Numerical Analysis Of Degenerate Kolmogorov Equations of Constrained Stochastic Hamiltonian Systems

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Abstract

In this work, we propose a method to compute numerical approximations of the invariant measures and Rice's formula (frequency of threshold crossings) for a certain type of stochastic Hamiltonian system constrained by an obstacle and subjected to white or colored noise. As an alternative to probabilistic Monte-Carlo simulations, our approach relies on solving a class of degenerate partial differential equations with non-local Dirichlet boundary conditions, as derived in [Mertz, Stadler, Wylie; 2018]. A functional analysis framework is presented; regularisation and approximation by the finite element method is applied; numerical experiments on these are performed and show good agreement with probabilistic simulations.

Keywords: Constrained stochastic Hamiltonian system, Rice's formula, Partial differential equations, nonlocal boundary conditions.

1. Introduction

The wikipedia page `en.wikipedia.org/wiki/Rice%27s_formula` mentions Rice's formula as "one of the most important results in the applications of smooth stochastic processes" for engineering. Rice's formula [9] is indeed a very powerful tool to compute the frequency of threshold crossing for processes in the class of stochastic hamiltonian systems,

$$\dot{Y}_t + F(X_t, Y_t)\partial_y H(X_t, Y_t) + \partial_x H(X_t, Y_t) = Z_t, \quad \dot{X}_t = \partial_y H(X_t, Y_t). \qquad (\mathcal{SHS})$$

Here $(X, Y) \in \mathbb{R}^2$, H, the Hamiltonian, and F, are smooth functions in the sense of [11]. We consider two types of noise :1) Z_t, white noise where $Z_t = \sigma\dot{W}$ with $\sigma \in \mathbb{R}^+$ and W is a real-valued Wiener process and 2) "colored noise", i.e. an Ornstein-Uhlenbeck process:

$$Z_t = Z_0 \exp(-\alpha t) + \sigma \int_0^t \exp(-\alpha(t-s))dW_s \text{ where } \alpha, \sigma \in \mathbb{R}^+. \qquad (1)$$

Such problems arise in mechanical engineering in the context of earthquakes [3].

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Rice's formula asserts that if (X,Y) has a unique invariant probability density $m(x,y)$ then the frequency $f(s)$ of X crossing a given threshold s is given by $f = f_+ + f_-$, with

$$f_+(s) = \int_0^\infty \partial_y H(s,y) 1_{\{\partial_y H(s,y)\geq 0\}} m(s,y)\mathrm{d}y,\ \ f_-(s) = \int_{-\infty}^0 |\partial_y H(s,y)|\, 1_{\{\partial_y H(s,y)\leq 0\}} m(s,y)\mathrm{d}y. \tag{$\mathcal{R}$}$$

An extension of Rice's formula can be established when the stochastic process is also constrained by an obstacle located at $X = \pm L_x$ in the sense that,

$$\forall t \geq 0,\ |X_t| \leq L_x \text{ and whenever } |X_t| = L_x \text{ then } \dot{X}_{t^+} = -e\dot{X}_{t^-}. \tag{$\mathcal{BC}$}$$

Here $e \in (0,1]$ is called a restitution coefficient. In general, there is no closed form expression for m, except when $e = 1$. Note that existence and uniqueness of the invariant measure, together with asymptotic formulae related to the probabilities of threshold-crossing for smooth approximations of a broad class of mechanical systems under white or colored noise, have been discussed in [5].

2. Kolmogorov equations

We denote by $p(x,y,t;\xi,\eta,s)$ the transition probability density for the ($\mathcal{SHS}$)+($\mathcal{BC}$)-"random walker" (X,Y) to move from $(x,y) \in (-Ł_x, Ł_x)\times\mathbb{R}$ at time t to $(\xi,\eta) \in (-Ł_x, Ł_x)\times\mathbb{R}$ at a later time $s > t$. By definition the density of the invariant measure, denoted m^e on $D = (-1,1)\times\mathbb{R}$ and defined by

$$\mathbb{E}(g(X_t,Y_t)) = \int_D g(\xi,\eta) m_2^e(\xi,\eta) d\xi d\eta,$$

for all continuous functions g and all time t, is related to p by

$$\lim_{s\to\infty} p(x,y,t;\xi,\eta,s) = m_2^e(\xi,\eta).$$

2.1. White noise case

Let

$$B_1(x,y) = -[F(x,y)\partial_y H(x,y) + \partial_x H(x,y)] \quad \text{and} \quad B_2(x,y) = \partial_y H(x,y).$$

From the theory of Markov processes, when the triple (ξ,η,s) is fixed as a parameter, the backward Kolmogorov equation characterises p as the solution of

$$\partial_t p + \frac{\sigma^2}{2}\partial_{yy} p + B_1(x,y)\partial_y p + B_2(x,y)\partial_x p = 0,\ \forall (x,y) \in (-Ł_x, Ł_x)\times\mathbb{R},\ \forall t < s, \tag{$\mathcal{B}$}$$

$$\lim_{t\to s, t<s} p(x,y,t;\xi,\eta,s) = \delta_{(\xi,\eta)},\ \ p(Ł_x,y,t;\xi,\eta,s) = p(Ł_x,-ey,t;\xi,\eta,s),\ \forall y > 0,\ \forall t < s$$

$$\lim_{t\to s, t<s} p(x,y,t;\xi,\eta,s) = \delta_{(\xi,\eta)},\ \ p(-Ł_x,y,t;\xi,\eta,s) = p(-Ł_x,-ey,t;\xi,\eta,s),\ \forall -y > 0,\ \forall t < s.$$

The last 2 equations are symbolically written as

$$\lim_{t\to s, t<s} p(x,y,t;\xi,\eta,s) = \delta_{(\xi,\eta)},\ \ p(\pm Ł_x,y,t;\xi,\eta,s) = p(\pm Ł_x,-ey,t;\xi,\eta,s),\ \forall \pm y > 0,\ \forall t < s,$$

meaning by this that every $\pm$ are assigned to $+$ together or $-$ together.

2.2. Colored noise case (Z is an Ornstein-Uhlenbeck process)

To remain in a markovian framework, the ($\mathcal{SHS}$)+($\mathcal{BC}$)-"random walker" must be extended to (X, Y, Z) with (1). The transition probability density for the "random walker" (X, Y, Z) to move from $(x, y, z) \in (-Ł_x, Ł_x) \times \mathbb{R} \times \mathbb{R}$ at time t to $(\xi, \eta, \zeta) \in (-Ł_x, Ł_x) \times \mathbb{R} \times \mathbb{R}$ at a later time $s > t$ is solution of the backward Kolmogorov equation

$$\partial_t p + \frac{\sigma^2}{2}\partial_{zz} p - \alpha z \partial_z p + (z + B_1(x, y))\,\partial_y p + B_2(x, y)\partial_x p = 0, \text{ in } (-Ł_x, Ł_x) \times \mathbb{R} \times \mathbb{R}, \quad (\mathcal{B}_c)$$

$$\lim_{t \to s, t<s} p(x, y, z, t; \xi, \eta, \zeta, s) = \delta_{(\xi,\eta,\zeta)},$$

$$p(\pm Ł_x, y, z, t; \xi, \eta, \zeta, s) = p(\pm Ł_x, -ey, z, t; \xi, \eta, \zeta, s), \ \forall \pm y > 0, \ \forall t < s.$$

2.3. Computation of the Invariant Probability Measure

To compute any quantity of the form

$$\int_D g(\xi, \eta) m_2^e(\xi, \eta) d\xi d\eta$$

we consider the problem

$$\lambda u^\lambda - \frac{\sigma^2}{2}\partial_{yy} u^\lambda - B_1(x, y)\partial_y u^\lambda - B_2(x, y)\partial_x u^\lambda = g \quad \text{in} \quad D \tag{2}$$

with the boundary condition

$$u^\lambda(\pm 1, y) = u^\lambda(\pm 1, -ey), \ \forall \pm y > 0. \tag{3}$$

This is motivated by the fact that from ergodic theory

$$\forall (x, y), \ \lim_{\lambda \to 0} \lambda u^\lambda(x, y) = \int_D g(\xi, \eta) m_2^{\boldsymbol{e}}(\xi, \eta) d\xi d\eta.$$

Such a problem is easier than the time-dependent problem. Thus in order to approach $f(s)$, in the right-hand side of (2), we will consider $g_s(x, y) = y^\pm \delta(x - s)$, possibly approximated by $\frac{y^\pm}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-s)^2}{2\sigma^2}\right)$, with $\sigma > 0$ small. Here, $y^\pm = \max(0, \pm y)$. Such a numerical method has been used in [2] and [3] for the statistics of an elasto-plastic problem excited by white noise, an altogether different problem.

3. The Problem

The purpose of this section is to study (2),(3) with $F \equiv \gamma \in \mathbb{R}^+$, $H(x, y) = \frac{1}{2}y^2 + \frac{k}{2}x^2$, $k \in \mathbb{R}^+$, in a domain $D = (-1, 1) \times \mathbb{R}$. Thus we consider the problem

$$\lambda u - \frac{\sigma^2}{2}\partial_{yy} u + (\gamma y + kx)\partial_y u - y\partial_x u = g, \tag{4}$$

$$u(1, y) = u(1, -ey) \ \forall y > 0, \quad u(-1, y) = u(-1, -ey) \ \forall y < 0. \tag{5}$$

where g is at least in $L^2(D)$.

Definition 1. *We shall say that u is e-symmetric when (5) holds.*

Remark 1. *Notice that (5) implies also that*

$$u(1,y) = u\left(1, -\frac{y}{e}\right) \ \forall y < 0, \quad u(-1,y) = u\left(-1, -\frac{y}{e}\right) \ \forall y > 0.$$

In (4),(5) the parameters are the variance, $\sigma > 0$, the reflection coefficient e on $\pm X$, the material properties γ and k, and the source term $(x,y) \mapsto g(x,y)$. However by changing $(x,y) \to (\bar{x},\bar{y}) = (Ł_x x, Ł_x y)$, $\bar{\sigma} = \frac{\sigma}{Ł_x}$, the problem becomes (4),(5) with bars, posed on $(-Ł_x, Ł_x) \times \mathbb{R}$ instead of D.

3.1. Asymptotic Behaviour When $y \to \infty$

A frequent case in practice is $g(x,y) = y^n,\ n \geq 1$; but note that as

$$-\frac{\sigma^2}{2}\partial_y\left(\mathrm{e}^{-\frac{1}{\gamma\sigma^2}(\gamma y+kx)^2}\partial_y u\right) = \mathrm{e}^{-\frac{1}{\gamma\sigma^2}(\gamma y+kx)^2}\left(-\frac{\sigma^2}{2}\partial_{yy}u + (\gamma y + kx)\partial_y u\right),$$

the PDE can be rewritten as

$$\mathrm{e}^{-\frac{1}{\gamma\sigma^2}(\gamma y+kx)^2}(\lambda u - y\partial_x u) - \frac{\sigma^2}{2}\partial_y\left(\mathrm{e}^{-\frac{1}{\gamma\sigma^2}(\gamma y+kx)^2}\partial_y u\right) = g\mathrm{e}^{-\frac{1}{\gamma\sigma^2}(\gamma y+kx)^2}. \tag{6}$$

Here we see that a polynomial growth in y of g does not endanger existence provided we use weighted Sobolev spaces with a weight such that g be square-weighted integrable.
Let m be an integer; let

$$\begin{aligned}
\mu_m(y) &= \frac{1}{2}(1+y^2)^{-\frac{m}{2}},\\
\nu_m(y,e) &= \mu_m(y)1_{y>0} + \mu_m(-ey)1_{y\leq 0},\\
\rho_m(x,y) &= (1-x)\nu_m(y,e) + (1+x)\nu_m(y,\frac{1}{e}).
\end{aligned} \tag{7}$$

Denote by $H^1_m(D)$ the Sobolev space of order 1 with weight ρ_m. It has been shown in [1] that a similar problem with different boundary conditions but the same PDE is well-posed in that space.
In any case, for simplicity and with the numerical approximation in mind, we begin with an analysis of the problem localized in the domain of fig.1. Moreover, as the differential operator is not coercive in x we regularize it with a small second-order term.

3.2. The Regularized Problem in a Finite Domain

Let $L_y > 0$ be a large number and let $\mathbb{D}$ be the parallelogram of vertices $(-1,-L_y)$, $(1,-eL_y)$, $(1,L_y)$, $(-1,eL_y)$ (fig.1). Its boundary is denoted by $\partial\mathbb{D}$. The left vertical part is $\partial\mathbb{D}^-$ and the right one $\partial\mathbb{D}^+$; the union of both is denoted by $\partial\mathbb{D}^\pm$.
The part of $\partial\mathbb{D}^\pm$ on which the e-symmetry (5) holds is called Σ:

$$\Sigma = (\partial\mathbb{D}^- \cap \{y \leq 0\}) \cup (\partial\mathbb{D}^+ \cap \{y \geq 0\}).$$

Remark 2. *Notice that the e-symmetry makes sense only if $\mathbb{D}$ is the parallelogram shown in fig.1 with vertices at $(-1,-L_y)$, $(1,-eL_y)$, $(1,L_y)$, $(-1,eL_y)$. The slanted sides need not be straight segments but the vertical sides must have the length ratio of the parallelogram.*

Localization in a finite domain requires an additional boundary condition on the border $\partial\mathbb{D}^\infty$ which approximates infinity. We shall specify a function $\mu(x,y)$ and assume that

$$\frac{\sigma^2}{2}\partial_y u + \mu(x,y)u = 0 \text{ on } \partial\mathbb{D}^\infty. \tag{8}$$

The simplest case is $\mu = 0$ but to be compatible with a polynomial asymptotic behavior of the type $u \sim y^n$ for large y, then (8) implies:

$$\frac{\sigma^2}{2}\partial_y(y^n) + \mu(x,y)y^n = 0 \quad \Rightarrow \quad \mu \sim -\frac{n\sigma^2}{2y}.$$

Notation: $H^1_e(\mathbb{D})$. Consider the linear map $u \in H^1(\mathbb{D}) \mapsto Lu \in H^{\frac{1}{2}}(\Sigma)$ defined by

$$Lu(x,y) = u(x,y) - u(x,-ey), \quad \forall (x,y) \in \Sigma. \tag{9}$$

Let $H^1_e(\mathbb{D})$ be the kernel of L, i.e. the set of $u \in H^1(\mathbb{D})$ such that $u(x,y) - u(x,-ey) = 0$ for all $(x,y) \in \Sigma$, i.e. the subset of $H^1(\mathbb{D})$ of e-periodic functions.
As the trace operator is continuous from $H^1(\mathbb{D})$ to $H^{\frac{1}{2}}(\partial\mathbb{D})$, the kernel of L is a closed subspace of $H^1(\mathbb{D})$. This makes $H^1_e(\mathbb{D})$ a closed linear subspace of $H^1(\mathbb{D})$.

3.3. Regularization

Furthermore, consider the regularized PDE

$$\begin{aligned} &\lambda u^\varepsilon - \varepsilon\partial_{xx}u^\varepsilon - \frac{\sigma^2}{2}\partial_{yy}u^\varepsilon + (\gamma y + kx)\partial_y u^\varepsilon - y\partial_x u^\varepsilon = g, \\ &u(1,y) = u(1,-ey)\ \forall y > 0, \quad u(-1,y) = u(-1,-ey)\ \forall y < 0. \end{aligned} \tag{10}$$

with e-compatible Neumann boundary conditions on $\partial\mathbb{D}^\pm$, the vertical left and right sides of $\mathbb{D}$. Here $\partial\mathbb{D}^\pm = \partial\mathbb{D}^- \cup \partial\mathbb{D}^+$. More precisely, u^ϵ is the solution in $H^1_e(\mathbb{D})$ of the variational equation (14), below, which contains the e-compatible Neumann conditions (11).

3.4. Interpretation of the e-Compatible Neumann Conditions

In a variational setting, a Neumann condition on $\partial\mathbb{D}^\pm$ for (10) with e-symmetry means

$$\int_{\partial\mathbb{D}^\pm} v\partial_x u^\varepsilon = 0 \text{ for all } v \in H^1_e(\mathbb{D}). \tag{11}$$

So on $\partial\mathbb{D}^+$, for instance,

$$\int_{-eL_y}^{L_y} (v\partial_x u^\varepsilon)_{|x=1} = \int_{-eL_y}^{0} (v\partial_x u^\varepsilon)_{|x=1} + \int_0^{L_y} (v\partial_x u^\varepsilon)_{|x=1}$$

$$= \int_{-eL_y}^{0} (v(1,y)\partial_x u^\varepsilon(1,y))dy + \frac{1}{e}\int_{-eL_y}^{0} v(1,z)\partial_x u^\varepsilon(1,z\frac{-1}{e})dz, \quad (12)$$

because $v(1,y)\partial_x u^\varepsilon(1,y) = v(1,-ey)\partial_x u^\varepsilon(1,-ey\frac{-1}{e})$ when $y \geq 0$. This leads to

$$\partial_x u^\varepsilon(1,y) = -\frac{1}{e}\partial_x u^\varepsilon(1,-\frac{y}{e}) \quad \forall y < 0,$$

and a similar condition on $\partial\mathbb{D}^-$. Following remark 1, it implies also

$$\partial_x u^\varepsilon(1,y) = -e\partial_x u^\varepsilon(1,-ey) \quad \forall y > 0, \ \partial_x u^\varepsilon(-1,y) = -e\partial_x u^\varepsilon(-1,-ey) \quad \forall y < 0. \quad (13)$$

which is an e-antisymmetric Neumann condition.

3.5. Variational Formulations

So let us establish variational formulations for problem (10),(13). For clarity we set $X = 1$. Multiplying (10) by $\hat{u}$ and integrating by parts the second-derivatives and using (11) leads to the first variational formulation: find $u^\epsilon \in H^1_e(\mathbb{D})$, such that, $\forall \hat{u} \in H^1_e(\mathbb{D})$,

$$\begin{aligned}\int_{\mathbb{D}} \Big(\lambda u^\epsilon \hat{u} \ &+\frac{\sigma^2}{2}\partial_y u^\epsilon \partial_y \hat{u} + \epsilon \partial_x u^\epsilon \partial_x \hat{u}\Big) + \int_{\partial\mathbb{D}^\infty} \mu n_y u^\epsilon \hat{u} \\ &+ \int_{\mathbb{D}} ((\gamma y + kx)\partial_y u^\epsilon - y\partial_x u^\epsilon)\,\hat{u} = \int_{\mathbb{D}} g\hat{u}, \end{aligned} \quad (14)$$

where $(n_x, n_y)^T$ is the outer normal of $\mathbb{D}$ and the stabilizing term is $\epsilon\partial_x u^\epsilon \partial_x \hat{u}$. The rest of the boundary of $\mathbb{D}$ is called $\partial\mathbb{D}^\infty$.

If the first-order terms are also integrated by parts then

$$\begin{aligned}\int_{\mathbb{D}} ((\gamma y + kx)\partial_y u^\epsilon - y\partial_x u^\epsilon)\,\hat{u} \ &= -\int_{\mathbb{D}} (\gamma u^\epsilon \hat{u} + (\gamma y + kx)u^\epsilon \partial_y \hat{u} - y u^\epsilon \partial_x \hat{u}) \\ &+ \int_{\partial\mathbb{D}^\infty} (\gamma y + kx) n_y u^\epsilon \hat{u} - \int_{\partial\mathbb{D}^\pm} y n_x u^\epsilon \hat{u}. \end{aligned} \quad (15)$$

Hence, by using the above multiplied by one half,

$$\begin{aligned}&\int_{\mathbb{D}} ((\gamma y + kx)\partial_y u^\epsilon - y\partial_x u^\epsilon)\,\hat{u} = \frac{1}{2}\int_{\mathbb{D}} ((\gamma y + kx)\partial_y u^\epsilon - y\partial_x u^\epsilon)\,\hat{u} \\ &+\frac{1}{2}\Big(-\int_{\mathbb{D}} (\gamma u^\epsilon \hat{u} + (\gamma y + kx)u^\epsilon \partial_y \hat{u} - y u^\epsilon \partial_x \hat{u}) + \int_{\partial\mathbb{D}^\infty} (\gamma y + kx) n_y u^\epsilon \hat{u} - \int_{\partial\mathbb{D}^\pm} y n_x u^\epsilon \hat{u}\Big). \end{aligned} \quad (16)$$

This leads to a second variational formulation:

3.6. Problem Statement

Assume $\lambda, \sigma, \mu, \epsilon \in \mathbb{R}^+$, $\sigma > 0$, $k, \gamma \in \mathbb{R}$ and $g \in L^2(\mathbb{D})$. Find $u^\epsilon \in H^1_e(\mathbb{D})$, such that, $\forall \hat{u} \in H^1_e(\mathbb{D})$,

$$\int_{\mathbb{D}} \Big((\lambda - \frac{\gamma}{2})u^\epsilon \hat{u} \ +\frac{\sigma^2}{2}\partial_y u^\epsilon \partial_y \hat{u} + \epsilon \partial_x u^\epsilon \partial_x \hat{u}$$

$$+\frac{1}{2}(\gamma y+kx)(\hat{u}\partial_y u^\epsilon - u^\epsilon\partial_y\hat{u})+\frac{y}{2}(u^\epsilon\partial_x\hat{u}-\hat{u}\partial_x u^\epsilon)\Big)$$
$$+\int_{\partial\mathbb{D}^\infty}(\gamma y+kx+2\mu)n_y\frac{u^\epsilon\hat{u}}{2}-\int_{\partial\mathbb{D}^\pm} yn_x\frac{u^\epsilon\hat{u}}{2}=\int_{\mathbb{D}} g\hat{u}. \tag{17}$$

Of course, the two formulations are equivalent provided $u^\epsilon \in H^1(\mathbb{D})$, a condition which insures that all the integrals exist. Note that the same problem with $\epsilon = 0$: find $u \in H^1_e(\mathbb{D})$ such that,

$$\int_{\mathbb{D}}\Big((\lambda-\frac{\gamma}{2})u\hat{u} \quad +\frac{\sigma^2}{2}\partial_y u\partial_y\hat{u}+\frac{1}{2}(\gamma y+kx)(\hat{u}\partial_y u-u\partial_y\hat{u})+\frac{y}{2}(u\partial_x\hat{u}-\hat{u}\partial_x u)\Big)$$
$$+\int_{\partial\mathbb{D}^\infty}(\gamma y+kx+2\mu)n_y\frac{u\hat{u}}{2}-\int_{\partial\mathbb{D}^\pm} yn_x\frac{u\hat{u}}{2}=\int_{\mathbb{D}} g\hat{u}. \tag{18}$$

for all $\hat{u} \in H^1_e(\mathbb{D})$, makes sense also, provided $g \in L^2(\mathbb{D})$.

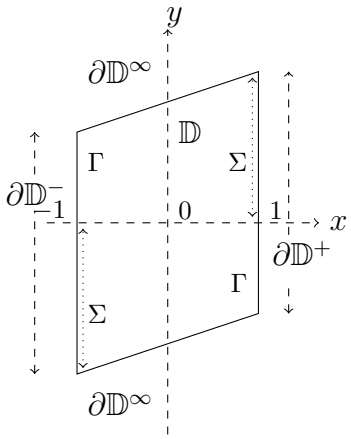


Fig. 1: *Domain of definition of the PDE localized. The vertices of the parallelogram are* $(-1,-L_y)$, $(1,-eL_y)$, $(1,L_y)$, $(-1,eL_y)$. Here $e<1$.

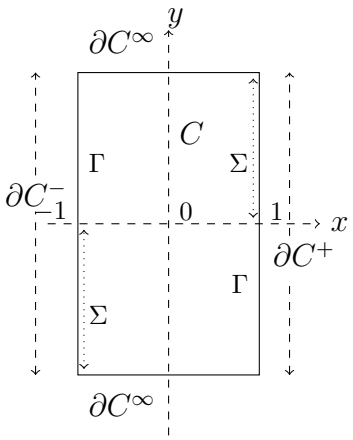


Fig. 2: *Domain of definition of the PDE localized in a finite domain. A change of variable maps* $\mathbb{D}$ *(left) into* $\mathbb{C}$ *(right).*

4. Analysis and Discretization

4.1. The Case $e \geq 1$

Lemma 1. *For any solution* $u^\epsilon \in H^1_e(D)$ *of (17) , the following energy conservation holds*

$$\int_D \left((\lambda-\frac{\gamma}{2})u^{\epsilon 2}+\frac{\sigma^2}{2}(\partial_y u^\epsilon)^2+\epsilon(\partial_x u^\epsilon)^2\right)+\int_{\partial\mathbb{D}^\infty}(\gamma y+kx+2\mu)n_y\frac{u^{\epsilon 2}}{2}$$
$$+\frac{1}{2}(1-\frac{1}{e^2})\left(\int_0^{eL_y} yu^{\epsilon 2}(-1,y)+\int_{-eL_y}^0(-y)u^{\epsilon 2}(1,y)\right)=\int_D gu^\epsilon. \tag{19}$$

Proof
Let us choose $\hat{u} = u^\epsilon$ in (17). Then, assuming the existence of a unique solution $u^\epsilon \in H^1_e(D)$,

$$\int_D \left(\left(\lambda - \frac{\gamma}{2}\right) u^{\epsilon 2} + \frac{\sigma^2}{2}(\partial_y u^\epsilon)^2 + \epsilon(\partial_x u^\epsilon)^2 \right) + \int_{\partial\mathbb{D}^\infty} (\gamma y + kx + 2\mu) n_y \frac{u^{\epsilon 2}}{2} - \int_{\partial\mathbb{D}^\pm} y n_x \frac{u^{\epsilon 2}}{2} = \int_D g u^\epsilon.$$

When $yn_x < 0$ the last term on the left is positive, i.e. $y > 0$ when $x = -1$ and $y < 0$ when $x = 1$.
To show that the other terms are also positive, we let $y' = \frac{y}{e}$ and use the boundary condition $u^\epsilon(-1, y') = u^\epsilon(-1, -ey')$ for all $y' < 0$; so with $y'' = -y = -ey'$,

$$\int_{y'<0} y' u^\epsilon(-1, y')^2 dy' = -\frac{1}{e^2} \int_{y''>0} y'' u^\epsilon(-1, y'')^2 dy'',$$

and similarly for the integral on $y > 0, x = 1$:

$$\int_{y>0} y u^\epsilon(1, y)^2 dy = -\frac{1}{e^2} \int_{y<0} y u^\epsilon(1, y)^2 dy.$$

Therefore

$$-\int_{\partial D^\pm} y n_x u^{\epsilon 2} = (1 - \frac{1}{e^2}) \left(\int_0^{eL_y} y u^{\epsilon 2}(-1, y) + \int_{-eL_y}^0 (-y) u^{\epsilon 2}(1, y) \right). \tag{20}$$

Q.E.D.

Theorem 1. *Assume that $e \geq 1$, $\epsilon > 0$, $\lambda > \frac{\gamma}{2}$ and $\gamma > \frac{k+\mu}{L_y}$ and $g \in L^2(\mathbb{D})$. Then Problem (17) has one and only one solution in $H^1_e(\mathbb{D})$.*

Proof Denote by $a^\epsilon(\cdot, \cdot)$ the bilinear form of the problem

$$\begin{aligned} a^\epsilon(u, \hat{u}) = \int_{\mathbb{D}} & \left((\lambda - \frac{\gamma}{2}) u\hat{u} + \frac{\sigma^2}{2} \partial_y u \partial_y \hat{u} + \epsilon \partial_x u \partial_x \hat{u} \right. \\ & \left. + \frac{1}{2}(\gamma y + kx)(\hat{u}\partial_y u - u\partial_y \hat{u}) - \frac{y}{2}(\hat{u}\partial_x u - u\partial_x \hat{u}) \right) \\ & - \int_{\partial\mathbb{D}^\pm} y n_x \frac{u\hat{u}}{2} + \int_{\partial\mathbb{D}^\infty} (\gamma y + kx + \mu) n_y \frac{u\hat{u}}{2}. \end{aligned} \tag{21}$$

By Lemma 1 and the fact that the last integral is always positive when $\hat{u} = u^\epsilon$, the bilinear form of (4) is coercive and continuous in $H^1_e(\mathbb{D})$. So the solution exists and is unique in $H^1_e(\mathbb{D})$ by the Lax-Milgram theorem.

Theorem 2. *If $g \in L^\infty(\mathbb{D})$, $e \geq 1$, $g \geq 0$, $\epsilon > 0$, $\lambda > \max\{1, \frac{\gamma}{2}\}$, $n_y \mu \geq 0$ on $\partial\mathbb{D}^\infty$ and $\gamma > (k + \mu)/L_y$, then: $0 \leq u^\epsilon \leq \|g\|_\infty / (\lambda - \frac{\gamma}{2})$.*

Proof With standard notations $u^\epsilon = u^+ - u^-$ with $u^\pm \geq 0$. Let $\hat{u} = -u^-$ in (17); since $u^+u^- = 0$ a.e., we have

$$\begin{aligned}\int_{\mathbb{D}} \Big((\lambda - \frac{\gamma}{2})(u^-)^2 \quad &+ \frac{\sigma^2}{2}(\partial_y u^-)^2 + \epsilon(\partial_x u^-)^2\Big) \\ &+\frac{1}{2}\int_{\partial\mathbb{D}^\infty} (\gamma y + kx + \mu)(u^-)^2 n_y \\ &+\frac{1}{2}(1 - \frac{1}{e^2})\left(\int_\Gamma |y|(u^-)^2\right) = -\int_{\mathbb{D}} g u^-. \end{aligned} \tag{22}$$

As $yn_y > 0$ on $\partial\mathbb{D}^\infty$, everything is positive on the left, hence $u^- = 0$.
Similarly let us derive from (17) a variational equation for $u_m = u - g_m$ with $g_m = \|g\|_\infty/(\lambda - \frac{\gamma}{2})$.
Note that u_m is solution of (4),(5) with $g - \lambda g_m$ instead of g and (8) with right-hand side equal to $-\mu g_m$; hence $u_m \in H^1_e(\mathbb{D})$ and

$$a^\epsilon(u_m, \hat{u}) = \int_{\mathbb{D}} (g - \lambda g_m)\hat{u} - \int_{\partial\mathbb{D}^\infty} \mu n_y g_m \hat{u}, \quad \forall \hat{u} \in H^1_e(\mathbb{D}).$$

By choosing $\hat{u} = u_m^+$ we obtain:

$$\begin{aligned}\int_{\mathbb{D}} &\left((\lambda - \frac{\gamma}{2})(u_m^+)^2 + \frac{\sigma^2}{2}(\partial_y u_m^+)^2 + \epsilon(\partial_x u_m^+)^2\right) \\ &+\int_{\partial\mathbb{D}^\infty} \left((\gamma y + kx + \mu) n_y \frac{(u_m^+)^2}{2} + \mu n_y g_m u_m^+\right) \\ &+\frac{1}{2}(1 - \frac{1}{e^2})\left(\int_\Gamma |y|(u_m^+)^2\right) = \int_{\mathbb{D}} (g - \lambda g_m) u_m^+. \end{aligned} \tag{23}$$

The integral on the right is negative while those on the left are positive, so $u_m^+ = 0$.

4.2. The limit case $\epsilon \to 0$

Theorem 3. *If g and $\partial_x g$ are in $L^2(\mathbb{D})$, the unique solution to (17) is bounded in $H^1(\mathbb{D})$ independently of ϵ and when $\epsilon \to 0$ any H^1-weakly converging subsequence satisfies (4),(5) and (17) with $\epsilon = 0$. Hence the unique solution, u^ϵ, of (17) tends to the unique solution, u, in $H^1(\mathbb{D})$ of (18).*

Proof
For clarity the proof is given when $\mathbb{D} = (-1, 1) \times \mathbb{R}$ because the difficulty is on the vertical boundaries and not on $\partial\mathbb{D}^\infty$. From (19) in lemma 1 we see that u^ϵ, and $\partial_y u^\epsilon$ are $L^2(\mathbb{D})$-bounded independently of ϵ. It is the solution in $H^1_e(\mathbb{D})$ of

$$\lambda u^\epsilon - \epsilon\partial_{xx} u^\epsilon - \frac{\sigma^2}{2}\partial_{yy} u^\epsilon + (\gamma y + kx)\partial_y u^\epsilon - y\partial_x u^\epsilon = g,$$

in the sense of distribution. To find an estimate for $|\partial_x u^\epsilon|_0$ uniform in ϵ we differentiate the PDE for u^ϵ above. It leads to a similar PDE for $v^\epsilon := \partial_x u^\epsilon$,

$$\lambda v^\epsilon - \epsilon\partial_{xx} v^\epsilon - \frac{\sigma^2}{2}\partial_{yy} v^\epsilon + (\gamma y + kx)\partial_y v^\epsilon - y\partial_x v^\epsilon = \partial_x g - k\partial_y u^\epsilon$$

with the e-antisymmetric Neumann condition (13), namely, in terms of v^ϵ,

$$v^\varepsilon(1,y) = -ev^\varepsilon(1,-ey) \quad \forall y > 0, \quad v^\varepsilon(-1,y) = -ev^\varepsilon(-1,-ey) \quad \forall y < 0.$$

Let $H^1_{*e}(\mathbb{D})$ be the space of functions of $H^1(\mathbb{D})$ which satisfy the above. Then we seek for $v^\varepsilon \in H^1_{*e}(\mathbb{D})$ such that $\forall v \in H^1_{*e}(\mathbb{D})$,

$$\int_{\mathbb{D}} \left(\lambda v^\epsilon \hat{v} + \frac{\sigma^2}{2}\partial_y v^\epsilon \partial_y \hat{v} + \epsilon\partial_x v^\epsilon \partial_x \hat{v}\right) + \int_{\mathbb{D}} \left((\gamma y + kx)\partial_y v^\epsilon - y\partial_x v^\epsilon\right)\hat{v} = \int_{\mathbb{D}} (\partial_x g - k\partial_y u^\epsilon)\hat{v}.$$

An energy estimate for this equation, derived in the same way as (19), is

$$\int_D \left((\lambda - \frac{\gamma}{2})v^{\epsilon 2} + \frac{\sigma^2}{2}(\partial_y v^\epsilon)^2 + \epsilon(\partial_x v^\epsilon)^2\right) = \int_D (\partial_x g - k\partial_y u^\epsilon)\, v^\epsilon.$$

The integrals on $\partial\mathbb{D}^\pm$ disapear because in (20), $(1 - \frac{1}{e^2})$ becomes $(1 - \frac{e^2}{e^2}) = 0$. Now we apply theorem 1 again; it shows that the solution $v^\epsilon \in H^1_{*e}(\mathbb{D})$ exists and is unique when $\partial_x g - k\partial_y u^\epsilon \in L^2(\mathbb{D})$ and by lemma 1 v^ϵ is bounded in $L^2(\mathbb{D})$ independently of ϵ.
Consequently u^ϵ is uniformly bounded in $H^1(\mathbb{D})$. Hence, weak converging sequences exist and all terms in (17) pass to their limits. Uniqueness is a consequence of (19).

4.3. Analysis when $e < 1$

Equation (19) indicates that the operator of the problem is no longer strongly elliptic.
Let us first recall a similar situation in linear algebra: finding $x \in \mathbb{R}^d$ that solves $Ax = b$, for $b \in \mathbb{R}^d$, for a singular $d \times d$ matrix A, (det$A$=0) requires a compatibility condition: $b \in (\mathrm{Ker}A)^\perp$. Hence if A is positive definite, there is a solution to $\Lambda Ix - Ax = b$ for any b if $\Lambda \in \mathbb{R}$ is not in the spectrum of A. Otherwise b must be orthogonal to the eigensubspace.
Here (14) is built from a positive definite operator inside $\mathbb{D}$ (see (10)). More precisely (10),(13) define a compact operator from $H^1_e(\mathbb{D})$ to the dual $H^*_e(\mathbb{D})$. Hence the Ritz-Schauder theorem applies, namely there are either a finite number of eigenvalues to the problem or a countable number clustering near zero (see for example Wloka [12], page 166).
Consequently, let us look at the following eigenvalue problem, built from (17): for all $\hat{u} \in H^1_e(\mathbb{D})$,

$$\begin{aligned}\int_{\mathbb{D}} \Big((\lambda - \frac{\gamma}{2})u^\epsilon \hat{u} \quad &+ \frac{\sigma^2}{2}\partial_y u^\epsilon \partial_y \hat{u} + \epsilon\partial_x u^\epsilon \partial_x \hat{u} + \frac{1}{2}(\gamma y + kx)(\hat{u}\partial_y u^\epsilon - u^\epsilon \partial_y \hat{u}) + \frac{y}{2}(u^\epsilon \partial_x \hat{u} - \hat{u}\partial_x u^\epsilon)\Big) \\ &+ \int_{\partial\mathbb{D}^\infty} (\gamma y + kx + 2\mu) n_y \frac{u^\epsilon \hat{u}}{2} = \frac{\Lambda}{2}\int_\Gamma y n_x u^\epsilon \hat{u}. \end{aligned} \tag{24}$$

If Λ, i.e. $\dfrac{1}{e^2} - 1$, is not in the spectrum of the operator, problem (10),(5),(13) has a solution and the solution is unique. If it is in the spectrum then there will be conditions on g to have a solution and uniqueness may not hold.
Uniqueness is a straightforward consequence of the linearity of the problem and the definition of an eigenvalue. Indeed, denote by $\lambda I - A$ the operator defined by (10),(5),(13). Let u_1, u_2 be two solutions. Then, when Λ is not an eigenvalue,

$$\Lambda u_i - A u_i = g,\ i = 1, 2 \quad \Rightarrow \quad \Lambda(u_1 - u_2) - A(u_1 - u_2) = 0 \quad \Rightarrow \quad u_1 - u_2 = 0.$$

Unfortunately the theoretical study of the spectrum of (24) is hard. On the other hand numerically, with the finite element method of degree 2, presented in the next section, and the library ARPACK the results indicate that there is no non-zero solution to this eigenvalue problem (fig.3).

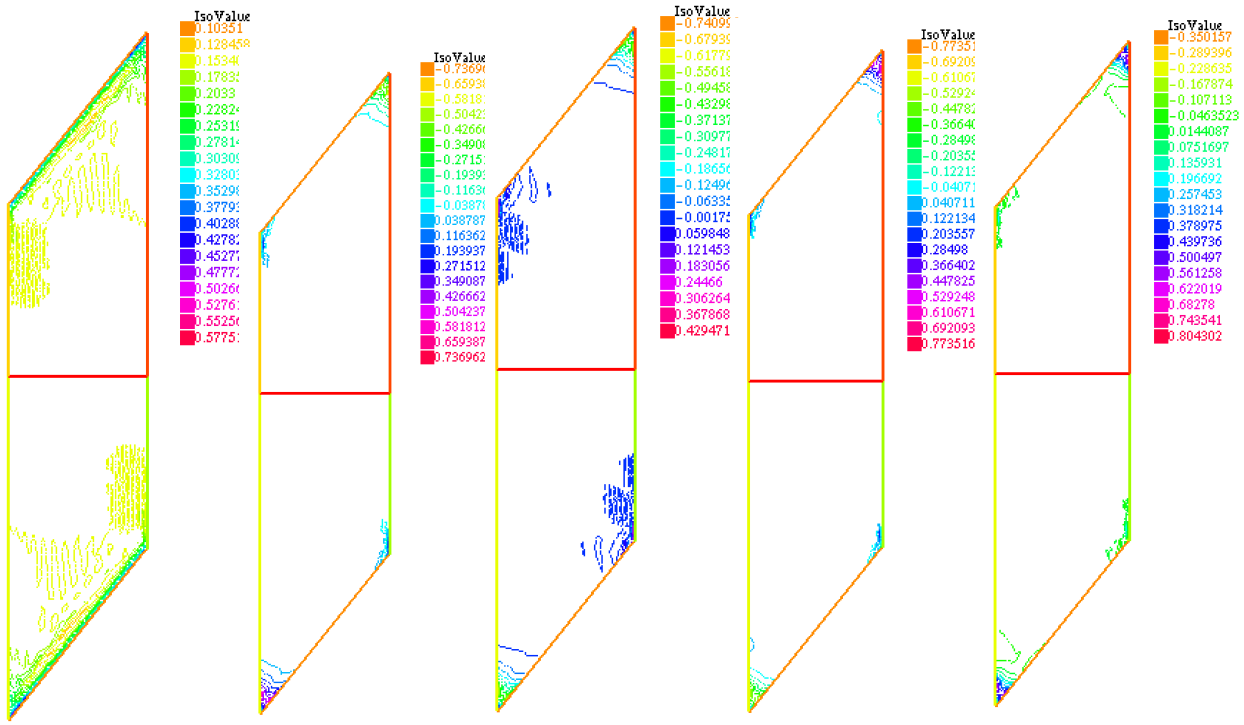


Fig. 3: Using a mesh with 854 vertices, ARPACK computed 5 real eigenvalues to (24) very close to $\Lambda = 11$. The 5 eigenvectors are shown here. As they are zero almost everywhere except at the corners, we can tentatively conclude that they exist only for the discrete system and not for the continuous one. The symmetry line $y = 0$ and the borders are shown by fat color lines.

4.4. Discretization

4.4.1. Finite Element Approximation

Consider a standard finite element approximation $U_h \subset H^1(\mathbb{D})$ with P^1 triangular conforming elements on a mesh $\mathcal{T}_h$; let T_h be the discretization of $\partial\mathbb{D}^{\pm}$ made by the boundary edges of $\mathcal{T}_h$ on $\partial\mathbb{D}^{\pm}$; let $q^j = (q_1^j, q_2^j)^T,\ j = 1, ..J$ be the vertices of T_h. We will work with the following approximation of $H_e^1(\mathbb{D})$:

$$V_h = \{v_h \in U_h \ : \ u_h(\pm 1, q^j) = u(\pm 1, -eq^j)\ \forall j = 1, \ldots, J \text{ such that } \pm q_2^j > 0\}.$$

The discrete system is:

$$\text{Find } u_h^\epsilon \in V_h \text{ such that: } \forall \hat{u}_h \in V_h\ , \qquad a^\epsilon(u_h^\epsilon, \hat{u}_h) = \int_{\mathbb{D}} g\hat{u}_h, \tag{25}$$

where $a^\epsilon(\cdot,\cdot)$ is defined in (21).

Theorem 4. *Assume that $\epsilon > 0$, $\lambda > \frac{\gamma}{2}$ and $\gamma > \frac{k+\mu}{L_y}$, then the solution to (25) exist and is unique. Furthermore, provided the solution of (17) is in $H^2(\mathbb{D})$,*

$$(\lambda - \frac{\gamma}{2})|u^\epsilon - u_h^\epsilon|_0^2 + \frac{\sigma^2}{2}|\partial_y(u^\epsilon - u_h^\epsilon)|_0^2 + \epsilon|\partial_x(u^\epsilon - u_h^\epsilon)|_0^2 \le C^\epsilon h^2, \tag{26}$$

where C^ϵ is a positive constant, dependent on the $H^2(\mathbb{D})$ norm of u^ϵ.

Proof

As $V_h \subset H_e^1(\mathbb{D})$, we may subtract (17) from (25) and obtain

$$a^\epsilon(u_h^\epsilon - u^\epsilon, \hat{u}_h) = 0 \quad \forall \hat{u}_h \in V_h. \tag{27}$$

Let v_h be the function of V_h which is equal to u^ϵ at the vertices of $\mathcal{T}_h$. Now by lemma 1,

$$\begin{aligned}&(\lambda - \frac{\gamma}{2})|u_h^\epsilon - u^\epsilon|_0^2 + \frac{\sigma^2}{2}|\partial_y(u_h^\epsilon - u^\epsilon)|_0^2 + \epsilon|\partial_x(u_h^\epsilon - u^\epsilon)|_0^2 \\ &\le a^\epsilon(u_h^\epsilon - u^\epsilon, u_h^\epsilon - u^\epsilon) = a^\epsilon(u_h^\epsilon - u^\epsilon, u_h^\epsilon - v_h) + a^\epsilon(u_h^\epsilon - u^\epsilon, v_h - u^\epsilon)\end{aligned} \tag{28}$$

The next to last term is 0 by (27) and the last term is bounded by $a^\epsilon(u_h^\epsilon - u^\epsilon)^{\frac{1}{2}} a^\epsilon(v_h - u^\epsilon)^{\frac{1}{2}}$ where $a^\epsilon(v) := a^\epsilon(v, v)$. Finally $a^\epsilon(v_h - u^\epsilon)^{\frac{1}{2}} \le Ch$ because, $u^\epsilon \in H^2(\mathbb{D})$ implies that $|v_h - u^\epsilon|_0 = O(h^2)$, $|\nabla^\epsilon(v_h - u^\epsilon)|_0 = O(h)$.

Q.E.D.

4.5. Change of Variable

For large and small values of e the parallelogram $\mathbb{D}$ (see fig.1) is very difficult to mesh properly.

With structured meshes in mind we wish to transform $\mathbb{D}$ into $\mathbb{C}$ by a change of variable.

Let $e \in (0, 1)$. Let u be the solution of (4),(5) in $\mathbb{D}$ of fig.1. Let $v(x, z) = u(x, y)$ with $y = \varphi(x, z))$ and

$$\varphi(x, z) = (\frac{1}{e} - 1)|z|\frac{x}{2} + (\frac{1}{e} + 1)\frac{z}{2}.$$

Notice that v is defined in $C = [-1, 1] \times [-L_y, L_y]$.

Lemma 2. *With v defined as above,*

$$v(\pm 1, z) = v(\pm 1, -z), \ \forall z \in [-L_y, L_y].$$

Proof There are 4 cases.

$$\begin{aligned}
z > 0 \ : & \\
v(-1,z) &= u(-1,\varphi(-1,z)) = u(-1,z) = u(-1,\frac{-z}{e}) = u(-1,\varphi(-1,-z)) \\
&= v(-1,-z), \\
v(1,z) &= u(1,\varphi(1,z)) = u(1,\frac{z}{e}) = u(1,-z) = u(1,\varphi(1,-z)) \\
&= v(1,-z); \\
z < 0 \ : & \\
v(1,z) &= u(1,\varphi(1,z)) = u(1,z) = u(1,\frac{-z}{e}) = u(1,\varphi(1,-z)) \\
&= v(1,-z), \\
v(-1,z) &= u(-1,\varphi(-1,z)) = u(-1,\frac{z}{e}) = u(-1,-z) = u(-1,\varphi(-1,-z)) \\
&= v(-1,-z).
\end{aligned}$$

Q.E.D.

Proposition 1. *Problem (4),(5) is equivalent to finding $v \in H^1_1(C)$ such that*

$$\begin{aligned}
\int_{\mathbb{C}} \Big(\lambda \partial_z\varphi \, v\hat{v} + \frac{\sigma^2}{2\partial_z\varphi} \partial_z v \partial_z \hat{v} \quad & + \partial_z\varphi \left((\gamma + \partial_x\varphi)\,\varphi + kx \right) \hat{v} \partial_z v - \hat{v}\varphi \partial_x v \Big) \\
&= \int_{\mathbb{C}} g \partial_z\varphi \, \hat{v}, \forall \hat{v} \in H^1_1(\mathbb{C}).
\end{aligned} \tag{29}$$

where $H^1_1(\mathbb{C})$ is $H^1_e(\mathbb{C})$ with $e = 1$.

Proof With v as in Lemma 2,

$$\partial_z v = \partial_y u \partial_z\varphi, \quad \partial_x v = \partial_x u + \partial_y u \partial_x\varphi \ \Rightarrow \ \partial_y u = \frac{\partial_z v}{\partial_z\varphi}, \ \partial_x u = \partial_x v - \partial_z v \frac{\partial_x\varphi}{\partial_z\varphi}.$$

Consequently (4),(5) written at $x, y = \varphi(x,z)$ becomes

$$\lambda v - \frac{\sigma^2}{2\partial_z\varphi} \partial_z \left(\frac{\partial_z v}{\partial_z\varphi} \right) + (\gamma\varphi + kx) \frac{\partial_z v}{\partial_z\varphi} - \varphi \left(\partial_x v - \partial_z v \frac{\partial_x\varphi}{\partial_z\varphi} \right) = g,$$

which, multiplied by $\hat{v}\partial_z\varphi$ and integrated over $\mathbb{C}$ leads to the variational formulation (29).

Q.E.D.

Notice that

$$\partial_z\varphi(x,z) = (\frac{1}{e} - 1)\frac{x}{2}\text{sign}(z) + \frac{1}{2}(\frac{1}{e} + 1), \qquad \partial_x\varphi(x,z) = (\frac{1}{e} - 1)\frac{|z|}{2}.$$

These are bounded and positive in $\mathbb{C}$ when $e \in (0,1]$, hence (29) is a problem of type: find v, 1-symmetric in $\mathbb{C}$, with, for all $\hat{v} \in H^1_1(\mathbb{C})$,

$$\int_{\mathbb{C}} \left(\tilde{\lambda} \, v\hat{v} + \frac{\tilde{\sigma}^2}{2} \partial_z v \partial_z \hat{v} + \tilde{c}\hat{v}\partial_z v - \hat{v}\varphi\partial_x v \right) = \int_{\mathbb{C}} \tilde{g} \, \hat{v}, \tag{30}$$

where $\tilde{\lambda}$, $\tilde{\sigma}$, $\tilde{c}$, are bounded and positive, bounded away from 0. Furthermore $|\varphi(x,z)| \geq |z|$ and sign$(\varphi(x,z))$ =sign(z).
An energy estimate, similar to (19), is derived by letting $\hat{v} = v$ in (30):

$$\int_{\mathbb{C}} \Big((\tilde{\lambda} \quad -\frac{1}{2}\partial_z \tilde{c} + \frac{1}{2}\partial_x \varphi)\, v^2 + \frac{\tilde{\sigma}^2}{2}(\partial_z v)^2 \Big) + \int_{\partial \mathbb{C}^\infty} \tilde{c} n_z \frac{v^2}{2} + \frac{1}{2}\int_{\Gamma} |\varphi| v^2 = \frac{1}{2}\int_{\Sigma} |\varphi| v^2 + \int_{\mathbb{C}} \tilde{g}\, v. \tag{31}$$

But the same difficulty remains, because,

$$\frac{1}{2}\int_{\Gamma} |\varphi| v^2 - \frac{1}{2}\int_{\Sigma} |\varphi| v^2 = (1 - \frac{1}{e}) \int_{\partial \mathbb{C}^\pm} |z| v^2.$$

So we cannot assert semi-ellipticity when $e \in (0,1)$.

5. Numerical Examples

First we reproduce the results obtained in Mertz et al. [6]. We use the $\mathbb{P}_2$ Lagrangian finite element method in $\mathbb{C}$.
So we take $g = y^2$, $e = 0.5$ in (25). Table 5 shows results for various values of X using $\mathbb{D}$ and using the change of variable in $\mathbb{C}$. We also reproduce table 3 of [6] obtained by a finite difference method with one million points and a Monte-Carlo simulation; it is compared here with computations on the finite element meshes shown on fig.4. The other parameters are $L_y = 5$, $\lambda = 10^{-6}$, $\gamma = 1$, $k = 1$, $\sigma = 1$, $\epsilon = 0$.
In an attempt to study the error we took as reference solution the result of a simulation with 5500 vertices. Then we computed the error at $x = 0, y = 0$ for 7 meshes with $X = 0.5$, $e = 0.5$ and $e = 1.5$. The right side of table 2 shows these errors and fig.5 displays the convergence. It appears that the convergence is at most linear for $e = 0.5$ and at least cubic for $e = 1.5$.

5.1. Rice's formula

In the context of the Hamiltonian system studied in this article, Rice's formula $(\mathcal{R})$ for $s \mapsto f_+$ is obtained by computing the solution of (25) with $g = \delta(x - s)y_+$ and then plot $\lambda u(0,0)$ versus s. Each point of the plot requires a new solution of the PDE $(\mathcal{R})$. The singularity of g is not a difficulty because the right-hand side of $(\mathcal{R})$ is

$$\int_{\mathbb{D}} \delta(x - s) y_+ \hat{u}_h = \int_{\mathbb{D} \cap \{x = s\}} y_+ u_h dy.$$

Fig.6 shows the numerical result and a comparison with a computation by a Monte-Carlo method. In the case of white noise, the PDE is two dimensional and the mesh is fine enough

X	FEM/$\mathbb{D}$	FEM/$\mathbb{C}$	FDM [6]	MC [6]
0.1	0.17853	0.17846	0.179	0.179
0.2	0.24925	0.24912	0.250	0.250
0.3	0.29706	0.29691	0.298	0.297
0.4	0.33334	0.33318	0.347	0.334
0.5	0.36245	0.36231	0.364	0.364
0.6	0.38656	0.38644	0.388	0.389
0.7	0.40685	0.40676	0.409	0.408
0.8	0.42405	0.42400	0.426	0.423
0.9	0.43865	0.43864	0.441	0.437
1.0	0.45099	0.45102	0.453	0.450

Table 1: Left: Computations of $\lambda u(0,0)$ for $e = 0.5$ and $g = y^2$, various values of X and various methods. Convergence as the number of vertices N_f increases, when $X = 0.5$ using FEM on $\mathbb{D}$.

N_f	$e = 0.5$	$e = 1.5$
109	0.361776	32.0478
393	0.362102	37.6635
1389	0.362495	37.7466
5507	0.362596	37.7981
22422	0.362666	37.7975
84440	0.362699	37.79879

Table 2: Left: Computations of $\lambda u(0,0)$ when $X = 0.5$, for $e = 0.5$ or $e = 1.5$ and $g = y^2$, with meshes refined roughly by a factor 2 each time. Convergence as the number of vertices N_f increases, using FEM on $\mathbb{D}$. Fig.5 is the same in graphic form.

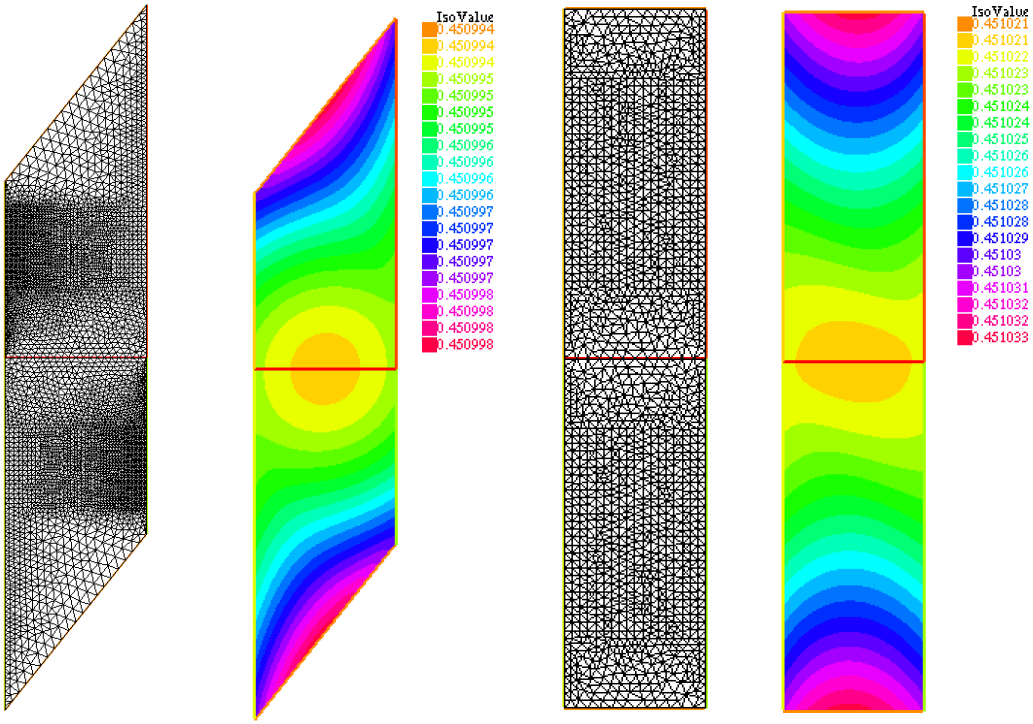


Fig. 4: Finite element meshes used for $\mathbb{D}$ and $\mathbb{C}$ and corresponding results depicting the color map of the values of u. Notice that for such small values of λ ,u is almost constant.

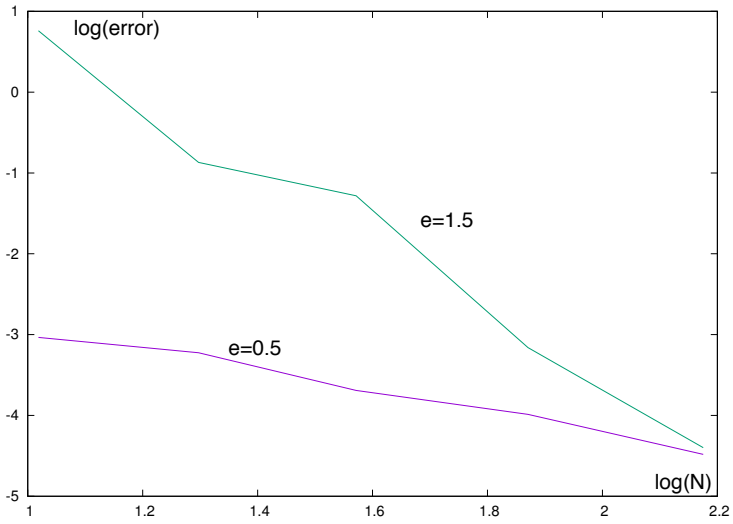


Fig. 5: Log of the error versus the log of N_f, the number of vertices in the mesh, when $e = 1.5$ (top curve) and $e = 0.5$ (lower curve). A very fine mesh has been used to compute a reference solution to obtain the error at the point (0,0). The lowest curve corresponds to $e = 0.5$ and has slope -1 indicating an error of order 1. The highest curve has a slope around -4 and has been obtained with $e = 1.5$.

(8350 vertices). For colored noise the PDE is in $\mathbb{R}^3$ (see (33) below) and the mesh has 51686 vertices; we have observed an embarrassing dependence on L_y with the PDE set in $\mathbb{D}$ and not so with the PDE set in $\mathbb{C}$ after a change of variable.
For the probabilistic numerical scheme of $\{(X_t, Y_t),\ t \geq 0\}$ of $(\mathcal{SHS})$, we use a standard Monte-Carlo method with time step δt. We consider $T = N\delta t$ large enough, $t_n = n\delta t$ and we construct random variables $\{(X_n, Y_n), 1 \leq n \leq N_{\delta t}\}$ to approximate (X_{t_n}, Y_{t_n}). In fig.6, we consider $g(x, y) = y^2$ and proceed with the following approximation

$$\mathbb{E}g(X_T, Y_T) \approx \frac{1}{M} \sum_{m=1}^{M} g(X_N^m, Y_N^m), \tag{32}$$

where $\{(X^m, Y^m), m = 1, \ldots, M\}$ is an i.i.d. sequence of trajectories produced by the algorithm.
In fig.6, for each threshold $s \in [0, L)$ we read from a sufficiently long numerical trajectory the frequency of X crossing s with positive velocities.

6. Extension to Three Dimensions

Consider now the colored noise case with $\mathbb{D} = (-1, 1) \times \mathbb{R}^2$:

$$\begin{aligned} &\lambda u - \frac{\sigma^2}{2}\partial_{zz}u + (\gamma y + kx - z)\partial_y u + \alpha z \partial_z u - y\partial_x u = g, \text{ in } \mathbb{D}, \\ &u(1, y, z) = u(1, -ey, z)\ \forall y > 0, \quad u(-1, y, z) = u(-1, -ey, z)\ \forall y < 0,\ \forall z \in \mathbb{R}. \end{aligned} \tag{33}$$

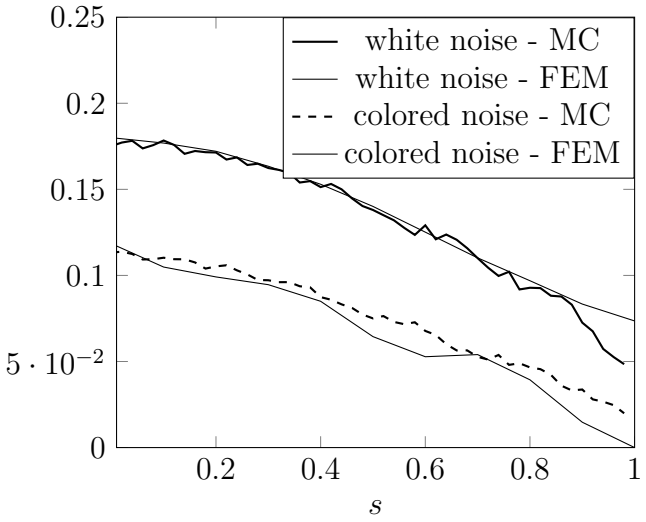


Fig. 6: $s \mapsto f_+(s)$ computed with Rice's formula ($\mathcal{R}$). Comparison between a finite element solution and a Monte-Carlo simulation, with $n = 10^4$. Left : $\delta t = 10^{-3}$, middle : $\delta t = 10^{-4}$, right : $\delta t = 10^{-5}$.

The symmetrised variational formulation is,

$$\int_{\mathbb{D}} \Big[(\lambda - \frac{\gamma+\alpha}{2})u\,\hat{u} + \frac{\sigma^2}{2}\partial_z u \partial_z \hat{u} + \frac{1}{2}(\gamma y + kx - z)(\hat{u}\partial_y u - u\partial_y \hat{u}) + \frac{\alpha}{2} z(\hat{u}\partial_z u - u\partial_z \hat{u}) - \frac{y}{2}(\hat{u}\partial_x u - u\partial_x \hat{u})\Big] - \int_{\partial\mathbb{D}^\pm} n_x \frac{y}{2} u\,\hat{u} = \int_{\mathbb{D}} g\,\hat{u}, \text{ in } \mathbb{D}. \tag{34}$$

Now $\partial\mathbb{D}^\pm$ denotes the planes $x = \pm 1$.
Everything that was said for the bi-dimensional case can be reproduced for the tri-dimensional case.
Discretization can be done using quadratic tetrahedral elements. As a test case we chose all parameters as in the 2D case and $\alpha = 1$ on a mesh with 64800 elements. The results are shown on fig.7. The value of $\lambda u(0,0,0)$ found is 0.66912; it also agrees to 4 digits with the mean value of u on $\mathbb{D}$, 0.669124; computing time is 10 seconds on a MacBook Pro Core i7 2.5GHz.

7. The Time-Dependent Case

The time dependent problem (see (35) below) has λu replaced by $\partial_t u$ and a zero right-hand side. Naturally, an initial condition must be given: $u(t = 0) = g$. Then $t \to u(0,0,t)$ is asymptotic to λu of the stationary case. Fig.8 illustrates this property. It is computed with the same parameter as the stationary case on $\mathbb{D}$ with an implicit Euler scheme and $\delta t = 0.0125$, $X = 1$.

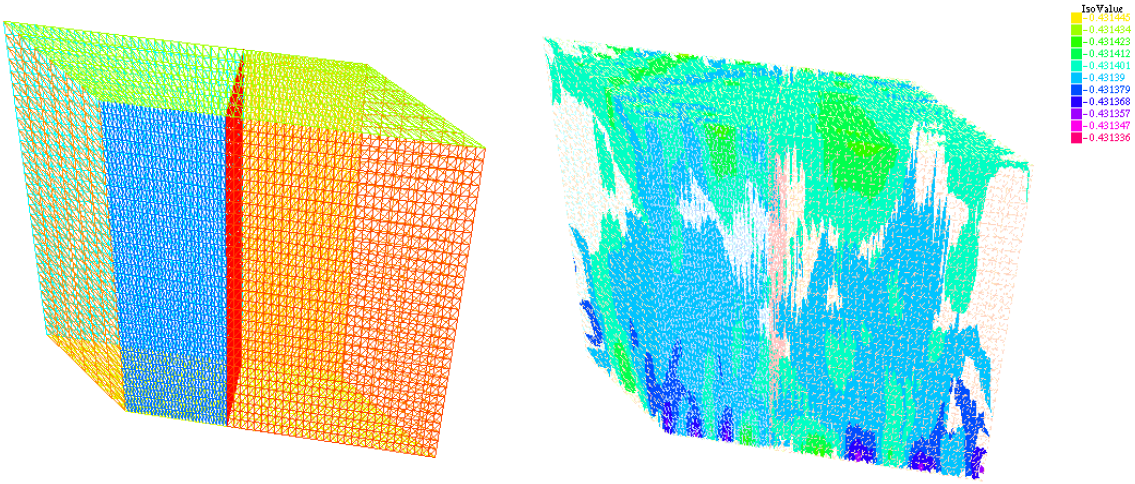


Fig. 7: Finite element solution of the 3D case. Notice that here too u is almost constant.

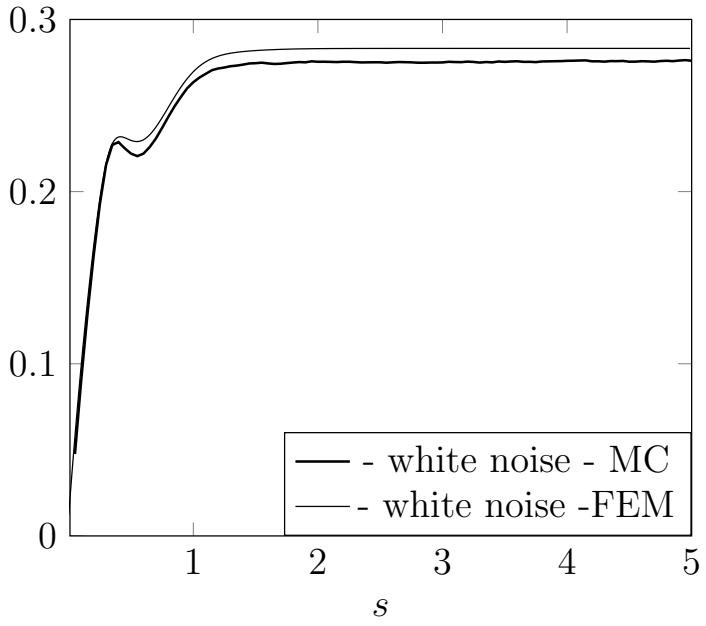


Fig. 8: Energy: Finite element solution of the time dependent case showing $u(0,0,t)$ versus time t. This curve is to be compared with Figure 4 of [6], reproduced here as (MC).

The variational setting of the problem is as follows.

Theorem 5. *If $e \geq 1$, $n_y\mu \geq 0$ on $\partial\mathbb{D}^\infty$, $\gamma L_y > k + \mu$, $\varepsilon > 0$, $g \in L^2(\mathbb{D})$, the following problem has one and only one solution: find $u^\epsilon \in L^2(0,T;H^1_e(\mathbb{D}))$ such that, for all $\hat{u} \in$*

$H_e^1(\mathbb{D})$,

$$\int_{\mathbb{D}} (\partial_t u^\epsilon \, \hat{u}) + a_t^\epsilon(u^\epsilon, \hat{u}) = 0, \quad t \in (0,T), \quad u(0) = g \tag{35}$$

where

$$\begin{aligned} a_t^\epsilon(u^\epsilon, \hat{u}) = \int_{\mathbb{D}} \; & \Big(-\frac{\gamma}{2} u^\epsilon \hat{u} + \frac{\sigma^2}{2} \partial_y u^\epsilon \partial_y \hat{u} + \epsilon \partial_x u^\epsilon \partial_x \hat{u} \\ & + \frac{1}{2}(\gamma y + kx)(\hat{u}\partial_y u^\epsilon - u^\epsilon \partial_y \hat{u}) - \frac{y}{2}(\hat{u}\partial_x u^\epsilon - u^\epsilon \partial_x \hat{u}) \Big) \\ & - \int_{\partial\mathbb{D}^\pm} y n_x \frac{u^\epsilon \hat{u}}{2} + \int_{\partial\mathbb{D}^\infty} (\gamma y + kx + \mu) n_y \frac{u^\epsilon \hat{u}}{2}. \end{aligned} \tag{36}$$

Proof: All the conditions of Theorem 3.2 of [4] are met. Gårding's inequality holds due to the coercivity of a_t without the term $-\frac{\gamma}{2} u\hat{u}$, shown in theorem 1.

Remark 3. *The time-dependent case with $e \leq 1$ is as difficult as the stationary case.*

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