

The work of Lewis Bowen on the entropy theory of non-amenable group actions

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1 Introduction

We present the achievements of Lewis Bowen, or, more precisely, his breakthrough works after which a theory started to develop. The focus will therefore be made here on the isomorphism problem for Bernoulli actions of countable non amenable groups which he solved brilliantly in two remarkable papers. Here two invariants were introduced, which led to many developments.

2 The f-invariant

Definition 2.1. *Given a countable group G and a measure space (X, \mathcal{A}, μ) an action of G is a mapping $(X \times G) \rightarrow X$, $(x, g) \rightarrow \tau_g(x)$, where τ_g is a measure preserving transformation for every $g \in G$ and $\tau_{gh} = \tau_g \tau_h$. Given a discrete probability space (P, π) , the Bernoulli action is defined on the product space $X = P^G$ equipped with the measure $\mu = \pi^{\otimes G}$ in the following way:*

for $x \in X$, $x = (x_g)$, $g \in G$, and $h \in G$, $(\tau_h(x))_g = (x_{h^{-1}g})$. We shall use the notation $B(\pi, G)$ to mean the system $(X, \mathcal{A}, \mu, G, \tau)$ which has just been constructed. We call P the partition corresponding to the coordinate 0 in X .

We recall that if $\pi = (p_1, p_2, \dots, p_k)$,

$$h(\pi) = \sum_{i=1}^{i=k} p_i \log p_i$$

the entropy of the distribution π , which was first introduced by Shannon.

We introduce the essential notion of isomorphism.

Definition 2.2. *We say that two actions $(X, \mathcal{A}, \mu, G, \tau)$. and $(Y, \mathcal{B}, \nu, G, \sigma)$ of the same group G are isomorphic if there exists an invertible measure preserving mapping Φ from (X, \mathcal{A}, μ) to (Y, \mathcal{B}, ν) such that*

$$\sigma_g \Phi = \Phi \tau_g$$

for almost every $x \in X$, for all $g \in G$.

In case the mapping Φ is not invertible, we say that $(Y, \mathcal{B}, \nu, G, \sigma)$ is a factor of $(X, \mathcal{A}, \mu, G, \tau)$.

The theory started, for \mathbb{Z} , with the proof, by Kolmogorov and Sinai, that $B(\pi_1, \mathbb{Z})$ and $B(\pi_2, \mathbb{Z})$ were not isomorphic when $h(\pi_1) \neq h(\pi_2)$.

Then D. Ornstein proved that if $h(\pi_1) = h(\pi_2)$, $B(\pi_1, \mathbb{Z})$ is isomorphic to $B(\pi_2, \mathbb{Z})$.

For G a countable amenable group, J Kieffer proved that, in the same way as what was taking place for \mathbb{Z} , $B(\pi_1, G)$ and $B(\pi_2, G)$ are not isomorphic when $h(\pi_1) \neq h(\pi_2)$. Ornstein and Weiss proved later, in complete analogy with the \mathbb{Z} case, that two Bernoulli actions of the same amenable group G with the same entropies were isomorphic.

In the paper where they developed the isomorphism theory for amenable groups, considering Bernoulli actions, D. Ornstein and B. Weiss [24] asked the following: What happens for non amenable groups? At the same time they provided a famous example showing that $B(1/4, 1/4, 1/4, 1/4, F_2)$ is a factor of $B(1/2, 1/2, F_2)$. Here F_2 is the free group on two generators. In the classical theory, entropy decreases under the taking of factors, but here, $h(1/4, 1/4, 1/4, 1/4) > h(1/2, 1/2)$. This example happened to be for many years a deterrent to serious investigations, as it meant that there could not be a "good entropy theory" for actions of the free group. In the positive direction a nice result of Stepin [31], was that, with the only condition that G contains as a subgroup a copy of \mathbb{Z} , if $h(\pi_1) = h(\pi_2)$, then $B(\pi_1, G)$ is isomorphic to $B(\pi_2, G)$. However the question whether this condition was necessary for the isomorphism of Bernoulli actions remained open until 2010 when Lewis Bowen proved [5]:

Theorem 2.3. *Consider G a free group on finitely many generators and the Bernoullis $B(\pi_1, G)$ and $B(\pi_2, G)$. They are isomorphic if and only if $h(\pi_1) = h(\pi_2)$.*

The "if" part is clearly a consequence of the Stepin theorem. The strategy of the proof of the "only if" part of the theorem follows, in a way, the same path as the historical approach of Kolmogorov: also relying on Shannon's entropy, it is to first define a numerical isomorphism invariant for general actions of the free group, and then to compute it for Bernoulli shifts.

Definition 2.4. *Given an action $(X, \mathcal{A}, \mu, G, \tau)$ as in the previous definition, a measurable partition P of X with countably many atoms is said to be a generating partition, or to be a generator, if the smallest σ -algebra which contains all the images $\tau_g(P)$ of P for all the elements $g \in G$ is \mathcal{A} .*

Definition 2.5. *For a partition P of the space (X, \mathcal{A}, μ) , with π corresponding to the distribution of its atoms, $h(P) = h(\pi)$*

Let us note that it follows immediately from the Definition 2.1. of a Bernoulli action that the family of partitions $\tau_h(P)$, $h \in G$, is independent. We call then P an independent generator.

The f-invariant of Lewis Bowen, is defined as follows. Given an action σ of F_k , the free group on k generators, on (X, \mathcal{A}, m) with a generating partition P , let

$$F(P) = \sum_{i=1}^{i=k} \left(h(P \vee \sigma_{s_i} P) - 2h(P) \right) + h(P)$$

where the s_i , $1 \leq i \leq k$ are the generators of the group F_k .

Let B_n be the ball of radius n with respect to the word metric associated to the generators of F_k and let

$$P^n = \bigvee_{g \in B_n} \sigma_g P$$

Let then

$$f(P) = \inf_n F(P^n)$$

The theorem 2.3. is therefore a consequence of the following (Bowen [5])

Theorem 2.6. *With the above definitions:*

(1) *Given an action of F_k on (X, \mathcal{A}, m) , for any two finite generating partitions P and Q , $f(P) = f(Q)$.*

(2) *For a Bernoulli shift with independent generator P , $f(P) = h(P)$.*

The proof of (1) given by Lewis Bowen rests on a beautiful and unconventional coding argument.

The conclusion (2) is a consequence of the fact, which rests on a simple entropy computation, that $F(P^n) = h(P)$ for all n .

There is another equivalent way to present the f-invariant. Defining

$$F_*(P) = (1 - k)h(P) + \sum_{i=1}^{i=k} h(P, \sigma_{s_i})$$

where $h(P, \sigma_{s_i})$ is the classical entropy of the action of σ_{s_i} on P and letting, as before, $f_*(P) = \inf_n F_*(P^n)$,

$$f_*(P) = f(P)$$

Therefore, clearly, when $k = 1$ the f-invariant is exactly the usual entropy of the corresponding \mathbb{Z} action. As Lewis Bowen indicates himself, the f-invariant can be thought of as the limit of the sums of the entropies in the s_i directions applied to suitable "n step Markov approximations" of the P process generated by the free group action. The f-invariant can be computed for Markov chains; however, it can increase under the taking of factors and can take negative values (just consider $\tau = Id$ acting on the space of the Bernoulli $B(\pi_1, F_d)$, then $f(Id) = -(d - 1)h(\pi)$).

There are nevertheless several interesting features of the f-invariant which show that it shares a lot, functorially, with the usual entropy theory of \mathbb{Z} actions.

- (1) the f-invariant possesses a satisfactory relative version.
- (2) the f-invariant satisfies an ergodic decomposition formula

3 The sofic entropy

It is quite remarkable that, almost simultaneously, Lewis Bowen devised another invariant with which he could solve the same questions for a much larger class of groups than the ones which he had considered at first.

As an essential tool lies the noticeable and important fact, due to M. Gromov [14], that, for a wide class of non amenable groups, there is a way, given any finite set Γ in G , to produce an "almost faithful" representation of Γ in the set S_N of permutations of the interval $[1, N]$. A closely related property had been defined by E. Gordon and A. Vershik [13].

We equip the interval $[1, N]$ of integers between 1 and N with the normalized counting measure ν . We also say that, in a probability measure space, a property holds ϵ almost everywhere if it holds on a set measure $> 1 - \epsilon$.

To be precise,

Definition 3.1. *A group G is sofic if given ϵ , given Γ a finite set in G , there is N and an injective mapping τ from Γ to S_N , $\gamma \rightarrow \tau_\gamma$ such that for all γ_1 and γ_2 in Γ such that $\gamma_1\gamma_2$ stays in Γ , $\tau_{\gamma_1\gamma_2}(x) = \tau_{\gamma_1}(\tau_{\gamma_2}(x))$ for ϵ almost every $x \in [1, N]$ together with the fact that for all x , the cardinality of its τ_Γ orbit is ϵ close to the cardinality of Γ , that is,*

$$1/|\Gamma| \times \left(|\Gamma| - |\tau_\Gamma(x)| \right) < \epsilon.$$

We shall refer to the preceding objects as sofic approximations. This terminology was introduced by B. Weiss. Let us note that the Γ action defined before is ϵ -measure preserving, meaning that for all A outside of a set of measure $< \epsilon$, $\left| \frac{\nu(\tau_\gamma(A))}{\nu(A)} - 1 \right| < \epsilon$

Clearly \mathbb{Z} is sofic: Consider $\Gamma = [1, K]$. Given ϵ , for all N large enough, and $k < K$, let $\tau_k(x) = x + k$ when $x < N - k$, and, $\tau_k(x) = x + k - N$ for $x \geq N - k$. In the same way, it is seen that amenable groups are also sofic: given Γ a finite subset of the amenable group G equipped with the Følner sequence A_n , and $\epsilon > 0$, extend in any way to the whole of A_n the action of Γ on $A_n \cap \Gamma A_n$ with n sufficiently large to imply that $|\Gamma A_n \Delta A_n| / |A_n| < \epsilon$. Residually finite groups, and in particular free groups, are sofic, as was shown by Gromov.

We are now given an action σ of a countable sofic group G on a Lebesgue space (X, \mathcal{A}, m) equipped with a finite generating partition P . Let Γ be a finite subset of G . By P^Γ we mean the partition of X which is spanned by the $\sigma_g(P)$, $g \in \Gamma$. Given a sofic approximation τ associated to Γ and N , a partition P' of $[1, N]$ labelled in the same way as P is exactly a word ω of length N in the alphabet of P . If the Γ action is sufficiently close to be measure preserving, we can define with arbitrary accuracy the distribution of the span of the $\tau_g P'$, $g \in \Gamma$. We consider the weak topology distance between (P^Γ, m) and (P'^Γ, ν) (P'^Γ is the partition of $[1, N]$ which is the span of the $\tau_g P'$, $g \in \Gamma$): $d_\Gamma(P, P') = \sum_{p \in P^\Gamma} |m(p) - \nu(p')|$. (Here $\nu(p')$ is the measure of the atom of P'^Γ which has the same name as p given by the action of Γ by σ). The ω 's such that $d_\Gamma(P, P') < \alpha$ have been called by Lewis Bowen (P, Γ, N, α) microstates. Let $M_N(P, \Gamma, \alpha)$ be the cardinality of this set.

Clearly, when N , P and Γ are fixed, $M_N(P, \Gamma, \epsilon)$ decreases when ϵ decreases. Also $M_N(P, \Gamma, \epsilon)$ decreases when Γ increases. The exponential rate at which $M_N(P, \Gamma, \epsilon)$ grows with N when $\epsilon \downarrow 0$ and $\Gamma \uparrow G$ is the Bowen entropy. More precisely [6] :

Theorem 3.2. *Let G be a group which admits a sofic approximation and an action $(X, \mathcal{A}, m, \sigma_g)$ of G on X . Let P be a finite partition of X . Let $h_\Sigma(P, \sigma_g)$ be*

$$h_\Sigma(P, \sigma_g) = \inf_{\Gamma} \inf_{\epsilon > 0} \limsup_{N \rightarrow \infty} \frac{1}{N} \log M_N(P, \Gamma, \epsilon)$$

For any two finite generators P and Q , $h_\Sigma(P, \tau_g) = h_\Sigma(Q, \tau_g)$.

Proof. (in this sketch, we follow a presentation first developed by B. Weiss [32]). Let $\alpha = h_P - h_Q$. (h_P and h_Q are the two preceding values). We consider two generating partitions P and Q for the action σ_g . For all $u > 0$, there is a finite subset of G , Γ_u , such that $P^{\Gamma_u} \supset \tilde{Q}$ with $d(\tilde{Q}, Q) < u$. We say that P u codes Q and we write $P^{\Gamma_u} \supset^u Q$. Given any finite set $\Gamma \subset G$, there exist convenient finite sets Γ_1 and Γ_2 , numbers ϵ_1 with $h(2\epsilon_1) = \alpha/10$, ϵ_2 and associated codes such that $P^{\Gamma_2} \supset^{\epsilon_2} Q^{\Gamma_1} \supset^{\epsilon_1} P^\Gamma$ (by which we mean that P codes to \tilde{Q} such that $d_{\Gamma_1}(\tilde{Q}, Q) < \epsilon_2$ where ϵ_2 has been chosen so that \tilde{P} coded from \tilde{Q} satisfies $d_\Gamma(\tilde{P}, P) < \epsilon_1$).

Given a $(P, \Gamma_2, \epsilon_3, N)$ microstate ω where we choose ϵ_3 such that the successive codings of the associated partition P' of $[1, N]$ will produce first ω' and a corresponding partition Q' of $[1, N]$ such that Q'^{Γ_1} is still ϵ_2 close to Q^{Γ_1} (so that ω' will be a $(Q, \Gamma_1, \epsilon_2, N)$ microstate) and then ω'' with corresponding partition P'' with $P''^{\Gamma} \epsilon_1$ close to P^Γ , we see in particular that P'' will be ϵ_1 close to P' which implies that the corresponding P -names of length N are $2\epsilon_1$ close in \bar{d} which means that ω'' is $2\epsilon_1$ close to ω for the Hamming metric.

It is no restriction to assume that Γ_1 , Γ and ϵ_1 are also such such that $M_N(Q, \Gamma_1, \epsilon_2) < 2^{N(h_Q + \alpha/10)}$ and $M_N(P, \Gamma, \epsilon_1) > 2^{N(h_P - \alpha/10)}$. This is clearly impossible.

Note that what we have been doing is just to imitate in the approximate actions given by the sofic approximation and the microstates the relations between generators which were taking place in the original action. This was possible because only finitely many coordinates had to be considered. Lastly, the important fact was that in the approximate models, close partitions correspond to \bar{d} close names. ■

In fact, Lewis Bowen has extended his definition of the sofic entropy to countable generators which have finite entropy in the following way: let P be a countable partition, let P_n an increasing sequence of finite P measurable partitions which increase towards P . The sofic entropy is then:

$$h_{\Sigma}(P, \sigma_g) = \inf_{\Gamma} \inf_{P_n} \inf_{\epsilon > 0} \limsup_{N \rightarrow \infty} \frac{1}{N} \log M_N(P_n, \Gamma, \epsilon)$$

Here P_n vary among all possible sequences of finite partitions increasing towards P . With this definition, Bowen in fact proved the theorem 3.2. for generators with finite entropy.

There is a nice interpretation of the f-invariant as an average sofic entropy in the paper by Lewis Bowen. In the same way as it was for the f-invariant, there are groups for which the identity has sofic entropy $-\infty$. To complete the extension of the theorem 2.4. to sofic groups, Bowen has proved [6].

Theorem 3.3. *Consider a Bernoulli action $B(\pi, G)$ of the group G which admits a sofic approximation. We call P the partition which makes measurable the first coordinate of the Bernoulli action. Then,*

$$h_{\Sigma}(P, \tau_g) = h(\pi)$$

Proof. it is classical that, when considering the product measure on P^N , given any finite Γ , given ϵ , the (P, Γ, N, ϵ) microstates will fill a set of arbitrarily big measure, for N large enough. The MacMillan theorem therefore implies that the sofic entropy satisfies $h_{\Sigma}(P, \tau_g) \geq h(\pi)$. If the previous inequality were strict, we would get arbitrarily large N and an excessive number of microstates, every one of them, as a consequence of the assumption, generic for a distribution with one-marginal very close to π . Let τ be the distribution on names of length N of P giving equal mass to everyone of these microstates. If we build a stack of N levels above a Bernoulli with distribution τ (call S the corresponding transformation) and if we let \tilde{P} be the partition obtained by labelling the levels with the names of the microstates, we get that $\tilde{P} \vee \Phi$ will be a generator for S (Φ the two sets partition which is the base of the stack and its complement) which contradicts Abramov's formula (as the distribution of \tilde{P} is going to be as close as we want to π and $h(\Phi)$ can be made as small as we want provided N is sufficiently large). ■

Although there are presently no examples known of countable non amenable groups which are not sofic, there are several results which have already been shown to hold for all non amenable countable groups. In another spectacular paper, which adds to the weirdness of the entropy theory of actions of non amenable groups, Lewis Bowen has proved [11] :

Theorem 3.4. *Let Γ be a non amenable countable group, any two Bernoulli shifts over that group are weakly isomorphic.*

(Two actions of a group are weakly isomorphic if each one is a factor of the other).

(In a previous paper [8], L. Bowen had proved this result with the restriction that the acting group contained a copy of a free group).

At the opposite end, we have seen that amenable groups were sofic. It was proved by L. Bowen [10] and D. Kerr and H. Li [22], that the sofic entropy, when applied to amenable groups coincided with the classical entropy theory (initiated by Kieffer [23], in the general amenable group case). The proofs rely deeply on the Ornstein Weiss Rokhlin lemma for general countable amenable groups [25].

As a beautiful extension of the Stepin theorem, Lewis Bowen proved [9] that

Theorem 3.5. *Let Γ be a countable non amenable group. Let π_1 and π_2 be two probability distributions, both with at least 3 states, such that $h(\pi_1) = h(\pi_2)$, then $B(\pi_1, \Gamma)$ is isomorphic to $B(\pi_2, \Gamma)$.*

Brandon Seward [27] improved the same result and showed it to be true without the "3 states restriction". He also proved there that the isomorphism could be made finitary.

The world of Bernoulli actions of countable non amenable group actions brings a lot of surprises. D. Kerr [19] proved:

Theorem 3.6. *A Bernoulli action of a sofic group has completely positive sofic entropy. (That is, all its non trivial factors have positive sofic entropy for any sofic approximation)*

However S. Popa has proved [26]: (see also Tim Austin [3])

Theorem 3.7. *For some non amenable groups (namely for a class of groups which satisfy Kazhdan's property T) Bernoulli actions have factors which are not isomorphic to a Bernoulli action.*

4 The continuations of the theory

The theory initiated by Lewis Bowen has evolved in different directions. The first one is the extension of the definition of sofic measure theoretic entropy to the case where the action is defined on a topological space (that is without an a priori generator). The second one concerns the extensions of the sofic entropy to the topological sofic entropy and to the corresponding variational principles. All this was mainly worked out by D. Kerr and H. Li. [18], [21].

The third one is the appearance of an other invariant, first introduced by Brandon Seward [28], the Rokhlin entropy. For Bernoulli actions of sofic groups, the Rokhlin entropy coincides with the sofic entropy. Although it is only known in general to be greater than the sofic entropy, it has the advantage to be defined for all countable groups. It has been the source of important, beautiful and unexpected results. In particular, the following, due to Brandon Seward [30] which is really a Sinai theorem for non amenable group actions: Given a countable group action with positive Rokhlin entropy, it has as a factor a Bernoulli action (and, due to the theorem 3.4. above of Bowen, any Bernoulli action). Note that, due to the domination result of B. Seward, the same theorem holds under the assumption that the sofic entropy of the action is positive.

There are also nice implications related to spectral theory: if the Koopman representation associated to an action of Γ is singular with respect to the left regular representation, then the Rokhlin entropy (hence the sofic entropy) of the action is 0. This is due to B. Hayes [16] and B. Seward [29]. This is obtained through the definition of important new objects: the Pinsker algebra and the outer Pinsker algebra. See also Alpeev [1]. These are deep extensions to the non amenable case of the theorem which states, for \mathbb{Z} actions, that the spectrum, on the orthocomplement of L^2 of the Pinsker algebra is countable Lebesgue.

Another direction has been initiated by T. Austin [4]: to define invariants related to the sofic entropy which would behave well (that is be additive) under the operation of taking products.

Many examples have been the object of study: Markov actions, actions of algebraic origin [15], Gaussian actions [17]. T. Austin and P. Burton [2] have constructed uncountably many actions with the same entropy with completely positive sofic entropy, pairwise not isomorphic (and none of them being a factor of a Bernoulli shift).

Important questions remain in the theory. It is not known whether the

sofic entropy depends upon the choice of the sofic approximations. There are no examples known presently of groups which are not sofic. In case such exist, computing the Rokhlin entropy for their Bernoulli actions becomes a new challenge.

Lastly, I cannot resist to mention the recent work of Lewis Bowen [12]: "Sofic homological invariants and the weak Pinsker property", in which he produces, for sufficiently large r actions of F_r for which the weak Pinsker property fails (as well for the Rokhlin entropy as for any sofic entropy). This is in sharp contrast with the fact, recently proved by T. Austin, that, for \mathbb{Z} , the weak Pinsker property is universal.

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