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Approximation of Sweeping Processes and Controllability for a Set Valued Evolution

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Abstract

We consider a controlled evolution problem for a set \( \Omega(t) \in \mathbb{R}^d \), originally motivated by a model where a dog controls a flock of sheep. Necessary conditions and sufficient conditions are given, in order that the evolution be completely controllable. Similar techniques are then applied to the approximation of a sweeping process. Under suitable assumptions, we prove that there exists a control function such that the corresponding evolution of the set \( \Omega(t) \) is arbitrarily close to the one determined by the sweeping process.

1 Introduction

In this paper we consider a controllability problem for the evolution of a set \( \Omega(t) \subset \mathbb{R}^d \). This was originally motivated by the model introduced in [4], describing the evolution of a flock of sheep, who tend to scatter around but also react to the presence of a dog. The region \( \Omega(t) \subset \mathbb{R}^2 \) occupied by the sheep is described as the reachable set for a differential inclusion, while the position of the dog is regarded as a control function. As in [4], we consider a “scare function” \( \varphi = \varphi(r) > 0 \), describing the speed at which sheep run away from the dog, depending on the distance \( r \). Further results and extensions can be found in [9, 10]. For more general models of crowd dynamics we refer to [3]. A general theory of evolution problems in metric spaces, also describing the evolution of a set, was developed in [1, 11].

In the following we consider the evolution of a set in \( \mathbb{R}^d \), and assume

(A1) The function \( r \mapsto \varphi(r) \) is continuously differentiable for \( r > 0 \), and satisfies

\[
\varphi' < 0, \quad \lim_{r \to 0^+} \varphi(r) = +\infty, \quad \lim_{r \to +\infty} \varphi(r) = 0. \tag{1.1}
\]
Given a function \( t \mapsto \xi(t) \in \mathbb{R}^d \) describing the position of a repelling agent, we define the velocity field
\[
\mathbf{v}(x,\xi) = \psi(|x-\xi|) \frac{x-\xi}{|x-\xi|}.
\] (1.2)

For a given initial set \( \Omega_0 \), we denote by \( \Omega^\xi(t) \) the set reached by trajectories of
\[
x(t) \in \mathbf{v}(x,\xi(t)), \quad x(0) \in \Omega_0.
\] (1.3)

In other words, for any \( t \geq 0 \),
\[
\Omega^\xi(t) = \left\{ x(t); \quad x(0) \in \Omega_0, \quad x(\cdot) \text{ is absolutely continuous,} \quad \dot{x}(\tau) = \mathbf{v}(x(\tau),\xi(\tau)) \text{ for a.e. } \tau \in [0,t] \right\}.
\] (1.4)

Throughout the following we write \( \partial \Omega, \overline{\Omega}, \text{ and } \text{int} \Omega \), for the boundary, the closure, and the interior of a set \( \Omega \subset \mathbb{R}^d \), respectively. By \( B(\Omega, r) \) we denote the open neighborhood of radius \( r \) around the set \( \Omega \), while \( d_H \) denotes Hausdorff distance [2].

To avoid any difficulty about uniqueness of solutions of (1.2)-(1.3), we assume that the control \( \xi(\cdot) \) is chosen so that
\[
\inf_{t \in [0,\tau]} d(\xi(t),\Omega^\xi(t)) > 0 \quad \text{for all } 0 \leq \tau < T.
\] (1.5)

We wish to understand how the function \( \psi \) affects the controllability properties of the evolution equation (1.3). Roughly speaking, given an initial set \( \Omega_0 \) and a terminal set \( \Omega_1 \), we seek a control \( \xi(\cdot) \) such that, at the terminal time \( T \), the set \( \Omega^\xi(T) \) in (1.4) is arbitrary close to \( \Omega_1 \).

**Definition 1.** We say that the set-valued evolution (1.3) satisfies the **Global Approximate Confinement** property (GAC) if the following holds. Let \( \Omega_0, \Omega_1 \subset \mathbb{R}^d \) be any two compact domains, with \( \Omega_1 \) simply connected and such that \( \Omega_1 \subset \text{int} \Omega_0 \). Then for any \( T, \varepsilon > 0 \), there exists a Lipschitz continuous control \( \xi : [0,T] \mapsto \mathbb{R}^d \) satisfying (1.5) and such that the set (1.4) satisfies
\[
\Omega_1 \subseteq \Omega^\xi(T) \subseteq B(\Omega_1,\varepsilon).
\] (1.6)

If there exists a locally Lipschitz control \( \xi : [0,T] \mapsto \mathbb{R}^d \) satisfying (1.5) and such that
\[
\Omega^\xi(T) = \Omega_1,
\] (1.7)
we then say that the set-valued evolution (1.3) satisfies the **Global Exact Confinement** property (GEC).

The primary goal of this paper is to find conditions which are necessary, or sufficient, to achieve the (GAC) or (GEC) properties. Indeed, we will show that these properties are determined by the asymptotic behavior of the function \( \psi \) as \( r \to 0^+ \). Our first main result is

**Theorem 1 (necessary condition).** Let \( \psi \) satisfy (A1). If the (GAC) property holds, then the function \( \psi \) must satisfy
\[
\int_0^1 r^{d-2} \psi(r) \, dr = +\infty.
\] (1.8)

To state a sufficient condition, we introduce the assumption

(A2) For every \( \kappa > 0 \) one has

\[
\lim_{r \to 0^+} \frac{r^{d/2} \cdot \varphi(\kappa r^{1/2}) + 1}{r^d \cdot \varphi(r)} = 0.
\] (1.9)

**Theorem 2 (sufficient condition).** If the function \( \varphi \) satisfies (A1)-(A2), then the (GAC) and (GEC) properties hold.

This result applies, in particular, to the function \( \varphi(r) = r^{-\beta} \) for any \( \beta > d \).

A proof of Theorem 1 will be given in Section 2, while Theorem 2 will be proved in Section 3.

The controllability of the set-valued evolution (1.4) is closely related to a result on the approximation of a sweeping process. Indeed, let \( t \mapsto V(t) \) describe a moving set in \( \mathbb{R}^d \). We assume that each \( V(t) \) is a compact set with nonempty interior and smooth boundary, smoothly depending on time. More precisely:

\[
V(t) = \{ x \in \mathbb{R}^d ; \ \psi(t,x) \leq 0 \},
\] (1.10)

where \( \psi : \mathbb{R} \times \mathbb{R}^d \mapsto \mathbb{R} \) has \( C^2 \) regularity and satisfies the nondegeneracy condition

\[
\psi(t,x) = 0 \implies \nabla_x \psi(t,x) \neq 0.
\] (1.11)

As usual, we denote by \( N_{V(t)}(x) \) the outer normal cone to \( V(t) \) at a boundary point \( x \in \partial V(t) \). In the case of an interior point \( x \in \text{int} V(t) \), we simply define \( N_{V(t)}(x) = \{0\} \). By the well known theory of sweeping processes [5, 6, 7, 8, 12], for any initial point \( x_0 \in V(0) \), the differential inclusion

\[
\dot{x}(t) \in - N_{V(t)}(x(t)), \quad x(0) = x_0
\] (1.12)

has a unique solution \( t \mapsto x(t,x_0) \in V(t) \). In turn, for a given initial set \( \Omega_0 \subset V(0) \), one can consider the sets

\[
\Omega(t) \doteq \{ x(t,x_0) ; \ x_0 \in \Omega_0 \}.
\] (1.13)

A natural question is whether there exists a control \( \xi(\cdot) \) such that the corresponding sets \( \Omega^\xi(t) \) in (1.4) remain uniformly close to the sets \( \Omega(t) \), for all \( t \in [0,T] \). It turns out that this is true, under an assumption which slightly strengthens (A2), namely:

(A2') For some \( \beta > 1/2 \) one has

\[
\lim_{r \to 0^+} \frac{r^{\beta d} \cdot \varphi(r^{\beta}) + 1}{r^d \cdot \varphi(r)} = 0.
\] (1.14)

In the following, \( t \mapsto x^\xi(t,x_0) \) denotes the solution to

\[
\dot{x}(t) = v(x(t),\xi(t)), \quad x(0) = x_0,
\] (1.15)

with \( v \) as in (1.2), while \( t \mapsto x(t,x_0) \) is the trajectory of the sweeping process (1.12), with the same initial condition.
**Theorem 3 (approximation of a sweeping process).** Assume that the function $\varphi$ satisfies (A1) and (A2'). As in (1.10)-(1.11), let $t \mapsto V(t)$ be a family of sets with $C^2$ boundaries. Then, for any $T, \varepsilon > 0$ there exists a measurable control $t \mapsto \xi(t)$ such that

$$|x^\xi(t, x_0) - x(t, x_0)| \leq \varepsilon \quad \text{for all } x_0 \in V(0), \ t \in [0, T].$$

(1.16)

An immediate consequence of (1.16) is that, for any initial subset $\Omega_0 \subseteq V(0)$, the corresponding sets $\Omega^\xi(t)$ in (1.4) and $\Omega(t)$ in (1.13) satisfy

$$d_H(\Omega^\xi(t), \Omega(t)) \leq \varepsilon \quad \text{for all } t \in [0, T].$$

(1.17)

A proof of Theorem 3 will be worked out in Section 4.

2 Proof of the necessary condition

In this section we give a proof of Theorem 1. The main idea is that, if (1.8) fails, then for any choice of the control $\xi(\cdot)$ the volume of the set $\Omega^\xi$ cannot shrink too fast. This puts a constraint on the sets that can be approximately reached at time $T$.

Let $t \mapsto \xi(t)$ be any admissible control. Fix any time $t \geq 0$ and let $\nu = \nu(\cdot, \xi(t))$ be the vector field in (1.2). Then, calling $\delta = d(\xi(t), \Omega^\xi(t)), we compute

$$\frac{d}{dt} \meas(\Omega^\xi(t)) = \int_{\Omega^\xi(t) \cap B(\xi(t), 1)} \div \nu \, dx + \int_{\Omega^\xi(t) \setminus B(\xi(t), 1)} \div \nu \, dx$$

$$\geq \omega_{d-1} \int_{\delta} r^{d-1} \varphi'(r) \, dr + \varphi'(1) \cdot \meas(\Omega^\xi(t)).$$

(2.1)

Here and in the sequel $\omega_{d-1}$ denotes the $(d-1)$-dimensional measure of the surface of the unit ball in $\mathbb{R}^d$. An integration by parts yields

$$\int_{\delta} r^{d-1} \varphi'(r) \, dr = - \int_{\delta} (d-1) r^{d-2} [\varphi(r) - \varphi(1)] \, dr - \delta^{d-1} [\varphi(\delta) - \varphi(1)]$$

$$\geq - \int_{\delta} (d-1) r^{d-2} \varphi(r) \, dr - \delta^{d-1} \varphi(\delta).$$

(2.2)

If (1.8) fails, then

$$\int_{\delta} r^{d-2} \varphi(r) \, dr \leq M = \int_{0}^{1} r^{d-2} \varphi(r) \, dr < \infty.$$

Since $\varphi$ is decreasing, one has

$$\delta^{d-1} \varphi(\delta) = \frac{1}{d-1} \cdot \int_{0}^{\delta} s^{d-2} \cdot \varphi(s) \, ds \leq \frac{1}{d-1} \cdot \int_{0}^{\delta} s^{d-2} \cdot \varphi(s) \, ds \leq \frac{M}{d-1}.$$

This implies that the right hand side of (2.2) is bounded below by a constant. Therefore, (2.1) yields

$$\frac{d}{dt} \meas(\Omega^\xi(t)) \geq \varphi'(1) \cdot \meas(\Omega^\xi(t)) - \frac{\omega_{d-1} Md}{d-1}.$$ (2.3)
By Gronwall’s inequality, for all times $t \geq 0$ we conclude that
\[
\text{meas}(\Omega^{\xi}(t)) \geq e^{\varphi'(1)t} \text{meas}(\Omega_0) - \frac{\omega_{d-1}Md}{d-1} \cdot t. \tag{2.4}
\]
This a priori lower bound on the measure of the set $\Omega^{\xi}(t)$ shows that approximate controllability cannot be achieved. \hfill \square

**Remark.** Assume that the divergence of the vector field $v(\cdot, \xi)$ remains negative for $x$ close to $\xi$, that is
\[
\varphi'(r) + \frac{d - 1}{r} \varphi(r) \leq 0 \quad \text{for all } 0 < r < \bar{r}, \tag{2.5}
\]
for some $\bar{r} > 0$. In this case, the Global Approximate Confinement property implies
\[
\limsup_{r \to 0^+} r^{d-1} \varphi(r) = +\infty. \tag{2.6}
\]
Indeed, one can replace the estimate (2.1) with
\[
\frac{d}{dt} \text{meas}(\Omega^{\xi}(t)) = \int_{\Omega^{\xi}(t) \cap B(\xi(t), \bar{r})} \text{div} v \, dx + \int_{\Omega^{\xi}(t) \setminus B(\xi(t), \bar{r})} \text{div} v \, dx \\
\geq \int_{\delta < |x| < \bar{r}} \text{div} v \, dx + \varphi'(\bar{r}) \cdot \text{meas}(\Omega^{\xi}(t)) \\
= \omega_{d-1} \left[ \bar{r}^{d-1} \varphi(\bar{r}) - \delta^{d-1} \varphi(\delta) \right] + \varphi'(\bar{r}) \cdot \text{meas}(\Omega^{\xi}(t)) \\
\geq - \omega_{d-1} \delta^{d-1} \varphi(\delta) + \varphi'(\bar{r}) \cdot \text{meas}(\Omega^{\xi}(t)). \tag{2.7}
\]
If (2.6) fails, then
\[
\frac{d}{dt} \text{meas}(\Omega^{\xi}(t)) \geq \varphi'(\bar{r}) \cdot \text{meas}(\Omega^{\xi}(t)) - C_1
\]
for some constant $C_1$, and it leads again a priori lower bound on the measure of the set $\Omega^{\xi}(t)$.

### 3 Proof of the sufficient condition

Aim of this section is to provide a proof of Theorem 2. As a preliminary, consider a bounded open set $\Omega \subset \mathbb{R}^d$ with $C^2$ boundary $\Sigma = \partial \Omega$. On the complement $\mathbb{R}^d \setminus \Sigma$ we consider the vector field
\[
v(x) = \int_{\Sigma} \varphi(|x - \xi|) \frac{x - \xi}{|x - \xi|} \, d\sigma(\xi), \tag{3.1}
\]
where $\sigma$ denotes the $(d - 1)$-dimensional surface measure on $\Sigma$. Since $\Sigma$ has $C^2$ regularity, for every $x$ sufficiently close to $\Sigma$ there exists a unique perpendicular projection $y_x \in \Sigma$ such that
\[
|x - y_x| = d(x, \Sigma) = \min_{y \in \Sigma} |x - y|. \tag{3.2}
\]
To fix the ideas, assume that this perpendicular projection $x \mapsto y_x$ is well defined whenever $d(x, \Sigma) < r_0$, for some curvature radius $r_0 > 0$. In the following, $n_x = \frac{x - y_x}{|x - y_x|}$ denotes the unit normal to the surface $\Sigma$ at the point $y_x$. 


Lemma 1. Let the function \( \varphi \) satisfy (A1)-(A2). Then the vector field \( \mathbf{v} \) in (3.1) satisfies

\[
\lim_{d(x, \Sigma) \to 0} \langle \mathbf{n}_x, \mathbf{v}(x) \rangle = +\infty. \quad (3.3)
\]

Figure 1: The portion of \( \Sigma \) near the point \( y_x \) can be represented as the graph of a function \( f \), as in (4.3).

Proof. 1. Consider any point \( x \) sufficiently close to \( \Sigma \) so that the perpendicular projection \( y_x = \pi(x) \) at (3.2) is well defined. Let \( V_x \) be the hyperplane tangent to \( \Sigma \) at \( y_x \) and let \( \mathbf{n}_x \) be the unit normal vector. As shown in Fig. 1, in a neighborhood of \( y_x \), the surface \( \Sigma \) can be expressed as the graph of a function \( f : V_x \mapsto \mathbb{R} \). More precisely, call \( \varepsilon = d(x, \Sigma) = |x - y_x| \).

Without loss of generality, we can choose a system of coordinates such that \( y_x = 0 \). Notice that, by the regularity and compactness of the surface \( \Sigma \), we can assume that the radius \( \delta_0 \) of the ball where the function \( f \) is defined is independent of \( y_x \in \Sigma \). Moreover, the \( C^2 \) norm of \( f \) remains uniformly bounded. By construction we have

\[
f(0) = 0, \quad \nabla f(0) = 0, \quad \|f\|_{C^2(B(0,\delta_0))} \leq C_0, \quad (3.4)
\]

for some uniform constant \( C_0 \). Defining the constant \( \kappa = \sqrt{2/C_0} > 0 \), by (3.4) one has the implication

\[
|y - y_x| < \kappa|x - y_x|^{\frac{1}{2}} \quad \implies \quad \langle \mathbf{n}_x, x - y - f(y)\mathbf{n}_x \rangle \geq 0. \quad (3.5)
\]

Figure 2: The estimates (3.10)-(3.11).
2. Next, consider the decomposition $\Sigma = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3$, where
\[
\begin{align*}
\Sigma_1 &= \{ y + f(y)n_x \mid |y| \leq \kappa |x|^{1/2} \}, \\
\Sigma_2 &= \{ y + f(y)n_x \mid \kappa |x|^{1/2} < y < \delta_0 \}, \\
\Sigma_3 &= \Sigma \setminus (\Sigma_1 \cup \Sigma_2).
\end{align*}
\] (3.6)

Based on (3.6), we shall estimate the vector field $v$ by splitting the integral in (3.1) in three parts:
\[
v_i(x) = \int_{\Sigma_i} \varphi(|x-\xi|) \frac{x-\xi}{|x-\xi|} \, d\sigma(\xi), \quad i = 1, 2, 3,
\] (3.7)
so that
\[
v(x) = v_1(x) + v_2(x) + v_3(x).
\]

In the following, for $y \in V_x$, we use the bound $|f(y)| \leq \frac{C_0}{\varepsilon} |y|^2$ and the identities
\[
|x| = \varepsilon, \quad |y| = r, \quad |x-y| = \sqrt{\varepsilon^2 + r^2}, \quad n_x = \frac{x}{|x|}.
\] (3.8)

As long as $|y| \leq \delta_0$, the above implies
\[
|x-y-f(y)n_x|^2 \in \left[ \varepsilon^2 + (1-C_0\varepsilon)r^2, \varepsilon^2 + (1+C_0\varepsilon)r^2 + C_0^2 r^4/4 \right].
\] (3.9)

Using (3.9) and the monotonicity of $\varphi$ we obtain the estimates
\[
\langle n_x, v_1(x) \rangle \geq c_0 \int_{\kappa \varepsilon^{1/2}}^{\delta_0} r^{d-2} \left( \varepsilon - \frac{C_0 r^2}{2} \right) \cdot \frac{\varphi \left( \sqrt{\varepsilon + C_0 r^2} \right)}{\sqrt{\varepsilon + (C_0 r^2/2)^2 + r^2}} \, dr
\]
\[
\geq c_0 \int_{\kappa \varepsilon^{1/2}}^{\delta_0} r^{d-2} \left( \varepsilon - \frac{C_0 r^2}{2} \right) \cdot \frac{\varphi \left( \sqrt{\varepsilon^2 + (1+2C_0\varepsilon)r^2} \right)}{\sqrt{\varepsilon^2 + (1+2C_0\varepsilon)r^2}} \, dr
\]
\[
\geq \frac{3}{4} c_0 \int_{\kappa \varepsilon^{1/2}}^{\delta_0} \varepsilon r^{d-2} \cdot \frac{\varphi \left( \sqrt{\varepsilon^2 + (1+2C_0\varepsilon)r^2} \right)}{\sqrt{\varepsilon^2 + (1+2C_0\varepsilon)r^2}} \, dr,
\] (3.10)

\[
|\langle n_x, v_2(x) \rangle| \leq c_0 \int_{\kappa \varepsilon^{1/2}}^{\delta_0} r^{d-2} \varphi \left( \sqrt{\varepsilon^2 + (1-C_0\varepsilon)r^2} \right) \cdot \frac{\varepsilon + C_0 r^2/2}{\sqrt{\varepsilon^2 + (1-C_0\varepsilon)r^2}} \, dr
\]
\[
\leq C_1 \cdot \int_{\kappa \varepsilon^{1/2}}^{\delta_0} \frac{\varphi \left( \sqrt{\varepsilon^2 + (1-C_0\varepsilon)r^2} \right)}{\sqrt{\varepsilon^2 + (1-C_0\varepsilon)r^2}} \cdot r^d \, dr.
\] (3.11)

for some constants $C, C_1, c_0 > 0$. Performing the change of variable $s = r \sqrt{1+2C_0 \varepsilon}$ in (3.10), one finds
\[
\langle n_x, v_1(x) \rangle \geq C_2 \cdot \int_{\kappa \varepsilon^{1/2}}^{\delta_0} \varepsilon s^{d-2} \cdot \frac{\varphi \left( \sqrt{\varepsilon^2 + s^2} \right)}{\sqrt{\varepsilon^2 + s^2}} \, ds
\]
\[
\geq \frac{4C_2}{\kappa^2} \cdot \int_{\kappa \varepsilon^{1/2}}^{\delta_0} s^d \cdot \frac{\varphi \left( \sqrt{\varepsilon^2 + s^2} \right)}{\sqrt{\varepsilon^2 + s^2}} \, ds.
\]
for some constant $C_2 > 0$. Setting $t = \sqrt{\varepsilon^2 + s^2}$, we estimate
\[
\langle \mathbf{n}_x, \mathbf{v}_1(x) \rangle \geq \frac{4C_2}{\kappa^2} \cdot \int_\varepsilon^{\sqrt{\varepsilon^2 + \frac{1}{2} \kappa^2}} (t^2 - \varepsilon^2)^{\frac{d-1}{2}} \cdot \varphi(t) \, dt \\
\geq \frac{4C_2}{\kappa^2} \cdot \int_2^{2\varepsilon^{1/2}} (t^2 - \varepsilon^2)^{\frac{d-1}{2}} \cdot \varphi(t) \, dt \\
\geq \frac{4C_2}{\kappa^2} \cdot \left( \frac{3}{4} \right)^{\frac{d-1}{2}} \cdot \int_2^{2\varepsilon^{1/2}} r^{d-1} \cdot \varphi(t) \, dt.
\]
Similarly, using the variable $s = \sqrt{\varepsilon^2 + (1 - C_0\varepsilon)r^2}$ in (3.11), we have
\[
\int_{\kappa \varepsilon^{1/2}}^{\delta_0} \frac{\varphi\left(\sqrt{\varepsilon^2 + (1 - C_0\varepsilon)r^2}\right)}{\sqrt{\varepsilon^2 + (1 - C_0\varepsilon)r^2}} \cdot r^d \, dr = \frac{1}{1 - C_0\varepsilon} \cdot \int_{\varepsilon^2 + (1 - C_0\varepsilon)\kappa^2}^{\varepsilon^2 + (1 - C_0\varepsilon)\kappa^2} \varphi(s) \cdot r^{d-1} \, ds \\
\leq \frac{1}{(1 - C_0\varepsilon)^{\frac{d+1}{2}}} \cdot \int_{\varepsilon^2 + (1 - C_0\varepsilon)\kappa^2}^{\varepsilon^2 + (1 - C_0\varepsilon)\kappa^2} \varphi(s) \cdot s^{d-1} \, ds.
\]
Thus, for $\varepsilon > 0$ sufficiently small, it holds
\[
|\langle \mathbf{n}_x, \mathbf{v}_2(x) \rangle| \leq C_3 \cdot \int_{\frac{\kappa \varepsilon^{1/2}}{2}}^{\delta_0} s^{d-1} \cdot \varphi(s) \, ds
\]
for some constant $C_3 > 0$. Setting $\tilde{\varepsilon} = 2\varepsilon$ we obtain
\[
\langle \mathbf{n}_x, \mathbf{v}_1(x) \rangle \geq C_4 \cdot \int_{\tilde{\varepsilon}}^{\kappa \tilde{\varepsilon}^{1/2}} s^{d-1} \cdot \varphi(s) \, ds
\]
and
\[
|\langle \mathbf{n}_x, \mathbf{v}_2(x) \rangle| \leq C_3 \cdot \int_{\kappa_1 \tilde{\varepsilon}^{1/2}}^{\delta_0} s^{d-1} \cdot \varphi(s) \, ds,
\]
for $\kappa_1 = \frac{\kappa}{2\sqrt{2}}$ and some constant $C_4 > 0$. In particular,
\[
\frac{\langle \mathbf{n}_x, \mathbf{v}_1(x) \rangle}{|\langle \mathbf{n}_x, \mathbf{v}_2(x) \rangle|} \geq \frac{C_4}{C_3} \cdot \frac{\int_{\tilde{\varepsilon}}^{\kappa_1 \tilde{\varepsilon}^{1/2}} s^{d-1} \cdot \varphi(s) \, ds}{\int_{\kappa_1 \tilde{\varepsilon}^{1/2}}^{\delta_0} s^{d-1} \cdot \varphi(s) \, ds} = \frac{C_4}{C_3} \left[ \frac{\int_{\tilde{\varepsilon}}^{\delta_0} s^{d-1} \cdot \varphi(s) \, ds}{\int_{\kappa_1 \tilde{\varepsilon}^{1/2}}^{\delta_0} s^{d-1} \cdot \varphi(s) \, ds} - 1 \right].
\]
If the assumption (A2) holds, then one has
\[
\lim_{r \to 0^+} \frac{r^d \cdot \varphi(r)}{r^d} = + \infty.
\]
This implies
\[
\lim_{r \to 0^+} \int_r^{\delta_0} s^{d-1} \cdot \varphi(s) \, ds \geq \lim_{r \to 0^+} \sup_{r}^{2r} s^{d-1} \cdot \varphi(s) \, ds \geq \frac{2^d - 1}{d2^d} \cdot \lim_{r \to 0^+} (2r)^{d} \cdot \varphi(2r) = + \infty.
\]
Applying L’Hopital’s rule and the assumption (A2), we obtain
\[
\lim_{\tilde{\varepsilon} \to 0^+} \int_{\tilde{\varepsilon}}^{\delta_0} s^{d-1} \cdot \varphi(s) \, ds = \frac{\kappa_1^d}{2} \cdot \lim_{\tilde{\varepsilon} \to 0^+} \frac{\tilde{\varepsilon}^{d/2} \cdot \varphi(\kappa_1 \tilde{\varepsilon}^{1/2})}{\tilde{\varepsilon}^{d-1} \varphi(\tilde{\varepsilon})} = \frac{\kappa_1^d}{2} \cdot \lim_{\tilde{\varepsilon} \to 0^+} \frac{\tilde{\varepsilon}^{d/2} \cdot \varphi(\kappa_1 \tilde{\varepsilon}^{1/2})}{\tilde{\varepsilon}^{d} \varphi(\tilde{\varepsilon})} = 0.
\]
This yields
\[
\left| \langle \mathbf{n}_x, \mathbf{v}_1(x) \rangle \right| \leq \frac{C_3}{C_4} \cdot \int_{\varepsilon \varepsilon}^{\delta_0} s^{d-1} \cdot \varphi(s) \, ds \to 0 \quad \text{as} \quad \varepsilon \to 0.
\]

Finally, observing that
\[
|\langle \mathbf{n}_x, \mathbf{v}_3(x) \rangle| \leq C,
\]
the limit behavior (3.3) is clear. This achieves the proof, because the constants \(C_0, \delta_0, \kappa\) are independent of the point \(y_x \in \Sigma\).

In the following, for \(x \notin \Sigma(t)\) we denote by \(\pi(t, x)\) the perpendicular projection of \(x\) on \(\Sigma(t)\), and call \(\mathbf{n}(t, x) = \frac{x - \pi(t, x)}{|x - \pi(t, x)|}\) the unit normal vector.

**Corollary 1.** Consider a family of compact \(C^2\) surfaces \(\Sigma(t)\), continuously depending on \(t \in [0, T]\), with uniformly bounded curvature. Define the vector fields
\[
\mathbf{v}(t, x) = \int_{\Sigma(t)} \varphi(|x - \xi|) \frac{x - \xi}{|x - \xi|} \, d\sigma(\xi). \quad (3.16)
\]
Then for any \(N\) there exists \(\delta > 0\) such that, for all \(t \in [0, T]\),
\[
d(x, \Sigma(t)) < \delta \quad \implies \quad \langle \mathbf{n}(t, x), \mathbf{v}(t, x) \rangle \geq N. \quad (3.17)
\]
Indeed, this follows from Lemma 1, observing that the limit in (3.3) is uniform over all surfaces \(\Sigma(t)\).

**Proof of Theorem 2.**

1. Let the compact sets \(\Omega_1 \subset \text{int} \Omega_0\) be given, with \(\Omega_1\) simply connected. To fix the ideas, assume
\[
B(\Omega_1, \rho) \subset \Omega_0, \quad (3.18)
\]
for some radius \(\rho > 0\). Given \(T > 0\) and \(0 < \varepsilon < \rho\), we can find a decreasing family of compact sets \(t \mapsto V(t)\) with \(C^2\) boundary, as in (1.10)-(1.11), such that
\[
B(\Omega_0, \varepsilon) \subset V(0), \quad B(\Omega_1, \varepsilon/2) \subset V(T) \subset B(\Omega_1, 3\varepsilon/4). \quad (3.19)
\]
Call \(\Sigma(t) = \partial V(t)\) the boundaries of these sets and define the vector fields
\[
\mathbf{w}(t, x) = \delta_0 \cdot \int_{\Sigma(t)} \varphi(|x - \xi|) \frac{x - \xi}{|x - \xi|} \, d\sigma(\xi). \quad (3.20)
\]
Here the constant \(\delta_0 > 0\) is chosen small enough so that
\[
\delta_0 \cdot \int_{\Sigma(t)} d\sigma \leq 1 \quad \text{for all } t \in [0, T], \quad (3.21)
\]
\[ |w(t, x)| < \frac{\varepsilon}{8T} \quad \text{for all } x \in B(\Omega_1, \varepsilon/2), \quad t \in [0, T]. \quad (3.22) \]

2. For any point \( x_0 \in \Omega_0 \), denote by \( t \mapsto x(t, x_0) \) the solution of
\[
\dot{x}(t) = w(t, x(t)), \quad x(0) = x_0. \quad (3.23)
\]

We claim that
\[ x(t, x_0) \in \text{int} V(t) \quad \text{for all } t \in [0, T]. \quad (3.24) \]

To prove (3.24), let \( L \) be a Lipschitz constant for the multifunction \( t \mapsto \Sigma(t) \), so that the Hausdorff distance between the two boundaries satisfies
\[ d_H(\Sigma(s), \Sigma(t)) \leq L(t - s) \quad \text{for any } 0 < s < t < T. \quad (3.25) \]

For any trajectory \( t \mapsto x(t) \) of (3.23), (3.20), consider the distance
\[ d(t) = \text{dist} \left( x(t), \Sigma(t) \right) \]
of \( x(t) \) from the boundary \( \Sigma(t) = \partial V(t) \). By Corollary 1 there is a constant \( \varepsilon_1 \in [0, \varepsilon] \) such that, for any \( x \in \text{int} V(t) \) with \( \text{dist}(x, \Sigma(t)) \leq \varepsilon_1 \), one has
\[ \left\langle n(t, x), \int_{\Sigma(t)} \varphi(|x - \xi|) \frac{x - \xi}{|x - \xi|} d\sigma(\xi) \right\rangle > \frac{L}{\delta_0}. \quad (3.26) \]

In view of (3.26) and (3.25), if \( d(x(t), \Sigma(t)) \leq \varepsilon_1 \), then the time derivative of the distance \( d(t) \) satisfies
\[ d(t) \geq -L + \left\langle n(t, x(t)), w(t, x(t)) \right\rangle \\
= -L + \left\langle n(t, x(t)), \delta_0 \cdot \int_{\Sigma(t)} \varphi(|x(t) - \xi|) \frac{x(t) - \xi}{|x(t) - \xi|} d\sigma(\xi) \right\rangle \\
> -L + \delta_0 \cdot \frac{L}{\delta_0} = 0. \quad (3.27) \]

If \( x(t) \) is a trajectory starting inside \( \Omega_0 \), then \( d(0) \geq \varepsilon \geq \varepsilon_1 \). By (3.27) we thus have \( d(t) \geq \varepsilon_1 \) for all \( t \in [0, T] \).

We conclude this step by observing that, for every \( x_0 \in \overline{B}(\Omega_1, \varepsilon/4) \), by (3.22) the corresponding trajectory satisfies
\[ |x(t, x_0) - x_0| \leq \frac{\varepsilon t}{8T} \quad \text{for all } t \in [0, T]. \quad (3.28) \]

3. Relying on the approximation procedure developed in [4], we claim that there exists a Lipschitz control \( t \mapsto \xi(t) \) for (1.2)-(1.3) that produces almost the same trajectories as (3.23). More precisely, calling \( t \mapsto x^\xi(t, x_0) \) the solution to
\[ \dot{x} = v(x, \xi(t)), \quad x(0) = x_0, \quad (3.29) \]
for every \( t \in [0, T] \) and \( x_0 \in \Omega_0 \) one has
\[ |x^\xi(t, x_0) - x(t, x_0)| < \frac{\varepsilon}{8}. \quad (3.30) \]
Toward this goal, for any $t \in [0,T]$, define $\mu^t$ to be the $(d-1)$-dimensional measure supported on $\Sigma(t)$, so that

$$\mu^t(A) = \int_{A \cap \Sigma(t)} d\sigma$$

for every open set $A \subset \mathbb{R}^d$. By (3.21) it follows

$$\delta_0 \mu^t(\mathbb{R}^d) = \delta_0 \cdot \text{[surface area of } \Sigma(t)\text{]} \leq 1$$

for all $t \in [0,T]$.

We now choose a point $\bar{x} \in \mathbb{R}^d$ very far from the origin and define the probability measure

$$\tilde{\mu}^t = \delta_0 \mu^t + \left(1 - \delta_0 \mu^t(\mathbb{R}^d)\right)m_{\bar{x}}$$

where $m_{\bar{x}}$ denotes a unit Dirac mass at $\bar{x}$.

Notice that, as $|\bar{x}| \to +\infty$, by the second limit in (1.1) the vector

$$\tilde{w}(t, x) = \int \phi(|x - \xi|) \frac{x - \xi}{|x - \xi|} d\tilde{\mu}^t(\xi)$$

approaches $\delta_0 w(t, x)$, uniformly on compact subsets of $\mathbb{R}^d \setminus \Sigma(t)$.

Given an integer $n \geq 1$, we split the interval $[0,T]$ into $n$ equal subintervals, inserting the points $t_i = iT/n$, $i = 0, 1, \ldots, n$. For each $i$, the probability measure $\tilde{\mu}^{t_i}$ can now be approximated by the sum of $N$ equal masses, say located at $\xi_{i1}, \ldots, \xi_{iN}$. Defining the time step $h = T/nN$, we then consider the control function

$$\xi(t) = \xi_{ij} \quad \text{if} \quad t_i + (j - 1)h < t \leq t_i + jh.\quad (3.32)$$

The same arguments used in [4] now show that, as $n,N \to \infty$, by suitably choosing the points $\xi_{ij}$, trajectories of the ODE (1.3), (3.32) converge to the corresponding trajectories of $\dot{x} = \tilde{v}(t, x)$. Moreover, the convergence is uniform for all initial data in the compact set $\Omega_0 \subset \mathbb{R}^d \setminus \Sigma(0)$.

Finally, we can replace the piecewise constant function $\xi(\cdot)$ by a Lipschitz function $\tilde{\xi}(\cdot)$. If $\|\tilde{\xi} - \xi\|_{L^1}$ is sufficiently small, the corresponding trajectories still satisfy the same estimate (3.30).

4. Recalling that

$$x(T, x_0) \in V(T) \subseteq B(\Omega_1, 3\varepsilon/4),$$

by (3.30) we now conclude

$$\Omega^\xi(T) = \{x^\xi(T, x_0); \ x_0 \in \Omega_0\} \subseteq B(\Omega_1, \varepsilon).\quad (3.33)$$

This establishes the second inclusion in (1.6).

To prove the first inclusion, consider the continuous map $x_0 \mapsto x^\xi(T, x_0)$ from the compact set $\overline{B}(\Omega_1, \varepsilon/4)$ into $\mathbb{R}^d$. By (3.28) and (3.30) it follows

$$|x^\xi(T, x_0) - x_0| \leq \frac{\varepsilon}{4} \quad \text{for all } x_0 \in \overline{B}(\Omega_1, \frac{3\varepsilon}{4}).\quad (3.34)$$

For any given $y \in \Omega_1$, define the continuous map $g^y : \overline{B}(0, \varepsilon/4) \mapsto \mathbb{R}^d$ by setting

$$g^y(z) = x^\xi(T, y - z) - (y - z) \quad \text{for all } z \in \overline{B}(0, \varepsilon/4).$$
By (3.34) one has
\[ g^y(B(0,\varepsilon/4)) \subseteq B(0,\varepsilon/4). \]
Therefore, Brouwer’s fixed point theorem implies
\[ g^y(z_0) = z_0 \quad \text{for some } z_0 \in B(0,\varepsilon/4). \]
This yields
\[ y = x^\xi(T,x_0) \quad \text{with} \quad x_0 = y - z_0 \in B(y,\varepsilon/4). \]
Hence \( \Omega_1 \subseteq \Omega^\xi(T) \).

5. Finally, to pass from approximate controllability to exact controllability one can split the interval \([0,T]\), inserting an increasing sequence of times \( \tau_j \) with \( \tau_j \to T^- \) as \( j \to \infty \). Then construct Lipschitz controls \( t \mapsto \xi(t) \) on each subinterval \([\tau_{j-1}, \tau_j]\) such that the corresponding sets \( \Omega^\xi(\tau_j) \) satisfy
\[ B(\Omega_1,2^{-j}) \subseteq \Omega^\xi(\tau_j) \subseteq B(\Omega_1,2^{1-j}). \]

4 Approximating a sweeping process

The key tool for the proof of Theorem 3 is the following lemma, which improves on Lemma 1 under the stronger assumption \((A2')\).

**Lemma 2.** Let \( \Omega \subset \mathbb{R}^d \) be a compact set with \( C^2 \) boundary \( \Sigma = \partial \Omega \). Let \( \mathbf{v} \) be the vector field in (3.1). If the function \( \varphi \) satisfies \((A1)-(A2')\), then
\[
|\mathbf{v}(x)| \to +\infty \quad \text{as} \quad d(x,\Sigma) \to 0, \quad (4.1)
\]
\[
\frac{|\mathbf{v}(x) - x - \pi(x)|}{|\mathbf{v}(x)|} \to 0 \quad \text{as} \quad d(x,\Sigma) \to 0. \quad (4.2)
\]

**Proof. 1.** Consider any point \( x \) sufficiently close to \( \Sigma \) so that the perpendicular projection \( y_x = \pi(x) \) at (3.2) is well defined. As in the proof of Lemma 1, in a neighborhood of \( y_x \), the surface \( \Sigma \) can be expressed as the graph of a function \( f : V_x \to \mathbb{R} \). More precisely, call \( \varepsilon = d(x,\Sigma) = |x - y_x| \). Then, given \( \frac{1}{2} < \alpha < 1 \), we can write \( \Sigma = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \), where
\[
\begin{cases}
\Sigma_1 = \{ y + f(y) \mathbf{n}_x ; \ y \in V_x, \ |y| \leq \varepsilon^\alpha \}, \\
\Sigma_2 = \{ y + f(y) \mathbf{n}_x ; \ y \in V_x, \ \varepsilon^\alpha < |y| \leq \delta_0 \}, \\
\Sigma_3 = \Sigma \setminus (\Sigma_1 \cup \Sigma_2). 
\end{cases} \quad (4.3)
\]
Notice that, by the regularity and compactness of the surface \( \Sigma \), we can assume that the radius \( \delta_0 \) of the ball where the function \( f \) is defined is independent of \( y_x \in \Sigma \). Moreover, the \( C^2 \) norm of \( f \) remains uniformly bounded.
Without loss of generality, in the following computations we shall assume $y_x = 0 \in \mathbb{R}^d$. By construction we again have the bounds (3.4), valid for some constant $C_0$, uniform w.r.t. $y_x \in \Sigma$. We shall estimate the vector field

$$\mathbf{v}(x) = \mathbf{v}_1(x) + \mathbf{v}_2(x) + \mathbf{v}_3(x)$$

by splitting the integral (3.1) in three parts, as in (3.7). Notice, however, that now we refer to the different decomposition (4.3) of the surface $\Sigma$.

2. Calling $J(y) = \sqrt{1 + |\nabla f(y)|^2}$ the Jacobian determinant of the map $y \mapsto y + f(y)n_x$ from $V_x \cap B(y_x, \delta_0)$ into $\Sigma$, we have

$$\mathbf{v}_1(x) = \int_{|y| < \varepsilon} \varphi(|x - y - f(y)n_x|) \cdot \frac{x - y - f(y)n_x}{|x - y - f(y)n_x|} J(y) \, dy. \tag{4.4}$$

We write

$$\mathbf{v}_1(x) = \mathbf{v}_{11}(x) + \mathbf{v}_{12}(x),$$

where

$$\mathbf{v}_{11}(x) = \int_{|y| < \varepsilon} \varphi(|x - y|) \frac{x - y}{|x - y|} \, dy, \quad \mathbf{v}_{12}(x) = \mathbf{v}_1(x) - \mathbf{v}_{11}(x). \tag{4.5}$$

Notice that $\mathbf{v}_{11}(x)$ is a vector parallel to $n_x$ and is computed as

$$\mathbf{v}_{11}(x) = \left( c_0 \int_0^{\varepsilon} \varepsilon^{d-2} \varphi \left( \sqrt{\varepsilon^2 + r^2} \right) \cdot \frac{\varepsilon}{\sqrt{\varepsilon^2 + r^2}} \, dr \right) \cdot n_x, \tag{4.6}$$

for some constant $c_0 > 0$. Hence, in order to obtain the limit (4.2), let’s first prove that

$$\frac{|\mathbf{v}_{12}(x)|}{|\mathbf{v}_{11}(x)|} \to 0 \quad \text{as} \quad \varepsilon \to 0. \tag{4.7}$$

The vector $\mathbf{v}_{12}(x)$ satisfies

$$\begin{align*}
\mathbf{v}_{12}(x) &= \int_{|y| < \varepsilon} \left[ \varphi(|x - y - f(y)n_x|) - \varphi(|x - y|) \right] \cdot \frac{x - y - f(y)n_x}{|x - y - f(y)n_x|} J(y) \, dy \\
&\quad + \int_{|y| < \varepsilon} \varphi(|x - y|) \cdot \left\{ \frac{x - y - f(y)n_x}{|x - y - f(y)n_x|} - \frac{x - y}{|x - y|} \right\} J(y) \, dy \\
&\quad + \int_{|y| < \varepsilon} \varphi(|x - y|) \cdot \frac{x - y}{|x - y|} [J(y) - 1] \, dy \\
&= A_1 + A_2 + A_3.
\end{align*} \tag{4.8}$$

In the following, recalling (3.4), we use the bounds

$$|f(y)| \leq \frac{C_0}{2} |y|^2, \quad |J(y) - 1| = O(1) \cdot |y|^2, \tag{4.9}$$
and the identities (3.8). Since the function \( \varphi \) is decreasing and \( r \leq \varepsilon^\alpha \), using (3.9) one obtains the estimate

\[
|A_1| \leq C \cdot \int_0^{\varepsilon^\alpha} r^{d-2} \left[ \varphi\left( \sqrt{\varepsilon^2 + (1 - C_0\varepsilon)r^2} \right) - \varphi\left( \sqrt{\varepsilon^2 + r^2} \right) \right] dr \\
+ C \cdot \int_0^{\varepsilon^\alpha} r^{d-2} \left[ \varphi\left( \sqrt{\varepsilon^2 + r^2} \right) - \varphi\left( \sqrt{\varepsilon^2 + (1 + 2C_0\varepsilon)r^2} \right) \right] dr
\]

\[
= A_{11} + A_{12},
\]

for some constant \( C \) and all \( \varepsilon > 0 \) sufficiently small. In addition, we have

\[
| \frac{x - y - f(y)n_x}{|x - y - f(y)n_x|} - \frac{x - y}{|x - y|} | \leq | \frac{|x - y| - |x - y - f(y)n_x|}{|x - y|} | + | \frac{f(y)}{|x - y|} | \\
\leq 2 \cdot | \frac{f(y)}{|x - y|} | \leq C_0 \cdot | \frac{|y|^2}{|x - y|} | .
\]

Consequently,

\[
|A_2| \leq C \cdot \int_{|y|<\varepsilon^\alpha} \varphi\left( \frac{|x - y|}{|x - y|} \right) \cdot \frac{|y|^2}{|x - y|} dy = C \cdot c_0 \int_0^{\varepsilon^\alpha} r^{d-2} \varphi\left( \sqrt{\varepsilon^2 + r^2} \right) \cdot \frac{r^2}{\sqrt{\varepsilon^2 + r^2}} dr \\
\leq C \cdot c_0 \int_0^{\varepsilon^\alpha} r^{d-2} \varphi\left( \sqrt{\varepsilon^2 + r^2} \right) \cdot \frac{\varepsilon^{2\alpha}}{\sqrt{\varepsilon^2 + r^2}} dr
\]

and

\[
|A_3| \leq C \cdot c_0 \int_0^{\varepsilon^\alpha} r^{d-2} \varphi\left( \sqrt{\varepsilon^2 + r^2} \right) r^2 dr \leq C \cdot c_0 \int_0^{\varepsilon^\alpha} r^{d-2} \varphi\left( \sqrt{\varepsilon^2 + r^2} \right) \cdot \frac{\varepsilon^{2\alpha}}{\sqrt{\varepsilon^2 + r^2}} dr
\]

for a suitable constant \( C \).

Since we are choosing \( \alpha > 1/2 \), comparing (4.12) and (4.13) with (4.6), it is clear that

\[
\frac{|A_2| + |A_3|}{|v_{11}(x)|} \leq 2C\varepsilon^{2\alpha-1} \to 0 \quad \text{as} \quad \varepsilon \to 0.
\]

Proving a similar estimate for \( A_1 \) requires more work. Performing the variable change

\[
s = \sqrt{1 - C_0 \varepsilon} \cdot r,
\]

one obtains

\[
0 \leq \int_0^{\varepsilon^\alpha} r^{d-2} \left[ \varphi\left( \sqrt{\varepsilon^2 + (1 - C_0\varepsilon)r^2} \right) - \varphi\left( \sqrt{\varepsilon^2 + r^2} \right) \right] dr \\
\leq (1 - C_0\varepsilon)^{-\frac{d-1}{2}} \int_0^{\varepsilon^\alpha} \sqrt{1 - C_0\varepsilon} s^{d-2} \varphi\left( \sqrt{\varepsilon^2 + s^2} \right) ds - \int_0^{\varepsilon^\alpha} r^{d-2} \varphi\left( \sqrt{\varepsilon^2 + r^2} \right) dr \\
\leq [(1 - C_0\varepsilon)^{-\frac{d-1}{2}} - 1] \int_0^{\varepsilon^\alpha} r^{d-2} \varphi\left( \sqrt{\varepsilon^2 + r^2} \right) dr \\
\leq C_1 \varepsilon \int_0^{\varepsilon^\alpha} r^{d-2} \varphi\left( \sqrt{\varepsilon^2 + r^2} \right) dr,
\]

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for some constant $C_1$. Recalling (4.10) and comparing with (4.6), we thus obtain

$$A_{11} \leq C_2 \int_0^{\epsilon^\alpha} r^{d-2} \varphi \left( \sqrt{\epsilon^2 + r^2} \right) dr$$

$$\leq C_2 \sqrt{\epsilon^2 + \epsilon^{2\alpha}} \int_0^{\epsilon^\alpha} r^{d-2} \varphi \left( \sqrt{\epsilon^2 + r^2} \right) \frac{\epsilon}{\sqrt{\epsilon^2 + r^2}} dr$$

$$= \frac{C_2}{c_0} \sqrt{\epsilon^2 + \epsilon^{2\alpha}} \cdot |v_{11}(x)| \leq \frac{2C_2 \epsilon^\alpha}{c_0} \cdot |v_{11}(x)|$$

for some constant $C_2$. A similar argument yields

$$A_{12} \leq C \left[ \int_0^{\epsilon^\alpha} r^{d-2} \varphi \left( \sqrt{\epsilon^2 + r^2} \right) dr - \left( \frac{1}{1 + 2C_0 \epsilon} \right)^{d-1} \int_0^{\epsilon^\alpha} r^{d-2} \varphi \left( \sqrt{\epsilon^2 + r^2} \right) dr \right]$$

$$\leq C_3 \epsilon \int_0^{\epsilon^\alpha} r^{d-2} \varphi \left( \sqrt{\epsilon^2 + r^2} \right) dr \leq \frac{2C_3 \epsilon^\alpha}{c_0} \cdot |v_{11}(x)|.$$  \hspace{1cm} (4.17)

Putting together (4.10), (4.14), (4.16), and (4.17), we can compare the sizes of the vectors $v_{11}$ and $v_{12}$ in (4.5)–(4.8). Indeed, the previous analysis shows that

$$\frac{|v_{12}(x)|}{|v_{11}(x)|} \leq \frac{|A_1| + |A_2| + |A_3|}{|v_{11}(x)|} \leq \frac{A_{11} + A_{12} + |A_2| + |A_3|}{|v_{11}(x)|} \to 0 \quad \text{as} \quad \epsilon \to 0. \hspace{1cm} (4.18)$$

3. In a similar fashion we now compute

$$v_2(x) = v_{21}(x) + v_{22}(x),$$

where

$$v_{21}(x) = \int_{\epsilon^\alpha < |y| < \delta_0} \varphi(|x-y|) \cdot \frac{x-y}{|x-y|} dy$$

$$= \left( c_0 \int_{\epsilon^\alpha}^{\delta_0} r^{d-2} \varphi \left( \sqrt{\epsilon^2 + r^2} \right) \cdot \frac{\epsilon}{\sqrt{\epsilon^2 + r^2}} dr \right) n_x,$$ \hspace{1cm} (4.20)

and

$$v_{22}(x) = \int_{\epsilon^\alpha < |y| < \delta_0} \left[ \varphi(|x-y - f(y)n_x|) - \varphi(|x-y|) \right] \cdot \frac{x-y - f(y)n_x}{|x-y - f(y)n_x|} \cdot J(y) dy$$

$$+ \int_{\epsilon^\alpha < |y| < \delta_0} \varphi(|x-y|) \cdot \left\{ \frac{x-y - f(y)n_x}{|x-y - f(y)n_x|} - \frac{x-y}{|x-y|} \right\} J(y) dy$$

$$+ \int_{\epsilon^\alpha < |y| < \delta_0} \varphi(|x-y|) \cdot \frac{x-y}{|x-y|} [J(y) - 1] dy$$

$$= B_1 + B_2 + B_3.$$ \hspace{1cm} (4.21)

As in (4.12)-(4.13), we have

$$|B_2| + |B_3| \leq C_3 \int_{\epsilon^\alpha}^{\delta_0} r^{d-1} \varphi \left( \sqrt{\epsilon^2 + r^2} \right) dr \leq C_3 \int_{\epsilon^\alpha}^{\delta_0} r^{d-1} \varphi \left( \sqrt{\epsilon^2 + r^2} \right) dr. \hspace{1cm} (4.22)$$
Using again (3.9) and the fact that $\varphi$ is a decreasing function, we obtain
\[
|B_1| \leq C \cdot \int_{\epsilon}^{\delta_0} r^{d-2} \left[ \varphi\left(\sqrt{\varepsilon^2 + (1 - C_0 \varepsilon)r^2}\right) - \varphi\left(\sqrt{\varepsilon^2 + r^2}\right) \right] dr
+ \int_{\epsilon}^{\delta_0} r^{d-2} \left[ \varphi\left(\sqrt{\varepsilon^2 + r^2}\right) - \varphi\left(\sqrt{\varepsilon^2 + (1 + C_0 \varepsilon)r^2 + C_0^2 r^4/4}\right) \right] dr
\]
\[
= B_{11} + B_{12}.
\]
As in (4.15), performing the variable change $s = \sqrt{1 - C_0 \varepsilon} \cdot r$, we obtain
\[
B_{11} = C \cdot \int_{\epsilon}^{\delta_0} r^{d-2} \left[ \varphi\left(\sqrt{\varepsilon^2 + (1 - C_0 \varepsilon)r^2}\right) - \varphi\left(\sqrt{\varepsilon^2 + r^2}\right) dr \right]
= C \cdot \int_{\epsilon}^{\delta_0} r^{d-2} \varphi\left(\sqrt{\varepsilon^2 + s^2}\right) ds - \int_{\epsilon}^{\delta_0} r^{d-2} \varphi\left(\sqrt{\varepsilon^2 + r^2}\right) dr
\]
\[
\leq C_3 \varepsilon \int_{\epsilon}^{\delta_0} r^{d-2} \varphi\left(\sqrt{\varepsilon^2 + r^2}\right) dr + C_3 \int_{\epsilon}^{\delta_0} r^{d-2} \varphi\left(\sqrt{\varepsilon^2 + r^2}\right) dr
\]
for a suitable constant $C_3$. Since $\varphi$ is decreasing, for $\varepsilon$ sufficiently small we have
\[
\int_{0}^{\varepsilon} r^{d-2} \varphi\left(\sqrt{\varepsilon^2 + r^2}\right) dr \geq \int_{\varepsilon}^{\delta_0 (1 - C_3 \varepsilon)} r^{d-2} \varphi\left(\sqrt{\varepsilon^2 + r^2}\right) dr
\]
\[
\geq (1 - C_3 \varepsilon)^{d-2} \cdot \int_{\varepsilon}^{\delta_0} \varepsilon^{d-2} \varphi\left(\sqrt{\varepsilon^2 + r^2}\right) dr
\]
\[
\geq (1 - C_3 \varepsilon)^{d-2} \cdot \frac{C_3 \varepsilon^{a_0 + \frac{1}{2}}}{C_3 \varepsilon^{a_0 + 1}} \cdot \int_{\varepsilon}^{\delta_0 (1 - C_3 \varepsilon)} r^{d-2} \varphi\left(\sqrt{\varepsilon^2 + r^2}\right) dr
\]
\[
\geq \frac{1}{2 \sqrt{\varepsilon}} \cdot \int_{\varepsilon}^{\delta_0 (1 - C_3 \varepsilon)} r^{d-2} \varphi\left(\sqrt{\varepsilon^2 + r^2}\right) dr.
\]
In turn, this yields
\[
B_{11} \leq C_3 \varepsilon \int_{\varepsilon}^{\delta_0} r^{d-2} \varphi\left(\sqrt{\varepsilon^2 + r^2}\right) dr + 2C_3 \varepsilon \int_{0}^{\varepsilon} r^{d-2} \varphi\left(\sqrt{\varepsilon^2 + r^2}\right) dr
\]
\[
\leq C_3 \varepsilon \int_{\varepsilon}^{\delta_0} r^{d-2} \varphi\left(\sqrt{\varepsilon^2 + r^2}\right) dr + 2C_3 \varepsilon \int_{\varepsilon}^{\delta_0} \varepsilon^{d-2} \cdot \varphi\left(\sqrt{\varepsilon^2 + r^2}\right) dr
\]
\[
\leq C_3 \varepsilon \int_{\varepsilon}^{\delta_0} r^{d-2} \varphi\left(\sqrt{\varepsilon^2 + r^2}\right) dr + \int_{\varepsilon}^{\delta_0} r^{d-2} \cdot \varphi\left(\sqrt{\varepsilon^2 + r^2 + C_0^2 r^4/4}\right) dr
\]
\[
\geq \int_{\varepsilon}^{\delta_0} \varphi\left(\sqrt{\varepsilon^2 + (1 + C_0 \varepsilon)r^2 + C_0^2 r^4/4}\right) dt.
\]
for a suitable constant $C_4$. This implies that

$$ B_{12} \leq C \left[ \int_{\varepsilon}^{\delta_0} r^{d-2} \varphi \left( \sqrt{\varepsilon^2 + r^2} \right) dr - \int_{\varepsilon}^{\delta_0} \frac{r^{d-2}}{1 + 2C_0\sqrt{t}} \cdot \varphi \left( \sqrt{\varepsilon^2 + r^2} \right) dr \right] $$

$$ \leq C \left[ \int_{\varepsilon}^{\delta_0} \left( 1 - \frac{1}{1 + C_4(\varepsilon + r^2)} \right) r^{d-2} \varphi \left( \sqrt{\varepsilon^2 + r^2} \right) dr + \int_{\varepsilon}^{\delta_0} \varepsilon^{(1+2C_0\sqrt{t})} r^{d-2} \varphi \left( \sqrt{\varepsilon^2 + r^2} \right) dr \right] $$

$$ \leq CC_4 \int_{\varepsilon}^{\delta_0} (\varepsilon + r^2) r^{d-2} \varphi \left( \sqrt{\varepsilon^2 + r^2} \right) dr + C \int_{\varepsilon}^{\delta_0} \varepsilon^{(1+2C_0\sqrt{t})} r^{d-2} \varphi \left( \sqrt{\varepsilon^2 + r^2} \right) dr. $$

As in (4.25), one estimates

$$ \int_{\varepsilon}^{\delta_0} (\varepsilon + r^2) r^{d-2} \varphi \left( \sqrt{\varepsilon^2 + r^2} \right) \leq 2\sqrt{\varepsilon} \int_{0}^{\delta_0} r^{d-2} \varphi \left( \sqrt{\varepsilon^2 + r^2} \right) dr $$

for $\varepsilon$ sufficiently small. Thus, as in (4.23), we obtain

$$ B_{12} \leq CC_4 \int_{\varepsilon}^{\delta_0} (\varepsilon + r^2) r^{d-2} \varphi \left( \sqrt{\varepsilon^2 + r^2} \right) dr + 2C\sqrt{\varepsilon} \int_{0}^{\delta_0} r^{d-2} \varphi \left( \sqrt{\varepsilon^2 + r^2} \right) dr $$

$$ \leq CC_4 \int_{\varepsilon}^{\delta_0} (\varepsilon + r^2) r^{d-2} \varphi \left( \sqrt{\varepsilon^2 + r^2} \right) dr + 4C\varepsilon^{\alpha - \frac{1}{2}} \cdot |v_{11}(x)|. $$

Combining (4.21), (4.22), (4.23), (4.26), and (4.27), we finally obtain

$$ |v_{22}(x)| \leq C_5 \cdot \int_{\varepsilon}^{\delta_0} (\varepsilon + r^2) r^{d-2} \varphi \left( \sqrt{\varepsilon^2 + r^2} \right) dr + C_5\varepsilon^{\alpha - \frac{1}{2}} \cdot |v_{11}(x)| $$

for some constant $C_5$.

4. Finally, since in (4.8) the integral over $\Sigma_3$ involves functions which are uniformly bounded over $\Sigma$, we have a trivial bound of the form

$$ |v_3(x)| \leq C_6. $$

5. We now compare the sizes of $v_{22}(x)$ and $v_3(x)$ with $v_{11}(x)$. From (4.6) it follows

$$ |v_{11}(x)| \geq c_0 \int_{0}^{\varepsilon} r^d \cdot \varphi \left( \frac{\sqrt{\varepsilon^2 + r^2}}{\sqrt{\varepsilon^2 + r^2}} \right) dr. $$

Performing the change of variable $t = \sqrt{\varepsilon^2 + r^2}$ one obtains

$$ |v_{11}(x)| \geq c_0 \int_{\varepsilon}^{\sqrt{\varepsilon^2 + r^2}} (t^2 - \varepsilon^2)^{\alpha - \frac{d}{2}} \varphi(t) dt \geq c_0 \left( \frac{3}{4} \right)^{\alpha - \frac{d}{2}} \cdot \int_{2\varepsilon}^{\delta_0} r^{d-1} \varphi(r) dr. $$

Recalling (4.28) and (4.29), we obtain

$$ \frac{|v_{22}(x)| + |v_3(x)|}{|v_{11}(x)|} \leq C_7 \cdot \int_{\varepsilon}^{\delta_0} r^{d-1} \varphi(r) dr + \frac{C_5 \varepsilon^{\alpha - \frac{1}{2}} + C_7}{|v_{11}(x)|}, $$

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for some constant \( C_7 \).

By (4.18) we already know that the ratio \(|v_{12}|/|v_{11}|\) approaches zero as \( \varepsilon \to 0 \). Moreover, by (4.6) and (4.20) one has

\[
v_{21}(x) = C \varepsilon v_{11}(x)
\]

for some constant \( C \varepsilon > 0 \).

Therefore, in view of (4.31), we can conclude that (4.2) holds true provided that

\[
\lim_{\varepsilon \to 0^+} \frac{\int_{\varepsilon}^{1} r^{d-1} \varphi(r)\, dr}{\int_{\varepsilon}^{1} r^{d-1} \varphi(r)\, dr} = 0.
\]

We show that (4.32) is satisfied if \( \varphi \) satisfies the assumption \((A2')\). Indeed, let \( \beta > \frac{1}{2} \) be as in \((A2')\) and choose \( \frac{1}{2} < \alpha < \beta \). Setting \( \tilde{\varepsilon} = 2\varepsilon \), we have \( \varepsilon^\alpha > \tilde{\varepsilon}^\beta \) if \( \varepsilon \) is sufficiently small. Consequently,

\[
\int_{\varepsilon}^{1} r^{d-1} \varphi(r)\, dr \leq \int_{\tilde{\varepsilon}}^{1} r^{d-1} \varphi(r)\, dr \quad \text{and} \quad \int_{\varepsilon}^{\tilde{\varepsilon}} r^{d-1} \varphi(r)\, dr \geq \int_{\tilde{\varepsilon}}^{\varepsilon} r^{d-1} \varphi(r)\, dr.
\]

On the other hand, as in the proof of Lemma 1, it holds

\[
\lim_{r \to 0^+} \int_{r}^{1} s^{d-1} \varphi(s)\, ds = +\infty.
\]

Using L'Hopital’s rule and the assumption \((A2')\), we obtain

\[
\lim_{\varepsilon \to 0^+} \frac{\int_{\tilde{\varepsilon}}^{1} r^{d-1} \varphi(r)\, dr}{\int_{\varepsilon}^{\varepsilon} r^{d-1} \varphi(r)\, dr} = \lim_{\varepsilon \to 0^+} \frac{\tilde{\varepsilon}^\beta (d-1) \varphi(\tilde{\varepsilon}) \cdot (\beta \tilde{\varepsilon}^{\beta-1})}{\varepsilon^{d-1} \cdot \varphi(\tilde{\varepsilon})} = \beta \cdot \lim_{\varepsilon \to 0^+} \frac{\tilde{\varepsilon}^{d} \cdot \varphi(\tilde{\varepsilon})}{\tilde{\varepsilon}^{d} \cdot \varphi(\varepsilon)} = 0.
\]

This implies

\[
\lim_{\varepsilon \to 0^+} \frac{\int_{\varepsilon}^{1} r^{d-1} \varphi(r)\, dr}{\int_{\varepsilon}^{\varepsilon} r^{d-1} \varphi(r)\, dr} \leq \lim_{\varepsilon \to 0^+} \frac{\int_{\tilde{\varepsilon}}^{1} r^{d-1} \varphi(r)\, dr}{\int_{\tilde{\varepsilon}}^{\varepsilon} r^{d-1} \varphi(r)\, dr} = 0,
\]

proving (4.32).

\[\square\]

**Corollary 2.** Consider a family of compact \( C^2 \) surfaces \( \Sigma(t) \), continuously depending on \( t \in [0, T] \), with uniformly bounded curvature. Define the vector fields \( \mathbf{v}(t, \cdot) \) as in (3.16). Then for any \( N, \varepsilon > 0 \) there exists \( \delta > 0 \) such that, for all \( t \in [0, T] \),

\[
d(x, \Sigma(t)) \leq \delta \quad \implies \quad |\mathbf{v}(t, x)| \geq N,
\]

\[
d(x, \Sigma(t)) \leq \delta \quad \implies \quad \left| \frac{\mathbf{v}(t, x)}{|\mathbf{v}(t, x)|} - \frac{x - \pi(t, x)}{|x - \pi(t, x)|} \right| \leq \varepsilon.
\]

Indeed, the proof of Lemma 2 shows that the limits (4.1)-(4.2) are uniformly valid over a family of surfaces \( \Sigma(t) \) with uniformly bounded curvature.
5 Proof of Theorem 3.

Relying on Lemma 2, we can now give a proof of the convergence to the sweeping process, stated in (1.16). We recall that this sweeping process keeps all trajectories inside a moving compact set \( V(t) \subset \mathbb{R}^d \) with smooth boundary \( \Sigma(t) \). To fix the ideas, we assume that this set is defined in terms of a \( C^2 \) function \( \psi \), as in (1.10)-(1.11). The argument relies on three main properties.

(P1) There exists a radius \( \rho_0 > 0 \) such that, if \( t \in [0, T] \) and \( d(x, \Sigma(t)) \leq \rho_0 \), then the perpendicular projection \( \pi(t, x) \) of \( x \) on \( \Sigma(t) \) is well defined. In this case we denote by \( n(t, x) \) the unit normal vector to \( \Sigma(t) \) at the point \( \pi(t, x) \), as in (3.15)

\[
\text{To see why this is true, assume } d(x^{\delta}(t), \Sigma(t)) < \rho_0 \text{ and consider the unit normal vector }
\]

\[
\mathbf{n}(t, x) = \frac{x^{\delta}(t) - \pi(t, x^{\delta}(t))}{|x^{\delta}(t) - \pi(t, x^{\delta}(t))|}.
\]  

(P2) Setting

\[
v(t, x) = \int_{\Sigma(t)} \varphi(|x - \xi|) \frac{x - \xi}{|x - \xi|} d\sigma(\xi), \quad (5.1)
\]

for any \( \delta > 0 \) the solution \( t \mapsto x^{\delta}(t) \) to

\[
\dot{x} = \delta v(t, x), \quad x(0) = x_0. \quad (5.2)
\]

satisfies \( x^{\delta}(t) \in V(t) \), for every \( x_0 \in int(V(0)) \) and \( t \in [0, T] \).

To see why this is true, assume \( d(x^{\delta}(t), \Sigma(t)) < \rho_0 \) and consider the unit normal vector

\[
\mathbf{n}(t, x^{\delta}) = \frac{x^{\delta}(t) - \pi(t, x^{\delta}(t))}{|x^{\delta}(t) - \pi(t, x^{\delta}(t))|}. \quad (5.3)
\]

Then

\[
\frac{d}{dt} \left( d(x^{\delta}(t), \Sigma(t)) \right) \geq \left\langle \delta v(t, x^{\delta}(t)), \mathbf{n}(t, x^{\delta}) \right\rangle - L_{\Sigma},
\]

where \( L_{\Sigma} \) is a Lipschitz constant for the multifunction \( t \mapsto \Sigma(t) \). By (4.1)-(4.2), it follows that

\[
\left\langle \mathbf{v}(t, y), \mathbf{n}(t, y) \right\rangle \to +\infty \quad \text{as} \quad d(y, \Sigma(t)) \to 0.
\]

Hence the distance \( d(x^{\delta}(t), \Sigma(t)) \) remains uniformly positive in time, for every fixed \( \delta > 0 \) and all initial points \( x_0 \) at a uniformly positive distance from \( \Sigma(0) \).

Finally, by the properties (1.1) of \( \varphi \) it follows

(P3) For every \( 0 < \varepsilon < \frac{1}{4} \), by choosing \( 0 < \delta < \delta_0 < \varepsilon \) sufficiently small, for every \( x \in V(t) \) one has the implication

\[
0 < d(x, \Sigma(t)) \leq \delta_0 \implies \left| \mathbf{v}(t, x) - \frac{x - \pi(t, x)}{|\mathbf{v}(t, x)|} \frac{|x - \pi(t, x)|}{|x - \pi(t, x)|} \right| < \varepsilon. \quad (5.4)
\]

\[
d(x, \Sigma(t)) > \delta_0 \implies \delta |\mathbf{v}(t, x)| < \varepsilon. \quad (5.5)
\]
Indeed, (5.4) follows from Corollary 2 and the properties (1.11) of the function \( \psi \), defining the boundary \( \Sigma(t) \). The implication (5.5) trivially holds, choosing \( \delta > 0 \) sufficiently small.

In the following, using the properties (P1)–(P3), we estimate the distance between \( x^\delta(t) \) and the solution \( x(t, x_0) \) of the sweeping process (1.12). The proof will be given in several steps.

1. For a given initial condition \( x_0 \in \text{int} V(0) \), let \( t \mapsto x^\delta(t) \) be the solution to

\[
\dot{x} = \delta v(t, x), \quad x(0) = x_0,
\]

and let \( t \mapsto x(t) \) be the corresponding solution to the sweeping process driven by the set \( V(t) \).

For every \( t \in [0, T] \) such that \( d(x(t), \Sigma(t)) \leq \rho_0/2 \), let \( n(t) \) be the unit normal vector to \( \Sigma(t) \) at the point \( \pi(t, x(t)) \). By the regularity of \( \Sigma(\cdot) \), we can extend \( n(\cdot) \) to a Lipschitz function defined on the entire time interval \([0, T]\). For simplicity, this extension will still be denoted by \( t \mapsto n(t) \).

We now split the difference as

\[
w(t) = x^\delta(t) - x(t) = w_1(t) + w_2(t), \tag{5.6}
\]

where the vector \( w_1(t) \) is parallel to \( n(t) \) while \( w_2(t) \) is orthogonal to \( n(t) \). Namely,

\[
w_1(t) = \theta(t) n(t), \quad \theta(t) = \langle w(t), n(t) \rangle, \quad w_2(t) = w(t) - w_1(t). \tag{5.7}
\]

For future use, we observe that, if \( d(x^\delta(t), \Sigma(t)) \leq \rho_0 \), then the unit normal vector \( n(t, x^\delta) \) at (5.3) is well defined and

\[
|n(t) - n(t, x^\delta)| \leq C_n |w(t)|, \tag{5.8}
\]

for a suitable constant \( C_n \).

Our main goal is to show that \( w(t) \) remains small. This will be achieved by estimating the time derivatives \( \dot{w}_1(t), \dot{w}_2(t) \), considering two possible alternatives (see Fig. 3).

CASE 1: \( d(x^\delta(t), \Sigma(t)) \geq \delta_0 \).

CASE 2: \( d(x^\delta(t), \Sigma(t)) < \delta_0 \).

We observe that, by the \( C^2 \) regularity of the boundaries \( \Sigma(t) = \partial V(t) \), there exists a constant \( C_\Sigma \) such that

\[
\langle n(t, x), y - x \rangle \geq -C_\Sigma \cdot |y - x|^2 \quad \text{for all} \quad t \in [0, T], \quad x \in \Sigma(t), \quad y \in V(t). \tag{5.9}
\]

In particular,

\[
x(t) \in \Sigma(t) \implies \theta(t) \geq -C_\Sigma |w_2(t)|^2. \tag{5.10}
\]

2. In this step we consider the Case 1: \( d(x^\delta(t), \Sigma(t)) \geq \delta_0 \).

On the interval \([0, T]\), let \( L_n \geq 1 \) be a Lipschitz constant for the map \( t \mapsto n(t) \), and let \( L_\Sigma \) be a Lipschitz constant for the multifunction \( t \mapsto \Sigma(t) \), w.r.t. the Hausdorff distance. Observe
Figure 3: Left and center: the two different cases considered in the proof of Theorem 3, depending on the distance of \( x^\delta(t) \) from the boundary \( \Sigma(t) \). Right: if \( d(x^\delta(t), \Sigma(t)) < \delta_0 \), then the speed \( \delta v(t, x^\delta) \) can be very large, and the same is true of \( \hat{\theta} \). To handle this situation, we need to insert a weight function \( W \) in our estimates.

that \( L_{\Sigma} \) provides a common Lipschitz constant for all trajectories \( t \mapsto x(t) \) of the sweeping process.

We claim that

\[
\varepsilon + C_2 |w_2(t)| \geq \hat{\theta}(t) \geq \begin{cases} 
-\varepsilon - L_n |w_2(t)| & \text{if } x(t) \notin \Sigma(t) \\
-C_1 & \text{if } x(t) \in \Sigma(t)
\end{cases}, \quad |\dot{w}_2(t)| \leq \varepsilon + C_2 |w(t)|,
\]

where \( C_1 = 1 + L_{\Sigma} + L_n \cdot \max_{t \in [0,T]} \{\text{diam}(V(t))\} \) and \( C_2 = 2L_n \). Indeed, recalling (5.5) and (5.7), we have

\[
\hat{\theta}(t) = \langle \dot{w}(t), n(t) \rangle + \langle w(t), \dot{n}(t) \rangle \\
= \langle \dot{x}^\delta(t), n(t) \rangle - \langle \dot{x}(t), n(t) \rangle + \theta(t) \langle n(t), \dot{n}(t) \rangle + \langle w_2(t), \dot{n}(t) \rangle \\
\geq -|\dot{x}^\delta(t)| - |\dot{x}(t)| - |w_2(t)| \cdot |\dot{n}(t)|
\]

(5.12)

Moreover,

\[
\hat{\theta}(t) \leq \langle \dot{x}^\delta(t), n(t) \rangle - \langle \dot{x}(t), n(t) \rangle + |w_2(t)| \cdot |\dot{n}(t)| \leq \varepsilon + L_n |w_2(t)|.
\]

(5.13)

Together, (5.12)-(5.13) yield the upper and lower bounds on \( \hat{\theta} \) in (5.11).

Next, by (5.7) one has

\[
|\dot{w}_2(t)| = \left| \frac{d}{dt} [w(t) - w_1(t)] \right| = \left| \frac{d}{dt} [w(t) - \langle w(t), n(t) \rangle \cdot n(t)] \right| \\
= |\dot{w}(t) - \langle \dot{w}(t), n(t) \rangle n(t) - \langle \dot{w}(t), \dot{n}(t) \rangle n(t) - \langle w(t), n(t) \rangle \dot{n}(t)| \\
\leq |\dot{w}(t) - \langle \dot{w}(t), n(t) \rangle n(t)| + 2L_n |w(t)|.
\]

Since \( \dot{x}(t) \) is either zero or parallel to \( n(t) \), by (5.5) it follows

\[
|\dot{w}(t) - \langle \dot{w}(t), n(t) \rangle n(t)| = |\delta v(t, x^\delta(t)) - \dot{x}(t) - \langle \delta v(t, x^\delta(t)) - \dot{x}(t), n(t) \rangle n(t)|
\]

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and generality we can assume $\delta$ for some constants $C$
In the following, we thus assume $d$ by choosing a constant $C$ defined. Recalling (5.6)–(5.9) we obtain

$$\frac{d}{dt}|w(t)| \leq 3(\varepsilon + L_n |w(t)|).$$

(5.14)

3. In this step we consider Case 2: $d(x^\delta(t), \Sigma(t)) < \delta_0$. As long as

$$|w(t)| \leq \frac{1}{4C_n},$$

(5.15)

we claim that

$$\theta(t) \leq \delta_0 + C_3(\delta_0 + |w(t)|)^2,$$

(5.16)

$$\dot{\theta}(t) \geq \begin{cases} -L_n|w_2(t)| & \text{if } x(t) \notin \Sigma(t), \\ -C_1 & \text{if } x(t) \in \Sigma(t), \end{cases}$$

(5.17)

for some constants $C_3, C_4$.

Notice that, if $d(x(t), \Sigma(t)) \geq \rho_0/2$, we then have $|w(t)| \geq \rho_0/4$, because without loss of generality we can assume $\delta_0 < \varepsilon < \rho_0/4$. In this case the estimate (5.16) is trivially satisfied, by choosing a constant $C_3$ large enough.

In the following, we thus assume $d(x(t), \Sigma(t)) \geq \rho_0/2$, so that the projection $\pi(t, x(t))$ is well defined. Recalling (5.6)–(5.9) we obtain

$$\dot{\theta}(t) = \langle \dot{x}(t) - x(t), n(t) \rangle = \langle x^\delta(t) - x(t), n(t) \rangle + \langle \pi(t, x^\delta) - x(t), n(t) \rangle$$

$$\leq \left| x^\delta(t) - x(t) \right| + \langle \pi(t, x^\delta) - x(t), n(t) \rangle$$

$$\leq \delta_0 + \left| \pi(t, x^\delta) - x(t), n(t) - n(t, x^\delta) \right| + \langle \pi(t, x^\delta) - x(t), n(t, x^\delta) \rangle$$

$$\leq \delta_0 + C_n |w(t)| \left| \pi(t, x^\delta) - x(t) \right| + C_\Sigma \left| \pi(t, x^\delta) - x(t) \right|^2$$

$$\leq \delta_0 + C_n |w(t)| (\delta_0 + |w(t)|) + C_\Sigma (\delta_0 + |w(t)|)^2$$

$$\leq \delta_0 + C_3 (\delta_0 + |w(t)|)^2$$

for a suitable constant $C_3$. This implies (5.16).

Using (5.4), the time derivative $\dot{\theta}$ can be estimated as

$$\dot{\theta}(t) = \langle \dot{w}(t), n(t) \rangle + \langle \dot{x}(t), n(t) \rangle \geq \langle \dot{x}(t), n(t) \rangle - |\dot{x}(t)| - |w_2(t)||\dot{n}(t)|$$

$$= \delta |v(t, x^\delta)| \left( \left( \frac{v(t, x^\delta)}{|v(t, x^\delta)|} - n(t, x^\delta) \right) + \left( n(t, x^\delta) - n(t) \right) + n(t) \right)$$

$$- |\dot{x}(t)| - |w_2(t)||\dot{n}(t)|$$

(5.18)

$$\geq \begin{cases} \delta |v(t, x^\delta)| \left( - \varepsilon - C_n |w(t)| + 1 \right) - L_n |w_2(t)| & \text{if } x(t) \notin \Sigma(t), \\ \delta |v(t, x^\delta)| \left( - \varepsilon - C_n |w(t)| + 1 \right) - L_\Sigma - L_n |w_2(t)| & \text{if } x(t) \in \Sigma(t). \end{cases}$$
As long as (5.15) holds, we have $1 - \varepsilon - C_n|w(t)| \geq \frac{1}{2}$. This already yields the first inequality in (5.17). From (5.18) we also deduce

$$|\delta v(t, x^\delta)| \leq 2(C_1 + |\dot{\theta}(t)|).$$

(5.19)

In turn, this yields

$$|\dot{w}_2(t)| \leq |\delta v(t, x^\delta)| \leq \frac{|v(t, x^\delta)|}{|v(t, x)|} - n(t) + 2L_n |w(t)|$$

$$\leq |\delta v(t, x^\delta)| \left\{ \frac{|v(t, x^\delta)|}{|v(t, x)|} - n(t, x^\delta) + |n(t, x^\delta) - n(t)| \right\} + 2L_n |w(t)|$$

$$\leq |\delta v(t, x^\delta)| (\varepsilon + C_n|w(t)|) + 2L_n |w(t)|$$

$$\leq 2(C_1 + |\dot{\theta}(t)|)(\varepsilon + C_n|w(t)|) + 2L_n |w(t)|,$$

establishing the second inequality in (5.17).

**4.** In this step we prove that there exists a constant $C_5 > 0$ such that, for every $t \in [0, T]$, $|\theta(t)| \leq C_5(\varepsilon + \bar{w}_2(t))$, (5.20)

where

$$\bar{w}_2(t) \equiv \max_{s \in [0,t]} |w_2(s)| \text{ for all } t \in [0, T].$$

(5.21)

Notice that, if $x(t) \in \Sigma(t)$, then (5.10) applies. Let us assume that $x(t) \notin \Sigma(t)$ and define the time

$$t_0 \equiv \inf\{s \in [0,t] \ ; \ x(r) \notin \Sigma(r) \text{ for all } r \in [s,t]\}.$$

From (5.11) and (5.17) it follows

$$\dot{\theta}(s) \geq -\varepsilon - L_n |w_2(s)| \geq -\varepsilon - L_n \bar{w}_2(s) \text{ for a.e. } s \in [t_0, t].$$

This yields

$$\theta(t) = \theta(t_0) + \int_{t_0}^{t} \dot{\theta}(s) \, ds \geq \theta(t_0) - \int_{t_0}^{t} \varepsilon + L_n \bar{w}_2(s) \, ds$$

$$\geq -C_\Sigma |w_2(t_0)|^2 - T(\varepsilon + L_n \bar{w}_2(t)) \geq -C_\Sigma \bar{w}_2^2(t) - T(\varepsilon + L_n \bar{w}_2(t)).$$

As long as $\bar{w}_2(t) < 1$, recalling that $L_n \geq 1$ we have

$$\theta(t) \geq - (C_\Sigma + TL_n) \cdot (\varepsilon + \bar{w}_2(t)).$$

Thus, to obtain (5.20), one only needs to consider the case where $\theta(t) > 0$. Observe that, if $d(x^\delta(t), \Sigma(t)) \leq \delta_0$, as long as

$$\varepsilon + |w(t)| \leq \frac{1}{2C_3},$$

(5.22)
from (5.16) it follows
\[ \theta(t) \leq \bar{C}_5(\varepsilon + \tilde{w}_2(t)) \]  
for some constant \( \bar{C}_5 \).

We claim that (5.20) holds, for a suitable constant \( C_5 \). Indeed, consider the time
\[ t_1 \doteq \inf \left\{ s \in [0, t) : \theta(r) > 0 \quad \text{and} \quad d(x^\delta(r), \Sigma(r)) > \delta_0 \quad \text{for all} \quad r \in (s, t) \right\}. \]

We then have \( \theta(t_1) \leq \bar{C}_5(\varepsilon + \tilde{w}_2(t_1)) \). Therefore, by (5.23) and (5.11) it follows
\[
\theta(t) = \theta(t_1) + \int_{t_1}^{t} \dot{\theta}(s) \, ds \leq \bar{C}_5(\varepsilon + \tilde{w}_2(t_1)) + \int_{t_1}^{t} (\varepsilon + C_2|w_2(s)|) \, ds \\
\leq \bar{C}_5(\varepsilon + \tilde{w}_2(t)) + \int_{t_1}^{t} (\varepsilon + C_2\tilde{w}_2(t)) \, ds \leq C_5(\varepsilon + \tilde{w}_2(t)).
\]

5. In the following we shall assume \( |w(t)| \leq \rho_0/3 \) for all \( t \in [0, T] \). In this case, when \( d(x(t), \Sigma(t)) > \rho_0 \) we have \( \dot{x}(t) = 0, |x^\delta(t)| < \varepsilon \) and the estimates are trivial. Without loss of generality, we can thus assume that the normal vectors \( n(t) \) and \( n(t, x^\delta) \) are well defined.

For a suitable constant \( \kappa \) (to be determined later), define the weight
\[
W(t) \doteq \begin{cases} 
\exp \left\{ -\kappa d(x^\delta(t), \Sigma(t)) \right\} & \text{if} \quad d(x^\delta(t), \Sigma(t)) \leq \delta_0, \\
\exp \left\{ -\kappa \delta_0 \right\} & \text{if} \quad d(x^\delta(t), \Sigma(t)) \geq \delta_0.
\end{cases}
\]

We now analyze how the weighted distance
\[ \Lambda(t) \doteq |\theta(t)| + W(t) \tilde{w}_2(t) \]
changes in time. The heart of the matter is to provide a bound on \( \tilde{w}_2 \). Indeed, by (5.20) the component \( w_1(t) = \theta(t)n(t) \) can be bounded in terms of \( \tilde{w}_2(t) \).

6. At any point \( t \in [0, T] \) where \( w_2(\cdot) \) is differentiable, by the definition of \( \tilde{w}_2 \) it follows
\[ 0 \leq \frac{d}{dt} \tilde{w}_2(t) \leq |\tilde{w}_2(t)|. \]

We first consider Case 1, where \( d(x^\delta(t), \Sigma(t)) \geq \delta_0 \). By (5.11) and (5.20), we have that \( \dot{W}(t) = 0 \) and
\[ |\tilde{w}_2(t)| \leq \varepsilon + C_2 |w(t)| \leq C_2 (\varepsilon + |\theta(t)| + \tilde{w}_2(t)) \leq C_2 (1 + C_5)(\varepsilon + \tilde{w}_2(t)). \]

Therefore, (5.25) yields
\[
\frac{d}{dt} (W(t)\tilde{w}_2(t)) = W(t) \frac{d}{dt} \tilde{w}_2(t) \leq W(t) \cdot |\tilde{w}_2(t)| \leq C_2 (1 + C_5)W(t)(\varepsilon + \tilde{w}_2(t)).
\]
On the other hand, in the case where \( d(x^\delta(t), \Sigma(t)) < \delta_0 \), by (5.17) and (5.20) one has

\[
|\dot{w}_2(t)| \leq C_4(\varepsilon + |w(t)|)(1 + |\dot{\theta}(t)|)
\]

\[
\leq C_4(\varepsilon + |\theta(t)| + \dot{w}_2(t))(1 + |\dot{\theta}(t)|) \leq (C_4 + C_4C_5)(\varepsilon + \ddot{w}_2(t))(1 + |\dot{\theta}(t)|)
\]

\[
\leq \begin{cases} (C_4 + C_4C_5)(1 + C_1)(\varepsilon + \ddot{w}_2(t)) & \text{if } \dot{\theta}(t) \leq 0, \\ (C_4 + C_4C_5)(\varepsilon + \ddot{w}_2(t))(1 + \dot{\theta}(t)) & \text{if } \dot{\theta}(t) > 0, \end{cases}
\]

\[
\leq \begin{cases} C_6(\varepsilon + \ddot{w}_2(t)) & \text{if } \dot{\theta}(t) \leq 0, \\ C_6(\varepsilon + \ddot{w}_2(t))(1 + \dot{\theta}(t)) & \text{if } \dot{\theta}(t) > 0, \end{cases}
\]

By (5.25), for a.e. \( t \in [0, T] \) one has

\[
\frac{d}{dt} \ddot{w}_2(t) \leq \begin{cases} C_6(\varepsilon + \ddot{w}_2(t)) & \text{if } \dot{\theta}(t) \leq 0, \\ C_6(\varepsilon + \ddot{w}_2(t))(1 + \dot{\theta}(t)) & \text{if } \dot{\theta}(t) > 0. \end{cases}
\]

(5.26)

As long as \( \varepsilon + |w(t)| \leq \frac{1}{2C_7(2C_3+C_4^2)} \), from (5.8), (5.12), and (5.19), it follows

\[
\frac{d}{dt} d(x^\delta(t), \Sigma(t)) \geq \langle \delta v(t, x^\delta(t)), n(t, x^\delta) \rangle - L_\Sigma
\]

\[
\geq \langle \delta v(t, x^\delta(t)), n(t) \rangle - C_n|w(t)| \cdot |\delta v(t, x^\delta(t))| - L_\Sigma
\]

\[
\geq \langle \dot{w}_1(t), n(t) \rangle + \langle \dot{w}_2(t), n(t) \rangle + \langle \dot{z}(t), n(t) \rangle - C_n|w(t)| \cdot |\delta v(t, x^\delta(t))| - L_\Sigma
\]

\[
\geq \dot{\theta}(t) - |\dot{w}_2(t)| - C_n|w(t)| \cdot |\delta v(t, x^\delta(t))| - L_\Sigma
\]

\[
\geq \dot{\theta}(t) - C_4(\varepsilon + |w(t)|) \cdot (1 + |\dot{\theta}(t)|) - 2C_n|w(t)| \cdot (C_1 + |\dot{\theta}(t)|) - L_\Sigma
\]

\[
\geq \dot{\theta}(t) - (2C_n + C_4)(\varepsilon + |w(t)|) \cdot (C_1 + |\dot{\theta}(t)|) - L_\Sigma \geq \frac{1}{2}\dot{\theta}(t) - C_7
\]

for some constant \( C_7 > 0 \). Inserting the weight, we now estimate

\[
\frac{d}{dt} (W(t) \ddot{w}_2(t)) = -\kappa W(t) \ddot{w}_2(t) \cdot \frac{d}{dt} d(x^\delta(t), \Sigma(t)) + W(t) \cdot \frac{d}{dt} \ddot{w}_2(t)
\]

\[
\leq -\frac{\kappa}{2} W(t) \ddot{w}_2(t) \dot{\theta}(t) + \kappa C_7 W(t) \ddot{w}_2(t) + W(t) \cdot \frac{d}{dt} \ddot{w}_2(t).
\]

Two cases can occur:

- **If** \( \dot{\theta}(t) \leq 0 \), then (5.26) and (5.17) yield

\[
\frac{d}{dt} (W(t) \ddot{w}_2(t)) \leq C_8 W(t) (\varepsilon + \ddot{w}_2(t))
\]

for some constant \( C_8 \).
• If $\dot{\theta}(t) > 0$, then (5.26) yields

$$
\frac{d}{dt} (W(t) \bar{w}_2(t)) \leq -\frac{\kappa}{2} W(t) \bar{w}_2(t) \dot{\theta}(t) + W(t) \cdot \frac{d}{dt} \bar{w}_2(t) + \kappa C_7 W(t) \bar{w}_2(t)
$$

$$
\leq -\frac{\kappa}{2} W(t) \bar{w}_2(t) \dot{\theta}(t) + C_6 W(t)(\varepsilon + \bar{w}_2(t))(1 + \dot{\theta}(t)) + \kappa C_7 W(t) \bar{w}_2(t)
$$

$$
\leq W(t) \cdot \left[ \left( -\frac{\kappa}{2} \bar{w}_2(t) + C_6 \cdot (\varepsilon + \bar{w}_2(t)) \right) \cdot \dot{\theta}(t) + (C_6 + \kappa C_7)(\varepsilon + \bar{w}_2(t)) \right].
$$

We now choose the constant $\kappa$ in (5.24) so that

$$
\frac{\kappa}{2} \geq 2C_6.
$$

In this case, either $\bar{w}_2(t) < \varepsilon$, or else

$$
\frac{d}{dt} (W(t) \bar{w}_2(t)) \leq W(t) \cdot \left[ \left( -\frac{\kappa}{2} + 2C_6 \right) \dot{\theta}(t) \bar{w}_2(t) + (C_6 + \kappa C_7)(\varepsilon + \bar{w}_2(t)) \right]
$$

$$
\leq W(t) (C_6 + \kappa C_7)(\varepsilon + \bar{w}_2(t)).
$$

(5.27)

Combining both Cases 1 and 2, we obtain that either $\bar{w}_2(t) \leq \varepsilon$ or else

$$
\frac{d}{dt} (W(t) \bar{w}_2(t)) \leq C_9 W(t) \bar{w}_2(t),
$$

(5.28)

provided that

$$
\varepsilon + |\theta(t)| + \bar{w}_2(t) \leq \min \left\{ \frac{1}{4}, \frac{1}{4C_n}, \frac{1}{2C_3}, \frac{1}{2C_1(2C_n + C_4)}, \frac{\rho_0}{3} \right\}.
$$

(5.29)

7. To complete the argument, consider the time

$$
\bar{t} \doteq \sup \left\{ \tau \in [0, T] ; \ (5.29) \text{ holds for all } t \in [0, \tau] \right\}.
$$

Since $\bar{w}_2(t)$ is continuous and non-decreasing, there exists $t_\varepsilon \in [0, \bar{t}]$ such that

$$
\begin{cases}
\bar{w}_2(t) \leq \varepsilon & \text{for all } t \in [0, t_\varepsilon], \\
\bar{w}_2(t) > \varepsilon & \text{for all } t \in [t_\varepsilon, \bar{t}].
\end{cases}
$$

Hence (5.28) implies

$$
W(t) \bar{w}_2(t) \leq e^{C_9(t-t_\varepsilon)} W(t_\varepsilon) \bar{w}_2(t_\varepsilon) \quad \text{for all } t_\varepsilon \leq t \leq \bar{t}.
$$

Since $e^{-\kappa\delta_0} \leq W(t) \leq 1$, we have

$$
\bar{w}_2(t) \leq \exp (C_9 T + \kappa \cdot \delta_0) \cdot \varepsilon \quad \text{for all } t \in [0, \bar{t}].
$$

Recalling (5.20), we obtain

$$
|\theta(t)| \leq C_5 [\exp (C_9 T + \kappa \cdot \delta_0) + 1] \cdot \varepsilon \quad \text{for all } t \in [0, \bar{t}].
$$
This yields
\[ \varepsilon + |\theta(t)| + \bar{w}_2(t) \leq C_{10} \varepsilon \quad \text{for all } t \in [0, \bar{t}], \]
where
\[ C_{10} = (1 + C_5) \left[ \exp \left( C_9 T + \kappa \cdot \delta_0 \right) + 1 \right]. \]
Therefore, for any \( \varepsilon > 0 \) such that
\[ \varepsilon < \frac{1}{C_{10}} \min \left\{ \frac{1}{4}, \frac{1}{4C_n}, \frac{1}{2C_3}, \frac{1}{2C_1(2C_n + C_4)}, \frac{\rho_0}{3} \right\}, \]
we conclude that
\[ \bar{t} = T, \quad |w(t)| \leq |\theta(t)| + \bar{w}_2(t) \leq C_{10} \varepsilon \quad \text{for all } t \in [0, T]. \quad (5.30) \]

8. The previous analysis has shown that, by choosing \( \delta > 0 \) small enough, the sweeping process can be arbitrarily well approximated by the evolution generated by the vector field \( \delta v(t, x) \). Repeating the argument in step 3 of the proof of Theorem 2, we now construct a control function \( t \mapsto \xi(t) \) such that trajectories of the ODE
\[ \dot{x}(t) = \varphi(|x - \xi(t)|) \frac{x - \xi(t)}{|x - \xi(t)|} \]
approximate the trajectories of \( \dot{x} = \delta v(t, x) \), uniformly for \( t \in [0, T] \) and for all initial data in the compact set \( \Omega_0 \subset \mathbb{R}^d \setminus \Sigma(0) \). This completes the proof.

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References


