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Abstract

We consider a controlled evolution problem for a set $\Omega(t) \in \mathbb{R}^d$, originally motivated by a model where a dog controls a flock of sheep. Necessary conditions and sufficient conditions are given, in order that the evolution be completely controllable. Similar techniques are then applied to the approximation of a sweeping process. Under suitable assumptions, we prove that there exists a control function such that the corresponding evolution of the set $\Omega(t)$ is arbitrarily close to the one determined by the sweeping process.

1 Introduction

In this paper we consider a controllability problem for the evolution of a set $\Omega(t) \subset \mathbb{R}^d$. This was originally motivated by the model introduced in [4], describing the evolution of a flock of sheep, who tend to scatter around but also react to the presence of a dog. The region $\Omega(t) \subset \mathbb{R}^2$ occupied by the sheep is described as the reachable set for a differential inclusion, while the position of the dog is regarded as a control function. As in [4], we consider a "scare function" $\varphi = \varphi(r) > 0$, describing the speed at which sheep run away from the dog, depending on the distance r. Further results and extensions can be found in [9, 10]. For more general models of crowd dynamics we refer to [3]. A general theory of evolution problems in metric spaces, also describing the evolution of a set, was developed in [1, 11].

In the following we consider the evolution of a set in $\mathbb{R}^d$, and assume

(A1) *The function $r \mapsto \varphi(r)$ is continuously differentiable for $r > 0$, and satisfies*

$$\varphi' < 0, \qquad \lim_{r\to 0+} \varphi(r) = +\infty, \qquad \lim_{r\to+\infty} \varphi(r) = 0. \tag{1.1}$$

Given a function $t \mapsto \xi(t) \in I\!R^d$ describing the position of a repelling agent, we define the velocity field

$$\mathbf{v}(x,\xi) \doteq \varphi(|x-\xi|)\,\frac{x-\xi}{|x-\xi|}\,. \tag{1.2}$$

For a given initial set Ω_0, we denote by $\Omega^\xi(t)$ the set reached by trajectories of

$$\dot{x} \in \mathbf{v}(x,\xi(t))\,, \qquad\qquad x(0) \in \Omega_0\,. \tag{1.3}$$

In other words, for any $t \geq 0$,

$$\begin{aligned}\Omega^\xi(t) \doteq \Big\{x(t)\,; \quad x(0) \in \Omega_0\,, \qquad x(\cdot) \text{ is absolutely continuous,}\\ \dot{x}(\tau) = \mathbf{v}\,(x(\tau),\,\xi(\tau)) \quad \text{for a.e. } \tau \in [0,t]\Big\}.\end{aligned} \tag{1.4}$$

Throughout the following we write $\partial\Omega$, $\overline{\Omega}$, and $int\,\Omega$, for the boundary, the closure, and the interior of a set $\Omega \subset I\!R^d$, respectively. By $B(\Omega,r)$ we denote the open neighborhood of radius r around the set Ω, while d_H denotes Hausdorff distance [2].

To avoid any difficulty about uniqueness of solutions of (1.2)-(1.3), we assume that the control $\xi(\cdot)$ is chosen so that

$$\inf_{t\in[0,\tau]} d(\xi(t),\Omega^\xi(t)) > 0 \qquad\qquad \text{for all } 0 \leq \tau < T. \tag{1.5}$$

We wish to understand how the function φ affects the controllability properties of the evolution equation (1.3). Roughly speaking, given an initial set Ω_0 and a terminal set Ω_1, we seek a control $\xi(\cdot)$ such that, at the terminal time T, the set $\Omega^\xi(T)$ in (1.4) is arbitrary close to Ω_1.

Definition 1. *We say that the set-valued evolution (1.3) satisfies the* **Global Approximate Confinement** *property* **(GAC)** *if the following holds. Let $\Omega_0, \Omega_1 \subset I\!R^d$ be any two compact domains, with Ω_1 simply connected and such that $\Omega_1 \subset int\,\Omega_0$. Then for any $T, \varepsilon > 0$, there exists a Lipschitz continuous control $\xi : [0,T] \mapsto I\!R^d$ satisfying (1.5) and such that the set (1.4) satisfies*

$$\Omega_1 \subseteq \Omega^\xi(T) \subseteq B(\Omega_1,\varepsilon) \tag{1.6}$$

If there exists a locally Lipschitz control $\xi : [0,T[\mapsto I\!R^d$ satisfying (1.5) and such that

$$\Omega^\xi(T) = \Omega_1\,, \tag{1.7}$$

we then say that the set-valued evolution (1.3) satisfies the **Global Exact Confinement** *property* **(GEC)***.*

The primary goal of this paper is to find conditions which are necessary, or sufficient, to achieve the **(GAC)** or **(GEC)** properties. Indeed, we will show that these properties are determined by the asymptotic behavior of the function φ as $r \to 0+$. Our first main result is

Theorem 1 (necessary condition). *Let φ satisfy* **(A1)***. If the* **(GAC)** *property holds, then the function φ must satisfy*

$$\int_0^1 r^{d-2}\varphi(r)\,dr = +\infty\,. \tag{1.8}$$

To state a sufficient condition, we introduce the assumption

(A2) *For every $\kappa > 0$ one has*

$$\lim_{r\to 0+} \frac{r^{d/2}\cdot \varphi(\kappa r^{1/2})+1}{r^d\cdot \varphi(r)} ~=~ 0\,. \tag{1.9}$$

Theorem 2 (sufficient condition). *If the function φ satisfies* **(A1)-(A2)***, then the* **(GAC)** *and* **(GEC)** *properties hold.*

This result applies, in particular, to the function $\varphi(r) = r^{-\beta}$ for any $\beta > d$.

A proof of Theorem 1 will be given in Section 2, while Theorem 2 will be proved in Section 3.

The controllability of the set-valued evolution (1.4) is closely related to a result on the approximation of a sweeping process. Indeed, let $t\mapsto V(t)$ describe a moving set in $I\!R^d$. We assume that each $V(t)$ is a compact set with nonempty interior and smooth boundary, smoothly depending on time. More precisely:

$$V(t) ~=~ \{x\in I\!R^d\,;~~\psi(t,x)\leq 0\}, \tag{1.10}$$

where $\psi : I\!R\times I\!R^d \mapsto I\!R$ has $\mathcal{C}^2$ regularity and satisfies the nondegeneracy condition

$$\psi(t,x) ~=~ 0 \qquad \Longrightarrow \qquad \nabla_x \psi(t,x) ~\neq~ 0. \tag{1.11}$$

As usual, we denote by $N_{V(t)}(x)$ the outer normal cone to $V(t)$ at a boundary point $x\in\partial V(t)$. In the case of an interior point $x \in intV(t)$, we simply define $N_{V(t)}(x) = \{0\}$. By the well known theory of sweeping processes [5, 6, 7, 8, 12], for any initial point $x_0\in V(0)$, the differential inclusion

$$\dot x(t) ~\in~ -N_{V(t)}(x(t)), \qquad\qquad x(0)=x_0 \tag{1.12}$$

has a unique solution $t\mapsto x(t,x_0)\in V(t)$. In turn, for a given initial set $\Omega_0\subset V(0)$, one can consider the sets

$$\Omega(t) ~\doteq~ \{x(t,x_0)\,;~~x_0\in \Omega_0\}. \tag{1.13}$$

A natural question is whether there exists a control $\xi(\cdot)$ such that the corresponding sets $\Omega^\xi(t)$ in (1.4) remain uniformly close to the sets $\Omega(t)$, for all $t\in [0,T]$. It turns out that this is true, under an assumption which slightly strengthens **(A2)**, namely:

(A2′) *For some $\beta > 1/2$ one has*

$$\lim_{r\to 0+} \frac{r^{\beta d}\cdot \varphi(r^\beta)+1}{r^d\cdot \varphi(r)} ~=~ 0. \tag{1.14}$$

In the following, $t\mapsto x^\xi(t,x_0)$ denotes the solution to

$$\dot x(t) ~=~ \mathbf{v}(x(t),\xi(t)), \qquad\qquad x(0)=x_0\,, \tag{1.15}$$

with $\mathbf{v}$ as in (1.2), while $t\mapsto x(t,x_0)$ is the trajectory of the sweeping process (1.12), with the same initial condition.

Theorem 3 (approximation of a sweeping process). *Assume that the function φ satisfies* **(A1)** *and* **(A2′)**. *As in (1.10)-(1.11), let $t \mapsto V(t)$ be a family of sets with $\mathcal{C}^2$ boundaries. Then, for any $T, \varepsilon > 0$ there exists a measurable control $t \mapsto \xi(t)$ such that*

$$|x^\xi(t,x_0) - x(t,x_0)| \ \leq\ \varepsilon \qquad \textit{for all } x_0 \in V(0), \quad t \in [0,T]. \tag{1.16}$$

An immediate consequence of (1.16) is that, for any initial subset $\Omega_0 \subseteq V(0)$, the corresponding sets $\Omega^\xi(t)$ in (1.4) and $\Omega(t)$ in (1.13) satisfy

$$d_H(\Omega^\xi(t),\, \Omega(t)) \ \leq\ \varepsilon \qquad \text{for all } t \in [0,T]. \tag{1.17}$$

A proof of Theorem 3 will be worked out in Section 4.

2 Proof of the necessary condition

In this section we give a proof of Theorem 1. The main idea is that, if (1.8) fails, then for any choice of the control $\xi(\cdot)$ the volume of the set Ω^ξ cannot shrink too fast. This puts a constraint on the sets that can be approximately reached at time T.

Let $t \mapsto \xi(t)$ be any admissible control. Fix any time $t \geq 0$ and let $\mathbf{v} = \mathbf{v}(\cdot, \xi(t))$ be the vector field in (1.2). Then, calling $\delta \doteq d(\xi(t), \Omega^\xi(t))$, we compute

$$\begin{aligned} \frac{d}{dt}\text{meas}\Big(\Omega^\xi(t)\Big) \ &=\ \int_{\Omega^\xi(t)\cap B(\xi(t),1)} \text{div}\,\mathbf{v}\,dx + \int_{\Omega^\xi(t)\setminus B(\xi(t),1)} \text{div}\,\mathbf{v}\,dx \\ &\geq\ \omega_{d-1}\int_\delta^1 r^{d-1}\varphi'(r)\,dr + \varphi'(1)\cdot \text{meas}(\Omega^\xi(t))\,. \end{aligned} \tag{2.1}$$

Here and in the sequel ω_{d-1} denotes the $(d-1)$-dimensional measure of the surface of the unit ball in $\mathbb{R}^d$. An integration by parts yields

$$\begin{aligned} \int_\delta^1 r^{d-1}\varphi'(r)\,dr \ &=\ -\int_\delta^1 (d-1)r^{d-2}[\varphi(r)-\varphi(1)]\,dr - \delta^{d-1}[\varphi(\delta)-\varphi(1)] \\ &\geq\ -\int_\delta^1 (d-1)r^{d-2}\varphi(r)\,dr - \delta^{d-1}\varphi(\delta). \end{aligned} \tag{2.2}$$

If (1.8) fails, then

$$\int_\delta^1 r^{d-2}\varphi(r)\ dr \ \leq\ M \ \doteq\ \int_0^1 r^{d-2}\varphi(r)dr \ <\ \infty\,.$$

Since φ is decreasing, one has

$$\delta^{d-1}\varphi(\delta) \ =\ \frac{1}{d-1}\cdot\int_0^\delta s^{d-2}\cdot\varphi(\delta)\ ds \ \leq\ \frac{1}{d-1}\cdot\int_0^\delta s^{d-2}\cdot\varphi(s)\ ds \ \leq\ \frac{M}{d-1}\,.$$

This implies that the right hand side of (2.2) is bounded below by a constant. Therefore, (2.1) yields

$$\frac{d}{dt}\text{meas}\Big(\Omega^\xi(t)\Big) \ \geq\ \varphi'(1)\cdot\text{meas}(\Omega^\xi(t)) - \frac{\omega_{d-1}Md}{d-1}. \tag{2.3}$$

By Gronwall's inequality, for all times $t \geq 0$ we conclude that

$$\text{meas}\Big(\Omega^\xi(t)\Big) \ \geq \ e^{\varphi'(1)\cdot t}\,\text{meas}(\Omega_0) - \frac{\omega_{d-1}Md}{d-1}\cdot t\,. \tag{2.4}$$

This a priori lower bound on the measure of the set $\Omega^\xi(t)$ shows that approximate controllability cannot be achieved. □

Remark. Assume that the divergence of the vector field $\mathbf{v}(\cdot,\xi)$ remains negative for x close to ξ, that is

$$\varphi'(r) + \frac{d-1}{r}\,\varphi(r) \ \leq \ 0 \qquad\qquad \text{for all } 0 < r < \bar r\,, \tag{2.5}$$

for some $\bar r > 0$. In this case, the Global Approximate Confinement property implies

$$\limsup_{r\to 0+} \ r^{d-1}\varphi(r) \ = \ +\infty. \tag{2.6}$$

Indeed, one can replace the estimate (2.1) with

$$\begin{aligned}\frac{d}{dt}\text{meas}\Big(\Omega^\xi(t)\Big) \ &= \ \int_{\Omega^\xi(t)\cap B(\xi(t),\bar r)} \text{div}\,\mathbf{v}\,dx + \int_{\Omega^\xi(t)\setminus B(\xi(t),\bar r)} \text{div}\,\mathbf{v}\,dx \\ &\geq \ \int_{\delta<|x|<\bar r} \text{div}\,\mathbf{v}\,dx + \varphi'(\bar r)\cdot \text{meas}(\Omega^\xi(t)) \\ &= \ \omega_{d-1}\Big[\bar r^{d-1}\varphi(\bar r) - \delta^{d-1}\varphi(\delta)\Big] + \varphi'(\bar r)\cdot\text{meas}(\Omega^\xi(t)) \\ &\geq \ -\,\omega_{d-1}\delta^{d-1}\varphi(\delta) + \varphi'(\bar r)\cdot\text{meas}(\Omega^\xi(t))\,.\end{aligned} \tag{2.7}$$

If (2.6) fails, then

$$\frac{d}{dt}\text{meas}\Big(\Omega^\xi(t)\Big) \ \geq \ \varphi'(\bar r)\cdot\text{meas}(\Omega^\xi(t)) - C_1$$

for some constant C_1, and it leads again a priori lower bound on the measure of the set $\Omega^\xi(t)$.

3 Proof of the sufficient condition

Aim of this section is to provide a proof of Theorem 2. As a preliminary, consider a bounded open set $\Omega\subset I\!R^d$ with $\mathcal{C}^2$ boundary $\Sigma = \partial\Omega$. On the complement $I\!R^d\setminus\Sigma$ we consider the vector field

$$\mathbf{v}(x) \ \doteq \ \int_\Sigma \varphi(|x-\xi|)\,\frac{x-\xi}{|x-\xi|}\,d\sigma(\xi), \tag{3.1}$$

where σ denotes the $(d-1)$-dimensional surface measure on Σ. Since Σ has $\mathcal{C}^2$ regularity, for every x sufficiently close to Σ there exists a unique perpendicular projection $y_x\in\Sigma$ such that

$$|x-y_x| \ = \ d(x,\Sigma) \ \doteq \ \min_{y\in\Sigma}|x-y|\,. \tag{3.2}$$

To fix the ideas, assume that this perpendicular projection $x\mapsto y_x$ is well defined whenever $d(x,\Sigma) < r_0$, for some curvature radius $r_0 > 0$. In the following, $\mathbf{n}_x = \frac{x-y_x}{|x-y_x|}$ denotes the unit normal to the surface Σ at the point y_x.

Lemma 1. *Let the function φ satisfy* **(A1)-(A2)**. *Then the vector field* $\mathbf{v}$ *in (3.1) satisfies*

$$\lim_{d(x,\Sigma)\to 0} \langle \mathbf{n}_x\,,\ \mathbf{v}(x)\rangle \;=\; +\infty. \tag{3.3}$$

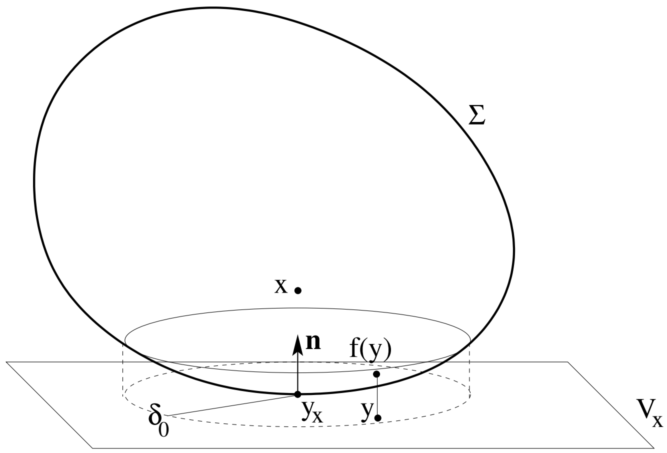

Figure 1: The portion of Σ near the point y_x can be represented as the graph of a function f, as in (4.3).

Proof. 1. Consider any point x sufficiently close to Σ so that the perpendicular projection $y_x \doteq \pi(x)$ at (3.2) is well defined. Let V_x be the hyperplane tangent to Σ at y_x and let $\mathbf{n}_x$ be the unit normal vector. As shown in Fig. 1, in a neighborhood of y_x, the surface Σ can be expressed as the graph of a function $f : V_x \mapsto I\!\!R$. More precisely, call $\varepsilon = d(x,\Sigma) = |x - y_x|$. Without loss of generality, we can choose a system of coordinates such that $y_x = 0$. Notice that, by the regularity and compactness of the surface Σ, we can assume that the radius δ_0 of the ball where the function f is defined is independent of $y_x \in \Sigma$. Moreover, the $\mathcal{C}^2$ norm of f remains uniformly bounded. By construction we have

$$f(0) \;=\; 0, \qquad \nabla f(0) \;=\; 0, \qquad \|f\|_{\mathcal{C}^2(B(0,\delta_0))} \;\leq\; C_0, \tag{3.4}$$

for some uniform constant C_0. Defining the constant $\kappa = \sqrt{2/C_0} > 0$, by (3.4) one has the implication

$$|y - y_x| \;<\; \kappa |x - y_x|^{\frac{1}{2}} \qquad \Longrightarrow \qquad \langle \mathbf{n}_x\,,\ x - y - f(y)\mathbf{n}_x\rangle \;\geq\; 0. \tag{3.5}$$

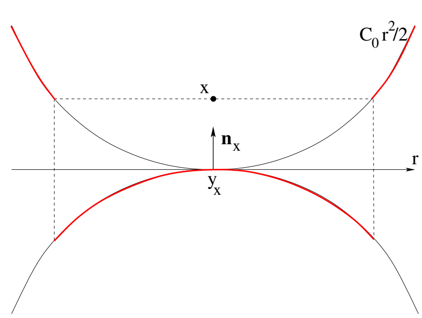

Figure 2: The estimates (3.10)-(3.11).

2. Next, consider the decomposition $\Sigma = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3$, where

$$\begin{cases} \Sigma_1 &= \{y + f(y)\mathbf{n}_x\,;\ \ |y| \leq \kappa |x|^{1/2}\}, \\ \Sigma_2 &= \{y + f(y)\mathbf{n}_x\,;\ \ \kappa |x|^{1/2} < y < \delta_0\}, \\ \Sigma_3 &= \Sigma \setminus (\Sigma_1 \cup \Sigma_2). \end{cases} \tag{3.6}$$

Based on (3.6), we shall estimate the vector field $\mathbf{v}$ by splitting the integral in (3.1) in three parts:

$$\mathbf{v}_i(x) \ = \ \int_{\Sigma_i} \varphi(|x-\xi|)\,\frac{x-\xi}{|x-\xi|}\,d\sigma(\xi), \qquad\qquad i = 1,2,3, \tag{3.7}$$

so that

$$\mathbf{v}(x) \ = \ \mathbf{v}_1(x) + \mathbf{v}_2(x) + \mathbf{v}_3(x).$$

In the following, for $y \in V_x$, we use the bound $|f(y)| \ \leq \ \frac{C_0}{2}|y|^2$ and the identities

$$|x| \ = \ \varepsilon\,, \qquad |y| \ = \ r\,, \qquad |x-y| \ = \ \sqrt{\varepsilon^2 + r^2}\,, \qquad \mathbf{n}_x \ = \ \frac{x}{|x|}\,. \tag{3.8}$$

As long as $|y| \leq \delta_0$, the above implies

$$|x - y - f(y)\mathbf{n}_x|^2 \ \in \ \Big[\varepsilon^2 + (1 - C_0\varepsilon) r^2\,,\ \varepsilon^2 + (1 + C_0\varepsilon) r^2 + C_0^2 r^4/4\Big]. \tag{3.9}$$

Using (3.9) and the monotonicity of φ we obtain the estimates

$$\begin{aligned} \langle \mathbf{n}_x\,,\,\mathbf{v}_1(x)\rangle \ &\geq \ c_0 \int_0^{\kappa\varepsilon^{1/2}} r^{d-2}\Big(\varepsilon - \frac{C_0}{2} r^2\Big)\cdot \frac{\varphi\Big(\sqrt{(\varepsilon + C_0 r^2/2)^2 + r^2}\Big)}{\sqrt{(\varepsilon + C_0 r^2/2)^2 + r^2}}\,dr \\ &\geq \ c_0 \int_0^{\kappa\varepsilon^{1/2}} r^{d-2}\Big(\varepsilon - \frac{C_0}{2} r^2\Big)\cdot \frac{\varphi\Big(\sqrt{\varepsilon^2 + (1 + 2C_0\varepsilon) r^2}\Big)}{\sqrt{\varepsilon^2 + (1 + 2C_0\varepsilon) r^2}}\,dr \\ &\geq \ \frac{3}{4} c_0 \int_0^{\frac{\kappa\varepsilon^{1/2}}{2}} \varepsilon r^{d-2}\cdot \frac{\varphi\Big(\sqrt{\varepsilon^2 + (1 + 2C_0\varepsilon) r^2}\Big)}{\sqrt{\varepsilon^2 + (1 + 2C_0\varepsilon) r^2}}\,dr\,, \end{aligned} \tag{3.10}$$

$$\begin{aligned} |\langle \mathbf{n}_x\,,\,\mathbf{v}_2(x)\rangle| \ &\leq \ c_0 \int_{\kappa\varepsilon^{1/2}}^{\delta_0} r^{d-2}\varphi\Big(\sqrt{\varepsilon^2 + (1 - C_0\varepsilon) r^2}\Big)\cdot \frac{\varepsilon + C_0 r^2/2}{\sqrt{\varepsilon^2 + (1 - C_0\varepsilon) r^2}}\,dr \\ &\leq \ C_1 \cdot \int_{\kappa\varepsilon^{1/2}}^{\delta_0} \frac{\varphi\Big(\sqrt{\varepsilon^2 + (1 - C_0\varepsilon) r^2}\Big)}{\sqrt{\varepsilon^2 + (1 - C_0\varepsilon) r^2}}\cdot r^d\,dr\,, \end{aligned} \tag{3.11}$$

$$|\langle \mathbf{n}_x\,,\,\mathbf{v}_3(x)\rangle| \ \leq \ C\,, \tag{3.12}$$

for some constants $C, C_1, c_0 > 0$. Performing the change of variable $s = r\sqrt{1 + 2C_0\varepsilon}$ in (3.10), one finds

$$\begin{aligned} \langle \mathbf{n}_x\,,\,\mathbf{v}_1(x)\rangle \ &\geq \ C_2 \cdot \int_0^{\frac{1}{2}\kappa\varepsilon^{1/2}} \varepsilon s^{d-2}\cdot \frac{\varphi\left(\sqrt{\varepsilon^2 + s^2}\right)}{\sqrt{\varepsilon^2 + s^2}}\,ds \\ &\geq \ \frac{4C_2}{\kappa^2}\cdot \int_0^{\frac{1}{2}\kappa\varepsilon^{1/2}} s^d\cdot \frac{\varphi\left(\sqrt{\varepsilon^2 + s^2}\right)}{\sqrt{\varepsilon^2 + s^2}}\,ds \end{aligned}$$

for some constant $C_2 > 0$. Setting $t = \sqrt{\varepsilon^2 + s^2}$, we estimate

$$\begin{aligned}
\langle \mathbf{n}_x\,,\,\mathbf{v}_1(x)\rangle \;&\geq\; \frac{4C_2}{\kappa^2}\cdot\int_{\varepsilon}^{\sqrt{\varepsilon^2+\frac{1}{4}\kappa^2\varepsilon}}\Big(t^2-\varepsilon^2\Big)^{\frac{d-1}{2}}\cdot\varphi(t)\;dt\\
&\geq\; \frac{4C_2}{\kappa^2}\cdot\int_{2\varepsilon}^{\frac{1}{2}\kappa\varepsilon^{1/2}}\Big(t^2-\varepsilon^2\Big)^{\frac{d-1}{2}}\cdot\varphi(t)\;dt\\
&\geq\; \frac{4C_2}{\kappa^2}\cdot\left(\frac{3}{4}\right)^{\frac{d-1}{2}}\cdot\int_{2\varepsilon}^{\frac{1}{2}\kappa\varepsilon^{1/2}} t^{d-1}\cdot\varphi(t)\;dt\,.
\end{aligned}$$

Similarly, using the variable $s = \sqrt{\varepsilon^2 + (1-C_0\varepsilon)r^2}$ in (3.11), we have

$$\begin{aligned}
\int_{\kappa\varepsilon^{1/2}}^{\delta_0}\frac{\varphi\Big(\sqrt{\varepsilon^2+(1-C_0\varepsilon)r^2}\Big)}{\sqrt{\varepsilon^2+(1-C_0\varepsilon)r^2}}\cdot r^d\,dr \;&=\; \frac{1}{1-C_0\varepsilon}\cdot\int_{\sqrt{\varepsilon^2+(1-C_0\varepsilon)\kappa^2\varepsilon}}^{\sqrt{\delta^2+\varepsilon(\varepsilon-C_0\delta^2)}}\varphi(s)\cdot r^{d-1}\;ds\\
&\leq\; \frac{1}{(1-C_0\varepsilon)^{\frac{d+1}{2}}}\cdot\int_{\sqrt{\varepsilon^2+(1-C_0\varepsilon)\kappa^2\varepsilon}}^{\sqrt{\delta^2+\varepsilon(\varepsilon-C_0\delta^2)}}\varphi(s)\cdot s^{d-1}\;ds\,.
\end{aligned}$$

Thus, for $\varepsilon > 0$ sufficiently small, it holds

$$|\langle \mathbf{n}_x\,,\,\mathbf{v}_2(x)\rangle| \;\leq\; C_3\cdot\int_{\frac{1}{2}\kappa\varepsilon^{1/2}}^{\delta_0} s^{d-1}\cdot\varphi(s)\;ds$$

for some constant $C_3 > 0$. Setting $\tilde{\varepsilon} = 2\varepsilon$ we obtain

$$\langle \mathbf{n}_x\,,\,\mathbf{v}_1(x)\rangle \;\geq\; C_4\cdot\int_{\tilde{\varepsilon}}^{\kappa_1\tilde{\varepsilon}^{1/2}} s^{d-1}\cdot\varphi(s)\;ds \tag{3.13}$$

and

$$|\langle \mathbf{n}_x\,,\,\mathbf{v}_2(x)\rangle| \;\leq\; C_3\cdot\int_{\kappa_1\tilde{\varepsilon}^{1/2}}^{\delta_0} s^{d-1}\cdot\varphi(s)\;ds\,, \tag{3.14}$$

for $\kappa_1 = \frac{\kappa}{2\sqrt{2}}$ and some constant $C_4 > 0$. In particular,

$$\frac{\langle \mathbf{n}_x\,,\,\mathbf{v}_1(x)\rangle}{|\langle \mathbf{n}_x\,,\,\mathbf{v}_2(x)\rangle|} \;\geq\; \frac{C_4}{C_3}\cdot\frac{\displaystyle\int_{\tilde{\varepsilon}}^{\kappa_1\tilde{\varepsilon}^{1/2}} s^{d-1}\cdot\varphi(s)\;ds}{\displaystyle\int_{\kappa_1\tilde{\varepsilon}^{1/2}}^{\delta_0} s^{d-1}\cdot\varphi(s)\;ds} \;=\; \frac{C_4}{C_3}\cdot\left[\frac{\displaystyle\int_{\tilde{\varepsilon}}^{\delta_0} s^{d-1}\cdot\varphi(s)\;ds}{\displaystyle\int_{\kappa_1\tilde{\varepsilon}^{1/2}}^{\delta_0} s^{d-1}\cdot\varphi(s)} - 1\right].$$

If the assumption **(A2)** holds, then one has

$$\limsup_{r\to 0+} r^d\cdot\varphi(r) \;=\; +\infty\,.$$

This implies

$$\lim_{r\to 0+}\int_r^{\delta_0} s^{d-1}\cdot\varphi(s)\;ds \;\geq\; \limsup_{r\to 0+}\int_r^{2r} s^{d-1}\cdot\varphi(s)\;ds \;\geq\; \frac{2^d-1}{d2^d}\cdot\limsup_{r\to 0+}\;(2r)^d\cdot\varphi(2r) \;=\; +\infty\,.$$

Applying L'Hopital's rule and the assumption **(A2)**, we obtain

$$\begin{aligned}
\lim_{\tilde{\varepsilon}\to 0+}\frac{\displaystyle\int_{\kappa_1\tilde{\varepsilon}^{1/2}}^{\delta_0} s^{d-1}\cdot\varphi(s)\;ds}{\displaystyle\int_{\tilde{\varepsilon}}^{\delta_0} s^{d-1}\cdot\varphi(s)\;ds} \;&=\; \frac{\kappa_1^d}{2}\cdot\lim_{\tilde{\varepsilon}\to 0+}\frac{\tilde{\varepsilon}^{d/2-1}\cdot\varphi(\kappa_1\tilde{\varepsilon}^{1/2})}{\tilde{\varepsilon}^{d-1}\varphi(\tilde{\varepsilon})}\\
&=\; \frac{\kappa_1^d}{2}\cdot\lim_{\tilde{\varepsilon}\to 0+}\frac{\tilde{\varepsilon}^{d/2}\cdot\varphi(\kappa_1\tilde{\varepsilon}^{1/2})}{\tilde{\varepsilon}^{d}\varphi(\tilde{\varepsilon})} \;=\; 0.
\end{aligned}$$

This yields

$$\left|\frac{\langle \mathbf{n}_x\,,\,\mathbf{v}_2(x)\rangle}{\langle \mathbf{n}_x\,,\,\mathbf{v}_1(x)\rangle}\right| \;\leq\; \frac{C_3}{C_4}\cdot\frac{\displaystyle\int_{\kappa_1\tilde\varepsilon^{1/2}}^{\delta_0} s^{d-1}\cdot\varphi(s)\;ds}{\displaystyle\int_{\tilde\varepsilon}^{\kappa_1\tilde\varepsilon^{1/2}} s^{d-1}\cdot\varphi(s)\;ds}\;\to\;0 \qquad \text{as} \qquad \varepsilon\;\to\;0\,.$$

Finally, observing that

$$|\langle \mathbf{n}_x\,,\,\mathbf{v}_3(x)\rangle|\;\leq\;C,$$

the limit behavior (3.3) is clear. This achieves the proof, because the constants C_0, δ_0, κ are independent of the point $y_x \in \Sigma$. □

In the following, for $x \notin \Sigma(t)$ we denote by $\pi(t,x)$ the perpendicular projection of x on $\Sigma(t)$, and call

$$\mathbf{n}(t,x)\;\doteq\;\frac{x-\pi(t,x)}{|x-\pi(t,x)|} \tag{3.15}$$

the unit normal vector.

Corollary 1. *Consider a family of compact $\mathcal{C}^2$ surfaces $\Sigma(t)$, continuously depending on $t\in[0,T]$, with uniformly bounded curvature. Define the vector fields*

$$\mathbf{v}(t,x)\;\doteq\;\int_{\Sigma(t)}\varphi(|x-\xi|)\,\frac{x-\xi}{|x-\xi|}\,d\sigma(\xi). \tag{3.16}$$

Then for any N there exists $\delta>0$ such that, for all $t\in[0,T]$,

$$d(x,\Sigma(t))<\delta \qquad\Longrightarrow\qquad \langle \mathbf{n}(t,x)\,,\,\mathbf{v}(t,x)\rangle\;\geq\;N. \tag{3.17}$$

Indeed, this follows from Lemma 1, observing that the limit in (3.3) is uniform over all surfaces $\Sigma(t)$.

Proof of Theorem 2.

1. Let the compact sets $\Omega_1\subset int\,\Omega_0$ be given, with Ω_1 simply connected. To fix the ideas, assume

$$B(\Omega_1,\rho)\;\subset\;\Omega_0\,, \tag{3.18}$$

for some radius $\rho>0$. Given $T>0$ and $0<\varepsilon<\rho$, we can find a decreasing family of compact sets $t\mapsto V(t)$ with $\mathcal{C}^2$ boundary, as in (1.10)-(1.11), such that

$$B(\Omega_0,\varepsilon)\;\subset\;V(0), \qquad\qquad B(\Omega_1,\varepsilon/2)\;\subset\;V(T)\;\subset\;B(\Omega_1,3\varepsilon/4). \tag{3.19}$$

Call $\Sigma(t)\doteq\partial V(t)$ the boundaries of these sets and define the vector fields

$$\mathbf{w}(t,x)\;\doteq\;\delta_0\cdot\int_{\Sigma(t)}\varphi(|x-\xi|)\,\frac{x-\xi}{|x-\xi|}\,d\sigma(\xi). \tag{3.20}$$

Here the constant $\delta_0>0$ is chosen small enough so that

$$\delta_0\cdot\int_{\Sigma(t)}d\sigma\;\leq\;1 \qquad\qquad \text{for all } t\in[0,T], \tag{3.21}$$

$$|\mathbf{w}(t,x)| \;<\; \frac{\varepsilon}{8T} \qquad \text{for all } x\in B(\Omega_1,\varepsilon/2), \quad t\in[0,T]. \tag{3.22}$$

2. For any point $x_0\in\Omega_0$, denote by $t\mapsto x(t,x_0)$ the solution of

$$\dot x(t) \;=\; \mathbf{w}(t,x(t)), \qquad\qquad x(0)=x_0\,. \tag{3.23}$$

We claim that

$$x(t,x_0)\;\in\; int\,V(t) \qquad\qquad \text{for all } t\in[0,T]. \tag{3.24}$$

To prove (3.24), let L be a Lipschitz constant for the multifunction $t\mapsto\Sigma(t)$, so that the Hausdorff distance between the two boundaries satisfies

$$d_H\Big(\Sigma(s),\,\Sigma(t)\Big)\;\leq\; L\,(t-s) \qquad \text{for any}\quad 0<s<t<T. \tag{3.25}$$

For any trajectory $t\mapsto x(t)$ of (3.23), (3.20), consider the distance

$$d(t)\;=\;\text{dist}\,\Big(x(t)\,,\;\Sigma(t)\Big)$$

of $x(t)$ from the boundary $\Sigma(t)=\partial V(t)$. By Corollary 1 there is a constant $\varepsilon_1\in\,]0,\varepsilon]$ such that, for any $x\in int\,V(t)$ with $\text{dist}(x,\Sigma(t))\leq\varepsilon_1$, one has

$$\left\langle \mathbf{n}(t,x),\int_{\Sigma(t)}\varphi(|x-\xi|)\,\frac{x-\xi}{|x-\xi|}\,d\sigma(\xi)\right\rangle \;>\; \frac{L}{\delta_0}\,. \tag{3.26}$$

In view of (3.26) and (3.25), if $d(x(t),\Sigma(t))\leq\varepsilon_1$, then the time derivative of the distance $d(t)$ satisfies

$$\begin{aligned}\dot d(t) \;&\geq\; -L+\Big\langle \mathbf{n}(t,x(t)),\,\mathbf{w}(t,x(t))\Big\rangle\\ &=\; -L+\left\langle \mathbf{n}(t,x(t)),\,\delta_0\cdot\int_{\Sigma(t)}\varphi(|x(t)-\xi|)\,\frac{x(t)-\xi}{|x(t)-\xi|}\,d\sigma(\xi)\right\rangle\\ &>\; -L+\delta_0\cdot\frac{L}{\delta_0}\;=\;0\,.\end{aligned} \tag{3.27}$$

If $x(t)$ is a trajectory starting inside Ω_0, then $d(0)\geq\varepsilon\geq\varepsilon_1$. By (3.27) we thus have $d(t)\geq\varepsilon_1$ for all $t\in[0,T]$.

We conclude this step by observing that, for every $x_0\in\overline{B}(\Omega_1,\varepsilon/4)$, by (3.22) the corresponding trajectory satisfies

$$|x(t,x_0)-x_0|\;\leq\;\frac{\varepsilon t}{8T} \qquad\qquad \text{for all } t\in[0,T]. \tag{3.28}$$

3. Relying on the approximation procedure developed in [4], we claim that there exists a Lipschitz control $t\mapsto\xi(t)$ for (1.2)-(1.3) that produces almost the same trajectories as (3.23). More precisely, calling $t\mapsto x^\xi(t,x_0)$ the solution to

$$\dot x\;=\;\mathbf{v}(x,\xi(t)), \qquad\qquad x(0)=x_0\,, \tag{3.29}$$

for every $t\in[0,T]$ and $x_0\in\Omega_0$ one has

$$|x^\xi(t,x_0)-x(t,x_0)|\;<\;\frac{\varepsilon}{8}\,. \tag{3.30}$$

Toward this goal, for any $t \in [0,T]$, define μ^t to be the $(d-1)$-dimensional measure supported on $\Sigma(t)$, so that

$$\mu^t(A) \;=\; \int_{A\cap\Sigma(t)} d\sigma$$

for every open set $A \subset I\!\!R^d$. By (3.21) it follows

$$\delta_0\,\mu^t(I\!\!R^d) \;=\; \delta_0\cdot[\text{surface area of } \Sigma(t)] \;\leq\; 1 \qquad\qquad \text{for all } t\in[0,T].$$

We now choose a point $\bar x \in I\!\!R^d$ very far from the origin and define the probability measure

$$\tilde\mu^t \;=\; \delta_0\mu^t + \Big(1-\delta_0\,\mu^t(I\!\!R^d)\Big)m_{\bar x}$$

where $m_{\bar x}$ denotes a unit Dirac mass at $\bar x$.

Notice that, as $|\bar x|\to+\infty$, by the second limit in (1.1) the vector

$$\tilde{\mathbf{w}}(t,x) \;\doteq\; \int \varphi(|x-\xi|)\,\frac{x-\xi}{|x-\xi|}\,d\tilde\mu^t(\xi) \tag{3.31}$$

approaches $\delta_0\mathbf{w}(t,x)$, uniformly on compact subsets of $I\!\!R^d\setminus\Sigma(t)$.

Given an integer $n\geq 1$, we split the interval $[0,T]$ into n equal subintervals, inserting the points $t_i = iT/n$, $i=0,1,\ldots,n$. For each i, the probability measure $\tilde\mu^{t_i}$ can now be approximated by the sum of N equal masses, say located at $\xi_{i1},\ldots,\xi_{iN}$. Defining the time step $h\doteq \frac{T}{n\,N}$, we then consider the control function

$$\xi(t) \;=\; \xi_{ij} \qquad \text{if} \quad t_i+(j-1)h < t \;\leq\; t_i+jh\,. \tag{3.32}$$

The same arguments used in [4] now show that, as $n,N\to\infty$, by suitably choosing the points ξ_{ij}, trajectories of the ODE (1.3), (3.32) converge to the corresponding trajectories of $\dot x = \tilde{\mathbf{v}}(t,x)$. Moreover, the convergence is uniform for all initial data in the compact set $\Omega_0\subset I\!\!R^d\setminus\Sigma(0)$.

Finally, we can replace the piecewise constant function $\xi(\cdot)$ by a Lipschitz function $\widetilde\xi(\cdot)$. If $\|\widetilde\xi-\xi\|_{\mathbf{L}^1}$ is sufficiently small, the corresponding trajectories still satisfy the same estimate (3.30).

4. Recalling that

$$x(T,x_0)\;\in\;V(T)\;\subseteq\;B(\Omega_1,\,3\varepsilon/4),$$

by (3.30) we now conclude

$$\Omega^\xi(T) \;=\; \{x^\xi(T,x_0)\,;\;\; x_0\in\Omega_0\} \;\subseteq\; B(\Omega_1,\,\varepsilon). \tag{3.33}$$

This establishes the second inclusion in (1.6).

To prove the first inclusion, consider the continuous map $x_0\mapsto x^\xi(T,x_0)$ from the compact set $\overline{B}(\Omega_1,\,\varepsilon/4)$ into $I\!\!R^d$. By (3.28) and (3.30) it follows

$$|x^\xi(T,x_0)-x_0| \;\leq\; \frac{\varepsilon}{4} \qquad \text{for all } x_0\in\overline{B}\Big(\Omega_1,\,\frac{\varepsilon}{4}\Big). \tag{3.34}$$

For any given $y\in\Omega_1$, define the continuous map $g^y:\overline{B}(0,\varepsilon/4)\mapsto I\!\!R^d$ by setting

$$g^y(z) \;\doteq\; x^\xi(T,y-z)-(y-z) \qquad \text{for all } z\in\overline{B}(0,\varepsilon/4)\,.$$

By (3.34) one has

$$g^y(\overline{B}(0,\varepsilon/4)) \subseteq \overline{B}(0,\varepsilon/4).$$

Therefore, Brouwer's fixed point theorem implies

$$g^y(z_0) = z_0 \qquad \text{for some } z_0 \in \overline{B}(0,\varepsilon/4).$$

This yields

$$y = x^\xi(T,x_0) \qquad \text{with} \qquad x_0 = y - z_0 \in \overline{B}(y,\varepsilon/4).$$

Hence $\Omega_1 \subseteq \Omega^\xi(T)$.

5. Finally, to pass from approximate controllability to exact controllability one can split the interval $[0,T]$, inserting an increasing sequence of times τ_j with $\tau_j \to T-$ as $j \to \infty$. Then construct Lipschitz controls $t \mapsto \xi(t)$ on each subinterval $[\tau_{j-1},\tau_j]$ such that the corresponding sets $\Omega^\xi(\tau_j)$ satisfy

$$\overline{B}(\Omega_1, 2^{-j}) \subseteq \Omega^\xi(\tau_j) \subseteq \overline{B}(\Omega_1, 2^{1-j}).$$

□

4 Approximating a sweeping process

The key tool for the proof of Theorem 3 is the following lemma, which improves on Lemma 1 under the stronger assumption **(A2′)**.

Lemma 2. *Let $\Omega \subset I\!R^d$ be a compact set with $\mathcal{C}^2$ boundary $\Sigma = \partial\Omega$. Let $\mathbf{v}$ be the vector field in (3.1). If the function φ satisfies* **(A1)-(A2′)**, *then*

$$|\mathbf{v}(x)| \to +\infty \qquad \text{as} \quad d(x,\Sigma) \to 0, \tag{4.1}$$

$$\left|\frac{\mathbf{v}(x)}{|\mathbf{v}(x)|} - \frac{x-\pi(x)}{|x-\pi(x)|}\right| \to 0 \qquad \text{as} \quad d(x,\Sigma) \to 0. \tag{4.2}$$

Proof. 1. Consider any point x sufficiently close to Σ so that the perpendicular projection $y_x \doteq \pi(x)$ at (3.2) is well defined. As in the proof of Lemma 1, in a neighborhood of y_x, the surface Σ can be expressed as the graph of a function $f : V_x \mapsto I\!R$. More precisely, call $\varepsilon = d(x,\Sigma) = |x - y_x|$. Then, given $\frac{1}{2} < \alpha < 1$, we can write $\Sigma = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3$, where

$$\begin{cases} \Sigma_1 = \Big\{ y + f(y)\,\mathbf{n}_x\,; \quad y \in V_x\,,\ |y| \le \varepsilon^\alpha \Big\}, \\ \Sigma_2 = \Big\{ y + f(y)\,\mathbf{n}_x\,; \quad y \in V_x\,,\ \varepsilon^\alpha < |y| \le \delta_0 \Big\}, \\ \Sigma_3 = \Sigma \setminus (\Sigma_1 \cup \Sigma_2)\,. \end{cases} \tag{4.3}$$

Notice that, by the regularity and compactness of the surface Σ, we can assume that the radius δ_0 of the ball where the function f is defined is independent of $y_x \in \Sigma$. Moreover, the $\mathcal{C}^2$ norm of f remains uniformly bounded.

Without loss of generality, in the following computations we shall assume $y_x = 0 \in I\!R^d$. By construction we again have the bounds (3.4), valid for some constant C_0, uniform w.r.t. $y_x \in \Sigma$. We shall estimate the vector field

$$\mathbf{v}(x) = \mathbf{v}_1(x) + \mathbf{v}_2(x) + \mathbf{v}_3(x)$$

by splitting the integral (3.1) in three parts, as in (3.7). Notice, however, that now we refer to the different decomposition (4.3) of the surface Σ.

2. Calling $J(y) = \sqrt{1 + |\nabla f(y)|^2}$ the Jacobian determinant of the map $y \mapsto y + f(y)\mathbf{n}_x$ from $V_x \cap B(y_x, \delta_0)$ into Σ, we have

$$\mathbf{v}_1(x) \;=\; \int_{|y|<\varepsilon^\alpha} \varphi(|x - y - f(y)\mathbf{n}_x|) \cdot \frac{x - y - f(y)\mathbf{n}_x}{|x - y - f(y)\mathbf{n}_x|}\, J(y)\, dy. \tag{4.4}$$

We write

$$\mathbf{v}_1(x) \;=\; \mathbf{v}_{11}(x) + \mathbf{v}_{12}(x),$$

where

$$\mathbf{v}_{11}(x) \;=\; \int_{|y|<\varepsilon^\alpha} \varphi(|x-y|)\, \frac{x-y}{|x-y|}\, dy\,, \qquad\qquad \mathbf{v}_{12}(x) \;=\; \mathbf{v}_1(x) - \mathbf{v}_{11}(x). \tag{4.5}$$

Notice that $\mathbf{v}_{11}(x)$ is a vector parallel to $\mathbf{n}_x$ and is computed as

$$\mathbf{v}_{11}(x) \;=\; \left(c_0 \int_0^{\varepsilon^\alpha} r^{d-2} \varphi\Big(\sqrt{\varepsilon^2 + r^2}\Big) \cdot \frac{\varepsilon}{\sqrt{\varepsilon^2 + r^2}}\, dr \right) \cdot \mathbf{n}_x\,, \tag{4.6}$$

for some constant $c_0 > 0$. Hence, in order to obtain the limit (4.2), let's first prove that

$$\frac{|\mathbf{v}_{12}(x)|}{|\mathbf{v}_{11}(x)|} \;\to\; 0 \qquad \text{as} \quad \varepsilon \to 0\,. \tag{4.7}$$

The vector $\mathbf{v}_{12}(x)$ satisfies

$$\begin{aligned}
\mathbf{v}_{12}(x) \;&=\; \int_{|y|<\varepsilon^\alpha} \Big[\varphi(|x - y - f(y)\mathbf{n}_x|) - \varphi(|x-y|)\Big] \cdot \frac{x - y - f(y)\mathbf{n}_x}{|x - y - f(y)\mathbf{n}_x|}\, J(y)\, dy \\
&\quad + \int_{|y|<\varepsilon^\alpha} \varphi(|x-y|) \cdot \left\{ \frac{x - y - f(y)\mathbf{n}_x}{|x - y - f(y)\mathbf{n}_x|} - \frac{x-y}{|x-y|} \right\} J(y)\, dy \\
&\quad + \int_{|y|<\varepsilon^\alpha} \varphi(|x-y|) \cdot \frac{x-y}{|x-y|}\, [J(y) - 1]\, dy \\
&=\; A_1 + A_2 + A_3\,.
\end{aligned} \tag{4.8}$$

In the following, recalling (3.4), we use the bounds

$$|f(y)| \;\leq\; \frac{C_0}{2} |y|^2, \qquad\qquad |J(y) - 1| \;=\; \mathcal{O}(1) \cdot |y|^2, \tag{4.9}$$

and the identities (3.8). Since the function φ is decreasing and $r \le \varepsilon^\alpha$, using (3.9) one obtains the estimate

$$\begin{aligned}
|A_1| \;&\le\; C\cdot\int_0^{\varepsilon^\alpha} r^{d-2}\left[\varphi\Big(\sqrt{\varepsilon^2+(1-C_0\varepsilon)r^2}\Big)-\varphi\Big(\sqrt{\varepsilon^2+r^2}\Big)\right]dr\\
&\qquad\qquad +C\cdot\int_0^{\varepsilon^\alpha} r^{d-2}\left[\varphi\Big(\sqrt{\varepsilon^2+r^2}\Big)-\varphi\Big(\sqrt{\varepsilon^2+(1+2C_0\varepsilon)r^2}\Big)\right]dr\\
&\doteq\; A_{11}+A_{12}\,,
\end{aligned} \tag{4.10}$$

for some constant C and all $\varepsilon > 0$ sufficiently small. In addition, we have

$$\begin{aligned}
\left|\frac{x-y-f(y)\mathbf{n}_x}{|x-y-f(y)\mathbf{n}_x|}-\frac{x-y}{|x-y|}\right| \;&\le\; \left|\frac{|x-y|-|x-y-f(y)\mathbf{n}_x|}{|x-y|}\right|+\frac{|f(y)|}{|x-y|}\\
&\le\; 2\cdot\frac{|f(y)|}{|x-y|} \;\le\; C_0\cdot\frac{|y|^2}{|x-y|}\,.
\end{aligned} \tag{4.11}$$

Consequently,

$$\begin{aligned}
|A_2| \;&\le\; C\cdot\int_{|y|<\varepsilon^\alpha}\varphi\left(|x-y|\right)\cdot\frac{|y|^2}{|x-y|}\,dy \;=\; C\cdot c_0\int_0^{\varepsilon^\alpha} r^{d-2}\varphi\Big(\sqrt{\varepsilon^2+r^2}\Big)\cdot\frac{r^2}{\sqrt{\varepsilon^2+r^2}}\,dr\\
&\le\; C\cdot c_0\int_0^{\varepsilon^\alpha} r^{d-2}\varphi\Big(\sqrt{\varepsilon^2+r^2}\Big)\cdot\frac{\varepsilon^{2\alpha}}{\sqrt{\varepsilon^2+r^2}}\,dr
\end{aligned} \tag{4.12}$$

and

$$|A_3| \;\le\; C\cdot c_0\int_0^{\varepsilon^\alpha} r^{d-2}\varphi\Big(\sqrt{\varepsilon^2+r^2}\Big)\,r^2\,dr \;\le\; C\cdot c_0\int_0^{\varepsilon^\alpha} r^{d-2}\varphi\Big(\sqrt{\varepsilon^2+r^2}\Big)\cdot\frac{\varepsilon^{2\alpha}}{\sqrt{\varepsilon^2+r^2}}\,dr \tag{4.13}$$

for a suitable constant C.

Since we are choosing $\alpha > 1/2$, comparing (4.12) and (4.13) with (4.6), it is clear that

$$\frac{|A_2|+|A_3|}{|\mathbf{v}_{11}(x)|} \;\le\; 2C\varepsilon^{2\alpha-1} \;\to\; 0 \qquad \text{as} \quad \varepsilon\to 0. \tag{4.14}$$

Proving a similar estimate for A_1 requires more work. Performing the variable change

$$s \;=\; \sqrt{1-C_0\varepsilon}\cdot r\,,$$

one obtains

$$\begin{aligned}
0 \;&\le\; \int_0^{\varepsilon^\alpha} r^{d-2}\left[\varphi\Big(\sqrt{\varepsilon^2+(1-C_0\varepsilon)r^2}\Big)-\varphi\Big(\sqrt{\varepsilon^2+r^2}\Big)\right]dr\\
&\le\; (1-C_0\varepsilon)^{-\frac{d-1}{2}}\int_0^{\varepsilon^\alpha\cdot\sqrt{1-C_0\varepsilon}} s^{d-2}\varphi\left(\sqrt{\varepsilon^2+s^2}\right)\,ds-\int_0^{\varepsilon^\alpha} r^{d-2}\varphi\Big(\sqrt{\varepsilon^2+r^2}\Big)\,dr\\
&\le\; \left[(1-C_0\varepsilon)^{-\frac{d-1}{2}}-1\right]\int_0^{\varepsilon^\alpha} r^{d-2}\varphi\left(\sqrt{\varepsilon^2+r^2}\right)\,dr\\
&\le\; C_1\varepsilon\int_0^{\varepsilon^\alpha} r^{d-2}\varphi\left(\sqrt{\varepsilon^2+r^2}\right)\,dr,
\end{aligned} \tag{4.15}$$

for some constant C_1. Recalling (4.10) and comparing with (4.6), we thus obtain

$$\begin{aligned} A_{11} &\leq C_2\varepsilon \int_0^{\varepsilon^\alpha} r^{d-2}\varphi\left(\sqrt{\varepsilon^2+r^2}\right)\,dr \\ &\leq C_2\sqrt{\varepsilon^2+\varepsilon^{2\alpha}}\cdot \int_0^{\varepsilon^\alpha} r^{d-2}\varphi\left(\sqrt{\varepsilon^2+r^2}\right)\,\frac{\varepsilon}{\sqrt{\varepsilon^2+r^2}}\,dr \\ &= \frac{C_2}{c_0}\sqrt{\varepsilon^2+\varepsilon^{2\alpha}}\cdot|\mathbf{v}_{11}(x)| \;\leq\; \frac{2C_2\varepsilon^\alpha}{c_0}\cdot|\mathbf{v}_{11}(x)| \end{aligned} \tag{4.16}$$

for some constant C_2. A similar argument yields

$$\begin{aligned} A_{12} &\leq C\cdot\left[\int_0^{\varepsilon^\alpha} r^{d-2}\varphi\left(\sqrt{\varepsilon^2+r^2}\right)\,dr \;-\left(\frac{1}{1+2C_0\varepsilon}\right)^{\frac{d-1}{2}}\cdot\int_0^{\varepsilon^\alpha\sqrt{1+2C_0\varepsilon}} r^{d-2}\varphi\left(\sqrt{\varepsilon^2+r^2}\right)\,dr\right] \\ &\leq C_3\varepsilon\int_0^{\varepsilon^\alpha} r^{d-2}\varphi\left(\sqrt{\varepsilon^2+r^2}\right)\,dr \;\leq\; \frac{2C_3\varepsilon^\alpha}{c_0}\cdot|\mathbf{v}_{11}(x)|\,. \end{aligned} \tag{4.17}$$

Putting together (4.10), (4.14), (4.16), and (4.17), we can compare the sizes of the vectors $\mathbf{v}_{11}$ and $\mathbf{v}_{12}$ in (4.5)–(4.8). Indeed, the previous analysis shows that

$$\frac{|\mathbf{v}_{12}(x)|}{|\mathbf{v}_{11}(x)|} \;\leq\; \frac{|A_1|+|A_2|+|A_3|}{|\mathbf{v}_{11}(x)|} \;\leq\; \frac{A_{11}+A_{12}+|A_2|+|A_3|}{|\mathbf{v}_{11}(x)|} \;\to\; 0 \qquad \text{as} \qquad \varepsilon\;\to\;0\,. \tag{4.18}$$

3. In a similar fashion we now compute

$$\mathbf{v}_2(x) \;=\; \mathbf{v}_{21}(x)+\mathbf{v}_{22}(x), \tag{4.19}$$

where

$$\begin{aligned} \mathbf{v}_{21}(x) &= \int_{\varepsilon^\alpha<|y|<\delta_0} \varphi\Big(|x-y|\Big)\cdot\frac{x-y}{|x-y|}\,dy \\ &= \left(c_0\int_{\varepsilon^\alpha}^{\delta_0} r^{d-2}\varphi\Big(\sqrt{\varepsilon^2+r^2}\Big)\cdot\frac{\varepsilon}{\sqrt{\varepsilon^2+r^2}}\,dr\right)\mathbf{n}_x\,, \end{aligned} \tag{4.20}$$

and

$$\begin{aligned} \mathbf{v}_{22}(x) \;&=\; \int_{\varepsilon^\alpha<|y|<\delta_0}\Big[\varphi(|x-y-f(y)\mathbf{n}_x|)-\varphi(|x-y|)\Big]\cdot\frac{x-y-f(y)\mathbf{n}_x}{|x-y-f(y)\mathbf{n}_x|}\,J(y)\,dy \\ &\quad+\int_{\varepsilon^\alpha<|y|<\delta_0}\varphi(|x-y|)\cdot\left\{\frac{x-y-f(y)\mathbf{n}_x}{|x-y-f(y)\mathbf{n}_x|}-\frac{x-y}{|x-y|}\right\}J(y)\,dy \\ &\quad+\int_{\varepsilon^\alpha<|y|<\delta_0}\varphi(|x-y|)\cdot\frac{x-y}{|x-y|}\,[J(y)-1]\,dy \\ &=\; B_1+B_2+B_3\,. \end{aligned} \tag{4.21}$$

As in (4.12)-(4.13), we have

$$|B_2|+|B_3| \;\leq\; C_3\cdot\int_{\varepsilon^\alpha}^{\delta_0}\frac{r^d}{\sqrt{\varepsilon^2+r^2}}\cdot\varphi\left(\sqrt{\varepsilon^2+r^2}\right)\,dr \;\leq\; C_3\int_{\varepsilon^\alpha}^{\delta_0} r^{d-1}\varphi(\sqrt{\varepsilon^2+r^2})\,dr\,. \tag{4.22}$$

Using again (3.9) and the fact that φ is a decreasing function, we obtain

$$\begin{aligned}
|B_1| \;&\leq\; C\cdot\Big[\int_{\varepsilon^\alpha}^{\delta_0} r^{d-2}\left[\varphi\left(\sqrt{\varepsilon^2+(1-C_0\varepsilon)r^2}\right)-\varphi\Big(\sqrt{\varepsilon^2+r^2}\Big)\right]dr \\
&\qquad\qquad + \int_{\varepsilon^\alpha}^{\delta_0} r^{d-2}\left[\varphi\Big(\sqrt{\varepsilon^2+r^2}\Big)-\varphi\Big(\sqrt{\varepsilon^2+(1+C_0\varepsilon)r^2+C_0^2r^4/4}\Big)\right]dr \\
&\doteq\; B_{11}+B_{12}\,.
\end{aligned} \tag{4.23}$$

As in (4.15), performing the variable change $s=\sqrt{1-C_0\varepsilon}\cdot r$, we obtain

$$\begin{aligned}
B_{11} \;&=\; C\cdot\int_{\varepsilon^\alpha}^{\delta_0} r^{d-2}\left[\varphi\left(\sqrt{\varepsilon^2+(1-C_0\varepsilon)r^2}\right)-\varphi\Big(\sqrt{\varepsilon^2+r^2}\Big)\,dr\right] \\
&=\; C\cdot\left[(1-C_0\varepsilon)^{-\frac{d-1}{2}}\int_{\varepsilon^\alpha\sqrt{1-C_0\varepsilon}}^{\delta_0\sqrt{1-C_0\varepsilon}} s^{d-2}\varphi\Big(\sqrt{\varepsilon^2+s^2}\Big)\,ds-\int_{\varepsilon^\alpha}^{\delta_0} r^{d-2}\varphi\Big(\sqrt{\varepsilon^2+r^2}\Big)\,dr\right] \\
&\leq\; C_3\varepsilon\int_{\varepsilon^\alpha}^{\delta_0} r^{d-2}\varphi\Big(\sqrt{\varepsilon^2+r^2}\Big)\,dr+C_3\int_{\varepsilon^\alpha\sqrt{1-C_0\varepsilon}}^{\varepsilon^\alpha} r^{d-2}\varphi\Big(\sqrt{\varepsilon^2+r^2}\Big)\,dr
\end{aligned} \tag{4.24}$$

for a suitable constant C_3. Since φ is decreasing, for ε sufficiently small we have

$$\begin{aligned}
\int_0^{\varepsilon^\alpha} r^{d-2}\varphi\Big(\sqrt{\varepsilon^2+r^2}\Big)\,dr \;&\geq\; \int_{\varepsilon^\alpha(1-C_3\sqrt{\varepsilon})}^{\varepsilon^\alpha} r^{d-2}\varphi\Big(\sqrt{\varepsilon^2+r^2}\Big)\,dr \\
&\geq\; (1-C_3\sqrt{\varepsilon})^{d-2}\cdot\int_{\varepsilon^\alpha(1-C_3\sqrt{\varepsilon})}^{\varepsilon^\alpha} \varepsilon^{d-2}\varphi\Big(\sqrt{\varepsilon^2+r^2}\Big)\,dr \\
&\geq\; (1-C_3\sqrt{\varepsilon})^{d-2}\cdot\frac{C_3\varepsilon^{\alpha+\frac12}}{C_3\varepsilon^{\alpha+1}}\cdot\int_{\varepsilon^\alpha(1-C_3\varepsilon)}^{\varepsilon^\alpha} r^{d-2}\varphi\Big(\sqrt{\varepsilon^2+r^2}\Big)\,dr \\
&\geq\; \frac{1}{2\sqrt{\varepsilon}}\cdot\int_{\varepsilon^\alpha(1-C_3\varepsilon)}^{\varepsilon^\alpha} r^{d-2}\varphi\Big(\sqrt{\varepsilon^2+r^2}\Big)\,dr.
\end{aligned} \tag{4.25}$$

In turn, this yields

$$\begin{aligned}
B_{11} \;&\leq\; C_3\varepsilon\int_{\varepsilon^\alpha}^{\delta_0} r^{d-2}\varphi\Big(\sqrt{\varepsilon^2+r^2}\Big)\,dr+2C_3\sqrt{\varepsilon}\int_0^{\varepsilon^\alpha} r^{d-2}\varphi\Big(\sqrt{\varepsilon^2+r^2}\Big)\,dr \\
&\leq\; C_3\varepsilon\int_{\varepsilon^\alpha}^{\delta_0} r^{d-2}\varphi\Big(\sqrt{\varepsilon^2+r^2}\Big)\,dr+2C_3\sqrt{\varepsilon+\varepsilon^{2\alpha-1}}\cdot\int_0^{\varepsilon^\alpha}\frac{\varepsilon r^{d-2}}{\sqrt{\varepsilon^2+r^2}}\cdot\varphi\left(\sqrt{\varepsilon^2+r^2}\right)\,dr \\
&\leq\; C_3\varepsilon\int_{\varepsilon^\alpha}^{\delta_0} r^{d-2}\varphi\Big(\sqrt{\varepsilon^2+r^2}\Big)\,dr+\frac{4C_3\varepsilon^{\alpha-\frac12}}{c_0}\cdot|\mathbf{v}_{11}(x)|\,.
\end{aligned} \tag{4.26}$$

Next, performing the change of variable $t=\sqrt{(1+C_0\varepsilon)r^2+C_0^2r^4/4}$, we estimate

$$\int_{\varepsilon^\alpha}^{\delta_0} r^{d-2}\cdot\varphi\Big(\sqrt{\varepsilon^2+(1+C_0\varepsilon)r^2+C_0^2r^4/4}\Big)\,dr \;\;\geq\;\; \int_{\varepsilon^\alpha(1+2C_0\sqrt{\varepsilon})}^{\delta_0}\frac{t^{d-2}}{1+C_4(\varepsilon+t^2)}\cdot\varphi\left(\sqrt{\varepsilon^2+t^2}\right)dt$$

for a suitable constant C_4. This implies that

$$\begin{aligned} B_{12} &\leq C \cdot \left[\int_{\varepsilon^\alpha}^{\delta_0} r^{d-2} \varphi\left(\sqrt{\varepsilon^2+r^2}\right) dr - \int_{\varepsilon^\alpha(1+2C_0\sqrt{\varepsilon})}^{\delta_0} \frac{r^{d-2}}{1+C_4(\varepsilon+r^2)} \cdot \varphi\left(\sqrt{\varepsilon^2+r^2}\right) dr \right] \\ &\leq C \cdot \left[\int_{\varepsilon^\alpha}^{\delta_0} \left(1 - \frac{1}{1+C_4(\varepsilon+r^2)}\right) r^{d-2} \varphi\left(\sqrt{\varepsilon^2+r^2}\right) dr + \int_{\varepsilon^\alpha}^{\varepsilon^\alpha(1+2C_0\sqrt{\varepsilon})} r^{d-2} \varphi\left(\sqrt{\varepsilon^2+r^2}\right) dr \right] \\ &\leq CC_4 \int_{\varepsilon^\alpha}^{\delta_0} (\varepsilon+r^2) r^{d-2} \varphi\left(\sqrt{\varepsilon^2+r^2}\right)\ dr + C \int_{\varepsilon^\alpha}^{\varepsilon^\alpha(1+2C_0\sqrt{\varepsilon})} r^{d-2} \varphi\left(\sqrt{\varepsilon^2+r^2}\right)\ dr\,. \end{aligned}$$

As in (4.25), one estimates

$$\int_{\varepsilon^\alpha}^{\varepsilon^\alpha(1+2C_0\sqrt{\varepsilon})} r^{d-2} \varphi\left(\sqrt{\varepsilon^2+r^2}\right) \ \leq\ 2\sqrt{\varepsilon} \int_0^{\varepsilon^\alpha} r^{d-2} \varphi\left(\sqrt{\varepsilon^2+r^2}\right) dr$$

for ε sufficiently small. Thus, as in (4.23), we obtain

$$\begin{aligned} B_{12} &\leq CC_4 \int_{\varepsilon^\alpha}^{\delta_0} (\varepsilon+r^2) r^{d-2} \varphi\left(\sqrt{\varepsilon^2+r^2}\right)\ dr + 2C\sqrt{\varepsilon} \int_0^{\varepsilon^\alpha} r^{d-2} \varphi\left(\sqrt{\varepsilon^2+r^2}\right) dr \\ &\leq CC_4 \int_{\varepsilon^\alpha}^{\delta_0} (\varepsilon+r^2) r^{d-2} \varphi\left(\sqrt{\varepsilon^2+r^2}\right)\ dr + 4C\varepsilon^{\alpha-\frac{1}{2}} \cdot |\mathbf{v}_{11}(x)|\,. \end{aligned} \tag{4.27}$$

Combining (4.21), (4.22), (4.23), (4.26), and (4.27), we finally obtain

$$|\mathbf{v}_{22}(x)| \ \leq\ C_5 \cdot \int_{\varepsilon^\alpha}^{\delta_0} (\varepsilon+r+r^2) r^{d-2} \varphi\left(\sqrt{\varepsilon^2+r^2}\right)\ dr + C_5 \varepsilon^{\alpha-\frac{1}{2}} \cdot |\mathbf{v}_{11}(x)| \tag{4.28}$$

for some constant C_5.

4. Finally, since in (4.8) the integral over Σ_3 involves functions which are uniformly bounded over Σ, we have a trivial bound of the form

$$|\mathbf{v}_3(x)| \ \leq\ C_6\,. \tag{4.29}$$

5. We now compare the sizes of $\mathbf{v}_{22}(x)$ and $\mathbf{v}_3(x)$ with $\mathbf{v}_{11}(x)$. From (4.6) it follows

$$|\mathbf{v}_{11}(x)| \ \geq\ c_0 \int_0^{\varepsilon^\alpha} r^d \cdot \frac{\varphi\left(\sqrt{\varepsilon^2+r^2}\right)}{\sqrt{\varepsilon^2+r^2}}\ dr\,.$$

Performing the change of variable $t = \sqrt{\varepsilon^2+r^2}$ one obtains

$$|\mathbf{v}_{11}(x)| \ \geq\ c_0 \int_\varepsilon^{\sqrt{\varepsilon^2+\varepsilon^{2\alpha}}} (t^2-\varepsilon^2)^{\frac{d-1}{2}} \varphi(t)\ dt \ \geq\ c_0 \left(\frac{3}{4}\right)^{\frac{d-1}{2}} \cdot \int_{2\varepsilon}^{\varepsilon^\alpha} r^{d-1} \varphi(r)\ dr. \tag{4.30}$$

Recalling (4.28) and (4.29), we obtain

$$\frac{|\mathbf{v}_{22}(x)| + |\mathbf{v}_3(x)|}{|\mathbf{v}_{11}(x)|} \ \leq\ C_7 \cdot \frac{\displaystyle\int_{\varepsilon^\alpha}^{\delta_0} r^{d-1} \varphi(r)\, dr}{\displaystyle\int_{2\varepsilon}^{\varepsilon^\alpha} r^{d-1} \varphi(r)\ dr} + C_5 \cdot \varepsilon^{\alpha-\frac{1}{2}} + \frac{C_7}{|\mathbf{v}_{11}(x)|}\,, \tag{4.31}$$

for some constant C_7.

By (4.18) we already know that the ratio $|\mathbf{v}_{12}|/|\mathbf{v}_{11}|$ approaches zero as $\varepsilon \to 0$. Moreover, by (4.6) and (4.20) one has

$$\mathbf{v}_{21}(x) \;=\; C_\varepsilon\,\mathbf{v}_{11}(x) \qquad \text{for some constant } C_\varepsilon > 0\,.$$

Therefore, in view of (4.31), we can conclude that (4.2) holds true provided that

$$\lim_{\varepsilon\to 0+} \frac{\displaystyle\int_{\varepsilon^\alpha}^1 r^{d-1}\varphi(r)\,dr}{\displaystyle\int_{2\varepsilon}^{\varepsilon^\alpha} r^{d-1}\varphi(r)\,dr} \;=\; 0\,. \tag{4.32}$$

We show that (4.32) is satisfied if φ satisfies the assumption **(A2′)**. Indeed, let $\beta > \frac{1}{2}$ be as in **(A2′)** and choose $\frac{1}{2} < \alpha < \beta$. Setting $\tilde\varepsilon = 2\varepsilon$, we have $\varepsilon^\alpha > \tilde\varepsilon^\beta$ if ε is sufficiently small. Consequently,

$$\int_{\varepsilon^\alpha}^1 r^{d-1}\varphi(r)dr \;\le\; \int_{\tilde\varepsilon^\beta}^1 r^{d-1}\varphi(r)dr \qquad \text{and} \qquad \int_{2\varepsilon}^{\varepsilon^\alpha} r^{d-1}\varphi(r)dr \;\ge\; \int_{\tilde\varepsilon}^{\tilde\varepsilon^\beta} r^{d-1}\varphi(r)dr\,.$$

On the other hand, as in the proof of Lemma 1, it holds

$$\lim_{r\to 0+} \int_r^1 s^{d-1}\varphi(s)\,ds \;=\; +\infty\,.$$

Using L'Hopital's rule and the assumption (**A2′**), we obtain

$$\lim_{\tilde\varepsilon\to 0+} \frac{\displaystyle\int_{\tilde\varepsilon^\beta}^1 r^{d-1}\varphi(r)dr}{\displaystyle\int_{\tilde\varepsilon}^1 r^{d-1}\varphi(r)dr} \;=\; \lim_{\tilde\varepsilon\to 0} \frac{\tilde\varepsilon^{\beta(d-1)}\varphi(\tilde\varepsilon^\beta)\cdot(\beta\tilde\varepsilon^{\beta-1})}{\tilde\varepsilon^{d-1}\cdot\varphi(\tilde\varepsilon)} \;=\; \beta\cdot\lim_{\tilde\varepsilon\to 0} \frac{\tilde\varepsilon^{\beta d}\cdot\varphi(\tilde\varepsilon^\beta)}{\tilde\varepsilon^d\cdot\varphi(\tilde\varepsilon)} \;=\; 0\,.$$

This implies

$$\lim_{\varepsilon\to 0^+} \frac{\displaystyle\int_{\varepsilon^\alpha}^1 r^{d-1}\varphi(r)\,dr}{\displaystyle\int_{2\varepsilon}^{\varepsilon^\alpha} r^{d-1}\varphi(r)\,dr} \;\le\; \lim_{\tilde\varepsilon\to 0+} \frac{\displaystyle\int_{\tilde\varepsilon^\beta}^1 r^{d-1}\varphi(r)dr}{\displaystyle\int_{\tilde\varepsilon}^{\tilde\varepsilon^\beta} r^{d-1}\varphi(r)dr} \;=\; 0,$$

proving (4.32). □

Corollary 2. *Consider a family of compact $\mathcal{C}^2$ surfaces $\Sigma(t)$, continuously depending on $t \in [0,T]$, with uniformly bounded curvature. Define the vector fields $\mathbf{v}(t,\cdot)$ as in (3.16). Then for any $N, \varepsilon > 0$ there exists $\delta > 0$ such that, for all $t \in [0,T]$,*

$$\begin{aligned} d(x,\Sigma(t)) \le \delta \qquad &\Longrightarrow \qquad |\mathbf{v}(t,x)| \ge N, \\ d(x,\Sigma(t)) \le \delta \qquad &\Longrightarrow \qquad \left|\frac{\mathbf{v}(t,x)}{|\mathbf{v}(t,x)|} - \frac{x-\pi(t,x)}{|x-\pi(t,x)|}\right| \le \varepsilon. \end{aligned} \tag{4.33}$$

Indeed, the proof of Lemma 2 shows that the limits (4.1)-(4.2) are uniformly valid over a family of surfaces $\Sigma(t)$ with uniformly bounded curvature.

5 Proof of Theorem 3.

Relying on Lemma 2, we can now give a proof of the convergence to the sweeping process, stated in (1.16). We recall that this sweeping process keeps all trajectories inside a moving compact set $V(t) \subset I\!R^d$ with smooth boundary $\Sigma(t)$. To fix the ideas, we assume that this set is defined in terms of a $\mathcal{C}^2$ function ψ, as in (1.10)-(1.11). The argument relies on three main properties.

(P1) There exists a radius $\rho_0 > 0$ such that, if $t \in [0,T]$ and $d(x,\Sigma(t)) \le \rho_0$, then the perpendicular projection $\pi(t,x)$ of x on $\Sigma(t)$ is well defined. In this case we denote by $\mathbf{n}(t,x)$ the unit normal vector to $\Sigma(t)$ at the point $\pi(t,x)$, as in (3.15)

(P2) Setting

$$\mathbf{v}(t,x) \;\doteq\; \int_{\Sigma(t)} \varphi(|x-\xi|)\,\frac{x-\xi}{|x-\xi|}\,d\sigma(\xi), \tag{5.1}$$

for any $\delta > 0$ the solution $t \mapsto x^\delta(t)$ to

$$\dot x \;=\; \delta\,\mathbf{v}(t,x), \qquad\qquad x(0) = x_0\,. \tag{5.2}$$

satisfies $x^\delta(t) \in V(t)$, for every $x_0 \in int(V(0))$ and $t \in [0,T]$.

To see why this is true, assume $d(x^\delta(t),\Sigma(t)) < \rho_0$ and consider the unit normal vector

$$\mathbf{n}(t,x^\delta) \;\doteq\; \frac{x^\delta(t) - \pi(t,x^\delta(t))}{|x^\delta(t) - \pi(t,x^\delta(t))|}. \tag{5.3}$$

Then

$$\frac{d}{dt}\Big(d(x^\delta(t),\Sigma(t))\Big) \;\ge\; \Big\langle \delta\mathbf{v}(t,x^\delta(t)),\, \mathbf{n}(t,x^\delta)\Big\rangle - L_\Sigma\,,$$

where L_Σ is a Lipschitz constant for the multifunction $t \mapsto \Sigma(t)$. By (4.1)-(4.2), it follows that

$$\langle \mathbf{v}(t,y),\, \mathbf{n}(t,y)\rangle \;\to\; +\infty \qquad \text{as} \qquad d(y,\Sigma(t)) \to 0.$$

Hence the distance $d(x^\delta(t),\Sigma(t))$ remains uniformly positive in time, for every fixed $\delta > 0$ and all initial points x_0 at a uniformly positive distance from $\Sigma(0)$.

Finally, by the properties (1.1) of φ it follows

(P3) For every $0 < \varepsilon < \frac{1}{4}$, by choosing $0 < \delta < \delta_0 < \varepsilon$ sufficiently small, for every $x \in V(t)$ one has the implication

$$0 \;<\; d(x,\Sigma(t)) \;\le\; \delta_0 \qquad \Longrightarrow \qquad \left|\frac{\mathbf{v}(t,x)}{|\mathbf{v}(t,x)|} - \frac{x-\pi(t,x)}{|x-\pi(t,x)|}\right| \;<\; \varepsilon. \tag{5.4}$$

$$d(x,\Sigma(t)) \;>\; \delta_0 \qquad \Longrightarrow \qquad \delta|\mathbf{v}(t,x)| \;<\; \varepsilon. \tag{5.5}$$

Indeed, (5.4) follows from Corollary 2 and the properties (1.11) of the function ψ, defining the boundary $\Sigma(t)$. The implication (5.5) trivially holds, choosing $\delta > 0$ sufficiently small.

In the following, using the properties **(P1)–(P3)**, we estimate the distance between $x^\delta(t)$ and the solution $x(t, x_0)$ of the sweeping process (1.12). The proof will be given in several steps.

1. For a given initial condition $x_0 \in intV(0)$, let $t \mapsto x^\delta(t)$ be the solution to

$$\dot{x} \;=\; \delta \mathbf{v}(t,x), \qquad\qquad x(0) = x_0\,,$$

and let $t \mapsto x(t)$ be the corresponding solution to the sweeping process driven by the set $V(t)$.

For every $t \in [0,T]$ such that $d(x(t), \Sigma(t)) \le \rho_0/2$, let $\mathbf{n}(t)$ be the unit normal vector to $\Sigma(t)$ at the point $\pi(t, x(t))$. By the regularity of $\Sigma(\cdot)$, we can extend $\mathbf{n}(\cdot)$ to a Lipschitz function defined on the entire time interval $[0,T]$. For simplicity, this extension will still be denoted by $t \mapsto \mathbf{n}(t)$.

We now split the difference as

$$w(t) \;\doteq\; x^\delta(t) - x(t) \;=\; w_1(t) + w_2(t), \tag{5.6}$$

where the vector $w_1(t)$ is parallel to $\mathbf{n}(t)$ while $w_2(t)$ is orthogonal to $\mathbf{n}(t)$. Namely,

$$w_1(t) \;\doteq\; \theta(t)\mathbf{n}(t), \qquad \theta(t) \;=\; \langle w(t), \mathbf{n}(t)\rangle, \qquad w_2(t) \;=\; w(t) - w_1(t). \tag{5.7}$$

For future use, we observe that, if $d(x^\delta(t), \Sigma(t)) \le \rho_0$, then the unit normal vector $\mathbf{n}(t, x^\delta)$ at (5.3) is well defined and

$$|\mathbf{n}(t) - \mathbf{n}(t, x^\delta)| \;\le\; C_\mathbf{n} |w(t)|, \tag{5.8}$$

for a suitable constant $C_\mathbf{n}$.

Our main goal is to show that $w(t)$ remains small. This will be achieved by estimating the time derivatives $\dot{w}_1(t), \dot{w}_2(t)$, considering two possible alternatives (see Fig. 3).

CASE 1: $d(x^\delta(t), \Sigma(t)) \ge \delta_0\,.$

CASE 2: $d(x^\delta(t), \Sigma(t)) < \delta_0\,.$

We observe that, by the $\mathcal{C}^2$ regularity of the boundaries $\Sigma(t) = \partial V(t)$, there exists a constant C_Σ such that

$$\langle \mathbf{n}(t,x),\, y - x\rangle \;\ge\; -\,C_\Sigma \cdot |y-x|^2 \qquad \text{for all} \quad t \in [0,T], \quad x \in \Sigma(t), \quad y \in V(t)\,. \tag{5.9}$$

In particular,

$$x(t) \in \Sigma(t) \qquad \Longrightarrow \qquad \theta(t) \;\ge\; -\,C_\Sigma\, |w_2(t)|^2. \tag{5.10}$$

2. In this step we consider the Case 1: $d(x^\delta(t), \Sigma(t)) \ge \delta_0\,.$

On the interval $[0,T]$, let $L_\mathbf{n} \ge 1$ be a Lipschitz constant for the map $t \mapsto \mathbf{n}(t)$, and let L_Σ be a Lipschitz constant for the multifunction $t \mapsto \Sigma(t)$, w.r.t. the Hausdorff distance. Observe

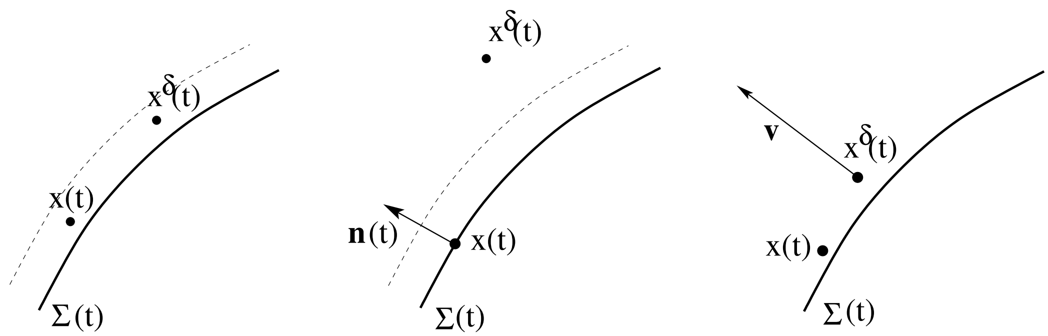

Figure 3: Left and center: the two different cases considered in the proof of Theorem 3, depending on the distance of $x^\delta(t)$ from the boundary $\Sigma(t)$. Right: if $d(x^\delta(t),\Sigma(t))<\delta_0$, then the speed $\delta\,\mathbf{v}(t,x^\delta)$ can be very large, and the same is true of $\dot\theta$. To handle this situation, we need to insert a weight function W in our estimates.

that L_Σ provides a common Lipschitz constant for all trajectories $t\mapsto x(t)$ of the sweeping process.

We claim that

$$\varepsilon+C_2|w_2(t)|\ \geq\ \dot\theta(t)\ \geq\ \begin{cases}-\varepsilon-L_{\mathbf{n}}|w_2(t)| & \text{if}\quad x(t)\notin\Sigma(t)\\ \quad -C_1 & \text{if}\quad x(t)\in\Sigma(t)\end{cases},\qquad |\dot w_2(t)|\ \leq\ \varepsilon+C_2\,|w(t)|, \tag{5.11}$$

where $C_1=1+L_\Sigma+L_{\mathbf{n}}\cdot\max_{t\in[0,T]}\{\mathrm{diam}(V(t))\}$ and $C_2=2L_{\mathbf{n}}$. Indeed, recalling (5.5) and (5.7), we have

$$\begin{aligned}\dot\theta(t)\ &=\ \langle\dot w(t),\mathbf{n}(t)\rangle+\langle w(t),\dot{\mathbf{n}}(t)\rangle\\ &=\ \langle\dot x^\delta(t),\mathbf{n}(t)\rangle-\langle\dot x(t),\mathbf{n}(t)\rangle+\theta(t)\langle\mathbf{n}(t),\dot{\mathbf{n}}(t)\rangle+\langle w_2(t),\dot{\mathbf{n}}(t)\rangle\\ &\geq\ -|\dot x^\delta(t)|-|\dot x(t)|-|w_2(t)|\cdot|\dot{\mathbf{n}}(t)|\\ &\geq\ \begin{cases}\quad -\varepsilon-L_{\mathbf{n}}|w_2(t)| & \text{if}\quad x(t)\notin\Sigma(t),\\ -\varepsilon-L_\Sigma-L_{\mathbf{n}}|w_2(t)| & \text{if}\quad x(t)\in\Sigma(t).\end{cases}\end{aligned} \tag{5.12}$$

Moreover,

$$\dot\theta(t)\ \leq\ \langle\dot x^\delta(t),\mathbf{n}(t)\rangle-\langle\dot x(t),\mathbf{n}(t)\rangle+|w_2(t)|\cdot|\dot{\mathbf{n}}(t)|\ \leq\ \varepsilon+L_{\mathbf{n}}\,|w_2(t)|. \tag{5.13}$$

Together, (5.12)-(5.13) yield the upper and lower bounds on $\dot\theta$ in (5.11).

Next, by (5.7) one has

$$\begin{aligned}|\dot w_2(t)|\ &=\ \left|\frac{d}{dt}\,[w(t)-w_1(t)]\right|\ =\ \left|\frac{d}{dt}\Big[w(t)-\langle w(t),\mathbf{n}(t)\rangle\,\mathbf{n}(t)\Big]\right|\\ &=\ \Big|\dot w(t)-\langle\dot w(t),\mathbf{n}(t)\rangle\,\mathbf{n}(t)-\langle w(t),\dot{\mathbf{n}}(t)\rangle\,\mathbf{n}(t)-\langle w(t),\mathbf{n}(t)\rangle\,\dot{\mathbf{n}}(t)\Big|\\ &\leq\ \Big|\dot w(t)-\langle\dot w(t),\mathbf{n}(t)\rangle\,\mathbf{n}(t)\Big|+2L_{\mathbf{n}}\,|w(t)|\,.\end{aligned}$$

Since $\dot x(t)$ is either zero or parallel to $\mathbf{n}(t)$, by (5.5) it follows

$$\Big|\dot w(t)-\langle\dot w(t),\mathbf{n}(t)\rangle\,\mathbf{n}(t)\Big|\ =\ \Big|\delta\,\mathbf{v}(t,x^\delta(t))-\dot x(t)-\langle\delta\,\mathbf{v}(t,x^\delta(t))-\dot x(t),\mathbf{n}(t)\rangle\,\mathbf{n}(t)\Big|$$

$$
\begin{aligned}
&= \; |\delta\,\mathbf{v}(t,x^\delta(t)) - \langle \delta\,\mathbf{v}(t,x^\delta(t)), \mathbf{n}(t)\rangle\,\mathbf{n}(t)| \\
&\leq \; |\delta\,\mathbf{v}(t,x^\delta(t))| \;\leq\; \varepsilon\,.
\end{aligned}
$$

This implies the second inequality in (5.11). In particular, if $x(t) \in int(V(t))$ then $\dot{x}(t) = 0$ and

$$
\frac{d}{dt}|w(t)| \;\leq\; 3(\varepsilon + \mathbf{L_n}\,|w(t)|)\,. \tag{5.14}
$$

3. In this step we consider Case 2: $d(x^\delta(t), \Sigma(t)) < \delta_0$. As long as

$$
|w(t)| \;\leq\; \frac{1}{4C_{\mathbf{n}}}, \tag{5.15}
$$

we claim that

$$
\theta(t) \;\leq\; \delta_0 + C_3(\delta_0 + |w(t)|)^2, \tag{5.16}
$$

$$
\dot{\theta}(t) \;\geq\; \begin{cases} -L_{\mathbf{n}}|w_2(t)| & \text{if} \quad x(t) \notin \Sigma(t), \\ \quad -C_1 & \text{if} \quad x(t) \in \Sigma(t), \end{cases} \qquad |\dot{w}_2(t)| \;\leq\; C_4(\varepsilon + |w(t)|)(1 + |\dot{\theta}(t)|), \tag{5.17}
$$

for some constants C_3, C_4.

Notice that, if $d(x(t), \Sigma(t)) \geq \rho_0/2$, we then have $|w(t)| \geq \rho_0/4$, because without loss of generality we can assume $\delta_0 < \varepsilon < \rho_0/4$. In this case the estimate (5.16) is trivially satisfied, by choosing a constant C_3 large enough.

In the following, we thus assume $d(x(t), \Sigma(t)) \geq \rho_0/2$, so that the projection $\pi(t, x(t))$ is well defined. Recalling (5.6)–(5.9) we obtain

$$
\begin{aligned}
\theta(t) \;&=\; \Big\langle x^\delta(t) - x(t), \mathbf{n}(t)\Big\rangle \;=\; \Big\langle x^\delta(t) - \pi(t,x^\delta), \mathbf{n}(t)\Big\rangle + \Big\langle \pi(t,x^\delta) - x(t), \mathbf{n}(t)\Big\rangle \\
&\leq\; \Big|x^\delta(t) - \pi(t,x^\delta)\Big| + \Big\langle \pi(t,x^\delta) - x(t), \mathbf{n}(t)\Big\rangle \\
&\leq\; \delta_0 + \Big\langle \pi(t,x^\delta) - x(t),\, \mathbf{n}(t) - \mathbf{n}(t,x^\delta)\Big\rangle + \Big\langle \pi(t,x^\delta) - x(t),\, \mathbf{n}(t,x^\delta)\Big\rangle \\
&\leq\; \delta_0 + C_{\mathbf{n}}\,|w(t)|\,\Big|\pi(t,x^\delta) - x(t)\Big| + C_\Sigma\,\Big|\pi(t,x^\delta) - x(t)\Big|^2 \\
&\leq\; \delta_0 + C_{\mathbf{n}}\,|w(t)|\,(\delta_0 + |w(t)|) + C_\Sigma\,(\delta_0 + |w(t)|)^2 \\
&\leq\; \delta_0 + C_3\,(\delta_0 + |w(t)|)^2
\end{aligned}
$$

for a suitable constant C_3. This implies (5.16).

Using (5.4), the time derivative $\dot{\theta}$ can be estimated as

$$
\begin{aligned}
\dot{\theta}(t) \;&=\; \langle \dot{w}(t), \mathbf{n}(t)\rangle + \langle w(t), \dot{\mathbf{n}}(t)\rangle \;\geq\; \langle \dot{x}^\delta(t), \mathbf{n}(t)\rangle - |\dot{x}(t)| - |w_2(t)|\,|\dot{\mathbf{n}}(t)| \\
&=\; \delta|\mathbf{v}(t,x^\delta)|\,\Big\langle \Big(\tfrac{\mathbf{v}(t,x^\delta)}{|\mathbf{v}(t,x^\delta)|} - \mathbf{n}(t,x^\delta)\Big) + \Big(\mathbf{n}(t,x^\delta) - \mathbf{n}(t)\Big) + \mathbf{n}(t),\, \mathbf{n}(t)\Big\rangle \\
&\qquad - |\dot{x}(t)| - |w_2(t)|\,|\dot{\mathbf{n}}(t)| \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (5.18) \\
&\geq\; \begin{cases} \delta|\mathbf{v}(t,x^\delta)|\Big(-\varepsilon - C_{\mathbf{n}}|w(t)| + 1\Big) - L_{\mathbf{n}}\,|w_2(t)| & \text{if} \quad x(t) \notin \Sigma(t), \\ \delta|\mathbf{v}(t,x^\delta)|\Big(-\varepsilon - C_{\mathbf{n}}|w(t)| + 1\Big) - L_\Sigma - L_{\mathbf{n}}\,|w_2(t)| & \text{if} \quad x(t) \in \Sigma(t). \end{cases}
\end{aligned}
$$

As long as (5.15) holds, we have $1-\varepsilon-C_{\mathbf{n}}|w(t)|\geq \frac{1}{2}$. This already yields the first inequality in (5.17). From (5.18) we also deduce

$$|\delta\mathbf{v}(t,x^\delta)| \;\leq\; 2(C_1+|\dot\theta(t)|). \tag{5.19}$$

In turn, this yields

$$\begin{aligned}
|\dot w_2(t)| &\leq \Big|\delta\mathbf{v}(t,x^\delta)-\langle\delta\,\mathbf{v}(t,x^\delta),\mathbf{n}(t)\rangle\,\mathbf{n}(t)\Big| + 2L_{\mathbf{n}}\,|w(t)|\\
&\leq |\delta\mathbf{v}(t,x^\delta)|\,\left|\frac{\mathbf{v}(t,x^\delta)}{|\mathbf{v}(t,x^\delta)|}-\mathbf{n}(t)\right| + 2L_{\mathbf{n}}\,|w(t)|\\
&\leq |\delta\mathbf{v}(t,x^\delta)|\,\left\{\left|\frac{\mathbf{v}(t,x^\delta)}{|\mathbf{v}(t,x^\delta)|}-\mathbf{n}(t,x^\delta)\right| + \Big|\mathbf{n}(t,x^\delta)-\mathbf{n}(t)\Big|\right\} + 2L_{\mathbf{n}}\,|w(t)|\\
&\leq |\delta\mathbf{v}(t,x^\delta)|\,(\varepsilon+C_{\mathbf{n}}|w(t)|)+2L_{\mathbf{n}}\,|w(t)|\\
&\leq 2(C_1+|\dot\theta(t)|)\,(\varepsilon+C_{\mathbf{n}}|w(t)|)+2L_{\mathbf{n}}\,|w(t)|,
\end{aligned}$$

establishing the second inequality in (5.17).

4. In this step we prove that there exists a constant $C_5>0$ such that, for every $t\in[0,T]$,

$$|\theta(t)| \;\leq\; C_5(\varepsilon+\widetilde w_2(t))\,, \tag{5.20}$$

where

$$\widetilde w_2(t) \;\doteq\; \max_{s\in[0,t]}\,|w_2(s)| \qquad \text{for all } t\in[0,T]. \tag{5.21}$$

Notice that, if $x(t)\in\Sigma(t)$, then (5.10) applies. Let us assume that $x(t)\notin\Sigma(t)$ and define the time

$$t_0 \;\doteq\; \inf\{s\in[0,t]\ ;\ x(r)\notin\Sigma(r)\quad \text{for all } r\in[s,t]\}\,.$$

From (5.11) and (5.17) it follows

$$\dot\theta(s) \;\geq\; -\,\varepsilon - L_{\mathbf{n}}|w_2(s)|) \;\geq\; -\,\varepsilon-L_{\mathbf{n}}\widetilde w_2(s) \qquad \text{for a.e. } s\in[t_0,t]\,.$$

This yields

$$\begin{aligned}
\theta(t) &= \theta(t_0)+\int_{t_0}^t\dot\theta(s)\,ds \;\geq\; \theta(t_0)-\int_{t_0}^t\varepsilon+L_{\mathbf{n}}\widetilde w_2(s)\;ds\\
&\geq -C_\Sigma\,|w_2(t_0)|^2-T(\varepsilon+L_{\mathbf{n}}\widetilde w_2(t)) \;\geq\; -\,C_\Sigma\,\widetilde w_2^2(t)-T(\varepsilon+L_{\mathbf{n}}\widetilde w_2(t))\,.
\end{aligned}$$

As long as $\widetilde w_2(t)<1$, recalling that $L_{\mathbf{n}}\geq 1$ we have

$$\theta(t) \;\geq\; -\,(C_\Sigma+TL_{\mathbf{n}})\cdot(\varepsilon+\widetilde w_2(t))\,.$$

Thus, to obtain (5.20), one only needs to consider the case where $\theta(t)>0$. Observe that, if $d(x^\delta(t),\Sigma(t))\leq\delta_0$, as long as

$$\varepsilon+|w(t)| \;\leq\; \frac{1}{2C_3}\,, \tag{5.22}$$

from (5.16) it follows

$$\theta(t) \;\leq\; \widetilde{C}_5(\varepsilon + \widetilde{w}_2(t)) \tag{5.23}$$

for some constant $\widetilde{C}_5$.

We claim that (5.20) holds, for a suitable constant C_5. Indeed, consider the time

$$t_1 \;\doteq\; \inf\Big\{ s \in [0,t)\,;\ \theta(r) > 0 \quad \text{and} \quad d(x^\delta(r), \Sigma(r)) > \delta_0 \quad \text{for all } r \in (s,t) \Big\}\,.$$

We then have $\theta(t_1) \leq \widetilde{C}_5(\varepsilon + \widetilde{w}_2(t))$. Therefore, by (5.23) and (5.11) it follows

$$\begin{aligned}
\theta(t) \;&=\; \theta(t_1) + \int_{t_1}^t \dot\theta(s)\,ds \;\leq\; \widetilde{C}_5(\varepsilon + \widetilde{w}_2(t_1)) + \int_{t_1}^t \Big(\varepsilon + C_2|w_2(s)|\Big)\,ds \\
&\leq\; \widetilde{C}_5(\varepsilon + \widetilde{w}_2(t)) + \int_{t_1}^t \Big(\varepsilon + C_2\widetilde{w}_2(t)\Big)\,ds \;\leq\; C_5(\varepsilon + \widetilde{w}_2(t)).
\end{aligned}$$

5. In the following we shall assume $|w(t)| \leq \rho_0/3$ for all $t \in [0,T]$. In this case, when $d(x(t),\Sigma(t)) > \rho_0$ we have $\dot x(t) = 0$, $|\dot x^\delta(t)| < \varepsilon$ and the estimates are trivial. Without loss of generality, we can thus assume that the normal vectors $\mathbf{n}(t)$ and $\mathbf{n}(t,x^\delta)$ are well defined.

For a suitable constant κ (to be determined later), define the weight

$$W(t) \;\doteq\; \begin{cases} \exp\Big\{ -\kappa\, d(x^\delta(t), \Sigma(t))\Big\} & \text{if} \quad d(x^\delta(t),\Sigma(t)) \leq \delta_0\,, \\[2mm] \exp\{-\kappa\delta_0\} & \text{if} \quad d(x^\delta(t),\Sigma(t)) \geq \delta_0\,. \end{cases} \tag{5.24}$$

We now analyze how the weighted distance

$$\Lambda(t) \;\doteq\; |\theta(t)| + W(t)\,\widetilde{w}_2(t)$$

changes in time. The heart of the matter is to provide a bound on $\widetilde{w}_2$. Indeed, by (5.20) the component $w_1(t) = \theta(t)\mathbf{n}(t)$ can be bounded in terms of $\widetilde{w}_2(t)$.

6. At any point $t \in [0,T]$ where $w_2(\cdot)$ is differentiable, by the definition of $\widetilde{w}_2$ it follows

$$0 \;\leq\; \frac{d}{dt}\,\widetilde{w}_2(t) \;\leq\; |\dot w_2(t)|. \tag{5.25}$$

We first consider Case 1, where $d(x^\delta(t),\Sigma(t)) \geq \delta_0$. By (5.11) and (5.20), we have that $\dot W(t) = 0$ and

$$|\dot w_2(t)| \;\leq\; \varepsilon + C_2\,|w(t)| \;\leq\; C_2\,(\varepsilon + |\theta(t)| + \widetilde{w}_2(t)) \;\leq\; C_2\,(1 + C_5)(\varepsilon + \widetilde{w}_2(t))\,.$$

Therefore, (5.25) yields

$$\frac{d}{dt}\,(W(t)\widetilde{w}_2(t)) \;=\; W(t)\,\frac{d}{dt}\widetilde{w}_2(t) \;\leq\; W(t)\cdot|\dot w_2(t)| \;\leq\; C_2\,(1+C_5)W(t)(\varepsilon + \widetilde{w}_2(t))\,.$$

On the other hand, in the case where $d(x^\delta(t),\Sigma(t))<\delta_0$, by (5.17) and (5.20) one has

$$\begin{aligned}
|\dot w_2(t)| &\le C_4(\varepsilon+|w(t)|)(1+|\dot\theta(t)|)\\
&\le C_4(\varepsilon+|\theta(t)|+\widetilde w_2(t))(1+|\dot\theta(t)|) \ \le\ (C_4+C_4C_5)(\varepsilon+\widetilde w_2(t))(1+|\dot\theta(t)|)\\
&\le \begin{cases} (C_4+C_4C_5)(1+C_1)\,(\varepsilon+\widetilde w_2(t)) & \text{if}\quad \dot\theta(t)\le 0,\\ (C_4+C_4C_5)(\varepsilon+\widetilde w_2(t))(1+\dot\theta(t)) & \text{if}\quad \dot\theta(t)>0\,,\end{cases}\\
&\le \begin{cases} C_6(\varepsilon+\widetilde w_2(t)) & \text{if}\quad \dot\theta(t)\le 0,\\ C_6(\varepsilon+\widetilde w_2(t))(1+\dot\theta(t)) & \text{if}\quad \dot\theta(t)>0\,,\end{cases}
\end{aligned}$$

By (5.25), for a.e. $t\in[0,T]$ one has

$$\frac{d}{dt}\widetilde w_2(t)\ \le\ \begin{cases} C_6(\varepsilon+\widetilde w_2(t)) & \text{if}\quad \dot\theta(t)\le 0,\\ C_6(\varepsilon+\widetilde w_2(t))(1+\dot\theta(t)) & \text{if}\quad \dot\theta(t)>0\,.\end{cases} \tag{5.26}$$

As long as $\varepsilon+|w(t)|\le\frac{1}{2C_1(2C_{\mathbf n}+C_4)}$, from (5.8), (5.12), and (5.19), it follows

$$\begin{aligned}
\frac{d}{dt}\,d(x^\delta(t),\Sigma(t)) &\ge \Big\langle \delta\mathbf v(t,x^\delta(t)),\,\mathbf n(t,x^\delta)\Big\rangle - L_\Sigma\\
&\ge \Big\langle \delta\mathbf v(t,x^\delta(t)),\,\mathbf n(t)\Big\rangle - C_{\mathbf n}|w(t)|\cdot|\delta\mathbf v(t,x^\delta(t))| - L_\Sigma\\
&\ge \Big\langle \dot w_1(t),\,\mathbf n(t)\Big\rangle + \Big\langle \dot w_2(t),\,\mathbf n(t)\Big\rangle + \Big\langle \dot x(t),\,\mathbf n(t)\Big\rangle - C_{\mathbf n}|w(t)|\cdot|\delta\mathbf v(t,x^\delta(t))| - L_\Sigma\\
&\ge \dot\theta(t) - |\dot w_2(t)| - C_{\mathbf n}|w(t)|\cdot|\delta\mathbf v(t,x^\delta(t))| - L_\Sigma\\
&\ge \dot\theta(t) - C_4(\varepsilon+|w(t)|)\cdot(1+|\dot\theta(t)|) - 2C_{\mathbf n}|w(t)|\cdot(C_1+|\dot\theta(t)|) - L_\Sigma\\
&\ge \dot\theta(t) - (2C_{\mathbf n}+C_4)(\varepsilon+|w(t)|)\cdot(C_1+|\dot\theta(t)|) - L_\Sigma \ \ge\ \tfrac12\dot\theta(t) - C_7
\end{aligned}$$

for some constant $C_7>0$. Inserting the weight, we now estimate

$$\begin{aligned}
\frac{d}{dt}\Big(W(t)\widetilde w_2(t)\Big) &= -\kappa W(t)\widetilde w_2(t)\cdot\frac{d}{dt}d(x^\delta(t),\Sigma(t)) + W(t)\cdot\frac{d}{dt}\widetilde w_2(t)\\
&\le -\frac{\kappa}{2}W(t)\widetilde w_2(t)\dot\theta(t) + \kappa C_7 W(t)\widetilde w_2(t) + W(t)\cdot\frac{d}{dt}\widetilde w_2(t)\,.
\end{aligned}$$

Two cases can occur:

- If $\dot\theta(t)\le 0$, then (5.26) and (5.17) yield

$$\frac{d}{dt}\Big(W(t)\widetilde w_2(t)\Big)\ \le\ C_8\,W(t)\,(\varepsilon+\widetilde w_2(t))$$

  for some constant C_8.

- If $\dot\theta(t) > 0$, then (5.26) yields

$$\begin{aligned}\frac{d}{dt}\ (W(t)\widetilde w_2(t)) &\leq -\frac{\kappa}{2}W(t)\widetilde w_2(t)\dot\theta(t) + W(t)\cdot\frac{d}{dt}\widetilde w_2(t) + \kappa C_7 W(t)\widetilde w_2(t)\\ &\leq -\frac{\kappa}{2}W(t)\widetilde w_2(t)\dot\theta(t) + C_6 W(t)(\varepsilon+\widetilde w_2(t))(1+\dot\theta(t)) + \kappa C_7 W(t)\widetilde w_2(t)\\ &\leq W(t)\cdot\left[\left(-\frac{\kappa}{2}\widetilde w_2(t) + C_6\cdot(\varepsilon+\widetilde w_2(t))\right)\cdot\dot\theta(t) + (C_6+\kappa C_7)(\varepsilon+\widetilde w_2(t))\right].\end{aligned}$$

We now choose the constant κ in (5.24) so that

$$\frac{\kappa}{2}\ \geq\ 2C_6\,.$$

In this case, either $\widetilde w_2(t) < \varepsilon$, or else

$$\begin{aligned}\frac{d}{dt}\Big(W(t)\,\widetilde w_2(t)\Big) &\leq W(t)\cdot\left[\left(-\frac{\kappa}{2}+2C_6\right)\dot\theta(t)\widetilde w_2(t) + (C_6+\kappa C_7)(\varepsilon+\widetilde w_2(t))\right]\\ &\leq W(t)\,(C_6+\kappa C_7)(\varepsilon+\widetilde w_2(t))\,.\end{aligned} \tag{5.27}$$

Combining both Cases 1 and 2, we obtain that either $\widetilde w_2(t)\leq\varepsilon$ or else

$$\frac{d}{dt}\Big(W(t)\,\widetilde w_2(t)\Big)\ \leq\ C_9\,W(t)\,\widetilde w_2(t), \tag{5.28}$$

provided that

$$\varepsilon + |\theta(t)| + \widetilde w_2(t)\ \leq\ \min\left\{\frac14,\frac{1}{4C_{\mathbf n}},\frac{1}{2C_3},\frac{1}{2C_1(2C_{\mathbf n}+C_4)},\frac{\rho_0}{3}\right\}. \tag{5.29}$$

7. To complete the argument, consider the time

$$\bar t\ \doteq\ \sup\Big\{\tau\in[0,T]\,;\ (5.29)\text{ holds for all } t\in[0,\tau]\Big\}.$$

Since $\widetilde w_2(t)$ is continuous and non-decreasing, there exists $t_\varepsilon\in[0,\bar t]$ such that

$$\begin{cases}\widetilde w_2(t)\ \leq\ \varepsilon & \text{for all } t\in[0,t_\varepsilon],\\[2mm] \widetilde w_2(t)\ >\ \varepsilon & \text{for all } t\in\,]t_\varepsilon,\bar t]\,.\end{cases}$$

Hence (5.28) implies

$$W(t)\widetilde w_2(t)\ \leq\ e^{C_9(t-t_\varepsilon)}W(t_\varepsilon)\widetilde w_2(t_\varepsilon)\qquad\text{for all } t_\varepsilon\leq t\leq\bar t\,.$$

Since $e^{-\kappa\delta_0}\leq W(t)\leq 1$, we have

$$\widetilde w_2(t)\ \leq\ \exp\left(C_9T+\kappa\cdot\delta_0\right)\cdot\varepsilon\qquad\text{for all } t\in[0,\bar t]\,.$$

Recalling (5.20), we obtain

$$|\theta(t)|\ \leq\ C_5\ [\exp\left(C_9T+\kappa\cdot\delta_0\right)+1]\cdot\varepsilon\qquad\text{for all } t\in[0,\bar t]\,.$$

This yields

$$\varepsilon + |\theta(t)| + \widetilde{w}_2(t) \;\leq\; C_{10}\,\varepsilon \qquad \text{for all } t \in [0, \bar{t}],$$

where

$$C_{10} \;=\; (1 + C_5)\left[\exp\left(C_9 T + \kappa \cdot \delta_0\right) + 1\right] .$$

Therefore, for any $\varepsilon > 0$ such that

$$\varepsilon \;<\; \frac{1}{C_{10}} \min\left\{\frac{1}{4}, \frac{1}{4C_{\mathbf{n}}}, \frac{1}{2C_3}, \frac{1}{2C_1(2C_{\mathbf{n}} + C_4)}, \frac{\rho_0}{3}\right\},$$

we conclude that

$$\bar{t} = T, \qquad\qquad |w(t)| \;\leq\; |\theta(t)| + \widetilde{w}_2(t) \;\leq\; C_{10}\,\varepsilon \qquad \text{for all } t \in [0, T]\,. \tag{5.30}$$

8. The previous analysis has shown that, by choosing $\delta > 0$ small enough, the sweeping process can be arbitrarily well approximated by the evolution generated by the vector field $\delta\mathbf{v}(t, x)$. Repeating the argument in step **3** of the proof of Theorem 2, we now construct a control function $t \mapsto \xi(t)$ such that trajectories of the ODE

$$\dot{x}(t) \;=\; \varphi(|x - \xi(t)|)\,\frac{x - \xi(t)}{|x - \xi(t)|}$$

approximate the trajectories of $\dot{x} = \delta\,\mathbf{v}(t, x)$, uniformly for $t \in [0, T]$ and for all initial data in the compact set $\Omega_0 \subset I\!R^d \setminus \Sigma(0)$. This completes the proof. □

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