Sierpiński Gasket versus Arrowhead curve

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Abstract

In the sequel, we extend the theoretical comparison of the Arrowhead Curve and the Sierpiński Gasket, on a numerical point of view, by studying diffusion on both objects, in the purpose of understanding the part played by the initial topological differences.

1 Introduction

In [Dav19a], Cl. David has put the light on a very interesting theoretical problem, by exhibiting two singular objects, with completely different topologies, obtained thus by two different processes, which, however, lead to the same limit: the Sierpiński Gasket, and the Arrowhead Curve. The first one can be obtained by means of an iterated function system, the second, by means of a L-system where, moreover, the same self-similarity no longer seems to hold. The problem at stake was the building of a Laplacian: would those two a priori different structures lead to the same operator ?

To handle the specific geometric features of the Curve, Cl. David has used a discrete approach, by means of a sequence of prefractal graphs. One of the difficulties was to dispose of a well-suited measure. As exposed in [?], for singular sets $\mathcal{F}$ of dimension $d < n$, the existing works rely on what is classically called a d-measure, i.e. a Radon measure μ with support $\mathcal{F}$ such that there exist two strictly positive constants c_1 and c_2 satisfying, for any strictly positive number r, and any ball $\mathcal{B}(X,r)$ the center of which belongs to $\mathcal{F}$:

$$c_1\, r^d \leqslant \mu\left(\mathcal{B}(X,r)\right) \leqslant c_2\, r^d$$

Usually, one works with the Hausdorff measure (or equivalent ones). Yet, such measures are not adapted to the very specific configuration of Weierstrass spaces, in so far as euclidean geometric conditions are required, for instance, as concerns the Markov Inequality [Mar48], may one want to use trace theorems [JW84]. Cl. David has thus generalized what she had been done in the specific case of the Weierstrass Curve [Dav18], [Dav19b], by sticking to a n-dimensional measure, defined by means of a a sequence of trapezoidal domains $(\mathcal{T}_m)_{m\in\mathbb{N}}$ playing the part of a trapezoidal neighborhood of the Curve. Interestingly, she has shown that this choice enabled one to retrieve existing results and was in perfect accordance with both the Kigami and Strichartz approaches [Kig03] [Str06], and the Lagrangian based Mosco one [Mos02].

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In the sequel, we propose to go a step further, and compare numerical results for the heat equation, for both structures, the Curve and the Gasket.

2 Gasket vs Curve

2.1 Frame of the study

We place ourselves, in the following, in the euclidian plane of dimension 2, referred to a direct orthonormal frame. The usual Cartesian coordinates are (x, y).

Notations. We set:

$$P_1 = (0,0) \quad , \quad P_2 = (1,0) \quad , \quad P_3 = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) \cdot$$

Notations. For $1 \leqslant i \leqslant 3$, we denote by f_i the contraction map, of fixed point $P_i \in \mathbb{R}^2$, such that:

$$\forall X \in \mathbb{R}^2 : \quad f_i(X) = \frac{1}{2}\,(X + P_i) \cdot$$

Definition 2.1 (**Sierpiński Gasket**).

The **Sierpiński Gasket** is $\mathfrak{S}\mathfrak{G}$ the unique set such that:

$$\mathfrak{S}\mathfrak{G} = \bigcup_{i=1}^{3} f_i(\mathfrak{S}\mathfrak{G}) \cdot$$

Notations.

i. For any real number θ, we denote by $\mathcal{R}_{O,\theta}$ the following rotation matrix:

$$\mathcal{R}_{O,\theta} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \cdot$$

ii. For $1 \leqslant i \leqslant 3$, we denote by g_i the contraction map, of fixed point $P_i \in \mathbb{R}^2$, such that:

$$\forall X \in \mathbb{R}^2 : \quad g_1(X) = \frac{1}{2}\,\mathcal{R}_{O,-\frac{2\pi}{3}}\,X + \frac{P_3}{2} \quad , \quad g_2(X) = \frac{1}{2}\,\mathcal{R}_{O,0}\,X + \frac{P_3}{2} \quad , \quad g_3(X) = \frac{1}{3}\,\mathcal{R}_{O,2\frac{\pi}{3}}\,X + P_2$$

Definition 2.2 (**Sierpiński Arrowhead Curve**).

We will call **Sierpiński Arrowhead Curve** $\mathfrak{SAC}$ the unique curve such that:

$$\mathfrak{SAC} = \bigcup_{i=1}^{3} g_i\left(\mathfrak{SAC}\right) \cdot$$

Notations. We will denote by V_0 (respectively by V_0') the ordered sets of points:

$$V_0 = \{P_1, P_2, P_3\} \quad , \quad V_0' = \{P_1, P_2\} \cdot$$

Notations. For any strictly positive integer m, we set:

$$V_m = \bigcup_{i=1}^{3} f_i\left(V_{m-1}\right) \quad , \quad V_m' = \bigcup_{i=1}^{3} g_i\left(V_{m-1}'\right) \cdot$$

Property 2.1. *For any strictly positive integer m, we set:*

$$V_m' \subset V_m \cdot$$

Proof. It can be proved by induction, starting from the fact that, by construction: $V_0' \subset V_0$.

□

Property 2.2 ([Dav19a]).

The set $V_\star' = \bigcup\limits_{m\in\mathbb{N}} V_m'$ is dense in $\mathfrak{SAC}$.

Proposition 2.3.

Given a natural integer m, we will denote by $\mathcal{N}_m$ (resp. $\mathcal{N}_m'$) the number of vertices of V_m (resp. V_m'). One has for any strictly positive integer m:

$$\mathcal{N}_0 = 3 \quad , \quad \mathcal{N}_m = \frac{3^{m+1}+3}{2} \quad , \quad \mathcal{N}_0' = 2 \quad , \quad \mathcal{N}_m' = 3^m + 1 \cdot$$

Proof. It simply comes from the fact that, for any strictly positive integer m:

$$\mathcal{N}_m = 3\,\mathcal{N}_{m-1} - 3 \quad , \quad \mathcal{N}_m' = 3\,\mathcal{N}_{m-1}' - 2 \cdot$$

□

2.2 Iterative construction

2.2.1 Sierpiński Gasket

Definition 2.3 (m^{th} **order graph,** $m \in \mathbb{N}^\star$)**.**

We will denote by $\mathfrak{S}\mathfrak{G}_0$ the complete graph of the set of points V_0, where, for any integer i belonging to $\{1, 2, 3\}$, the point P_i is linked to P_j, $j \neq i$.

For any strictly positive integer m, the set of points V_m, where the points of an m^{th}-order cell are linked in the same way as $\mathfrak{S}\mathfrak{G}_0$, is an oriented graph, which we will denote by $\mathfrak{S}\mathfrak{G}_m$ (see figures 1-6).

By extension, we will write:

$$\mathfrak{S}\mathfrak{G}_m = \bigcup_{i=1}^{3} f_i \left(\mathfrak{S}\mathfrak{G}_{m-1}\right) \cdot$$

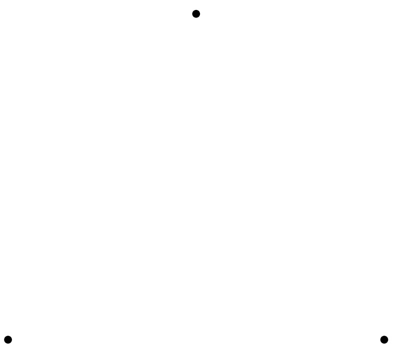

Figure 1 – V_0.

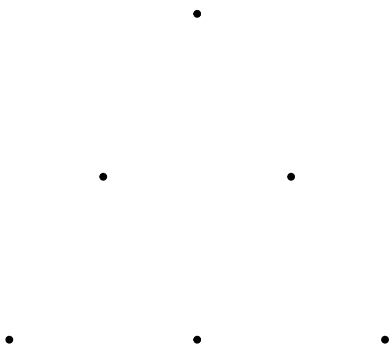

Figure 2 – V_1.

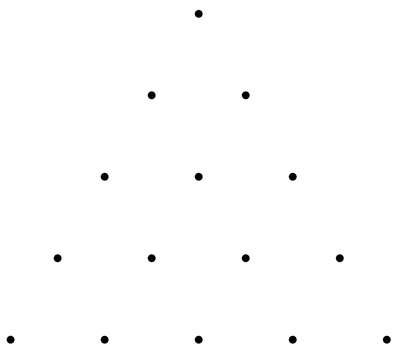

Figure 3 – V_2.

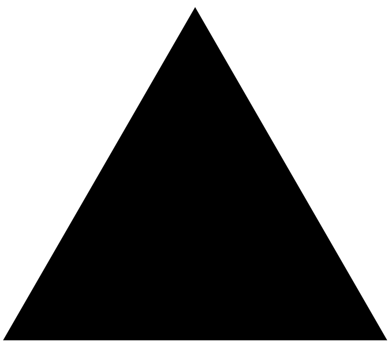

Figure 4 – $\mathfrak{S}\mathfrak{G}_0$.

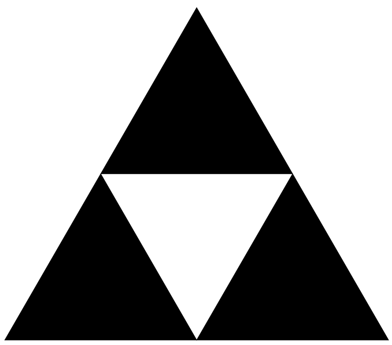

Figure 5 – $\mathfrak{S}\mathfrak{G}_1$.

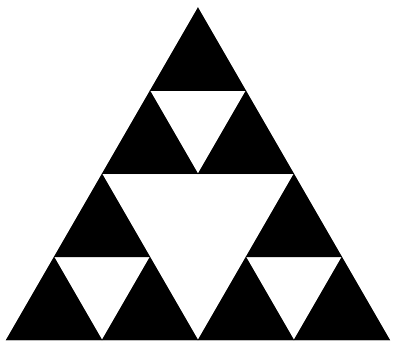

Figure 6 – $\mathfrak{S}\mathfrak{G}_2$.

The sequence $\left(\mathfrak{S}\mathfrak{G}_m\right)_{m\in\mathbb{N}}$**. The** m^{th} **order cells are in black.**

Property 2.4.

For any natural integer m:

$$V_m \subset V_{m+1}\cdot$$

Property 2.5 (**[BD85], or [Str99]**).

The set $V_\star = \bigcup\limits_{m\in\mathbb{N}} V_m$ *is dense in* $\mathfrak{SG}$.

2.2.2 Arrowhead Curve

The Curve is obtained by means of a L-system, as described in [Dav19a]. For the sake of clarity, we recall the construction.

Notation. Given a point $X \in \mathbb{R}^2$, we will denote by:

i. $\mathcal{S}im_{X,\frac{1}{2},\frac{\pi}{3}}$ the similarity of ratio $\dfrac{1}{2}$, the center of which is X, and the angle, $\dfrac{\pi}{3}$;

ii. $\mathcal{S}im_{X,\frac{1}{2},-\frac{\pi}{3}}$ the similarity of ratio $\dfrac{1}{2}$, the center of which is X, and the angle, $-\dfrac{\pi}{3}$.

Definition 2.4. Let us consider the following points of $\mathbb{R}^2$:

$$A=(0,0) \quad , \quad D=(1,0) \quad , \quad B=\mathcal{S}im_{A,\frac{1}{2},\frac{\pi}{3}}(D) \quad , \quad C=\mathcal{S}im_{D,\frac{1}{2},-\frac{\pi}{3}}(A)\cdot$$

One has:

$$V_1'=\{A,B,C,D\}\cdot$$

The set of points V_1', where A is linked to B, B is linked to C, and where C is linked to D, constitutes an oriented graph, that we will denote by $\mathfrak{SAC}_1$. V_0' is called the set of vertices of the graph $\mathfrak{SAC}_1$.

Let us build by induction the sequence of points:

$$\left(V_m'\right)_{m\in\mathbb{N}^\star}=\left(X_j^m\right)_{1\leqslant j\leqslant \mathcal{N}_m^{\mathcal{S}},\, m\in\mathbb{N}^\star} \quad , \quad \mathcal{N}_m^{\mathcal{S}}\in\mathbb{N}^\star$$

such that:

$$X_1^1=A \quad , \quad X_2^1=B \quad , \quad X_3^1=A \quad , \quad X_4^1=D$$

and for any integers $m \geqslant 2$, $1 \leqslant j \leqslant \mathcal{N}_m^{\mathcal{S}}$, $k\in\mathbb{N}$, $\ell\in\mathbb{N}$:

$$X_{j+k}^m = X_j^{m-1} \quad \text{if} \quad k\equiv 0\,[3]$$

$$X_{j+k+\ell}^m=\mathcal{S}im_{X_{j+\ell}^{m-1},\frac{1}{2},(-1)^{m+j+\ell+k+1}\,\frac{\pi}{3}}\left(X_{j+\ell+1}^{m-1}\right) \quad \text{if} \quad k\equiv 1\,[3] \quad \text{and} \quad \ell\in 2\,\mathbb{N}$$

$$X_{j+k+\ell}^m=\mathcal{S}im_{X_{j+\ell+1}^{m-1},\frac{1}{2},(-1)^{m+j+\ell+k+1}\,\frac{\pi}{3}}\left(X_{j+\ell}^{m-1}\right) \quad \text{if} \quad k\equiv 2\,[3] \quad \text{and} \quad \ell\in \mathbb{N}\setminus 2\,\mathbb{N}$$

The set of points V_m', where two consecutive points are linked, is an oriented graph, which we will denote by $\mathfrak{SAC}_m$. V_m is called the set of vertices of the graph $\mathfrak{SAC}_m$.

Property 2.6. *For any strictly positive integer m:*

$$V'_m \subset V'_{m+1}$$

Property 2.7. *If one denotes by $(\mathfrak{SAC}_m)_{m\in\mathbb{N}}$ the sequence of graphs which approximate the Sierpiński gasket $\mathcal{SG}$, then, for any strictly positive integer m:*

$$\mathfrak{SAC}_m \subsetneq \mathfrak{SG}_m$$

Definition 2.5. Consecutive vertices on the graph $\mathfrak{SAC}$

Two points X and Y of $\mathfrak{SAC}$ will be called ***consecutive vertices*** of the graph $\mathfrak{SAC}$ if there exists a natural integer m, and an integer j of $\left\{1,\ldots,\mathcal{N}_m^{\mathcal{S}}-1\right\}$, such that:

$$X = X_j^m \quad \text{and} \quad Y = X_{j+1}^m$$

or:

$$Y = X_j^m \quad \text{and} \quad X = X_{j+1}^m$$

Definition 2.6. For any positive integer m, the $\mathfrak{SAC}_m$ consecutive vertices of the graph $\mathcal{SG}_m^{\mathcal{C}}$ are, also, the vertices of 3^{m-1} trapezes $\mathcal{T}_{m,j}$, $1 \leqslant j \leqslant 3^{m-1}$. For any integer j such that $1 \leqslant j \leqslant 3^{m-1}$, one obtains each trapeze by linking the point number j to the point number $j+1$ if $j = i \bmod 4$, $0 \leqslant i \leqslant 2$, and the point number j to the point number $j-3$ if $j = -1 \bmod 4$.
One has to consider those polygons as semi-closed ones, since, for any of those 4−gons, the starting vertex, i.e. the point number j, is not connected, on the graph $\mathcal{SG}_m^{\mathcal{C}}$, to the extreme one, i.e. the point number $j-3$, if $j = -1 \bmod 4$. These trapezes generate a Borel set of $\mathbb{R}^2$.
In the sequel, we will denote by $\mathcal{T}_1$ the initial trapeze, the vertices of which are, respectively:

$$A \quad , \quad B \quad , \quad C \quad , \quad D$$

Definition 2.7 (m^{th} **order polygonal domain delimited by the Arrowhead curve,** $m \in \mathbb{N}^\star$**).**

We denote by $\mathcal{T}_0$ the polygonal domain delimited by the set of points V_0.

For any strictly positive integer m, the polygonal domain $\mathcal{T}_m$ formed by the union of the three copies of $\{g_1(\mathcal{T}_{m-1}), g_2(\mathcal{T}_{m-1}), g_3(\mathcal{T}_{m-1})\}$ (see figures 7-12).

We will write:

$$\mathcal{T}_m = \bigcup_{i=1}^{3} g_i\left(\mathcal{T}_{m-1}\right) \cdot$$

Property 2.8.

$$\mathcal{T} = \lim_{m \to +\infty} \mathcal{T}_m = \mathfrak{S}\mathfrak{A}\mathfrak{C} \cdot$$

Proof. This simply comes from properties 2.1 and 2.2.

□

Figure 7 – V_0'.

Figure 8 – V_1'.

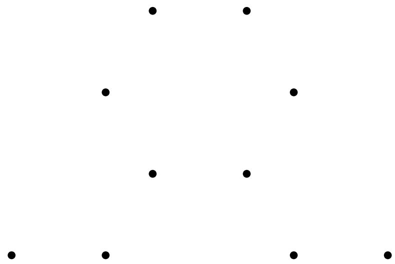

Figure 9 – V_2'.

Figure 10 – $\mathcal{T}_0$.

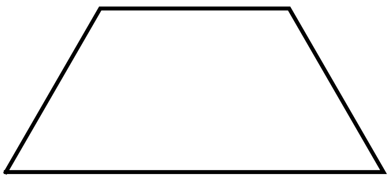

Figure 11 – $\mathcal{T}_1$.

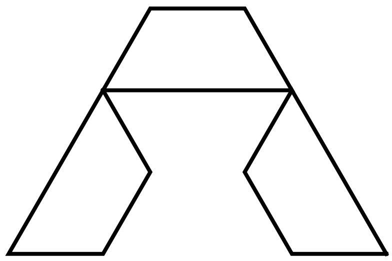

Figure 12 – $\mathcal{T}_2$.

The sequence $(\mathcal{T}_m)_{m\in\mathbb{N}}$ of polygonal domains delimited by $\mathfrak{S}\mathfrak{A}\mathfrak{C}_m$.

Property 2.9.

The Arrowhead Curve is dense in the Sierpiński Gasket.

Proof.

For any X in $\mathfrak{S}\mathfrak{G}$, there exists a sequence $(X_m)_{m\in\mathbb{N}}$ of points such that, for any natural integer m, X_m belongs to $\mathfrak{S}\mathfrak{G}_m$, and where:

$$\lim_{m \to +\infty} X_m = X \cdot$$

Moreover, for any natural integer m, the triangle $T_m^{\mathfrak{S}\mathfrak{G}} \subset \mathfrak{S}\mathfrak{G}_m$ contains a trapeze $\mathcal{T}_{m,j}$, $1 \leqslant j \leqslant 3^{m-1}$. One can choose a point Y_m belonging to $T_m^{\mathcal{T}}$ such that:

$$d\left(X_m, Y_m\right) = \mathcal{O}\left(2^{-m}\right)$$

and consider the sequence $(Y_m)_{m\in\mathbb{N}}$ of $(\mathcal{T}_m)_{m\in\mathbb{N}}$ such that:

$$\begin{aligned}
\lim_{m\to+\infty} d(X, Y_m) &\leqslant \lim_{m\to+\infty} d(X, X_m) + d(X_m, Y_m) \\
&\leqslant \lim_{m\to+\infty} d(X, X_m) + \mathcal{O}(2^{-m}) \\
&= 0
\end{aligned}$$

□

Notations. The Hausdorff dimensions of the Gasket and Curve, which are equals, are:

$$D_{\mathfrak{S}\mathfrak{A}\mathfrak{C}} = D_{\mathfrak{S}\mathfrak{G}} = \frac{\ln 3}{\ln 2} \cdot$$

3 Partial differential equations on Sierpiński Gasket and Arrowhead curve

3.1 Laplacian, on the Sierpiński Gasket

For the sake of clarity, we recall the construction Kigami construction of the Laplacian [Kig01].

Definition 3.1 (**Self-similar measure on $\mathfrak{S}\mathfrak{G}$ [Str06]**).

We define the **self-similar measure** μ on $\mathfrak{S}\mathfrak{G}$ to be the measure supported by $\mathfrak{S}\mathfrak{G}$ such that:

$$\mu = \frac{1}{3} \sum_{i=1}^{3} \mu \circ f_i^{-1} \cdot$$

Given a continuous function u on $\mathfrak{S}\mathfrak{G}$, we set:

$$\int_{\mathfrak{S}\mathfrak{G}} u \, d\mu = \lim_{m\to+\infty} \sum_{X_m \in f_{\mathcal{W}}\mathfrak{S}\mathfrak{G}} u(X_m)\, \mu\left(f_{\mathcal{W}_m}\mathfrak{S}\mathfrak{G}\right))$$

where $f_{\mathcal{W}_m} = f_1 \circ \cdots \circ f_m$, $\{f_1, \ldots, f_m\} \in \{f_1, f_2, f_3\}^m$. Moreover, the self-similarity of the measure yields:

$$\int_{\mathfrak{S}\mathfrak{G}} u \, d\mu = \frac{1}{3} \sum_{i=1}^{3} \int_{\mathfrak{S}\mathfrak{G}} u \circ f_i^{-1} \, d\mu \cdot$$

Notation. In the sequel, μ will thus denote a self-similar measure on $\mathfrak{SG}$.

Definition 3.2 (**Dirichlet form on Sierpiński Gasket [Str06]**).

Given a natural integer m, and a real-valued function u, defined on the set V_m of vertices of $\mathfrak{SG}_m$, the map, which, to any pair of real-valued, functions (u,v) defined on V_m, associates:

$$\mathcal{E}_{\mathfrak{SG}_m}(u,v) = \left(\frac{5}{3}\right)^m \sum_{X \underset{m}{\sim} Y} (u(X)-u(Y))\,(v(X)-v(Y))$$

is a Dirichlet form on $\mathfrak{SG}_m$.
Moreover:

$$\mathcal{E}_{\mathfrak{SG}_m}(u,u) = 0 \Leftrightarrow u \text{ is constant}$$

It makes sense to define the following Dirichlet form $\mathcal{E}$ on $\mathfrak{SG}$ through:

$$\mathcal{E}(u) = \lim_{m\to+\infty} \mathcal{E}_{\mathfrak{SG}_m}(u)\cdot$$

Notations. We will denote by:

i. $\operatorname{dom}\mathcal{E}$ the subspace of continuous functions defined on $\mathfrak{SG}$, such that:

$$\mathcal{E}(u) < +\infty$$

ii. $\operatorname{dom}_0\mathcal{E}$ the subspace of continuous functions defined on $\mathfrak{SG}$, which take the value zero on V_0, and such that:

$$\mathcal{E}(u) < +\infty\cdot$$

Definition 3.3 (**Laplacian, on the Sierpiński Gasket [Str06]**).

For $u \in \operatorname{dom}\mathcal{E}$, $f \in C(\mathfrak{SG})$, u belongs to $\operatorname{dom}\Delta_\mu$ and is such $\Delta_\mu u = f$, if

$$\mathcal{E}(u,v) = -\int_{\mathfrak{SG}} f\,v\,d\mu\cdot$$

If $f \in L^2_\mu(\mathfrak{SG})$, the same definition holds with $u \in \operatorname{dom}_{L^2_\mu}\Delta_\mu$.

Theorem 3.1 (**Pointwise formula [Str06]**).

Given a strictly positive integer m, a vertex $X \in V_\star \setminus V_0$, and $\psi_X^m \in \mathcal{S}(\mathcal{H}_0, V_m)$ a spline function such that:

$$\psi_X^m(Y) = \begin{cases} \delta_{XY} & \forall \quad Y \in V_m \\ 0 & \forall \quad Y \notin V_m \end{cases} \quad , \quad \textit{where} \quad \delta_{XY} = \begin{cases} 1 & \textit{if} \quad X = Y \\ 0 & \textit{else} \end{cases}$$

The Laplacian Δ_μ of a function u exists at $X \in \mathfrak{SG}$ if and only if the sequence

$$\left(\frac{3}{2} \, 5^m \, \Delta_m u(X_m) \right)_{m \in \mathbb{N}}$$

converges uniformly towards:

$$\Delta_\mu u(X)$$

where:

$$\Delta_m u(X) = \sum_{Y \underset{m}{\sim} X} \left(u(Y) - u(X) \right)$$

$Y \underset{m}{\sim} X$ means that (X, Y) is an edge of $\mathcal{F}_m$, and $(X_m)_{m \in \mathbb{N}}$ is a sequence of $(V_m)_{m \in \mathbb{N}}$ converging twards X.

Integration by parts enables one to define normal derivatives on fractal sets. This definition is valid either both on the boundary V_0, or in the **interior** of the considered self-similar set. This provides an equivalent formulation of the **Gauss-Green formula** :

Theorem 3.2 (**Green-Gauss formula [Str06]**).

Given $u \in dom_{\Delta_\mu}$ for a measure μ, $\partial_n u$ exists for all $X \in V_0$, and:

$$\mathcal{E}(u, v) = - \int_{\mathcal{F}} \Delta_\mu u \, v \, d\mu + \sum_{X \in V_0} \partial_n u(X) \, v$$

holds for all $v \in dom \, \mathcal{E}$, where:

$$\partial_n u(X) = \lim_{m \to +\infty} \left(\frac{5}{3} \right)^m \sum_{Y \underset{m}{\sim} X} \left(u(X) - u(Y) \right) \cdot$$

3.2 Laplacian, on the Arrowhead Curve

Definition 3.4 (**Laplacian of order $m \in \mathbb{N}^\star$ [Dav19a]**).

For any strictly positive integer m, and any real-valued function u, defined on the set V'_m of the vertices of the graph $\mathfrak{SAC}_m$, we introduce the Laplacian of order m, $\Delta_m(u)$, by:

$$\Delta_m u(X) = \sum_{Y \in V'_m, \, Y \underset{m}{\sim} X} c_m \, \frac{u(Y) - u(X)}{\ell_m^2} \quad \forall \, X \in V'_m \setminus V'_0$$

where:

$$\ell_m = \frac{1}{2^m} \quad , \quad \frac{c_m}{\ell_m^2} = 2^{2 \, m \, D_{\mathcal{SAC}}} \, .$$

Definition 3.5 (**Laplacian [Dav19a]**).

A real valued function u, continuous on the Arrowhead Curve, will be said to be in the domain of the Laplacian $\operatorname{dom}\Delta$ if, for any $X \notin V_0'$:

$$\Delta u(X) = \lim_{m \to +\infty} \Delta_m u(X) < +\infty \cdot$$

4 Numerical results

We hereafter present results obtained using finite difference computations. We refer to [?] for prooves of the following results.

4.1 The heat equation

In the sequel, u denotes the solution of:

$$\left\{ \begin{array}{rclr} \dfrac{\partial u}{\partial t}(t,x) - \Delta u(t,x) & = & 0 & \forall\,(t,x) \,\in\,]0,T[\, \times \mathcal{F} \\ u(t,x) & = & 0 & \forall\,(x,t) \,\in\, \partial\mathcal{F} \times [0,T[\\ u(0,x) & = & g(x) & \forall\, x \,\in\, \mathcal{F} \end{array} \right.$$

where $\mathcal{F}$ is a generic notation that refers to the Gasket $\mathfrak{S}\mathfrak{G}$, or the curve $\mathfrak{S}AC$.

As in [?], we use a first order forward difference scheme to approximate the time derivative $\dfrac{\partial u}{\partial t}$; the Laplacian is approximated by means of sequence of graph Laplacians $(\Delta_m\, u)_{m \in \mathbb{N}^\star}$.

Notation ($p \times p$ **Identity matrix,** $p \in \mathbb{N}^\star$).

Given a strictly positive integer p, we will denote by I_p denotes the $p \times p$ identity matrix.

Notations. Given a strictly positive integer N, we denote by $h = \dfrac{T}{N}$ the related time step.

4.1.1 Sierpiński Gasket

The finite difference scheme is given by [?].

For any integer k belonging to $\{0, \ldots, N-1\}$, any strictly positive integer m, and any point X in the set $V_m \setminus V_0$, the scheme writes:

$$(\mathcal{S}_{\mathcal{H}}) \quad \begin{cases} \dfrac{u_h^m((k+1)\,h, X) - u_h^m(k\,h, X)}{h} & = & \dfrac{3}{2}\, 5^{-m} \left(\displaystyle\sum_{X \underset{m}{\sim} Y} u_h^m(k\,h, Y) - u_h^m(k\,h, X) \right) \\ u_h^m(k\,h, P_j) & = & 0 \\ u_h^m(0, X) & = & g(X) \end{cases}$$

The solution vector

$$U_h^m(k) = \begin{pmatrix} u_h^m(k\,h, X_1) \\ \vdots \\ u_h^m(k\,h, X_{\mathcal{N}_m - 3}) \end{pmatrix}$$

is such that:

$$U_h^m(k+1) = A\, U_h^m(k)$$

where

$$A = I_{\mathcal{N}_m - 3} - h\, \tilde{\Delta}_m$$

and where $\tilde{\Delta}_m$ denotes the$(\mathcal{N}_m - 3) \times (\mathcal{N}_m - 3)$ normalized Laplacian matrix.

Theorem 4.1 (**Consistency [?]**).

The finite difference scheme is consistent, the scheme error being given, for α-Hölder continuous functions, by:

$$\varepsilon_{k,i}^m = \mathcal{O}(h) + \mathcal{O}(2^{-m\alpha}) \quad 0 \leqslant k \leqslant N-1,\, 1 \leqslant i \leqslant \mathcal{N}_m - 3 \cdot$$

Theorem 4.2 (**CFL condition for the convergence [?]**).

Under the stability condition

$$h\, 5^m \leqslant \frac{2}{9}$$

the scheme is also convergent for the norm $\|\cdot\|_{2,\infty}$, such that:

$$\left\| (u_h^m(k\,h, X_i))_{0 \leqslant k \leqslant N, X_i \in V_m \setminus V_0} \right\|_{2,\infty} = \max_{0 \leqslant k \leqslant N} \left(3^{-m} \sum_{1 \leqslant i \leqslant \mathcal{N}_m} |u_h^m(k\,h, X_i)|^2) \right)^{\frac{1}{2}} \cdot$$

4.1.2 Arrowhead Curve

For any integer k belonging to $\{0, \ldots, N-1\}$, any strictly positive integer m, and any point X in the set $V'_m \setminus V'_0$, the scheme writes:

$$(\mathcal{S}_{\mathcal{H}}) \quad \left\{ \begin{array}{ccc} \dfrac{u_h^m((k+1)\,h, X) - u_h^m(k\,h, X)}{h} & = & \beta(X)\, 3^{-2m} \left(\displaystyle\sum_{X \underset{m}{\sim} Y} u_h^m(k\,h, Y) - u_h^m(k\,h, X) \right) \\ u_h^m(k\,h, P_j) & = & 0 \\ u_h^m(0, X) & = & g(X) \end{array} \right.$$

The solution vector

$$U_h^m(k) = \begin{pmatrix} u_h^m(k\,h, X_1) \\ \vdots \\ u_h^m(k\,h, X_{\mathcal{N}'_m-2}) \end{pmatrix}$$

is such that:

$$U_h^m(k+1) = A\, U_h^m(k)$$

where

$$A = I_{\mathcal{N}'_m-2} - h\,\tilde{\Delta}_m$$

and where $\tilde{\Delta}_m$ denotes the $(\mathcal{N}'_m - 2) \times (\mathcal{N}'_m - 2)$ normalized Laplacian matrix.

Theorem 4.3 (**Consistency**).

The finite difference scheme is consistent, the scheme error being given, for α-Hölder continuous functions, by:

$$\varepsilon_{k,i}^m = \mathcal{O}(h) + \mathcal{O}(2^{-m\alpha}) \quad 0 \leqslant k \leqslant N-1,\, 1 \leqslant i \leqslant \mathcal{N}'_m - 2\,\cdot$$

Proof. One has to prove the space discretization error. As in To this purpose, as in [**?**], one may consider a strictly positive integer m, a point $X \in V'_m \setminus V'_0$, and a harmonic function $\psi_X^{(m)}$ on the m^{th}-order cell, taking the value 1 on X, and 0 on the others vertices, and take mean value formula

$$\begin{aligned} \Delta_\mu u(X) - \beta(X)\, 3^{2m} \Delta_m u(X) &= \beta(X)\, 3^m \int_{\mathfrak{SAC}} \psi_X^{(m)}(Y)(\Delta u(X) - \Delta u(Y))\, d\mu(Y) \\ &= \Delta_\mu u(X) - \Delta_\mu u(c_m) \\ &\lesssim |X - c_m|^\alpha \\ &\lesssim \left(\frac{1}{2}\right)^{m\,\alpha} \end{aligned}$$

for some c_m in the m^{th}-order cell containing X nd $\beta(X)$ taking the values $\dfrac{4}{2}$ or 4, depending if X belongs to two different m^{th}-cells or not. □

Theorem 4.4 (CFL condition).

Under the stability condition

$$\beta(X)\,h\,3^{2m} \leqslant \frac{1}{2}$$

the scheme is also convergent for the norm $\|\cdot\|_{2,\infty}$, *such that:*

$$\left\|(u_h^m(k\,h,X_i))_{0\leqslant k\leqslant N, X_i\in V'_m\setminus V'_0}\right\|_{2,\infty} = \max_{0\leqslant k\leqslant N}\left(d^{-m}\sum_{1\leqslant i\leqslant \mathcal{N}'_m}|u_h^m(k\,h,X_i)|^2)\right)^{\frac{1}{2}}\cdot$$

Proof. For for $i=1,\ldots,\mathcal{N}'_m-2$, the i^{th} eigenvalue λ_i of the matrix A is given by:

$$\lambda_i = 1-2\,h\,\beta\,3^{2m}+h\,\beta\,3^{2m}\gamma_i$$

where, for $i=1,\ldots,\mathcal{N}'_m-2$, γ_i denotes the i^{th} eigenvalue of the matrix:

$$B = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 1 & 0 & 1 & \ldots & 0 & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & 0 & \ldots & 1 & 0 \end{pmatrix}\cdot$$

Proposition 4.5. *For* $i=1,\ldots,\mathcal{N}'_m-2$*:*

$$\gamma_i = 2\,\cos\left(\pi\,\frac{i}{\mathcal{N}'_m-1}\right)\cdot$$

Proof. Given an eigenvalue γ of the matrix B, there exists a real vector v of the form

$$v = \begin{pmatrix} V'_0 \\ \vdots \\ v_{\mathcal{N}'_m-1} \end{pmatrix}$$

such that, for any integer $i \in \{1,\ldots,\leqslant \mathcal{N}'_m-2\}$:

$$v_{i-1}+v_{i+1} = \gamma v_i\,\cdot$$

Let us search the roots of the scond order polynomial equation $X^2-\gamma X+1=0$:

1. If $\gamma^2=4$, then $X_1=X_2=\pm 1$ and $v_i=(a\,i+b)\,(\pm 1)^i$ for $i \in \{1,\ldots,\leqslant \mathcal{N}'_m-2\}$. Since $v_0=v_{\mathcal{N}'_m-1}=0$, one has:

$$b=0 \quad\text{and}\quad a\,\left(\mathcal{N}'_m-1\right)(\pm)^{\mathcal{N}'_m-1} = a\,3^m\,\left(1+(-1)^{3^m}\right)=0\,\cdot$$

then

$$a = 0$$

This ensures the nullity of the vector v.

2. If $\gamma^2 \neq 4$, then

$$X_1, X_2 = \frac{\gamma \pm \sqrt{\gamma^2 - 4}}{2}$$

and $v_i = a(X_1)^i + b(X_2)^i$ for all i. Since $v_0 = v_{\mathcal{N}'_m - 1} = 0$, $X_1 X_2 = 1$ and $X_1 + X_2 = \gamma$, one has:

$$X_1 \neq 0, \quad X_2 = (X_1)^{-1}, \quad \gamma = X_1 + (X_1)^{-1}$$

and, for $i \in \{1, \ldots, \leqslant \mathcal{N}'_m - 2\}$:

$$v_i = a(X_1)^i + b(X_1)^{-i}$$

On the other hand, we have :

$$a + b = 0, \quad a\,(X_1)^{\mathcal{N}'_m - 1} + b(X_1)^{-(\mathcal{N}'_m - 1)} = 0$$

Thus:

$$a = -b \quad , \quad a\left((X_1)^{\mathcal{N}'_m - 1} - (X_1)^{-(\mathcal{N}'_m - 1)}\right) = 0$$

To avoid the case of null eigenvector, one must have :

$$\left((X_1)^{\mathcal{N}'_m - 1} - (X_1)^{-(\mathcal{N}'_m - 1)}\right) = 0 \cdot$$

For $X_1 \neq 0$, we have

$$(X_1)^{1 - \mathcal{N}'_m}\left((X_1)^{2\mathcal{N}'_m - 2} - 1\right) = 0$$

i.e.

$$(X_1)^{2\mathcal{N}'_m - 2} = 1$$

and, for $0 \leqslant k \leqslant 2\mathcal{N}'_m - 3$:

$$X_1 = \exp\left(i\,\pi\,\frac{2k}{2\mathcal{N}'_m - 2}\right)$$

$$\gamma = \exp\left(i\,\pi\,\frac{2k}{2\,\mathcal{N}'_m - 2}\right) + \exp\left(-i\,\pi\,\frac{2\,k}{2\,\mathcal{N}'_m - 2}\right) = 2\,\cos\left(\pi\,\frac{k}{\mathcal{N}'_m - 1}\right)$$

For $k=0,\ldots,\mathcal{N}'_m-2,\mathcal{N}'_m,\ldots,2\,\mathcal{N}'_m-3$, we set:

$$\gamma_k = 2\,\cos\left(\pi\,\frac{k}{\mathcal{N}'_m-1}\right)\cdot$$

We have:

$$\begin{aligned}\gamma_{(2\,\mathcal{N}'_m-3)-(k-1)} &= 2\,\cos\left(\pi\,\frac{2\,\mathcal{N}'_m-3-k+1}{\mathcal{N}'_m-1}\right)\\ &= 2\,\cos\left(2\,\pi-\pi\,\frac{k}{\mathcal{N}'_m-1}\right)\\ &= 2\,\cos\left(\pi\,\frac{k}{\mathcal{N}'_m-1}\right)\\ &= \gamma_k\end{aligned}$$

The eigenvalue set of the matrix is thus:

$$\left\{2\,\cos\left(\pi\,\frac{k}{\mathcal{N}'_m-1}\right)\right\}_{0\leqslant k\leqslant \mathcal{N}'_m-2}\cdot$$

□

Let us go back to the eigenvalues λ_k, $k=1,\ldots,\mathcal{N}'_m-2$:

$$\begin{aligned}\lambda_k &= 1-2\,h\,\beta\,3^{2m}\left(1-\cos\left(\pi\frac{k}{\mathcal{N}'_m-1}\right)\right)\\ &= 1-4\,h\,\beta\,3^{2m}\,\sin^2\left(\pi\,\frac{k}{2\,\mathcal{N}'_m-2}\right)\end{aligned}$$

For $k=1,\ldots,\mathcal{N}'_m-2$:

$$1-4\,h\,\beta\,3^{2m}\leqslant\lambda_k\leqslant 1\cdot$$

The stability condition is then:

$$h\,3^{2m}\leqslant\frac{1}{2\,\beta}\Longrightarrow|\lambda_k|\leqslant 1\cdot$$

The convergence follows by applying the same idea as in [**?**]. □

4.2 Numerical results

In the following, we simulate a heat transfer, in the cases of the Sierpiński Gasket and of the Arrowhead curve. The initial value condition is the polynomial function defined, for any pair (x,y) of real numbers, by:

$$g(x,y)=-3\,x^2-y^2+x\,y+3\,x-\frac{1}{2}y$$

As regards the CFL stability conditions, the Arrowhead Curve appears as more demanding $(N\sim\mathcal{O}(9^m))$ than the Gasket $(N\sim\mathcal{O}(5^m))$.

4.2.1 Arrowhead Curve (see figures 13-19)

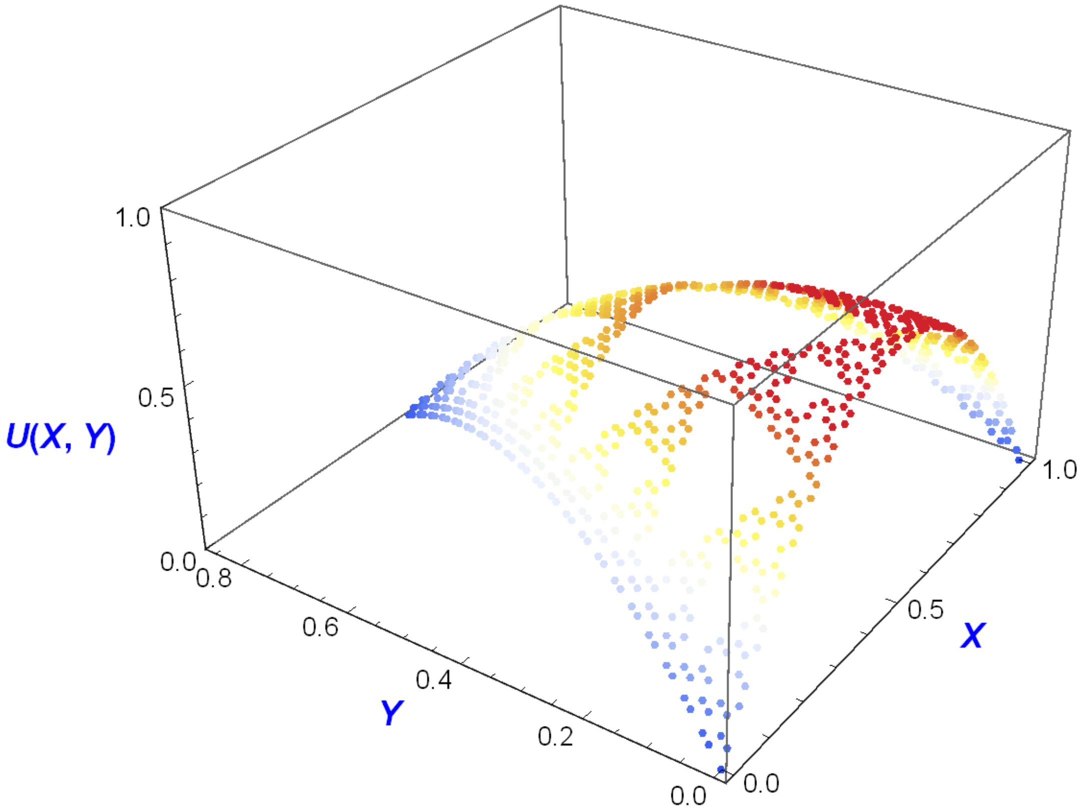

Figure 13 – The graph of the approached solution for $T = 1$, $k = 0$, $m = 6$.

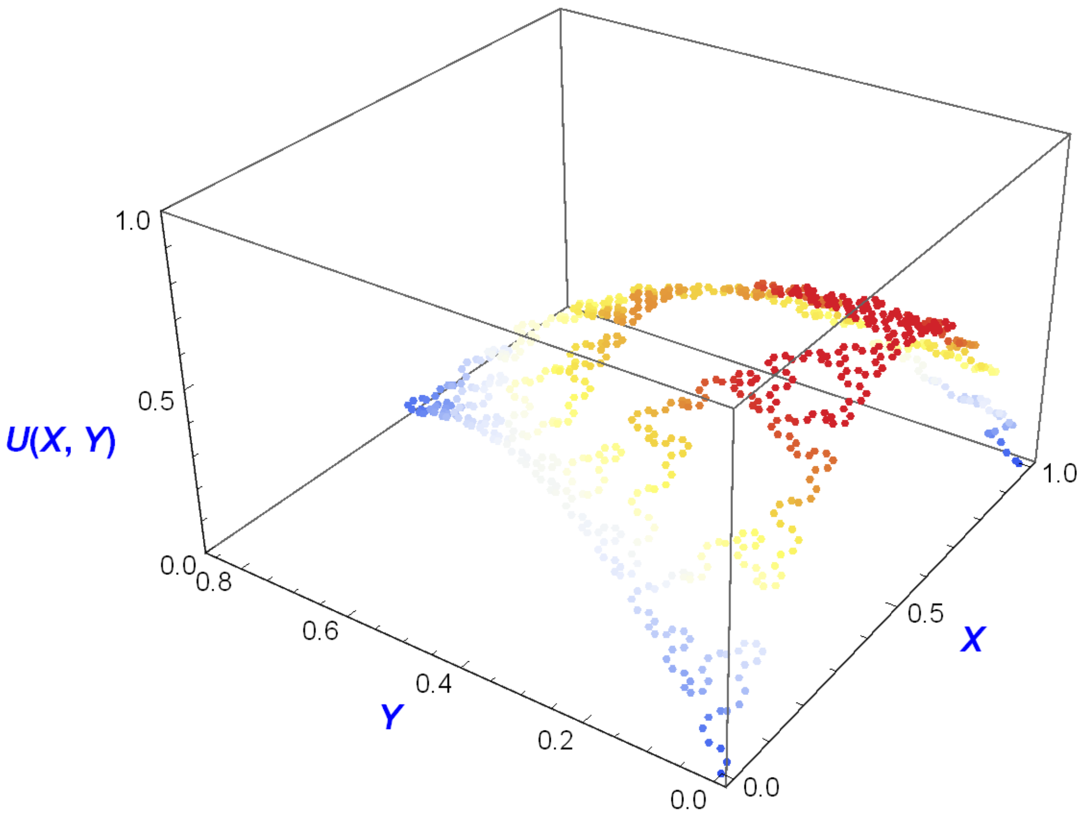

Figure 14 – The graph of the approached solution for $T = 1$, $k = 10$, $m = 6$.

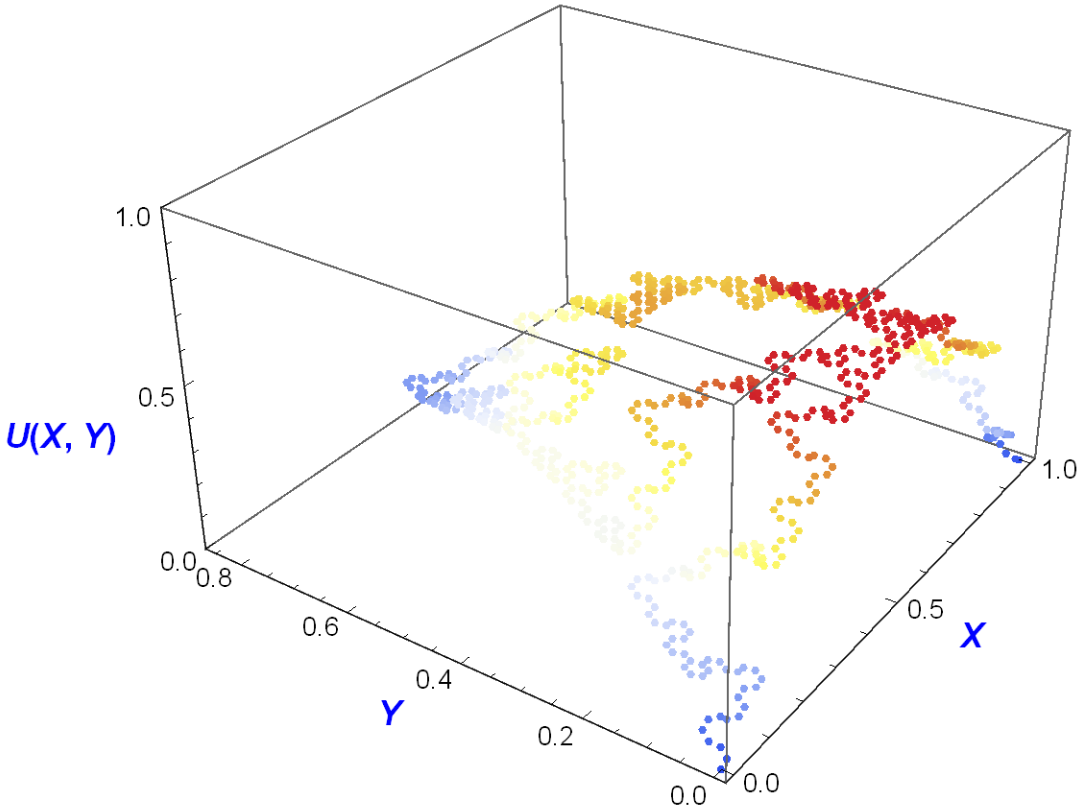

Figure 15 – The graph of the approached solution for $T = 1$, $k = 100$, $m = 6$.

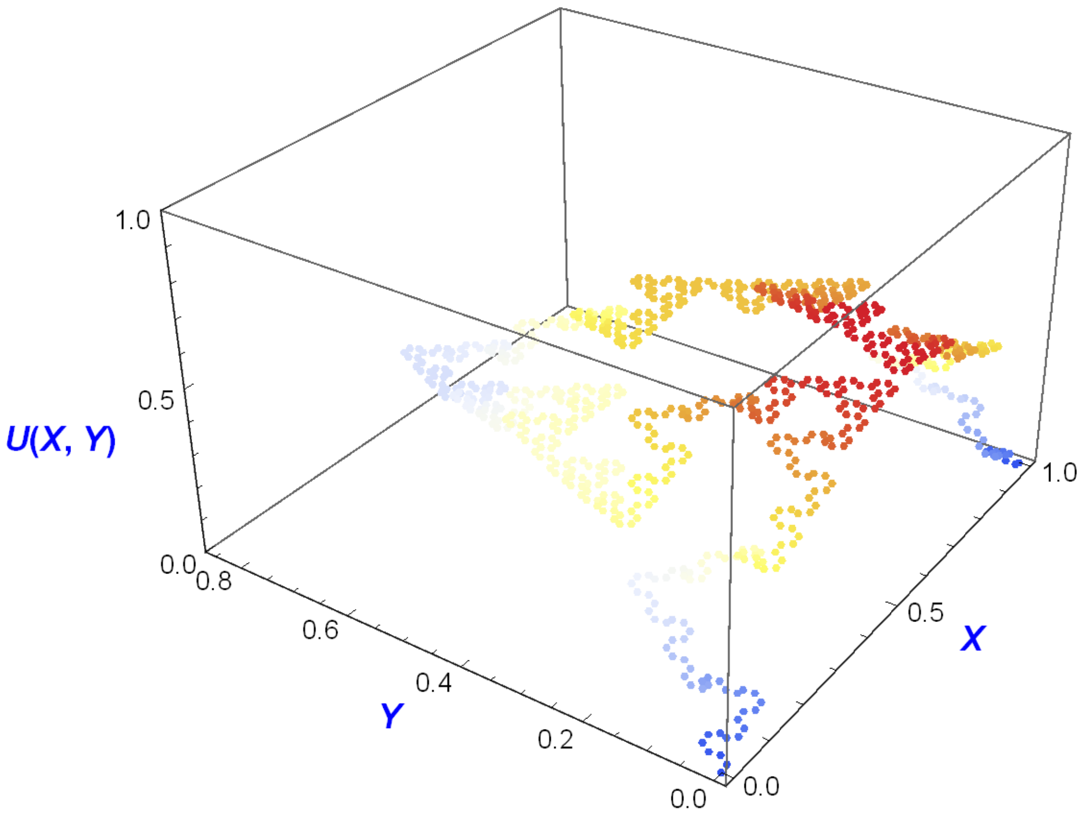

Figure 16 – The graph of the approached solution for $T = 1$, $k = 1000$, $m = 6$.

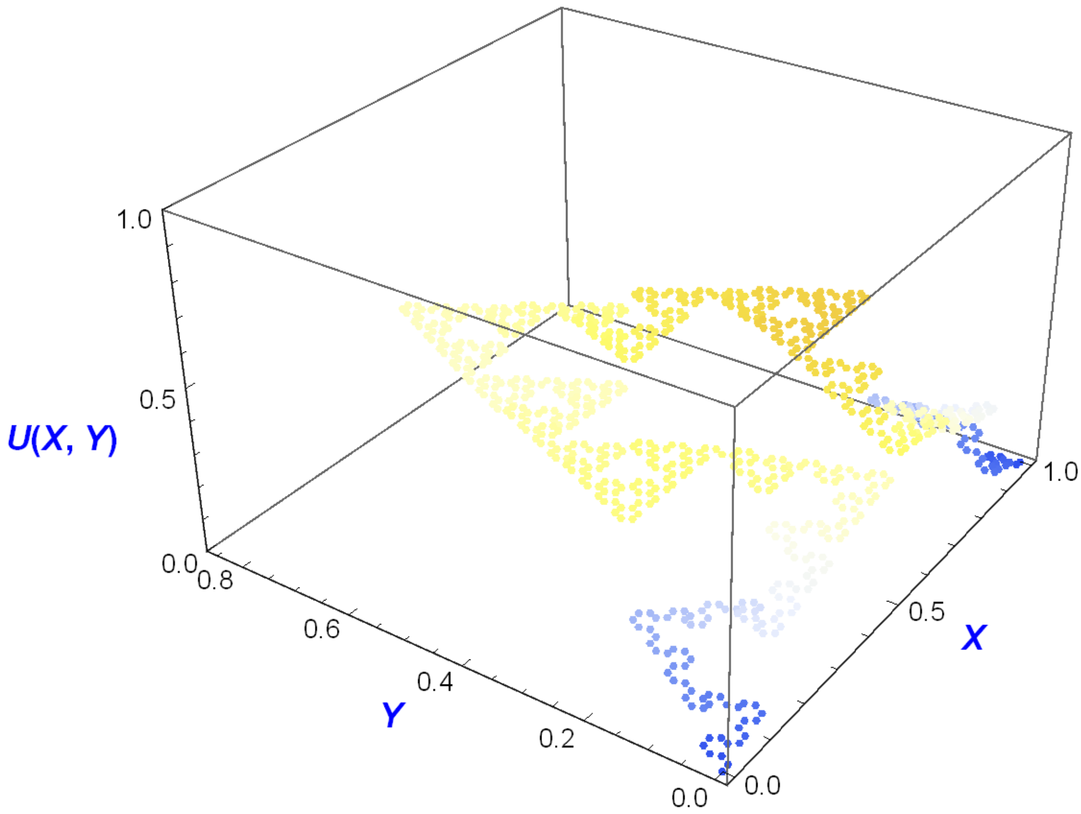

Figure 17 – The graph of the approached solution for $T = 1$, $k = 10000$, $m = 6$.

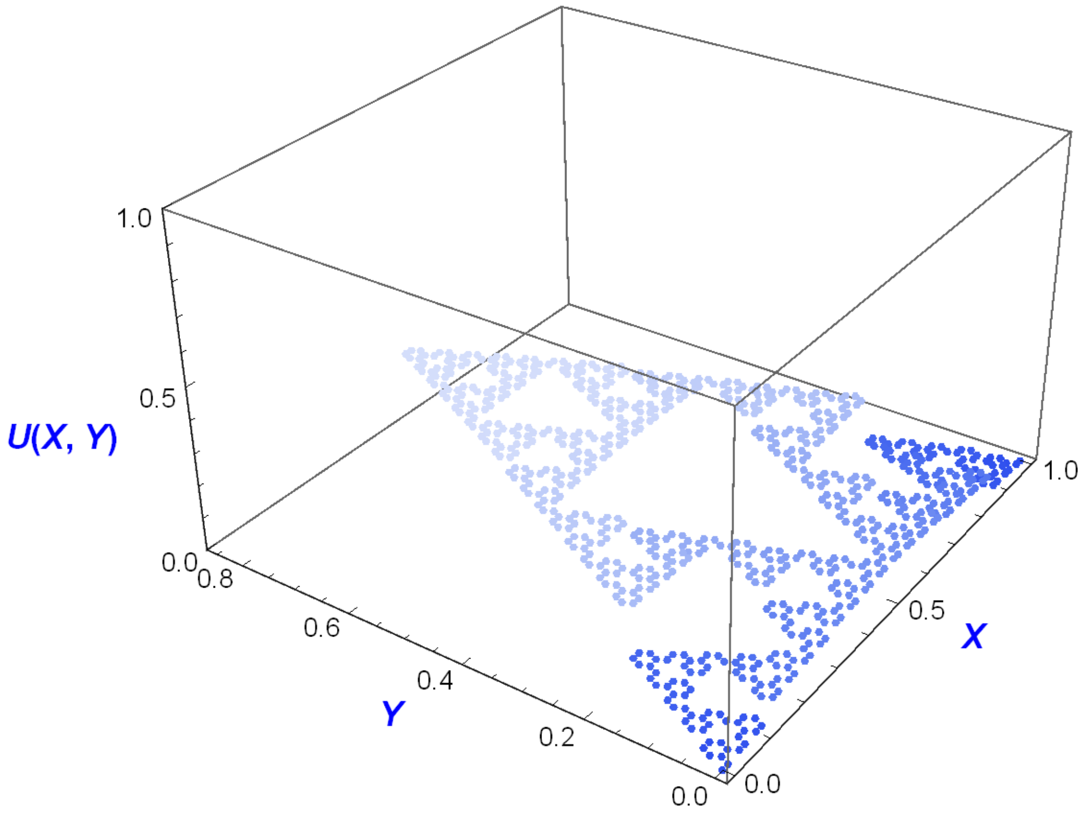

Figure 18 – The graph of the approached solution for $T = 1$, $k = 1000000$, $m = 6$.

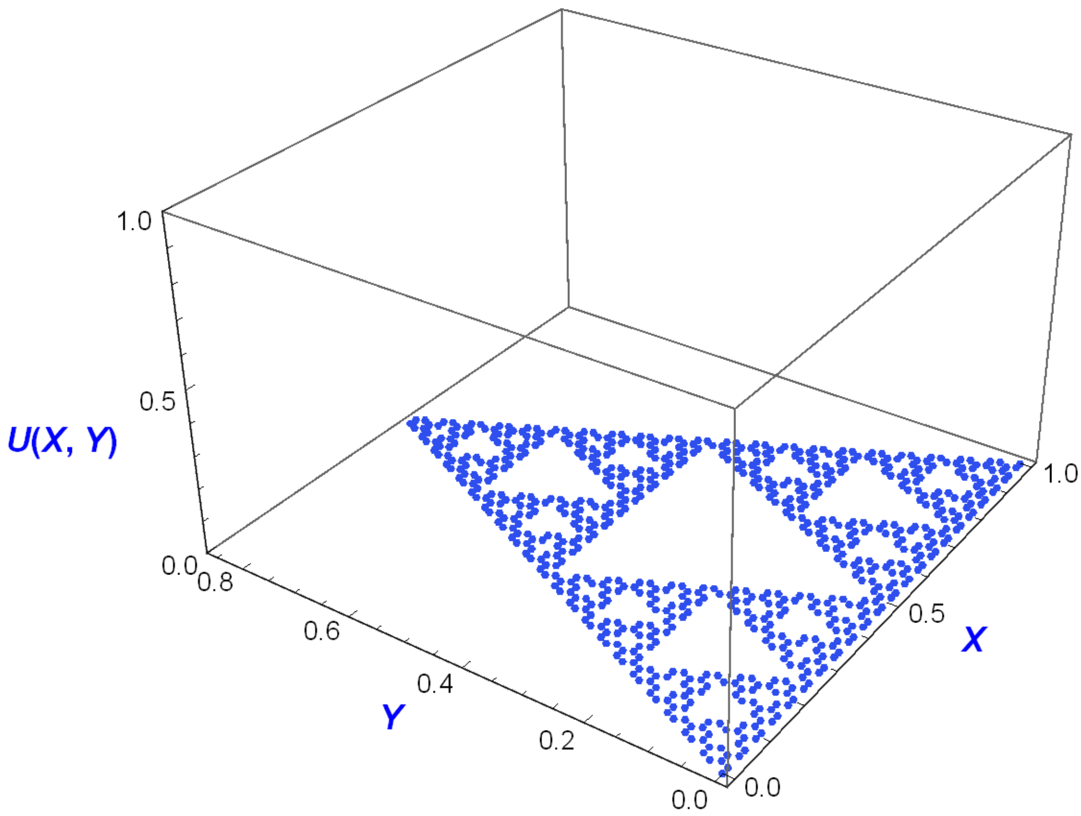

Figure 19 – The graph of the approached solution for $T = 1$, $k = 1000000$, $m = 6$.

4.2.2 Sierpiński Gasket (see figures 20-26)

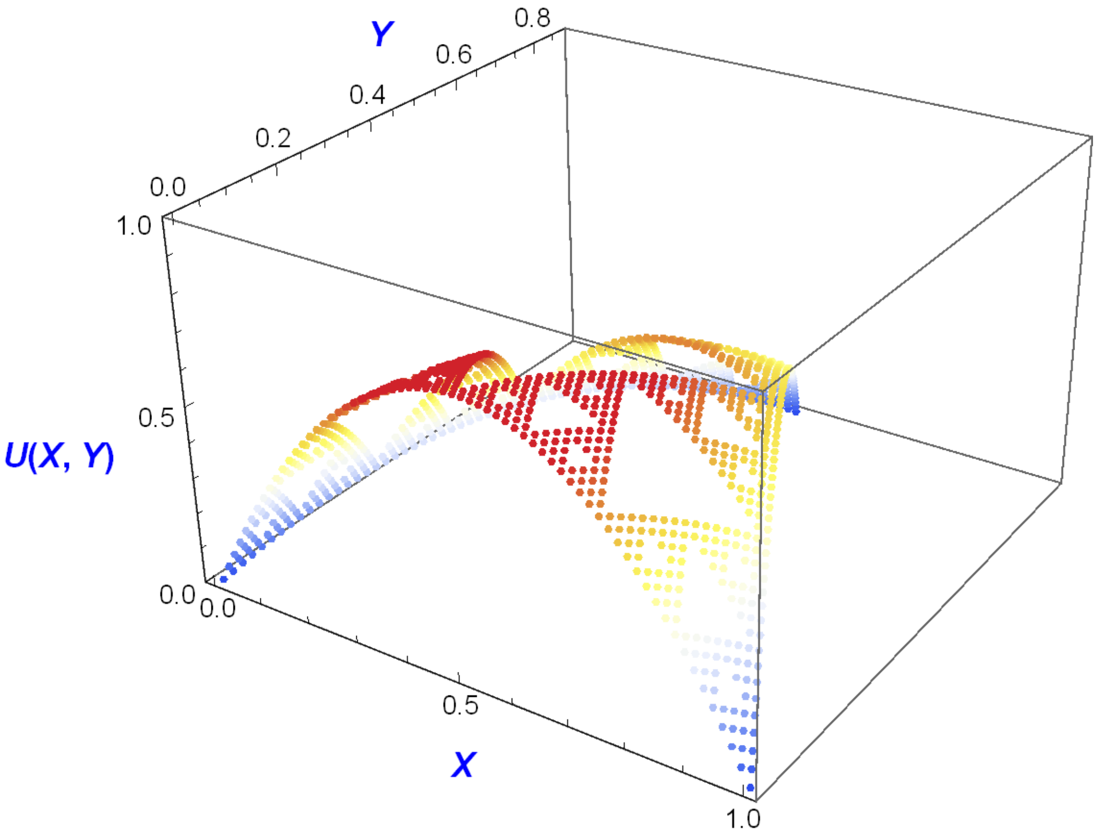

Figure 20 – The graph of the approached solution for $T = 1$, $k = 0$, $m = 6$.

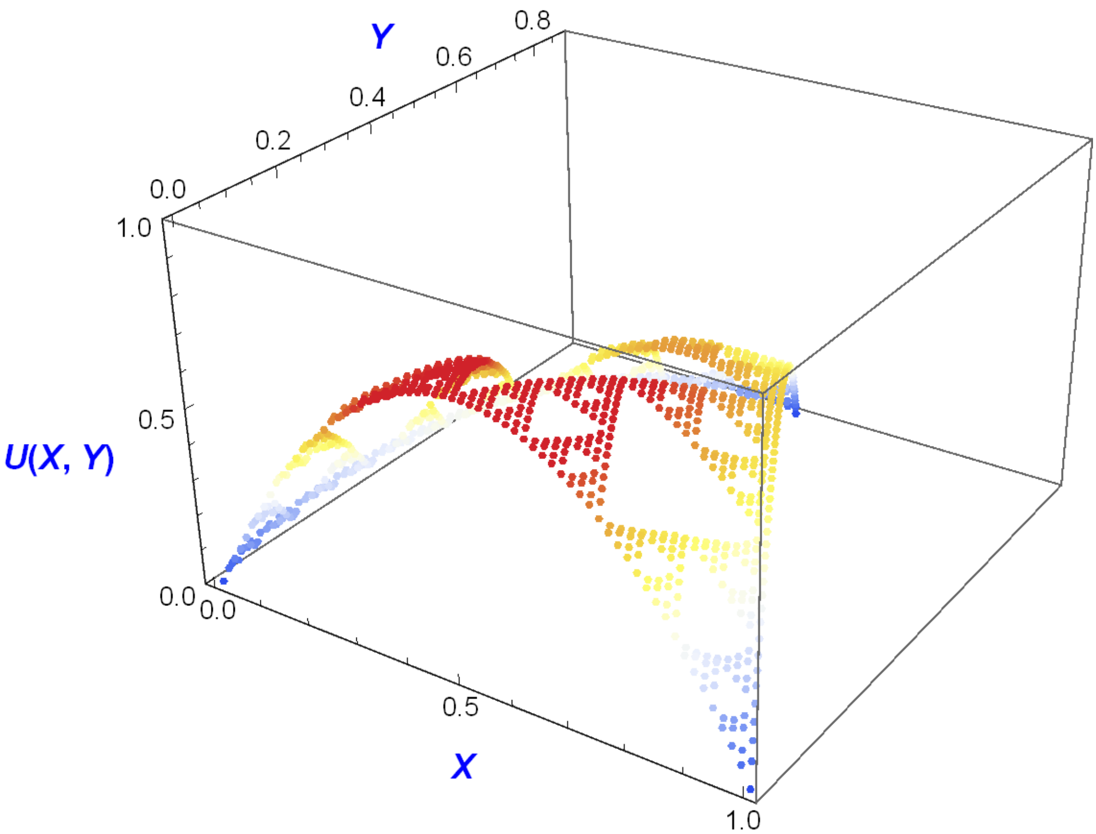

Figure 21 – The graph of the approached solution for $T = 1$, $k = 10$, $m = 6$.

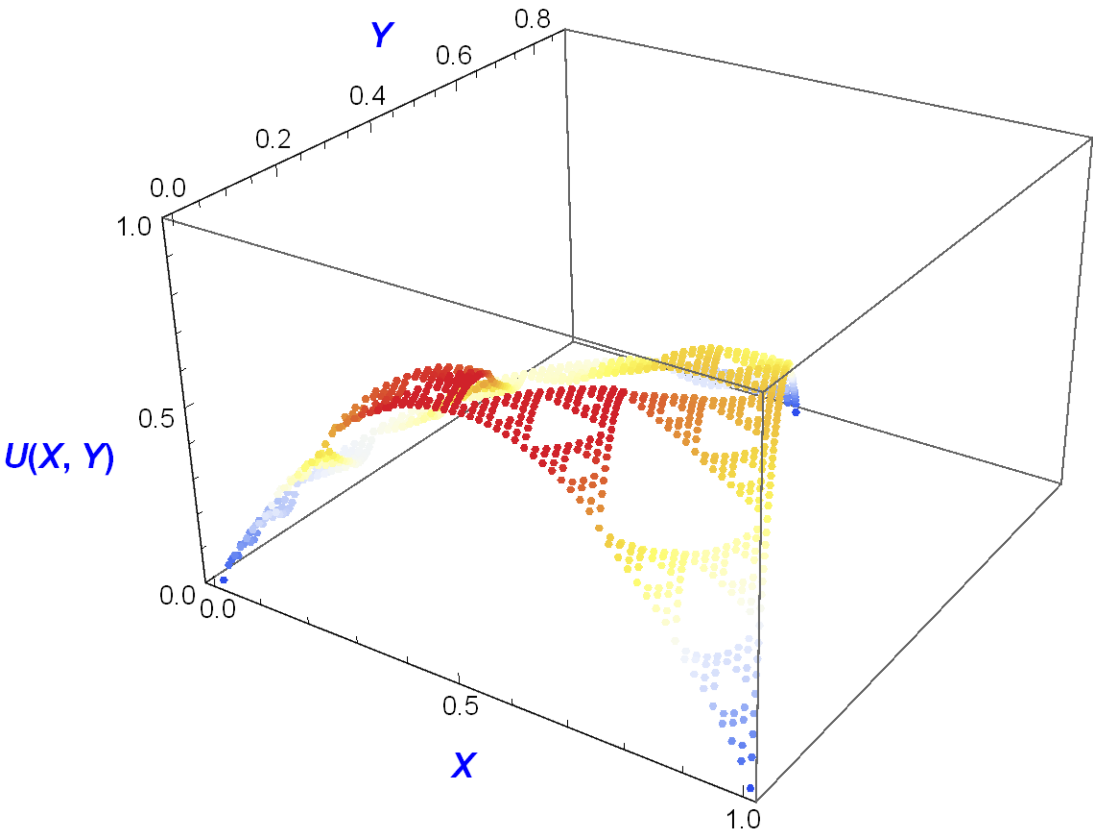

Figure 22 – The graph of the approached solution for $T = 1$, $k = 100$, $m = 6$.

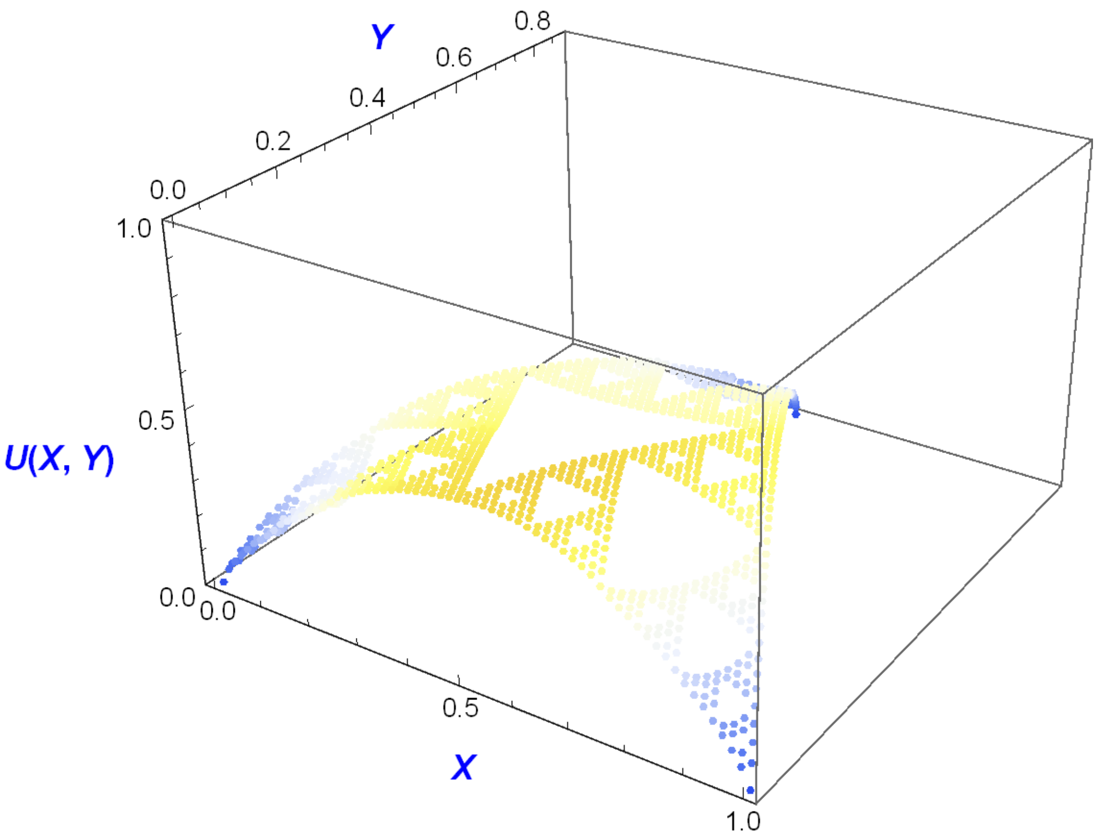

Figure 23 – The graph of the approached solution for $T = 1$, $k = 1000$, $m = 6$.

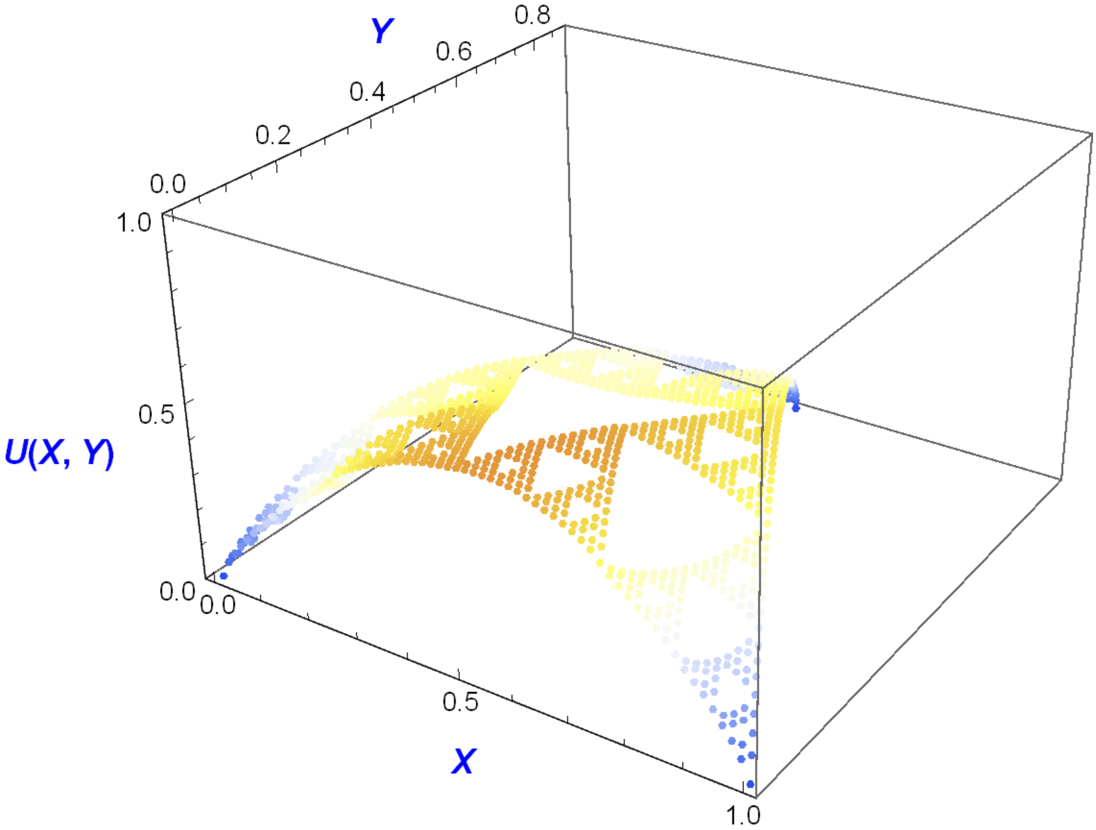

Figure 24 – The graph of the approached solution for $T = 1$, $k = 10000$, $m = 6$.

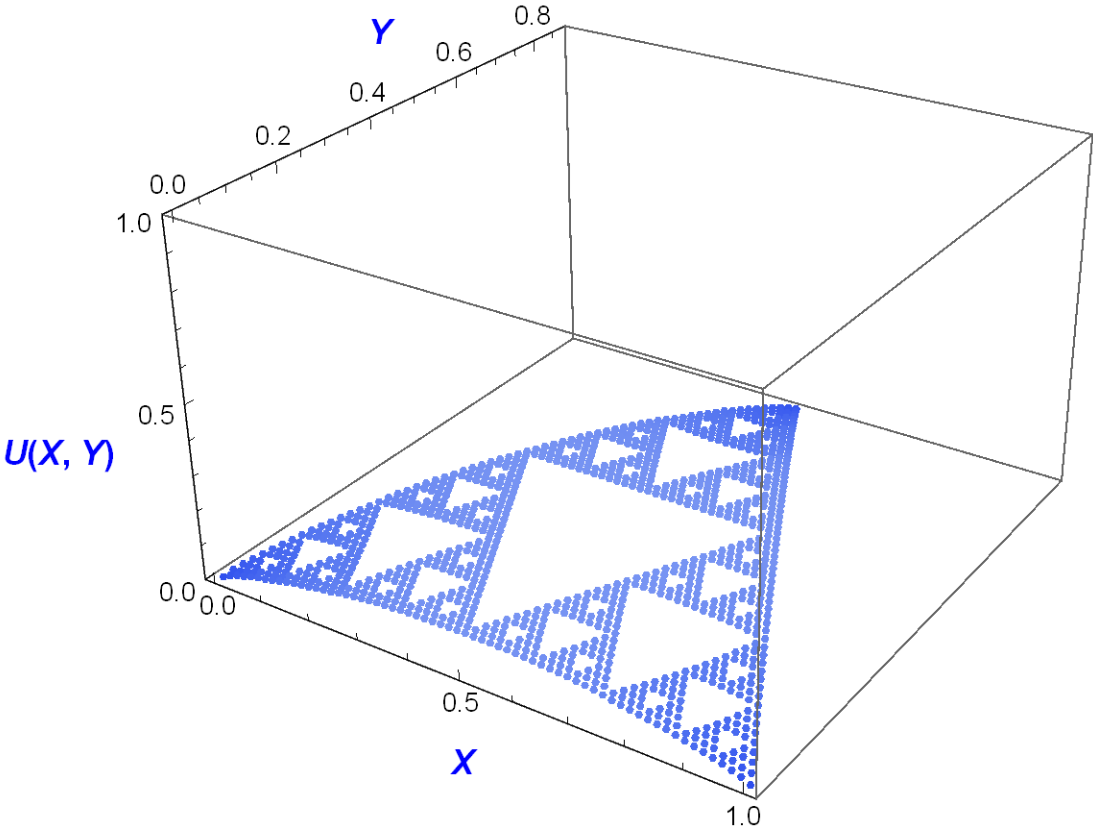

Figure 25 – The graph of the approached solution for $T = 1$, $k = 100000$, $m = 6$.

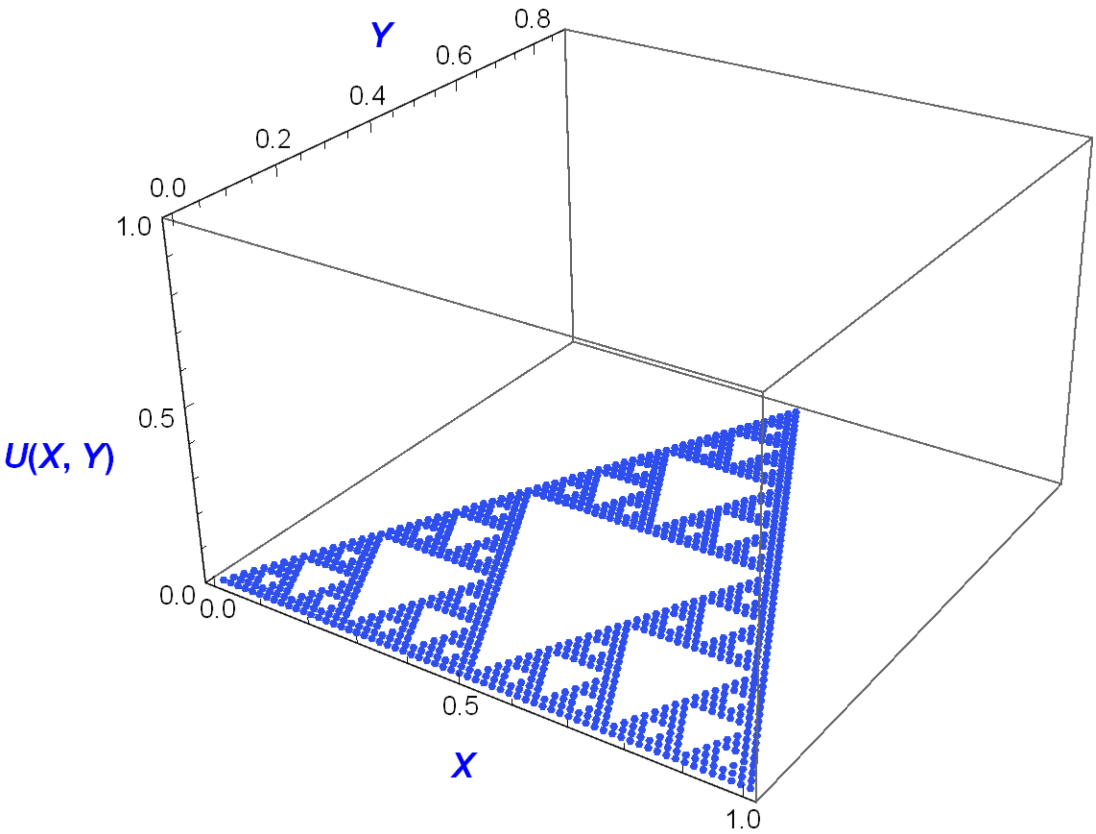

Figure 26 – The graph of the approached solution for $T = 1$, $k = 1000000$, $m = 6$.

5 Discussion

Propagation in both case appears as quite different, starting from the fact that the CFL stability condition is more demanding for the Curve than the Gasket. One can note a difference in the diffusion process, where the trail is off first in the neighborhood of the boundary V_0', and of the point P_3 at the end, where it reaches the zero value. In the case of the Gasket, the boundary $V_0 \neq V_0'$ includes the point P_3.

At $T = 1$, the situation is the same for the both cases, and the solution reaches the zero value. It is important to remark that the diffusion process in the curve case follows a snake pattern, resulting from the chain structure, contrary to the Gasket, where every points has four neighbours.

To go further and understand the difference between both processes, let us recall the **Einstein relation** [?], between the walk dimension D_W, the Hausdorff dimension D_H, and the spectral dimension D_S:

$$D_H = \frac{D_S \, D_W}{2}$$

where:

$$D_S = \frac{2 \, \ln N}{\ln(N \times \rho)}$$

and where N denotes the number of initial points, ρ being the energy scaling factor.

One has:

$$\begin{array}{rclcrcl} D_H\left(\mathfrak{S}\mathfrak{G}\right) & = & \dfrac{\ln(3)}{\ln(2)} & , & D_H\left(\mathfrak{S}\mathfrak{A}\mathfrak{C}\right) & = & \dfrac{\ln(3)}{\ln(2)} \\ D_S\left(\mathfrak{S}\mathfrak{G}\right) & = & \dfrac{2\ln(3)}{\ln(5)} & , & D_S\left(\mathfrak{S}\mathfrak{A}\mathfrak{C}\right) & = & 1 \\ D_W\left(\mathfrak{S}\mathfrak{G}\right) & = & \dfrac{\ln(5)}{\ln(2)} & , & D_W\left(\mathfrak{S}\mathfrak{A}\mathfrak{C}\right) & = & \dfrac{2\ln(3)}{\ln(2)} \end{array}$$

The walk dimension, which describes the time-space-scaling of a random walk on the set, is higher in the case of the Arrowhead Curve (the mean exit time from balls is bigger).

As for the spectral dimension, which describes the eigenvalue counting function of the Laplacian, it is of course different in both cases. It reflects the fact that spectral asymptotics on fractals does not only depend on the Hausdorff dimension (geometry), but also on the topology (ramification properties).

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