Degree of a polynomial ideal and Bézout inequalities

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Abstract

A complete theory of the degree of a polynomial ideal is presented, with a systematic use of the rational form of the Hilbert function in place of the (more commonly used) Hilbert polynomial.

This is used for a simple algebraic proof of classical Bézout theorem, and for proving a ”strong Bézout inequality”, which has as corollaries all previously known Bézout inequalities, and is much sharper than all of them in the case of a non-equidimensional ideal.

Key words: Degree of an algebraic variety, degree of a polynomial ideal, Bézout theorem, primary decomposition.

1 Introduction

Bézout's theorem states: *if n polynomials in n variables have a finite number of common zeros, including those at infinity, then the number of these zeros, counted with their multiplicities, is the product of the degrees of the polynomials.* Despite its apparently simple statement, this theorem needed almost a century for being completely proved. The main difficulty was to give an accurate definition of multiplicities, and this requires some machinery of commutative algebra. Most proofs of this theorem proceed by recurrence on the number of polynomials, and use the concept of degree of a polynomial ideal. These proofs obtain the theorem as a corollary of: *if a homogeneous polynomial f of degree d is not a zero divisor modulo a homogeneous ideal I of degree D, then the degree of the ideal $I + \langle f \rangle$* is dD. If some hypotheses are relaxed, such as counting multiplicities or working with homogeneous polynomials, one gets only inequalities, commonly called *Bézout inequalities*. For example, if n polynomials in n variables have a finite number of common zeros, then the number of these zeros, counted with their multiplicities is

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at most the product of the degrees of the polynomials. Surprisingly, this Bézout inequality is not a corollary of Bézout's theorem, and seems to have not been proved before 1983 [7].

All these results require the definition of the degree of a polynomial ideal. Several definitions have been given, either in terms of algebraic geometry or in terms of commutative algebra [2, 3, 4, 6]. Most deal only with homogeneous ideals. One of the objectives of this article is to give a general definition in terms of basic commutative algebra and to prove that all other definitions are special cases. In fact, we give several definitions that are all equal for equidimensional ideals, but not in general.

Basically, the degree of an algebraic variety is the number of points of its intersection with a generic linear variety of a convenient dimension. This definition is not algebraic but can easily be translated into an algebraic one. However the resulting definition is not intrinsic, involving auxiliary generic polynomials, which make proofs unnecessarily complicated. Therefore, we use the definition through Hilbert series, which provides a simpler presentation of the theory. Many authors use instead the Hilbert polynomial, and the fact that the coefficients of the Hilbert series are, for large degrees, the values of a polynomial (the Hilbert polynomial). The proof of this polynomial property relies generally on Hilbert's syzygy theorem. Here we prefer using the fact that the Hilbert series is a rational function, and we define the degree directly from its rational form. In our opinion, this gives simpler proofs, and, in particular, avoids using the syzygy theorem. This approach has other advantages. Firstly, the Hilbert series carries more information than the Hilbert polynomial. In particular, it includes the degree from which the Hilbert function starts to be polynomial. Also, all known algorithms for computing the degree of an explicitly given ideal use Gröbner bases; the Hilbert series, which can be computed from the leading monomials of a Gröbner basis, is nowadays the simplest and the more efficient way for computing the degree. In the case of homogeneous ideals, our approach is not new, although is seems unknown by many specialists of algebraic geometry, and we do not know any published presentation of it. We have learnt it from an early version of [1], by Carlo Traverso alone. In Traverso's article, the definition of the degree is also extended to the case of non-homogeneous ideals.

In the case of non-equidimensional ideals, the classical definition of the degree does not depend on the components of lower dimension and the embedded components. For taking all isolated components into account, Masser and Wüstholz [7] have introduced another notion of degree, which is the sum of the degrees of the isolated components of any dimension. We call this degree the *total degree*. Masser and Wüstholz have proved that, for an ideal generated by polynomials of degrees $d_2 \geq \cdots \geq d_k \geq d_1$, the degree of the intersection of the isolated components of height h is at most $d_1 d_2 \ldots d_h$. So, if the height of the ideal is h, the total degree is at most $h d_1 d_2 \ldots d_h$.

The main new result of this article is the following.

Theorem 1. *Let I be a polynomial ideal generated by polynomials of degrees $d_2 \geq \cdots \geq d_k \geq d_1$. Define the divided degree of an isolated primary component of height h of I as the quotient of its usual degree by $d_1 d_2 \ldots d_h$. Then, the sum of the divided degrees of the isolated components of I is at most one.*

This inequality implies immediately Masser-Wüstholz inequality, and all other previoulsy known Bézout inequalities. It is much stronger for ideals that are not equidimensional. Moreover, although Masser-Wüstholz proof involves analytic geometry, our proof is purely algebraic.

The article is structured as follows. Section 2 is a short section where the basic notation that is used through the article is defined. In section 3, the Hilbert series is defined, and its main properties are given. These properties are used in section 4 for defining the degree of a homogeneous ideal and proving its main properties. There are several ways to extend this definition to non homogeneous ideals, that are presented and shown to be equivalent in section 5. In section 6 these definitions of the degree of an ideal are shown to be a generalization of other earlier definitions, that require often some further constraints, such as equidimensionality of the ideal or a ground field that is algebraically closed. In section 7, the classical Bézout's theorem is proved proved by using our definition of the degree. Section 9 is devoted to the statement of the main result of this paper, the "strong Bézout inequality", and its comparison with previous Bézout inequalities. Its proof is the object of section 10.

2 Notation and terminology

In this article, we work with the polynomial ring $R_n = K[X_1, \ldots, X_n]$ in n indeterminates over a field K. This ring is a graded by the degree, and this gradation extends to homogeneous ideals and quotients by such ideals. In all these graded modules, the homogeneous part of degree d is a finite-dimensional K-vector space. For all modules that are considered, all homogeneous parts of negative degree are zero.

We call *algebra* the quotient of R_n by a homogeneous ideal I. If A is such an algebra, the homogeneous elements of positive degree generate the unique homogeneous maximal ideal, which will be denoted A_+.

A homogeneous element of an algebra is *sufficiently generic* or simply *generic* if it does not belong to the union of the associated primes (of zero) that are different from A_+. A homogeneous polynomial of R_n is *generic* if it does not belong to the union of the associated primes different from $(R_n)_+$ of the ideals that are under consideration. Generic elements exist always if the field K is infinite, since, in this case a vector space cannot be the union of a finite number of proper vector subspaces. When generic elements are used in this article, this is always for proving a property that is invariant under extension of the ground field K. So, some of our proofs are incomplete in the case of finite fields, but can easily be completed by saying that if K is finite, the result follows by extension/restriction of the ground field.

3 Gradation and Hilbert series

Definition 1. *If*

$$A = \bigoplus_{d \geq 0} A_d$$

is a graded object, where A_d is a finite-dimensional K-vector space for every d, then the Hilbert series of A is the formal power series

$$\mathrm{HS}_A(t) = \sum_{d=0}^{\infty} t^d \dim_K(A_d).$$

The main property of Hilbert series is to be additive under exact sequences,

Proposition 1. *If*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is an exact sequence of homomorphisms of graded modules, then

$$\mathrm{HS}_B(t) = \mathrm{HS}_A(t) + \mathrm{HS}_C(t)$$

Proof. Results immediately from the similar formula for the dimension of vector spaces. □

Proposition 2. *If A is a graded algebra, and f a homogeneous element of degree d that is not a zero divisor in A, then*

$$\mathrm{HS}_{A/\langle f \rangle}(t) = (1 - t^d)\, \mathrm{HS}_A(t).$$

Proof. Results immediately from the exact sequence.

$$0 \longrightarrow A^{[d]} \longrightarrow A \longrightarrow A/\langle f \rangle \longrightarrow 0,$$

where $A^{[d]}$ is A with its gradation shifted by d, (this shift multiplies the Hilbert series by t^d). □

Corollary 3. *The Hilbert series of $R_n = K[x_1, \ldots, x_n]$ is*

$$\mathrm{HS}_{R_n}(t) = \frac{1}{(1-t)^n}.$$

Proof. Proof by recurrence on n, using the preceding proposition with $f = x_n$. The recurrence starts with the trivial result $\mathrm{HS}_{R_0}(t) = \mathrm{HS}_K(t) = 1$. $\square$

Corollary 4. *If $f_1, \ldots, f_k$ is a regular sequence of homogeneous elements in R_n, of respective degrees $d_1, \ldots, d_k$ (this means that each f_i is not a zero divisor modulo the preceding $f_j s$), then*

$$\mathrm{HS}_{R_n/\langle f_1,\ldots,f_k\rangle}(t) = \frac{\prod_{i=1}^{k}(1-t^{d_i})}{(1-t)^k}.$$

Proof. Results from proposition 2 by recurrence on k, starting from the case $k = 0$, which is the preceding corollary. $\square$

For proving further basic properties of Hilbert series, we have to deal with the fact that, in an algebra A, it is possible that all elements of A_+ are zero divisors. This occurs when A_+ is an associate prime of the zero ideal. For solving this problem, we have to consider the annihilator $\mathrm{ann}_A(f)$ of a sufficiently generic element f of A, and to prove that its Hilbert series is a polynomial. This allows proving theorem 2, below, which is a generalization of proposition 2.

Lemma 5. *If f is a generic element in a graded algebra A, then the Hilbert series of the annihilator $\mathrm{ann}_A(f)$ is a polynomial. This annihilator and its Hilbert series are zero if and only if the maximal homogeneous ideal A_+ is not associated to 0 in A.*

Proof. The zero divisors are the elements of the union of the associated primes of zero. The above definition of a generic element implies thus the second assertion, and we may suppose that A_+ is associated to 0 in A.

By definition of a primary ideal, the equality $f\ \mathrm{ann}_A(f) = 0$, implies that $\mathrm{ann}_A(f)$ is contained in all primary component of 0, except the one corresponding to A_+. If $\mathfrak{q}$ is this primary component, one has $\mathfrak{q}\ \mathrm{ann}_A(f) = 0$, as this product in contained is all primary components of zero. One has $A_+^m \subset \mathfrak{q}$ for some integer m. Let k be an upper bound of the degrees of a generating set of $\mathrm{ann}_A(f)$. For every $j \geq m+k$, the homogeneous part of degree j of $\mathrm{ann}_A(f)$ is thus contained in $\mathfrak{q}\ \mathrm{ann}_A(f) = 0$, and its dimension as a vector space is 0. Therefore the Hilbert series is a finite sum, and thus a polynomial. $\square$

Theorem 2. *Let A be a graded algebra (typically the quotient of R_n by a homogeneous ideal I), and f be a homogeneous element in A of degree d that is sufficiently* *generic (see section 2). Then*

$$\mathrm{HS}_{A/\langle f\rangle}(t) - (1-t^d)\,\mathrm{HS}_A(t)$$

is a polynomial with positive integer coefficients.

If A_+ is not associated to 0 in A, this polynomial is 0 (this is proposition 2).

Let us consider the exact sequence

$$0 \longrightarrow \mathrm{ann}_A(f)^{[d]} \longrightarrow A^{[d]} \xrightarrow{f} A \longrightarrow A/fA \longrightarrow 0$$

where exponents $[d]$ denote a shift of d of the gradation, and $\xrightarrow{f}$ represents the product by f. From the additive property of the Hilbert series, we get

$$t^d\,\mathrm{HS}_{\mathrm{ann}_A(f)} - t^d\,\mathrm{HS}_A(t) + \mathrm{HS}_A(t) - \mathrm{HS}_{A/fA}(t) = 0.$$

This proves the desired property, by lemma 5. □

Theorem 3. *If I is an ideal of R_n, and $A = R_n/I$, the Hilbert series of A has the form*

$$\mathrm{HS}_A(t) = \frac{N(t)}{(1-t)^d},$$

where d is the Krull dimension of A, and N is a polynomial such that $N(1) \neq 0$ (thus the fraction is irreducible).

Proof. The proof proceeds by recurrence on d. If the dimension of A is zero, then A_+ is the only homogeneous prime ideal of A, and $A_+^k = 0$ for some integer k. This implies that the Hilbert series has at most k nonzero terms, and is a polynomial.

If $d > 0$, let f be a sufficiently generic element of degree one in A. Theorem 2 shows that

$$\mathrm{HS}_A(t) = \frac{\mathrm{HS}_{A/\langle f\rangle}(t) + P(t)}{1-t},$$

where $P(t)$ is a polynomial with positive coefficients.

Krull's principal ideal theorem implies that the dimension of $A/\langle f\rangle$ is $d-1$, and thus, by recurrence hypothesis that

$$\mathrm{HS}_{A/\langle f\rangle}(t) = \frac{N(t)}{(1-t)^{d-1}},$$

and thus

$$\mathrm{HS}_A(t) = \frac{N(t) + (1-t)^{d-1}P(t)}{(1-t)^d}.$$

If $d = 1$, the numerator is the sum of two polynomials with nonnegative coefficients, and thus has not 1 as a root. If $d > 1$, the value at 1 of the numerator is $N(1)$, and this proves the desired result. $\square$

4 Degree of homogeneous ideals

Classically, the degree of an affine or projective algebraic variety of dimension d is the number of its intersection points with a generic linear variety of a dimension $n - d$, where n is the dimension of the ambient space.

This can easily be translated in algebraic terms for defining the the degree of an ideal in R_n. However, the resulting definition of the degree of an ideal has several practical issues. Firstly, for dealing with non-radical ideals, it must involve the multiplicity of a local Artinian ring. Also, this definition is not intrinsic, as depending on the (arbitrary) choice of generic linear polynomials. For this reason, it seems more convenient to choose the following definition, which is direct and purely algebraic, and then to prove that it is equivalent the classical geometric one.

Definition 2. *Let I be a homogeneous polynomial in $R_n = K[X_1, \ldots, X_n]$, such the algebra $A = R_n/I$ has the Krull dimension d. The* degree *of I, denoted* $\deg(I)$*, is $P(1)$, where P is the numerator of the Hilbert series of A:*

$$\mathrm{HS}_A(t) = \frac{P(t)}{(1-t)^d}$$

The dimension *of I, denoted* $\dim(I)$ *is d.*

Before proving that this definition is equivalent with other classical definitions, we list some useful properties that are simple consequences of the properties of the Hilbert series given in the preceding section.

Proposition 6. *The degree of a principal ideal equals the degree of its generator. More precisely, if $f \in R_n$ is a homogeneous polynomial, then*

$$\deg(\langle f \rangle) = \deg(f).$$

Proof. As there are no zero divisors in R_n, proposition 2 and corollary 3 implies

$$\mathrm{HS}_{R_n/\langle f \rangle}(t) = (1 - t^d)\,\mathrm{HS}_{R_n}(t) = \frac{1 - t^d}{(1-t)^n} = \frac{1 + t + \cdots + t^{d-1}}{(1-t)^{n-1}}.$$

The results follows, since the numerator has d terms with coefficients 1. $\square$

The following proposition is a basic tool for the proof of Bézout's theorem, and may be viewed as a generalization of it.

Proposition 7. *If the homogeneous polynomial f is not a zero divisor modulo a homogeneous ideal I, then*

$$\deg(I + \langle f \rangle) = \deg(I) \cdot \deg(f).$$

Proof. As $R_n/(I + \langle f \rangle) = (R_n/I)/\langle f \rangle$, theorem 2 implies

$$\mathrm{HS}_{R_n/(I+\langle f \rangle)}(t) = (1 - t^d) HS_{R_n/I}(t),$$

where d is the degree of f. The hypothesis implies that the dimension of I is positive. So, $1 - t$ can be factored out from the numerator and the denominator of this Hilbert series, and the numerator becomes the product by $1 + \cdots + t^{d-1}$ of the numerator of $HS_{R_n/I}(t)$. The result follows by substituting 1 for t in this numerator. □

Lemma 8. *If $I \subset J$ are ideals of R_n, either* $\dim(J) < \dim(I)$*, or* $\dim(J) = \dim(I)$ *and* $\deg(I) \geq \deg(J)$.

Proof. Because of the ideal inclusion, one has $\delta = \dim(I) \geq \dim(J)$. Each coefficient of the power series $\mathrm{HS}_{R_n/J}(t)$ is nonnegative and not greater than the corresponding coefficient of $\mathrm{HS}_{R_n/J}(t)$, since the algebra R_n/J is a quotient of R_n/J. It follows that for every $0 < \alpha < 1$, the Hilbert series are convergent, and $\mathrm{HS}_{R_n/J}(\alpha) \leqslant \mathrm{HS}_{R_n/I}(\alpha)$. If the dimensions are equal, multiplying by $(1 - \alpha)^\delta$, and taking the limit for $\alpha = 1$ gives the same equality for the values at 1 of the numerators. □

Proposition 9 (Subadditivity under intersection)**.** *Let I and J be two ideals in $R_n = K[X_1, \ldots, X_n]$ such that* $\dim(I) \geq \dim(J)$. *Then* $\dim(I \cap J) = \dim(I)$, *and*

- $\deg(I \cap J) = \deg(I)$ *if* $\dim(I) > \dim(J)$,
- $\deg(I \cap J) \leq \deg(I) + \deg(J)$ *if* $\dim(I) = \dim(J)$
- $\deg(I \cap J) = \deg(I) + \deg(J)$, *if* $\dim(I) = \dim(J)$, *and I and J have no common associated primes of dimension* $\dim(I)$.

Proof. From the exact sequence

$$0 \longrightarrow R_n/(I \cap J) \longrightarrow R_n/I \oplus R/J \longrightarrow R_n/(I + J) \longrightarrow 0$$

we get

$$\mathrm{HS}_{R_n/(I\cap J)}(t) = \mathrm{HS}_{R_n/I}(t) + \mathrm{HS}_{R/J}(t) - \mathrm{HS}_{R/(I+J)}(t).$$

The right-hand side has thus the form

$$\frac{P_I(t)}{(1-t)^{\dim(I)}} + \frac{P_J(t)}{(1-t)^{\dim(J)}} - \frac{P_{I+J}(t)}{(1-t)^{\dim(I+J)}} =$$
$$\frac{P_I(t) + (1-t)^{\dim(I)-\dim(J)} P_J(t) - (1-t)^{\dim(I)-\dim(I+J)} P_{I+J}(t)}{(1-t)^{\dim(I)}}.$$

By hypothesis and lemma 8, one has $\dim(I) \geq \dim(J) \geq \dim(I+J)$. So the numerator of the last fraction is a polynomial. Its value at 1 cannot be zero, since this would implies $\dim(I \cap J) < \dim(I)$, which is excluded by lemma 8. This proves the assertion on dimensions. The degree of $I \cap J$ is the value at 1 of the numerator, which is

- $P_I(1) = \deg(I)$ if $\dim(I) > \dim(J)$,
- $P_I(1) + P_J(1) = \deg(I) + \deg(J)$ if $\dim(I) = \dim(J) > \dim(I+J)$, or, equivalently, if $\dim(I) = \dim(J)$, and there is no prime ideal of dimension $\dim(I)$ that contain both I and J.
- $P_I(1) + P_J(1) - P_{I+J}(1) = \deg(I) + \deg(J) - \deg(I+J) < \deg(I) + \deg(J)$ if $\dim(I) = \dim(J) = \dim(I+J)$.

If $\dim(I) = \dim(J) = \delta$, one has $\dim(I+J) = \delta$ if and only if I has a minimal prime of dimension δ. As such a prime contains both I and J, it must be a minimal prime of both I and J. Conversely, if I and J have a common minimal prime of dimension δ, this prime contains $I+J$, and is thus a minimal prime of $I+J$. This shows $\dim(I+J) = \delta$ and completes the proof. □

Corollary 10. *The degree of an ideal of dimension δ is the sum of the degrees of its primary components of dimension δ.*

5 Non-homogeneous ideals

For extending the definition of the degree to non-homogeneous ideals, there are two natural ways that give the same value to the degree. Both consist to associate a homogeneous ideal to the given ideal, and to consider the degree of this homogeneous ideal. The degree of a homogeneous polynomial does not depends whether it is considered as homogeneous or not.

5.1 Homogenization

The *homogenization* of an ideal is the algebraic counterpart of the projective completion of an algebraic variety. It consists to add a new indeterminate x_{n+1} to the polynomial ring R_n, and to use it for homogenizing all polynomials of the ideal.

More precisely, if $f(x_1, \dots, x_n) \in R_n$ is a polynomial of degree d, it is homogenized into the homogeneous polynomial

$$^{H}f(x_1, \dots, x_{n+1}) = x_{n+1}^d \, f\left(\frac{x_1}{x_{n+1}}, \dots, \frac{x_n}{x_{n+1}}\right).$$

Given an ideal I of R_n, the corresponding homogenized ideal ^{H}I is the ideal generated by all ^{H}f for $f \in I$ (it is not sufficient to homogenize only the elements of a generating set of I).

The following lemma is standard, and its easy proof is left to the reader.

Lemma 11. *Every homogeneous polynomial g in R_{n+1} can be written $g = x_{n+1}^k \, {}^{H}f$, with $f \in R_n$, and this factorization is unique. The polynomial f is obtained by substituting 1 for x_{n+1} in g.*

Corollary 12. *A homogeneous ideal I in R_{n+1} can be obtained by homogenization from an ideal in R_n if and only if x_{n+1} is not a zero divisor modulo I.*

5.2 Homogeneous part of highest degree

Another homogenous ideal can be associated to an ideal I of R_n. This is the ideal ^{h}I generated by the homogeneous parts of highest degree of the elements of I. It can be obtained by substituting 0 for x_{n+1} in the elements of ^{H}I. So, corollary 12 and proposition 2 imply

$$\mathrm{HS}_{R_n/{}^{h}I}(t) = \mathrm{HS}_{R_{n+1}/({}^{h}I + \langle x_{n+1}\rangle)}(t) = (1-t)\,\mathrm{HS}_{R_{n+1}/{}^{H}I}(t)$$

.

It results that the degree of the ideal I is defined as

$$\deg(I) = \deg({}^{H}I) = \deg({}^{h}I).$$

It is sometimes useful to interpret ^{h}I and $R_{n+1}/{}^{h}I$ in term of filtrations and gradations.

A *filtration* of a K-vector space V is a sequence $F_0(V) \subset F_1(V) \subset \cdots \subset V$. In the case of a K-algebra A and of a module M over a filtered algebra, the filtration must satisfy $F_i(A) \cdot F_j(A) \subset F_{i+j}(A)$ and $F_i(A) \cdot F_j(M) \subset F_{i+j}(M)$. The polynomial ring R_n is naturally filtered by the degree, $F_d(R_n)$ being the vector space of the polynomials of degree at most d. An ideal I of R_n is filtered by the induced filtration $F_d(I) = I \cap F_d(R_n)$, and the quotient R_n/I is filtered by the quotients $F_d(R_n/I) = F_d(R_n)/F_d(I)$.

The graded vector space associated to a filtered vector space is the direct sum

$$G(V) = \bigoplus_{i=0}^{\infty} G_i(V),$$

where

$$G_i(V) = F_i(V)/F_{i-1}(V),$$

and $F_{-1}(V) = (0)$.

In our case, $G(R_n)$ is a ring which is isomorphic to R_n. In fact, the equivalence class modulo $F_{d-1}(R_n)$ of an element $f \in F_d(R_n)$ consists of all polynomials of degree d that have the same homogeneous part of degree d as f. This class contains a unique homogeneous polynomial of degree d, and this defines an isomorphism from $G(R_n)$ to the vector space of homogeneous polynomials of degree d. the direct sum of these isomorphisms is thus an isomorphism of graded algebras from $G(R_n)$ to R_n.

If I is an ideal, $G(I)$ is a homogeneous ideal of $G(R_n)$, and the quotient $G(R_n/I) = G(R_n)/G(I)$ is a graded algebra. The image of $G(I)$ under the above isomorphism is ${}^hI$.

The exact sequence

$$0 \longrightarrow F_{d-1}(R_n/I) \longrightarrow F_d(R_n/I) \longrightarrow G_d(R_n/I) \longrightarrow 0$$

implies, by a simple recurrence, the following lemma.

Lemma 13. *If I is an ideal of R_n, one has*

$$\dim_K(F_d(R_n/I)) = \sum_{i=0}^{d} \dim_K(G_i(R_n/I))$$

6 Equivalence with other definitions

The other definitions of the degree of a polynomial that have been given in the literature split into geometric ones and algebraic ones. Except for the *total degree that is studied in section 8*, they are equivalent to the definition given in this article, or to the restriction of this definition to some ideals. For example, the purely geometric definition of the degree of an (irreducible) algebraic variety, is equivalent to our definition restricted to prime ideals.

Because all definitions either apply only to homogeneous ideals, or define the same degree for an ideal and its homogenization (geometrically, an affine algebraic set and its projective completion), we prove the equivalence between definitions only in the homogeneous case.

Let us recall that the dimension of a projective algebraic set is one less that the dimension of the ideal that define it.

The classical geometric definition of the degree of a projective algebraic set of dimension $d - 1$ in the projective space of dimension $n - 1$ is the sum of the multiplicities of the intersection of the algebraic set with $d - 1$ generic hyperplane.

We have to prove that this degree is the degree equals the degree of the ideal that defines the algebraic set. The proof will proceed in several steps.

The first step reduces the proof to the case of an ideal of dimension one (that is, a projective algebraic set of dimension zero). This is an immediate corollary of theorem 2, which shows that adding to an ideal a generic linear homogeneous ideal, one does not change the degree. So, if the ideal has dimension d, by adding to it $d-1$ generic linear polynomials, one gets an ideal of dimension one, with the same degree, which defines an algebraic set of dimension zero.

We can extend the ground field to an algebraically closed extension; this does not change the Hilbert series of the ideal nor the algebraic set, as the points of an algebraic set are supposed to be defined over an algebraically closed field. So, the points of the algebraic set are in one to one correspondence with the homogeneous prime ideals of R_n/I that are different from $(R_n)_+/I$. As the algebraic set has dimension zero, all these ideals are maximal (among those that are different from $(R_n)_+/I$), and the multiplicity of a point is the length of the local ring at the corresponding prime.

However, using localization at primes would complicate the proof, and we prefer another approach. So, let I be our homogeneous ideal of dimension one in R_n. We first do a linear change of variables in order that x_n will be sufficiently generic, that is, does not belong to any associated prime of I, except possibly $(R_n)_+$. Let

$$J = \{f \in R_n \mid \exists k; x_n^k \in I\} = I_{x_n} \cap R_n,$$

where the index x_n means the localization by the multiplicative set of the powers of x_n. So, J is the intersection of the primary components of J, except the one that has $(R_n)_+$ as an associated prime, if it exists. This implies that I and J define the same algebraic set (with the same points and the same multiplicities), and, by corollary 10, $\deg(J) = \deg(I)$.

By corollary 12, one can write $J = {}^H J_1$ for some non-homogeneous ideal J_1 of dimension zero in R_{n-1}. As $\dim(J_1) = 0$, the ring R_{n-1}/J_1 is Artinian, It is thus a finite dimensional vector space, and there is an integer k such that one has $F_k(R_{n-1}/J_1) = R_{n-1}/J_1$ for any filtration. Considering the gradation associated to the filtration by the degree, one has thus $G_i(R_{n-1}/J_1) = (0)$ for $i > k$. The corresponding Hilbert series is thus a polynomial, and, by lemma 13, its value at 1 is $\dim_K(R_{n-1}/J_1)$; that is $\deg(J_1) = \dim_K(R_{n-1}/J_1)$.

On the other hand, being a commutative Artinian ring, R_{n-1}/J_1 is a direct product of local Artinian rings which correspond each to a point of the corresponding algebraic set (as we have extended K for having an algebraically closed ground field). The multiplicity of such a point is the dimension (as a vector space) of the corresponding local ring [8]. So the sum of the multiplicities of the point of the algebraic set is $\dim_K(R_{n-1}/J_1) = \deg(J_1)$. This finishes the proof that our definition of the degree is equivalent to the geometric one.

6.2 Hilbert polynomial

Many authors use the Hilbert polynomial for defining the degree. Here we show that this definition is equivalent with ours.

Let us redefine the binomial coefficient as

$$\binom{x+d-1}{d-1} = \begin{cases} \dfrac{(x+1)(x+2)\dots(x+d-1)}{(d-1)!} & \text{if } x \geq 1-d \\ 0 \quad \text{otherwise.} \end{cases}$$

This a polynomial function of x for $x \geq 1-d$. It is standard that, if $d > 1$ is an integer, one has

$$\frac{1}{(1-t)^d} = \sum_{i=0}^{\infty} \binom{i+d-1}{d-1} t^i.$$

This formula extends to the case $d = 1$, with the convention that the empty product equals one.

If $P(t) = \sum_{i=0}^{k} c_i x^i$ is a polynomial, the coefficient of t^δ in the series expansion of $\frac{P(t)}{(1-t)^d}$ is

$$\sum_{i=0}^{k} c_i \binom{\delta - i + d - 1}{d-1}.$$

This is thus a polynomial function of δ for $\delta \geq k + 1 - d$, called the *Hilbert polynomial*. The degree of this polynomial is $d-1$ and its leading coefficient is $\frac{P(1)}{(d-1)!}$.

Thus, our degree of an ideal is the same as the degree that is defined by using the Hilbert polynomial.

7 Classical Bézout theorem

Original Bézout's theorem concerns only plane curves intersections. Nevertheless, its generalization to any dimension is straightforward, and is commonly called *Bézout theorem*. We call it *classical Bézout theorem* for distinguishing it from variants. It states:

Theorem 4 (Classical Bézout theorem)**.** *In the projective space of dimension n, if the intersection of n hypersurfaces consists only of isolated points, the sum of the multipliciies of these points equals the product of the degrees of the hypersurfaces.*

Before stating and proving it in algebraic terms, let us recall some standard facts of commutative algebra (see [5], for example). The *height* h of a prime ideal is the maximal length of increasing chains of prime ideals contained in it: $\mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_h$. The height of an ideal is the minimal height of the prime ideals that contain it. In R_n, the sum of the height and the dimension of any ideal is n. Bézout theorem is strongly related with the so called unmixedness theorem.

Theorem 5 (Unmixedness theorem)**.** *Let $f_1, \ldots, f_h$ be homogeneous polynomials in R_n. The following two conditions are equivalent.*

- $\mathrm{height}(\langle f_1, \ldots, f_h \rangle) = h$
- *for $i = 2, \ldots, h$, the polynomial f_i is not a zero divisor modulo $\langle f_1, \ldots, f_{i-1} \rangle$*

Moreover, if these conditions are satisfied, all associated prime of $\langle f_1, \ldots, f_h \rangle$ have the same height h. Thus the ideal is equidimensional *and has no embedded primary component.*

An algebraic generalization of Bézout's theorem is the following.

Theorem 6 (Algebraic Bézout theorem)**.** *If the homogeneous polynomials $f_1, \ldots, f_h$ generate an ideal of height h, the degree of this ideal is the product of the degrees of the polynomials.*

If the polynomials belong to R_n, Bézout's theorem is the case $h = n - 1$: in this case, the dimension is one, and the projective algebraic set defined by the ideal has the dimension zero. This occurs if and only if this algebraic set has only a finite number of points. So, the classical Bézout's theorem results from the geometric interpretation of the degree (section 6.1).

Proof. Proof by recurrence on h: the case $h = 1$ is proposition 6, and the induction step is proposition 7. $\square$

8 Total and weighted degrees

By corollary 10, the degree of an ideal depends only on the primary components of maximal dimension. When components of lower dimension needs to be taken in consideration, on needs a degree that involve them. Such a degree appears in several articles ([7], in particular). It consists in summing the degrees of all isolated primary components. We call it the *total degree* of an ideal, for distinguishing it from the degree that has been considered until now.

Above algebraic Bézout theorem (theorem 6) is an equality. If the ideal is not equidimensional, it becomes an inequality. The total degree was introduced for getting a sharper inequality. However, the resulting Bézout inequality remains very coarse. When writing this article, it appeared that a much sharper inequality can be obtained by introducing a divided degree and a weighted degrees.

Definition 3. *Given a sequence of positive integers* $\mathrm{d}_1, \ldots, d_n$*, the* divided degree *of an ideal* I *is the sum*

$$\operatorname{divdeg}(I) = \sum \frac{\deg(\mathfrak{q})}{d_1 \ldots d_{\operatorname{height}(\mathfrak{q})}},$$

where the sum runs over the isolated primary components of I*. For every* $k \geq \operatorname{height}(I)$*, the weighted degree is*

$$\operatorname{wdeg}_k(I) = d_1 \ldots d_k \operatorname{divdeg}(I).$$

The total degree $\operatorname{Deg}(I)$ *is the special case of the weighted degree, where all* d_i *equal* 1.

Proposition 14. *If* I *is an ideal of height* h*, one has, for every* $k \geq h$

$$\deg(I) \leqslant \operatorname{Deg}(I) \leqslant \operatorname{wdeg}_h(I) \leqslant \operatorname{wdeg}_k(I).$$

The first inequality is an equality if and only if I *is weakly equidimensional (that is all isolated primary components have the same dimension). The second inequality is an equality if and only if* $d_{m+1} = d_{m+2} = \cdots = d_h = 1$*, where* m *is the minimal heigh of an isolated primary component of* I*. Finally, the last inequality is an equality if and only if* $d_{h+1} = \cdots = d_k = 1$.

Proof. This results immediately from corollary 10. □

Proposition 15 (Additivity of weighted degrees)**.** *One has*

$$\operatorname{Deg}(I \cap J) \leqslant \operatorname{Deg}(I) + \operatorname{Deg}(J),$$

with equality if and only if there is no inclusion between a minimal prime of I and a minimal prime of J. The same is true if Deg *is replaced by* divdeg *or* wdeg.

Proof. The union of the sets of primary components of I and J is a primary decomposition of $I \cap J$, which may be redundant. The way of making irredundant a primary decomposition shows several possibilities. Let $\mathfrak{q}$ be an isolated primary component of I with $\mathfrak{p}$ as associated prime. If $\mathfrak{p}$ does not contain any minimal prime of J, then $\mathfrak{q}$ is an isolated component of $I \cap J$, and $\deg(\mathfrak{q})$ counts in $\mathrm{Deg}(I \cap J)$. If $\mathfrak{p}$ contains strictly a minimal prime of J, it is not a minimal prime of $I \cap J$, and $\deg(\mathfrak{q})$ does not count at all in $\mathrm{Deg}(I \cap J)$. If $\mathfrak{p}$ is a minimal prime of both I and J, let $\mathfrak{q}'$ be the corresponding primary component of J. So, $\mathfrak{p}$ is a minimal prime of $I \cap J$, with $\mathfrak{q} \cap \mathfrak{q}'$ as the corresponding primary component. As $\mathfrak{q} + \mathfrak{q}' \subset \mathfrak{p}$, the proof of proposition 9 shows that $\deg(\mathfrak{q} \cap \mathfrak{q}') < \deg(\mathfrak{q}) + \deg(\mathfrak{q}')$. The result follows, as all possibilities have been considered. The proof is exactly the same for divdeg or wdeg. $\square$

9 Strong Bézout inequality

The strong Bézout inequality bounds the total degree of an ideal in terms of the degrees of its generators

Theorem 7 (Strong Bézout inequality)**.** *Let $I = \langle f_1, \ldots, f_k \rangle$ be an ideal that is generated by nonzero polynomials in R_n, that are of respective degrees $d_1, \ldots, d_k$, and are sorted in order that $\deg(f_2) \geq \deg(f_3) \geq \cdots \geq \deg(f_k) \geq \deg(f_1)$. Then, for the divided degree associated to $d_1, \ldots, d_k$, one has*

$$\mathrm{divdeg}(I) \leqslant 1.$$

For comparing with usual formulations of Bézout inequalities, this theorem can be restated as follows.

Corollary 16. *With the same hypotheses as above, if i is any integer such that $\mathrm{height}(I) \leqslant i \leqslant k$, one has*

$$\mathrm{wdeg}_i(I) \leqslant f_1 \ldots f_i,$$

for the weighted degree associated to $d_1, \ldots, d_k$.

As stated, the theorem is true in the homogeneous case as well as in the non-homogeneous one, but in the non-homogeneous case, one gets a sharper inequality by homogenizing the input. More precisely, in the formulation of the corollary, one can subtract the weighted degree of the components at infinity from the product of the degrees.

Theorem 7 improves all known Bézout inequalities. For example:

Corollary 17. *With notation of theorem 7,* $\text{height}(I) \leqslant i \leqslant k$*, and, in particular if* $i = \min(k, n)$*, one has*

$$\text{Deg}(I) \leqslant f_1 \dots f_i.$$

This results immediately from proposition 14. This theorem seems new, although a slightly coarser bound has been proved by Masser and Wüstholz (see below).

Corollary 18 (Heintz's Bézout inequality [**?**])**.** *In the preceding corollary,* $\text{Deg}(I)$ *can be replaced by the number of isolated components of* I*.*

Corollary 19 (Improved Masser-Wüstholz theorem)**.** *With above notation, for every* $h \leqslant n$*, the sum of the degrees of the isolated components of* I *whose height is at most* h *is not greater than* $d_1 \cdots d_h$*.*

Proof. The sum is a partial sum of the sum involved in the definition of $\deg(I)$. $\square$

Masser and Wüstholz [7] have proved the weaker result with "equal to" instead of "at most". As an important consequence, Masser-Wüstholz theorem is the first proof of the following *affine Bézout inequality*, which does not result of common proofs of classical Bézout theorem, because of the possibility of components of positive dimension at infinity.

Corollary 20 (Affine Bézout inequality)**.** *If* n *polynomials in* n *indeterminates have a finite number of common zeros in an algebraically closed field, the sum of the multiplicities of these zeros is at most the product of the degrees of the polynomials.*

9.1 Example of gaps

Strong Bézout inequality is the same as previous Bézout inequalities in the equidimensional case. In the non-equidimensional case, it is much sharper, but remains far from an equality. This is clear when considering the simplest non-equidimensional case.

Let g_0, g_1, and g_2 be three pairwise-coprime polynomials of respective degrees e_0, e_1, and e_2, such that g_0 is not a zero divisor modulo $\langle g_1, g_2 \rangle$.

Let $f_1 = g_0 g_1$ of degree $d_1 = e_0 + e_1$, and $f_2 = g_0 g_2$ of degree $d_2 = e_0 + e_2$.

The primary decomposition of $I = \langle f_1, f_2 \rangle$ is $\langle g_0 \rangle \cap \langle g_1, g_2 \rangle$. It follows that

$$\begin{aligned}\deg(I) = e_0 < \mathrm{Deg}(I) = e_0 + e_1 e_2 < \mathrm{wdeg}_2(I) &= e_0(e_0 + e_2) + e_1 e_2 \\ &< d_1 d_2 = (e_0 + e_1)(e_0 + e_2).\end{aligned}$$

The successive gaps are thus e_1e_2, $e_0(e_0 + e_2 - 1)$, and e_0e_1. If $e_0 = e_1 = e_2 = d$, the inequalities become

$$d < d + d^2 < 3d^2 < 4d^2.$$

It seems that the gap between the weighted degree and the Bézout bound may be due, at least partially, to the fact that the latter is symmetric in the degrees of input polynomials, while the former is not. We do not know any way for taking this symmetry into account for getting a sharper inequality.

10 Proof of the strong Bézout inequality

10.1 Regularity condition

Proofs of Bézout inequalities require that the input polynomials are as close as possible to form a regular sequence. This is the *regularity condition*.

Definition 4 (Regularity condition)**.** *A sequence $f_1, \ldots, f_k$ satisfy the* regularity condition *if, for $i = 2, \ldots, k$, the following condition is satisfied:* If f_i belongs to some associated prime $\mathfrak{p}$ of the ideal $f_1, \ldots, f_{i-1}$, then $f_j \in \mathfrak{p}$. *This condition can also be stated as:* if $f_j \notin \mathfrak{p}$ for some $j > i$ and some associated prime $\mathfrak{p}$, then $f_i \notin \mathfrak{p}$

The regularity condition can classically be obtained by a linear change of polynomials (see [5], for example).

Proposition 21. *If the ground field is infinite, and the degrees of $f_1, \ldots, f_k$ satisfy $d_2 \geq \cdots \geq d_k \geq d_1$, there are polynomials $g_1, \ldots, g_k$ of the same degrees that satisfy the regularity condition, and verify $g_1 = f_1$, and*

$$g_i = f_i + \sum_{j>i} c_{i,j} f_j,$$

for $i > 1$, where $c_{i,j}$ is either a homogeneous polynomial of degree $d_i - d_j$ (in the homogeneous case) or a constant (in the non-homogeneous case).

Proof. We proceed by increasing values of i. Let V be the finite-dimensional vector space of polynomials of degree at most d_i, and W be the subspace of V formed by the polynomials $\lambda f_i + \sum_{j>i} c_{i,j} f_j$ where λ is a scalar, and the $c_{i,j}$ are as in the statement of the proposition. The elements of W such that $\lambda = 0$ form a subspace $W_0 \subset W$. For each associated prime $\mathfrak{p}$ of $\langle g_1, \ldots, g_{i-1}\rangle$ such that $f_j \notin \mathfrak{p}$ for some $j \geq i$, the vector space $W_\mathfrak{p} = \mathfrak{p} \cap W$ is a proper subspace of W, since there is a $c_{i,j}$ such that $c_{i,j} f_j \notin \mathfrak{p}$. Over an infinite field, a vector space cannot be the union of a finite number of proper subspaces. This implies that, if $W_0 \neq W$, there is an element of W that does not belong to the union of W_0 and the $W_\mathfrak{p}$; in this case, the quotient of this element by λ is the desired g_i. If $W_0 = W$, there is a relation $f_i + \sum_{j>i} c'_{i,j} f_j = 0$. Adding this relation to an element of W_0 that does not belong to any $W_\mathfrak{p}$ gives also the desired g_i. □

Changing the f_i to the g_i does not change the generated ideal, nor the degrees of the generators. This implies that over an infinite field, for proving the strong Bézout inequality, one can suppose that the regularity condition is satisfied.

The case of a finite ground field can be reduced to the case of an infinite ground field by an extension of the scalars. This does not change the f_i. A primary component either gives a primary component of the same degree and height, or factors into several conjugate components which have the same height as the initial component and whose degrees sum is the degree of the initial component. This implies that all quantities appearing in the strong Bézout inequality remain unchanged by extension of the scalars.

In summary, in all cases, we have reduced the proof to the case of a sequence of polynomials that satisfy the regularity condition.

10.2 Technical lemmas

Lemma 22. *If I and J are two ideals, and f is a polynomial, then $(I \cap J) + \langle f\rangle$ and $(I + \langle f\rangle) \cap (J + \langle f\rangle)$ have the same minimal primes.*

Moreover, if there is no inclusion between the associated primes of $I + \langle f\rangle$ and $J + \langle f\rangle$, (that is, if no associated prime of one ideal contains the other ideal), then $(I \cap J) + \langle f\rangle = (I + \langle f\rangle) \cap (J + \langle f\rangle)$, and a minimal primary decomposition of $(I \cap J) + \langle f\rangle$ is the union of minimal primary decompositions of $I + \langle f\rangle$ and $J + \langle f\rangle$.

Proof. One has

$$(I + \langle f\rangle) \cdot (J + \langle f\rangle) \subset (I \cap J) + \langle f\rangle \subset (I + \langle f\rangle) \cap (J + \langle f\rangle).$$

As the intersection and the product of two ideals have the same radical, $(I \cap J) + \langle f\rangle$ and $(I + \langle f\rangle) \cap (J + \langle f\rangle)$ have the same radical and thus the same minimal primes.

If the two above inclusions are equalities, the second assertion results immediately from its hypothesis. It remains thus to prove that if two ideals H and K satisfy the hypothesis of the last assertion, then $H \cap K = H \cdot K$. In fact, no associated prime of either ideal can contain $H + K$. So, there is an element $z \in H + K$ that does not belong to either associated prime. If $x \in H \cap K$, then $zx \in H \cdot K$. If $S = \{1, z, z^2, \dots)$, it follows that $S^{-1}(H \cap K) = S^{-1}(H \cdot K)$. The hypothesis implies thus that the inverse image in R of a primary decomposition of this localized ideal is a primary decomposition of both $H \cap K$ and $H \cdot K$, which are thus equal. □

Lemma 23. *Let $\mathfrak{p}$ be a minimal prime of an ideal I. There is an element $t \in R \setminus \mathfrak{p}$ that belongs to all other associated primes of I. If S is the multiplicative set of the powers of t, then the $\mathfrak{p}$-primary component of I is $R \cap S^{-1}I$.*

Proof. As $\mathfrak{p}$ is a minimal prime, each other associated prime contains an element that is not in $\mathfrak{p}$. The product of these elements is the desired element t. The last assertion results of the classical property of stability of primary decompositions under localization. □

Lemma 24. *Let $I \subset J$ be two ideals that have a common minimal prime $\mathfrak{p}$. Let $\mathfrak{q}$ and $\mathfrak{q}'$ be their respective $\mathfrak{p}$-primary components. Then $\mathfrak{q} \subset \mathfrak{q}'$, and $\deg(\mathfrak{q}) \geq \deg(\mathfrak{q}')$.*

Proof. The first assertion results from Lemma 23, since localization and intersection preserve inclusion.

By definition of Hilbert series, the inclusion $\mathfrak{q} \subset \mathfrak{q}'$ implies that each coefficient of the Hilbert series $\mathrm{HS}_{R/\mathfrak{q}}(t)$ is not smaller than the corresponding coefficient of $\mathrm{HS}_{R/\mathfrak{q}'}(t)$. Thus $\mathrm{HS}_{R/\mathfrak{q}}(t) \geq \mathrm{HS}_{R/\mathfrak{q}'}(t)$ for $t < 1$. Writing these series as a rational fractions with denominator $(1-t)^\delta$, one see that the same inequality applies to the numerators, to their limits when $t \to 1$, and thus to the degrees of the ideals. □

Lemma 25. *Let I be an ideal, and f be a polynomial. If $\mathfrak{p}$ is a minimal prime of I such that $f \in \mathfrak{p}$, then $\mathfrak{p}$ is a minimal prime of $I + \langle f \rangle$. Moreover, if $\mathfrak{q}$ and $\mathfrak{q}'$ are the $\mathfrak{p}$-primary components of I and $I + \langle f \rangle$, respectively, then $\deg(\mathfrak{q}) \geq \deg(\mathfrak{q}')$.*

Proof. By hypothesis, $I + \langle f \rangle \subset \mathfrak{p}$. Any minimal prime $\mathfrak{m}$ of $I + \langle f \rangle$ contains I, and thus some minimal prime of I. Therefore, $\mathfrak{m}$ cannot be strictly included in $\mathfrak{p}$; that is, $\mathfrak{p}$ is a minimal prime of $I + \langle f \rangle$. Then, the inequality on the degrees follows directly from lemma 24.

10.3 Recurrence

As shown in section 10.1, for completing the proof, we can work with k polynomials $f_1, \ldots, f_k$ of degrees $d_1, \ldots, d_k$ that satisfy the *regularity condition,* such that, for $i < k$, and every associated prime $\mathfrak{p}$ of the ideal $\langle f_1, \ldots, f_i \rangle$, if $f_{i+1} \in \mathfrak{p}$, then $f_j \in \mathfrak{p}$ for all $j > i$.

As remarked just after corollary 16, we can also suppose that all polynomials and ideals are homogeneous. However, this hypothesis is not really used, so it will not appear explicitly in proofs that follow.

Let $I_r = \langle f_1, \ldots, f_r \rangle$, for $r = 1, \ldots, k$. The proof proceeds by relating the primary components of I_r and those of $I_r + \langle f_{r+1} \rangle$. For this purpose, we consider several case.

10.3.1 Components of lower height

We have first to study the minimal primes of I_r. By Krull's height theorem, the heights of these minimal primes are at most r.

Lemma 26. *If $\mathfrak{p}_r$ if a minimal prime of I_r of height $h < r$, then $\mathfrak{p}_r$ is a minimal prime of I_h, and $f_i \in \mathfrak{p}_r$ for $i > h$.*

Proof. Let us choose recursively, for $i = r, r-1, \ldots, 1$, a minimal prime $\mathfrak{p}_i$ of I_i that is contained in $\mathfrak{p}_{i+1}$. As the height of $\mathfrak{p}_r$ is less than r, these minimal primes cannot be all distinct. Thus, let i be the lowest index such $\mathfrak{p}_i = \mathfrak{p}_{i+1}$. This implies that $f_{i+1} \in \mathfrak{p}_i$, and, by regularity condition, $f_j \in \mathfrak{p}_i$ for $j > i$. Therefore $\mathfrak{p}_r = \mathfrak{p}_i$. Finally, $i = h$, since the height of $\mathfrak{p}_i$ is at most i, and it is at least i, as $\mathfrak{p}_1, \ldots, \mathfrak{p}_i$ is a strictly increasing sequence of primes. □

Lemma 27. *Let $\mathfrak{p}$ be a minimal prime of I_r whose height is less than r. It is also a minimal prime of I_{r-1}. Let $\mathfrak{q}_r$ and $\mathfrak{q}_{r-1}$ be the corresponding primary components of I_r and I_{r-1}. Then $\deg(\mathfrak{q}_r) \leqslant \deg(q_{r-1})$, and $\operatorname{divdeg}(\mathfrak{q}_r) \leqslant \operatorname{divdeg}(q_{r-1})$.*

Proof. The first assertion results immediately from lemma 26. The first inequality is a special case of lemma 24, and the second inequality follows immediately, since both primary components have the same height. □

Corollary 28. *If $h := \operatorname{height}(I_k) < k$, then $\operatorname{Deg} I_k \leqslant \operatorname{Deg} I_h$.*

Now, we have to study, for $r > 1$ the isolated primary components of I_r whose height is r (the case $r = 1$ being trivial). So, let $r > 1$ such that $I_r = \langle f_1, \dots, f_r \rangle$ has a minimal prime of height r. Let s be a polynomial that does not belong to any minimal prime of height r of I_r, but belongs to all other associated primes of I_r. Let $S_r = \{1, s, s^2, \dots, s^i, \dots\}$ the multiplicative set generated by s.

The ideal $J_r = R \cap S_r^{-1} I_r$ is the intersection of the isolated primary components of height r of I_r. We will prove the following lemma by recursion on j.

Lemma 29. *For every $j < r$, the sequence $(f_1, \dots, f_j)$ is a regular sequence in $S_r^{-1} R$, and the ideal $J_{j,r} = R \cap S_r^{-1} I_j$ is unmixed of height j (that is, all its primary components have height j).*

Moreover, $J_{j,r}$ is the intersection of some isolated primary components of height j of I_j whose associated prime does not contain f_{j+1}.

Proof. The cases $j = 0$ and $j = 1$ are trivial. So we suppose that the assertions are true for some $j < r$, and we prove them for $j + 1$.

The definition of $J_{j,r}$ as the inverse image of a localization implies that $J_{j,r}$ is the intersection of some primary components of I_j, and that the minimal primes of $J_{j,r}$ are also minimal primes of I_j. None of these minimal primes can contain f_{j+1}. In fact, if such a prime would contains f_{j+1}, it would contain $f_{j+1}, \dots, f_r$ by the regularity condition, and thus it would be a minimal prime of I_r of height $< r$; this is impossible since S has been chosen for having a non-empty intersection with such a prime. As $J_{j,r}$ is unmixed, we can deduce that f_{j+1} is not a zero divisor modulo $J_{j,r}$. As, by recurrence hypothesis, $f_1, \dots, f_j$, the same is thus true for $f_1, \dots, f_{j+1}$.

This shows that the primary components of $J_{j,r}$ are primary components of I_j, that they have height j and that their associated primes do not contain f_{j+1}. □

Lemma 30. *Let J_r the intersection of all primary components of height r of I_r, and J'_{r-1} be the intersection of the isolated primary components of height $r - 1$ of I_{r-1} whose associated prime do not contain f_r. Then $\deg(J_r) \leqslant \deg(f_r) \cdot \deg(J'_{r-1})$, and $\operatorname{divdeg}_i(J_r) \leqslant \operatorname{divdeg}_i(J'_{r-1})$.*

Proof. Using previous notations, and the last assertion of lemma 29, we have $J'_{r-1} \subset J_{r-1,r}$, and the two ideals have the same height. Thus, by lemma 8,

$$\deg(J_{r-1,r}) \leqslant \deg(J'_{r-1}).$$

In the preceding lemma, we have proved that f_r is not a zero divisor modulo $\deg(J_{r-1,r})$. So,

$$\deg(J'_{r-1} + \langle f_r \rangle) \leqslant \deg(f_r) \cdot \deg(J_{r-1,r}).$$

It results from the proof of the preceding lemma that $J'_{r-1} + \langle f_r \rangle \subset J_r$, and that these ideals have the same height. So

$$\deg J_r \leqslant \deg(J'_{r-1} + \langle f_r \rangle).$$

The result follows immediately, by combining these inequalities, and using the fact that J'_{r-1} and J_r are equidimensional of respective heights $r-1$ and r. $\square$

10.5 End of the proof

Let us recall that we denote

$$\operatorname{divdeg}(I) = \sum_{\mathfrak{q}} \operatorname{divdeg}(\mathfrak{q}) = \sum_{\mathfrak{q}} \frac{\deg(\mathfrak{q})}{d_1 \dots d_{\operatorname{height}(\mathfrak{q})}},$$

where the sums runs on the isolated primary components of I, and d_i is the degree of f_i. We have to prove that $\operatorname{divdeg}(I_r) \leqslant 1$ for every r. It suffices to prove that $\operatorname{divdeg}(I_r) \leqslant \operatorname{divdeg}_k(I_{r-1})$ for $r > 1$, since it is trivial that $\operatorname{divdeg}(I_1) = 1$.

Using notation of lemma 30, we have

$$\operatorname{divdeg}(I_r) = \operatorname{divdeg}(J_r) + \sum_{\mathfrak{q}} \operatorname{divdeg}(\mathfrak{q}),$$

where the sum is restricted to isolated primary components of I_r whose height is less than r. Lemmas 30 and 27 implies thus that

$$\operatorname{divdeg}(I_r) \leqslant \operatorname{divdeg}(J'_{r-1}) + \sum_{\mathfrak{q}'} \operatorname{divdeg}(\mathfrak{q}'),$$

where the sum runs on the isolated primary components of I_{r-1} that contain a power of f_r. As the divided degree of $J'_{r-1})$ is the sum of the divided degrees of some isolated primary components of I_{r-1} that do not contain any power of I_{r-1}, we can deduce that $\operatorname{divdeg}(I_r) \leqslant \operatorname{divdeg}_k(I_{r-1})$, which finishes the proof.

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