(\cdot) & \text{in } \Omega\ , \end{cases}$$

where $F : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ are given functions and $y_0 \in L^\infty(\Omega)$. For more details about this subject, see for instance [2], [13] and [16] and the references therein. In addition, in [9], the authors studied the boundary null controllability of a fourth order parabolic semi-linear equation where the control acts on the whole boundary. On the other hand, several controllability results for semi-linear fourth order parabolic equations in dimension 1 were proved in [5], [7], [8] by using Carleman estimates and an inverse mapping theorem. Additionally, based on a Carleman estimate and Kakutani's fix point theorem, the author in [34] deduced a null controllability result for the semi-linear system

$$\begin{cases} \partial_t y + \partial_{xxxx} y + f(y) = \chi_\omega v & \text{in } (0,T) \times (0,1) \ , \\ y(\cdot,0) = y(\cdot,1) = 0 & \text{in } (0,T) \ , \\ \partial_x y(\cdot,0) = \partial_x y(\cdot,1) = 0 & \text{in } (0,T) \ , \\ y(0,\cdot) = y_0(\cdot) & \text{in } (0,1) \ , \end{cases}$$

where $y_0 \in L^2(0,1)$ and f is a globally Lipschitz continuous function. As far as we know, in higher dimensions, there are no results on the null controllability of semi-linear fourth order parabolic equations.

Let us remark that all three main results of this paper (Theorem 1.3, Theorem 1.4 and Theorem 1.7 below) are the first controllability results for fourth-order semilinear parabolic equations in dimension $N \geq 2$. As we explained below, the only previous controllability result concerning a semilinear fourth-order parabolic equation is [34], where the case of $N = 1$ and a nonlinear term $f(y)$ is treated (only depending on y) with f globally Lipschitz.

Concerning the new tools used in this paper, we prove a new Carleman inequality for a (very) weak fourth-order parabolic equation which is defined by transposition. This is the first time such an inequality is proved for a fourth-order parabolic equation with non-homogeneous boundary conditions and for source terms which do not belong to L^p spaces. The source terms we consider belong to $H^{-1}(0,T;L^2(\Omega)) \cap L^2(0,T;H^{-4}(\Omega))$ and certain general first, second and third order terms in the spatial variable. In the literature, the authors had never considered $L^2(0,T;H^{-1}(\Omega))$ terms, not even in dimension $N = 1$.

The rest of this paper is organised as follows. In the next section, we prove a new Carleman estimate for a fourth order parabolic equation with non-regular source terms. Furthermore, in Section 3, we prove the existence of controls in $L^\infty((0,T) \times \omega)$ such that an associated linear system is null controllable. In the last section, we perform a Kakutani fixed point theorem and we prove Theorem 1.3.

2 New Carleman estimate for a linear fourth order parabolic equation with general zero, first and second order terms.

Let us introduce the following backwards in time system:

$$\begin{cases} \partial_t q - \Delta^2 q = F_0 + \nabla \cdot F_1 + \displaystyle\sum_{i,j=1}^{N} \partial_{ij} \hat{F}_{ij} & \text{in } Q \ , \\ q = f_0 \ , \ \dfrac{\partial q}{\partial \vec{n}} = \tilde{f} & \text{on } \Sigma \ , \\ q(T,\cdot) = q_0(\cdot) & \text{in } \Omega \ . \end{cases} \tag{24}$$

Let us assume the following conditions on the data:

$$\begin{aligned} &q_0 \in H^{-2}(\Omega), \quad F_0 \in L^2(Q), \quad F_1 \in L^2(Q)^N, \\ &\hat{F} \in L^2(Q)^{N^2}, \quad f_0 \in L^2(\Sigma), \qquad \tilde{f} \in L^2(\Sigma). \end{aligned} \tag{25}$$

Remark 2.1. *For the proof of the null controllability of the system (1), it suffices to take* $f_0 = \tilde{f} = 0$ *but we do the general case of any* $f_0,\ \tilde{f} \in L^2(\Sigma)$ *so our Carleman inequality can be applied for other (more general) boundary conditions.*

Under the previous assumptions, it will be said that $q \in L^2(Q)$ is the unique solution by transposition of system (24) if for every $h \in L^2(Q)$, we have

$$\begin{aligned} \iint_Q h\, q\, dxdt \;&=\; -\iint_Q \Big(F_0\, z - F_1 \cdot \nabla z + \sum_{i,j=1} \hat{F}_{ij}\, \partial_{ij} z\Big) dxdt \\ &\quad + \iint_\Sigma \frac{\partial \Delta z}{\partial \vec{n}}\, f_0\, d\sigma dt - \iint_\Sigma \Delta\, z\, \tilde{f}\, d\sigma dt + < q_0, z(T,\cdot) >_{H^{-1}(\Omega)\times H_0^1(\Omega)}\ , \end{aligned} \tag{26}$$

where z verifies

$$\begin{cases} \partial_t z + \Delta^2 z = h & \text{in } Q\ , \\ z = \dfrac{\partial z}{\partial \vec{n}} = 0 & \text{on } \Sigma\ , \\ z(0,\cdot) = 0 & \text{in } \Omega\ . \end{cases} \tag{27}$$

Observe that this definition makes sense since $h \in L^2(Q)$ implies $z \in L^2(0,T;H^4(\Omega)) \cap H^1(0,T;L^2(\Omega))$. Let us explain in which sense the boundary conditions in (24) are satisfied :

Remark 2.2. *From (26), by taking* $z \in D(Q) \subset \{z \in L^2(0,T;H^4(\Omega)) \cap H^1(0,T;L^2(\Omega));\ z = \dfrac{\partial z}{\partial \vec{n}} = 0\ \text{on}\ \Sigma,\ z(0,\cdot) = 0\ \text{in}\ \Omega\}$*, we can deduce that*

$$\partial_t q - \Delta^2 q = F_0 + \nabla \cdot F_1 + \sum_{i,j=1}^{N} \partial_{ij} \hat{F}_{ij} \ \ in\ \ D'(Q).$$

Then we deduce that $\Delta^2 q \in H^{-1}(0,T;H^{-2}(\Omega))$*. Combining this with the fact that* $q \in L^2(Q)$*, we deduce that* $q_{|\Sigma} \in H^{-1}(0,T;H^{-1/2}(\partial\Omega))$ *and* $\dfrac{\partial q}{\partial \vec{n}}|_\Sigma \in H^{-1}(0,T;H^{-3/2}(\partial\Omega))$*.*

Let us introduce the following weight functions used in [17] :

$$\alpha(x,t) = \frac{e^{4\lambda||\eta||_\infty} - e^{\lambda(2||\eta||_\infty + \eta(x))}}{t^{1/2}(T-t)^{1/2}},\ \xi(x,t) = \frac{e^{\lambda(2||\eta||_\infty + \eta(x))}}{t^{1/2}(T-t)^{1/2}},$$

where η satisfies:

$$\eta \in C^4(\bar{\Omega}), \quad \eta_{|\partial\Omega} = 0, \quad |\nabla \eta| \geq C_0 > 0 \text{ in } \Omega \setminus \overline{\omega''},$$

with $\omega'' \subset\subset \omega$ an open set and $C_0 = C_0(\Omega, \omega)$. For the existence of η, see [16]. Let us notice some essential properties on the weight functions :

Remark 2.3. *We have*

$$\nabla \xi = \lambda \xi \nabla \eta \ \ in\ Q, \quad \xi^{-1} \leq \frac{T}{2} \ \ in\ Q, \quad \nabla \eta = \frac{\partial \eta}{\partial \vec{n}} \vec{n} \ \ on\ \Sigma.$$

The main result in this section is a global Carleman inequality for the solution by transposition of (24) :

Proposition 2.4. *Let us assume* (25) *and let ω' be an open set satisfying $\omega'' \subset\subset \omega' \subset\subset \omega$. Then, there exists a positive constant $C_0 = C_0(\Omega, \omega')$ such that*

$$\begin{aligned} s^6\lambda^8 \iint_Q e^{-2s\alpha}\xi^6|q|^2 dxdt \leq C_0\Big(s^7\lambda^8 \iint_{(0,T)\times\omega'} e^{-2s\alpha}\xi^7|q|^2 dxdt + \iint_Q e^{-2s\alpha}|F_0|^2 dxdt \\ + s^2\lambda^2 \iint_Q e^{-2s\alpha}\xi^2|F_1|^2 dxdt + s^4\lambda^4 \iint_Q e^{-2s\alpha}\xi^4 \sum_{i,j=1}^N |\hat{F}_{ij}|^2 dxdt \\ + s^5\lambda^5 \iint_\Sigma e^{-2s\alpha}\xi^3|\tilde{f}|^2 d\sigma dt + s^7\lambda^7 \iint_\Sigma e^{-2\alpha}\xi^7|f_0|^2 d\sigma dt \Big), \end{aligned} \tag{28}$$

for any $\lambda \geq C_0$ and any $s \geq C_0(T^{1/2} + T)$.

Remark 2.5. *Let us denote*

$$Y = \overline{C^\infty(\overline{Q})}^{||\cdot||_Y} \quad \text{with } ||G||_Y = \left(||G||^2_{L^2(Q)} + ||G(T,\cdot)||^2_{H^{-2}(\Omega)} \right)^{1/2}$$

and

$$\tilde{X} = \overline{C^\infty(\overline{Q})}^{||\cdot||_{\tilde{X}}} \quad \text{with } ||F||_{\tilde{X}} = \left(\iint_Q |F|^2 dxdt + \iint_\Sigma |F|^2 d\sigma dt \right)^{1/2}.$$

Assume that we add in the right-hand side of $(24)_1$ *the following three terms :*

$$\sum_{i,j,k=1}^N \partial_{ijk}\tilde{F}_{ijk}, \quad \sum_{i,j=1} \partial_{ij}\Delta F^*_{ij}, \quad \partial_t G$$

where $\tilde{F} \in \tilde{X}^{N^3}$, $F^* \in \tilde{X}^{N^2}$, $G \in Y$ and that we change the boundary conditions in (24) *to*

$$\begin{cases} q = f_0 & \text{on } \Sigma \ , \\ \dfrac{\partial q}{\partial \vec{n}} + \displaystyle\sum_{i,j=1}^N \partial_i F^*_{ij}\, n_j = \tilde{f} & \text{on } \Sigma \ . \end{cases}$$

Assume also that we add the following terms

$$\iint_Q \sum_{i,j=1}^N \partial_{ij}\Delta\, z\, \tilde{F}_{ijk} dxdt\ , \quad -\iint_Q \sum_{i,j,k=1}^N \partial_{ijk} z\, \tilde{F}_{ijk} dxdt\ , \quad -\iint_Q \partial_t z\, G\, dxdt,$$

$$\iint_\Sigma \sum_{i,j,k=1}^N \partial_{ik} z\, \tilde{F}_{ijk}\, n_k\, d\sigma\, dt, \quad -\iint_\Sigma \sum_{i,j=1}^N \partial_j \Delta\, z\, F^*_{ij}\, n_k\, d\sigma\, dt, \quad \langle G(T,\cdot), z(T,\cdot)\rangle$$

in the right-hand side of (26). Then Lemma 2.4 remains true if we add the terms

$$s^6\lambda^6 \iint_Q e^{-2s\alpha}\xi^6 \sum_{i,j,k=1}^N |\tilde{F}_{ijk}|^2 dxdt\ ,\ s^8\lambda^8 \iint_Q e^{-2s\alpha}\xi^8 \left(\sum_{i,j=1}^N |F^*_{ij}|^2 + |G|^2 \right) dxdt$$

and

$$s^5\lambda^5 \iint_\Sigma e^{-2s\alpha}\xi^5 \sum_{i,j,k=1}^N |\tilde{F}_{ijk}\, n_j|^2 d\sigma dt\ ,\ s^7\lambda^7 \iint_\Sigma e^{-2s\alpha}\xi^7 \sum_{i,j=1}^N |F^*_{ij}\, n_i|^2 d\sigma dt$$

to the right-hand side of (28).

To give a sense to $q_{|\Sigma}$ and $\dfrac{\partial q}{\partial \vec{n}}|_\Sigma + \displaystyle\sum_{i,j=1}^N (\partial_i F^*_{ij})_{|\Sigma}\, n_j$, we give the following remark :

Remark 2.6. *Let* $\tilde{q} \in L^2(Q)$ *satisfy*

$$\partial_t \tilde{q} - \Delta^2 \tilde{q} = \sum_{i,j=1}^N \partial_{ij} \Delta F^*_{ij} \;\; \text{in} \;\; D'(Q).$$

Then we deduce that $\displaystyle\sum_{i,j=1}^N \partial_{ij} \Delta \Big(\tilde{q}\, \delta_{ij} + F^*_{ij} \Big) \in H^{-1}(0,T;L^2(\Omega))$. *This, combined with the fact that* $\tilde{q} + F^*_{ij} \in L^2(Q)$, $\forall (i,j) \in \{1,...,N\}^2$, *implies that* $\tilde{q}_{|\Sigma} \in H^{-1}(0,T;H^{-1/2}(\partial\Omega))$ *and* $\dfrac{\partial \tilde{q}}{\partial \vec{n}} + \displaystyle\sum_{i,j=1}^N \partial_i F^*_{ij}\, n_j \in H^{-1}(0,T;H^{-3/2}(\partial\Omega))$.

Let us recall the Carleman estimate proved in [17] (see Theorem 1.2 and Open Problem 3.2 in that reference).

Lemma 2.7. *There exists a positive constant* $C_0 = C_0(\Omega, \omega')$ *such that*

$$\begin{aligned}\iint_Q e^{-2s\alpha} \Big(& s^6 \lambda^8 \xi^6 |\varphi|^2 + s^4 \lambda^6 \xi^4 |\nabla \varphi|^2 + s^3 \lambda^4 \xi^3 |\Delta \varphi|^2 \\ & + s^2 \lambda^4 \xi^2 |\nabla^2 \varphi|^2 + s \lambda^2 \xi |\nabla \Delta \varphi|^2 + s^{-1} \xi^{-1} (|\partial_t \varphi|^2 + |\Delta^2 \varphi|^2) \Big) dxdt \\ & \leq C_0 \Big(s^7 \lambda^8 \iint_{\omega' \times (0,T)} e^{-2s\alpha} \xi^7 |\varphi|^2 dxdt + \iint_Q e^{-2s\alpha} |f|^2 dxdt \Big),\end{aligned}$$

for any $\lambda \geq C_0$, *any* $s \geq C_0(T^{1/2} + T)$ *and where* φ *satisfies*

$$\begin{cases} -\partial_t \varphi + \Delta^2 \varphi = f & \text{in } Q \;, \\ \varphi = \dfrac{\partial \varphi}{\partial \vec{n}} = 0 & \text{on } \Sigma \;, \\ \varphi(T, \cdot) = \varphi_0(\cdot) & \text{in } \Omega \;. \end{cases}$$

Let us now start the proof of Proposition 2.4.

Proof. This proof is inspired by [24]. Let s and λ be as in Lemma 2.7 and let us introduce the following maps

$$\kappa(p,p') = \iint_Q e^{-2s\alpha} \mathfrak{L}^* p\, \mathfrak{L}^* p' dxdt + s^7 \lambda^8 \iint_{(0,T)\times\omega'} e^{-2s\alpha} \xi^7\, p\, p' dxdt \;,\; \forall p, p' \in P_0 \tag{29}$$

and

$$l(p') = s^6 \lambda^8 \iint_Q e^{-2s\alpha} \xi^6\, p'\, q dxdt \;,\; \forall p' \in P_0, \tag{30}$$

where $P_0 = \{z \in C^4(\overline{Q}); z = \dfrac{\partial z}{\partial \vec{n}} = 0 \text{ on } \Sigma\}$, $\mathfrak{L}p = \partial_t p + \Delta^2 p$ and $\mathfrak{L}^* p = -\partial_t p + \Delta^2 p$. Then $\kappa(\cdot,\cdot)$ is a definite positive and symmetric bilinear form in P_0.

Let P be the completion of P_0 for the norm $||.||_P = (\kappa(\cdot,\cdot))^{1/2}$. Thanks to Lemma 2.7, P is a Hilbert space for the scalar product $\kappa(\cdot,\cdot)$. It is clear that by using Lemma 2.7, we have that l is a continuous linear form on P :

$$|l(p')| \leq C_0 \left(s^6\lambda^8 \iint_Q e^{-2s\alpha}\xi^6|q|^2 dxdt \right)^{1/2} ||p'||_P, \;\; \forall p' \in P.$$

Consequently, from the Lax-Milgram lemma, the following variational equation possesses exactly one solution $\hat{p} \in P$:

$$\kappa(\hat{p}, p') = l(p'), \;\; \forall p' \in P.$$

Let us denote $\hat{z} = e^{-2s\alpha}\mathfrak{L}^*\hat{p}$ and $\hat{u} = s^7\lambda^8 e^{-2s\alpha}\xi^3\,\hat{p}\chi_{\omega'}$. Then it is not difficult to see that $(\hat{z}, \hat{u})$ satisfies

$$\begin{cases} \partial_t \hat{z} + \Delta^2 \hat{z} = s^6\lambda^8 e^{-2s\alpha}\xi^6\, q - \hat{u}\chi_{\omega'} & \text{in } Q\ , \\ \hat{z} = \dfrac{\partial \hat{z}}{\partial \vec{n}} = 0 & \text{on } \Sigma\ , \\ \hat{z}(0,\cdot) = \hat{z}(T,\cdot) = 0 & \text{in } \Omega\ . \end{cases} \tag{31}$$

Let us prove the following lemma :

Lemma 2.8. *There exists a positive constant $C_0 = C_0(\Omega, \omega')$ such that*

$$\begin{aligned} & s^{-7}\lambda^{-8} \iint_{(0,T)\times\omega'} e^{2s\alpha}\xi^{-7}|\hat{u}|^2 dxdt + \iint_Q e^{2s\alpha}|\hat{z}|^2 dxdt + s^{-2}\lambda^{-2} \iint_Q e^{2s\alpha}\xi^{-2}|\nabla \hat{z}|^2 dxdt \\ & + s^{-4}\lambda^{-4} \iint_Q e^{2s\alpha}\xi^{-4} \sum_{i,j=1}^N |\partial_{ij}\hat{z}|^2 dxdt + s^{-5}\lambda^{-5} \iint_\Sigma e^{2s\alpha}\xi^{-5} \sum_{i,j=1}^N |\partial_{ij}\,\hat{z}|^2\, d\sigma dt \\ & + s^{-7}\lambda^{-7} \iint_\Sigma e^{2s\alpha}\xi^{-7} \sum_{i,j,k=1}^N |\partial_{ijk}\,\hat{z}|^2\, d\sigma dt + s^{-6}\lambda^{-6} \iint_Q e^{2s\alpha}\xi^{-6} \sum_{i,j,k=1}^N |\partial_{ijk}\hat{z}|^2 dxdt \\ & + s^{-8}\lambda^{-8} \iint_Q e^{2s\alpha}\xi^{-8} \left(\sum_{i,j,k,l=1}^N |\partial_{ijkl}\hat{z}|^2 + |\partial_t\,\hat{z}|^2 \right) dxdt \\ & \leq C_0\, s^6\lambda^8 \iint_Q e^{-2s\alpha}\xi^6|q|^2 dxdt. \end{aligned} \tag{32}$$

for $\lambda \geq C_0$ and $s \geq C_0(T + T^{1/2})$.

Remark 2.9. *For the proof of Proposition 2.4, there is no need to introduce the last three terms in the left-hand side of (32), but we will add them to justify Remark 2.5.*

Proof of Lemma 2.8. We divide it in several steps.

- Step 1. Estimate of the first four terms in the left-hand side of (32) .

By using the definition of κ and l for $p' = \hat{p}$ and combining this with the Carleman inequality presented in Lemma 2.7, we can estimate the first two terms in the left-hand side of (32). In other words, we have

$$s^{-7}\lambda^{-8} \iint_{(0,T)\times\omega'} e^{2s\alpha}\xi^{-7}|\hat{u}|^2 dxdt + \iint_Q e^{2s\alpha}|\hat{z}|^2 dxdt \leq C_0\, s^6\lambda^8 \iint_Q e^{-2s\alpha}\xi^6|q|^2 dxdt. \tag{33}$$

To estimate the fourth term in the left-hand side of (32), we multiply $(31)_1$ by $s^{-4}\lambda^{-4}e^{2s\alpha}\xi^{-4}\hat{z}$. Then by integrating by parts, we get

$$\begin{aligned} s^{-4}\lambda^{-4}\iint_Q e^{2s\alpha}\xi^{-4}|\Delta\hat{z}|^2dxdt \quad \leq \quad & C_0\bigg(s^{-2}\lambda^{-2}\iint_Q e^{2s\alpha}\xi^{-2}|\nabla\hat{z}|^2dxdt+\iint_Q e^{2s\alpha}|\hat{z}|^2dxdt \\ & +s^{-7}\lambda^{-8}\iint_{(0,T)\times\omega'} e^{2s\alpha}\xi^{-7}|\hat{u}|^2dxdt+s^6\lambda^8\iint_Q e^{-2s\alpha}\xi^6|q|^2dxdt\bigg) \\ & +\frac{1}{2}s^{-4}\lambda^{-4}\iint_Q e^{2s\alpha}\xi^{-4}|\Delta\hat{z}|^2dxdt, \end{aligned} \tag{34}$$

for $\lambda \geq C_0$ and $s \geq C_0(T+T^{1/2})$. For the first term in the right-hand side of (34) by integrating by parts, we obtain

$$s^{-2}\lambda^{-2}\iint_Q e^{2s\alpha}\xi^{-2}|\nabla\hat{z}|^2dxdt \quad \leq \quad \varepsilon\, s^{-4}\lambda^{-4}\iint_Q e^{2s\alpha}\xi^{-4}|\Delta\hat{z}|^2dxdt+C_0\iint_Q e^{2s\alpha}|\hat{z}|^2dxdt, \tag{35}$$

for all $\varepsilon > 0$. Combining (34) and (35) with (33) and taking ε small enough, we get

$$s^{-4}\lambda^{-4}\iint_Q e^{2s\alpha}\xi^{-4}|\Delta\hat{z}|^2dxdt+s^{-2}\lambda^{-2}\iint_Q e^{2s\alpha}\xi^{-2}|\nabla\hat{z}|^2dxdt \leq C_0\, s^6\lambda^8\iint_Q e^{-2s\alpha}\xi^6|q|^2dxdt, \tag{36}$$

for $\lambda \geq C_0$ and $s \geq C_0\,(T+T^{1/2})$. Next, we set $\tilde{w} = s^{-2}\lambda^{-2}e^{s\alpha}\xi^{-2}\hat{z}$. We observe that $\|\Delta\tilde{w}\|^2_{L^2(Q)}$ is bounded by the term in the right-hand side of (32) , which means that $||\tilde{w}||^2_{L^2(0,T;H^2(\Omega))}$ also is, since $\hat{z} = 0$ on Σ. This allows to deduce that

$$\begin{aligned} s^{-7}\lambda^{-8}\iint_{(0,T)\times\omega'} e^{2s\alpha}\xi^{-7}|\hat{u}|^2dxdt+\sum_{|\tau|\leq 2}\left\|s^{-|\tau|}\lambda^{-|\tau|}e^{s\alpha}\xi^{-|\tau|}D^\tau\hat{z}\right\|^2_{L^2(Q)} \\ \leq C_0\, s^6\lambda^8\iint_Q e^{-2s\alpha}\xi^6|q|^2dxdt, \end{aligned} \tag{37}$$

for $\lambda \geq C_0$ and $s \geq C_0(T+T^{1/2})$. Before we estimate the following terms in (32), let us present a Lemma which will be useful for the sequel :

Lemma 2.10. *There exists a constant $C_0(\Omega) > 0$ such that*

$$\begin{aligned} \left\|e^{s\alpha}\xi^{-\gamma/2}g\right\|^2_{L^2(\partial\Omega)} \quad \leq \quad & C_0\bigg((1+\frac{1}{\mu})s\lambda\int_\Omega e^{2s\alpha}\xi^{-\gamma+1}|g|^2\,dx \\ & +\mu s^{-1}\lambda^{-1}\int_\Omega e^{2s\alpha}\xi^{-\gamma-1}|\nabla g|^2\,dx\bigg)\ ,\ \forall g\in H^1(\Omega) \end{aligned} \tag{38}$$

for $s \geq C_0T$ and where $\gamma \in \mathbb{N}$ and $\mu > 0$.

Proof. Let us introduce $\theta \in C^2(\overline{\Omega})$ such that

$$\begin{cases} \theta = 0 & \text{on } \partial\Omega, \\ \dfrac{\partial\theta}{\partial\vec{n}} = 1 & \text{on } \partial\Omega. \end{cases}$$

Let us notice that by integrating by parts, we have

$$\begin{aligned} I \quad := \quad & 2\int_\Omega e^{2s\alpha}\xi^{-\gamma}\sum_{i=1}^N \partial_i g\, g\,\partial_i\theta dx \\ = \quad & -\int_\Omega \nabla\cdot(e^{2s\alpha}\xi^{-\gamma}\nabla\theta)|g|^2dx+\int_{\partial\Omega} e^{2s\alpha}\xi^{-\gamma}|g|^2\,\frac{\partial\theta}{\partial\vec{n}}d\sigma. \end{aligned}$$

We deduce that

$$\int_{\partial\Omega} e^{2s\alpha}\xi^{-\gamma}|g|^2\,d\sigma = I + \int_\Omega \nabla\cdot(e^{2s\alpha}\xi^{-\gamma}\nabla\theta)|g|^2dx. \tag{39}$$

On the other hand, we have

$$I \leq C_0\Big(\frac{1}{\mu}s\lambda\int_\Omega e^{2s\alpha}\xi^{-\gamma+1}|g|^2\,dx + \mu s^{-1}\lambda^{-1}\int_\Omega e^{2s\alpha}\xi^{-\gamma-1}|\nabla g|^2\,dx\Big).$$

Combining the last estimate with (39), we deduce (38). □

• Step 2. Estimate of the last four terms in the left-hand side of (32) .

Let us now prove the following estimate

$$\begin{aligned} s^{-8}\lambda^{-8}\iint_Q e^{2s\alpha}\xi^{-8}\sum_{i,j,k,l=1}^N|\partial_{ijkl}\,\hat z|^2dxdt + s^{-6}\lambda^{-6}\iint_Q e^{2s\alpha}\xi^{-6}\sum_{i,j,k=1}^N|\partial_{ijk}\,\hat z|^2dxdt \\ +s^{-7}\lambda^{-7}\iint_\Sigma e^{2s\alpha}\xi^{-7}|\nabla\Delta\hat z|^2d\sigma dt \leq C_0\,s^6\lambda^8\iint_Q e^{-2s\alpha}\xi^6|q|^2dxdt, \end{aligned} \tag{40}$$

for $\lambda\geq C_0$ and $s\geq C_0(T+T^{1/2})$. Observe that, from (31), $\hat z$ satisfies also

$$\begin{cases} \partial_t\nabla\hat z + \nabla(\Delta^2\hat z - s^6\lambda^8e^{-2s\alpha}\xi^6\,q) = -\nabla(\hat u\chi_{\omega'}) & \text{in } Q\ , \\ \hat z = \dfrac{\partial\hat z}{\partial\vec n} = \Delta^2\hat z - s^6\lambda^8e^{-2s\alpha}\xi^6\,q = 0 & \text{in } \Sigma\ , \\ \nabla\hat z(0,\cdot) = \nabla\hat z(T,\cdot) = 0 & \text{in } \Omega\ . \end{cases} \tag{41}$$

We multiply $(41)_1$ by $s^{-8}\lambda^{-8}e^{2s\alpha}\xi^{-8}\nabla\Delta\hat z$, we integrate by parts and we obtain

$$\begin{aligned} s^{-8}\lambda^{-8}\iint_Q e^{2s\alpha}\xi^{-8}|\Delta^2\hat z|^2dxdt \quad\leq\quad & C_0\Big(s^{-4}\lambda^{-4}\iint_Q e^{2s\alpha}\xi^{-4}|\Delta\hat z|^2dxdt \\ & +s^{-6}\lambda^{-6}\iint_Q e^{2s\alpha}\xi^{-6}|\nabla\Delta\hat z|^2dxdt \\ & +s^{-7}\lambda^{-8}\iint_{(0,T)\times\omega'} e^{2s\alpha}\xi^{-7}|\hat u|^2dxdt \\ & +s^6\lambda^8\iint_Q e^{-2s\alpha}\xi^6|q|^2dxdt \\ & +s^{-7}\lambda^{-7}\Big|\iint_Q e^{2s\alpha}\xi^{-7}\partial_t\nabla\hat z\cdot\nabla\eta\,\Delta\,\hat zdxdt\Big| \\ & +s^{-8}\lambda^{-7}\Big|\iint_Q e^{2s\alpha}\xi^{-8}\partial_t\nabla\hat z\cdot\nabla\eta\,\Delta\,\hat zdxdt\Big|\Big), \end{aligned} \tag{42}$$

for $\lambda\geq C_0$ and $s\geq C_0(T+T^{1/2})$. In order to estimate the last two terms in the right-hand side of (42), we multiply $(41)_1$ by $s^{-k}\lambda^{-7}e^{2s\alpha}\xi^{-k}\nabla\eta\,\Delta\hat{z}$ $(k=7,8)$ and we integrate by parts, we obtain

$$\begin{aligned} s^{-k}\lambda^{-7}\left|\iint_Q e^{2s\alpha}\xi^{-k}\partial_t\,\nabla\hat{z}\cdot\nabla\eta\,\Delta\,\hat{z}dxdt\right| \;\leq\; & C_0\left(\varepsilon s^{-8}\lambda^{-8}\iint_Q e^{2s\alpha}\xi^{-8}|\Delta^2\hat{z}|^2dxdt\right.\\ &+\frac{1}{\varepsilon}s^{-6}\lambda^{-6}\iint_Q e^{2s\alpha}\xi^{-6}|\nabla\Delta\hat{z}|^2dxdt\\ &\left.+\frac{1}{\varepsilon}s^{-4}\lambda^{-4}\iint_Q e^{2s\alpha}\xi^{-4}|\Delta\,\hat{z}|^2dxdt\right),\\ &+s^6\lambda^8\iint_Q e^{-2s\alpha}\xi^6|q|^2dxdt\\ &+s^{-7}\lambda^{-8}\iint_Q e^{2s\alpha}\xi^{-7}|\hat{u}|^2dxdt \end{aligned} \tag{43}$$

for $s\geq C_0\,(T+T^{1/2})$ and $\varepsilon>0$. Combining (42) and (37) with (43), we deduce

$$\begin{aligned} s^{-8}\lambda^{-8}\iint_Q e^{2s\alpha}\xi^{-8}|\Delta^2\hat{z}|^2dxdt \;\leq\; & C_0\left(s^{-6}\lambda^{-6}\iint_Q e^{2s\alpha}\xi^{-6}|\nabla\Delta\hat{z}|^2dxdt\right.\\ &\left.+s^6\lambda^8\iint_Q e^{-2s\alpha}\xi^6|q|^2dxdt\right) \end{aligned} \tag{44}$$

for $\lambda\geq C_0$ and $s\geq C_0(T+T^{1/2})$. For the first term in the right-hand side of (44), by integrating by parts and using (37), we obtain

$$\begin{aligned} s^{-6}\lambda^{-6}\iint_Q e^{2s\alpha}\xi^{-6}|\nabla\Delta\hat{z}|^2dxdt \;\leq\; & C_0\left(\varepsilon\, s^{-8}\lambda^{-8}\iint_Q e^{2s\alpha}\xi^{-8}|\Delta^2\hat{z}|^2dxdt\right.\\ &+\frac{1}{\varepsilon}s^{-4}\lambda^{-4}\iint_Q e^{2s\alpha}\xi^{-4}|\Delta\hat{z}|^2dxdt\\ &+\varepsilon\, s^{-7}\lambda^{-7}\iint_\Sigma e^{2s\alpha}\xi^{-7}\left|\frac{\partial\Delta\hat{z}}{\partial\vec{n}}\right|^2d\sigma dt\\ &\left.+\frac{1}{\varepsilon}s^{-5}\lambda^{-5}\iint_\Sigma e^{2s\alpha}\xi^{-5}|\Delta\hat{z}|^2d\sigma dt\right), \end{aligned} \tag{45}$$

for $s\geq C_0\,T$ and $\varepsilon>0$. For the last two terms in the right-hand side of (45), by applying Lemma 2.10, we deduce that

$$\begin{aligned} s^{-7}\lambda^{-7}\iint_\Sigma e^{2s\alpha}\xi^{-7}|\nabla\Delta\hat{z}|^2dxdt \;\leq\; & C_0\left(s^{-8}\lambda^{-8}\iint_Q e^{2s\alpha}\xi^{-8}|\nabla^2\Delta\hat{z}|^2dxdt\right.\\ &\left.+s^{-6}\lambda^{-6}\iint_Q e^{2s\alpha}\xi^{-6}|\nabla\Delta\hat{z}|^2dxdt\right) \end{aligned} \tag{46}$$

and

$$\begin{aligned} s^{-5}\lambda^{-5}\iint_\Sigma e^{2s\alpha}\xi^{-5}|\Delta\hat{z}|^2dxdt \;\leq\; & C_0\left(\varepsilon^2 s^{-6}\lambda^{-6}\iint_Q e^{2s\alpha}\xi^{-6}|\nabla\Delta\hat{z}|^2dxdt\right.\\ &\left.+\frac{1}{\varepsilon^2}s^{-4}\lambda^{-4}\iint_Q e^{2s\alpha}\xi^{-4}|\Delta\hat{z}|^2dxdt\right), \end{aligned} \tag{47}$$

for $s \geq C_0 T$ and $\varepsilon > 0$. Combining the last two estimates with (45) and (37), we deduce

$$\begin{aligned} s^{-6}\lambda^{-6} \iint_Q e^{2s\alpha}\xi^{-6}|\nabla\Delta\hat{z}|^2 dxdt \quad &\leq \quad C_0 \Big(\varepsilon\, s^{-8}\lambda^{-8} \iint_Q e^{2s\alpha}\xi^{-8}|\nabla^2\Delta\hat{z}|^2 dxdt \\ &\qquad + \frac{1}{\varepsilon} s^6\lambda^8 \iint_Q e^{-2s\alpha}\xi^6|q|^2 dxdt\Big), \end{aligned} \tag{48}$$

for $\lambda \geq C_0$, $s \geq C_0(T + T^{1/2})$ and for $\varepsilon > 0$. Arguing similarly as in (45)-(48), we obtain also an estimate for all the derivatives of order 3 :

$$\begin{aligned} s^{-6}\lambda^{-6} \iint_Q e^{2s\alpha}\xi^{-6} \sum_{i,j,k=1}^N |\partial_{ijk}\,\hat{z}|^2 dxdt \quad &\leq \quad C_0 \Big(\varepsilon\, s^{-8}\lambda^{-8} \iint_Q e^{2s\alpha}\xi^{-8} \sum_{i,j,k,l=1}^N |\partial_{ijkl}\,\hat{z}|^2 dxdt \\ &\qquad + \frac{1}{\varepsilon} s^6\lambda^8 \iint_Q e^{-2s\alpha}\xi^6|q|^2 dxdt\Big), \end{aligned} \tag{49}$$

for $\lambda \geq C_0$, $s \geq C_0(T + T^{1/2})$ and $\varepsilon > 0$. Then, we deduce

$$\begin{aligned} & s^{-6}\lambda^{-6} \iint_Q e^{2s\alpha}\xi^{-6} \sum_{i,j,k=1}^N |\partial_{ijk}\,\hat{z}|^2 dxdt + s^{-8}\lambda^{-8} \iint_Q e^{2s\alpha}\xi^{-8}|\Delta^2\,\hat{z}|^2 dxdt \\ & \leq C_0 \Big(\varepsilon\, s^{-8}\lambda^{-8} \iint_Q e^{2s\alpha}\xi^{-8} \sum_{i,j,k,l=1}^N |\partial_{ijkl}\,\hat{z}|^2 dxdt + s^6\lambda^8 \iint_Q e^{-2s\alpha}\xi^6|q|^2 dxdt\Big), \end{aligned} \tag{50}$$

for $\lambda \geq C_0$, $s \geq C_0(T + T^{1/2})$ and $\varepsilon > 0$. To finish the proof of (40), we introduce the following system :

$$\begin{cases} h := \Delta^2\hat{w} & \text{in } Q \ , \\ \hat{w} = \dfrac{\partial\hat{w}}{\partial\vec{n}} = 0 \ , & \text{on } \Sigma \ , \end{cases} \tag{51}$$

where $\hat{w} = s^{-4}\lambda^{-4}e^{s\alpha}\xi^{-4}\hat{z}$. From (37) and (50), we deduce

$$\|h\|^2_{L^2(Q)} \quad \leq \quad C_0 \Big(\varepsilon\, s^{-8}\lambda^{-8} \iint_Q e^{2s\alpha}\xi^{-8} \sum_{i,j,k,l=1}^N |\partial_{ijkl}\,\hat{z}|^2 dxdt + s^6\lambda^8 \iint_Q e^{-2s\alpha}\xi^6|q|^2 dxdt\Big), \tag{52}$$

for $\lambda \geq C_0$, $s \geq C_0(T + T^{1/2})$ and $\varepsilon > 0$. Using the ellipticity of system (51) and combining this with (50) and (52) for ε small enough, we deduce (40).

- Step 3. Conclusion.

Putting together (37) and (40), we find

$$s^{-7}\lambda^{-8} \iint_{(0,T)\times\omega'} e^{2s\alpha}\xi^{-7}|\hat{u}|^2 dxdt + \sum_{|\tau|\leq 4} \left\| s^{-|\tau|}\lambda^{-|\tau|}e^{s\alpha}\xi^{-|\tau|}D^\tau\hat{z}\right\|^2_{L^2(Q)} \leq C_0\, s^6\lambda^8 \iint_Q e^{-2s\alpha}\xi^6|q|^2 dxdt, \tag{53}$$

for $\lambda \geq C_0$ and $s \geq C_0(T + T^{1/2})$. On the other hand, by applying Lemma 2.10 and using (53), we deduce

$$\sum_{2\leq|\tau|\leq 3} \left\| s^{-|\tau|-\frac{1}{2}}\lambda^{-|\tau|-\frac{1}{2}}e^{s\alpha}\xi^{-|\tau|-\frac{1}{2}}D^\tau\hat{z}\right\|^2_{L^2(\Sigma)} \leq C_0\, s^6\lambda^8 \iint_Q e^{-2s\alpha}\xi^6|q|^2 dxdt, \tag{54}$$

for $\lambda \geq C_0$ and $s \geq C_0(T + T^{1/2})$. The proof of Lemma 2.8 is finished. □

Finally, the desired inequality (28) is readily deduced from (26) for $h = s^6\lambda^8 e^{-2s\alpha}\xi^6\, q - \hat{u}\chi_{\omega'}$ and (32). □

3 A null controllability result for the linear system

In this section, we prove the existence of a control in $L^\infty((0,T)\times\omega)$ for a linear system with zero, first and second order terms. More precisely, we consider :

$$\begin{cases}\partial_t y + \Delta^2 y + a\,y + B\cdot\nabla y + D : \nabla^2 y + M \therefore \nabla^3 y = \chi_\omega v & \text{in } Q\ ,\\ y = \dfrac{\partial\, y}{\partial \vec{n}} = 0 & \text{on } \Sigma\ ,\\ y(0,\cdot) = y_0(\cdot) & \text{in } \Omega\ ,\end{cases} \tag{55}$$

where

$$a \in L^\infty(Q),\quad B\in L^\infty(Q)^N\ ,\quad D\in L^\infty(Q)^{N^2}\quad ,\quad M\in C^0(Q)^{N^3} \tag{56}$$

and we suppose that $y_0 \in W^{4-4/p,p}(\Omega)\cap H^2_0(\Omega)$ for p large enough (see Remark 1.2). In order to simplify the notation, let us denote for the rest of the paper

$$\begin{aligned}\overline{C}_1 = \overline{C}_1(T,\|a\|_\infty,\|B\|_\infty,\|D\|_\infty,\|M\|_\infty) := (1+\|a\|_\infty^{1/3}+\|B\|_\infty^{1/2}\\ +\|D\|_\infty)e^{\|M\|_\infty^{1/2}} + T(1+\|a\|_\infty+\|B\|_\infty^{4/3}+\|D\|^2_\infty+\|M\|_\infty^{8/5})\end{aligned} \tag{57}$$

and

$$\overline{C}_2 = \overline{C}_2(T,\|a\|_\infty,\|B\|_\infty,\|D\|_\infty,\|M\|_\infty) := \overline{C}_1(T,\|a\|_\infty,\|B\|_\infty,\|D\|_\infty,\|M\|_\infty) + e^{\|M\|_\infty^{1/2}}\frac{1}{\sqrt{T}}. \tag{58}$$

Let us now present the following result :

Proposition 3.1. *For every $T>0$, system (55) is null controllable at time T, with controls in $L^\infty((0,T)\times\omega)$. Furthermore, the controls v can be found satisfying*

$$\|v\|_{L^\infty((0,T)\times\omega)} \le e^{C_0(\Omega,\omega)\overline{C}_2}\|y_0\|_{W^{4-4/p,p}(\Omega)},$$

where $\overline{C}_2$ is given in (58).

Before giving the proof of this result, we present some technical results in the following subsection.

3.1 Technical results.

Let us consider the following system :

$$\begin{cases}\partial_t z + \Delta^2 z + a\,z + B\cdot\nabla z + D : \nabla^2 z + M \therefore \nabla^3 z = F & \text{in } Q\ ,\\ z = \dfrac{\partial z}{\partial \vec{n}} = 0 & \text{on } \Sigma\ ,\\ z(0,\cdot) = z_0(\cdot) & \text{in } \Omega\ ,\end{cases} \tag{59}$$

where z_0 and F are given. Let us present a first result on system (59) :

Lemma 3.2. *Assume that $z_0\in H^2_0(\Omega)$ and $F\in L^2(Q)$. Then the solution z of (59) satisfies*

$$z \in\ H^1(0,T;L^2(\Omega))\cap L^2(0,T;H^4(\Omega))\cap C^0([0,T];H^2(\Omega)).$$

Moreover, there exists $C_0(\Omega)$ such that

$$\|z\|_{H^1(0,T;L^2(\Omega))} + \|z\|_{L^2(0,T;H^4(\Omega))} + \|z\|_{C^0([0,T];H^2(\Omega))} \le e^{C_0(\Omega)\overline{C}_1}\bigg(\|F\|_{L^2(Q)} + \|z_0\|_{H^2_0(\Omega)}\bigg), \tag{60}$$

where $\overline{C}_1$ is given in (57).

Proof. At first, let us prove the following estimate :

$$\|z\|_{L^2(0,T;H^4(\Omega)} \le C_0(\|F\|_{L^2(Q)} + \|z_0\|_{H^2(\Omega)}). \tag{61}$$

By multiplying (59)$_1$ by $\Delta^2 z$, we integrate by parts and we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega |\Delta z(t)|^2\,dx - C_{0,\varepsilon}(\|a\|_\infty^2 + \|B\|_\infty^2 + \|D\|_\infty^2 + \|M\|_\infty^4)\|\Delta z(t)\|^2_{L^2(\Omega)} + \int_\Omega |\Delta^2 z(t)|^2\,dx \\ \le \varepsilon\,\|z\|^2_{H^4(\Omega)} + C_{0,\varepsilon}\|F(t)\|^2_{L^2(\Omega)},\end{aligned} \tag{62}$$

for $\varepsilon > 0$. In fact, we have used Young's inequality in order to deduce

$$\left|\int_\Omega M \therefore \nabla^3 z\,\Delta^2 z dx\right| \le \|M\|_\infty \|z\|_{H^3(\Omega)}\|z\|_{H^4(\Omega)} \le \|M\|_\infty \|z\|^{1/2}_{H^2(\Omega)}\|z\|^{3/2}_{H^4(\Omega)} \le C_{0,\varepsilon}\|M\|^4_\infty\|z(t)\|^2_{H^2(\Omega)} + \varepsilon\|z\|^2_{H^4(\Omega)}.$$

Then, we deduce

$$\begin{aligned}\frac{d}{dt}\left(exp[-tC_0(\|a\|_\infty^2 + \|B\|_\infty^2 + \|D\|_\infty^2 + \|M\|_\infty^4)]\int_\Omega |\Delta z(t)|^2\,dx\right) + \int_\Omega |\Delta^2 z(t)|^2\,dx \\ \le \varepsilon\,\|z(t)\|^2_{H^4(\Omega)} + C_{0,\varepsilon}\|F(t)\|^2_{L^2(\Omega)},\end{aligned}$$

for $\varepsilon > 0$. Using the fact that there exists $\lambda > 0$ such that for any $u \in H^4(\Omega) \cap H^2_0(\Omega)$, we have

$$\int_\Omega |\Delta^2 u|^2\,dx \ge \lambda\,\|u\|^2_{H^4(\Omega)} \tag{63}$$

and integrating in $(0,T)$, we deduce (61). Let us now prove the following estimate :

$$sup_{t\in[0,T]}\|z(t)\|_{L^2(\Omega)} \le exp[C_0(1 + T(\|a\|_\infty + \|B\|_\infty^{4/3} + \|D\|_\infty^2 + \|M\|_\infty^{8/5}))]\Big(\|F\|_{L^2(Q)} + \|z_0\|_{H^2(\Omega)}\Big). \tag{64}$$

By multiplying (59)$_1$ by z, integrating by parts, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega |z(t)|^2\,dx + \int_\Omega |\Delta z(t)|^2\,dx - \varepsilon\|z\|^2_{H^2(\Omega)} \le C_{0,\varepsilon}\Big(\|F(t)\|^2_{L^2(\Omega)} \\ +(\|a\|_\infty + \|B\|_\infty^{4/3} + \|D\|_\infty^2 + \|M\|_\infty^{8/5})\int_\Omega |z(t)|^2\,dx + \|z(t)\|_{H^4(\Omega)}\Big),\end{aligned} \tag{65}$$

for $\varepsilon > 0$. In fact, we have used Young's inequality in order to deduce

$$\left|\int_\Omega B\cdot\nabla z\,z dx\right| \le \|B\|_\infty\|z\|_{H^1(\Omega)}\|z\|_{L^2(\Omega)} \le \|B\|_\infty\|z\|^{3/2}_{L^2(\Omega)}\|z\|^{1/2}_{H^2(\Omega)} \le C_{0,\varepsilon}\|B\|_\infty^{4/3}\|z\|^2_{L^2(\Omega)} + \varepsilon\|z\|^2_{H^2(\Omega)}$$

and

$$\left|\int_\Omega M \therefore \nabla^3 z\,z dx\right| \le \|M\|_\infty\|z\|_{H^3(\Omega)}\|z\|_{L^2(\Omega)} \le \|M\|_\infty\|z\|^{5/4}_{L^2(\Omega)}\|z\|^{3/4}_{H^4(\Omega)} \le C_0\Big(\|M\|_\infty^{8/5}\|z(t)\|^2_{L^2(\Omega)} + \|z(t)\|^2_{H^4(\Omega)}\Big).$$

Let us notice that there exists $\lambda > 0$ such that for any $u \in H^2(\Omega) \cap H^1_0(\Omega)$, we have

$$\int_\Omega |\Delta u|^2\,dx \ge \lambda\,\|u\|^2_{H^2(\Omega)}. \tag{66}$$

Combining this with (65), we deduce

$$\frac{d}{dt}\left(exp[-tC_0(\|a\|_\infty + \|B\|_\infty^{4/3} + \|D\|_\infty^2 + \|M\|^{8/5})]\int_\Omega |z(t)|^2\,dx\right) \le C_0\Big(\int_\Omega |F(t)|^2\,dx + \|z(t)\|^2_{H^4(\Omega)}\Big).$$

Integrating in $(0,t)$ and using (61), we have

$$\|z(t)\|_{L^2(\Omega)} \leq exp[C_0(1+T(\|a\|_\infty + \|B\|_\infty^{4/3} + \|D\|_\infty^2 + \|M\|^{8/5}))]\Big(\|F\|_{L^2(Q)} + \|z_0\|_{H^2(\Omega)}\Big). \tag{67}$$

So, we deduce (64) by taking $sup_{t\in[0,T]}$. In order to prove (60), we multiply $(59)_1$ by $\Delta^2 z$, we integrate by parts and we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\Delta z(t)|^2\,dx + \int_\Omega |\Delta^2 z(t)|^2\,dx \leq C_0\Big(\|F(t)\|^2_{L^2(\Omega)} + (\|a\|_\infty + \|B\|_\infty + \|D\|_\infty + \|M\|_\infty)\|z\|^2_{H^4(\Omega)}\Big), \tag{68}$$

Integrating in $(0,t)$, using (61) and taking $sup_{t\in[0,T]}$ we deduce (60).

□

Lemma 3.3. *Assume that $2 \leq p < \infty$ and $\varepsilon > 0$. There exists $C_{0,\varepsilon}(\Omega)$ such that, for any $F \in L^p(Q)$ and any $z_0 \in W^{4-4/p,p}(\Omega) \cap H^2_0(\Omega)$, the solution z of (59) satisfies*

$$\|z\|_{L^p(0,T;W^{4-\varepsilon,p}(\Omega))} + \|z\|_{W^{1,p}(0,T;W^{-\varepsilon,p}(\Omega))} \leq e^{C_{0,\varepsilon}\,\overline{C_1}}(\|z_0\|_{W^{4-4/p,p}(\Omega)} + \|F\|_{L^p(Q)}), \tag{69}$$

where the constant $\overline{C}_1$ is given by (57).

Proof. We divide it in several steps.

Step 1. Introduction of the problem.

Let us define the linear operator $A := \Delta^2$:

$$A\,u := \Delta^2\,u\ ,\ \ D(A) = \{u \in W^{4,p}(\Omega)\,;\,u_{|\partial\Omega} = \frac{\partial\,u}{\partial\vec{n}}_{|\partial\Omega} = 0\}.$$

From Lemma 2.1 in [29] we deduce that $\left(\Delta^2\,u,\,u_{|\partial\Omega},\,\frac{\partial\,u}{\partial\vec{n}}_{|\partial\Omega}\right)$ is normally elliptic. We obtain that A is the infinitesimal generator of an analytic semigroup $T(t) = e^{-t\Delta^2}$ and we have the following estimate (see Lemma 2.1 in [29]):

$$\|\nabla^j(T(t)u)\|_{L^p(\Omega)} \leq C(\Omega,p)t^{-j/4}\|u\|_{L^p(\Omega)}, \tag{70}$$

$\forall u \in L^p(\Omega)$, for any $t \in (0,T)$ and $j \in \{0,1,2,3,4\}$. By using an interpolation argument, we deduce

$$\|T(t)u\|_{W^{r,p}(\Omega)} \leq C(\Omega,p)(1+t^{-r/4})\|u\|_{L^p(\Omega)}, \tag{71}$$

for $0 \leq r \leq 4$ and any $t \in (0,T)$. Let us now decompose the solution of (59) as follows :

$$z = z_1 + z_2$$

where $z_1 := T(t)z_0$ is the solution of

$$\begin{cases} \partial_t z_1 + \Delta^2 z_1 = 0 & \text{in } Q\ , \\ z_1 = \dfrac{\partial\,z_1}{\partial\vec{n}} = 0 & \text{on } \Sigma\ , \\ z_1(0,\cdot) = z_0(\cdot) & \text{in } \Omega \end{cases}$$

and $z_2 := \int_0^t T(t-s)(F(s) - az(s) - B\cdot\nabla\,z(s) - D : \nabla^2\,z(s) - M \therefore \nabla^3\,z(s))\,ds$ is the solution of

$$\begin{cases} \partial_t z_2 + \Delta^2 z_2 = F - az - B\cdot\nabla\,z - D : \nabla^2\,z - M \therefore \nabla^3\,z & \text{in } Q\ , \\ z_2 = \dfrac{\partial\,z_2}{\partial\vec{n}} = 0 & \text{on } \Sigma\ , \\ z_2(0,\cdot) = 0 & \text{in } \Omega\ . \end{cases}$$

Step 2. Estimate of z_1.

Let us denote $X := L^p(\Omega)$ and $X_1 := W^{4,p}(\Omega)$. We write $(X, X_1)_\theta$ the real interpolation space between X and X_1 of exponent $0 < \theta < 1$ (for instance see [30] Section 1.3). From [30] (Section 1.14.5, page 96), using the fact that $(T(t))_{t\geq 0}$ is an analytic semigroup, we deduce that the following norm

$$\|\cdot\|_\theta := \|\cdot\|_X + \left(\int_0^T \|t^{1-\theta} AT(t)\cdot\|_X^p \frac{dt}{t}\right)^{1/p}$$

is an equivalent norm on $(X, X_1)_\theta$. By taking $\theta = 1 - 1/p$, we deduce that

$$\|z_1\|_{L^p(0,T;W^{4,p}(\Omega))} + \|z_1\|_{W^{1,p}(0,T;L^p(\Omega))} \leq C_0 \|z_0\|_{W^{4-4/p,p}(\Omega)}. \tag{72}$$

Step 3. Estimate of z_2.

Let us notice that from the definition of z_2 we have

$$\|z_2(t)\|_{W^{r,p}(\Omega)} \leq \int_0^t \|T(t-s)(F(s) - az(s) - B\cdot\nabla z(s) - D : \nabla^2 z(s) - M \therefore \nabla^3 z(s))\|_{W^{r,p}(\Omega)}\, ds,$$

where $3 < r < 4$. Combining the last estimate with (71), we have

$$\|z_2(t)\|_{W^{r,p}(\Omega)} \leq C_0 \int_0^t (1 + (t-s)^{-r/4})\, \|F(s) - az(s) - B\cdot\nabla z(s) - D : \nabla^2 z(s) - M \therefore \nabla^3 z(s)\|_{L^p(\Omega)}\, ds.$$

Applying Young's inequality, we deduce for $r = 4 - \varepsilon$,

$$\|z_2(t)\|_{L^p(0,T;W^{4-\varepsilon,p}(\Omega))} \leq C_{0,\varepsilon}\, (T + T^{\varepsilon/4}) \|F - az - B\cdot\nabla z - D : \nabla^2 z - M \therefore \nabla^3 z\|_{L^p(Q)}\, ds.$$

At the end, we deduce

$$\begin{aligned} \|z_2\|_{L^p(0,T;W^{4-\varepsilon,p}(\Omega))} \quad &\leq \quad C_{0,\varepsilon}\Big[(1+T)\Big(\|F\|_{L^p(Q)} + (\|a\|_\infty + \|B\|_\infty \\ &\qquad + \|D\|_\infty + \|M\|_\infty)\|z\|_{L^p(0,T;W^{3,p}(\Omega))}\Big]. \end{aligned} \tag{73}$$

Step 4. Conclusion.

From (73) and (72), we deduce

$$\begin{aligned} \|z\|_{L^p(0,T;W^{4-\varepsilon,p}(\Omega))} \quad &\leq \quad C_{0,\varepsilon}\Big[(1+T)\Big(\|F\|_{L^p(Q)} + (\|a\|_\infty + \|B\|_\infty \\ &\qquad + \|D\|_\infty + \|M\|_\infty)\|z\|_{L^p(0,T;W^{3,p}(\Omega))}\Big) + \|z_0\|_{W^{4-4/p,p}(\Omega)}\Big]. \end{aligned} \tag{74}$$

Using $(59)_1$, we obtain

$$\begin{aligned} \|z\|_{L^p(0,T;W^{4-\varepsilon,p}(\Omega))} + \|z\|_{W^{1,p}(0,T;W^{-\varepsilon,p}(\Omega))} \quad &\leq \quad C_{0,\varepsilon}\Big[(1+T)\Big(\|F\|_{L^p(Q)} + (\|a\|_\infty \\ &\qquad + \|B\|_\infty + \|D\|_\infty + \|M\|_\infty)\|z\|_{L^p(0,T;W^{3,p}(\Omega))}\Big) \\ &\qquad + \|z_0\|_{W^{4-4/p,p}(\Omega)}\Big]. \end{aligned} \tag{75}$$

To finish the proof, let us also notice that

$$\|z\|_{L^p(0,T;W^{3,p}(\Omega))} \leq C_{0,\varepsilon}\, T^{\gamma_0} \|z\|^{\theta}_{L^\infty(0,T;H^2(\Omega))} \|z\|^{1-\theta}_{L^p(0,T;W^{4-\varepsilon,p}(\Omega))} \tag{76}$$

for some $0 < \theta < 1$ and some $\gamma_0 > 0$. Combining (76) with (75) and using Young's inequality, we deduce

$$\begin{aligned}\|z\|_{L^p(0,T;W^{4-\varepsilon,p}(\Omega))} + \|z\|_{W^{1,p}(0,T;W^{-\varepsilon,p}(\Omega))} \quad &\leq \quad C_{0,\varepsilon}\Big[(1+T)\|F\|_{L^p(Q)} + T^{\gamma_0/\theta}(1+T)^{1/\theta}(\|a\|_\infty \\ &\quad + \|B\|_\infty + \|D\|_\infty + \|M\|_\infty)^{1/\theta}\|z\|_{L^\infty(0,T;H^2(\Omega))} + \|z_0\|_{W^{4-4/p,p}(\Omega)}\Big].\end{aligned}$$

Using Lemma 3.2 with the last estimate, we deduce (69). □

Let us now introduce the adjoint system of (59) :

$$\begin{cases} -\partial_t q + \Delta^2 q + a\, q - \nabla\cdot(B\, q) + \displaystyle\sum_{i,j=1}^{N} \frac{\partial^2}{\partial x_i x_j}(D_{ij}\, q) - \sum_{i,j,k=1}^{N} \frac{\partial^3}{\partial x_i x_j x_k}(M_{ijk}\, q) = 0 & \text{in } Q\ , \\ q = \dfrac{\partial\, q}{\partial \vec{n}} = 0 & \text{on } \Sigma\ , \\ q(T,\cdot) = q_0(\cdot) & \text{in } \Omega\ , \end{cases} \tag{77}$$

where $q_0 \in H^{-2}(\Omega)$. We can deduce from the Carleman estimates proved in Section 2 an observability inequality for (77), as follows.

Lemma 3.4. *There exists a postive constant $C_0 = C_0(\Omega, \omega')$ such that for any $q_0 \in H^{-2}(\Omega)$, we have*

$$\|q(0,\cdot)\|^2_{H^{-2}(\Omega)} \leq e^{C_0\, \overline{C}_2} \iint_{(0,T)\times\omega'} |q|^2\, dxdt, \tag{78}$$

where the constant $\overline{C}_2$ was defined in (58) and q is the solution to the corresponding system (77).

Proof. Let us start by proving the following estimate

$$\iint_{(\frac{T}{4},\frac{3T}{4})\times\Omega} e^{-2s\alpha}\xi^6|q|^2 dxdt \leq C_0\, s \iint_{(0,T)\times\omega'} e^{-2s\alpha}\xi^7|q|^2 dxdt. \tag{79}$$

for $s \geq C_0\Big(T(1 + \|a\|^{1/3}_\infty + \|B\|^{1/2}_\infty + \|D\|_\infty) + T^{1/2}\Big)$ and $\lambda \geq C_0\|M\|^{1/2}_\infty$.

It is clear that

$$\iint_Q e^{-2s\alpha}\Big(|a\, q|^2 + s^2\lambda^2\xi^2|B\, q|^2 + s^4\lambda^4\xi^4|D\, q|^2 + s^6\lambda^6\xi^6|M\, q|^2\Big)dxdt \leq s^6\lambda^8 \iint_Q e^{-2s\alpha}\xi^6|q|^2 dxdt$$

for $s \geq C_0 T(\|a\|^{1/3}_\infty + \|B\|^{1/2}_\infty + \|D\|_\infty)$ and $\lambda \geq C_0\|M\|^{1/2}_\infty$. By taking $F_0 = -a\, q$, $F_1 = B\, q$, $\hat{F} = -q\, D$, $\tilde{F} = q\, M$ and $f_0 = f_1 = 0$ in (28) we can easily deduce (79).

Now from the definition of ξ and α we have

$$\begin{cases} e^{-2s\alpha}\xi^7 \leq \dfrac{C_0}{T^7}\exp[-C_0(1 + \dfrac{1}{\sqrt{T}} + \|a\|^{1/3}_\infty + \|B\|^{1/2}_\infty + \|D\|_\infty)], & \forall (t,x) \in Q, \\ e^{-2s\alpha}\xi^6 \geq \dfrac{C_0}{T^6}\exp[-2C_0(1 + \dfrac{1}{\sqrt{T}} + \|a\|^{1/3}_\infty + \|B\|^{1/2}_\infty + \|D\|_\infty)e^{4C_0\|\eta\|_\infty\|M\|^{1/2}_\infty}], & \forall (t,x) \in (\dfrac{T}{4}, \dfrac{3T}{4}) \times \Omega. \end{cases}$$

By replacing η by $\eta/4C_0\|\eta\|_\infty$, we can deduce that

$$\iint_{(\frac{T}{4},\frac{3T}{4})\times\Omega} |q|^2\,dxdt \le \exp[C_0(1+\frac{1}{\sqrt{T}}+\|a\|_\infty^{1/3}+\|B\|_\infty^{1/2}+\|D\|_\infty)e^{\|M\|_\infty^{1/2}}]\iint_{(0,T)\times\omega'} |q|^2 dxdt. \tag{80}$$

To finish, it suffices to combine the last estimate with the following inequality :

$$\begin{aligned} \|q(0,\cdot)\|^2_{H^{-2}(\Omega)} &\le \|\theta q\|^2_{C^0(0,T;H^{-2}(\Omega))} \\ &\le \overline{C}_1\|\theta' q\|^2_{L^2(Q)} \\ &\le \overline{C}_1 \iint_{(\frac{T}{4},\frac{3T}{4})\times\Omega} |q|^2\,dxdt, \end{aligned} \tag{81}$$

where $\overline{C}_1$ was defined in (57) and $\theta \in C^1([0,T])$ such that

$$\begin{cases} 1 & \text{if } \ t\in[0,T/2]\ , \\ 0 & \text{if } \ t\in[3T/4,T]\ . \end{cases}$$

$\square$

3.2 Proof of Proposition 3.1

We divide it in several steps. Let us consider a family of open sets $(\omega_i)_{i=0}^n$ for $n\in\mathbb{N}^*$ large enough such that $\omega' \subset\subset \omega_0 \subset\subset \omega_1 ... \subset\subset \omega_{n-1} \subset\subset \omega_n = \omega$, where ω' was defined in Proposition 2.4.

Step 1. First computations in ω_0.

From Lemma 3.4, we can deduce in a classical way the existence of a control $\tilde v \in L^2((0,T)\times\omega')$ such that the associated solution to (55) verifies $\tilde y(T,\cdot)=0$ and the following estimate holds :

$$\|\tilde v\|_{L^2((0,T)\times\omega')} \le exp\{C_0\,\overline{C}_2\}\|y_0\|_{L^2(\Omega)},$$

where $\overline{C}_2$ is given in (58).

We introduce now a cut-off function $\delta=\delta(t)$ satisfying

$$\delta\in C^\infty([0,T]),\quad \delta(t)=1 \text{ in } (0,T/4),\quad \delta(t)=0 \text{ in } (3T/4,T)$$

and

$$0\le\delta(t)\le 1\ ,\ |\delta'(t)|\le\frac{C}{t} \text{ in } (0,T)$$

and we denote by ϑ the solution to the system

$$\begin{cases} \partial_t\vartheta+\Delta^2\vartheta+a\,\vartheta+B\cdot\nabla\vartheta+D:\nabla^2\vartheta+M\,\therefore\,\nabla^3\vartheta=0 & \text{in } Q\ , \\ \vartheta=\dfrac{\partial\,\vartheta}{\partial\vec n}=0 & \text{on } \Sigma\ , \\ \vartheta(0,\cdot)=y_0(\cdot) & \text{in } \Omega\ . \end{cases} \tag{82}$$

Then, the function $\tilde w=\tilde y-\delta\vartheta$ satisfies

$$\begin{cases} \partial_t\tilde w+\Delta^2\tilde w+a\,\tilde w+B\cdot\nabla\tilde w+D:\nabla^2\tilde w+M\,\therefore\,\nabla^3\tilde w=\chi_{\omega'}\tilde v-\delta'\,\vartheta & \text{in } Q\ , \\ \tilde w=\dfrac{\partial\,\tilde w}{\partial\vec n}=0 & \text{on } \Sigma\ , \\ \tilde w(0,\cdot)=0 & \text{in } \Omega\ . \end{cases} \tag{83}$$

Let us now consider a cut-off function β, with

$$\beta \in C_0^4(\omega_0) \ , \ \beta \equiv 1 \ \text{ in } \ \omega'$$

and let us set $w = (1-\beta)\tilde{w}$. Then we have :

$$\begin{cases} \partial_t w + \Delta^2 w + a\, y + B \cdot \nabla w + D : \nabla^2 w + M \therefore \nabla^3 w = \chi_{\omega_0} v_0 - \delta' \vartheta & \text{in } Q \ , \\ w = \dfrac{\partial w}{\partial \vec{n}} = 0 & \text{on } \Sigma \ , \\ w(0,\cdot) = 0 & \text{in } \Omega \ , \end{cases} \tag{84}$$

with

$$v_0 = \beta\delta'\vartheta - \Delta^2\beta\,\tilde{w} - 4\nabla\Delta\beta\cdot\nabla\tilde{w} - 2\Delta\beta\Delta\tilde{w} - 4\nabla^2\beta : \nabla^2\tilde{w} - 4\nabla\Delta\tilde{w}\cdot\nabla\beta - B\cdot\nabla\beta\,\tilde{w} - D : \nabla^2\beta\,\tilde{w} - 2D\cdot\nabla\beta\nabla\tilde{w}$$
$$- \sum_{ijk=1}^{N} M_{ijk}\Big(\partial_{ijk}\beta\tilde{w} + \partial_{jk}\beta\partial_i\tilde{w} + \partial_{ik}\beta\partial_j\tilde{w} + \partial_{ij}\beta\partial_k\tilde{w} + \partial_k\beta\partial_{ij}\tilde{w} + \partial_i\beta\partial_{jk}\tilde{w} + \partial_j\beta\partial_{ik}\tilde{w}\Big).$$

Let us remark that $supp\, v_0 \subset [0,T]\times\omega_0$ and $y := w + \delta\vartheta$ where y fulfills (55) with (ω, v) replaced by (ω_0, v_0). On the other hand, from Lemma 3.2, we have that $\tilde{y}$ belongs to $L^2(0,T;H^4(\Omega)) \cap H^1(0,T;L^2(\Omega))$. Let us introduce $p_0 = \dfrac{2(N+4)}{N+2} > 2$. Then $L^2(0,T;H^1(\Omega)) \cap H^1(0,T;H^{-3}(\Omega)) \hookrightarrow L^{p_0}(Q)$ which implies that $v_0 \in L^{p_0}((0,T)\times\omega_0)$ with the following estimate :

$$\|v_0\|_{L^{p_0}((0,T)\times\omega_0)} \le e^{C_0\,\overline{C}_2}\|y_0\|_{L^2(\Omega)},$$

(recall that $\overline{C}_2$ was defined in (58)). Using Lemma 3.3 for $\varepsilon = 1/2$, we obtain that the solution of (55) associated to v_0 satisfies

$$\|y\|_{L^{p_0}(0,T;W^{7/2,p_0}(\Omega))} + \|y\|_{W^{1,p_0}(0,T;W^{-1/2,p_0}(\Omega))} \le e^{C_0\,\overline{C}_2}\|y_0\|_{W^{4-4/p_0,p_0}(\Omega)}. \tag{85}$$

Step 2. Computations in ω_i for $i = 1...n-1$.

Let us introduce the following sequence :

$$\begin{cases} p_0 = \dfrac{2(N+4)}{N+2} > 2, \\ \dfrac{1}{p_{i+1}} = \dfrac{1}{p_i} - \dfrac{1}{2(N+4)} \ \text{ for } i = 0...n-2 \\ p_{n-1} = 2(N+4)+1. \end{cases}$$

Using the same argument as in Step 1, we obtain from (85) that there exists $v_1 \in L^{p_1}((0,T)\times\omega_1)$ such that the associated solution to (55) with (ω, v) replaced by (ω_1, v_1) satisfies

$$\|y\|_{L^{p_1}(0,T;W^{7/2,p_1}(\Omega))} + \|y\|_{W^{1,p_1}(0,T;W^{-1/2,p_1}(\Omega))} \le e^{C_0\,\overline{C}_2}\|y_0\|_{W^{4-4/p_1,p_1}(\Omega)}, \tag{86}$$

where $\overline{C}_2$ is given in (58). In order to prove this, we have used the fact that $L^{p_0}(0,T;W^{1/2,p_0}(\Omega)) \cap W^{1,p_0}(0,T;W^{-7/2,p_0}(\Omega))$ is continuously embedded into $L^{p_1}(Q)$ and we applied Lemma 3.3 for $\varepsilon = 1/2$.

After $n-2$ steps, we can construct a control $v_{n-2} \in L^{p_{n-2}}((0,T)\times\omega_{n-2})$ such that the associated solution to (55) with (ω, v) replaced by (ω_{n-2}, v_{n-2}) satisfies $y(T,\cdot) = 0$ in Ω and

$$\|y\|_{L^{p_{n-2}}(0,T;W^{7/2,p_{n-2}}(\Omega))} + \|y\|_{W^{1,p_{n-2}}(0,T;W^{-1/2,p_{n-2}}(\Omega))} \le e^{C_0\,\overline{C}_2}\|y_0\|_{W^{4-4/p_{n-2},p_{n-2}}(\Omega)}. \tag{87}$$

Step 3. Computations in ω_{n-1} and ω_n.

By using again a bootstrap processus, we deduce the existence of a control $v_{n-1} \in L^{p_{n-1}}((0,T)\times\omega_{n-1})$ with $p_{n-1} > 2(N+4)$ and we deduce the following estimate :

$$\|v_{n-1}\|_{L^{p_{n-1}}((0,T)\times\omega_{n-1})} \leq e^{C_0\,\overline{C}_2}\|y_0\|_{W^{4-4/p_{n-2},p_{n-2}}(\Omega)}.$$

On the other hand, using again Lemma 3.3 we deduce that the solution y associated to v_{n-1} belongs to $L^{p_{n-1}}(0,T;W^{7/2,p_{n-1}}(\Omega)) \cap W^{1,p_{n-1}}(0,T;W^{-1/2,p_{n-1}}(\Omega))$. By using the Sobolev injections we deduce

$$\begin{aligned}\|y\|_{L^\infty(0,T;W^{3,\infty}(\Omega))} &\leq C_0(\|y\|_{L^{p_{n-2}}(0,T;W^{7/2,p_{n-2}}(\Omega))} + \|y\|_{W^{1,p_{n-2}}(0,T;W^{-1/2,p_{n-2}}(\Omega))})\\ &\leq e^{C_0\,\overline{C}_2}\|y_0\|_{W^{4-4/p_{n-1},p_{n-1}}(\Omega)(\Omega)}.\end{aligned} \tag{88}$$

By using for the last time a bootstrap processus, we deduce the existence of a control $v := v_n \in L^\infty((0,T)\times\omega)$ with $\omega := \omega_n$ and the following estimate :

$$\|v\|_{L^\infty((0,T)\times\omega)} \leq e^{C_0\,\overline{C}_2}\|y_0\|_{W^{4-4/p_{n-1},p_{n-1}}(\Omega)}.$$

□

4 A null controllability result for the semi-linear system

Now we are ready to proof Theorem 1.3 by applying the same idea used in [11]. At the beginning, we suppose that $f_2 \equiv 0$, then we will add a remark to extend our proof to the case when $f \neq 0$. As we explained above, we will assume that $y_0 \in W^{4-4/p,p}(\Omega) \cap H_0^2(\Omega)$ for p large enough (see Remark 1.2). We will divide this section in two steps. First, we will assume that g, G and E are continuous and then we will finish the proof by a density argument.

4.1 Continuous case.

In this step, a fixed point argument will be used. We assume that

$$g_1 \in C^0(\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}^{N^2}),\quad G_1 \in C^0(\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}^{N^2})^N,\quad E_1 \in C^0(\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}^{N^2})^{N^2}$$

and that (7) is satisfied. From (7), we deduce that for every $\varepsilon > 0$, there exists C_ε such that

$$\begin{aligned}&|g_1(s,p,q)|^{1/3} + |G_1(s,p,q)|^{1/2} + |E_1(s,p,q)|\\ &\leq C_\varepsilon + \varepsilon\,\log(1+|s|+|p|+|q|)\quad \forall (s,p,q)\in\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}^{N^2}.\end{aligned} \tag{89}$$

Let us set $Z = L^\infty(0,T;W^{2,\infty}(\Omega))$. For each $z \in Z$, we consider the null controllability problem :

$$\begin{cases}\partial_t y + \Delta^2 y + g_1(z,\nabla z,\nabla^2 z)\,y + G_1(z,\nabla z,\nabla^2 z)\cdot\nabla y + E_1(z,\nabla z,\nabla^2 z) : \nabla^2 y = \chi_\omega v & \text{in } Q\ ,\\ y = \dfrac{\partial\, y}{\partial\vec{n}} = 0 & \text{on } \Sigma\ ,\\ y(0,\cdot) = y_0(\cdot) & \text{in } \Omega\ .\end{cases} \tag{90}$$

Let us set for each $z \in Z$

$$\begin{cases}a_z := g_1(z,\nabla z,\nabla^2 z) & \in L^\infty(Q),\\ B_z := G_1(z,\nabla z,\nabla^2 z) & \in L^\infty(Q)^N,\\ D_z := E_1(z,\nabla z,\nabla^2 z) & \in L^\infty(Q)^{N^2}.\end{cases}$$

Then, by applying Proposition 3.1 to (90) in the time interval $(0,T_z)$ where

$$T_z = \min\{T, \|a_z\|_\infty^{-2/3}, \|B_z\|_\infty^{-5/6}, \|D_z\|_\infty^{-1}\},$$

we deduce the existence of $\check{v}_z$ such that

$$\check{y}_z(T_z,\cdot)=0$$

and

$$\|\check{v}_z\|_{L^\infty((0,T)\times\omega)}\le e^{C_0\,\overline{C}_2(T_z,\|a_z\|_\infty,\|B_z\|_\infty,\|D_z\|_\infty)}\|y_0\|_{W^{4-4/p,p}(\Omega)}.$$

where $\overline{C}_2$ is given in (58) for $M\equiv 0$. From Lemma 3.3, we deduce

$$\|\check{y}_z\|_M\le e^{C_0\,\overline{C}_2(T_z,\|a_z\|_\infty,\|B_z\|_\infty,\|D_z\|_\infty)}\|y_0\|_{W^{4-4/p,p}(\Omega)},$$

where $M=L^p(0,T;W^{7/2,p}(\Omega))\cap W^{1,p}(0,T;W^{-1/2,p}(\Omega))$ and $\overline{C}_2$ is given in (58) for $M\equiv 0$. Let us denote $\tilde{v}_z$ (resp. $\tilde{y}_z$) the extension of $\check{v}_z$ (resp. $\check{y}_z$) to the whole cylinder $(0,T)\times\Omega$. It is not difficult to check that $\tilde{y}_z$ is the corresponding solution of (90) associated to $\tilde{v}_z$. Then, we deduce

$$\tilde{y}_z(\cdot,T)=0,$$

$$\|\tilde{v}_z\|_{L^\infty((0,T)\times\omega)}\le N(\Omega,\omega,T,z)\|y_0\|_{W^{4-4/p,p}(\Omega)}$$

and

$$\|\tilde{y}_z\|_M\le N(\Omega,\omega,T,z)\|y_0\|_{W^{4-4/p,p}(\Omega)}$$

where

$$N(\Omega,\omega,T,z)=e^{C(\Omega,\omega,T)(1+\|a_z\|_\infty^{1/3}+\|B_z\|_\infty^{1/2}+\|D_z\|_\infty)}$$

Let us introduce for each $z\in Z$

$$I(z)=\left\{\tilde{v}_z\in L^\infty((0,T)\times\omega)\,;\,\tilde{y}_z(T)=0,\ \|\tilde{v}_z\|_{L^\infty((0,T)\times\omega)}\le N(\Omega,\omega,T,z)\|y_0\|_{W^{4-4/p,p}(\Omega)}\right\}$$

and

$$\Gamma(z)=\left\{\tilde{y}_z\,;\,\tilde{v}_z\in I(z),\ \|\tilde{y}_z\|_M\le N(\Omega,\omega,T,z)\|y_0\|_{W^{4-4/p,p}(\Omega)}\right\},\tag{91}$$

The idea is to prove the existence of at least one fixed point y of the following mapping

$$z\mapsto\Gamma(z).$$

Thus, let us recall Kakutani's fixed point theorem :

Theorem 4.1. *Let Z be a Banach space and let $\Gamma:\,Z\mapsto Z$ be a set-valued mapping satisfying the following assumptions:*

1. *$\Gamma(z)$ is a nonempty closed convex set of Z for every $z\in Z$.*
2. *There exists a nonempty convex compact set $K_R\subset Z$ such that $\Gamma(K_R)\subset K_R$.*
3. *The mapping $z\mapsto\Gamma(z)$ is upper hemicontinuous, i.e., that the real-valued function*

$$z\in Z\mapsto sup_{y\in\Gamma(z)}\langle\mu,y\rangle$$

is upper semicontinuous for each bounded linear form $\mu\in Z'$.

Then Γ possesses a fixed point in the set K, i.e. there exists $z\in K$ such that $z\in\Gamma(z)$.

Let us check that Kakutani's theorem can be applied to Γ. From what we did above, we can deduce that $\Gamma(z)$ is a nonempty closed set of Z for every $z \in Z$ and convexity is easy to prove. Let us introduce for each $R > 0$, the following subspace :

$$K_R = \{z \in Z\,; \|z\|_M \leq R\}.$$

Then, we know that M is embedded compactly in Z so we deduce that K_R is a convex compact set of Z. On the other hand, from (89) and (91), we deduce the following estimate for any $z \in K_R$:

$$\begin{aligned}
\|\tilde{y}_z\|_Z &\leq \exp\Big(C(1 + \|a_z\|_\infty^{1/3} + \|B_z\|_\infty^{1/2} + \|D_z\|_\infty)\Big)\|y_0\|_{W^{4-4/p,p}(\Omega)} \\
&\leq \exp\Big(C(1 + C_\varepsilon + \varepsilon\log(1+3R))\Big)\|y_0\|_{W^{4-4/p,p}(\Omega)} \\
&\leq \exp\Big(C(1+C_\varepsilon)\Big)\Big(1+3R\Big)^{C\varepsilon}\|y_0\|_{W^{4-4/p,p}(\Omega)}.
\end{aligned}$$

By taking ε small enough such that $C\varepsilon < 1$ and R large enough we deduce that $\Gamma(K_R) \subset K_R$.

To verify the third assumption, it suffices to prove that the following set :

$$P_{\alpha,\mu} = \Big\{z \in Z\,;\, sup_{y\in\Gamma(z)}\langle \mu, y\rangle \geq \alpha\Big\}$$

is a closed set of Z for every $\alpha \in \mathbb{R}$ and every $\mu \in Z'$. So let $(z_n)_{n\in\mathbb{N}}$ be a sequence in $P_{\alpha,\mu}$ such that $z_n \to z$ in Z. From the fact that for every $n \in \mathbb{N}$ $\Gamma(z_n)$ is compact, we deduce that

$$\langle \mu, y_n\rangle = sup_{y\in\Gamma(z_n)}\langle \mu, y\rangle \geq \alpha, \tag{92}$$

for $y_n \in \Gamma(z_n)$. We deduce that there exists $v_n \in L^\infty((0,T)\times\omega)$ such that

$$\partial_t y_n + \Delta^2 y_n + g_1(z_n, \nabla z_n, \nabla^2 z_n)\, y_n + G_1(z_n, \nabla z_n, \nabla^2 z_n)\cdot \nabla y_n + E_1(z_n, \nabla z_n, \nabla^2 z_n) : \nabla^2 y_n = \chi_\omega v_n$$

and we have the following estimates :

$$\|v_n\|_{L^\infty((0,T)\times\omega)} \leq N(\Omega, \omega, T, z_n)\|y_0\|_{W^{4-4/p,p}(\Omega)}$$

and

$$\|y_n\|_M \leq N(\Omega, \omega, T, z)\|y_0\|_{W^{4-4/p,p}(\Omega)}.$$

By using the fact that $(y_n)_{n\in\mathbb{N}}$ is uniformly bounded in M and $(v_n)_{n\in\mathbb{N}}$ is uniformly bounded in $L^\infty((0,T)\times\omega)$ we deduce at least for a subsequence

$$y_n \to \breve{y} \text{ strongly in } Z$$

and

$$v_n \to \breve{v} \text{ weakly * in } L^\infty((0,T)\times\omega).$$

On the other hand, from the fact that g_1, G_1 and E_1 are continuous, we have

$$\begin{cases}
g_1(z_n, \nabla z_n, \nabla^2 z_n) \to g_1(z, \nabla z, \nabla^2 z) & \in L^\infty(Q), \\
G_1(z_n, \nabla z_n, \nabla^2 z_n) \to G_1(z, \nabla z, \nabla^2 z) & \in L^\infty(Q)^N, \\
E_1(z_n, \nabla z_n, \nabla^2 z_n) \to E_1(z, \nabla z, \nabla^2 z) & \in L^\infty(Q)^{N^2}.
\end{cases}$$

Passing to the limit, we deduce

$$\begin{cases}
\partial_t \breve{y} + \Delta^2 \breve{y} + g_1(z, \nabla z, \nabla^2 z)\,\breve{y} + G_1(z, \nabla z, \nabla^2 z)\cdot\nabla\breve{y} + E_1(z, \nabla z, \nabla^2 z) : \nabla^2\breve{y} = \chi_\omega \breve{v} & \text{in } Q\ , \\
\breve{y} = \dfrac{\partial\,\breve{y}}{\partial\vec{n}} = 0 & \text{on } \Sigma\ , \\
\breve{y}(0,\cdot) = y_0(\cdot)\,,\ \ \breve{y}(T,\cdot) = 0 & \text{in } \Omega\ .
\end{cases}$$

We deduce that $\breve{y} \in \Gamma(z)$ and $\breve{v} \in I(z)$. Then, we can take limits in (92) and deduce that

$$sup_{y\in\Gamma(z)}\langle \mu, y\rangle \ \geq \langle \mu, \breve{y}\rangle \ \geq \alpha.$$

This implies that $z \in P_{\mu,\alpha}$.

4.2 The general case.

We suppose now that f is a locally Lipschitz-continuous function satisfying assumptions (4) and (7) and we give a sketch of the proof. Let us introduce a positive function $\rho \in \mathcal{D}(\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}^{N^2})$ such that $supp\ \rho \subset \overline{B}(0,1)$ and

$$\iiint_{\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}^{N^2}} \rho(s,p,q)\,ds\,dp\,dq = 1.$$

Let us introduce the following function

$$\rho_n(s,p,q) = n^{N^2+N+1}\rho(ns,np,nq) \quad \forall (s,p,q) \in \mathbb{R}\times\mathbb{R}^N\times\mathbb{R}^{N^2}$$

and let us denote

$$g_{1,n} = \rho_n * g, \quad G_{1,n} = \rho_n * G, \quad E_{1,n} = \rho_n * E.$$

Let us denote $f_{1,n} := g_{1,n}(s,p,q)s + G_{1,n}(s,p,q)\cdot p + E_{1,n}(s,p,q) : q$. It is not difficult to check the following properties on these three functions :

1. $g_{1,n} \in C^0(\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}^{N^2})$, $G_{1,n} \in C^0(\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}^{N^2})^N$ and $E_{1,n} \in C^0(\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}^{N^2})^{N^2}$ for $n \geq 1$.
2. $f_{1,n} \to f$ uniformly in any compact set of $\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}^{N^2}$.
3. $\forall\lambda > 0$, $\exists\rho > 0$; $\|g_{1,n}\|_{L^\infty(B(0,\rho))} + \|G_{1,n}\|_{L^\infty(B(0,\rho))} + \|E_{1,n}\|_{L^\infty(B(0,\rho))} \leq \lambda$.
4. $\forall\varepsilon > 0$, $\exists C_\varepsilon > 0$; $\forall (s,p,q) \in \mathbb{R}\times\mathbb{R}^N\times\mathbb{R}^{N^2}$, $\forall n \geq 1$, $|g_{1,n}(s,p,q)|^{1/3} + |G_{1,n}(s,p,q)|^{1/2} + |E_{1,n}(s,p,q)| \leq C_\varepsilon + \varepsilon \log(1+|s|+|p|+|q|)$.

For every $n \in \mathbb{N}^*$, we deduce from the previous subsection, the existence of controls $v_n \in L^\infty((0,T)\times\omega)$ such that the solution y_n of the following system

$$\begin{cases} \partial_t y_n + \Delta^2 y_n + f_{1,n}(y_n, \nabla y_n, \nabla^2 y_n) = \chi_\omega v_n & \text{in } Q\ , \\ y_n = \dfrac{\partial y}{\partial \vec{n}} = 0 & \text{on } \Sigma\ , \\ y_n(0,\cdot) = y_0(\cdot) & \text{in } \Omega \end{cases} \tag{93}$$

satisfies

$$y_n(T,\cdot) = 0 \text{ in } \Omega.$$

On the other hand, from the previous subsection we deduce that

$$\|v_n\|_{L^\infty((0,T)\times\omega)} + \|y_n\|_M \leq C.$$

We deduce, for a subsequence, that

$$v_n \to v \ \text{ weakly * in } \ L^\infty((0,T)\times\omega)$$

and

$$y_n \to y \ \text{ in } \ Z.$$

Now, it is very easy to check that (y,v) fulfills the following system :

$$\begin{cases} \partial_t y + \Delta^2 y + f_1(y, \nabla y, \nabla^2 y) = \chi_\omega v & \text{in } Q\ , \\ y = \dfrac{\partial y}{\partial \vec{n}} = 0 & \text{on } \Sigma\ , \\ y(0,\cdot) = y_0(\cdot) & \text{in } \Omega\ , \end{cases}$$

with

$$y(T,\cdot) = 0 \text{ in } \Omega.$$

Before we finish the proof, let us give the following remark concerning the case when $f_2 \neq 0$:

Remark 4.2. *Let us notice that, the same computations and arguments can be used to prove the null controllability of system (1) when $f_2 \neq 0$. In fact, as in (89) we have for every $\varepsilon > 0$, there exists C_ε such that*

$$\begin{aligned} &|g_2(s,p,q,r)|^{\gamma/3} + |G_2(s,p,q,r)|^{\gamma/2} + |E_2(s,p,q,r)|^{\gamma} + e^{\frac{\gamma}{\gamma-1}|M_2(s,p,q,r)|^{1/2}} \\ &\leq C_\varepsilon + \varepsilon \log(1+|s|+|p|+|q|+|r|) \quad \forall (s,p,q,r) \in \mathbb{R}\times\mathbb{R}^N\times\mathbb{R}^{N^2}\times\mathbb{R}^{N^3}. \end{aligned} \tag{94}$$

On the other hand, we set this time $Z = L^\infty(0,T;W^{3,\infty})$. Then, for each $z \in Z$, we consider the null controllability problem :

$$\begin{cases} \partial_t y + \Delta^2 y + g_2(z,\nabla z,\nabla^2 z,\nabla^3 z)\, y + G_2(z,\nabla z,\nabla^2 z,\nabla^3 z)\cdot\nabla y + E_2(z,\nabla z,\nabla^2 z,\nabla^3 z) : \nabla^2 y & \\ \qquad\qquad + M_2(z,\nabla z,\nabla^2 z,\nabla^3 z) \therefore \nabla^3 y = \chi_\omega v & \text{in } Q\ , \\ y = \dfrac{\partial y}{\partial \vec{n}} = 0 & \text{on } \Sigma\ , \\ y(0,\cdot) = y_0(\cdot) & \text{in } \Omega\ , \end{cases} \tag{95}$$

and we denote

$$\begin{cases} a_z := g_2(z,\nabla z,\nabla^2 z,\nabla^3 z) & \in L^\infty(Q), \\ B_z := G_2(z,\nabla z,\nabla^2 z,\nabla^3 z) & \in L^\infty(Q)^N, \\ D_z := E_2(z,\nabla z,\nabla^2 z,\nabla^3 z) & \in L^\infty(Q)^{N^2}, \\ M_z := M_2(z,\nabla z,\nabla^2 z,\nabla^3 z) & \in L^\infty(Q)^{N^3}. \end{cases}$$

Then, by arguing as before and applying Proposition 3.1 to (95) in the time interval $(0,T_z)$ where

$$T_z = \min\{1, T, \|a_z\|_\infty^{\frac{\gamma}{3}-1}, \|B_z\|_\infty^{\frac{\gamma}{2}-\frac{4}{3}}, \|D_z\|_\infty^{\gamma-2}\},$$

and using the fact that for any $a,b\in\mathbb{R}$ we have $ab \leq \dfrac{a^\gamma}{\gamma} + \dfrac{\gamma-1}{\gamma} b^{\frac{\gamma}{\gamma-1}}$ where $\gamma > 1$, we deduce the existence of $\check{v}_z$ such that

$$\check{y}_z(T_z,\cdot) = 0$$

and

$$\|\check{v}_z\|_{L^\infty((0,T)\times\omega)} \leq e^{C_0\,\overline{C}_3}\|y_0\|_{W^{4-4/p,p}(\Omega)}$$

where

$$\overline{C}_3 = 1 + \|a_z\|_\infty^{\gamma/3} + \|B_z\|_\infty^{\gamma/2} + \|D_z\|_\infty^{\gamma} + e^{\frac{\gamma}{\gamma-1}\|M_z\|_\infty^{1/2}} + \frac{1}{T_z^{\gamma/2}} + T_z(1 + \|a_z\|_\infty + \|B_z\|_\infty^{4/3} + \|D_z\|_\infty^2 + \|M_z\|_\infty^{8/5}).$$

From Lemma 3.3, we deduce

$$\|\check{y}_z\|_M \leq e^{C_0\,\overline{C}_3}\|y_0\|_{W^{4-4/p,p}(\Omega)},$$

where $M = L^p(0,T;W^{7/2,p}(\Omega)) \cap W^{1,p}(0,T;W^{-1/2,p}(\Omega))$. By arguing as before, if we set $\tilde{v}_z$ (resp. $\tilde{y}_z$) the extension of $\check{v}_z$ (resp. $\check{y}_z$) to the whole cylinder $(0,T)\times\Omega$ then, we deduce

$$\tilde{y}_z(\cdot,T) = 0,$$

$$\|\tilde{v}_z\|_{L^\infty((0,T)\times\omega)} \leq N(\Omega,\omega,T,z)\|y_0\|_{W^{4-4/p,p}(\Omega)}$$

and

$$\|\tilde{y}_z\|_M \leq N(\Omega,\omega,T,z)\|y_0\|_{W^{4-4/p,p}(\Omega)}$$

where $\tilde{y}_z$ is the corresponding solution of (95) associated to $\tilde{v}_z$ and where

$$N(\Omega,\omega,T,z) = e^{C(\Omega,\omega,T)(1+\|a_z\|_\infty^{\gamma/3}+\|B_z\|_\infty^{\gamma/2}+\|D_z\|_\infty^{\gamma}+e^{\frac{\gamma}{\gamma-1}\|M_z\|_\infty^{1/2}})}.$$

To the rest of the proof, it suffices to argue the same as in last section.

□

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