THE PERIODIC ORBITS OF A DYNAMICAL SYSTEM ASSOCIATED WITH A FAMILY OF QRT-MAPS

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Abstract. We study the QRT-maps associated with the family of biquadratic curves $C_d(K)$ with equations $x^2y^2 - dxy - 1 + K(x^2 + y^2) = 0$. With the Prime Number Theorem and the geometry of elliptic cubics we determine the periods of periodic orbits of the dynamical systems defined by these QRT-maps, and prove sensitivity to its initial conditions.

1. Introduction

We consider the family of real *biquadratic* curves $\mathcal{C}_d(K)$ (that is algebraic curves which are of degree 2 in x and y) with equations

$$x^2y^2 - dxy - 1 + K(x^2 + y^2) = 0, \tag{1}$$

where $d > 0$ and $K \in \mathbb{R}$. Note that these curves are symmetric with respect the two diagonals. With this family we associate the family of QRT-maps geometrically defined by the following way, for d fixed: let $(x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\}$ be a point, we consider the curve $\mathcal{C}_d(K)$ which passes through this point; then the horizontal line through this point cuts the curve at a point (X, y), and the vertical line passing through this last point cuts again the curve at the point (X, Y) (some of these points may be at infinity). The map $(x, y) \mapsto (X, Y)$ is the QRT-map T_d associated with the parameter d. This type of maps was defined in [18], motivated by problems of physics, and studied in many papers: see in particular [14], [18], [15] and [16] for general results, [2] to [12] and [22] for particular studies, and [13] for another point of view. For [2] to [12], we present in Appendix 3 a summary of their approach and their relations with the present paper.

The explicit expression of the map T_d can be obtained in the following way: the value of K for the curve that passes through the point (x, y) is

$$K = \frac{1 + dxy - x^2y^2}{x^2 + y^2}, \tag{2}$$

so that we have

$$xX = \frac{Ky^2 - 1}{y^2 + K} \quad \text{and} \quad yY = \frac{KX^2 - 1}{X^2 + K} \tag{3}$$

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for this value of K (use the product of the roots of the quadratic equations). For $(x, y) \in \mathbb{R}^2 \setminus \{(0,0)\}$, the computation gives

$$X = \frac{dy^3 - x(1+y^4)}{(1+y^4)+xdy}, \quad \text{and then} \quad Y = \frac{dX^3 - y(1+X^4)}{(1+X^4)+ydX}. \tag{4}$$

Of course, the map T_d leaves globally invariant each of the curves $\mathcal{C}_d(K)$. For studying the restriction of T_d to these curves, we will make some birational transformations in order to change $\mathcal{C}_d(K)$ in a Weierstrass elliptic cubic $\Gamma_d(K)$ of the form $Y^2 = 4X^3 - g_2X - g_3$, the map T_d on $\mathcal{C}_d(K)$ being conjugated to the addition of a fixed point for the classical chord-tangent law on $\Gamma_d(K)$, which is also conjugated to a rotation on the unit circle (*via* Weierstrass function). This first step will use some tools of algebraic geometry of elliptic cubic curves.

Then we search the rotation number $\theta_d(K)$ of this rotation, what we denote "the rotation number of T_d on $\mathcal{C}_d(K)$". Of course, when this number is an irreducible rational $\frac{p}{q}$, then T_d will be q-periodic on $\mathcal{C}_d(K)$. Our goal is to find all the possible periods q. For this, we will prove that the rotation number is a continuous function of K, and determine an interval I where varies this number. So we have to find irreducible ratios $\frac{p}{q} \in I$. It is in this step that we will use the Prime Number Theorem.

Moreover, we will prove that the dynamical system has a form of sensitivity to initial conditions, as in previous papers (see for instance [7], [8]).

2. The results and the plan of proofs

The essential results are the two following ones.

Theorem 1. *(1) For any $d > 0$ and $K \neq 0$, the map $T_{d|\mathcal{C}_d(K)}$ is conjugated to a rotation on the unit circle. Moreover, both those levels $K \neq 0$ for wich all the initial conditions give rise to trajectories that fill densely $\mathcal{C}_d(K)$; and those levels for which this curve is filled by periodic points (with the same period), are dense. In consequence, the periodic points for T_d are dense in $\mathbb{R}^2$, and also the non-periodic ones.*
(2) For every $d > 0$, there exists an integer $N(d)$ such that every integer $n \geq N(d)$ is a minimal period of some point for the QRT-map T_d.
(3) Every integer $n \geq 3$ is the minimal period for T_d of some initial point for some d.
(4) There is no point of period 2, nor real finite fixed point, but it is possible to extend continuously T_d to $(0,0)$, with image $(0,0)$.

Theorem 2. *The dynamical system associated with the QRT-map T_d has a form of sensitivity to initial conditions. More precisely:*
(1) for every $0 < K_1 < K_2$, there exists a constant $k_d(K_1, K_2) > 0$ such that for every $K \in [K_1, K_2]$, for every point $M_0 \in \mathcal{C}_d(K)$ and every neighbourhood V of M_0 there exists a

point $M_1 \in V$ such that $d\big(T_d^n(M_1), T_d^n(M_0)\big) \geq k_d(K_1, K_2)$ for an infinite values of n (we denote this property as "uniform sensitivity");
(2) for $K < 0$, for every point M_0 of $\mathcal{C}_d(K)$ there exists a constant $k_d(M_0)$ such that for every neighbourhood V of M_0 there exists a point $M_1 \in V$ such that $d\big(T_d^n(M_1), T_d^n(M_0)\big) \geq k_d(M_0)$ for an infinite values of n (here we speak on "pointwise sensitivity").

So it remains a problem:

Problem. Is there uniform sensitivity to initial conditions for $K \in [K_1, K_2]$, if the condition $K_1 < K_2 < 0$ holds ?

For proving the previous two assertions, the following one plays a great rôle.

Proposition 3. *The limits of the rotation number $\theta_d(K)$ when K tends to $+\infty$, to $-\infty$, or to 0, are respectively $\frac{1}{2}$, $\frac{1}{2}$ and $\ell(d) := \dfrac{1}{\pi}\arctan\sqrt{1+\dfrac{4}{d^2}}$, which is a function of d decreasing between $\frac{1}{2}$ and $\frac{1}{4}$.*

In Section 3, we study some general properties of the curves $\mathcal{C}_d(K)$ and of the QRT-map T_d (with a substitution of a new parameter m to the parameter K), prove some parts of theorem 1, and define and study the first rational transformations of $\mathcal{C}_d(K)$ into elliptic cubic curves.

In Part 4, we make successive other rational transformations on the curve obtained in Section 3, with the aim to obtain the classical Weierstrass form for this curve.

In Section 5 we explain for being complete (as in [8] or [20]), why, with Weierstrass' function, the map $T_d(K)$ is conjugated to a rotation on the unit circle. We use Weierstrass' function for determining an expression of the rotation number $\theta_d(K)$ as the ratio of two integrals, and prove proposition 3.

In Part 6 we prove theorem 1, with the aid of a refinement of the prime number theorem.

In Part 7 we establish theorem 2, with arguments which soon are for some of them in [7], and for some others in [3] and in [8]. The two cases $K > 0$ and $K < 0$ requires different types of arguments.

We finish with an Appendix 1 on the 4 and 6-periodic orbits, an Appendix 2 on the forms of the curves $\mathcal{C}_d(K)$, and an Appendix 3 on a summary of previous results on QRT-maps.

3. The QRT-map and its invariant curves, first transformations

First, we present in the following figures some types of the curves $\mathcal{C}_d(K)$, when $K \neq 0$. In fact we represent the sets where $x^2y^2 - dxy - 1 + K(x^2+y^2) < 0$, the curve itself is the boundary of this set.

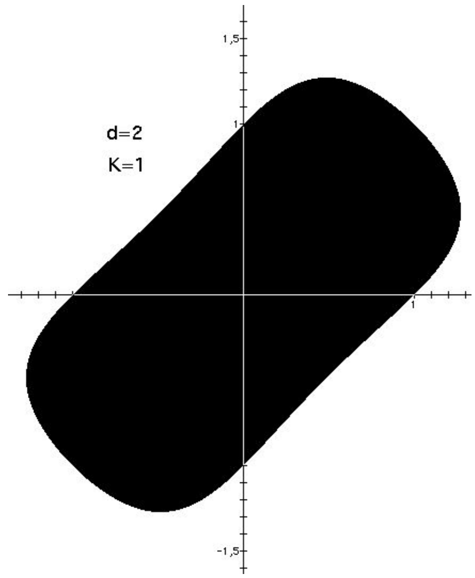

Figure 1: The curve $\mathcal{C}_d(K)$ for $K > 0$.

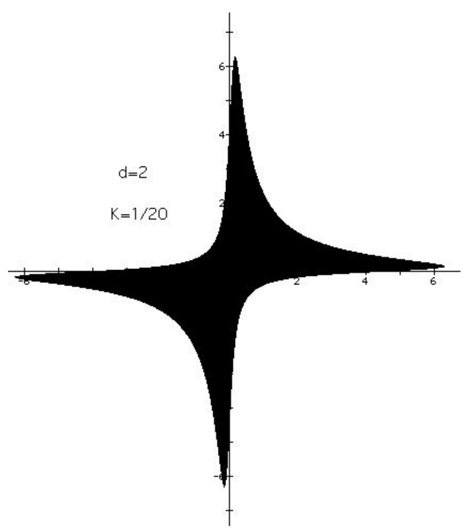

Figure 2: The curve $\mathcal{C}_d(K)$ for $K > 0$ small.

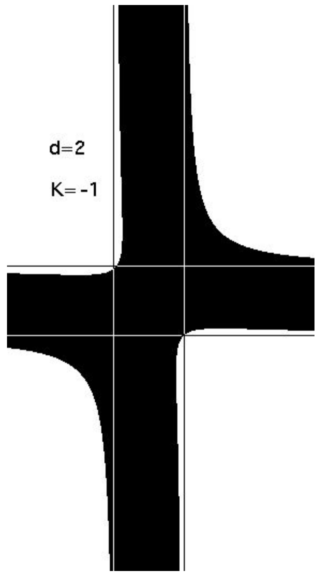

Figure 3: The curve $\mathcal{C}_d(K)$ for $K < 0$.

Next we present the figure of the surface S: $z = G_d(x, y) := \dfrac{1 + dxy - x^2y^2}{x^2 + y^2}$.

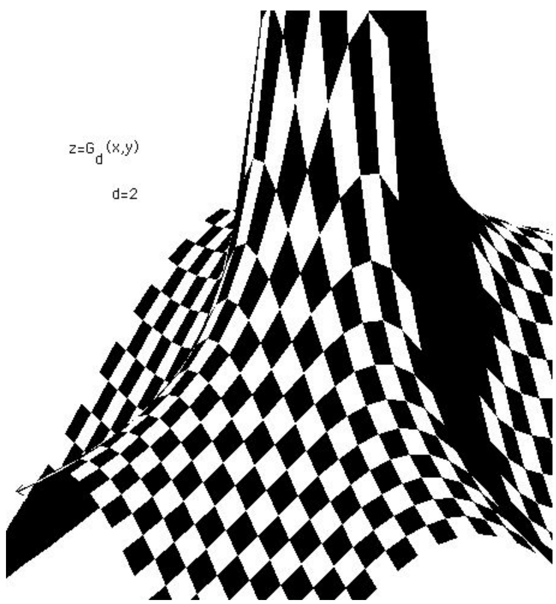

Figure 4: The surface $z = G_d(x, y)$.

For studying the curves $\mathcal{C}_d(K)$, which are the projections of the horizontal sections of the surface $z = G_d(x, y)$, we search the critical points of S and the limits of $G_d(x, y)$ when $(x, y) \to 0$ and (x, y) tends to the point at infinity of the plane. We have the following result.

Lemma 4. *(1) The surface S has no critical point;*
(2) $G_d(x, y) \to +\infty$ when $(x, y) \to (0, 0)$;

(3) $\inf G_d(x,y) = -\infty$ *on the plane, but on the axes* $\lim\limits_{x^2+y^2\to+\infty} G_d(x,y) = 0$.

Proof. Only the first point is not evident. The partial derivatives of G_d are

$$-dyx^2 + dy^3 - 2xy^4 - 2x = 0,$$

$$-dxy^2 + dx^3 - 2yx^4 - 2y = 0.$$

By equaling the two values of d given by these equations we obtain $(x^2 - y^2)[x^2(1+y^4) + y^2(1+x^4)] = 0$, and so we have $x = y$ or $x = -y$. In each case, by putting these values in the first equation, we find $-u(1+u^4) = 0$ (where $u := x$, $y = \pm u$), and this is impossible. ∎

With the previous results it is possible to prove the following results on the curves $\mathcal{C}_d(K)$ (see details in Appendix 2).

Lemma 5. *If* $K > 0$ *the curve* $\mathcal{C}_d(K)$ *is an oval which is bounded and connected; if* $K = 0$, $\mathcal{C}_d(K)$ *is the union of the two hyperbolas with equations* $xy = \frac{d}{2} \pm \sqrt{1 + \frac{d^2}{4}}$*; if* $K < 0$*, the curve* $\mathcal{C}_d(K)$ *has four connected but non bounded branches, with two vertical asymptotes* $x = \pm\sqrt{-K}$ *and two horizontal asymptotes* $y = \pm\sqrt{-K}$.

It is also easy, from formulas (3) or (4), to find the equation of the *locus* E_d of the points where X is infinite: on a curve $\mathcal{C}_d(K)$, with $K < 0$, there is exactly two points where the curve cuts its horizontal asymptotes. The set E_d is the curve with equation:

$$dxy + 1 + y^4 = 0. \tag{5}$$

If we put $K = -L^2$, we have a parametric equation for E_d:

$$x = -\frac{1 + L^4}{\varepsilon d L}, \quad y = \varepsilon L \quad \text{with } \varepsilon = \pm 1 : \tag{6}$$

There are the coordinates of the points where $\mathcal{C}_d(-L^2)$ cuts its horizontal asymptotes. At these points M_0 and M_1, X given by formulas (3) or (4) is infinite in horizontal direction. So, for extending the map T_d we shall introduce the extension of the curve $\mathcal{C}_d(K)$ in the projective real plane, with in particular the infinite points H in horizontal direction and V in vertical direction. We shall see that these points are singular points with distinct tangents (see Lemma 6), the horizontal and vertical asymptotes of the curve. In fact, we can give a symbolic representation in the projective real plane of this curve in Figure 5, where $t = 0$ is the line at infinity, when we take homogenous coordinates (x, y, t) for $\mathcal{C}_d(K)$.

Of course we have also an extension of $\mathcal{C}_d(K)$ in the complex projective plane. When $K > 0$, the asymptotes which are the tangents at the points H and V, are complex and distinct. Note that when $K = 0$, they are real but not distinct. For more simplicity we keep the same notation for the curve $\mathcal{C}_d(K)$ and its different extensions.

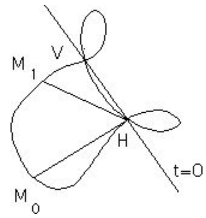

Figure 5: The real projective representation of $\mathcal{C}_d(K)$ when $K < 0$.

So it is not difficult to see that we can extend by continuity T_d so that $T_d(M_0) = T_d(M_1) = H$.

The reader can see himself the analogous results when Y is infinite, that is when the horizontals of points N_0 and N_1 of $\mathcal{C}_d(K)$ cut again the curve at one of the two intersections M'_0 and M'_1 of the vertical asymptotes with the curve.

Now we can see the algebraic nature of the curves $\mathcal{C}_d(K)$ when $K \neq 0$.

Lemma 6. *When $K \neq 0$ the complex projective curve $\mathcal{C}_d(K)$ has only two singular points, which are ordinary (with distinct tangents) and on the line at infinity: H and V. In this case, $\mathcal{C}_d(K)$ is an elliptic curve.*

Proof. (1) First we determine the singular points. We put $f(x, y, t) := x^2y^2 - dt^2xy - t^4 + Kt^2(x^2 + y^2)$, compute f'_x, f'_y, and the singular points are the solutions of $f'_x = 0$, $f'_y = 0$, $f = 0$. First, we have the solution $t = 0$ with $xy = 0$, which gives the points H and V.

Now we suppose $t = 1$, and have to solve the equations $f = 0$, $2xy^2 - dy + 2Kx = 0$ and $2yx^2 - dx + 2Ky = 0$. The two last ones give $(x - y)(2K - d - 2xy) = 0$.

First we have the solution $x = y := u$, and the second equation gives $u^2 = d/2 - K$; by puting in $f(u, u, 1) = 0$, we find $(d/2 - K)^2 + 1 = 0$, and this is impossible.

So we have the equations $-2xy+2K+d = 0$ and $2xy^2-dy+2Kx = 0$. This give $x = -y$, and $2x^2 + 2K + d = 0$. So we have the value of x^2 and y^2, and put them in $f(x, y, 1) = 0$. We obtain $-(K + d/2)^2 - 1 = 0$, and this is impossible.

Then, it is easy to compute the asymptotes of $\mathcal{C}_d(K)$ from its equation, for $K \neq 0$.

(2) For proving that the curve $\mathcal{C}_d(K)$ is elliptic if $K \neq 0$, we use the formula giving the genus of the curve: $g = \dfrac{(\delta-1)(\delta-2)}{2} - \sum_P \dfrac{\mu_p(\mu_p - 1)}{2}$, where δ is the degree of the curve (here $\delta = 4$), and P is the set of singular points p, and μ_p their order. Here $|P| = 2$ and the orders are 2. So we obtain $g = 1$, that is $\mathcal{C}_d(K)$ is elliptic.

But we have to verify that the curve is irreducible for using the previous formula giving the genus (see [17]).

First, suppose that we have the identity $x^2y^2-dxy-1+K(x^2+y^2) = (x+uy+v)f_1(x, y)$. If we make the division of the first member considered as a polynomial in x by $x + uy + v$, we obtain the remainder

$$Ky^2 - 1 + (uy + v)[uy^3 + vy^2 + (Ku + d)y + Kv],$$

which is a polynomial in y. The vanishing of it for all y gives the successive relations

$$u = 0,\ v^2 + K = 0,\ vd = 0,\ -1 = 0,$$

and this is impossible.

The only case which remains to study is if $x^2y^2 - dxy - 1 + K(x^2 + y^2)$ is the product of two irreducible polynomials of degree 2. But this situation gives for the curve $\mathcal{C}_d(K)$ more than two singular ordinary points (because $K \neq 0$), and this is not possible by the first part of the proof. ∎

So we can find a rational transformation which transforms $\mathcal{C}_d(K)$ in a regular cubic curve. For this, we split the point (m, m) of $\mathcal{C}_d(K)$ on the diagonal, with $m > 0$. This point satisfies the equation

$$K = \frac{d}{2} + \frac{1}{2m^2} - \frac{m^2}{2}. \tag{7}$$

The Figure 6 shows the graph of the function $m \mapsto K$: for $m > 0$, K decreases from $+\infty$ to $-\infty$.

The number m_d giving $K = 0$ is the positive solution of the equation:

$$m^4 - dm^2 - 1 = 0, \tag{8}$$

easy to solve. The function $m \mapsto K$ is one-to-one, so we can take for parameter of the family of curves $\mathcal{C}_d(K)$ (for d fixed) the number $m > 0$. We make this choice in the following computations, with $K \neq 0$, that is $m \neq m_d := \sqrt{\frac{d}{2} + \sqrt{1 + \frac{d^2}{4}}}$.

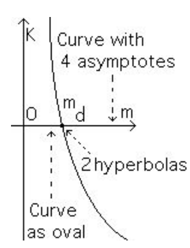

Figure 6: K function of m.

Now, the first transformation we use is $\phi_1(x, y) = (X, Y)$ defined, for $(x, y) \neq (m, m)$ and $(x, y) \neq (-m, -m)$, by

$$X = \frac{x - m}{xy - m^2}, \quad Y = \frac{y - m}{xy - m^2}. \tag{9}$$

In fact, it is easy to prove that the hyperbola $\mathcal{H}_m$ with equation $xy = m^2$ cuts the curve $\mathcal{C}_d(K)$ only at the points (m, m) and $(-m, -m)$, for K given by (7). So we can consider ϕ_1 defined on $\mathbb{R}^2 \setminus \mathcal{H}_m$.

By some computations, one see that the inverse of ϕ_1 is given by

$$x = \frac{1 - mX}{Y}, \quad y = \frac{1 - mY}{X}. \tag{10}$$

The pencil of horizontal lines, with equations $y = C$, becomes by the action of ϕ_1 the pencil of lines with equations $1 - mY - CX = 0$, that is the pencil of lines passing through the point

$$P = \Big(0, \frac{1}{m}\Big), \tag{11}$$

and the pencil of vertical lines becomes the one of lines passing through the point

$$P' = \Big(\frac{1}{m}, 0\Big). \tag{12}$$

We calculate the equation of the image of $\mathcal{C}_d(K)$ by ϕ_1, and see that it splits in this one of the line D

$$1 - m(X + Y) = 0, \tag{13}$$

and in this one of the symmetric cubic curve $A_d^1(m)$

$$Km(X^3+Y^3) - KmXY(X+Y) - K(X^2+Y^2) + XY(d-2m^2) + m(X+Y) - 1 = 0, \tag{14}$$

where K is the function of m given in (7).

Now, we make another transformation ϕ_2, by puting $(U, V) = \phi_2(X, Y)$, that is

$$X := \frac{U + V}{2}, \quad Y := \frac{U - V}{2}. \tag{15}$$

We obtain the cubic curve $A_d^2(m)$ with equation

$$V^2(KmU - a) - bU^2 + mU - 1 = 0 \tag{16}$$

where

$$a = \frac{K}{2} + \frac{d}{4} - \frac{m^2}{2} = \frac{d}{2} + \frac{1}{4m^2} - \frac{3m^2}{4}, \quad b = \frac{K}{2} - \frac{d}{4} + \frac{m^2}{2} = \frac{1}{4}\Big(m^2 + \frac{1}{m^2}\Big) \tag{17}$$

(K being given by relation (7)).

By the transformation $\phi_2 \circ \phi_1$ the pencils of horizontal and vertical lines become the pencils of lines passing respectively through the points

$$P_1 = \Big(\frac{1}{m}, -\frac{1}{m}\Big), \quad P_1' = \Big(\frac{1}{m}, \frac{1}{m}\Big). \tag{18}$$

So, we have results about the curve $A_d^2(m)$ and the conjugate of T_d on this curve.

Proposition 7. *(1) The curve $A_d^2(m)$ is symmetric with respect to the U-axis, V is defined for $U > \lambda_0$ if $K > 0$, and for $U < \lambda_0$ if $K < 0$, where*

$$\lambda_0 = \frac{a}{Km} = \frac{3m^4 - 2dm^2 - 1}{2m(m^4 - dm^2 - 1)}; \tag{19}$$

the curve has a vertical asymptote $U = \lambda_0$ and two parabolic branches for U tending to $\mathrm{sign}(K)\infty$ *(see Figures 7);*
(2) the map $\phi_2 \circ \phi_1$ is a homeomorphism of $\mathcal{C}_d(K) \setminus \{(m,m), (-m,-m)\}$ on $A_d^2(m)$; when $(x,y) \to (m,m)$, then $\phi_2 \circ \phi_1(x,y)$ tends to the infinite vertical point on the asymptote, and when $(x,y) \to (-m,-m)$, then $\phi_2 \circ \phi_1(x,y)$ tends to to the horizontal point at infinity on the parabolic branches of $\mathcal{A}_d^2$;
(3) the restriction of T_d to $\mathcal{C}_d(K) \setminus \{(m,m), (-m,-m)\}$ is conjugated by $\phi_2 \circ \phi_1$ to the addition of the point P_1 for the chord-tangent law $\underset{P_1'}{+}$ on $\mathcal{A}_d^2$ whose the zero point is P_1'.

Proof. (1) We can write

$$V^2 = \frac{bU^2 - mU + 1}{KmU - a}, \tag{20}$$

and remark that $bU^2 - mU + 1 > 0$ for every U; so the result is easy with $\lambda_0 = \dfrac{a}{Km}$.
(2) If we put $A(m) = \dfrac{3m^4 - 2dm^2 - 1}{m(m^4+1)}$, some computations with *Maple* show that one has, for $x = m+u$, $y = m - u + A(m)u^2 + O(u^3)$ when $u \to 0$. So we see that $U \to \frac{A(m)}{mA(m)-1} = \lambda_0$ when $u \to 0$, and $V \sim \frac{m^4+1}{m^4-dm^2-1}\frac{1}{u} \to \infty sign(u)$: this is the first part of point (2).

For the second part, some computations give that, if $x = -m+u$, then $U \sim \dfrac{m^4+1}{mK}\dfrac{1}{u^2}$ and $V \sim \dfrac{m^4+1}{2m^2K}\dfrac{1}{u}$; this gives easily the result.
(3) The conjugation is obvious with the previous results about the pencils of lines and the geometric definition of T_d : by the alignment of a point M with P_1, the line MP_1 cuts $A_d^2(m)$ at M_1, and the line M_1P_1' cuts $A_d^2(m)$ at the image of M by the conjugated of T_d by $\phi_2 \circ \phi_1$. And this is exactly the addition of P_1 to M for the law chord-tangent $\underset{P_1'}{+}$ on the curve, whose zero is the point P_1' (see [20] for the classical definition of this law). ∎

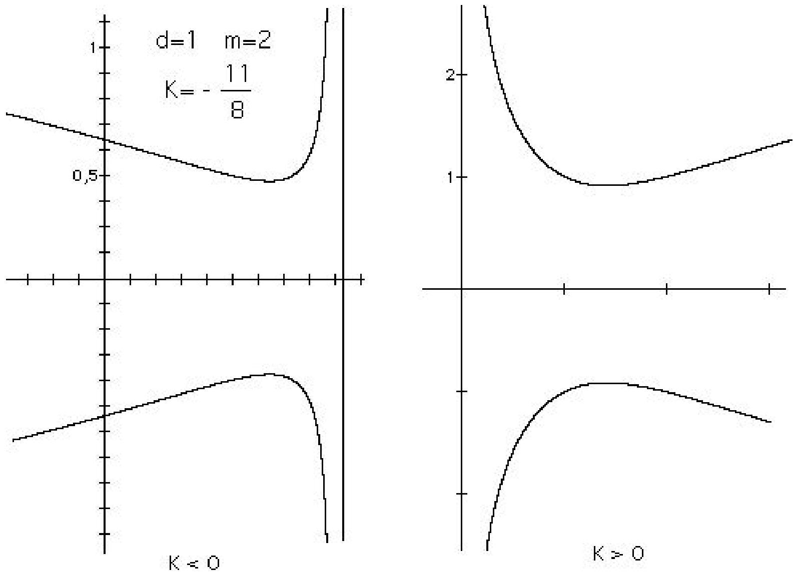

Figure 7: The curves $\mathcal{A}_d^2(m)$

4. Weierstrass form for the cubic curve $A_d^2(m)$

We start now from the cubic curve $A_d^2(m)$ whose equation may be written

$$V^2(KmU - a) = (bU^2 - mU + 1). \tag{21}$$

We put $KmU = \widetilde{U}$, so that to take into account the sign of K, and obtain

$$V^2(\widetilde{U} - a) = \frac{b}{K^2m^2}\widetilde{U}^2 - \frac{1}{K}\widetilde{U} + 1.$$

Now we multiply it by $\widetilde{U} - a$ and put $W := V(\widetilde{U} - a)$, and obtain a new curve $\mathcal{A}_d^3(m)$ with equation

$$W^2 = (\widetilde{U} - a)\Big(\frac{b}{K^2m^2}\widetilde{U}^2 - \frac{1}{K}\widetilde{U} + 1\Big). \tag{22}$$

With this form, we have a cubic curve which is almost in Weierstrass form. We see that the roots of the right hand member are simple. First, we introduce two notations: we put

$$\Delta := m^4 - dm^2 - 1, \quad \Omega := 3m^4 - 2dm^2 - 1. \tag{23}$$

Lemma 8. *The roots of the right hand member of (22), that is the abscissas of the intersection of $\mathcal{A}_d^3(m)$ with the $\widetilde{U}$-axis are*

$$a = -\frac{\Omega}{4m^2}, \ \textit{which is real, and} \ \frac{\Delta(-m^2 \pm i)}{m^4+1}, \ \textit{which are complex.} \tag{24}$$

Their sum is

$$\alpha := a - \frac{2m^2\Delta}{m^4+1} = -\frac{(m^4+1)\Omega + 8m^4\Delta}{4m^2(m^4+1)}. \tag{25}$$

We denote ϕ_3 the map $(U,V) \mapsto (\widetilde{U},W)$. Now, it is classical (but not obvious) that the map conjugated by $\phi_2 \circ \phi_1$ of T_d on $\mathcal{A}_d^2(m)$ is conjugated by ϕ_3 to the addition of $Q := \phi_3(P_1)$ for the chord-tangent law $\underset{Q'}{+}$ on the curve $\mathcal{A}_d^3(m)$ whose zero point is $Q' := \phi_3(P_1')$ (see[20]).

Now, to pass from the form (22) to the canonical Weierstrass form $\mathcal{A}_d^4(m)$ it suffices to make an affine transformation ϕ_4 on the variable $\widetilde{U}$ and W. But for identifying the map $\widetilde{T_d}$ conjugated to T_d by $\phi_4 \circ \phi_3 \circ \phi_2 \circ \phi_1$, it is necessary to make a group isomorphism ϕ_5 on the curve $\mathcal{A}_d^4(m)$, with the aim to send the point Q' to the infinite vertical point ω of the curve. This is similar to that is made in paper [7]. We recall that the new conjugated of $\widetilde{T_d}$ is now the map $\widetilde{\widetilde{T_d}}$ which is the addition of the point $R := \phi_5 \circ \phi_4(Q)$ for the classical chord-tangent law $\underset{\omega}{+}$ on the curve $\mathcal{A}_d^4(m)$ whose zero point is ω (see in [7] a computational proof, with *Maple*).

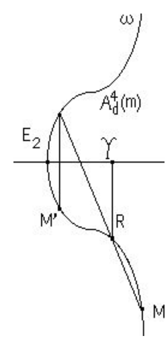

Figure 8: Addition of R on $\mathcal{A}_d^4(m)$ for $\underset{\omega}{+}$.

So the goal is to find the point R, and to interpret the addition of R for $\underset{\omega}{+}$ as a rotation. For this, the following lemma is useful. We will see (in Part 5) that it gives the abscissa of point R, if we take for $\mathcal{D}$ the curve $\mathcal{A}_d^4(m)$. The proof of this lemma is a simple computation which we leave to the reader.

Lemma 9. *Let $M := (u,v)$ with $v \neq 0$ a point on a canonical cubic curve $\mathcal{D}$ with equation*

$$Y^2 = 4X^3 - g_2X - g_3.$$

Then the tangent at $\mathcal{D}$ in M cuts the curve at a point whose abscissa Υ is given by

$$\Upsilon = \Big(\frac{12u^2 - g_2}{4v}\Big)^2 - 2u. \tag{26}$$

Now, we pass to the the interpretation of addition of R on $\mathcal{A}_d^4(m)$ as a rotation. Before, we assert a result which will be useful, and which is easy with formulas between (X,Y) and $(\widetilde{U},W)$.

Lemma 10. *The map $\phi_3 \circ \phi_2$ is a homeomorphism of $\mathcal{A}_d^1(m)$ onto $\mathcal{A}_d^3(m)$.*

5. An integral formula for the rotation number if $K \neq 0$

5.1. **Weierstrass function and the integral formula.** The map $\widetilde{\widetilde{T_d}}$ acts on the cubic curve $\mathcal{A}_d^4(m)$, and we will see that this action is conjugated to a rotation on the unit circle $\mathbb{T}$. For this, we parametrize the previous cubic in the affine complex plane by a Weierstrass function. For the proofs to be complete, we reproduce here arguments which are in [8], and are also in [1].

There exist two complex conjugated numbers denoted by $2\omega_1$ and $2\omega_3$ (with $\Re(\omega_3) > 0$), which depend on d and K (or m), with the following properties (see Figure 9): if Λ is the lattice in $\mathbb{C}$ defined by $\Lambda = \{2n\omega_1 + 2p\omega_3 | (n,p) \in \mathbb{Z}^2\}$, then the Weierstrass' elliptic function $\mathcal{P}$ is a meromorphic function defined on $\mathbb{C} \setminus \Lambda$ by

$$\mathcal{P}(z) := \frac{1}{z^2} + \sum_{\lambda \in \Lambda \setminus \{0\}} \Big[\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2}\Big]; \tag{27}$$

it is doubly periodic (its periods are its poles: the points of Λ), and gives the following parametrization of the entire cubic curve $\mathcal{A}_d^4(m)$ (its real and complex points in the projective plane)

$$X = \mathcal{P}(z), \quad Y = \mathcal{P}'(z). \tag{28}$$

The main properties of this parametrization are the following facts (see [1], [8]):
(1) it transforms the addition in $\mathbb{C}$ into the addition on $\mathcal{A}_d^4(m)$ for its group law $\underset{\omega}{+}$;
(2) it passes to quotient into a homeomorphism of the topological group $\mathbb{T}^2 \approx \mathbb{C}/\Lambda$ onto the compact complex projective curve $\mathcal{A}_d^4(m)$ which sends the quotient of segments $[0, 2\omega_2]$ (where $\omega_2 = \omega_1 + \omega_3$ is real) onto the compact projective *real* curve $\mathcal{A}_d^4(m)$;
(3) one has the relations: $\mathcal{P}(\omega_2) = e_2$, the abscissa of the intersection E_2 of the x-axis with $\mathcal{A}_d^4(m)$; $(\mathcal{P}(0), \mathcal{P}'(0)) = (\mathcal{P}(2\omega_2), \mathcal{P}'(2\omega_2)) = \omega$, the vertical point at infinity on the curve; $\mathcal{P}' < 0$ on $]0, \omega_2[$, $\mathcal{P}' > 0$ on $]\omega_2, 2\omega_2[$ (see figure 9).

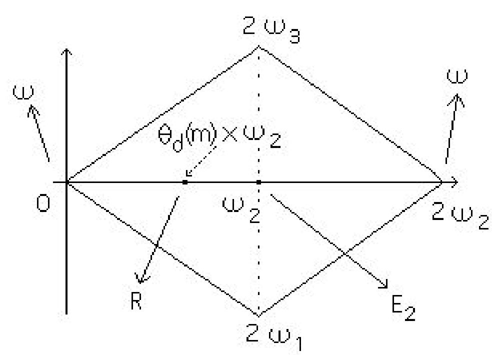

Figure 9: The lattice Λ.

So, if we identify $\mathbb{C}/\Lambda$ to the set of points $(e^{2i\pi u}, e^{2i\pi v})$ of $\mathbb{T}^2 \approx (\mathbb{R}/\mathbb{Z})^2$, the compact projective *real* cubic $\mathcal{A}_d^4(m)$ is group-homeomorphic to the diagonal $\Delta = \{(e^{2i\pi t}, e^{2i\pi t}) | 0 \leq$

$t \leq 1\}$ of $\mathbb{T}^2$. The parameters of E_2 and ω are $t = \frac{1}{2}$ and 0 or 1. But the point R of $\mathcal{A}_d^4(m)$ has then a well defined parameter $\theta_d(m) \in]0, \frac{1}{2}[$, and the addition of R to points of $\mathcal{A}_d^4(m)$ is traduced in Δ by the map $(e^{2i\pi t}, e^{2i\pi t}) \mapsto (e^{2i\pi(t+\theta_d(m))}, e^{2i\pi(t+\theta_d(m))})$. So we have proved the following result.

Proposition 11. *For $d > 0$ and $m > 0$ it exists a well-defined number $\theta_d(m) \in]0, \frac{1}{2}[$ such that the restriction of the map T_d to the curve $\mathcal{C}_d(K)$ is conjugated to a rotation of angle $2\pi\theta_d(m) \in]0, \pi[$ onto the unit circle, for K and m linked by (7). So this number $\theta_d(m)$ is the rotation number of the restriction of T_d to $\mathcal{C}_d(K)$.*

Corollary 12. *If $\theta_d(m)$ is rational with the form $\frac{p}{q}$ irreducible, then the restriction of T_d at $\mathcal{C}_d(K)$ is periodic, with minimal period q. When $\theta_d(m)$ is irrational, then the orbit of every point of $\mathcal{C}_d(K)$ in the action of $T_d(K)$ is dense in $\mathcal{C}_d(K)$.*

Now we can give the integral formula for $\theta_d(m)$. We start from the Weierstrass form of the equation of the curve $\mathcal{A}_d^4(m)$:

$$Y^2 = F(X) := 4X^3 - g_2X - g_3, \tag{29}$$

and denote e_2, e_1 and e_3 the real root (abscissa of E_2) and the complex conjugated roots of the polynomial F. We use the classical formula giving the inversion of function $\mathcal{P}$ for real values of the variable and the function (see [1]):

$$u = \int_{\mathcal{P}(u)}^{+\infty} \frac{dt}{\sqrt{F(t)}}.$$

With the relations $\mathcal{P}(\omega_2) = e_2$ and $\mathcal{P}(2\omega_2 \times \theta_d(m)) = \Upsilon$, we obtain

$$2\theta_d(m) = \frac{\int_{\Upsilon}^{+\infty} \frac{dt}{\sqrt{F(t)}}}{\int_{e_2}^{+\infty} \frac{dt}{\sqrt{F(t)}}}, \tag{30}$$

where Υ is the abscissa of point R given by lemma 9.

Now we make the change of variable in the two integrals: $t = e_2 + s^2$, and obtain the final result, by using the equality $F(t) = 4(t-e_2)(t-e_1)(t-e_3)$, with $e_3 = \bar{e_1}$, and putting

$$z := e_2 - e_1. \tag{31}$$

Proposition 13. *The rotation number is given by the following formula*

$$2\theta_d(m) = \frac{\int_{\sqrt{\Upsilon - e_2}}^{+\infty} \frac{ds}{|s^2+z|}}{\int_0^{+\infty} \frac{ds}{|s^2+z|}}, \tag{32}$$

where z is the complex number given by (31) and Υ the abscissa of R.

Remark that this formula proves easily that the map $m \mapsto \theta_d(m)$ is continuous, because the computations in section 4 prove that all the parameters Υ, $e_1,\ e_2,\ e_3,\ z$ are continuous functions of m.

Now it is time to precise the transformations ϕ_3, ϕ_4 and ϕ_5, and to evaluate the different parameters which are useful for studying the limits of rotation number when K tends to $\pm\infty$ and to 0.

5.2. **Limits of the rotation number when** $K \to +\infty$, $K \to -\infty$, $K \to 0$**: proof of proposition 3.** First we precise the map ϕ_4.

The curve $\mathcal{A}_d^3(m)$ has for equation

$$W^2 = \frac{m^4+1}{\Delta^2}\widetilde{U}^3 + \frac{\Omega(m^4+1)+8m^4\Delta}{4m^2\Delta^2}\widetilde{U}^2 + \frac{\Omega+2\Delta}{2\Delta}\widetilde{U} + \frac{\Omega}{4m^2}, \tag{33}$$

or

$$W^2 = \lambda\widetilde{U}^3 + \mu\widetilde{U}^2 + \nu\widetilde{U} + \rho. \tag{34}$$

Then, α being the sum of the roots of the second member of (34) (given by (25)), the affine transformation ϕ_4 is $(\widetilde{U}, W) \mapsto (X, Y)$ given by

$$X = \frac{\mu}{12} + \frac{\lambda}{4}\widetilde{U}, \quad Y = \frac{\lambda}{4}W. \tag{35}$$

So by action of ϕ_4 we pass to the Weierstrass cubic curve $\mathcal{A}_d^4(m)$ whose equation is $Y^2 = 4X^3 - g_2X - g_3$. The images of P_1 and P_1' by $\phi_4 \circ \phi_3$ are

$$Q_1 = (q, -r), \quad \text{and} \quad Q_1' = (q, r), \tag{36}$$

which are on the same vertical line, with

$$\begin{aligned} q &:= \frac{5m^8 - 4dm^6 - 6m^4 + 4dm^2 + 5}{48m^2\Delta^2} \\ r &:= \frac{(m^4+1)^2}{16m^3\Delta^2}. \end{aligned} \tag{37}$$

And the conjugated of T_d by $\phi_4 \circ \phi_3 \circ \phi_2 \circ \phi_1$ is the addition of Q_1 for the chord-tangent law $\underset{Q_1'}{+}$ on $\mathcal{A}_d^4(m)$ with zero element Q_1'. To pass to the classical chord-tangent law $\underset{\omega}{+}$, where ω is the point at infinity in vertical direction on $\mathcal{A}_d^4(m)$, we make the group isomorphism ϕ_5 defined on $\mathcal{A}_d^4(m)$ by

$$M \mapsto M \underset{Q_1'}{+} \omega. \tag{38}$$

It is not difficult to see that the image of point Q_1 by ϕ_5 is the point R where the tangent at the curve in Q_1' cuts again the curve. So its abscissa Υ is given by formula (26), where u and v are the coordinates of Q_1' given by (36) and (37).

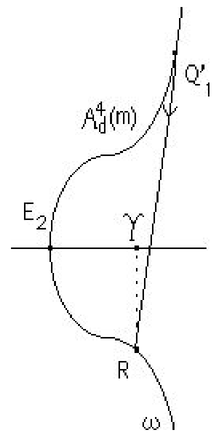

Figure 10: Building of point R

And the final conjugated $\widetilde{\widetilde{\widetilde{T_d}}}$ of T_d is now the addition of R for the classical chord-tangent law $\underset{\omega}{+}$ on $\mathcal{A}_d^4(m)$ (see Figure 8), which is conjugated to the rotation of angle $2\pi\theta_d(m)$ as explained in 5.2.

Now, we can compute, with the aid of *Maple*, the parameters in the integral formula given in Proposition 13 : $\varUpsilon$, e_2 and z (and also other useful parameters):

Lemma 14. *One has*

$$\begin{aligned} g_2 &= \frac{P_{16}(m)}{192m^4\Delta^4}, \\ e_2 &= \frac{m^8-2dm^6-6m^4+2dm^2+1}{24m^2\Delta^2}, \\ e_1 &= -\frac{m^8-2dm^6-6m^4+2dm^2+1}{48m^2\Delta^2}-\frac{i}{4\Delta}, \\ \varUpsilon &= \frac{2m^8-4dm^6+(3d^2-12)m^4+4dm^2+2}{48m^2\Delta^2}, \\ \sqrt{\varUpsilon-e_2} &= \frac{dm}{4|\Delta|}, \\ z &= \frac{m^8-2dm^6-6m^4+2dm^2+1}{16m^2\Delta^2}+\frac{i}{4\Delta}, \end{aligned} \tag{39}$$

where

$$P_{16}(m) = m^{16}-4dm^{14}+\cdots+4dm^2+1. \tag{40}$$

Corollary 15. *The map* $m \mapsto \theta_d(m)$ $\big[$*resp.* $K \mapsto \theta_d(K)\big]$ *is analytic on* $]0,+\infty[\setminus\{m_d\}$ $\big[$*resp. on* $\mathbb{R}\setminus\{0\}\big]$.

Now we can find the limits of the rotation number, from formula (32) in Proposition 13.

(a) Limit when $m \to +\infty$

We have

$$\begin{aligned} \sqrt{\Upsilon - e_2} &= \frac{c(m)}{m^3}, \quad \text{where} \quad c(m) \to \frac{d}{4}, \\ \mathrm{Re}(z) &= \frac{a(m)}{m^2}, \quad \text{where} \quad a(m) \to \frac{1}{16}, \\ \mathrm{Im}(z) &= \frac{b(m)}{m^4}, \quad \text{where} \quad b(m) \to \frac{1}{4}, \end{aligned} \tag{41}$$

and also

$$|s^2 + z| = \sqrt{\left(s^2 + \frac{a(m)}{m^2}\right)^2 + \frac{b(m)^2}{m^8}}. \tag{42}$$

We can write

$$2\theta_d(m) = 1 - \frac{\int_0^{\frac{c(m)}{m^3}} \frac{ds}{|s^2+z|}}{\int_0^{+\infty} \frac{ds}{|s^2+z|}}. \tag{43}$$

In the ratio of the two integrals, we shall show that the numerator tends to 0 and the denominator is greater than a number $D > 0$.

We have $\displaystyle\int_0^{\frac{c(m)}{m^3}} \frac{ds}{|s^2 + z|} \leq \frac{c(m)}{m^3} \frac{1}{\sqrt{\left(\frac{a(m)}{m^2}\right)^2 + \frac{b(m)^2}{m^8}}} = \frac{1}{m} \frac{c(m)d(m)}{a(m)}$, where $d(m) \to 1$ when $m \to +\infty$, and this ratio tends to 0.

Now, if $a(m) \leq A$, $b(m) \leq B$ and if $m \geq 10$, the denominator of (43) is minorized by

$$\int_0^{+\infty} \frac{ds}{\sqrt{\left(s^2 + \frac{A}{10^2}\right)^2 + \frac{B^2}{10^8}}} \geq \int_0^{+\infty} \frac{ds}{s^2 + \frac{A}{10^2} + \frac{B}{10^4}} := D > 0.$$

Then $\displaystyle\lim_{m \to +\infty} \theta_d(m) = \frac{1}{2}$.

(b) Limit when $m \to 0$

We have

$$\begin{aligned} z &= \frac{a'(m)}{m^2} + b'(m)i, \quad \text{where} \quad a'(m) \to \frac{1}{16} \quad \text{and} \quad b'(m) \to \frac{1}{4}, \\ \sqrt{\Upsilon - e_2} &= mc'(m), \quad \text{where} \quad c'(m) \to \frac{d}{4}. \end{aligned} \tag{44}$$

So we have also

$$|s^2 + z| = \sqrt{\left(s^2 + \frac{a'(m)}{m^2}\right)^2 + b'(m)^2}. \tag{45}$$

In the ratio of the two integrals of formula (43), we majorize the numerator

$$\int_0^{mc'(m)} \frac{ds}{|s^2 + z|} \leq mc'(m) \frac{1}{\sqrt{\left(\frac{a'(m)}{m^2}\right)^2 + b(m)^2}} = mc'(m) \frac{m^2}{a'(m)} d'(m),$$

where $d'(m) \to 1$ when $m \to 0$; and this tends to 0 with m.

Now we minor the denominator by

$$\int_0^{+\infty} \frac{ds}{\sqrt{\left(s^2+\frac{A}{m^2}\right)^2+B^2}} \geq \int_0^{+\infty} \frac{ds}{s^2+\frac{A}{m^2}+B},$$

where A is a constant larger than $a'(m)$ and B is a constant larger than $b'(m)$. So the denominator is minored by

$$\int_0^{+\infty} \frac{m^2 ds}{m^2 s^2 + A + Bm^2} = \int_0^{+\infty} \frac{dt}{t^2+A+Bm^2}$$

$$= \frac{1}{\sqrt{A+Bm^2}} \arctan \frac{t}{\sqrt{A+Bm^2}}\Big]_0^{+\infty} \geq \frac{\pi}{2\sqrt{A+B}} > 0$$

if $m \leq 1$. Then $\theta_d(m) \to \frac{1}{2}$ when $m \to 0$.

(c) Limit when $m \to m_d$ ($K \to 0$)

We put $u := m - m_d$. When $m \to m_d = \sqrt{\frac{d}{2}+\frac{1}{2}\sqrt{d^2+4}}$ we have, with some computations

$$\begin{aligned} \sqrt{\Upsilon - e_2} &= \frac{\gamma(m)}{|u|}, \\ z &= \frac{\alpha(m)}{u^2} + i\frac{\beta(m)}{u}, \\ |s^2+z| &= \sqrt{\left(s^2+\frac{\alpha(m)}{u^2}\right)^2 + \frac{\beta(m)^2}{u^2}} \end{aligned} \tag{46}$$

where

$$\begin{aligned} \lim_{u\to 0} \alpha(m) &= \frac{1}{64}, \\ \lim_{u\to 0} \beta(m) &= \frac{1}{8m_d\sqrt{d^2+4}}, \\ \lim_{u\to 0} \gamma(m) &= \frac{d}{8\sqrt{d^2+4}}. \end{aligned} \tag{47}$$

So, in $|s^2+z|$ given in (46), we put in factor $\frac{1}{u^4}$ in the two integrals of formula (32); then we change of variable by putting $s|u| = t$; after simplification, we obtain

$$2\theta_d(m) = \frac{\int_{\gamma(m)}^{+\infty} \frac{dt}{\sqrt{\left(t^2+\alpha(m)\right)^2+u^2\beta(m)^2}}}{\int_0^{+\infty} \frac{dt}{\sqrt{\left(t^2+\alpha(m)\right)^2+u^2\beta(m)^2}}}. \tag{48}$$

The limits of the integrands are easy, and we obtain by Lebesgue's theorem, as limit of $2\theta_d(m)$ when $m \to m_d$, that is when $u \to 0$, the quantity

$$\frac{8\Big[\arctan 8t\Big]_{\frac{d}{8\sqrt{d^2+4}}}^{+\infty}}{8\Big[\arctan 8t\Big]_{0}^{+\infty}}.$$

In fine, we have

$$\lim_{K\to 0}\theta_d(K) = \frac{\arctan\sqrt{1+\frac{4}{d^2}}}{\pi}. \tag{49}$$

Of course, the function $\ell(d) := \sqrt{1+\frac{4}{d^2}}$ decreases from $+\infty$ to 1 when d varies. So $\lim\limits_{K\to 0}\theta_d(K)$ varies between $\frac{1}{4}$ and $\frac{1}{2}$ when d varies. So we have proved Proposition 3.

Corollary 16. *The two maps $m \mapsto \theta_d(m)$ and $K \mapsto \theta_d(K)$ are not constant on every open interval of their domain.*

Proof. From corollary 15, these functions are analytical, and their limits on the boundary of the intervals are distinct, so there are not constant on every open interval. ∎

It is now easy to prove the following fact, *id est* **the point (1) of Theorem 1** (see [3] or use the previous corollary).

Proposition 17. *The set of periodic points is dense in $\mathbb{R}^2$; the set of points whose orbit is dense in the curve $\mathcal{C}_d(K)$ which passes through them is also dense in $\mathbb{R}^2$ (then these two sets form a partition of $\mathbb{R}^2 \setminus \{(0,0)\} \cup \mathcal{C}_d(0)$, each of these sets is an union of invariant and disjoint elliptic curves).*

6. The possible minimal periods of the QRT-map : proof of theorem 1

We know that the map $\theta : (d,K) \mapsto \theta_d(K)$ is continuous. We shall study its image, first for $d > 0$ fixed, and then for $d > 0$ and $K \in \mathbb{R}$.

For d fixed, the image of θ_d contains the interval

$$I_d := \Big]\frac{1}{\pi}\arctan\sqrt{1+\frac{4}{d^2}}, \frac{1}{2}\Big[= \Big]\ell(d), \frac{1}{2}\Big[. \tag{50}$$

So we shall find irreducible ratios $\frac{q}{n} \in I_d$. First, we search only this numbers with q a prime number not factor of the integer n. So we will find $q \in nI_d = \Big]n\ell(d), \dfrac{n}{2}\Big[$.

We use a refinement of the prime number theorem (PNT), which says that if $x \geq 52$, the number $\pi(x)$ of prime numbers not greater than x satisfies

$$\frac{x}{\ln x} \leq \pi(x) \leq \frac{x}{\ln x}\Big(1+\frac{3}{2\ln x}\Big); \tag{51}$$

For this result, due to Rosser and Schoenfeld, we refer to [21].

We use also an optimal majorization of G. Robin for the cardinal $\omega(x)$ of the set of distinct prime factors of the integer x (see [19]):

$$\omega(x) \leq 1.38402\frac{\ln x}{\ln(\ln x)}. \tag{52}$$

Now we use the same method as in [3]: if $n \geq 52$, the cardinal of the set of integers q relatively prime with n and strictly between $n\ell(d)$ and $\frac{n}{2}$ is at least

$$\frac{\frac{n}{2}-1}{\ln(\frac{n}{2}-1)} - \frac{n\ell(d)}{\ln(n\ell(d))}\Big(1+\frac{3}{2\ln(n\ell(d))}\Big) - 1.38402\frac{\ln n}{\ln(\ln n)}.$$

So, if the function

$$h_d(n) := \frac{\frac{n}{2}-1}{\ln(\frac{n}{2}-1)} - \frac{n\ell(d)}{\ln(n\ell(d))}\Big(1+\frac{3}{2\ln(n\ell(d))}\Big) - 1.38402\frac{\ln n}{\ln(\ln n)} - 1 \tag{53}$$

is positive, then n will be a minimal period of T_d.

Proof of part (2) of Theorem 1

The principal part of the asymptotic development of the function h_d when $n \to +\infty$ is $\frac{n}{\ln n}\big(\frac{1}{2}-\ell(d)\big)$. But $\frac{1}{2}-\ell(d) > 0$, so $h_d(n) > 0$ if $n \geq N(d)$ for some integer $N(d) > 0$: it was our goal !

For instances, when we take $d = \sqrt{2}$, then $\ell(d) = \frac{1}{\pi}\frac{\pi}{3} = \frac{1}{3}$, and if we find by some computations a number N such that $h_{\sqrt{2}}(n) > 0$ for $n \geq N$ we shall conclude that $N(\sqrt{2}) \leq N$. If we use a computer for looking to the graph of the function $h_{\sqrt{2}}(n)$, we see that it is negative for $n = 780$ and positive for $n \geq 781$, so we see that $N(\sqrt{2}) \leq 781$. We shall see in the next proof a method for improve this sort of result (see remark 19).

Proof of part (3) of Theorem 1

The image of function θ, when $d > 0$ and $K \in \mathbb{R}$ vary, contains $I = \bigcup_{d>0} I_d = \Big]\frac{1}{4}, \frac{1}{2}\Big[$, so we put $h(n) = h_{+\infty}(n)$. First, we prove that every $n \geq 7$ is a minimal period for some d and some K. Then we shall study the small values of n. The type of reasoning we use in this part is soon in [3].

We report the proof of the following lemma after this one of Theorem 1, points (3) and (4).

Lemma 18. *The function h is positive for $n \geq 265$.*

We put $J_n :=]\frac{n}{4}, \frac{n}{2}[$, and remark that if $n \in [x, 264]$, then $\Big]\frac{264}{4}, \frac{x}{2}\Big[\subset \bigcap_{x \leq n \leq 264} J_n$. We search such an x as small as possible, with a prime number q in the interval $]\frac{264}{4}, \frac{x}{2}[$. We have $\frac{264}{4} = 66$, and the smallest prime greater than 66 is $q = 67$. So we take $\frac{x}{2} = 68$. For this choice , and for $n \in [136, 264]$, we have $q \in]66, 68[$. So $\frac{q}{n} \in]\frac{1}{4}, \frac{1}{2}[$, with q prime. But in $[136, 264]$ we must exclude the multiples of $q = 67$, that is only 201. But it is easy to show that $\frac{97}{201}$ is irreducible and in $]\frac{1}{4},\frac{1}{2}[$. So every integer $n \in [136, 264]$ is a minimal period for some K and some d.

Now we shall make the same reasoning from the integer 135 instead of 264. We find that every $n \in [76, 135] \setminus \{111\}$ is a minimal period (with $q = 37$), but also the number 111, because $\frac{53}{111} \in]\frac{1}{4},\frac{1}{2}[$.

We continue, and obtain that every n in the following successive sets are minimal periods:

$$[40, 75] \setminus \{57\},\ [24, 39] \setminus \{33\},\ [16, 23] \setminus \{21\},\ [12, 15] \setminus \{15\}, [8, 11] \setminus \{9\}.$$

But the exceptional numbers 57, 33, 21, 15, 9 are easily minimal periods.

In fine, because $\frac{1}{3}$, $\frac{2}{5}$ and $\frac{3}{7} \in]\frac{1}{4},\frac{1}{2}[$, 3, 5 and 7 are also minimal periods.

We will see the case of the numbers 4 and 6 in an appendix: there are minimal periods for some d and some K.

Remark 19. *From the numerical result $N(\sqrt{2}) \leq 781$, the same method gives $N(\sqrt{2}) \leq 11$ and the fact that 5, 7, 8, 9 are minimal periods for $T_{\sqrt{2}}$. The cases of 3, 4, 6 and 10 remain to study.*

Proof of part (4) of Theorem 1

Of course, from formulas (4), it is clear that if $(x, y) \to (0, 0)$ then $X \to 0$ and then $Y \to 0$: we can extend continuously T_d to the point $(0, 0)$, with value the point $(0, 0)$.

Now, we prove that there is no fixed point (excepted $(0, 0)$). From relations (4), the equality $(X, Y) = (x, y)$ gives

$$\begin{aligned} &\text{(I)}: \quad x(1 + y^4 + dxy) + x(1 + y^4) - dy^3 = 0, \\ &\text{(II)}: \quad y(1 + x^4 + dxy) + y(1 + x^4) - dx^3 = 0. \end{aligned} \tag{54}$$

We make the equation $y \times \text{(I)} - x \times \text{(II)} = 0$, and obtain $(x^4 - y^4)(2xy - d) = 0$. First we suppose $y = \varepsilon x$, with $\varepsilon = \pm 1$, and then $x \neq 0$. With (II) we obtain, by simplifying by εx, the relation $2(1 + x^4) = 0$, which is impossible. Then, we suppose $2xy = d$, and substitute y by $\frac{d}{2x}$ in (II). We obtain $1 + \frac{d^2}{4} = 0$, and this is impossible.

In fine, with the symmetries of the curve $\mathcal{C}_d(K)$, it is easy to see geometrically that if T_d is 2-periodic on the curve, then the point (m, m) of the curve has for image the point $(-m, -m)$. With relations (4) we find (by addition) the relation $m(1 + m^4) = 0$, which is impossible. ∎

Proof of Lemma 18.

We first prove that, by choosing $k = 0.1$ and $A = 1000$, the function

$$g_k(x) = k\frac{x}{\ln x} - 1.38402\frac{\ln x}{\ln(\ln x)} - 1 \tag{55}$$

is increasing on the interval $[A, +\infty[$. It will result that $g_k(x) \geq g_k(A) > 8$ on this interval. We put $u = \ln x$, so that $g_k(x) = \widetilde{g}_k(u) = 0.1\dfrac{e^u}{u} - 1.38402\dfrac{u}{\ln u} - 1$. But we have, for

$u \geq 3\ln 10 > 6.9$,

$$\widetilde{g}_k'(u) = 0.1\frac{e^u(u-1)}{u^2} - 1.38402\frac{\ln u - 1}{\ln^2 u} \geq 0.1\frac{5.9e^u}{u^2} - \frac{1.38402}{\ln u} \geq 11 > 0. \tag{56}$$

Now, we put $\frac{x}{\ln x}$ in factor in the quantity

$$\frac{\frac{x}{2}-1}{\ln(\frac{x}{2}-1)} - \frac{\frac{x}{4}}{\ln\frac{x}{4}}\Big(1 + \frac{3}{2\ln\frac{x}{4}}\Big),$$

and prove by minoration that for $x \geq 1000$ this quantity is greater than $0.1012\frac{x}{\ln x}$.

So the function h is minored by $0.1012\frac{x}{\ln x} - 1.38402\frac{\ln x}{\ln(\ln x)} - 1 \geq g_{0.1}(x) \geq 8 > 0$ on $[1000, +\infty[$.

For $n \in [265, 999]$ we use a computer. ∎

7. Sensitivity to initial conditions: proof of theorem 2

First, we prove the following result.

Lemma 20. *If $K \neq 0$, then the map $\phi_4 \circ \phi_3 \circ \phi_2 \circ \phi_1$ is an homeomorphism of $\mathcal{C}_d(K) \setminus (-m, -m)$ on $\mathcal{A}_d^4(K)$, and when $M \to \omega$ on $\mathcal{A}_d^4(K)$, then $(\phi_4 \circ \phi_3 \circ \phi_2 \circ \phi_1)^{-1}(M) \to (-m, -m)$.*

Proof. If (x, y) are the variables on $\mathcal{C}_d(K)$ and (X, Y) the variables on $\mathcal{A}_d^4(K)$, then it is easy to see that we have

$$\begin{aligned} X &= \frac{\mu}{12} + \frac{\lambda}{4} mK \frac{x+y-2m}{xy-m^2}, \\ Y &= \frac{\lambda}{4}\frac{x-y}{xy-m^2}\Big[mK\frac{x+y-2m}{xy-m^2} - a\Big]. \end{aligned} \tag{57}$$

And with *maple* one see that we have

$$\begin{aligned} x &= -m\frac{P(m)X^2 + Q(m)X + R(m)Y + S(m)}{P(m)X^2 + Q_1(m)X + R_1(m)Y + S_1(m)}, \\ y &= -m\frac{P(m)X^2 + Q_2(m)X + R_2(m)Y + S_2(m)}{P(m)X^2 + Q_3(m)X + R_3(m)Y + S_3(m)} \end{aligned} \tag{58}$$

where $P, Q, \dots S_3$ are polynomials in m whose coefficients depend on d, with

$$P = 1152m^4(m^4 - dm^2 - 1)^4, \tag{59}$$

which is positive when $K \neq 0$.

So we have just to find the limit of (x, y) when $X \to \infty$, with $Y \sim \pm 2X^{3/2}$ (because the equation of $\mathcal{A}_d^4(K)$ is $Y^2 = 4X^3 - g_2X - g_3$). And the result is obvious. ∎

We have also the following lemma

Lemma 21. *When a point M on $\mathcal{C}_d(K)$, with $K \neq 0$, tends to the point (m, m), then $\phi_4 \circ \phi_3 \circ \phi_2 \circ \phi_1(M)$ tends to the summit $E_2 = (X_0, 0)$ of the cubic $\mathcal{A}_d^4(K)$, with*

$$X_0 = \frac{m^8 - 2dm^6 - 6m^4 + 2dm^2 + 1}{24m^2(m^4 - dm^2 - 1)^2}. \tag{60}$$

The proof is simple with an elementary computation from formulas (57), and using the limits of variables $\widetilde{U}$ and W linked by (22).

In fine, we use an argument which is in [7], and which is a modification of a result in [3]:

Lemma 22. *When $K \to K_0$, then $X = \mathcal{P}_K(t.2\omega_2(K)) \to \mathcal{P}_{K_0}(t.2\omega_2(K_0))$ uniformly for $t \in [\varepsilon, 1-\varepsilon]$, where $0 < \varepsilon < \frac{1}{2}$, and the same result is true for Y which is the derivative $\mathcal{P}'_K$ on the same point.*

(1) The sensitivity for $K > 0$: proof of Theorem 2(1)

Let be $0 < K_1 < K_2$. We represent in the following figures two domains: $\Gamma(K_1, K_2) = [K_1, K_2] \times \mathbb{T}$, which is the set of parameters (K, α) of the second domain $C(K_1, K_2)$. There is a one-to-one continuous map Φ from $\Gamma(K_1, K_2)$ onto $C(K_1, K_2)$ (one look at $\mathcal{P}_K$ as it was defined on the diagonal circle of $\mathbb{T}^2$ by is double periodicity: see part 5.1). The continuity of Φ results of Lemmas 20 and 22.

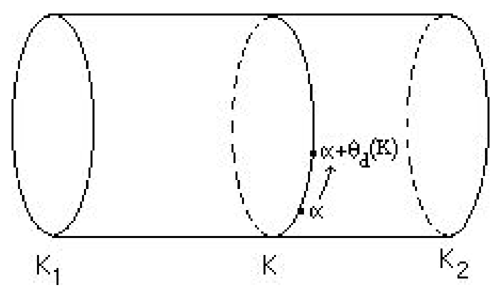

Figure 11: The set $\Gamma(K_1, K_2)$

But Φ is an homeomorphism of the two domains, because they are compact sets.

Now it is known that the dynamical system $(K, \alpha) \mapsto (K, \alpha + \theta_d(K))$ has uniform sensitivity to initial conditions with a constant δ if $\delta \in]0, \frac{1}{2}[$ (see [7], Proposition 25). So it is clear that Φ is uniformly continuous, and so the dynamical system $(C(K_1, K_2), T_d)$ has η-sensitivity to initial conditions for some constant $\eta > 0$, function of K_1 and K_2: it is Theorem 2(1).

(2) The sensitivity for $K < 0$: proof of Theorem 2(2)

In this case, the problem is more difficult, because the correspondant domain $C(K_1, K_2)$ is not compact, and if we make the domain compact in projective coordinates with the infinite points H and V, the correspondant map Φ were not well defined, because it is not difficult to show that then a point M in $\mathcal{C}_d(K)$ tends to H or to V, then $\phi_4 \circ \phi_3 \circ \phi_2 \circ \phi_1(M)$ tends to the infinite point ω of the cubic curve $\mathcal{A}_d^4(K)$. So we prove only the pointwise sensitivity to initial conditions, as defined in Theorem 2(2).

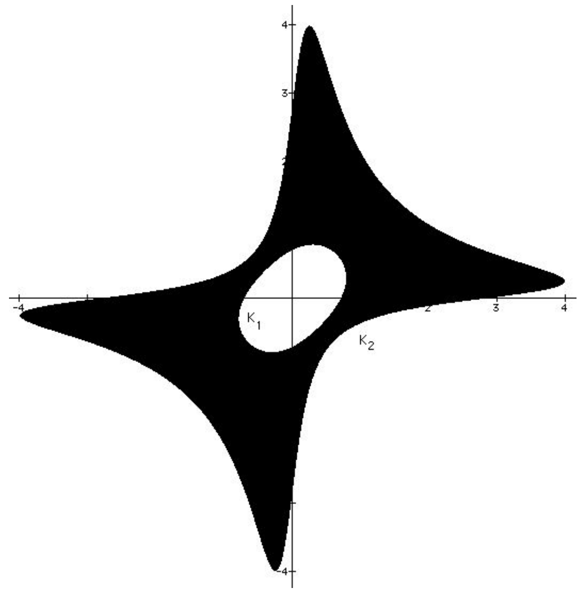

Figure 12: The set $C(K_1, K_2)$ for $K > 0$

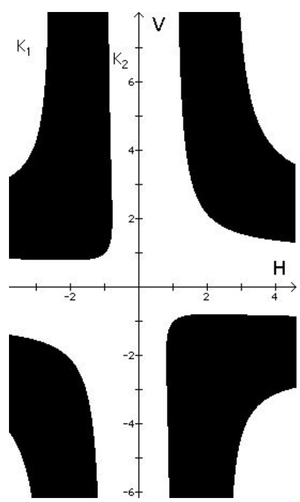

Figure 13: The set $C(K_1, K_2)$ for $K < 0$

For proving the pointwise sensitivity to initial conditions, the method is the same as in [3], with the same exception as in [8], because the Lemma 22 excludes the point ω of the cubic curve $\mathcal{A}_d^4(K)$. From Lemma 20, we can see that all the points of $C(K_1, K_2)\backslash\{(-m,-m)|K \in [K_1, K_2]\}$ have pointwise sensitivity to initial conditions. But it is the case of the points (m,m) for $K \in [K_1, K_2]$, and by symmetry the points $(-m,-m)$ have also pointwise sensitivity to initial conditions: it is the point (2) of Theorem 2.

8. Appendix 1: 4 and 6 are minimal periods

(a) We will find a necessary and sufficient condition for 4 being a minimal period.

Let $\mathcal{V}$ and $\mathcal{H}$ be the involutions defined by vertical [resp. horizontal] alignment of two points on the same curve $\mathcal{C}_d(K)$, so that we have $T_d = \mathcal{V} \circ \mathcal{H}$ and $T_d^{-1} = \mathcal{H} \circ \mathcal{V}$ (excepted for the points M_0, M_1, N_0, N_1).

From Proposition 11 or Corollary 12, we see that T_d is 4-periodic on $\mathcal{C}_d(K)$ if and only if (with an exception that we will see later) $A_0 := (m, m)$ is 4-periodic for T_d (m is given by relation (7)), or the same condition for $B_0 := (-m, -m)$. The condition $T_d^4(A_0) = A_0$ can be written

$$\mathcal{V} \circ \mathcal{H} \circ \mathcal{V} \circ \mathcal{H}(A_0) = \mathcal{H} \circ \mathcal{V} \circ \mathcal{H} \circ \mathcal{V}(A_0). \tag{61}$$

But these two points are symmetric with respect to the diagonal (because the curve is symmetric). So they can be equal if and only if they are both equal to A_0 or to B_0. If they are equal to A_0, we would have $T_d^2(A_0) = A_0$, and T_d would be 2-periodic, which is false. So we must have the equality

$$T_d(A_0) = T_d^{-1}(B_0). \tag{62}$$

If we put $T_d(A_0) = (X, Y)$ and $T_d^{-1}(B_0) = (X_1, Y_1)$, the condition (61) is $X - X_1 = 0$ and $Y - Y_1 = 0$, and the easy (with *Maple*) computation gives

$$d = \frac{\sqrt{2}(K^2 + 1)}{\sqrt{K^2 - 1}}. \tag{63}$$

As an example, for $d = 5$ and $K = 3$, T_5 is 4-periodic on the curve $\mathcal{C}_5(3)$.

An exception happens when the two denominators (which are the same ones) of $X - X_1$ and $Y - Y_1$ are 0, that is when

$$m^{12} - 5dm^{10} + 3(d^2 + 1)m^8 - (d^3 - 2d)m^6 + 3(d^2 + 1)m^4 + 3dm^2 + 1 = 0. \tag{64}$$

But it is easy to see that relations (64) and (62) are incompatible. So this exception is not a curve on which T_d is 4-periodic.

It is also interesting to note that the condition that $\mathcal{H}(A_0) = N_0$, so that $Y = \infty$, *id est* $T_d(A_0) = V$, is exactly the relation (61), and by symmetry it is also the relation for having $T_d^{-1}(B_0) = H$. So we find again the previous exception.

(b) Now we will find a condition for 6 being a minimal period.

With the same reasoning we find that the condition is

$$(\mathcal{V} \circ \mathcal{H})^{3\circ}(A_0) = (\mathcal{H} \circ \mathcal{V})^{3\circ}(A_0), \tag{65}$$

so that, by symmetry, these to points must be equal to A_0 or to B_0. If it would A_0, T_d would be 3-periodic, and 6 would not be minimal. So the good condition is

$$T_d^2(A_0) = T_d^{-1}(B_0). \tag{66}$$

It is now easy, with the use of *Maple*, and by computing $T_d^2(A_0) = (X_2, Y_2)$, to find the condition for 6 being a minimal period (at least, it is a sufficient condition)

$$K^2d^4 - (K^4-1)(K^2+1)d^2 + 3(K^2+1)^4 = 0, \tag{67}$$

which gives easily d as a function of K

$$d = \frac{(K^2+1)\sqrt{K^2-1 \pm \sqrt{K^4-14K^2+1}}}{K\sqrt{2}}. \tag{68}$$

As an example, for $d = 4\sqrt{3+2\sqrt{3}}$ and $K = 2+\sqrt{3}$, we obtain a curve on which 6 is a minimal period.

9. Appendix 2: on the forms of the curves $\mathcal{C}_d(K)$ (proof of lemma 5)

The plan of proof of Lemma 5 is simple :

(1) The curves are starlike with respect the point $(0,0)$: we put $x = \rho u$ and $y = \rho v$, with a unit vector (u, v). If $uv \neq 0$ we have with equation of $\mathcal{C}_d(K)$ a quadratic equation in ρ^2 with a unique positive solution, so $\rho = \pm\alpha$: the curve is starlike. If $uv = 0$, $\rho = \pm\frac{1}{\sqrt{K}}$ if $K > 0$, and if $K \leq 0$ and $uv = 0$, there is no solution: the curves $\mathcal{C}_d(K)$ does not cut the axes.

(2) There is no double point in finite distance (see Lemma 6).

(3) If $K > 0$, one see the inclusion $\mathcal{C}_d(K) \subset B\Big((0,0), \sqrt{\frac{1+d^2/4}{K}}\Big)$: one has $x^2+y^2 = \dfrac{1+dxy-x^2y^2}{K} \leq \dfrac{1+d^2/4}{K}$. So, from the three previous points, one see that the curves are homeomorphic to circles for $K > 0$.

(4) If $K < 0$, it is easy to find the asymptotes of the curve $\mathcal{C}_d(K)$. ∎

10. Appendix 3: Previous particular results of the authors about QRT-maps

The invention of QRT-maps was originally in [18], for physical reasons, and the essential exemple was on [22], but these papers were not easy to find. So the first papers ([3], [11] and [4]) were slightly different. They studied the now called "symmetric special QRT-maps", where the family of biquadratic curves $\mathcal{C}(K)$ with equations $Q_1(x,y) - KQ_2(x,y) = 0$ were symmetric with respect the diagonal, and with $Q_2(x,y) = xy$; and the QRT-map was defined by the following way: if $\mathcal{C}(K)$ is the curve passing through the point M, we cut it in M_1 by the horizontal line which contains M, and then $T(M)$ is the symmetric of M_1 with respect the diagonal (it is on $\mathcal{C}(K)$).

These cases correspond to the study of difference equations of the form

$$u_{n+2}u_n = \frac{au_{n+1}^2 + bu_{n+1} + c}{du_{n+1}^2 + eu_{n+1} + f}. \tag{69}$$

In [8] we studied the case of difference equations $u_{n+2} + u_n = \psi(u_{n+1})$ which is associated with "symmetric QRT-maps" defined by families of the forms $Q_1(x,y) + K = 0$.

In each of these papers, we determine the possible periods of periodic orbits, prove the density of periodic points and not periodic points, and a form of sensitivity to initial conditions.

In the other works, we studied classical QRT-maps, associated with a couple of difference equations of the form $u_{n+1}u_n = f(v_n), v_{n+1}v_n = g(u_{n+1})$. We give some examples :

[7] $u_{n+1}u_n = c + \dfrac{d}{v_n},\ v_{n+1}v_n = c + \dfrac{d}{v_{n+1}}$ with curves $xy(x+y) + xy + d - Kxy = 0$.

[5] $u_{n+1}u_n = v_n^2 - bv_n + c,\ v_{n+1}v_n = u_{n+1}^2 - au_{n+1} + c$ with curves $x^2+y^2-ax-by+c-Kxy = 0$.

[6] $u_{n+1}u_n = av_n + b,\ v_{n+1}v_n = u_{n+1} + \dfrac{b}{u_{n+1}}$ with curves $xy^2 + x^2 + ay + b - Kxy = 0$.

In [2] we studied the 2-periodic Lyness' equation $u_{n+2}u_n = u_{n+1} + a_n$, with a_n 2-periodic.

In [12] we study the particular case of the dynamical systems $(x, y) \mapsto T_d(x, y) = (X, Y)$ given by

$$(X, Y) = \Big(\frac{1}{x}\frac{dy^2 - 20y + 16}{y^2 - 5y + d}, \frac{1}{y}\frac{dX^2 - 20X + 16}{X^2 - 5X + d}\Big),$$

with curves $\mathcal{C}_d(K)$ equations of them are

$$x^2y^2 - 5xy(x+y) + d(x^2+y^2) - 20(x+y) + 16 - Kxy = 0.$$

In the present paper we begin study of non-special QRT-dynamical systems, associated with QRT-families of curves with equations $Q_1(x, y) - KQ_2(x, y) = 0$ with $Q_2(x, y)$ not of the form xy.

In fine, in [9] and [10] we present examples of QRT-families of degree four, but such that each of the curves of the family has genus zero, and studied the correspondent dynamical systems.

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