Global existence for a free boundary problem of Fisher-KPP type

Julien Berestycki, Éric Brunet, Sarah Penington

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Julien Berestycki*, Éric Brunet†, Sarah Penington‡

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Abstract

Motivated by the study of branching particle systems with selection, we establish global existence for the solution (u, μ) of the free boundary problem

$$\begin{cases} \partial_t u = \partial_x^2 u + u & \text{for } t > 0 \text{ and } x > \mu_t, \\ u(x,t) = 1 & \text{for } t > 0 \text{ and } x \le \mu_t, \\ \partial_x u(\mu_t, t) = 0 & \text{for } t > 0, \\ u(x,0) = v(x) & \text{for } x \in \mathbb{R}, \end{cases}$$

when the initial condition $v : \mathbb{R} \to [0,1]$ is non-increasing with $v(x) \to 0$ as $x \to \infty$ and $v(x) \to 1$ as $x \to -\infty$. We construct the solution as the limit of a sequence $(u_n)_{n\ge1}$, where each u_n is the solution of a Fisher-KPP equation with same initial condition, but with a different non-linear term.

Recent results of De Masi *et al.* [5] show that this global solution can be identified with the hydrodynamic limit of the so-called N-BBM, *i.e.* a branching Brownian motion in which the population size is kept constant equal to N by killing the leftmost particle at each branching event.

1 Main results and introduction

We establish global existence for a free boundary problem of Fisher-KPP type:

Theorem 1.1. *Let $v : \mathbb{R} \to [0,1]$ be a non-increasing function such that $v(x) \to 0$ as $x \to \infty$ and $v(x) \to 1$ as $x \to -\infty$. Let $\mu_0 = \inf\{x \in \mathbb{R} : v(x) < 1\} \in \{-\infty\} \cup \mathbb{R}$. Then there exists a unique classical solution (u, μ) with $u \in [0,1]$ to the following free boundary problem:*

$$\begin{cases} \partial_t u = \partial_x^2 u + u & \text{for } t > 0 \text{ and } x > \mu_t, \\ u(x,t) = 1 & \text{for } t > 0 \text{ and } x \le \mu_t, \\ \partial_x u(\mu_t, t) = 0 & \text{for } t > 0, \\ u(x,0) = v(x) & \text{for } x \in \mathbb{R}. \end{cases} \tag{FBP}$$

Furthermore, this unique solution satisfies the following properties:

- *For every $t > 0$, $u(\cdot, t) \in C^1(\mathbb{R})$, and $\partial_x u \in C(\mathbb{R} \times (0,\infty))$.*

* `julien.berestycki@stats.ox.ac.uk`, Department of Statistics, University of Oxford.

† `Eric.Brunet@lps.ens.fr`, Laboratoire de Physique Statistique, Ecole Normale Supérieure, PSL Research University; Université Paris Diderot Sorbonne Paris-Cité; Sorbonne Universités UPMC Univ Paris 06; CNRS.

‡ `penington@maths.ox.ac.uk`, Mathematical Institute, University of Oxford.

- *If $v^{(1)} \le v^{(2)}$ are two valid initial conditions and $(u^{(i)}, \mu^{(i)})$ is the solution with initial condition $v^{(i)}$, then $u^{(1)} \le u^{(2)}$ and $\mu^{(1)} \le \mu^{(2)}$.*

We say that (u, μ) is a *classical solution* to (FBP) above if $\mu_t \in \mathbb{R}\ \forall t > 0$, $t \mapsto \mu_t$ is continuous, $u : \mathbb{R} \times (0, \infty) \to [0, 1]$, $u \in C^{2,1}(\{(x, t) : t > 0,\ x > \mu_t\}) \cap C(\mathbb{R} \times (0, \infty))$, (u, μ) satisfies the equation (FBP), and $u(\cdot, t) \to v(\cdot)$ in L^1_{loc} as $t \searrow 0$.

Remark 1. *We will show that $u(t, x) \to v(x)$ at all points of continuity of v as $t \searrow 0$ (since v is non-increasing, it is differentiable almost everywhere).*

Remark 2. *If instead $v(x) \to l > 0$ as $x \to \infty$, then a classical solution (u, μ) of* (FBP) *exists for $t < t_c = -\log l$, with $\mu_t \to \infty$ as $t \nearrow t_c$.*

Remark 3. *As discussed below, the condition that v is non-increasing can be relaxed to some extent.*

Our motivation for studying the problem (FBP) stems from its connection with the so-called N-BBM, a variant of branching Brownian motion in $\mathbb{R}$ in which the number of active particles is kept constant (and equal to N) by removing the leftmost particle each time a particle branches. More details are given in Section 2 below, but in a nutshell, De Masi *et al.* [5] show that as $N \to \infty$, under appropriate conditions on the initial configuration of particles, the N-BBM has a hydrodynamic limit whose cumulative distribution can be identified with the solution of (FBP), provided such a solution exists.

The overall idea behind the proof is to construct u as the limit of a sequence of functions u_n, where, for each n, u_n satisfies an n-dependent non-linear equation, but where all the u_n have the same initial condition. More precisely, let $v : \mathbb{R} \to [0, 1]$ be a measurable function and, for $n \ge 2$, let $(u_n(x, t), x \in \mathbb{R}, t \ge 0)$ be the solution to

$$\begin{cases} \partial_t u_n = \partial_x^2 u_n + u_n - u_n^n & \text{for } x \in \mathbb{R} \text{ and } t > 0, \\ u_n(x, 0) = v(x) & \text{for } x \in \mathbb{R}. \end{cases} \tag{1.1}$$

For each $n \ge 2$, this is a version of the celebrated Fisher-KPP equation about which much is known (see e.g. [12, 1, 15, 17, 10, 16]). In particular,

- u_n exists and is unique,
- $u_n(x, t) \in (0, 1)$ for $x \in \mathbb{R}$ and $t > 0$ (unless $v \equiv 0$ or $v \equiv 1$).

Since the comparison principle applies, we see furthermore that for every $x \in \mathbb{R}, t > 0$ fixed, the sequence $n \mapsto u_n(x, t)$ is increasing. Therefore, the following pointwise limit is well defined:

$$u(x, t) := \lim_{n \to \infty} u_n(x, t), \tag{1.2}$$

with $u(x, t) \in (0, 1]$ for $t > 0$ (unless $v \equiv 0$). Indeed, in most of the cases we are interested in, there are regions where $u(x, t) = 1$.

We have the following results on u:

Theorem 1.2. *Let $v : \mathbb{R} \to [0, 1]$ be a measurable function. The function $u(x, t)$ as defined by* (1.1) *and* (1.2) *satisfies the following properties:*

- *u is continuous on $\mathbb{R} \times (0, \infty)$ and, for $t > 0$, $u(\cdot, t)$ is Lipschitz continuous.*
- *$u(\cdot, t) \to v(\cdot)$ in L^1_{loc} as $t \searrow 0$, and if v is continuous at x then $u(t, x) \to v(x)$ as $t \searrow 0$.*
- *At any (x, t) with $t > 0$ such that $u(x, t) < 1$, the function u is continuously differentiable in t and twice continuously differentiable in x, and satisfies*

$$\partial_t u = \partial_x^2 u + u.$$

- *u satisfies the following semigroup property: for any $t>0$ and any $t_0\geq 0$, $u(\cdot,t+t_0)$ can be obtained as the solution at time t to* (1.1) *and* (1.2) *with an initial condition $u(\cdot,t_0)$.*
- *If $v^{(1)}\leq v^{(2)}$ are two measurable functions and $u^{(i)}$ is the solution to* (1.1) *and* (1.2) *with initial condition $v^{(i)}$, then $u^{(1)}\leq u^{(2)}$.*

The existence result in Theorem 1.1 is then a consequence of the following result:

Proposition 1.3. *Suppose that v (and μ_0) is as in Theorem 1.1, and define $u(x,t)$ as in* (1.1) *and* (1.2)*. Then there exists a map $t\mapsto\mu_t$ with $\mu_t\in\mathbb{R}\ \forall t>0$ and $\mu_t\to\mu_0$ as $t\searrow 0$ such that*

$$u(x,t)=1\Leftrightarrow x\leq\mu_t\qquad \text{for } t>0. \tag{1.3}$$

Furthermore, $t\mapsto\mu_t$ is continuous and $u(\cdot,t)\in C^1(\mathbb{R})$ for $t>0$ with $\partial_x u\in C(\mathbb{R}\times(0,\infty))$.

By combining Theorem 1.2 and Proposition 1.3, we have that if v is as in Theorem 1.1 then (u,μ) is a classical solution of (FBP).

Remark 4. *For an arbitrary measurable initial condition v, for $t>0$, $u(\cdot,t)$ is obviously C^1 in the interior of the region where $u=1$, and by Theorem 1.2 it is C^1 in the region where $u<1$. The difficulty in proving Proposition 1.3 is to show that $u(\cdot,t)$ is also C^1 at the boundary between these two domains.*

Remark 5. *It turns out that the proof that $u(\cdot,t)$ is C^1 holds whenever the topological boundary between the (two-dimensional) domains $\{u=1\}$ and $\{u<1\}$ has measure zero. (In the case where v is non-increasing, this is implied by the existence of a continuous map $t\mapsto\mu_t$ satisfying* (1.3)*.) This means that it should be possible to show that $u(\cdot,t)$ is C^1 for any "reasonable" initial condition.*

Remark 6. *The condition that v is non-increasing in Theorem 1.1 is only used in the proof of Proposition 1.3 to show the existence of a continuous boundary $t\mapsto\mu_t$ as in* (1.3).

The idea of using the limit of $(u_n)_{n\geq 1}$ as the solution to (FBP) first appeared in [2] and the present article puts this intuition on a rigorous footing.

The rest of the article is organised as follows: the next section is devoted to putting our result in the context of several recent works on related problems. Next, in Section 3, we present the precise versions of the Feynman-Kac representation that we shall use in the rest of the proof. The proof of one of these Feynman-Kac results will be postponed until Section 7. We establish Theorem 1.2 in Section 4, and in Section 5 we prove Proposition 1.3. In Section 6, we complete the proof of Theorem 1.1 by proving the uniqueness of the classical solution of (FBP).

2 Context

Let ω be a probability measure on $\mathbb{R}$. Then define $v:\mathbb{R}\to[0,1]$ by setting

$$v(x)=\omega[x,\infty).$$

Note that v is non-increasing, and that $v(x)\to 0$ as $x\to\infty$ and $v(x)\to 1$ as $x\to-\infty$. Therefore, by Theorem 1.1, there exists a unique classical solution (u,μ) to the free boundary problem (FBP), and $\partial_x u$ is continuous on $\mathbb{R}\times(0,\infty)$.

Let $\rho=-\partial_x u$. The following result is an easy consequence of Theorem 1.1 and its proof.

Corollary 2.1. *Let ω be a probability measure on $\mathbb{R}$ and let $\mu_0=\inf\{x\in\mathbb{R}:\omega[x,\infty)<1\}\in\mathbb{R}\cup\{-\infty\}$. Then (ρ,μ) constructed as above from the solution of* (FBP) *with initial condition $v(x)=\omega[x,\infty)$ is the unique classical solution with $\rho\geq 0$ to the following free boundary problem:*

$$\begin{cases}\partial_t\rho=\partial_x^2\rho+\rho & \text{for } t>0 \text{ and } x>\mu_t,\\ \rho(\mu_t,t)=0,\quad \int_{\mu_t}^{\infty}\rho(y,t)\,\mathrm{d}y=1 & \text{for } t>0,\\ \rho(\cdot,t)\mathrm{d}\lambda\to\mathrm{d}\omega(\cdot) & \text{in the vague topology as } t\searrow 0.\end{cases} \tag{FBP$'$}$$

We say that (ρ,μ) is a *classical solution* to (FBP′) above if $\mu_t \in \mathbb{R}\ \forall t>0$, $t\mapsto \mu_t$ is continuous, $\rho:\mathbb{R}\times(0,\infty)\to[0,\infty)$, $\rho\in C^{2,1}(\{(x,t):t>0,\ x>\mu_t\})\cap C(\mathbb{R}\times(0,\infty))$, and (ρ,μ) satisfies the equation (FBP′).

This result improves on a recent result of Lee [14], where local existence of a solution to (FBP′) is shown (i.e. existence of a solution on a time interval $[0,T]$ for some $T>0$), under the additional assumptions that ω is absolutely continuous with respect to Lebesgue measure with probability density $\phi\in C_c^2(\mathbb{R})$, and that there exists $\mu_0\in\mathbb{R}$ such that $\phi(\mu_0)=0$, $\phi'(\mu_0)=1$ and $\int_{\mu_0}^{\infty}\phi(x)\,\mathrm{d}x=1$.

In [5], De Masi et al. study the hydrodynamic limit of the N-BBM and its relationship with the free boundary problem (FBP′). The N-BBM is a variant of branching Brownian motion in which the number of active particles is kept constant (and equal to N) by removing the leftmost particle each time a particle branches.

We shall now define this particle system more precisely. Suppose that $\phi\in L^1(\mathbb{R})$ is a probability density function which satisfies (a) $\|\phi\|_\infty<\infty$ and (b) $\int_r^\infty \phi(x)\,\mathrm{d}x=1$ for some $r\in\mathbb{R}$. Let $X_0^1,\dots,X_0^N$ be i.i.d. with density ϕ. At time 0, the N-BBM consists of N particles at locations $X_0^1,\dots,X_0^N$. These particles move independently according to Brownian motions, and each particle independently at rate 1 creates a new particle at its current location. Whenever a new particle is created, the leftmost particle is removed from the particle system.

Let $X_t=\{X_t^1,\dots,X_t^N\}$ denote the set of particle locations at time t. Let $\pi_t^{(N)}$ be the empirical distribution induced by the particle system at time t, i.e. for $A\subset\mathbb{R}$, let

$$\pi_t^{(N)}(A)=\frac{1}{N}|X_t\cap A|.$$

De Masi et al. prove in [5] that for each $t\geq 0$ there exists a probability density function $\psi(\cdot,t):\mathbb{R}\to[0,\infty)$ such that, for any $a\in\mathbb{R}$,

$$\lim_{N\to\infty}\pi_t^N[a,\infty)=\int_a^\infty \psi(r,t)\,\mathrm{d}r \qquad \text{a.s. and in } L^1.$$

Moreover, they show that if (ρ,μ) is a classical solution of (FBP′) with initial condition ω given by $\mathrm{d}\omega=\phi\,\mathrm{d}\lambda$ then $\psi=\rho$. The following result is then a direct consequence of Theorems 1 and 2 in [5] and our Corollary 2.1.

Corollary 2.2. *Suppose $\phi\in L^1(\mathbb{R})$ is a probability density function with $\|\phi\|_\infty<\infty$ and $\int_r^\infty\phi(x)\,\mathrm{d}x=1$ for some $r\in\mathbb{R}$. Construct an N-BBM with initial particle locations given by i.i.d. samples from ϕ, as defined above. Let $\pi_t^{(N)}$ denote the empirical distribution induced by the particle system at time t. Then for any $t\geq 0$ and $a\in\mathbb{R}$,*

$$\lim_{N\to\infty}\pi_t^N[a,\infty)=\int_a^\infty\rho(r,t)\,\mathrm{d}r=u(a,t) \qquad \textit{a.s. and in } L^1,$$

where (u,μ) is the solution of (FBP) *with initial condition v given by $v(x)=\int_x^\infty\phi(y)\,\mathrm{d}y$, and $\rho=-\partial_x u$.*

Lee [14] points out that (FBP′) can be reformulated as a variant of the Stefan problem; let (ρ,μ) be a solution of (FBP′) and define $w(x,t):=e^{-t}\partial_x\rho(x,t)$. Then under some regularity assumptions, (w,μ) solves

$$\begin{cases}\partial_t w=\partial_x^2 w & \text{for } t>0 \text{ and } x>\mu_t,\\ w(\mu_t,t)=e^{-t},\ \partial_t\mu_t=-\frac{1}{2}e^t\partial_x w(\mu_t,t) & \text{for } t>0.\end{cases} \tag{Stefan}$$

The Stefan problem describes the phase change of a material and is one of the most popular problems in the moving boundary problem literature. Typically, it requires solving heat equations for the temperature in the two phases (e.g. solid and liquid), while the position of the front separating them, the moving boundary, is determined from an energy balance referred to as the Stefan condition. The Stefan problem has been studied in great detail since Lamé and Clapeyron formulated it in the 19th century [13]. There are several reference books that the reader may consult such as the recent and up-to-date book [9].

The free boundary problem (FBP) was studied at a heuristic level in [2]. The main tool there was the following relation, which can be proved rigorously:

Lemma 2.3. *Let v be as in Theorem 1.1 with $\mu_0 \in \mathbb{R}$ and such that $\gamma := \sup\{r : \int_{\mu_0}^{\infty} v(x)e^{rx}\,\mathrm{d}x < \infty\} \geq 0$. Let (u, μ) be the classical solution to* (FBP)*. Then, for $r < \min(\gamma, 1)$, and for any $t \geq 0$,*

$$1 + r\int_0^{\infty} \mathrm{d}x\, u(\mu_t + x, t)e^{rx} = \int_0^{\infty} \mathrm{d}s\ e^{r(\mu_{t+s}-\mu_t)-(1+r^2)s} < \infty. \tag{2.1}$$

Although the proof is not very difficult, we omit it from the present work as it is not our main focus here; the main ideas can be found in [2, 3].

For instance, take a step initial condition $v(x) = \mathbb{1}_{\{x\leq 0\}}$. Using (2.1) with $t = 0$ and $r = 1-\epsilon$ gives

$$\int_0^{\infty} \mathrm{d}s\, e^{-\epsilon^2 s+(1-\epsilon)(\mu_s-2s)} = 1 \qquad \forall \epsilon > 0. \tag{2.2}$$

Whether (2.2) completely characterizes the function μ or not is an open question.

3 Feynman-Kac formulae

In this section, we state the versions of the Feynman-Kac formula which we shall use repeatedly in the rest of the paper. So as not to interrupt the flow of the main argument, the proof for Proposition 3.1 is postponed to Section 7.

We introduce the heat kernel

$$p_t(x) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}. \tag{3.1}$$

For $x \in \mathbb{R}$, we let $\mathbb{P}_x$ denote the probability measure under which $(B_t)_{t\geq 0}$ is a Brownian motion with diffusivity constant $\sqrt{2}$ started at x. We let $\mathbb{E}_x$ denote the corresponding expectation. The symbol $*$ denotes convolution; for instance,

$$p_t * v(x) = \int_{-\infty}^{\infty} \mathrm{d}y\, p_t(x-y)v(y) = \mathbb{E}_x[v(B_t)]$$

is the solution at time t to the heat equation on $\mathbb{R}$ with an initial condition v.

Proposition 3.1. *Suppose that $A \subseteq \mathbb{R} \times (0, \infty)$ is an open set, and that $w : A \to \mathbb{R}$ is $C^{2,1}$ and bounded, and satisfies*

$$\partial_t w = \partial_x^2 w + Kw + S \qquad \text{for } (x,t) \in A, \tag{3.2}$$

where $K : A \to \mathbb{R}$, $S : A \to \mathbb{R}$ are continuous and bounded. Then, if one of the conditions below is met, we have the following representation for $w(x, t)$ with $(x, t) \in A$:

$$w(x,t) = \mathbb{E}_x\left[w(B_\tau, t-\tau)e^{\int_0^\tau K(B_s,t-s)\,\mathrm{d}s} + \int_0^\tau \mathrm{d}r\, S(B_r, t-r)e^{\int_0^r K(B_s,t-s)\,\mathrm{d}s}\right], \tag{3.3}$$

where τ is a stopping time for $(B_s)_{s\geq 0}$.

For the representation (3.3) to hold, it is sufficient to have one the following:

1. *The stopping time τ is such that $(B_s, t-s) \in A$ for all $s \leq \tau$,*

2. *The set A is given by $A = \{(x,t) : t \in (0,T) \text{ and } x > \mu_t\}$ for some $T > 0$ and some continuous boundary $t \mapsto \mu_t$ with $\mu_t \in \mathbb{R} \cup \{-\infty\}$ $\forall t \in [0,T]$, the stopping time τ is given by $\tau = \inf\{s \geq 0 : B_s \leq \mu_{t-s}\} \wedge t$ (the first time at which $(B_\tau, t-\tau) \in \partial A$) and, furthermore, w is defined and bounded on $\bar{A}$, continuous on $\bar{A} \cap (\mathbb{R} \times (0,\infty))$ and satisfies $w(\cdot, t) \to w(\cdot, 0)$ in L^1_{loc} as $t \searrow 0$.*

Although this is a very classical result, we give a proof in Section 7 for the sake of completeness and because we could not find a exact statement with stopping times or a discontinuous initial condition in the literature. The proof that (3.3) holds under condition 1 essentially follows the proof of Theorem 4.3.2 in [8].

Proposition 3.1 gives some useful representations for the u_n defined in (1.1).

Corollary 3.2. *Let $v : \mathbb{R} \to [0,1]$ be measurable, let $n \geq 2$ and let $u_n(x,t)$ denote the solution to* (1.1)*. Then by Proposition 3.1:*

- *taking $K = 1 - u_n^{n-1}$ and $S = 0$, for τ a stopping time with $\tau < t$:*

$$u_n(x,t) = \mathbb{E}_x \left[u_n(B_\tau, t-\tau) e^{\int_0^\tau (1-u_n^{n-1}(B_s, t-s)) \mathrm{d}s} \right]. \tag{3.4}$$

- *taking $K = 1 - u_n^{n-1}$, $S = 0$, and $\tau = t$:*

$$u_n(x,t) = \mathbb{E}_x \left[v(B_t) e^{\int_0^t (1-u_n^{n-1}(B_s, t-s)) \mathrm{d}s} \right]. \tag{3.5}$$

- *taking $K = 0$, $S = u_n - u_n^n$ and $\tau = t$:*

$$\begin{aligned} u_n(x,t) &= \mathbb{E}_x \left[v(B_t) + \int_0^t \mathrm{d}r \left[u_n(B_r, t-r) - u_n^n(B_r, t-r) \right] \right] \\ &= p_t * v(x) + \int_0^t \mathrm{d}r\, p_r * \left[u_n(x, t-r) - u_n^n(x, t-r) \right] \end{aligned} \tag{3.6}$$

- *taking $K = 1$, $S = -u_n^n$ and $\tau = t$:*

$$\begin{aligned} u_n(x,t) &= \mathbb{E}_x \left[v(B_t) e^t - \int_0^t \mathrm{d}r\, e^r u_n^n(B_r, t-r) \right] \\ &= e^t p_t * v(x) - \int_0^t \mathrm{d}r\, e^r p_r * u_n^n(x, t-r). \end{aligned} \tag{3.7}$$

Proof. This is a direct consequence of the previous result. □

We will also use the following representation for solutions of the free boundary problem (FBP):

Corollary 3.3. *If v is as in Theorem 1.1 and (u, μ) is a classical solution of* (FBP) *with initial condition v, then for $t > 0$ and $x \in \mathbb{R}$,*

$$u(x,t) = \mathbb{E}_x \left[e^\tau \mathbb{1}_{\{\tau < t\}} + e^t v(B_t) \mathbb{1}_{\{\tau = t\}} \right], \tag{3.8}$$

where $\tau = \inf\{s \geq 0 : B_s \leq \mu_{t-s}\} \wedge t$.

Proof. This is a direct application of Proposition 3.1 under condition 2, with $K = 1$ and $S = 0$. □

Finally, we use the following result to recognise solutions to partial differential equations:

Lemma 3.4. *Suppose that $a < b$, $t_0 < t_1$, and that $g : [a,b] \times [t_0, t_1] \to [0, \infty)$ is continuous and for $x \in [a,b]$ and $t \in [t_0, t_1]$,*

$$g(x,t) = \mathbb{E}_x \left[g(B_\tau, t-\tau) e^\tau \right],$$

where $\tau = \inf\{s \geq 0 : B_s \in \{a,b\}\} \wedge (t - t_0)$. Then $g \in C^{2,1}((a,b) \times (t_0, t_1))$ with

$$\partial_t g = \partial_x^2 g + g \qquad \text{for } (x,t) \in (a,b) \times (t_0, t_1).$$

Proof. The proof is the same as the proof of Exercise 4.3.15 in [11], where an outline proof is given. □

4 Proof of Theorem 1.2

In this section, we suppose $v : \mathbb{R} \to [0,1]$ is measurable. Let u_n denote the solution of (1.1) and define u as in (1.2). We shall use the following basic results on the smoothing effect of convolution with the heat kernel p_t as introduced in (3.1).

Lemma 4.1. *Suppose $t > 0$.*

1. *If $x \mapsto a(x)$ is bounded, then $x \mapsto p_t * a(x)$ is C^∞ and $(p_t * a)^{(n)}(x) = p_t^{(n)} * a(x)$.*
2. *If $(x,s) \mapsto b(x,s)$ is such that $b_s := \|b(\cdot,s)\|_\infty < \infty$ for each $s \in (0,t)$, and the map $s \mapsto \frac{b_s}{\sqrt{s}}$ is integrable on $[0,t]$, then $f(x) := \int_0^t \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}y\, p_s(x-y)b(y,s) = \int_0^t \mathrm{d}s\, p_s * b(x,s)$ is C^1 and $f'(x) = \int_0^t \mathrm{d}s\, p_s' * b(x,s)$.*

Proof. The first statement holds since for every $n \in \mathbb{N}$ and $t > 0$, there exists a polynomial function $q_{n,t} : \mathbb{R} \to \mathbb{R}$ such that $|p_t^{(n)}(x-y)| \le |q_{n,t}(x-y)| e^{-(x-y)^2/(4t)}$ $\forall x, y \in \mathbb{R}$. Then for the second statement, we have that $f_s(x) := p_s * b(x,s)$ is smooth, with

$$|f_s'(x)| = |p_s' * b(x,s)| \le \int_{-\infty}^{\infty} \mathrm{d}y\, |p_s'(x-y)| b_s = \frac{b_s}{\sqrt{\pi s}}.$$

Since $s \mapsto \frac{b_s}{\sqrt{\pi s}}$ is integrable on $[0,t]$, the result follows. □

The following result of Uchiyama provides a useful bound on the spatial derivative of u_n.

Lemma 4.2 ([17], Section 4)**.** *For $x \in \mathbb{R}$ and $t > 0$,*

$$|\partial_x u_n(x,t)| \le \frac{1}{\sqrt{\pi t}} + \frac{\sqrt{8}}{\sqrt{\pi}}. \tag{4.1}$$

Proof. We briefly recall Uchiyama's proof. Using Lemma 4.1 to differentiate (3.6) with respect to x, and bounding the result (using $v \in [0,1]$ and $u_n - u_n^n \in [0,1]$) yields:

$$|\partial_x u_n(x,t)| \le \int_{-\infty}^{\infty} |p_t'(x-y)| \mathrm{d}y + \int_0^t \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}y\, |p_s'(x-y)| = \frac{1}{\sqrt{\pi t}} + 2\frac{\sqrt{t}}{\sqrt{\pi}}.$$

This bound reaches its minimum $\sqrt{8/\pi}$ at $t = 1/2$. For $t \le 1/2$, the result follows immediately. For $t \ge 1/2$, since $u_n(\cdot,t)$ is also the solution at time $1/2$ of (1.1) with initial condition $u_n(\cdot, t - 1/2)$, it follows that $|\partial_x u_n(x,t)| \le \sqrt{8/\pi}$ $\forall x \in \mathbb{R}$ and $t \ge 1/2$. □

In the following two lemmas, we prove the continuity of u.

Lemma 4.3. *For any $t > 0$, the map $x \mapsto u(x,t)$ is Lipschitz continuous, with Lipschitz constant $\frac{1}{\sqrt{\pi t}} + \frac{\sqrt{8}}{\sqrt{\pi}}$.*

Proof. For $x \in \mathbb{R}$ and $h > 0$, we can write, using (4.1),

$$\Big| u_n(x+h,t) - u_n(x,t) \Big| \le \Big(\frac{1}{\sqrt{\pi t}} + \frac{\sqrt{8}}{\sqrt{\pi}} \Big) h.$$

Then take the $n \to \infty$ limit to conclude. □

Lemma 4.4. *The map $(x,t) \mapsto u(x,t)$ is continuous on $\mathbb{R} \times (0,\infty)$. Furthermore, $u(\cdot,t) \to v$ in L^1_{loc} as $t \searrow 0$, and if v is continuous at x then $u(x,t) \to v(x)$ as $t \searrow 0$.*

Proof. Using the bound $u_n - u_n^n \in [0,1]$ in the expression for u_n in (3.6), we have that for $x \in \mathbb{R}$, $t_0 \geq 0$ and $t > 0$,

$$u_n(x,t) - p_t * v(x) \in [0,t] \qquad \text{and} \qquad u_n(x,t_0+t) - p_t * u_n(x,t_0) \in [0,t],$$

where for the second expression we used that $u_n(\cdot, t+t_0)$ is the solution at time t of (1.1) with initial condition $u_n(\cdot, t_0)$. Taking the $n \to \infty$ limit, it follows that

$$u(x,t) - p_t * v(x) \in [0,t] \qquad \text{and} \qquad u(x,t_0+t) - p_t * u(x,t_0) \in [0,t].$$

Since the solution to the heat equation $p_t * v$ converges to v in L^1_{loc} as $t \searrow 0$, we have that $u(\cdot,t) \to v$ in L^1_{loc}. If v is continuous at x, then $p_t * v(x) \to v(x)$ as $t \searrow 0$, and hence $u(x,t) \to v(x)$ as $t \searrow 0$.

It remains to prove that u is continuous. By Lemma 4.3, we have

$$\begin{aligned}
|p_t * u(x,t_0) - u(x,t_0)| &= \left|\mathbb{E}_x\Big[u(B_t,t_0) - u(x,t_0)\Big]\right| \\
&\leq \left(\frac{1}{\sqrt{\pi t_0}} + \frac{\sqrt{8}}{\sqrt{\pi}}\right) \mathbb{E}_x\left[|B_t - x|\right] = \left(\frac{1}{\sqrt{\pi t_0}} + \frac{\sqrt{8}}{\sqrt{\pi}}\right)\sqrt{\frac{4t}{\pi}}.
\end{aligned}$$

Therefore by the triangle inequality,

$$|u(x,t_0+t) - u(x,t_0)| \leq t + \left(\frac{1}{\sqrt{\pi t_0}} + \frac{\sqrt{8}}{\sqrt{\pi}}\right)\sqrt{\frac{4t}{\pi}}.$$

Hence by the triangle inequality and then by Lemma 4.3, for x_1, $x_2 \in \mathbb{R}$, $t_0 \geq 0$ and $t > 0$,

$$\begin{aligned}
|u(x_1,t_0+t) - u(x_2,t_0)| &\leq |u(x_1,t_0+t) - u(x_1,t_0)| + |u(x_1,t_0) - u(x_2,t_0)| \\
&\leq t + \left(\frac{1}{\sqrt{\pi t_0}} + \frac{\sqrt{8}}{\sqrt{\pi}}\right)\left(\sqrt{\frac{4t}{\pi}} + |x_1 - x_2|\right),
\end{aligned}$$

and the result follows. ☐

We now turn to the semigroup property.

Lemma 4.5. *Suppose $v : \mathbb{R} \to [0,1]$, take $t_0 \geq 0$ and, as throughout this section, let u_n and u denote the functions defined in* (1.1) *and* (1.2)*. Furthermore, for $t \geq t_0$, let $u_{n;t_0}(\cdot,t)$ denote the solution at time $t - t_0$ to* (1.1) *with the initial condition $v(\cdot)$ replaced by $u(\cdot,t_0)$. Then for $t \geq t_0$ and $x \in \mathbb{R}$,*

$$\lim_{n\to\infty} u_{n;t_0}(x,t) = \lim_{n\to\infty} u_n(x,t) = u(x,t).$$

Proof. Since $u_n(x,t_0) \leq u(x,t_0)$ $\forall x \in \mathbb{R}$, it follows by the comparison principle that $u_n(x,t) \leq u_{n;t_0}(x,t)$ $\forall x \in \mathbb{R}$, $t \geq t_0$. Then for $t \geq t_0$, by the Feynman-Kac formula (3.5),

$$\begin{aligned}
&u_{n;t_0}(x,t) - u_n(x,t) \\
&\quad = e^{t-t_0}\mathbb{E}_x\left[u(B_{t-t_0},t_0)e^{-\int_0^{t-t_0} u_{n;t_0}^{n-1}(B_s,t-s)\,\mathrm{d}s} - u_n(B_{t-t_0},t_0)e^{-\int_0^{t-t_0} u_n^{n-1}(B_s,t-s)\,\mathrm{d}s}\right] \\
&\quad = e^{t-t_0}\mathbb{E}_x\left[\Big(u(B_{t-t_0},t_0) - u_n(B_{t-t_0},t_0)\Big)e^{-\int_0^{t-t_0} u_{n;t_0}^{n-1}(B_s,t-s)\,\mathrm{d}s}\right] \\
&\qquad + e^{t-t_0}\mathbb{E}_x\left[u_n(B_{t-t_0},t_0)\left(e^{-\int_0^{t-t_0} u_{n;t_0}^{n-1}(B_s,t-s)\,\mathrm{d}s} - e^{-\int_0^{t-t_0} u_n^{n-1}(B_s,t-s)\,\mathrm{d}s}\right)\right] \\
&\quad \leq e^{t-t_0}\mathbb{E}_x\Big[u(B_{t-t_0},t_0) - u_n(B_{t-t_0},t_0)\Big],
\end{aligned}$$

where, in the last step, we used that $u_{n;t_0} \geq u_n$, $u_n \geq 0$, $u \geq u_n$ and $u_{n;t_0} \geq 0$. By dominated convergence, the right hand side converges to zero as $n \to \infty$, and this completes the proof. ☐

At this point, it is convenient to introduce the two sets

$$\begin{aligned} & U := \{(x,t) \in \mathbb{R} \times (0,\infty) : u(x,t) = 1\} \\ \text{and} \qquad & S := \{(x,t) \in \mathbb{R} \times (0,\infty) : u(x,t) < 1\}. \end{aligned} \tag{4.2}$$

By the continuity of u, the set S is open.

The next proposition focuses on the set S, while Proposition 4.7 below is about the behaviour of u_n in the set U.

Proposition 4.6. *The map u is $C^{2,1}$ on S and satisfies*

$$\partial_t u = \partial_x^2 u + u \qquad \text{on } S. \tag{4.3}$$

Proof. Choose $(x,t) \in S$. Let a, b, t_0 and t_1 be such that $x \in (a,b)$, $t \in (t_0,t_1)$ and $[a,b]\times[t_0,t_1] \subset S$. By (3.4), we have that for $(x',t') \in [a,b] \times [t_0,t_1]$,

$$u_n(x',t') = \mathbb{E}_{x'}\Big[u_n(B_\tau, t'-\tau) e^{\int_0^\tau \big(1-u_n^{n-1}(B_s,t'-s)\big)\mathrm{d}s}\Big], \tag{4.4}$$

where $\tau = (t'-t_0) \wedge \inf\{s \geq 0 : B_s \notin (a,b)\}$ is the time at which $(B_\tau, t'-\tau)$ hits the boundary of $[a,b] \times [t_0,t_1]$.

We now take the $n \to \infty$ limit. For a given Brownian path $(B_s)_{s\geq 0}$, since $(B_s, t'-s) \in S$ for $s \in [0,\tau]$, we have $u_n(B_s,t'-s) \to u(B_s,t'-s) < 1$ as $n \to \infty$ for $s \in [0,\tau]$ and so, since $\tau \leq t'-t_0$,

$$\int_0^\tau u_n^{n-1}(B_s,t'-s)\mathrm{d}s \to 0 \qquad \text{as } n \to \infty.$$

Hence by dominated convergence in (4.4),

$$u(x',t') = \mathbb{E}_{x'}\Big[u(B_\tau,t'-\tau)e^\tau\Big] \qquad \text{for any } (x',t') \in [a,b]\times[t_0,t_1]. \tag{4.5}$$

The result then follows by Lemma 3.4. □

To complete the proof of Theorem 1.2, it only remains to note that if $v^{(1)} \leq v^{(2)}$ are two measurable functions, and if $u_n^{(i)}$ is the solution to (1.1) with initial condition $v^{(i)}$, then by the comparison principle $u_n^{(1)} \leq u_n^{(2)}$ and hence $u^{(1)} \leq u^{(2)}$.

We finish this section by proving two more results on the behaviour of u_n which will be used in the proof of Proposition 1.3 in the next section, but which do not require any additional assumptions on v.

Proposition 4.7. *If (x,t) is in the interior of U, then*

$$\lim_{n\to\infty} u_n^n(x,t) = 1.$$

Before proving this result properly, we give a heuristic explanation. As in the proof of Proposition 4.6, choose a rectangle $[a,b]\times[t_0,t_1]$ in the interior of U, and write (4.4) for a point $(x',t') \in (a,b)\times(t_0,t_1)$. We take the limit $n \to \infty$ again. By construction, $u_n(x',t') \to 1$ and $u_n(B_\tau,t'-\tau) \to 1$, so we obtain

$$1 = \lim_{n\to\infty} \mathbb{E}_{x'}\Big[e^{\int_0^\tau \big(1-u_n^{n-1}(B_s,t'-s)\big)\mathrm{d}s}\Big].$$

This equation strongly suggests the result, because if there were a region where $\limsup_{n\to\infty} u_n^n < 1$ which was visited by the paths $(B_s, t'-s)$ with a strictly positive probability then the limiting expectation above would be larger than 1. However, we were not able to turn this heuristic into a proper proof of Proposition 4.7, so we used a completely different method.

Proof. Take (x,t) in the interior of U. For $\epsilon > 0$, let

$$A = [-\epsilon^{0.49}, \epsilon^{0.49}].$$

(The exponent 0.49 could be any positive number smaller than 1/2.) Choose ϵ sufficiently small that $[x-\epsilon^{0.49}, x+\epsilon^{0.49}] \times [t-\epsilon, t] \subset U$. Note that u_n is a monotone sequence and converges pointwise to 1 on $[x-\epsilon^{0.49}, x+\epsilon^{0.49}] \times [t-\epsilon, t]$. Therefore, by Dini's theorem, we can choose n_0 sufficiently large that $u_n(x+y, t-\epsilon) > 1 - \frac{\epsilon}{2}$ for all $y \in A$ and all $n \geq n_0$.

Let $w_n(y,s)$ denote the solution to

$$\begin{cases} \partial_s w_n = \partial_y^2 w_n + w_n - w_n^n & \text{for } y \in \mathbb{R} \text{ and } s > 0, \\ w_n(y,0) = \Big(1 - \dfrac{\epsilon}{2}\Big)\mathbb{1}_{\{y \in A\}} & \text{for } y \in \mathbb{R}. \end{cases} \tag{4.6}$$

Then, by the comparison principle, $u_n(x+y, t-\epsilon+s) \geq w_n(y,s)$ for $n \geq n_0$, $s \geq 0$ and $y \in \mathbb{R}$.

Heuristically, the domain A is so "large" that, for times $s \leq \epsilon$, the solution w_n behaves locally near $y=0$ as if started from a flat initial condition. This suggests that $\partial_y^2 w_n(0,s)$ is very small for $s \in [0,\epsilon]$. Indeed, starting from (3.7) we have

$$w_n(y,s) = e^s p_s * w_n(y,0) - \int_0^s \mathrm{d}r\, e^{s-r} \int_{-\infty}^{\infty} \mathrm{d}z\, p_{s-r}(y-z) w_n^n(z,r).$$

Taking the derivative with respect to y, using Lemma 4.1, yields

$$\partial_y w_n(y,s) = e^s p_s' * w_n(y,0) - \int_0^s \mathrm{d}r\, e^{s-r} \int_{-\infty}^{\infty} \mathrm{d}z\, p_{s-r}'(y-z) w_n^n(z,r).$$

Then integrating by parts with respect to z in the second term, we have that

$$\partial_y w_n(y,s) = e^s p_s' * w_n(y,0) - \int_0^s \mathrm{d}r\, e^{s-r} \int_{-\infty}^{\infty} \mathrm{d}z\, p_{s-r}(y-z) n \partial_z w_n(z,r) w_n^{n-1}(z,r).$$

Note that $|\partial_z w_n(z,r)| \leq \frac{1}{\sqrt{\pi r}} + \frac{\sqrt{8}}{\sqrt{\pi}}$ $\forall z \in \mathbb{R}$ by Lemma 4.2, and the map $r \mapsto e^{s-r}\frac{1}{\sqrt{s-r}}\Big(\frac{1}{\sqrt{\pi r}} + \frac{\sqrt{8}}{\sqrt{\pi}}\Big)$ is integrable on $[0,s]$. Hence by Lemma 4.1, we can take the derivative with respect to y again, to obtain, at $y=0$,

$$\partial_y^2 w_n(0,s) = e^s p_s'' * w_n(0,0) - n \int_0^s \mathrm{d}r\, e^{s-r} \int_{-\infty}^{\infty} \mathrm{d}z\, p_{s-r}'(-z) \partial_z w_n(z,r) w_n^{n-1}(z,r).$$

Clearly, $\partial_z w_n(z,r)$ has the opposite sign to z, while $p_{s-r}'(-z)$ has the same sign as z. Hence the double integral is negative and

$$\partial_y^2 w_n(0,s) \geq e^s p_s'' * w_n(0,0) = \Big(1 - \frac{\epsilon}{2}\Big) e^s 2 p_s'(\epsilon^{0.49}) = -\Big(1 - \frac{\epsilon}{2}\Big) e^s \frac{\epsilon^{0.49}}{2\sqrt{\pi} s^{\frac{3}{2}}} e^{-\frac{\epsilon^{0.98}}{4s}}.$$

The function $s \mapsto s^{-\frac{3}{2}} e^{-\epsilon^{0.98}/(4s)}$ reaches its maximum at $s = \epsilon^{0.98}/6$ and is increasing on $[0, \epsilon^{0.98}/6)$. Thus, for ϵ small enough, $s \mapsto s^{-\frac{3}{2}} e^{-\epsilon^{0.98}/(4s)}$ is increasing on $[0,\epsilon]$ and so

$$\partial_y^2 w_n(0,s) \geq -\epsilon^{-1.01} e^{-\frac{\epsilon^{-0.02}}{4}} \quad \text{for } s \in [0,\epsilon].$$

This bound is uniform in n and goes to zero faster than ϵ. Thus, we can choose ϵ small enough that $\partial_y^2 w_n(0,s) > -\epsilon/2$ $\forall s \in [0,\epsilon]$. We use this in (4.6) and obtain, by the comparison principle, $w_n(0,s) \geq y_n(s)$ for $s \in [0,\epsilon]$, where y_n is the solution of

$$\partial_s y_n(s) = -\frac{\epsilon}{2} + y_n(s) - y_n(s)^n, \qquad y_n(0) = 1 - \frac{\epsilon}{2}.$$

For n sufficiently large, $y_n(s)$ is an increasing function of s. Since, for $s \geq 0$, $y_n(s) \geq y_n(0) = 1 - \epsilon/2$ and $y_n(s)^n \leq e^{n(y_n(s)-1)}$, we see, again by the comparison principle, that $y_n(s) \geq z_n(s)$ $\forall s \geq 0$, where z_n is the solution of

$$\partial_s z_n(s) = 1 - \epsilon - e^{n(z_n(s)-1)}, \qquad z_n(0) = 1 - \frac{\epsilon}{2}.$$

This last equation can be solved explicitly, giving

$$e^{-n(z_n(s)-1)} = \frac{1 - e^{-n(1-\epsilon)s}}{1-\epsilon} + e^{-n\left((1-\epsilon)s - \frac{\epsilon}{2}\right)}.$$

Hence setting $s = \epsilon$ and letting $n \to \infty$, we obtain that for ϵ sufficiently small,

$$\lim_{n\to\infty} e^{-n(z_n(\epsilon)-1)} = \frac{1}{1-\epsilon}.$$

It follows that as $n \to \infty$,

$$z_n(\epsilon) = 1 + \frac{\log(1-\epsilon)}{n} + o\left(\frac{1}{n}\right).$$

Therefore

$$\lim_{n\to\infty} z_n(\epsilon)^n = 1 - \epsilon.$$

Since for ϵ sufficiently small and n sufficiently large we have $u_n(x,t) \geq w_n(0,\epsilon) \geq y_n(\epsilon) \geq z_n(\epsilon)$, this implies that for $\epsilon > 0$ sufficiently small,

$$\liminf_{n\to\infty} u_n^n(x,t) \geq 1 - \epsilon,$$

which yields the desired conclusion. □

Lemma 4.8. *If the topological boundary $\partial U = \partial S$ between U and S has measure zero, then $x \mapsto u(x,t)$ is C^1 for every $t > 0$, and $\partial_x u$ is continuous on $\mathbb{R} \times (0,\infty)$.*

Proof. Let

$$u^*(x,t) := \begin{cases} u(x,t) & \text{if } u(x,t) < 1, \\ 0 & \text{if } u(x,t) = 1. \end{cases}$$

Then we have almost everywhere

$$u_n(x,t) - u_n^n(x,t) \to u^*(x,t). \tag{4.7}$$

Indeed, this holds in S (obviously) and in the interior of U (by Proposition 4.7), and therefore holds almost everywhere by hypothesis. Hence by (3.6), letting $n \to \infty$ and applying dominated convergence, for $t > 0$,

$$u(x,t) = p_t * v(x) + \int_0^t \mathrm{d}r\, p_r * u^*(x,t-r). \tag{4.8}$$

Applying Lemma 4.1, we have that $u(\cdot,t)$ is C^1 with

$$\partial_x u(x,t) = p_t' * v(x) + \int_0^t \mathrm{d}r\, p_r' * u^*(x,t-r),$$

and hence, by dominated convergence, $\partial_x u$ is continuous on $\mathbb{R} \times (0,\infty)$, as required. □

5 Proof of Proposition 1.3

In this section, we suppose that $v : \mathbb{R} \to [0,1]$ is a non-increasing function such that $v(x) \to 0$ as $x \to \infty$ and $v(x) \to 1$ as $x \to -\infty$. Let $\mu_0 = \inf\{x \in \mathbb{R} : v(x) < 1\} \in \{-\infty\} \cup \mathbb{R}$. Let u_n denote the solution of (1.1), and define u as in (1.2). For $t > 0$, let

$$\mu_t = \inf\Big(\{x \in \mathbb{R} : u(x,t) < 1\} \cup \{\infty\}\Big) \in \mathbb{R} \cup \{\infty, -\infty\}.$$

Note that since v is non-increasing, by the comparison principle we have that $x \mapsto u_n(x,t)$ is non-increasing for each n and each $t \geq 0$, and therefore the same property holds for u. Hence, since u is continuous on $\mathbb{R} \times (0,\infty)$, we have that for $t > 0$

$$u(x,t) = 1 \Leftrightarrow x \leq \mu_t.$$

We first prove that $\mu_t \in \mathbb{R}$ for $t > 0$ and bound the increments of μ.

Proposition 5.1. *$\mu_t \in \mathbb{R}$ for any $t > 0$. Furthermore, there exists a non-negative continuous increasing function $\epsilon \mapsto a_\epsilon$ with $a_0 = 0$ such that for any $t > 0$ and any $\epsilon \geq 0$,*

$$\mu_{t+\epsilon} - \mu_t \geq -a_\epsilon. \tag{5.1}$$

If $\mu_0 \in \mathbb{R}$, the above also holds at $t = 0$.

Proof. By (3.5), we have that for $x \in \mathbb{R}$ and $t > 0$,

$$u(x,t) \leq e^t \mathbb{E}_0\left[v(B_t + x)\right], \tag{5.2}$$

and so, by dominated convergence, $u(x,t) \to 0$ as $x \to \infty$. Hence $\mu_t < \infty$ $\forall t > 0$.

We now turn to showing that (5.1) holds if $\mu_t \in \mathbb{R}$; we shall then use this to show that $\mu_t > -\infty$ for $t > 0$. Take $\underline{v} : \mathbb{R} \to \mathbb{R}$ measurable with $0 \leq \underline{v} \leq v$, and, for $x \in \mathbb{R}$ and $t \geq 0$, let

$$\underline{u}(x,t) = e^t \mathbb{E}_x\left[\underline{v}(B_t)\right]. \tag{5.3}$$

Let $T = \sup\{t \geq 0 : \underline{u}(x,t) < 1 \ \forall x \in \mathbb{R}\}$; (we call T the time at which $\underline{u}$ hits 1). Then

$$\underline{u}(x,t) \leq u(x,t) \qquad \forall x \in \mathbb{R},\, t \leq T. \tag{5.4}$$

Indeed, note that $\underline{u}(x,t)$ is the unique bounded solution to $\partial_t \underline{u} = \partial_x^2 \underline{u} + \underline{u}$ with initial condition $\underline{v}$. By Theorem 1.2, for $t < T$, $\underline{u}(\cdot,t)$ is equal to the solution of (1.1) and (1.2) with $\underline{v}$ as initial condition. Again by Theorem 1.2, it follows that $\underline{u}(\cdot,t) \leq u(\cdot,t)$ for $t < T$. By continuity, we now have (5.4).

Now fix $\epsilon > 0$. Let $\underline{v}(x) = \eta \mathbb{1}_{\{x \in [-a,a]\}}$ for some fixed $\eta \in (0,1)$ and $a > 0$ to be chosen later. For this choice of $\underline{v}$, define $\underline{u}(x,s)$ as in (5.3). For ϵ sufficiently small, the pair (η, a) can be chosen in such a way that $\underline{u}$ hits 1 at time ϵ (we shall explain below how this is done); by symmetry, the position where $\underline{u}$ hits 1 is $x = 0$.

Fix $t \geq 0$ such that $\mu_t \in \mathbb{R}$. Our definition of $\underline{v}$ ensures that $\underline{v}(x - \mu_t + a) \leq u(x,t)$ for all $x \in \mathbb{R}$. Then, by (5.4) and the semigroup property in Theorem 1.2, $\underline{u}(x - \mu_t + a, \epsilon) \leq u(x, t+\epsilon)$. In particular, $1 = \underline{u}(0,\epsilon) \leq u(\mu_t - a, t+\epsilon)$ and so $\mu_{t+\epsilon} \geq \mu_t - a$.

We now complete the proof of (5.1) by showing that it is possible, when ϵ is sufficiently small, to choose $a = a_\epsilon := \epsilon^{1/3}$ and to find $\eta \in (0,1)$ such that $\underline{u}$ hits 1 at time ϵ, as required.

Introduce

$$f(s) = \underline{u}(0,s) = \eta e^s \mathbb{P}_0\left(|B_s| < a_\epsilon\right) = \eta e^s \int_{-a_\epsilon}^{a_\epsilon} \frac{\mathrm{d}y}{\sqrt{4\pi s}} e^{-\frac{y^2}{4s}}.$$

Note that

$$\mathbb{P}_0\left(B_\epsilon \geq a_\epsilon\right) = \int_{a_\epsilon}^{\infty} \frac{\mathrm{d}y}{\sqrt{4\pi\epsilon}} e^{-\frac{y^2}{4\epsilon}} = \int_{a_\epsilon/\sqrt{\epsilon}}^{\infty} \frac{\mathrm{d}y}{\sqrt{4\pi}} e^{-y^2/4} \leq e^{-\epsilon^{-1/3}/4}.$$

Hence for ϵ sufficiently small,

$$e^{\epsilon}\int_{-a_\epsilon}^{a_\epsilon}\frac{\mathrm{d}y}{\sqrt{4\pi\epsilon}}e^{-\frac{y^2}{4\epsilon}}>1,$$

and we can find $\eta<1$ such that

$$\underline{u}(0,\epsilon)=f(\epsilon)=\eta e^{\epsilon}\int_{-a_\epsilon}^{a_\epsilon}\frac{\mathrm{d}y}{\sqrt{4\pi\epsilon}}e^{-\frac{y^2}{4\epsilon}}=1.$$

It only remains to show that $f(s)<1$ for $s<\epsilon$. To do this, we simply show that $f'(s)\geq 0$ for $s<\epsilon$. We have

$$f'(s)=\eta e^{s}\left(\int_{-a_\epsilon}^{a_\epsilon}\frac{\mathrm{d}y}{\sqrt{4\pi s}}e^{-\frac{y^2}{4s}}-\frac{a_\epsilon}{s}\frac{1}{\sqrt{4\pi s}}e^{-\frac{a_\epsilon^2}{4s}}\right). \tag{5.5}$$

Clearly, for ϵ sufficiently small and $s\leq\epsilon$, the first term in the parenthesis of (5.5) is arbitrarily close to 1 while the second term is arbitrarily close to 0. Hence, $f'(s)>0$ $\forall s\leq\epsilon$, which concludes the proof of (5.1).

Finally, we can now show that in fact $\mu_t>-\infty$ for $t>0$. Indeed, let $\underline{v}(x)=\eta\mathbb{1}_{\{x\in[-a,a]\}}$ where $a>0$ and $\eta\in(0,1)$ are such that $\underline{u}$ hits 1 at some time $s\leq t$. (By the above argument, such a pair (η,a) can always be found. By symmetry, the position where $\underline{u}$ hits 1 is $x=0$.) Now choose x_0 such that $\underline{v}(x-x_0)\leq v(x)$ $\forall x\in\mathbb{R}$ (this is always possible as we assumed $v(x)\to 1$ as $x\to-\infty$). Then by (5.4) we have $\underline{u}(x-x_0,s)\leq u(x,s)$ $\forall x\in\mathbb{R}$ and, in particular, $1\leq u(x_0,s)$, which implies that $\mu_s\geq x_0$. We now have that $\mu_s\in\mathbb{R}$ for some $s\leq t$ and therefore, by (5.1), $\mu_t>-\infty$. $\square$

Proposition 5.2. *The following left-limit exists for every $t>0$ and satisfies: (làg)*

$$\lim_{\epsilon\searrow 0}\mu_{t-\epsilon}\leq\mu_t.$$

Proof. Suppose that the left limit $\lim_{\epsilon\searrow 0}\mu_{t-\epsilon}$ does not exist for some $t>0$, and choose b and c such that

$$\liminf_{\epsilon\searrow 0}\mu_{t-\epsilon}<b<c<\limsup_{\epsilon\searrow 0}\mu_{t-\epsilon}.$$

Then for any $\epsilon>0$, there exists $\epsilon'\in(0,\epsilon)$ such that $\mu_{t-\epsilon'}>c$. There also exists $\epsilon''\in(0,\epsilon')$ such that $\mu_{t-\epsilon''}<b$, so that $\mu_{t-\epsilon''}-\mu_{t-\epsilon'}<b-c$. However, by Proposition 5.1, $\mu_{t-\epsilon''}-\mu_{t-\epsilon'}\geq -a_\epsilon$, which is a contradiction if ϵ is sufficiently small that $a_\epsilon<c-b$. Hence the left limit $\lim_{\epsilon\searrow 0}\mu_{t-\epsilon}$ exists. By Proposition 5.1 again, $\mu_{t-\epsilon}\leq\mu_t+a_\epsilon\to\mu_t$ as $\epsilon\to 0$, and so $\lim_{\epsilon\searrow 0}\mu_{t-\epsilon}\leq\mu_t$. $\square$

Proposition 5.3. *The map $t\mapsto\mu_t$ is right-continuous (càd and hence càdlàg), i.e. for every $t\geq 0$,*

$$\lim_{\epsilon\searrow 0}\mu_{t+\epsilon}=\mu_t.$$

Proof. Proposition 5.1 already implies that for $t\geq 0$, $\liminf_{\epsilon\searrow 0}\mu_{t+\epsilon}\geq\mu_t$. It now remains to prove that for any $t\geq 0$, $\limsup_{\epsilon\searrow 0}\mu_{t+\epsilon}\leq\mu_t$. Indeed, fix $t>0$ (we shall consider the case $t=0$ separately). For $z>0$, by the definition of μ_t, we have $u(\mu_t+z,t)<1$. Then since u is continuous on $\mathbb{R}\times(0,\infty)$, $u(\mu_t+z,t+\epsilon)<1$ for ϵ sufficiently small, and so $\mu_{t+\epsilon}\leq\mu_t+z$. Hence $\limsup_{\epsilon\searrow 0}\mu_{t+\epsilon}\leq\mu_t+z$, and the result follows since $z>0$ was arbitrary.

It remains to consider the case $t=0$. First suppose $\mu_0\in\mathbb{R}$ and take $z>0$. Since v is non-increasing, we have that $v(y)\leq v(\mu_0+z/2)<1$ $\forall y\geq\mu_0+z/2$. Since $u(\cdot,\epsilon)\to v$ in L^1_{loc} as $\epsilon\searrow 0$, and $u(\cdot,\epsilon)$ is non-increasing for $\epsilon>0$, it follows that $u(\mu_0+z,\epsilon)<1$ for ϵ sufficiently small, and so $\mu_\epsilon<\mu_0+z$. Hence for any $z>0$, $\limsup_{\epsilon\searrow 0}\mu_\epsilon\leq\mu_0+z$. By the same argument, if $\mu_0=-\infty$ then, for any $z\in\mathbb{R}$, $u(z,\epsilon)<1$ for ϵ small enough. Therefore $\mu_\epsilon<z$ and so for any $z\in\mathbb{R}$, $\limsup_{\epsilon\searrow 0}\mu_\epsilon<z$. $\square$

We can finally complete the following important step:

Proposition 5.4. *The map $t \mapsto \mu_t$ is continuous on* $[0, \infty)$.

Proof. By Propositions 5.3 and 5.2, we already have that $t \mapsto \mu_t$ is càdlàg, and that for $t > 0$, $\lim_{\epsilon \searrow 0} \mu_{t-\epsilon} \leq \mu_t$. Thus the only way in which μ could fail to be continuous would be if $\lim_{\epsilon \searrow 0} \mu_{t-\epsilon} < \mu_t$ for some $t > 0$. Suppose, for some $t > 0$, that $\lim_{\epsilon \searrow 0} \mu_{t-\epsilon} = a < b = \mu_t$, and take $c \in (a, b)$. Define $f(s) = u(c, s)$ and observe that f is continuous on $(0, \infty)$.

Since $\lim_{\epsilon \searrow 0} \mu_{t-\epsilon} = a$, we have $f(t - \epsilon) < 1$ for all $\epsilon > 0$ sufficiently small, but since $\mu_t = b$, we have $\lim_{s \to t} f(s) = f(t) = 1$. Fix $t_0 \in (0, t)$ such that $f(s) < 1$ $\forall s \in [t_0, t)$, and define $(\tilde{u}(x, s), x \in \mathbb{R}, s \geq t_0)$ as the solution of the boundary value problem

$$\begin{cases} \partial_t \tilde{u} = \partial_x^2 \tilde{u} + \tilde{u} & \text{for } x > c \text{ and } s > t_0, \\ \tilde{u}(c, s) = f(s) & \text{for } s > t_0, \\ \tilde{u}(x, t_0) = u(x, t_0) & \text{for } x \in \mathbb{R}. \end{cases} \tag{5.6}$$

By Theorem 1.2, and since $u(x, s) < 1$ for $s \in [t_0, t)$ and $x > c$, we have that $\partial_t u = \partial_x^2 u + u$ for $x > c$ and $s \in (t_0, t)$. Since the solution of the boundary value problem (5.6) is unique it follows that for all $s \in [t_0, t)$ and $x \geq c$ we have $\tilde{u}(x, s) = u(x, s)$. By taking $s \nearrow t$ we also have, by continuity, $\tilde{u}(x, t) = u(x, t)$ for $x \geq c$. But since $\mu_t = b$, we must have $\tilde{u}(x, t) = u(x, t) = 1$ $\forall x \in [c, b]$. Furthermore, $\lim_{x \to \infty} \tilde{u}(x, t) = \lim_{x \to \infty} u(x, t) = 0$. This is impossible because for each $s > t_0$, the solution $\tilde{u}(\cdot, s)$ of the boundary value problem is analytic (see Theorem 10.4.1 in [4]). □

The proof of Proposition 1.3 is now essentially complete. The map $t \mapsto \mu_t$ is continuous on $[0, \infty)$, whether μ_0 is finite or $-\infty$. Therefore, defining U and S as in (4.2), we see that the topological boundary between these two domains is simply $\partial U = \partial S = \{(\mu_t, t) : t > 0\}$. It has measure zero, and hence by Lemma 4.8, $u(\cdot, t)$ is C^1 for every $t > 0$ and $\partial_x u$ is continuous on $\mathbb{R} \times (0, \infty)$.

6 Proof of uniqueness

In this section we prove that the classical solution to (FBP) is unique. We start with the following very simple lemma.

Lemma 6.1. *If (u, μ) is a classical solution of* (FBP)*, then for $t > 0$,*

$$\mu_t = \inf\{x \in \mathbb{R} : u(x, t) < 1\}.$$

Proof. Suppose, for a contradiction, that $\mu_t < x := \inf\left(\{x \in \mathbb{R} : u(x, t) < 1\} \cup \{\infty\}\right)$ for some $t > 0$. Take $c \in (\mu_t, x)$ and $\epsilon > 0$ small enough that, by continuity, $\mu_{t+s} < c$ $\forall s \in [0, \epsilon]$. Then by Corollary 3.3, for $y \in (c, x)$ and $\delta \in (0, \epsilon]$,

$$u(y, t + \delta) \geq e^{\delta} \mathbb{P}_y\big(B_s \in [c, x] \ \forall s \leq \delta\big).$$

This is strictly larger than 1 for δ sufficiently small, which is a contradiction. □

This lemma implies that if $u_1 \leq u_2$ then $\mu_1 \leq \mu_2$, and so the proof of the comparison property of Theorem 1.1 will be a consequence of Theorem 1.2 and the uniqueness of classical solutions of (FBP). Furthermore, it implies that if (u, μ) and $(\tilde{u}, \tilde{\mu})$ are two classical solutions to (FBP) with the same initial condition v, it is sufficient to show that $u = \tilde{u}$ to obtain that $\mu = \tilde{\mu}$.

For $t > 0$, let G_t denote the Gaussian semigroup operator, so that for $f \in L^\infty(\mathbb{R}) \cup L^1(\mathbb{R})$,

$$G_t f(x) = p_t * f(x) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x-y)^2}{4t}} f(y) \, \mathrm{d}y.$$

For $m > 0$, let C_m denote the cut operator given by

$$C_m f(x) = \min(f(x), m).$$

Suppose that $v : \mathbb{R} \to [0,1]$ is as in Theorem 1.1, *i.e.* v is non-increasing, $v(x) \to 0$ as $x \to \infty$ and $v(x) \to 1$ as $x \to -\infty$. For $n \in \mathbb{Z}_{\geq 0}$ and $\delta > 0$, introduce

$$u^{n,\delta,-}(x) := \left[e^{\delta} G_{\delta} C_{e^{-\delta}}\right]^n v(x) \quad \text{ and } \quad u^{n,\delta,+}(x) := \left[C_1 e^{\delta} G_{\delta}\right]^n v^{\delta,+}(x),$$

where we now define $v^{\delta,+}$. Recall that $\mu_0 = \inf\{x \in \mathbb{R} : v(x) < 1\} \in \mathbb{R} \cup \{-\infty\}$;

$$\text{if } \mu_0 \in \mathbb{R}, \text{ let } \quad v^{\delta,+}(x) = \begin{cases} 1 & \text{if } x < \mu_0 + \delta \\ v(x) & \text{if } x \geq \mu_0 + \delta, \end{cases} \tag{6.1}$$

$$\text{and} \quad \text{if } \mu_0 = -\infty, \text{ let } \quad v^{\delta,+}(x) = \begin{cases} 1 & \text{if } v(x) > 1 - \delta \\ v(x) & \text{if } v(x) \leq 1 - \delta. \end{cases} \tag{6.2}$$

Our proof of uniqueness relies on the Feynman-Kac representation of Corollary 3.3 and the following two results.

Lemma 6.2. *Suppose* (u, μ) *is a classical solution of* (FBP) *with initial condition* v. *Then for* $n \in \mathbb{Z}_{\geq 0}$, $\delta > 0$ *and* $x \in \mathbb{R}$,

$$u^{n,\delta,-}(x) \leq u(x, n\delta) \leq u^{n,\delta,+}(x). \tag{6.3}$$

Lemma 6.3. *For any* $\delta > 0$, $n \in \mathbb{Z}_{\geq 0}$, *and* $A \geq \frac{1}{2}$,

$$\int_{-A}^{A} \left| u^{n,\delta,+}(x) - u^{n,\delta,-}(x) \right| \mathrm{d}x \leq 4A(1 + e^{\delta n})(e^{\delta} - 1).$$

Suppose that (u, μ) and $(\tilde{u}, \tilde{\mu})$ are classical solutions of (FBP) with initial condition v. Then by Lemmas 6.2 and 6.3, for $t > 0$, $n \in \mathbb{Z}_{\geq 0}$ and $A \geq \frac{1}{2}$,

$$\int_{-A}^{A} \left| u(x,t) - \tilde{u}(x,t) \right| \mathrm{d}x \leq \int_{-A}^{A} \left| u^{n,\frac{t}{n},+}(x) - u^{n,\frac{t}{n},-}(x) \right| \mathrm{d}x \leq 4A(1 + e^{t})(e^{\frac{t}{n}} - 1).$$

Since $n \in \mathbb{Z}_{\geq 0}$ can be taken arbitrarily large, it follows that $\int_{-A}^{A} |u(x,t) - \tilde{u}(x,t)| \,\mathrm{d}x = 0$. Letting $A \to \infty$, by continuity of $u(\cdot, t)$ and $\tilde{u}(\cdot, t)$ it follows that $u(x,t) = \tilde{u}(x,t)$ $\forall x \in \mathbb{R}$. Therefore (u, μ) is the unique classical solution to (FBP) with initial condition v.

It remains to prove Lemmas 6.2–6.3. We shall require the following preliminary result for the proof of Lemma 6.2.

Lemma 6.4. *Suppose* $v^+ : \mathbb{R} \to [0,1]$ *is non-increasing with* $v^+(x) \to 0$ *as* $x \to \infty$ *and* $v^+(x) = 1$ *for some* $x \in \mathbb{R}$. *For* $t \geq 0$, *let* $u^+(\cdot, t) = e^t G_t v^+(\cdot)$ *and let*

$$\mu_t^+ = \inf\{x \in \mathbb{R} : u^+(x,t) < 1\}.$$

Then $\mu_t^+ \in \mathbb{R}$ $\forall t \geq 0$ *and* $t \mapsto \mu_t^+$ *is continuous.*

This is a simple result about the heat equation, which can be proved, for instance, using the same techniques as in Section 5.

Proof of Lemma 6.2. We shall show the following result: suppose that $v : \mathbb{R} \to [0,1]$ is as in Theorem 1.1 and that (u, μ) is a classical solution of (FBP) with initial condition v. Let $\mu_0 = \inf\{x \in \mathbb{R} : v(x) < 1\} \in \mathbb{R} \cup \{-\infty\}$. Suppose v^- and v^+ are non-increasing functions with

$$0 \leq v^- \leq v \leq v^+ \leq 1,$$

and that $v^-(x) \to 1$ as $x \to -\infty$, $v^+(x) \to 0$ as $x \to \infty$ and $\mu_0^+ := \inf\{x \in \mathbb{R} : v^+(x) < 1\} > \mu_0$. Take $\delta > 0$. For $t \geq 0$ and $x \in \mathbb{R}$, let

$$u^+(x,t) = e^t G_t v^+(x) \quad \text{ and } \quad u^-(x,t) = e^t G_t C_{e^{-\delta}} v^-(x).$$

Let $\mu_t^+ = \inf\{x \in \mathbb{R} : u^+(x,t) < 1\}$. Then we shall prove that

$$u^-(x,\delta) \leq u(x,\delta) \leq u^+(x,\delta) \ \forall x \in \mathbb{R} \quad \text{ and } \quad \mu_\delta^+ > \mu_\delta. \tag{6.4}$$

Since $u(x,\delta) \in [0,1]$, it follows from (6.4) that

$$0 \leq e^\delta G_\delta C_{e^{-\delta}}\, v^-(x) \leq u(x,\delta) \leq C_1 e^\delta G_\delta\, v^+(x) \leq 1,$$

and (6.3) follows by the definition of $v^{\delta,+}$ and by induction on n.

We now prove (6.4). Let $\tau = \inf\{s \geq 0 : B_s \leq \mu_{\delta - s}\} \wedge \delta$. Then by Corollary 3.3, for $x \in \mathbb{R}$,

$$\begin{aligned}
u(x,\delta) &= \mathbb{E}_x\left[e^\delta v(B_\delta)\mathbb{1}_{\{\tau=\delta\}} + e^\tau \mathbb{1}_{\{\tau<\delta\}}\right] \\
&\geq \mathbb{E}_x\left[e^\delta \min(v(B_\delta), e^{-\delta})\mathbb{1}_{\{\tau=\delta\}} + e^\delta \min(v(B_\delta), e^{-\delta})\mathbb{1}_{\{\tau<\delta\}}\right] \\
&= \mathbb{E}_x\left[e^\delta \min(v(B_\delta), e^{-\delta})\right] \\
&= u^-(x,\delta).
\end{aligned}$$

Now let $t_0 := \inf\{t \geq 0 : \mu_t^+ \leq \mu_t\}$. By continuity of μ_t and μ_t^+ (from Lemma 6.4), we have $t_0 > 0$. We will show below that $t_0 = \infty$. Take $t < t_0$ and, again, let $\tau = \inf\{s \geq 0 : B_s \leq \mu_{t-s}\} \wedge t$. By Proposition 3.1 we have

$$u^+(x,t) = \mathbb{E}_x\left[e^t v^+(B_t)\mathbb{1}_{\{\tau=t\}} + e^\tau u^+(B_\tau, t-\tau)\mathbb{1}_{\{\tau<t\}}\right].$$

Then, again by Corollary 3.3,

$$\begin{aligned}
u(x,t) &= \mathbb{E}_x\left[e^t v(B_t)\mathbb{1}_{\{\tau=t\}} + e^\tau \mathbb{1}_{\{\tau<t\}}\right] \\
&\leq \mathbb{E}_x\left[e^t v^+(B_t)\mathbb{1}_{\{\tau=t\}} + e^\tau u^+(B_\tau, t-\tau)\mathbb{1}_{\{\tau<t\}}\right] \\
&= u^+(x,t),
\end{aligned}$$

where the second line follows since $v \leq v^+$ and since, on $\{\tau < t\}$, we have $B_\tau = \mu_{t-\tau} < \mu_{t-\tau}^+$ and so $u^+(B_\tau, t-\tau) \geq 1$. By continuity, the inequality also holds for $t = t_0$ and so

$$u^+(x,t) \geq u(x,t) \ \forall x \in \mathbb{R}, t \leq t_0. \tag{6.5}$$

Suppose, for a contradiction, that $t_0 < \infty$. Then, by continuity, $\mu_{t_0}^+ = \mu_{t_0}$. Hence $u(\mu_{t_0}, t_0) = 1 = u^+(\mu_{t_0}, t_0)$ and $\partial_x u(\mu_{t_0}, t_0) = 0$, and so by (6.5), $\partial_x u^+(\mu_{t_0}, t_0) = 0$.

Note that u^+ is smooth on $\mathbb{R} \times (0,\infty)$ and, by the same argument as in Lemma 4.2, for $t > 0$, $\partial_x u^+(\cdot, t/2)$ is bounded. Therefore for $x \in \mathbb{R}$,

$$\partial_x u^+(x,t) = e^{t/2} p_{t/2} * \partial_x u^+(x, t/2) < 0$$

since $u^+(\cdot, t/2)$ is a non-increasing non-constant function.

We now have a contradiction. Therefore $t_0 = \infty$ and we have $\mu_\delta^+ > \mu_\delta$ and $u^+(x,\delta) \geq u(x,\delta)$ $\forall x \in \mathbb{R}$ by (6.5). This completes the proof of (6.4). $\square$

Proof of Lemma 6.3. Some of the ideas in this proof are from Section 4.3 of [7].

In this proof, we use both the supremum norm $\|\ \|_\infty$ and the L^1 norm $\|\ \|_1$. When a property holds for both norms, we simply write it with $\|\ \|$.

Note the following basic properties of our operators: for $f, g \in L^\infty(\mathbb{R}) \cup L^1(\mathbb{R})$, $m > 0$ and $t > 0$, we have for either norm that

$$\|C_m f - C_m g\| \leq \|f - g\|, \quad \|G_t f\| \leq \|f\|. \tag{6.6}$$

For the supremum norm, we also have that

$$\|C_m f - f\|_\infty \leq \max\left(\|f\|_\infty - m, 0\right). \tag{6.7}$$

For $w : \mathbb{R} \to [0, \infty)$, $\delta > 0$ and $x \in \mathbb{R}$,

$$C_1 e^\delta w(x) = \min(e^\delta w(x), 1) = e^\delta \min(w(x), e^{-\delta}) = e^\delta C_{e^{-\delta}} w(x). \tag{6.8}$$

Using (6.8), we can rewrite $u^{n,\delta,-}$ as

$$u^{n,\delta,-} = [e^\delta G_\delta C_{e^{-\delta}}]^n\, v = [G_\delta C_1 e^\delta]^n\, v = G_\delta [C_1 e^\delta G_\delta]^{n-1} C_1 e^\delta\, v. \tag{6.9}$$

We can also write $u^{n,\delta,+}$ as

$$u^{n,\delta,+} = [C_1 e^\delta G_\delta]^n\, v^{\delta,+} = C_1 e^\delta G_\delta [C_1 e^\delta G_\delta]^{n-1}\, v^{\delta,+}. \tag{6.10}$$

Now let

$$f := e^\delta G_\delta [C_1 e^\delta G_\delta]^{n-1}\, v^{\delta,+} - e^\delta G_\delta [C_1 e^\delta G_\delta]^{n-1}\, v,$$

and let $g := u^{n,\delta,+} - u^{n,\delta,-} - f$. By the triangle inequality, we have

$$\|g\|_\infty \leq \left\|u^{n,\delta,+} - e^\delta G_\delta [C_1 e^\delta G_\delta]^{n-1}\, v^{\delta,+}\right\|_\infty + \left\|e^\delta G_\delta [C_1 e^\delta G_\delta]^{n-1}\, v - u^{n,\delta,-}\right\|_\infty.$$

By our expression for $u^{n,\delta,+}$ in (6.10) and the properties of G_δ and C_1 in (6.6) and (6.7) respectively, the first term on the right hand side is bounded above by $e^\delta - 1$. A second application of the triangle inequality then yields

$$\|g\|_\infty \leq e^\delta - 1 + (e^\delta - 1)\left\|G_\delta [C_1 e^\delta G_\delta]^{n-1}\, v\right\|_\infty + \left\|G_\delta [C_1 e^\delta G_\delta]^{n-1}\, v - u^{n,\delta,-}\right\|_\infty.$$

Clearly $\left\|G_\delta [C_1 e^\delta G_\delta]^{n-1}\, v\right\|_\infty \leq 1$ by (6.6). Replacing $u^{n,\delta,-}$ by its expression in (6.9) gives

$$\begin{aligned}\|g\|_\infty &\leq 2(e^\delta - 1) + \left\|G_\delta [C_1 e^\delta G_\delta]^{n-1} v - G_\delta [C_1 e^\delta G_\delta]^{n-1} C_1 e^\delta v\right\|_\infty \\ &\leq 2(e^\delta - 1) + e^{\delta(n-1)} \left\|v - C_1 e^\delta v\right\|_\infty,\end{aligned}$$

where (6.6) was used repeatedly in the second inequality. But $\|v - C_1 e^\delta v\|_\infty = \|C_1 v - C_1 e^\delta v\|_\infty \leq e^\delta - 1$ by (6.6), and so

$$\|g\|_\infty \leq 2(e^\delta - 1) + e^{\delta(n-1)}(e^\delta - 1) \leq (2 + e^{\delta n})(e^\delta - 1). \tag{6.11}$$

By (6.6) applied repeatedly, for either norm we have

$$\|f\| \leq e^{\delta n} \|v^{\delta,+} - v\|.$$

We now need to consider two cases.

- If $\mu_0 = -\infty$, then, by our definition of $v^{\delta,+}$ in (6.2), we have $\|v^{\delta,+} - v\|_\infty = \delta$ and so $\|f\|_\infty \leq e^{\delta n} \delta$.
- If $\mu_0 \in \mathbb{R}$, then by our definition of $v^{\delta,+}$ in (6.1), $\|v^{\delta,+} - v\|_1 \leq \delta$ and so $\|f\|_1 \leq e^{\delta n} \delta$.

In either case, if $A \geq \frac{1}{2}$ then

$$\int_{-A}^{A} |f(x)|\, \mathrm{d}x \leq 2A e^{\delta n} \delta \leq 2A e^{\delta n} (e^\delta - 1).$$

By (6.11), we also have

$$\int_{-A}^{A} |g(x)|\, \mathrm{d}x \leq 2A(2 + e^{\delta n})(e^\delta - 1).$$

By a final application of the triangle inequality to $u^{n,\delta,+} - u^{n,\delta,-} = f + g$, the result follows. $\square$

7 Proof of the Feynman-Kac results from Section 3

Before proving Proposition 3.1, we need the following result:

Lemma 7.1. *Let* $f : [0,1] \to \mathbb{R} \cup \{-\infty\}$ *be continuous with* $f(0) < 0$ *and* $f(1) < 0$. *Let* $(\xi_t)_{t\in[0,1]}$ *denote a Brownian bridge (with diffusivity* $\sqrt{2}$*) from 0 to 0 in time* 1. *Then*

$$\mathbb{P}\left(\min_{s\le 1}(\xi_s - f(s)) = 0\right) = 0.$$

Proof. By a union bound, we have that

$$\begin{aligned}
&\mathbb{P}\left(\min_{s\le 1}(\xi_s - f(s)) = 0\right) \\
&\quad \le \mathbb{P}\left(\min_{s\le 1/2}(\xi_s - f(s)) = 0\right) + \mathbb{P}\left(\min_{s\le 1/2}(\xi_{1-s} - f(1-s)) = 0\right).
\end{aligned}$$

Given any fixed continuous function $b : [0,\infty) \to \mathbb{R}$ with $b(0) = 0$, there is at most one value of $z \in \mathbb{R}$ such that

$$\min_{s\le 1/2}\{b(s) + 2sz - f(s)\} = 0.$$

Thus, recalling the definition of p_t in (3.1),

$$\begin{aligned}
&\mathbb{P}\left(\min_{s\le 1/2}(\xi_s - f(s)) = 0\right) \\
&\quad = \mathbb{E}\left[\mathbb{P}\left[\min_{s\le 1/2}(\xi_s - f(s)) = 0 \,\middle|\, \xi_{1/2}\right]\right] \\
&\quad = \int_{-\infty}^{\infty} \mathrm{d}z\, p_{1/4}(z)\mathbb{P}\left(\min_{s\le 1/2}\left\{\tfrac{1}{\sqrt{2}}\xi_{2s} + 2sz - f(s)\right\} = 0\right) \\
&\quad = \mathbb{E}\left[\int_{-\infty}^{\infty} \mathrm{d}z\, p_{1/4}(z)\mathbb{1}_{\{\min_{s\le 1/2}\{\frac{1}{\sqrt{2}}\xi_{2s}+2sz-f(s)\}=0\}}\right] \\
&\quad = 0,
\end{aligned}$$

where the second equality holds since $\xi_s \sim N(0, 2s(1-s))$, the third equality follows by Fubini's theorem and the last equality follows because for each realisation of $(\xi_s)_{s\in[0,1]}$ there is at most one value of z for which the integrand is non-zero. By the same argument,

$$\mathbb{P}\left(\min_{s\le 1/2}(\xi_{1-s} - f(1-s)) = 0\right) = 0,$$

and the result follows. ☐

Proof of Proposition 3.1. Fix $(x,t) \in A$. We begin by proving the result under condition 1. For $\sigma \in [0,\tau]$, let

$$M_\sigma = w(B_\sigma, t-\sigma)e^{I_\sigma} + \int_0^\sigma \mathrm{d}r\, S(B_r, t-r)e^{I_r}, \qquad \text{where } I_\sigma = \int_0^\sigma K(B_s, t-s)\,\mathrm{d}s.$$

Since w is $C^{2,1}$ on A, for $\sigma \le \tau$, we apply Itô's formula (with no leading $\frac{1}{2}$ in front of the ∂_x^2 term because $(B_s)_{s\ge 0}$ has diffusivity $\sqrt{2}$):

$$\begin{aligned}
\mathrm{d}M_\sigma = {} & \partial_x w(B_\sigma, t-\sigma)e^{I_\sigma}\mathrm{d}B_\sigma + \partial_x^2 w(B_\sigma, t-\sigma)e^{I_\sigma}\mathrm{d}\sigma \\
& - \partial_t w(B_\sigma, t-\sigma)e^{I_\sigma}\mathrm{d}\sigma + w(B_\sigma, t-\sigma)e^{I_\sigma}K(B_\sigma, t-\sigma)\mathrm{d}\sigma + S(B_\sigma, t-\sigma)e^{I_\sigma}\mathrm{d}\sigma \\
= {} & \partial_x w(B_\sigma, t-\sigma)e^{I_\sigma}\mathrm{d}B_\sigma + [-\partial_t w + \partial_x^2 w + Kw + S](B_\sigma, t-\sigma)e^{I_\sigma}\mathrm{d}\sigma \\
= {} & \partial_x w(B_\sigma, t-\sigma)e^{I_\sigma}\mathrm{d}B_\sigma,
\end{aligned}$$

where we used (3.2) in the last line, since $(B_\sigma, t-\sigma) \in A$ for $\sigma \le \tau$. We see that $(M_\sigma)_{\sigma\le\tau}$ is a local martingale. Therefore, since $(M_\sigma)_{\sigma\le\tau}$ is bounded, we have that

$$w(x,t) = \mathbb{E}_x[M_0] = \mathbb{E}_x[M_\tau],$$

which yields the result (3.3) under condition 1.

We now turn to condition 2, with $A = \{(x,t) : t \in (0,T), x > \mu_t\}$ and $\tau = \inf\{s \ge 0 : B_s \le \mu_{t-s}\} \wedge t$. The stopping time τ is the first time that $(B_\tau, t-\tau) \in \partial A$. For $\epsilon > 0$ and $\delta > 0$, introduce the stopping times

$$\tau_{\epsilon,\delta} = \inf\{s \ge 0 : B_s \le \mu_{t-s} + \delta\} \wedge (t-\epsilon), \quad \tau_\epsilon = \inf\{s \ge 0 : B_s \le \mu_{t-s}\} \wedge (t-\epsilon) = \tau \wedge (t-\epsilon).$$

By (3.3) under condition 1 with stopping time $\tau_{\epsilon,\delta}$ we have that for $x > \mu_t$,

$$w(x,t) = \mathbb{E}_x\left[w(B_{\tau_{\epsilon,\delta}}, t-\tau_{\epsilon,\delta})e^{\int_0^{\tau_{\epsilon,\delta}} K(B_s,t-s)\,\mathrm{d}s} + \int_0^{\tau_{\epsilon,\delta}} \mathrm{d}r\, S(B_r, t-r)e^{\int_0^r K(B_s,t-s)\,\mathrm{d}s}\right].$$

We now take the limit $\delta \to 0$. Since μ is continuous, $\tau_{\epsilon,\delta} \to \tau_\epsilon$ as $\delta \to 0$, and since w, K and S are bounded, and w is continuous on $\bar{A} \cap (\mathbb{R} \times (0,\infty))$, we obtain, by continuity and dominated convergence,

$$w(x,t) = \mathbb{E}_x\left[w(B_{\tau_\epsilon}, t-\tau_\epsilon)e^{\int_0^{\tau_\epsilon} K(B_s,t-s)\,\mathrm{d}s} + \int_0^{\tau_\epsilon} \mathrm{d}r\, S(B_r, t-r)e^{\int_0^r K(B_s,t-s)\,\mathrm{d}s}\right]. \tag{7.1}$$

We now take the limit $\epsilon \to 0$ to prove (3.3) under condition 2. Note that $\tau_\epsilon \nearrow \tau$ as $\epsilon \searrow 0$; as S and K are bounded, by dominated convergence

$$\mathbb{E}_x\left[\int_0^{\tau_\epsilon} \mathrm{d}r\, S(B_r,t-r)e^{\int_0^r K(B_s,t-s)\,\mathrm{d}s}\right] \to \mathbb{E}_x\left[\int_0^{\tau} \mathrm{d}r\, S(B_r,t-r)e^{\int_0^r K(B_s,t-s)\,\mathrm{d}s}\right] \qquad \text{as } \epsilon \searrow 0.$$

We now turn to the first term on the right hand side of (7.1). As above, for $r \ge 0$, let $I_r = \int_0^r K(B_s, t-s)\,\mathrm{d}s$. Write

$$\mathbb{E}_x\left[w(B_{\tau_\epsilon}, t-\tau_\epsilon)e^{I_{\tau_\epsilon}}\right] = \mathbb{E}_x\left[w(B_{\tau_\epsilon}, t-\tau_\epsilon)e^{I_{\tau_\epsilon}}\mathbb{1}_{\{\tau<t\}}\right] + \mathbb{E}_x\left[w(B_{t-\epsilon},\epsilon)e^{I_{t-\epsilon}}\mathbb{1}_{\{\tau=t\}}\right]$$

(we used that $\tau_\epsilon = t-\epsilon$ when $\tau = t$). Since w and K are bounded, and w is continuous on $\bar{A} \cap (\mathbb{R} \times (0,\infty))$, by continuity and dominated convergence the first term on the right hand side converges to $\mathbb{E}_x\left[w(B_\tau, t-\tau)e^{I_\tau}\mathbb{1}_{\{\tau<t\}}\right]$ as $\epsilon \searrow 0$. For the second term, write

$$\begin{aligned}
\mathbb{E}_x\Big[w(B_{t-\epsilon},\epsilon)e^{I_{t-\epsilon}}\mathbb{1}_{\{\tau=t\}}\Big] = \;& \mathbb{E}_x\Big[w(B_{t-\epsilon},\epsilon)e^{I_{t-\epsilon}}\Big(\mathbb{1}_{\{\tau=t\}} - \mathbb{1}_{\{\tau\ge t-\epsilon\}}\Big)\Big] \\
&+ \mathbb{E}_x\Big[w(B_{t-\epsilon},\epsilon)e^{I_{t-\epsilon}}\mathbb{1}_{\{\tau\ge t-\epsilon\}}\Big] - \mathbb{E}_x\Big[w(B_t,\epsilon)e^{I_t}\mathbb{1}_{\{\tau=t\}}\Big] \\
&+ \mathbb{E}_x\Big[w(B_t,\epsilon)e^{I_t}\mathbb{1}_{\{\tau=t\}}\Big].
\end{aligned} \tag{7.2}$$

(In this equation, we set $w(B_t,\epsilon)$ to an arbitrary bounded value when $B_t < \mu_\epsilon$.) It is clear by dominated convergence that the first line on the right hand side of (7.2) goes to 0 as $\epsilon \searrow 0$.

Let us now show that the second line of (7.2) goes to 0 as $\epsilon \searrow 0$. Define $\phi_r(y;x,t)$ as

$$\phi_r(y;x,t) = \mathbb{E}_x\Big[\delta(B_r - y)e^{I_r}\mathbb{1}_{\{\tau\ge r\}}\Big] = p_r(x-y)\mathbb{E}_x\Big[e^{I_r}\mathbb{1}_{\{\tau\ge r\}}\Big| B_r = y\Big]. \tag{7.3}$$

(This is the probability, weighted by e^{I_r}, that the path of length r started from (x,t) arrives at $(y, t-r)$ without touching the left boundary.) By integrating over the value of B_r, we have that for $r \le t$,

$$\mathbb{E}_x\Big[w(B_r,\epsilon)e^{I_r}\mathbb{1}_{\{\tau\ge r\}}\Big] = \int_{-\infty}^{\infty} \mathrm{d}y\, w(y,\epsilon)\phi_r(y;x,t).$$

Thus, the second line of (7.2) can be written as

$$\mathbb{E}_x\Big[w(B_{t-\epsilon},\epsilon)e^{I_{t-\epsilon}}\mathbb{1}_{\{\tau\geq t-\epsilon\}}\Big]-\mathbb{E}_x\Big[w(B_t,\epsilon)e^{I_t}\mathbb{1}_{\{\tau=t\}}\Big]=\int_{-\infty}^{\infty}\mathrm{d}y\,w(y,\epsilon)\Big(\phi_{t-\epsilon}(y;x,t)-\phi_t(y;x,t)\Big). \tag{7.4}$$

To show that the second line of (7.2) goes to zero as $\epsilon\searrow 0$, it is then sufficient to show that $\big|\phi_{t-\epsilon}(y;x,t)-\phi_t(y;x,t)\big|$ is bounded by an integrable function and goes to 0 as $\epsilon\searrow 0$.

Let $(\xi_t)_{t\in[0,1]}$ denote a Brownian bridge (with diffusivity $\sqrt{2}$) from 0 to 0 in time 1 and introduce, for x, y fixed,

$$F(\xi,s,r)=\sqrt{r}\xi+x+s\frac{y-x}{r},$$

so that $(F(\xi_{s/r},s,r))_{s\leq r}$ is a Brownian bridge from x to y in time r. Then by (7.3),

$$\begin{aligned}\phi_r(y;x,t)&=p_r(x-y)\mathbb{E}_x\Big[e^{I_r}\mathbb{1}_{\{\tau\geq r\}}\Big|B_r=y\Big]\\&=p_r(x-y)\mathbb{E}_x\Big[e^{\int_0^r K(B_s,t-s)\mathrm{d}s}\mathbb{1}_{\{\mu_{t-s}<B_s\ \forall s<r\}}\Big|B_r=y\Big]\\&=p_r(x-y)\mathbb{E}\Big[e^{\int_0^r K\big(F(\xi_{s/r},s,r),t-s\big)\mathrm{d}s}\mathbb{1}_{\{\mu_{t-s}<F(\xi_{s/r},s,r)\ \forall s<r\}}\Big].\end{aligned}$$

Set $r=t-\epsilon$ and take the $\epsilon\searrow 0$ limit. Now by the continuity of ξ,

$$\begin{aligned}\mathbb{1}_{\{\mu_{t-s}<F(\xi_{s/t},s,t)\ \forall s<t\}}&\leq\liminf_{\epsilon\searrow 0}\mathbb{1}_{\{\mu_{t-s}<F(\xi_{s/(t-\epsilon)},s,t-\epsilon)\ \forall s<t-\epsilon\}}\\&\leq\limsup_{\epsilon\searrow 0}\mathbb{1}_{\{\mu_{t-s}<F(\xi_{s/(t-\epsilon)},s,t-\epsilon)\ \forall s<t-\epsilon\}}\leq\mathbb{1}_{\{\mu_{t-s}\leq F(\xi_{s/t},s,t)\ \forall s<t\}}.\end{aligned}$$

Since $x>\mu_t$, for $y>\mu_0$ we can apply Lemma 7.1, which yields that the probability that the lower and upper bounds above are different is zero. We can conclude, by dominated convergence, that

$$\lim_{\epsilon\searrow 0}\phi_{t-\epsilon}(y;x,t)=\phi_t(y;x,t).$$

For $y<\mu_0$, we have that $y<\mu_\epsilon$ for ϵ sufficiently small, and so for ϵ sufficiently small,

$$\phi_{t-\epsilon}(y;x,t)=0=\phi_t(y;x,t).$$

Since $t>0$, and $\phi_{t-\epsilon}(y;x,t)\leq p_{t-\epsilon}(x-y)e^{t\|K\|_\infty}$ by (7.3), it is easy to see that $\phi_{t-\epsilon}(\cdot;x,t)$ can be uniformly bounded for $\epsilon<t/2$ by a function with Gaussian tails. Therefore, by dominated convergence we see that (7.4) (which is the second line of (7.2)) goes to 0 as $\epsilon\searrow 0$.

It only remains to consider the third line of (7.2). Using (7.3), we can write

$$\mathbb{E}_x\Big[w(B_t,\epsilon)e^{I_t}\mathbb{1}_{\{\tau=t\}}\Big]=\int_{-\infty}^{\infty}\mathrm{d}y\,w(y,\epsilon)\phi_t(y;x,t). \tag{7.5}$$

Since w is bounded, $\phi_t(y;x,t)\leq p_t(x-y)e^{t\|K\|_\infty}$ and $w(\cdot,\epsilon)\to w(\cdot,0)$ in L^1_{loc} as $\epsilon\searrow 0$, we have that

$$\int_{-\infty}^{\infty}\mathrm{d}y\,w(y,\epsilon)\phi_t(y;x,t)\to\int_{-\infty}^{\infty}\mathrm{d}y\,w(y,0)\phi_t(y;x,t)=\mathbb{E}_x\Big[w(B_t,0)e^{I_t}\mathbb{1}_{\{\tau=t\}}\Big]\quad\text{as }\epsilon\searrow 0.$$

This completes the proof. $\square$

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References

[1] D. G. Aronson and H. F. Weinberger. Nonlinear diffusion in population genetics, combustion, and nerve pulse propagation. In *Partial differential equations and related topics*, pages 5–49. Springer, 1975.

[2] J. Berestycki, É. Brunet, and B. Derrida. Exact solution and precise asymptotics of a Fisher–KPP type front. *Journal of Physics A: Mathematical and Theoretical*, 51(3):035204, 2017.

[3] É. Brunet and B. Derrida. An exactly solvable travelling wave equation in the Fisher–KPP class. *Journal of Statistical Physics*, 161(4):801–820, 2015.

[4] J. R. Cannon. *The One-Dimensional Heat Equation.* Cambridge University Press, 1984.

[5] A. De Masi, P. A. Ferrari, E. Presutti, and N. Soprano-Loto. Hydrodynamics of the N-BBM process. *arXiv preprint arXiv:1707.00799*, 2017.

[6] A. De Masi, P. A. Ferrari, E. Presutti, and N. Soprano-Loto. Non local branching Brownians with annihilation and free boundary problems. *arXiv preprint arXiv:1707.00799*, nov 2017.

[7] A. De Masi, P. A. Ferrari, E. Presutti, and N. Soprano-Loto. Non local branching Brownians with annihilation and free boundary problems. *arXiv preprint arXiv:1711.06390*, 2017.

[8] R. Durrett. *Stochastic calculus: a practical introduction.* CRC Press, 1996.

[9] S. C. Gupta. *The classical Stefan problem: basic concepts, modelling, and analysis.* Elsevier, 2003.

[10] F. Hamel, J. Nolen, J.-M. Roquejoffre, and L. Ryzhik. A short proof of the logarithmic Bramson correction in Fisher-KPP equations. *Netw. Heterog. Media*, 8:275–279, 2013.

[11] I. Karatzas and S. Shreve. *Brownian motion and stochastic calculus.* Springer Science & Business Media, 2012.

[12] A. N. Kolmogorov, I. G. Petrovsky, and N. S. Piskunov. Etude de l'équation de la diffusion avec croissance de la quantité de matiere et son application à un problème biologique. *Moscow Univ. Math. Bull*, 1:1–25, 1937.

[13] G. Lamé and B. Clapeyron. Mémoire sur la solidification par refroidissement d'un globe liquide. *Annales Chimie Physique*, 47:250–2561, 1831.

[14] J. Lee. A free boundary problem in biological selection models. *arXiv preprint arXiv:1707.01232*, 2017.

[15] H. P. McKean. Application of Brownian motion to the equation of Kolmogorov-Petrovskii-Piskunov. *Communications on Pure and Applied Mathematics*, 28(3):323–331, 1975.

[16] J. Nolen, J.-M. Roquejoffre, and L. Ryzhik. Refined long time asymptotics for the Fisher-KPP fronts. *arXiv preprint arXiv:1607.08802*, 2016.

[17] K. Uchiyama. The behavior of solutions of some non-linear diffusion equations for large time. *Journal of Mathematics of Kyoto University*, 18(3):453–508, 1978.