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Abstract. We establish some structural results for the Witt and Grothendieck–Witt groups of schemes over $\mathbb{Z}[1/2]$, including homotopy invariance for Witt groups and a formula for the Witt and Grothendieck–Witt groups of punctured affine spaces over a scheme. All these results hold for singular schemes and at the level of spectra.

1. Introduction

Let X be a quasi-projective scheme, or more generally a scheme with an ample family of line bundles, such that $2 \in \mathcal{O}(X)^\times$. In this paper, we show how new techniques can help calculate Balmer's 4-periodic Witt groups $W^n(X)$ of X, in particular when X is singular, including the classical Witt group $W^0(X)$ of [7]. For example, we establish homotopy invariance in Theorem 1.1: if $V \to X$ is a vector bundle then $W^*(X) \cong W^*(V)$. (If X is affine, this was proven by the first author in [5, 3.10]; if X is regular, it was proven by Balmer [2] and Gille [4].)

The formula $W^0(X \times \mathbb{G}_m) \cong W^0(X) \oplus W^0(X)$, which holds for nonsingular schemes by a result of Balmer–Gille [3], fails for curves with nodal singularities (see Example 6.5), but holds if $K_{-1}(X) = 0$; see Theorem 1.2. We show in *loc. cit.* that a similar result holds for the punctured affine space $X \times (\mathbb{A}^n - 0)$ over X.

All of this holds at the spectrum level. Recall from [11, 7.1] that there are spectra $L^{[r]}(X)$ whose homotopy groups are Balmer's 4-periodic triangular Witt groups: $\pi_i L^{[r]}(X) = L^{[r]}_i(X) \cong W^{r-i}(X)$. We write $L(X)$ for $L^{[0]}(X)$. Our first main theorem shows that the functors $L^{[r]}$ are homotopy invariant.

Theorem 1.1. *Let X be a scheme over $\mathbb{Z}[1/2]$, with an ample family of line bundles. If $V \to X$ is a vector bundle, then the projection induces*

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a stable equivalence of L-theory spectra $L(X) \xrightarrow{\simeq} L(V)$. On homotopy groups, $W^n(X) \xrightarrow{\cong} W^n(V)$ for all $n \in \mathbb{Z}$.

Our second main theorem generalizes a result of Balmer and Gille [3], from regular to singular schemes, because $K_{-1}(X) = 0$ when X is regular and separated. However, their proof uses Localization for Witt groups and Devissage, both of which fail for singular schemes.

Let $\widehat{\mathbf{H}}(K_{-1}X)$ denote the $\mathbb{Z}_2$-Tate spectrum of the abelian group $K_{-1}(X)$ with respect to the standard involution (sending a vector bundle to its dual). Its homotopy groups are the Tate cohomology groups $\widehat{\mathrm{H}}^*(\mathbb{Z}_2, K_{-1}X)$.

Theorem 1.2. *Let X be a scheme with an ample family of line bundles, such that $2 \in \mathcal{O}(X)^\times$. Then:*
(i) There is a homotopy fibration of spectra for each $n \geq 1$:

$$S^{-n-1} \wedge \widehat{\mathbf{H}}(K_{-1}X) \longrightarrow L(X) \oplus L^{[1-n]}(X) \longrightarrow L(X \times (\mathbb{A}^n - 0)).$$

(ii) Suppose that $K_{-1}X = 0$, or more generally that $\widehat{\mathrm{H}}^i(\mathbb{Z}_2, K_{-1}X) = 0$ for $i = 0, 1$. Then

$$W^r(X \times (\mathbb{A}^n - 0)) \cong W^r(X) \oplus W^{r+1-n}(X).$$

When $n = 1$ this becomes $W^r(X \times \mathbb{G}_m) \cong W^r(X) \oplus W^r(X)$, and specializes for $r = 0$ to: $W(X \times \mathbb{G}_m) \cong W(X) \oplus W(X)$.

Note that taking homotopy groups of part (i) yields part (ii). The main ingredient in proving Theorem 1.2 is Theorem 5.1 in the text which gives a direct sum decomposition of the hermitian K-theory of $X \times (\mathbb{A}^n - 0)$ into four canonical pieces generalising Bass' Fundamental Theorem for Hermitian K-theory [11, Theorem 9.13].

We also prove parallel results for the higher Witt groups $W_i(X)$ (and coWitt groups) defined by the second author (see [6]), and their variants $W_i^{[r]}(X)$. These results include homotopy invariance and:

Proposition 1.3. *Let X be a scheme with an ample family of line bundles, such that $2 \in \mathcal{O}(X)^\times$. Then the higher Witt groups satisfy:*

$$W_i^{[r]}(X \times (\mathbb{A}^n - 0)) \cong W_i^{[r]}(X) \oplus W_{i-1}^{[r-n]}(X).$$

In particular, the higher Witt groups $W_i^{[r]}(X \times (\mathbb{A}^n - 0))$ are 4-periodic in n.

Since $W_0^{[0]} = W$ is the classical Witt group and $W_i^{[r]}$ is 4-periodic in r we obtain the following.

Corollary 1.4. *For X as in Proposition 1.3, the classical Witt group $W(X \times (\mathbb{A}^n - 0))$ is 4-periodic in n.*

Here is a short overview of the contents of this paper. In Section 2 we establish homotopy invariance of Witt and stabilized Witt groups and prove Theorem 1.1. In Section 3 we give an elementary proof of Theorem 1.2 (ii) when $n = 1$ based on Bass' Fundamental Theorem for Grothendieck–Witt groups. In Section 4 we compute the K-theory of $X \times (\mathbb{A}^n - 0)$. In Section 5 we compute the Witt and Grothendieck–Witt groups of $X \times (\mathbb{A}^n - 0)$ and prove Theorem 1.2. In Section 6 we compute the Witt groups of a nodal curve over a field of characteristic not 2 and show that the formula of Balmer–Gille [3] does not hold for this curve. In Section 7 we generalise Theorems 1.1 and 1.2 to the higher Witt and coWitt groups of the first author, see Proposition 1.3. Finally, in the Appendix the second author computes the higher Grothendieck–Witt groups of $X \times \mathbb{P}^n$ in a form that is needed in the proofs in Section 5. This generalises some unpublished results of Walter [14].

Notation. Following [3], we write $C_n = \mathbb{A}^n - 0$ for the affine n-space minus the origin.

Here are the various spectra associated to a scheme X that we use.

For any abelian group A with involution (or more generally a spectrum with involution), we write $\widehat{\mathbf{H}}(A)$ for the (Tate) spectrum representing Tate cohomology of the cyclic group $\mathbb{Z}_2$ with coefficients in A. If A is a spectrum, then $\widehat{\mathbf{H}}(A)$ is the homotopy cofiber of the hypernorm map $A_{hG} \to A^{hG}$; if A is a group then $\pi_i \widehat{\mathbf{H}}(A) = \widehat{\mathrm{H}}^i(\mathbb{Z}_2, A)$.

We write $\mathbb{K}(X)$ for the nonconnective K-theory spectrum; the groups $K_i(X)$ are the homotopy groups $\pi_i \mathbb{K}(X)$ for all $i \in \mathbb{Z}$. In particular, $K_{-1}(X) = \pi_{-1}\mathbb{K}(X)$. (See [16, 12] for example.) We shall write $K^Q(X)$ for Quillen's connective K-theory spectrum, and $K_{<0}(X)$ for the cofiber of the natural map $K^Q(X) \to \mathbb{K}(X)$.

There is a standard involution on these K-theory spectra, and their homotopy groups $K_i(X)$, induced by the functor on locally free sheaves sending $\mathcal{E}$ to its dual sheaf $\mathcal{E}^* = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X)$; the corresponding Tate cohomology groups $\widehat{\mathrm{H}}^i(\mathbb{Z}_2, K_j X)$ are written as k_j and k'_j in the classical "clock" sequence [6, p. 278]. There are other involutions on K-theory; we shall write $K_i^{[r]}(X)$ for $K_i(X)$ endowed with the involution obtained using duality with values in $\mathcal{O}_X[r]$; see [11, 1.12].

The second author defined the *Grothendieck–Witt* spectra $GW^{[r]}(X)$ and the *Karoubi–Grothendieck–Witt* spectra $\mathbb{G}W^{[r]}(X)$; see [11, 5.7 and 8.6]; the element $\eta \in GW_{-1}^{[-1]}(\mathbb{Z}[1/2])$ plays an important role.

$L^{[r]}(X)$ denotes the spectrum obtained from $GW^{[r]}(X)$ by inverting η; see [11, Def. 7.1]. The homotopy groups $\pi_i L^{[r]}(X)$ are Balmer's 4-periodic triangular Witt groups $L_i^{[r]}(X) = W^{r-i}(X)$ [11, 7.2]. In particular, $W^0(X)$ is the classical Witt group of Knebusch [7] of symmetric bilinear forms on X and $W^2(X)$ is the classical Witt group of symplectic (that is, -1-symmetric) forms on X. The groups $W^1(X)$ and $W^3(X)$ are the Witt groups of formations (M, L_1, L_2) on X, where L_1 and L_2 are Lagrangians on the form M; see [8, p.147] or [13]. When $X = \operatorname{Spec} R$ is affine, then these groups are also Ranicki's L-groups $L_i(R) = W^{-i}(R)$; see [8].

The stabilized L-theory spectrum $\text{Ł}^{[r]}(X)$ is the spectrum obtained from $\mathbb{G}W^{[r]}(X)$ by inverting η; see [11, 8.12]. It is better behaved than $L^{[r]}(X)$, as $\text{Ł}^{[r]}(X)$ satisfies excision, as well as Zariski descent (for open subschemes U and V in $X = U \cup V$); see [11, 9.6].

These spectra fit into the following morphism of fibration sequences. (See [11, 7.6 and 8.13].)

$$\begin{tikzcd} K^Q(X)^{[r]}_{h\mathbb{Z}_2} \arrow[r] \arrow[d] & GW^{[r]}(X) \arrow[r] \arrow[d] & L^{[r]}(X) \arrow[d] \\ \mathbb{K}(X)^{[r]}_{h\mathbb{Z}_2} \arrow[r] & \mathbb{G}W^{[r]}(X) \arrow[r] & \text{Ł}^{[r]}(X). \end{tikzcd} \tag{1.5}$$

Lemma 1.6. *There is a fibration sequence*

$$S^{-1} \wedge \widehat{\mathbf{H}}(K^{[r]}_{<0}(X)) \to L^{[r]}(X) \to \text{Ł}^{[r]}(X) \to \widehat{\mathbf{H}}(K^{[r]}_{<0}(X)).$$

Proof. This follows from (1.5), since the cofiber of the left vertical map is $\mathbb{K}^{[r]}_{<0}(X)_{h\mathbb{Z}_2}$, and the cofiber of the middle vertical map is the homotopy fixed point spectrum $K^{[r]}_{<0}(X)^{h\mathbb{Z}_2}$; see [11, 8.14]. $\square$

2. Homotopy invariance of Witt groups

In order to prove homotopy invariance of $L(X)$ (Theorem 1.1), we first establish homotopy invariance of $\text{Ł}(X)$.

Lemma 2.1. *Let X be a scheme with an ample family of line bundles over $\mathbb{Z}[1/2]$. Then for every vector bundle V over X, the projection $V \to X$ induces an equivalence of spectra*

$$\text{Ł}^{[r]}(X) \xrightarrow{\simeq} \text{Ł}^{[r]}(V).$$

Proof. Since $\text{Ł}^{[r]}$ satisfies Zariski descent, we may assume that $X = \operatorname{Spec}(A)$ is affine, that V is trivial, and even that $V = \operatorname{Spec}(A[t])$. In this case, the homotopy groups $\text{Ł}^{[r]}_i(A)$ and $\text{Ł}^{[r]}_i(A[t])$ are the colimits of ordinary Witt groups $\text{Ł}^{[r]}_i(A) = \operatorname{colim} W^{j-i}(A_j)$ and $\text{Ł}^{[r]}_i(A[t]) = \operatorname{colim} W^{j-i}(A_j[t])$, by [11, 7.2, 8.12]. These colimits are isomorphic because $W^*(A) \cong W^*(A[t])$ by [5, 3.10]. Hence $\text{Ł}^{[r]}_i(A) \cong \text{Ł}^{[r]}_i(A[t])$. $\square$

Proof of Theorem 1.1. Write $L(V,X)$ for the cofiber of $L(X)\to L(V)$, and similarly for the cofibers of Ł and $K_{<0}$. By Lemma 1.6, we have a fiber sequence

$$L^{[r]}(V,X)\to Ł^{[r]}(V,X)\to \widehat{\mathbf{H}}(K^{[r]}_{<0}(V,X)).$$

The middle term is zero by Lemma 2.1. The right hand term is zero because $K_{<0}(V,X)$ has uniquely 2-divisible homotopy groups [15]; see [11, B.14]. This implies that $L^{[r]}(V,X)=0$, i.e., $L^{[r]}$ is homotopy invariant. □

3. Bass' Fundamental Theorem for Witt groups

The map $GW^{[r]}_i(X)\to \mathbb{G}W^{[r]}_i(X)$ is an isomorphism for all $i\geq 0$ and all n; see [11, 9.3]. For $i=-1$, we have the following result.

Lemma 3.1. *If $K_{-1}(X)=0$, then $W_0(X)\cong GW^{[-1]}_{-1}(X)\xrightarrow{\cong}\mathbb{G}W^{[-1]}_{-1}(X)$.*

Proof. Since $K^Q_i(X)\cong K_i(X)$ for $i\geq 0$, we see from [11, B.9] that $\pi_0(K^{h\mathbb{Z}_2}_{<0}X)=0$ and $\pi_{-1}(K^{h\mathbb{Z}_2}_{<0}X)\cong H^0(\mathbb{Z}_2,K_{-1}(X))$. Hence the middle column of (1.5) yields an exact sequence

$$0\to GW^{[-1]}_{-1}(X)\to \mathbb{G}W^{[-1]}_{-1}(X)\to H^0(\mathbb{Z}_2,K_{-1}(X))=0.$$

Finally, $W^0(X)\cong GW^{[-1]}_{-1}(X)$ by [11, 6.3]. □

Theorem 3.2. *Let X be a quasi-projective scheme over $\mathbb{Z}[1/2]$. If $K_{-1}(X)=0$, or more generally $H^*(\mathbb{Z}_2,K_{-1}(X))=0$, then*

$$W_0(X)\oplus W_0(X)\cong W_0(X\times\mathbb{G}_m).$$

Proof. The second author proved in [11, 9.13–14] that there is a natural split exact "contracted functor" sequence (for all n and i)

$$\begin{aligned}0\to \mathbb{G}W^{[r]}_i(X)\to \mathbb{G}W^{[r]}_i(X[t])\oplus \mathbb{G}W^{[r]}_i(X[1/t])\\ \to \mathbb{G}W^{[r]}_i(X[t,1/t])\to \mathbb{G}W^{[r-1]}_{i-1}(X)\to 0.\end{aligned}$$

Taking $n=i=0$ (so $GW_0\cong\mathbb{G}W_0$), we get a natural split exact sequence

$$\begin{aligned}0\to GW_0(X)\to GW_0(X[t])\oplus GW_0(X[1/t])\\ \to GW_0(X[t,1/t])\to \mathbb{G}W^{[-1]}_{-1}(X)\to 0.\end{aligned}$$

When $K_{-1}(X)=0$ we also have a split exact sequence [16, X.8.3]:

$$0\to K_0(X)\to K_0(X[t])\oplus K_0(X[\frac{1}{t}])\to K_0(X[t,\frac{1}{t}])\to 0.$$

Mapping this to the GW-sequence, we have a split exact sequence on cokernels:

$$0\to W_0(X)\to W_0(X[t])\oplus W_0(X[\frac{1}{t}])\to W_0(X[t,\frac{1}{t}])\to \mathbb{G}W^{[-1]}_{-1}(X)\to 0.$$

By Lemma 3.1, we have

$$GW^{[-1]}_{-1}(X)\cong \mathbb{G}W^{[-1]}_{-1}(X)\cong W^0(X).$$

Since $W_*(X)\cong W_*(X[t])$ by Theorem 1.1, the result follows. $\square$

4. K-theory of punctured affine space

The following result generalizes the "Fundamental Theorem" of K-theory, which says that there is an equivalence of spectra

$$\mathbb{K}(X)\ \oplus\ S^1\wedge\mathbb{K}(X)\ \oplus N\mathbb{K}(X)\oplus N\mathbb{K}(X)\xrightarrow{\simeq}\mathbb{K}(X\times\mathbb{G}_m).$$

Write $V(1)$ for the vector bundle $\mathbf{V}(\mathcal{O}(1))$ on $\mathbb{P}^{n-1}_X$ associated to the invertible sheaf $\mathcal{O}(1)$. For simplicity, we write $\mathbb{K}(\mathbb{A}^n_X, X)$ for the fiber of $\mathbb{K}(\mathbb{A}^n_X)\to\mathbb{K}(X)$ induced by the inclusion of X as the zero-section of $\mathbb{A}^n_X$, and write $\mathbb{K}(V(1),\mathbb{P}^{n-1}_X)$ for the fiber of the map $\mathbb{K}(V(1))\to\mathbb{K}(\mathbb{P}^{n-1}_X)$ which is induced by the inclusion of $\mathbb{P}^{n-1}_X$ as the zero-section of $V(1)$.

Theorem 4.1. *For every integer $n\geq 1$ and every quasi-compact and quasi-separated scheme X, there is an equivalence of spectra*

$$\mathbb{K}(X)\ \oplus\ S^1\wedge\mathbb{K}(X)\ \oplus\ \mathbb{K}(V(1),\mathbb{P}^{n-1}_X)\ \oplus\ \mathbb{K}(\mathbb{A}^n_X,X)\ \xrightarrow{\simeq}\mathbb{K}(X\times C_n)$$

functorial in X.

Remark 4.1.1. If X is regular, Theorem 4.1 is immediate from the fibration sequence $\mathbb{K}(X)\to\mathbb{K}(X\times\mathbb{A}^n)\to\mathbb{K}(X\times C_n)$; see [16, V.6].

If Z is a closed subscheme of a scheme X, we write $\mathbb{K}(X\ \text{ on }\ Z)$ for the homotopy fiber of $\mathbb{K}(X)\to\mathbb{K}(X-Z)$.

Proof. Consider the points $0=(0,...,0)$ of $\mathbb{A}^n$ and $z=[1:0:\cdots:0]$ of $\mathbb{P}^n$, and consider $\mathbb{A}^n$ embedded into $\mathbb{P}^n$ via the open immersion $(t_1,...,t_n)\mapsto[1:t_1,\cdots,t_n]$ sending 0 to z. We will write 0_X (resp., z_X) for the corresponding subschemes of $\mathbb{A}^n_X$ (resp., $\mathbb{P}^n_X$); both are isomorphic to X. We have a commutative diagram of spectra

$$\begin{tikzcd}
\mathbb{K}(\mathbb{P}^n_X\ \text{ on }\ z_X) \arrow[r] \arrow[d, "\wr"'] & \mathbb{K}(\mathbb{P}^n_X) \arrow[d] & \\
\mathbb{K}(\mathbb{A}^n_X\ \text{ on }\ 0_X) \arrow[r, "\varepsilon"] & \mathbb{K}(\mathbb{A}^n_X) \arrow[r] & \mathbb{K}(X\times C_n)
\end{tikzcd} \tag{4.1.1}$$

in which the lower row is a homotopy fibration, by definition, and the left vertical arrow is an equivalence, by Zariski-excision [12]. We will first show that $\varepsilon = 0$, that is, we will exhibit a null-homotopy of ε functorial in X.

Recall that $\mathbb{K}(\mathbb{P}^n_X)$ is a free $\mathbb{K}(X)$-module of rank $n+1$ on the basis

$$b_r = \bigotimes_{i=1}^{r} \left(\mathcal{O}_{\mathbb{P}^n}(-1) \xrightarrow{T_i} \mathcal{O}_{\mathbb{P}^n}\right), \quad r = 0, ..., n \tag{4.1.2}$$

where $\mathcal{O}_{\mathbb{P}^n}$ is placed in degree 0 and $\mathbb{P}^n = \mathrm{Proj}(\mathbb{Z}[T_0, ..., T_n])$. By convention, the empty tensor product b_0 is the tensor unit $\mathcal{O}_{\mathbb{P}^n}$. Note that the restriction of b_r to $\mathbb{A}^n$ is trivial in $K_0(\mathbb{A}^n)$ for $r = 1, ..., n$. This defines a null-homotopy of the middle vertical arrow of (4.1.1) on the components $\mathbb{K}(X) \cdot b_r$ of $\mathbb{K}(\mathbb{P}^n_X)$ for $r = 1, ..., n$. The remaining component $\mathbb{K}(X) \cdot b_0$ maps split injectively into $\mathbb{K}(X \times C_n)$ with retraction given by any rational point of C_n. Since the composition of the two lower horizontal maps is naturally null-homotopic, this implies $\varepsilon = 0$. Thus, we obtain a functorial direct sum decomposition

$$\mathbb{K}(X \times C_n) \simeq \mathbb{K}(\mathbb{A}^n_X) \ \oplus \ S^1 \wedge \mathbb{K}(\mathbb{P}^n_X \ \text{ on } \ z_X)$$

and it remains to exhibit the required decomposition of the two summands.

The composition $0 \to \mathbb{A}^n \to \mathrm{pt}$ is an isomorphism and induces the direct sum decomposition of $\mathbb{K}(\mathbb{A}^n_X)$ as $\mathbb{K}(X) \oplus \mathbb{K}(\mathbb{A}^n_X, X)$. For the other summand, note that b_n has support in $z = V(T_1, ..., T_n)$, as it is the Koszul complex for $(T_1, ..., T_n)$. In particular, the composition

$$\mathbb{K}(X) \xrightarrow{\otimes b_n} \mathbb{K}(\mathbb{P}^n_X \ \text{ on } \ z_X) \longrightarrow \mathbb{K}(\mathbb{P}^n_X)$$

is split injective, and defines a direct sum decomposition

$$\mathbb{K}(\mathbb{P}^n_X \ \text{ on } \ z_X) \cong \mathbb{K}(X) \oplus \widetilde{\mathbb{K}}(\mathbb{P}^n_X \ \text{ on } \ z_X).$$

It remains to identify $\widetilde{\mathbb{K}}(\mathbb{P}^n_X \text{ on } z_X)$ with $\Omega\mathbb{K}(V(1), \mathbb{P}^{n-1}_X)$. Consider the closed embedding $j : \mathbb{P}^{n-1} = \mathrm{Proj}(\mathbb{Z}[T_1, ..., T_n]) \subset \mathbb{P}^n$. Since $j(\mathbb{P}^{n-1})$ lies in $\mathbb{P}^n - \{z\}$, we have a commutative diagram of spectra

$$\begin{tikzcd}
\mathbb{K}(X) \arrow[r, "1"] \arrow[d, "b_n"'] & \mathbb{K}(X) \arrow[r] \arrow[d, "b_n"] & 0 \arrow[d] \\
\mathbb{K}(\mathbb{P}^n_X \ \text{ on } \ z_X) \arrow[r] \arrow[d] & \mathbb{K}(\mathbb{P}^n_X) \arrow[r] \arrow[d, "j^*"] & \mathbb{K}(\mathbb{P}^n_X - z_X) \arrow[d, "j^*"] \\
0 \arrow[r] & \mathbb{K}(\mathbb{P}^{n-1}_X) \arrow[r, "1"'] & \mathbb{K}(\mathbb{P}^{n-1}_X)
\end{tikzcd} \tag{4.1.3}$$

in which the rows are homotopy fibrations, and the middle column is split exact. It follows that we have a fibration

$$\widetilde{\mathbb{K}}(\mathbb{P}^n_X \text{ on } z_X) \longrightarrow \mathbb{K}(\mathbb{P}^n_X - z_X) \xrightarrow{j^*} \mathbb{K}(\mathbb{P}^{n-1}_X).$$

Since $V(1) \to \mathbb{P}^{n-1}_X$ is $\mathbb{P}^n_X - z_X \to \mathbb{P}^{n-1}_X$, it follows that

$$S^1 \wedge \widetilde{\mathbb{K}}(\mathbb{P}^n_X \text{ on } z_X) \simeq \mathbb{K}(\mathbb{P}^n_X - z_X, \mathbb{P}^{n-1}_X) = \mathbb{K}(V(1), \mathbb{P}^{n-1}_X). \qquad \square$$

Remark 4.2. The proof of Theorem 4.1 also applies to the homotopy K-theory spectrum KH of [16]. Since $KH(\mathbb{A}^n_X, X) = 0$ and $KH(V(1)_X, \mathbb{P}^{n-1}_X) = 0$, by homotopy invariance, we obtain an equivalence:

$$KH(X) \oplus S^1 \wedge KH(X) \simeq KH(X \times C_n).$$

The following fact will be needed in the next section.

Lemma 4.3. *Let V be a vector bundle over a scheme X defined over $\mathbb{Z}[1/2]$. Then the homotopy groups of $\mathbb{K}(V, X)$ are uniquely 2-divisible. In particular, for any involution on $\mathbb{K}(V, X)$ we have $\widehat{\mathbf{H}}(\mathbb{K}(V, X)) = 0$.*

Proof. This follows from [15] and Zariski-Mayer-Vietoris [12]. $\square$

5. GW and L-theory of punctured affine space

A modification of the argument in Theorem 4.1 yields a computation of the Hermitian K-theory of $X \times C_n$. This generalises the case $n = 1$ of [11, Theorem 9.13].

For any line bundle $\mathcal{L}$ on X, we write $\mathbb{K}^{[r]}(X; \mathcal{L})$ (resp., $\mathbb{G}W^{[r]}(X; \mathcal{L})$) for the K-theory spectrum of X (resp., $\mathbb{G}W$-spectrum) with involution $E \mapsto \mathrm{Hom}(E, \mathcal{L}[r])$. If $p : V \to X$ is a vector bundle on X, then p embeds $\mathbb{G}W^{[r]}(X; \mathcal{L})$ into $\mathbb{G}W^{[r]}(V; \mathcal{L})$ as a direct summand with retract given by the zero section. We write $\mathbb{G}W^{[r]}(V, X; \mathcal{L})$ for the complement of $\mathbb{G}W^{[r]}(X; \mathcal{L})$ in $\mathbb{G}W^{[r]}(V; p^*\mathcal{L})$. If $\mathcal{L} = \mathcal{O}_X$, we simply write $\mathbb{K}^{[r]}(X)$, $\mathbb{G}W^{[r]}(X)$ and $\mathbb{G}W^{[r]}(V, X)$.

Theorem 5.1. *For all integers r, n with $n \geq 1$ and every scheme X over $\mathbb{Z}[1/2]$ with an ample family of line bundles, there is a functorial equivalence of spectra*

$$\begin{aligned}\mathbb{G}W^{[r]}(X \times C_n) \simeq \quad & \mathbb{G}W^{[r]}(X) \oplus S^1 \wedge \mathbb{G}W^{[r-n]}(X) \\ & \oplus \mathbb{G}W^{[r]}(V(1), \mathbb{P}^{n-1}_X; \mathcal{O}(1-n)) \oplus \mathbb{G}W^{[r]}(\mathbb{A}^n_X, X).\end{aligned}$$

Proof. The proof is the same as that of Theorem 4.1 with the following modifications. For space reasons, we write $\mathcal{L}$ (resp., $\mathcal{L}'$) for the sheaf

$\mathcal{O}(1-n)$ on $\mathbb{P}^n$ (resp., on $\mathbb{P}^{n-1}$). Consider the commutative diagram of spectra, analogous to (4.1.1),

(5.1.1)
$$
\begin{tikzcd}
\mathbb{G}W^{[r]}(\mathbb{P}^n_X \text{ on } z_X;\mathcal{L}) \arrow[r] \arrow[d, "\wr"'] & \mathbb{G}W^{[r]}(\mathbb{P}^n_X;\mathcal{L}) \arrow[d] & \\
\mathbb{G}W^{[r]}(\mathbb{A}^n_X \text{ on } 0_X) \arrow[r, "\varepsilon"] & \mathbb{G}W^{[r]}(\mathbb{A}^n_X) \arrow[r] & \mathbb{G}W^{[r]}(X\times C_n)
\end{tikzcd}
$$

in which the lower row is a homotopy fibration (by definition), and the left vertical arrow is an equivalence, by Zariski-excision [10, Thm. 3], noting that $\mathcal{L}=\mathcal{O}(1-n)$ is trivial on $\mathbb{A}^n_X$. Again, we will first show that $\varepsilon = 0$.

Recall the complexes b_i from (4.1.2). We equip $b_0 = \mathcal{O}_{\mathbb{P}^n}$ with the unit form $\mathcal{O}_{\mathbb{P}^n}\otimes\mathcal{O}_{\mathbb{P}^n}\to\mathcal{O}_{\mathbb{P}^n} : x\otimes y\mapsto xy$. The target of the quasi-isomorphism

$$b_n \cong b_n\otimes\mathcal{O} \xrightarrow{1\otimes T_0} b_n\otimes\mathcal{O}(1)$$

is canonically isomorphic to $b_n^*\otimes\mathcal{L}[n]$, and endows b_n with a non-degenerate symmetric bilinear form with values in $\mathcal{L}[n]$. In detail, the complex $\beta_i = (T_i : \mathcal{O}(-1)\to\mathcal{O})$ with $\mathcal{O}$ placed in degree 0 is endowed with a symmetric form with values in $\mathcal{O}(-1)[1]$:

$$
\begin{tikzcd}
\beta_i\otimes\beta_i \arrow[d, "\varphi_i"] & \mathcal{O}(-2) \arrow[r, "\binom{T_i}{-T_i}"] \arrow[d] & \mathcal{O}(-1)\oplus\mathcal{O}(-1) \arrow[r, "(T_i,T_i)"] \arrow[d, "(1,1)"] & \mathcal{O} \arrow[d] \\
\mathcal{O}(-1)[1] & 0 \arrow[r] & \mathcal{O}(-1) \arrow[r] & 0.
\end{tikzcd}
$$

Hence the tensor product $b_n = \bigotimes_{i=1}^n\beta_i$ is equipped with a symmetric form with values in $\mathcal{L}[n]$:

$$\text{(5.1.2)}\qquad b_n\otimes b_n \xrightarrow{\bigotimes_{i=1}^n\varphi_i} \mathcal{O}(-n)[n] \xrightarrow{T_0} \mathcal{O}(1-n)[n] = \mathcal{L}[n].$$

Of course, to make sense of the map $\otimes_{i=1}^n\varphi_i$ we have to rearrange the tensor factors in $b_n\otimes b_n$ using the symmetry of the tensor product of complexes given by the Koszul sign rule. Note that b_n restricted to $\mathbb{A}^n$ is 0 in $GW_0^{[n]}(\mathbb{A}^n)$ since it is the external product of the restrictions of β_i to $\mathbb{A}^1 = \operatorname{Spec}\mathbb{Z}[T_i,1/2]$ which are trivial in $GW_0^{[1]}(\mathbb{A}^1)$, by [11, Lemma 9.12].

By Corollary A.5, the right vertical map of (5.1.1) is

$$\mathbb{G}W^{[r-n]}(X)\ \oplus\ \mathbb{K}(X)^{\oplus m}\ \oplus\ A \xrightarrow{(b_n,\ H(\oplus_{i=1}^m\mathcal{O}(-i)),\ a)} \mathbb{G}W^{[r]}(\mathbb{A}^n_X)$$

where $m = \lfloor \frac{n-1}{2} \rfloor$ and $a : A \to \mathbb{G}W^{[r]}(\mathbb{A}^n_X)$ is either $b_0 \circ H : \mathbb{K}(X) \to \mathbb{G}W^{[r]}(\mathbb{A}^n_X)$ or $b_0 : \mathbb{G}W^{[r]}(X) \to \mathbb{G}W^{[r]}(\mathbb{A}^n_X)$ depending on the parity of n. Since $\mathcal{O}(i)$ is isomorphic to b_0 over $\mathbb{A}^n_X$, this map is equal to $(b_n, b_0 \circ H, ..., b_0 \circ H, a)$. In other words, the map ε factors through

$$\mathbb{G}W^{[r-n]}(X) \ \oplus \mathbb{G}W^{[r]}(X)^{\oplus m} \ \oplus \mathbb{G}W^{[r]}(X) \xrightarrow{(b_n,\ b_0^m,\ b_0)} \mathbb{G}W^{[r]}(\mathbb{A}^n_X).$$

Changing basis and using the fact that $b_n = 0 \in \mathbb{G}W_0^{[n]}(\mathbb{A}^n)$, this map is isomorphic to

$$\mathbb{G}W^{[r-n]}(X) \ \oplus \mathbb{G}W^{[r]}(X)^{\oplus m} \ \oplus \mathbb{G}W^{[r]}(X) \xrightarrow{(0,\ 0,\ b_0)} \mathbb{G}W^{[r]}(\mathbb{A}^n_X).$$

In other words, the map ε factors through $b_0 : \mathbb{G}W^{[r]}(X) \to \mathbb{G}W^{[r]}(\mathbb{A}^n_X)$. Since the composition

$$\mathbb{G}W^{[r]}(X) \to \mathbb{G}W^{[r]}(\mathbb{A}^n_X) \to \mathbb{G}W^{[r]}(X \times C_n)$$

is split injective and the composition of the lower two horizontal arrows in (5.1.1) is zero, it follows that ε is null-homotopic functorially in X, and we obtain the functorial direct sum decomposition

$$\mathbb{G}W^{[r]}(X \times C_n) \simeq \mathbb{G}W^{[r]}(\mathbb{A}^n_X) \ \oplus \ S^1 \wedge \mathbb{G}W^{[r]}(\mathbb{P}^n_X \text{ on } z_X; \mathcal{L})$$

As before, the composition $0 \to \mathbb{A}^n \to \mathrm{pt}$ is an isomorphism and induces the direct sum decomposition

$$\mathbb{G}W^{[r]}(\mathbb{A}^n_X) = \mathbb{G}W^{[r]}(X) \oplus \mathbb{G}W^{[r]}(\mathbb{A}^n_X, X).$$

To decompose the other direct summand we use the analogue of diagram (4.1.3) which is:

$$\begin{tikzcd}
\mathbb{G}W^{[r-n]}(X) \arrow[r, "1"] \arrow[d, "\otimes b_n"'] & \mathbb{G}W^{[r-n]}(X) \arrow[r] \arrow[d, "\otimes b_n"] & 0 \arrow[d] \\
\mathbb{G}W^{[r]}(\mathbb{P}^n_X \text{ on } z_X; \mathcal{L}) \arrow[r] \arrow[d] & \mathbb{G}W^{[r]}(\mathbb{P}^n_X; \mathcal{L}) \arrow[r] \arrow[d, "j^*"] & \mathbb{G}W^{[r]}(\mathbb{P}^n_X - z_X; \mathcal{L}) \arrow[d, "j^*"] \\
0 \arrow[r] & \mathbb{G}W^{[r]}(\mathbb{P}^{n-1}_X; \mathcal{L}') \arrow[r, "1"] & \mathbb{G}W^{[r]}(\mathbb{P}^{n-1}_X; \mathcal{L}').
\end{tikzcd}$$

The rows are homotopy fibrations and the middle column is split exact, by Theorem A.1. It follows that $\mathbb{G}W^{[r-n]}(X)$ is a direct factor of $\mathbb{G}W^{[r]}(\mathbb{P}^n_X \text{ on } z_X;\mathcal{L})$ with complement equivalent to

$$\Omega\mathbb{G}W^{[r]}(\mathbb{P}^n_X - z_X, \mathbb{P}^{n-1}_X; \mathcal{L}) \cong \Omega\mathbb{G}W^{[r]}(V(1), \mathbb{P}^{n-1}_X; \mathcal{L}'),$$

since the restrictions of $\mathcal{L}$ and $\mathcal{L}'$ along the embedding $\mathbb{P}^n - z \subset \mathbb{P}^n$ and the projection $(\mathbb{P}^n - z) \to \mathbb{P}^{n-1}$ are isomorphic for $n \geq 1$. $\square$

When X is regular, the last two terms in Theorem 5.1 vanish. In this case, Theorem 5.1 gives the exact computation of $\mathbb{G}W^{[r]}(X \times C_n)$ and hence of $W^{[r]}(X \times C_n)$; the latter recovers a result of Balmer–Gille, cf. [3].

Corollary 5.2. *For all integers r, n with $n \geq 1$ and every noetherian regular separated scheme X over $\mathbb{Z}[1/2]$, there is an equivalence of spectra*

$$\mathbb{G}W^{[r]}(X) \oplus\ S^1 \wedge \mathbb{G}W^{[r-n]}(X) \xrightarrow{\sim} \mathbb{G}W^{[r]}(X \times C_n).$$

Proof. Recall that a noetherian regular separated scheme has an ample family of line bundles. The corollary follows from Theorem 5.1 since $\mathbb{G}W$ is homotopy invariant on such schemes [11, Thm. 9.8]. $\square$

Recall that the L-theory spectrum $L^{[r]}(X)$ and the stabilized L-theory spectrum $\text{Ł}^{[r]}(X)$ are obtained from $GW^{[r+*]}(X)$ and $\mathbb{G}W^{[r+*]}(X)$ by inverting the element

$$\eta \in GW^{[-1]}_{-1}(\mathbb{Z}[1/2]) = \mathbb{G}W^{[-1]}_{-1}(\mathbb{Z}[1/2]) = W(\mathbb{Z}[1/2])$$

corresponding to $\langle 1 \rangle \in W(\mathbb{Z}[1/2])$. See [11, Definitions 7.1 and 8.12].

Remark 5.3. All maps in Theorem 5.1 are $GW^{[*]}(\mathbb{Z}[1/2])$-module maps. Therefore, the map on the second factor,

$$S^1 \wedge \mathbb{G}W^{[r-n]}(X) \longrightarrow \mathbb{G}W^{[r]}(X \times C_n)$$

is the cup product with an element

$$\tilde{b}_n \in GW^{[n]}_1(\operatorname{Spec}(\mathbb{Z}[1/2]) \times C_n) = \mathbb{G}W^{[n]}_1(\operatorname{Spec}(\mathbb{Z}[1/2]) \times C_n).$$

Inverting η therefore yields canonical maps

$$(1, \tilde{b}_n) : L^{[r]}(X) \oplus\ S^1 \wedge L^{[r-n]}(X) \to L^{[r]}(X \times C_n),$$
$$(1, \tilde{b}_n) : \text{Ł}^{[r]}(X) \oplus\ S^1 \wedge \text{Ł}^{[r-n]}(X) \to \text{Ł}^{[r]}(X \times C_n).$$

Theorem 5.4. *Let X be a scheme over $\mathbb{Z}[1/2]$ with an ample family of line bundles. Then the following map is an equivalence of spectra*

$$(1, \tilde{b}_n) : \text{Ł}^{[r]}(X)\ \oplus\ S^1 \wedge \text{Ł}^{[r-n]}(X) \xrightarrow{\sim} \text{Ł}^{[r]}(X \times C_n)$$

Proof. This follows from the $\mathbb{G}W$ formula in Theorem 5.1 by inverting the element η of $GW^{-1}_{-1}(\mathbb{Z}[1/2])$ and noting that

$$\text{Ł}^{[r]}(V(1), \mathbb{P}^{n-1}_X; \mathcal{O}(1-n)) = \text{Ł}^{[r]}(\mathbb{A}^n_X, X) = 0$$

by homotopy invariance of Ł (Lemma 2.1). $\square$

We can now deduce Theorem 1.2 from Theorem 5.4. Recall that $\mathbb{K}^{[r]}(X)$ denotes the spectrum $\mathbb{K}(X)$ endowed with the involution obtained using duality with $\mathcal{O}_X[r]$.

Proof of Theorem 1.2. The proof of Theorem 5.1 shows that the equivalence in Theorem 4.1 is $\mathbb{Z}_2$-equivariant. In other words, the spectrum $\mathbb{K}^{[r]}(X \times C_n)$ with $\mathbb{Z}_2$-action is equivalent to

$$\mathbb{K}^{[r]}(X) \oplus \; S^1 \wedge \mathbb{K}^{[r-n]}(X) \oplus \mathbb{K}^{[r]}(V(1), \mathbb{P}^{n-1}_X; \mathcal{O}(1-n)) \oplus \mathbb{K}^{[r]}(\mathbb{A}^n_X, X).$$

We saw in Lemma 4.3 that the Tate spectrum $\widehat{\mathbf{H}}$ of the last two summands are trivial. Hence the map

$$\mathbb{K}^{[r]}_{<0}(X) \oplus (S^1 \wedge \mathbb{K}^{[r-n]}_{<0}(X)) \longrightarrow \mathbb{K}^{[r]}_{<0}(X \times C_n)$$

is an equivalence after applying $\widehat{\mathbf{H}}$. Suppressing X, consider the map of homotopy fibrations of spectra:

$$\begin{tikzcd}
L^{[r]} \oplus S^1 \wedge L^{[r-n]} \arrow[r] \arrow[d] & \text{Ł}^{[r]} \oplus S^1 \wedge \text{Ł}^{[r-n]} \arrow[r] \arrow[d, "\simeq"] & \widehat{\mathbf{H}}(\mathbb{K}^{[r]}_{<0}) \oplus S^1 \wedge \widehat{\mathbf{H}}(\mathbb{K}^{[r-n]}_{<0}) \arrow[d] \\
L^{[r]}(C_n) \arrow[r] & \text{Ł}^{[r]}(C_n) \arrow[r] & \widehat{\mathbf{H}}(\mathbb{K}^{[r]}_{<0}(C_n)).
\end{tikzcd}$$

The homotopy fiber of the right vertical map is

$$S^1 \wedge \widehat{\mathbf{H}}(S^{-1} \wedge K^{[r-n]}_{-1}) = \widehat{\mathbf{H}}(K^{[r-n]}_{-1}) = S^{r-n} \wedge \widehat{\mathbf{H}}(K_{-1}).$$

Hence the homotopy fiber of the left vertical map is $S^{r-n-1} \wedge \widehat{\mathbf{H}}(K_{-1})$. The statement in the theorem is the case $r = 0$. ☐

6. The Witt groups of a node and its Tate circle

To give an explicit example where $W(R[t, 1/t]) \neq W(R) \oplus W(R)$, i.e., where the conclusion of Theorem 1.2(ii) fails for $X = \mathrm{Spec}(R)$, we consider the coordinate ring of a nodal curve over a field of characteristic not 2.

The following lemma applies to the coordinate ring R of any curve, as it is well known that $K_n(R) = 0$ for $n \leq -2$. (See [16, Ex. III.4.4] for example.)

Lemma 6.1. *If R is a $\mathbb{Z}[1/2]$-algebra with $K_i(R) = 0$ for $i \leq -2$, then*

$$L^{[0]}(R[t, 1/t]) \cong L^{[0]}(R) \oplus \text{Ł}^{[0]}(R).$$

Proof. The assumption that $K_i(R) = 0$ for $i \leq -2$ implies that

$$K_{<0}(R[t, 1/t]) \cong K_{<0}(R) \oplus NK_{<0}(R) \oplus NK_{<0}(R).$$

Now $\widehat{\mathbf{H}}(NK_{<0}) = 0$, because the homotopy groups of $NK_{<0}(R)$ are 2-divisible (by [15]). Hence $\widehat{\mathbf{H}}(K_{<0}R) \simeq \widehat{\mathbf{H}}(K_{<0}R[t, \frac{1}{t}])$ The lemma now follows from the following diagram, whose rows and columns are fibrations by Lemma 1.6 and Theorem 5.4, and whose first two columns are split (by $t \mapsto 1$).

$$\begin{tikzcd}
L(R) \arrow[r] \arrow[d, "\text{split}"] & Ł(R) \arrow[r] \arrow[d, "\text{split}"] & \widehat{\mathbf{H}}(K_{<0}R) \arrow[d, "\simeq"] \\
L(R[t,\frac{1}{t}]) \arrow[r] \arrow[d] & Ł(R[t,\frac{1}{t}]) \arrow[r] \arrow[d] & \widehat{\mathbf{H}}(K_{<0}(R[t,\frac{1}{t}])) \arrow[d] \\
\text{cofiber} \arrow[r, "\simeq"] & Ł(R) \arrow[r] & 0.
\end{tikzcd}$$

∎

In what follows, R will denote the node ring ($y^2 = x^3 - x^2$) over a field k of characteristic $\neq 2$.

If F is any homotopy invariant functor from k-algebras to spectra satisfying excision, the usual Mayer-Vietoris argument for $R \subset k[t]$ yields $F(R) \simeq F(k) \oplus \Omega F(k)$; see [16, III.4.3]. In particular,

(6.2) $Ł(R) \simeq Ł(k) \oplus \Omega\, Ł(k)$, and $KH(R) \simeq KH(k) \oplus \Omega\, KH(k)$.

Since $KH_{<0}(k) = 0$, $KH_0(k) = \mathbb{Z}$ we have $K_{<0}(R) \simeq KH_{<0}(R) \simeq S^{-1} \wedge \mathbb{Z}$. It follows that $\pi_i \widehat{\mathbf{H}}(K_{<0}(R)) = \widehat{\mathrm{H}}^{i+1}(\mathbb{Z})$.

Remark 6.3. It is well known that $W^n(k) = 0$ for $n \not\equiv 0 \pmod 4$; the case $W^2(k) = 0$ (symplectic forms) is classical; a proof that $W^1(k) = W^3(k) = 0$ is given in [1, Thm. 5.6], although the result was probably known to Ranicki and Wall. Since $L(k) \simeq Ł(k)$, $Ł_n(R) = Ł^{-n}(R)$ is: $W(k)$ for $n \equiv 0, 3 \pmod 4$, and 0 otherwise.

Recall that the fundamental ideal $I(k)$ is the kernel of the (surjective) rank map $W(k) \to \mathbb{Z}/2$.

Lemma 6.4. *When R is the node, $W(R) \cong W(k) \oplus \mathbb{Z}/2$. In addition, $W^1(R) \cong I(k)$ and $W^2(R) = W^3(R) = 0$.*

Proof. Since $\widehat{\mathrm{H}}^0(\mathbb{Z}_2, \mathbb{Z}) = \mathbb{Z}/2$ and $\widehat{\mathrm{H}}^1(\mathbb{Z}_2, \mathbb{Z}) = 0$, Lemma 1.6 and Remark 6.3 yield the exact sequences:

$$0 \to L_3(R) \to Ł_3(R) \to \widehat{\mathrm{H}}^0(\mathbb{Z}_2, \mathbb{Z}) \to L_2(R) \to Ł_2(R) \to 0,$$
$$0 \to L_1(R) \to Ł_1(R) \to \widehat{\mathrm{H}}^0(\mathbb{Z}_2, \mathbb{Z}) \to L_0(R) \to Ł_0(R) \to 0.$$

Now the map $Ł_3(R) \cong W^0(k) \to \widehat{\mathrm{H}}^0(\mathbb{Z}_2, \mathbb{Z}) \cong \mathbb{Z}/2$ is the rank map $W(k) \to \mathbb{Z}/2$; it follows that $L_3(R) \cong I(k)$ and $L_2(R) = 0$. Since $Ł_1(R) = 0$, the second sequence immediately yields $W^3(R) = L_1(R) = 0$. Finally, the decomposition $W(R) \cong W(k) \oplus \mathbb{Z}/2$ follows because the map $L_0(R) \to Ł_0(R) \cong L_0(k)$ is a surjection, split by the natural map $L_0(k) \to L_0(R)$. ∎

Example 6.5. By Lemma 6.4, $W(R) \cong W(k) \oplus \mathbb{Z}/2$. Lemma 6.1 yields:

$$W^0(R[t, 1/t]) \cong W^0(R) \oplus \text{Ł}^0(R) \cong W(k) \oplus \mathbb{Z}/2 \oplus W(k).$$

In addition, $W^1(R) \cong I(k)$ but $W^1(R[t, 1/t]) \cong I(k) \oplus W(k)$.

7. Higher Witt groups

Recall that the *higher Witt group* $W_i(X)$ is defined to be the cokernel of the hyperbolic map $\mathbb{K}_i(X) \to \mathbb{G}W_i(X)$; see [6]. More generally, we can consider the cokernel $W_i^{[r]}(X)$ of $\mathbb{K}_i(X) \to \mathbb{G}W_i^{[r]}(X)$. Similarly, one can define the *higher coWitt group* $W'^{[r]}_i(X)$ to be the kernel of the forgetful map $\mathbb{G}W_i^{[r]}(X) \to \mathbb{K}_i(X)$. In this section we show that $W_i^{[r]}(X)$ and $W'^{[r]}_i(X)$ are homotopy invariant and we compute their values on $X \times C_n$.

We begin with an observation. As usual, for any functor F from schemes to spectra or groups, and any vector bundle $V \to X$, we write $F(V, X)$ for the cofiber (or cokernel) of $F(X) \to F(V)$.

Recall that the homotopy groups $K_i(V, X)$ are uniquely 2-divisible. Writing $K_i^{[r]}(V, X)$ for these groups, endowed with the involution arising from duality with $\mathcal{O}[r]$, we have a natural decomposition of $K_i^{[r]}(V, X)$ as the direct sum of its symmetric part $K_i^{[r]}(V, X)_+ = \pi_i(K(V, X)^{[r]}_{h\mathbb{Z}_2})$ and its skew-symmetric part $K_i^{[r]}(V, X)_-$.

Lemma 7.1. *For every vector bundle V over X, and for all i and r, $K_i^{[r]}(V, X)_+ \cong \mathbb{G}W_i^{[r]}(V, X)$.*

Proof. There is a fibration $\mathbb{K}(V, X)^{[r]}_{h\mathbb{Z}_2} \to \mathbb{G}W^{[r]}(V, X) \to \text{Ł}^{[r]}(V, X)$; see (1.5) and [11, 8.13]. Since we proved in Lemma 2.1 that $\text{Ł}^{[r]}(V, X) = 0$, we get an isomorphism of spectra $\mathbb{K}(V, X)^{[r]}_{h\mathbb{Z}_2} \xrightarrow{\simeq} \mathbb{G}W^{[r]}(V, X)$ and hence group isomorphisms $K_i^{[r]}(V, X)_+ \cong \mathbb{G}W_i^{[r]}(V, X)$. ☐

Theorem 7.2. *Let X be a scheme over $\mathbb{Z}[1/2]$ with an ample family of line bundles. The higher Witt and coWitt groups are homotopy invariant in the sense that for every vector bundle V over X, the projection $V \to X$ induces isomorphisms of higher Witt and coWitt groups:*

$$W_i^{[r]}(V) \cong W_i^{[r]}(X) \quad \text{and} \quad W'^{[r]}_i(V) \cong W'^{[r]}_i(X).$$

Proof. The hyperbolic map H is a surjection, as it factors:

$$K_i(V, X) \xrightarrow{\text{onto}} K_i^{[r]}(V, X)_+ \xrightarrow{\simeq} \mathbb{G}W_i^{[r]}(V, X).$$

Hence the cokernel $W_i^{[r]}(V,X)$ is zero. Similarly, the forgetful functor $\mathbb{G}W^{[r]}(V,X) \longrightarrow \mathbb{K}(V,X)^{[r]}$ factors as the equivalence $\mathbb{G}W^{[r]}(V,X) \simeq \mathbb{K}(V,X)^{[r]}_{h\mathbb{Z}_2}$ followed by the canonical map $\mathbb{K}(V,X)^{[r]}_{h\mathbb{Z}_2} \to \mathbb{K}(V,X)^{[r]}$. On homotopy groups, $\mathbb{G}W_i^{[r]}(V,X) \cong K_i^{[r]}(V,X)_+ \to K_i^{[r]}(V,X)$ is an inclusion, so the kernel $W_i'^{[r]}(V,X)$ is zero. □

A similar argument applies to $W_*(X \times C_n)$.

Theorem 7.3. *$W_i^{[r]}(X \times C_n) \cong W_i^{[r]}(X) \oplus W_{i-1}^{[r-n]}(X)$, and $W_i'^{[r]}(X \times C_n) \cong W_i'^{[r]}(X) \oplus W_{i-1}'^{[r-n]}(X)$.*

Proof. As we saw in Sections 4 and 5, the hyperbolic map

$$H : \mathbb{K}(X \times C_n) \longrightarrow \mathbb{G}W^{[r]}(X \times C_n)$$

is the sum of four maps. On homotopy groups, the cokernel $W_i^{[r]}(X \times C_n)$ of H is the sum of the corresponding cokernels. The first two cokernels are $W_i^{[r]}(X)$ and $W_{i-1}^{[r-n]}(X)$, while the last two are zero by Theorem 7.2.

A similar argument applies to the coWitt groups, which are the kernels of the map F. □

Appendix A. Grothendieck–Witt groups of $\mathbb{P}^n_X$

Marco Schlichting

The goal of this appendix is to prove Theorem A.1 which was used in the proof of Theorem 5.1. As a byproduct we obtain a computation of the $\mathbb{G}W^{[r]}$-spectrum of the projective space $\mathbb{P}^n_X$ over X. The π_0-versions are due to Walter [14], and the methods of *loc.cit.* could be adapted to give a proof of Theorem A.1. Here we will give a more direct proof. Using similar methods, a more general treatment of the Hermitian K-theory of projective bundles will appear in [9].

Recall from (4.1.2) and (5.1.2) that there is a strictly perfect complex b_n on $\mathbb{P}^n$ equipped with a symmetric form $b_n \otimes b_n \to \mathcal{L}[n]$, whose adjoint is a quasi-isomorphism; here $\mathcal{L}$ is the line bundle $\mathcal{O}(1-n)$ on $\mathbb{P}^n$. Let $j : \mathbb{P}^{n-1} \to \mathbb{P}^n$ denote the closed embedding as the vanishing locus of T_0.

Theorem A.1. *Let X be a scheme over $\mathbb{Z}[1/2]$ with an ample family of line bundles, and let $n \geq 1$ be an integer. Then the following sequence of Karoubi-Grothendieck–Witt spectra is a split fibration for all $r \in \mathbb{Z}$:*

$$\mathbb{G}W^{[r-n]}(X) \xrightarrow{\otimes b_n} \mathbb{G}W^{[r]}(\mathbb{P}^n_X, \mathcal{L}) \xrightarrow{j^*} \mathbb{G}W^{[r]}(\mathbb{P}^{n-1}_X, j^*\mathcal{L}).$$

The proof will use the following slight generalization of [11, Prop. 8.15] ("Additivity for $\mathbb{G}W$"), which was already used in the proof of the blow-up formula for $\mathbb{G}W$ in [11, Thm. 9.9]. If $(\mathcal{A}, w, \vee)$ is a dg category with weak equivalences and duality, we write $\mathcal{T}\mathcal{A}$ for $w^{-1}\mathcal{A}$, the associated triangulated category with duality obtained from $\mathcal{A}$ by formally inverting the weak equivalences; see [11, §1]. The associated hyperbolic category is $\mathcal{H}\mathcal{A} = \mathcal{A} \times \mathcal{A}^{op}$, and $\mathbb{G}W(\mathcal{H}\mathcal{A}, w \times w^{op}) \cong \mathbb{K}(\mathcal{A}, w)$; see [11, 4.7].

Lemma A.2. *Let $(\mathcal{U}, w, \vee)$ be a pretriangulated dg category with weak equivalences and duality such that $\frac{1}{2} \in \mathcal{U}$. Let $\mathcal{A}$ and $\mathcal{B}$ be full pretriangulated dg subcategories of $\mathcal{U}$ containing the w-acyclic objects of $\mathcal{U}$. Assume that: (i) $\mathcal{B}^\vee = \mathcal{B}$;*
(ii) $\mathcal{T}\mathcal{U}(X, Y) = 0$ for all (X, Y) in $\mathcal{A} \times \mathcal{B}$, $\mathcal{B} \times \mathcal{A}^\vee$ or $\mathcal{A} \times \mathcal{A}^\vee$; and
(iii) $\mathcal{T}\mathcal{U}$ is generated as a triangulated category by $\mathcal{T}\mathcal{A}$, $\mathcal{T}\mathcal{B}$ and $\mathcal{T}\mathcal{A}^\vee$. Then the exact dg form functor

$$\mathcal{B} \times \mathcal{H}\mathcal{A} \to \mathcal{U} : X, (Y, Z) \mapsto X \oplus Y \oplus Z^\vee$$

induces a stable equivalence of Karoubi–Grothendieck–Witt spectra:

$$\mathbb{G}W(\mathcal{B}, w) \times \mathbb{K}(\mathcal{A}, w) = \mathbb{G}W(\mathcal{B}, w) \times \mathbb{G}W(\mathcal{H}\mathcal{A}, w \times w^{op}) \xrightarrow{\sim} \mathbb{G}W(\mathcal{U}, w).$$

Proof. Let v be the class of maps in $\mathcal{U}$ which are isomorphisms in $\mathcal{TU}/\mathcal{TB}$. Then the sequence $(\mathcal{B}, w) \to (\mathcal{U}, w) \to (\mathcal{U}, v)$ induces a fibration of Grothendieck–Witt spectra

$$\mathbb{G}W(\mathcal{B}, w) \to \mathbb{G}W(\mathcal{U}, w) \to \mathbb{G}W(\mathcal{U}, v)$$

by the Localization Theorem [11, Thm. 8.10]. Let $\mathcal{A}' \subset \mathcal{U}$ be the full dg subcategory whose objects lie in the triangulated subcategory of $\mathcal{TU}$ generated by $\mathcal{TA}$ and $\mathcal{TB}$. By Additivity for $\mathbb{G}W$ [11, Prop. 8.15], the inclusion $(\mathcal{A}', v) \subset (\mathcal{U}, v)$ induces an equivalence of spectra $\mathbb{K}(\mathcal{A}', v) \simeq \mathbb{G}W(\mathcal{U}, v)$. Finally, the map $(\mathcal{A}, w) \to (\mathcal{A}', v)$ induces an equivalence of associated triangulated categories and thus a $\mathbb{K}$-theory equivalence: $\mathbb{K}(\mathcal{A}, w) \xrightarrow{\simeq} \mathbb{K}(\mathcal{A}', v) \simeq \mathbb{G}W(\mathcal{U}, v)$. The result follows. $\square$

For the proof of Theorem A.1, we shall need some notation. Recall that $\mathrm{sPerf}(X)$ is the dg category of strictly perfect complexes on X, and w is the class of quasi-isomorphisms; the localization $w^{-1}\,\mathrm{sPerf}(X)$ is the triangulated category $D^b\,\mathrm{Vect}(X)$.

Now consider the structure map $p\colon \mathbb{P}^m_X \to X$ for $m = n, n-1$, and $\mathcal{L} = \mathcal{O}(1-n)$. Recall that $D^b\,\mathrm{Vect}(\mathbb{P}^m_X)$ has a semi-orthongonal decomposition with pieces $\mathcal{O}(i) \otimes p^* D^b\,\mathrm{Vect}(X)$, $i = 0, -1, ..., -m$. Let $\mathcal{U}$ be the full dg subcategory of $\mathrm{sPerf}(\mathbb{P}^n_X)$ on the strictly perfect complexes on $\mathbb{P}^n_X$ lying in the full triangulated subcategory of $D^b\,\mathrm{Vect}(\mathbb{P}^n_X)$ generated by $\mathcal{O}(i) \otimes p^* D^b\,\mathrm{Vect}(X)$ for $i = 0, -1, ..., 1-n$. Note that $\mathcal{U}$ is closed under the duality $\vee$ with values in $\mathcal{L}$. Finally, let v denote the class of maps in $\mathrm{sPerf}(\mathbb{P}^n_X)$ which are isomorphisms in $D^b\,\mathrm{Vect}(\mathbb{P}^n_X)/\mathcal{TU}$.

Proof of Theorem A.1. Consider the following commutative diagram of Karoubi–Grothendieck–Witt spectra:

$$\begin{tikzcd}
 & \mathbb{G}W^{[r-n]}(X) \arrow[d, "\otimes b_n"] \arrow[dr, "\simeq"] & \\
\mathbb{G}W^{[r]}(\mathcal{U}, w, \vee) \arrow[r] \arrow[dr, "\simeq"'] & \mathbb{G}W^{[r]}(\mathbb{P}^n_X, \mathcal{L}) \arrow[r] \arrow[d, "j^*"] & \mathbb{G}W^{[r]}(\mathrm{sPerf}\,\mathbb{P}^n_X, v, \vee) \\
 & \mathbb{G}W^{[r]}(\mathbb{P}^{n-1}_X, j^*\mathcal{L}). &
\end{tikzcd}$$

The middle row is a homotopy fibration by Localization [11, Thm. 8.10]. The upper right diagonal arrow is a weak equivalence, because the standard semi-orthogonal decomposition on $\mathbb{P}^n_X$ yields an equivalence of triangulated categories $\otimes b_n : D^b\,\mathrm{Vect}(X) \xrightarrow{\simeq} D^b\,\mathrm{Vect}(\mathbb{P}^n_X)/\mathcal{TU}$. Finally, the lower left diagonal arrow is a weak equivalence by Lemma A.2, where we choose the full dg subcategories $\mathcal{A}$, $\mathcal{A}'$ and $\mathcal{B}$, $\mathcal{B}'$ of $\mathcal{U}$ and

$\mathcal{U}' = \mathrm{sPerf}(\mathbb{P}^{n-1}_X)$ as follows. They are determined by their associated triangulated categories.

When $n = 2m+1$ is odd, we let $\mathcal{TA} \subset \mathcal{TU}$, respectively $\mathcal{TA}' \subset \mathcal{TU}'$, be the triangulated subcategories generated by

$$\mathcal{O}(-2m) \otimes p^* D^b \operatorname{Vect}(X), ..., \mathcal{O}(-m-1) \otimes p^* D^b \operatorname{Vect}(X)$$

and we let $\mathcal{TB}$, resp. $\mathcal{TB}'$, be the subcategory $\mathcal{O}(-m) \otimes p^* D^b \operatorname{Vect}(X)$. By Lemma A.2, $\mathbb{G}W(\mathcal{U}, w, \vee)$ and $\mathbb{G}W(\mathcal{U}', w, \vee)$ are both equivalent to $\mathbb{G}W^{[r]}(X) \oplus \mathbb{K}(X)^{\oplus m}$. In particular,

$$\mathbb{G}W^{[r]}(\mathbb{P}^{2m}_X; \mathcal{O}(-2m)) \simeq \mathbb{G}W^{[r]}(X) \oplus \mathbb{K}(X)^{\oplus m}. \tag{A.3}$$

When $n = 2m$ is even, we let $\mathcal{TA} \subset \mathcal{TU}$, respectively $\mathcal{TA}' \subset \mathcal{TU}'$, be the triangulated subcategories generated by

$$\mathcal{O}(-2m) \otimes p^* D^b \operatorname{Vect}(X), ..., \mathcal{O}(-m-1) \otimes p^* D^b \operatorname{Vect}(X),$$

and $\mathcal{B} = \mathcal{B}' = 0$. In this case, $\mathbb{G}W(\mathcal{U}, w, \vee)$ and $\mathbb{G}W(\mathcal{U}', w, \vee)$ are both equivalent to $\mathbb{K}(X)^{\oplus m}$, by Lemma A.2. In particular,

$$\mathbb{G}W^{[r]}(\mathbb{P}^{2m-1}_X; \mathcal{O}(1-2m)) \simeq \mathbb{K}(X)^{\oplus m}. \tag{A.4}$$

□

As a result, we obtain a spectrum level version of some of Walter's calculations [14]:

Corollary A.5. *Let X be a scheme over $\mathbb{Z}[1/2]$ with an ample family of line bundles. For all integers r, n, i with $n \geq 0$, the Karoubi–Grothendieck–Witt spectrum $\mathbb{G}W^{[r]}(\mathbb{P}^n_X; \mathcal{O}(i))$ is equivalent to*

$$\begin{array}{ll} \mathbb{G}W^{[r]}(X) \oplus \mathbb{K}(X)^{\oplus m} & n = 2m,\ i \text{ even}, \\ \mathbb{K}(X)^{\oplus m} & n = 2m-1,\ i \text{ odd}, \\ \mathbb{G}W^{[r]}(X) \oplus \mathbb{K}(X)^{\oplus m} \oplus \mathbb{G}W^{[r-n]}(X) & n = 2m+1,\ i \text{ even}, \\ \mathbb{K}(X)^{\oplus m} \oplus \mathbb{G}W^{[r-n]}(X) & n = 2m,\ i \text{ odd}. \end{array}$$

Proof. The line bundle $\mathcal{O}(1)$ is canonically equipped with the non-degenerate symmetric bilinear form $\mathcal{O}(1) \otimes \mathcal{O}(1) \xrightarrow{=} \mathcal{O}(2)$ with values in $\mathcal{O}(2)$. Cup product with that form induces equivalences of Karoubi–Grothendieck–Witt spectra

$$\mathbb{G}W^{[r]}(\mathbb{P}^n_X, \mathcal{O}(i)) \simeq \mathbb{G}W^{[r]}(\mathbb{P}^n_X, \mathcal{O}(i+2)).$$

In particular, $\mathbb{G}W^{[r]}(\mathbb{P}^n_X, \mathcal{O}(i))$ only depends on the parity of i. Now, the first two computations of the corollary were already mentioned in the proof of Theorem A.1; see (A.3) and (A.4). The last two computations follow from those together with the statement of Theorem A.1. □

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