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NONEXISTENCE OF LEVI FLAT HYPERSURFACES WITH
POSITIVE NORMAL BUNDLE IN COMPACT KähLER
MANIFOLDS OF DIMENSION $\geq 3$

SÉVERINE BIARD AND ANDREI IORDAN

In memory of Gennadi M. Henkin

Abstract. Let $X$ be a compact connected Kähler manifold of dimension $\geq 3$ and $L$ a $C^\infty$ Levi flat hypersurface in $X$. Then the normal bundle to the Levi foliation does not admit a Hermitian metric with positive curvature along the leaves. This represents an answer to a conjecture of Marco Brunella.

1. Introduction

A classical theorem of Poincaré-Bendixson [28], [29], [5] states that every leaf of a foliation of the real projective plane accumulates on a compact leaf or on a singularity of the foliation. As a holomorphic foliation $\mathcal{F}$ of codimension 1 of $\mathbb{CP}_n$, $n \geq 2$, does not contain any compact leaf and its singular set $\text{Sing } \mathcal{F}$ is not empty, a major problem in foliation theory is the following: can $\mathcal{F}$ contain a leaf $F$ such that $\mathcal{F} \cap \text{Sing } \mathcal{F} = \emptyset$? If this is the case, then there exists a nonempty compact set $K$ called exceptional minimal, invariant by $\mathcal{F}$ and minimal for the inclusion such that $K \cap \text{Sing } \mathcal{F} = \emptyset$. The problem of the existence of an exceptional minimal in $\mathbb{CP}_n$, $n \geq 2$ is implicit in [12].

In [13] D. Cerveau proved a dichotomy under the hypothesis of the existence of a holomorphic foliation $\mathcal{F}$ of codimension 1 of $\mathbb{CP}_n$ which admits an exceptional minimal $\mathfrak{M}$: $\mathfrak{M}$ is a real analytic Levi flat hypersurface in $\mathbb{CP}_n$ (i.e. $T(\mathfrak{M}) \cap JT(\mathfrak{M})$ is integrable, where $J$ is the complex structure of $\mathbb{CP}_n$), or there exists $p \in \mathfrak{M}$ such that the leaf through $p$ has a hyperbolic holonomy and the range of the holonomy morphism is a linearisable abelian group. This gave rise to the conjecture of the nonexistence of smooth Levi flat hypersurface in $\mathbb{CP}_n$, $n \geq 2$.

The conjecture was proved for $n \geq 3$ by A. Lins Neto [22] for real analytic Levi flat hypersurfaces and by Y.-T. Siu [31] for $C^{12}$ smooth Levi flat hypersurfaces. The methods of proofs for the real analytic case are very different from the smooth case.

A real hypersurface of class $C^2$ in a complex manifold is Levi flat if its Levi form vanishes or equivalently, it admits a foliation by complex hypersurfaces. We say that a (non-necessarily smooth) real hypersurface $L$ in a complex manifold $X$ is Levi flat if $X \setminus L$ is pseudoconvex. An example of (non-smooth) Levi flat hypersurface in
$\mathbb{CP}^2$ is $L = \{(z_0, z_1, z_2) : |z_1| = |z_2|\}$, where $[z_0, z_1, z_2]$ are homogeneous coordinates in $\mathbb{CP}^2$ (see [19]).

In [21] Iordan and Mattthey proved the nonexistence of Lipschitz Levi flat hypersurfaces in $\mathbb{CP}^n$, $n \geq 3$, which are of Sobolev class $W^s$, $s > 9/2$. A principal element of the proof is that the Fubini-Study metric induces a metric of positive curvature on any quotient of the tangent space.

Nonexistence questions for the Levi flat hypersurfaces in compact Kähler manifolds were first discussed by T. Ohsawa in [24], who proved the nonexistence of real-analytic Levi flat hypersurfaces with Stein complement in compact Kähler manifolds of dimension $\geq 3$.

In [9], M. Brunella proved that the normal bundle to the Levi foliation of a closed real analytic Levi flat hypersurface in a compact Kähler manifold of dimension $n \geq 3$ does not admit any Hermitian metric with leafwise positive curvature. The real analytic hypothesis may be relaxed to the assumption of $C^{2,\alpha}$, $0 < \alpha < 1$, such that the Levi foliation extends to a holomorphic foliation in a neighborhood of the hypersurface.

The main step in his proof is to show that the existence of a Hermitian metric with leafwise positive curvature on the normal bundle to the Levi foliation of a compact Levi flat hypersurface $L$ in a Hermitian manifold $X$, implies that $X \setminus L$ is strongly pseudoconvex, i.e. there exists on $X \setminus L$ an exhaustion function which is strongly plurisubharmonic outside a compact set. This was generalized in [10] for invariant compact subsets of a holomorphic foliation of codimension one. Of course, if $X$ is the complex projective space, then every proper pseudoconvex domain in $X$ is Stein [34].

Brunella stated also the following conjecture [9]: Let $X$ be a compact connected Kähler manifold of dimension $n \geq 3$ and $L$ a $C^\infty$ compact Levi flat hypersurface in $X$. Then the normal bundle to the Levi foliation does not admit any Hermitian metric with leafwise positive curvature.

The assumption $n \geq 3$ is necessary in this conjecture (see Example 4.2 of [9]).

In [11] Brunella and Perrone proved that every leaf of a holomorphic foliation $\mathcal{F}$ of codimension one of a projective manifold $X$ of dimension at least 3 and such that $\text{Pic}(X) = \mathbb{Z}$ accumulates on the singular set of the foliation. In this case the normal bundle to the foliation is ample.

In [25], T. Ohsawa considered a $C^\infty$ Levi flat compact hypersurface $L$ in a compact Kähler manifold $X$ such that the normal bundle to the Levi foliation admits a fiber metric whose curvature is semipositive of rank $\geq k$ on the holomorphic tangent space to the leaves and proved that $X \setminus L$ admits an exhaustion plurisubharmonic function of logarithmic growth which is strictly $(n-k)$-convex. Then, if $\dim X \geq 3$, he proved that there are no Levi flat real analytic hypersurfaces such that the normal bundle to the Levi foliation admits a fiber metric whose curvature is semipositive of rank $\geq 2$ on the holomorphic tangent space to $L$. Some possibilities for generalization in the smooth case are also indicated.

In this paper we solve the above mentioned conjecture of Brunella for compact connected Kähler manifolds of dimension $n \geq 3$. The principal ingredient of the proof is a refinement of the proof of Brunella [9] of the strong pseudoconvexity of $X \setminus L$ : we show that there exist a neighborhood $U$ of $L$ and a function $v$ on $U$ vanishing on $L$, such that $-i\partial\overline{\partial}\ln v \geq \omega$ on $U \setminus L$, where $c > 0$ and $\omega$ is the $(1,1)$-form associated to the Kähler metric. Then we use the $L^2$ estimates [2], [1], [20],
[15] for the weighted \( \overline{\partial} \)-equation on \((n, q)\)-forms on \(X \setminus L\) endowed with a complete Kähler metric. These estimates together with the lower uniform boundedness of the eigenvalues of the Levi form and a duality method developed in [19], allow us to solve the \( \overline{\partial} \)-equation with compact support for \((0, q)\)-forms, \(1 \leq q \leq n - 1\), and this leads in dimensions \(\geq 3\) to the solution of Brunella’s conjecture.

2. Preliminaries

Let \(X\) be a complex \(n\)-dimensional manifold, \(\omega\) a Kähler metric on \(X\), \(\Omega\) a domain in \(X\) and \(\sigma\) a positive function on \(\Omega\). For \(\alpha \in \mathbb{R}\) denote

\[
L^2_{(p, q)}(\Omega, \sigma^\alpha, \omega) = \left\{ f \in L^2_{(p, q)\text{loc}}(\Omega) : \int_{\Omega} |f|^2 \sigma^{2\alpha} dV_\omega < \infty \right\}
\]

endowed with the norm

\[
N_{\alpha, \omega, \sigma}(f) = \left( \int_{\Omega} |f|^2 \sigma^{2\alpha} dV_\omega \right)^{1/2}.
\]

Let \(\Omega\) be a pseudoconvex domain in \(\mathbb{CP}_n\) and \(\delta_{\partial \Omega}\) the geodesic distance to the boundary for the Fubini-Study metric \(\omega_{FS}\). By using the \(L^2\) estimates for the \(\overline{\partial}\)-operator of Hörmander with the weight \(e^{-\varphi}\), \(\varphi = -\alpha \log \delta_{\partial \Omega}\) which is strongly plurisubharmonic by a theorem of Takeuchi [34], Henkin and Iordan proved in [19] the existence and regularity of the \(\overline{\partial}\) equation for \(\overline{\partial}\)-closed forms in \(L^2_{(p, q)}(\Omega, \delta_{\partial \Omega}^{-\alpha}, \omega_{FS})\) verifying the moment condition. This gives the regularity of the \(\overline{\partial}\)-operator in pseudoconcave domains with Lipschitz boundary [19] and, by using a method of Siu [31], [32], the nonexistence of smooth Levi flat hypersurfaces in \(\mathbb{CP}_n\), \(n \geq 3\) follows (see [21]). These techniques will be used in the 4th and the 5th paragraph.

We will use also the following theorem of regularity of \(\overline{\partial}\) equation of Brinkschulte [7]:

**Theorem 1.** Let \(\Omega\) be a relatively compact domain with Lipschitz boundary in a Kähler manifold \((X, \omega)\) and set \(\delta_{\partial \Omega}\) the geodesic distance to the boundary of \(\Omega\). Let \(f \in L^2_{(p, q)}(\Omega, \delta_{\partial \Omega}^{-k}, \omega) \cap C^\infty_{(p, q)}(\Omega) \cap C^\infty(\overline{\Omega})\), \(q \geq 1\), \(k \in \mathbb{N}\) and \(u \in L^2_{(p, q - 1)}(\Omega, \delta_{\partial \Omega}^{-k}, \omega)\) such that \(\overline{\partial}u = f\) and \(\overline{\partial}^{-k}u = 0\), where \(\overline{\partial}^{-k}\) is the Hilbert space adjoint of the unbounded operator \(\overline{\partial}^{-k} : L^2_{(p, q - 1)}(\Omega, \delta_{\partial \Omega}^{-k}, \omega) \to L^2_{(p, q)}(\Omega, \delta_{\partial \Omega}^{-k}, \omega)\). Then for \(k\) big enough \(u \in C^s(k) (\overline{\Omega})\) where \(s(k) \sim \sqrt{k}\).

3. Strong pseudoconvexity of the complement of a Levi flat hypersurface

Let \(L\) be a smooth Levi flat hypersurface in a Hermitian manifold \(X\). As was mentioned in [9] and in [25], by taking a double covering, we can assume that \(L\) is orientable and the complement of \(L\) has two connected components in a neighborhood of \(L\). This will be always supposed in the sequel and for an open neighborhood \(U\) of \(L\) we will denote by \(U^+\) and \(U^-\) the two connected components of \(U \setminus L\). We will denote by \(\delta_L\) the signed geodesic distance to \(L\).

In [9] Brunella proved that the complement of a closed Levi flat hypersurface in a compact Hermitian manifold of class \(C^{2, \alpha}\), \(0 < \alpha < 1\), having the property that the Levi foliation extends to a holomorphic foliation in a neighborhood of \(L\) and the normal bundle to the Levi foliation admits a \(C^2\) Hermitian metric with leafwise positive curvature is strongly pseudoconvex, i.e. there exists an exhaustion
function which is strongly plurisubharmonic outside a compact set. The following proposition strengthens this result:

**Proposition 1.** Let $L$ be a compact $C^3$ Levi flat hypersurface in a Hermitian manifold $X$ of dimension $n \geq 2$, such that the normal bundle $\mathcal{N}_L^{1,0}$ to the Levi foliation admits a $C^2$ Hermitian metric with leafwise positive curvature. Then there exist a neighborhood $U$ of $L$, $c > 0$ and a non-negative function $v \in C^2(U)$, vanishing on $L$ and positive on $U \setminus L$ such that $-i\partial\bar{\partial} \ln v \geq \omega$ on $U \setminus L$, where $\omega$ is the $(1,1)$-form associated to the metric. Moreover, there exists a nonvanishing continuous function $g$ in a neighborhood of $L$ such that $v = g\delta_L^1$.

**Proof.** Let $z_0 \in L$. There exist holomorphic coordinates $z = (z_1, \ldots, z_{n-1}, z_n) = (z', z_n)$ in a neighborhood of $z_0$ such that the local parametric equations for $L$ are of the form

$$z_j = w_j, \quad j = 1, \ldots, n-1, \quad z_n = \varphi(w', t)$$

where $\varphi$ is of class $C^3$ (see [4]) on a neighborhood of the origin in $\mathbb{C}^{n-1} \times \mathbb{R}$, holomorphic in $w'$ and $\partial_{\varphi} \ln (z_0) \in \mathbb{R}$. We consider a $C^3$ extension $\psi = (\psi_1, \ldots, \psi_n)$ of $\varphi$ on a neighborhood of the origin in $\mathbb{C}^{n-1} \times \mathbb{C}$, $\psi(w', t + is) = (w', \varphi(w', t) + is)$. Then $\psi$ is a $C^3$ diffeomorphism in a neighborhood of $z_0$ and holomorphic in $w'$. It follows that

$$L = \{(z', z_n) : \rho(z', z_n) = 0\},$$

where $\rho = \text{Im} \left( \psi^{-1} \right)_n$. We denote $f = (\psi^{-1})_n(z', z_n)$. Since $\partial_{\bar{\partial}} f = 0$ on $L$, where $\partial_{\bar{\partial}}$ is the tangential Cauchy-Riemann operator on $L$, there exists an extension $\tilde{f}$ of class $C^3$ in a neighborhood of $z_0$ such that $\partial_{\bar{\partial}} \tilde{f}$ vanishes to order greater than 2 on $L$, i.e. $D^l \partial_{\bar{\partial}} \tilde{f} = 0$ for $|l| \leq 2$ on $L$.

So there exists an open finite covering $\{\tilde{U}_j\}_{j \in J}$ by holomorphic charts of $L$ such that $\tilde{U}_j \setminus L = \tilde{U}_j^+ \cup \tilde{U}_j^-$ such that $U_j = L \cap \tilde{U}_j = \{z \in \tilde{U}_j : \text{Im} \tilde{f}_j = 0\}$, where $\partial_{\bar{\partial}} \tilde{f}_j$ vanishes to order greater than 2 on $L$ and the Levi foliation is given on $U_j$ by $\{z \in U_j : \tilde{f}_j (z) = c_j\}$, $c_j \in \mathbb{R}$. Thus $d\tilde{f}_j = \partial_{\bar{\partial}} \tilde{f}_j$ is a nonvanishing section of $\mathcal{N}_L^{1,0}$ on $U_j$ and by shrinking $\tilde{U}_j$, we may consider that $d\tilde{f}_j \neq 0$ on $\tilde{U}_j$.

We may suppose that $\mathcal{N}_L^{1,0}$ is represented by a cocycle $\{g_{jk}\}$ of class $C^2$ subordinated to the covering $\{(U_j)_{j \in J}\}$ and there exist closed $(1,0)$-forms $\alpha_j$ of class $C^2$ on $U_j$ holomorphic along the leaves such that $T^{1,0}(U_j) = \ker \alpha_j$ for every $j \in J$ and $\alpha_j = g_{jk} \alpha_k$ on $U_j \cap U_k$. So $\{\alpha_j\}_{j \in J}$ defines a global form $\alpha$ on $L$ with values in $\mathcal{N}_L^{1,0}$ such that locally on $U_j$ we have $\alpha(z) = \alpha_j(z) \otimes \alpha_j^*(z)$ where $\alpha_j^*$ is the dual frame of $\alpha_j$. In particular we have $\alpha_k^* = g_{jk} \alpha_j^*$.

Let $h$ be a $C^2$ Hermitian metric with positive leafwise curvature $\Theta_h \left( \mathcal{N}_L^{1,0} \right)$ on $\mathcal{N}_L^{1,0}$. $h$ is defined on each $U_j$ by a $C^2$ function $h_j = |\alpha_j^*|^2$ such that $h_k = |g_{jk}|^2 h_j$ on $U_j \cap U_k$.

Since $\alpha_j = \eta_j d\tilde{f}_j$ on $U_j$ for every $j$, where $\eta_j$ are nowhere vanishing functions of class $C^2$ on $U_j$ holomorphic along the leaves and

$$\frac{1}{\eta_k} \left( d\tilde{f}_k \right)^* = \frac{1}{\eta_j} g_{jk} \left( d\tilde{f}_j \right)^*$$
On $U_j \cap U_k$, it follows that
\[ |g_{jk}(z)|^2 = \left| \frac{\eta_j(z)}{\eta_k(z)} \right|^2 \left| \frac{df_k}{df_j} \right|^2 = \frac{h_k(\bar{z})}{h_j(\bar{z})}, \quad z \in U_j \cap U_k. \]

So
\[ h_j |\eta_j|^2 \left( \text{Im} \bar{f}_j \right)^2 - h_k |\eta_k|^2 \left( \text{Im} \bar{f}_k \right)^2 \]
vanishes to order greater than 2 on $U_j \cap U_k$ and \( \left( h_j |\eta_j|^2 \left( \text{Im} \bar{f}_j \right)^2 \right) \) defines a jet of order 2 on $L$. By Whitney extension theorem there exists a $C^2$ function $v$ on $X$ such that $v - h_j |\eta_j|^2 \left( \text{Im} \bar{f}_j \right)^2$ vanishes to order 2 on $U_j$ for every $j \in J$. Let $\tilde{\eta}_j, \tilde{h}_j$ be $C^2$ extensions of $\eta_j, h_j$ on $\tilde{U}_j$ and set $\tilde{\alpha}_j = \tilde{\eta}_j d \tilde{f}_j$, $\tilde{v} = \tilde{h}_j |\eta_j|^2 \left( \text{Im} \bar{f}_j \right)^2$.

For $z \in \tilde{U}_j$ denote $E'_z = \{ V' \in T^{1,0}_z(X) : \langle \partial \text{Im} \bar{f}_j, V' \rangle = 0 \}$ and $E''_z$ the orthogonal of $E'_z$ in $T^{1,0}_z(X)$. Then for every $V \in T^{1,0}_z(X)$ there exists $V' \in E'_z, V'' \in E''_z$ such that $V = V' + V''$. The curvature form $\Theta \left( \Lambda^{1,0}_{\omega} \right)$ is represented by $-i\partial \bar{\partial} \ln \left( h_j |\alpha_j|^2 \right)$ on $U_j$, so by shrinking $\tilde{U}_j$ we may suppose that there exists $\beta > 0$ such that $\left( -i\partial \bar{\partial} \ln \left( \tilde{h}_j |\tilde{\alpha}_j|^2 \right) \right) \left( V', \nabla V' \right) \geq \beta \omega \left( V', \nabla V' \right)$ for every $z \in \tilde{U}_j$ and $V \in T^{1,0}_z(X)$.

On $\tilde{U}_j \setminus L$ we have
\begin{align*}
-\partial \bar{\partial} \ln \tilde{v} & = -i\partial \bar{\partial} \ln \left( \frac{\tilde{h}_j |\tilde{\alpha}_j|^2}{d\tilde{f}_j} \right) \left( \text{Im} \bar{f}_j \right)^2 \\
& = -i\partial \bar{\partial} \ln \tilde{h}_j |\tilde{\alpha}_j|^2 + i\partial \bar{\partial} \ln \left| d\tilde{f}_j \right|^2 - i\partial \bar{\partial} \ln \left( \text{Im} \bar{f}_j \right)^2.
\end{align*}

Let $z \in \tilde{U}_j$ and $V \in T^{1,0}_z(X)$. Then $V = V' + V''$, $V' \in E'_z$ and $V'' \in E''_z$ and
\begin{align*}
-\partial \bar{\partial} \ln \tilde{h}_j |\tilde{\alpha}_j|^2 \left( V, \nabla V \right) & = \left( -i\partial \bar{\partial} \ln \left( \tilde{h}_j |\tilde{\alpha}_j|^2 \right) \right) \left( V', \nabla V \right) \\
& \quad + 2 \text{Re} \left( -i\partial \bar{\partial} \ln \left( \tilde{h}_j |\tilde{\alpha}_j|^2 \right) \right) \left( V', \nabla V'' \right) \\
& \quad + \left( -i\partial \bar{\partial} \ln \left( \tilde{h}_j |\tilde{\alpha}_j|^2 \right) \right) \left( V'', \nabla V'' \right)
\end{align*}

There exists a constant $C > 0$ depending on the eigenvalues of $-i\partial \bar{\partial} \ln \left( \tilde{h}_j |\tilde{\alpha}_j|^2 \right)$ with respect to $\omega$ such that for every $\varepsilon > 0$
\[ 2 \left| \text{Re} \left( -i\partial \bar{\partial} \ln \left( \tilde{h}_j |\tilde{\alpha}_j|^2 \right) \right) \left( V', \nabla V'' \right) \right| \leq C \left( \varepsilon \omega \left( V', \nabla V'' \right) + \frac{1}{\varepsilon} \omega \left( V'', \nabla V'' \right) \right), \]
so
\begin{align*}
-\partial \bar{\partial} \ln \tilde{h}_j |\tilde{\alpha}_j|^2 \left( V, \nabla V \right) & \geq \beta \omega \left( V', \nabla V \right) - C \left( \varepsilon \omega \left( V', \nabla V \right) - \frac{1}{\varepsilon} \omega \left( V'', \nabla V'' \right) \right) \\
& - \left\| -i\partial \bar{\partial} \ln \tilde{h}_j |\tilde{\alpha}_j|^2 \right\| \omega \left( V'', \nabla V'' \right) \end{align*}
Since $\partial\bar{f}_j$ vanishes to order greater than 2 on $L$, for every $\gamma > 0$ there exists a neighborhood of $L$ such that
\begin{equation}
(i\partial\bar{\partial} \ln |d\bar{f}_j|^2 (V,\overline{V}) \leq \gamma \omega (V,\overline{V}) \tag{3.3}
\end{equation}
and
\begin{equation}
(i\partial\partial \text{Im} \bar{f}_j (V,\overline{V}) \leq \gamma \left(\text{Im} \bar{f}_j \right) \omega (V,\overline{V}) \tag{3.4}
\end{equation}

Let $z \in \mathcal{U}_j \setminus L$. By (3.4) it follows that
\begin{equation}
-i\partial\bar{\partial} \text{ln} \left(\text{Im} \bar{f}_j \right)^2 (V,\overline{V}) = \left(-2 i\partial\bar{\partial} \text{Im} \bar{f}_j / \text{Im} \bar{f}_j + 2i \left(\text{Im} \bar{f}_j \wedge \bar{\partial} \text{Im} \bar{f}_j \right) / \left(\text{Im} \bar{f}_j \right)^2 \right) (V,\overline{V}) \tag{3.5}
\end{equation}

By choosing $0 < C\varepsilon < \beta$ and by shrinking $\mathcal{U}_j$ such that $2 / (\text{Im} \bar{f}_j)$ is big enough and $\gamma$ small enough, we obtain that there exists $c > 0$ such that $-i\partial\bar{\partial} \text{ln} \tilde{v} \geq c\omega$ on $\mathcal{U}_j \setminus L$. Finally, since $v - \tilde{v}$ vanishes to order greater than 2 on $L$, it follows that there exists a neighborhood $U'$ of $L$ such that $-\ln v$ is strongly plurisubharmonic on $U' \setminus L$. We can now take $U = \{ z \in U' : v(z) < \mu \}$ for $\mu > 0$ small enough.

$L$ is a $C^3$ manifold, so the signed distance function $\delta_L$ is a defining function of class $C^3$ for $L$. Since $v$ is of class $C^2$ on $U$ and vanishes to order greater than 2 on $L$, we have $v = g\delta_L^2$ with $g$ continuous in a neighborhood of $L$.

Suppose that there exists $x \in L$ such that $g(x) = 0$. Then $v = o(\delta_L^2)$ in a neighborhood of $x$. But there exists $j$ such that $x \in U_j$ and $v = h_j |h_j|^2 \left(\text{Im} \bar{f}_j \right)^2 + o(\delta_L^2)$. Since $\text{Im} \bar{f}_j = 0$ and $d \text{Im} \bar{f}_j \neq 0$ on $L$ it follows that $|\nabla^2 v| (x) \neq 0$. This contradiction shows that $g(x) \neq 0$ on $L$. [Q.E.D.]

4. Weighted estimates for the $\overline{\partial}$-equation

Remark 1. Under the hypothesis and conclusions of Proposition 1, we consider a positive extension $\tilde{v}$ of the restriction of $v$ on a neighborhood of $L$ to $X \setminus L$. Let $s > 0$ such that $\{ v < e^{-s} \} \subset U$ and let $\varphi$ be a smooth function on $\mathbb{R}$ such that $\varphi = 0$ on $[\varphi < s, \infty]$. Then $\psi = \varphi (-\ln \tilde{v})$ is a
plurisubharmonic exhaustion function of $X \setminus L$, which is strongly plurisubharmonic outside a compact subset of $X \setminus L$.

In the sequel, $L$ will be a compact $C^\infty$ Levi flat hypersurface in a compact Kähler manifold $X$ of dimension $n \geq 2$, verifying the hypothesis and the conclusions of Proposition 1. We denote $X^\pm$ the connected components of $\{z \in X : v > 0\}$ endowed with a complete Kähler metric $\bar{\omega}$ which will be defined later and we set

$$\mathcal{D}_{(p,q)}(X^\pm) = \left\{ f \in C^\infty_{(p,q)}(X^\pm) : \text{supp } f \subset X^\pm \right\}$$

and

$$\mathcal{H}_{(p,q)}(X^\pm, \bar{\omega}) = \ker \bar{\partial} \cap \ker \bar{\partial}^* \subset L^2_{(p,q)}(X^\pm, \bar{\omega})$$

where $\bar{\partial}^*$ is the Hilbert space adjoint of the operator $\bar{\partial} : L^2_{(p,q)}(X^\pm, \bar{\omega}) \to L^2_{(p,q+1)}(X^\pm, \bar{\omega})$.

**Proposition 2.** For every $\alpha > 0$, there exists a complete Kähler metric $\bar{\omega}$ on $X \setminus L$, $\omega \leq \bar{\omega} \leq C \bar{\omega}$, $C > 0$, such that the range $\mathcal{R}_{(n,q)}^\alpha(X^\pm)$ of the operator $\bar{\partial} : L^2_{(n,q-1)}(X^\pm, \bar{\omega}) \to L^2_{(n,q)}(X^\pm, \bar{\omega})$ is closed for $1 \leq q \leq n$.

**Proof.** The proof is based on methods of [16] (see also [14]).

Denote by $\omega$ the Kähler metric of $X$. Since $i\partial\bar{\partial}(-\ln v) \geq c\omega$ on $U \setminus L$, $c > 0$, by a method developed in [27] it follows that there exist a neighborhood $V$ of $L$ and $\eta > 0$ such that $-v^\eta$ is strongly plurisubharmonic on $V \setminus L$. Then for $0 < \beta < \eta$, we have the Donnelly-Fefferman estimate [17]

$$i\partial (-\ln v) \wedge \bar{\partial} (-\ln v) \leq i r \partial \bar{\partial} (-\ln v).$$

on $V \setminus L$, with $0 < r = \beta/\eta < 1$. This is equivalent to say that the norm of $\partial (-\ln v)$ measured in the metric $i\partial\bar{\partial}(-\ln v)$ is smaller than $r$ on $V \setminus L$ (see also [6] and [19]).

Let $\alpha > 0$. We consider the trivial line bundle $E$ on $X \setminus L$ endowed with the Hermitian metric $h_\alpha = e^{\alpha \ln \bar{v}}$. Set

$$\bar{\omega} = i\Theta(E) + K\omega = i\alpha \partial \bar{\partial} (-\ln \bar{v}) + K \omega$$

with $K$ a positive constant. Since $-\ln \bar{v}$ is an exhaustion function on $X \setminus L$, it follows by (4.1) that for $K$ big enough $\bar{\omega}$ is a complete Kähler metric on $X \setminus L$ such that $\omega \leq \bar{\omega} \leq C \bar{\omega}$, $C > 0$.

Denote $\lambda_j$ (respectively $\bar{\lambda}_j$) the eigenvalues of $i\Theta(E)$ with respect to $\omega$ (respectively $\bar{\omega}$), $1 \leq j \leq n$, in increasing order. By Proposition 1, there exists $c > 0$ such that $i\Theta(E) = i\alpha \partial \bar{\partial} (-\ln \bar{v}) \geq \alpha\omega$ on $\{\psi > b\}$ for $b$ big enough. So, as in [16] (1.6) we have

$$1 \geq \bar{\lambda}_j = \frac{\lambda_j}{\lambda_j + K} \geq \frac{\alpha c}{\alpha c + K} > 0, \quad 1 \leq j \leq n$$

on $\{\psi > b\}$. By Bochner-Kodaira-Nakano inequality (see for ex. [14]) we have

$$N_{\alpha,\bar{\omega}}(\partial u)^2 + N_{\alpha,\bar{\omega}}(\bar{\partial}^* u)^2 \geq \int_{X^{\pm}} (\langle [i\Theta(E), \Lambda_{\bar{\omega}}] u, u \rangle_{\alpha,\bar{\omega}}) dV_{\bar{\omega}}$$

for every $u \in D_{(n,q)}(X \setminus L)$, where $N_{\alpha,\bar{\omega}} = \int_{X^{\pm}} |u|^2_{\bar{\omega}} dV_{\bar{\omega}}$.

Let $\chi$ be a smooth function on $X$ such that $0 \leq \chi \leq 1$, $\chi = 0$ on a neighborhood of $\{\psi < b\}$ and $\chi = 1$ on a neighborhood $\{\psi > b'\}$ of $L$, $b' > b$. By (4.3) and (4.2),
for every \( u \in \mathcal{D}_{(n,q)}(X \setminus L) \) we have
\[
N_{\alpha,\bar{\omega},\bar{\nu}}(\partial (\chi u))^2 + N_{\alpha,\bar{\omega},\bar{\nu}}\left(\overline{\partial^*_{\alpha}} (\chi u)\right)^2 \geq \int_{X^\pm} \langle \left[ i \Theta(E), \Lambda_{\bar{\omega}} \right] \rangle \chi u, \chi u \rangle_{\alpha,\bar{\omega},\bar{\nu}} dV_{\bar{\omega}} \\
\geq \int_{\{\psi > b'\}} \langle \left[ i \Theta(E), \Lambda_{\bar{\omega}} \right] \rangle \chi u, \chi u \rangle_{\alpha,\bar{\omega},\bar{\nu}} dV_{\bar{\omega}} \\
\geq \int_{\{\psi > b'\}} (\lambda_1 + \cdots + \lambda_n) |\chi u|^2 \bar{\nu}^n dV_{\bar{\omega}} \\
\geq \frac{ac}{ac + K} \int_{\{\psi > b'\}} |u|^2 \bar{\nu}^n dV_{\bar{\omega}}.
\]
so there exists \( C, c' > 0 \) such that
\[
2N_{\alpha,\bar{\omega},\bar{\nu}}(\partial u)^2 + 2N_{\alpha,\bar{\omega},\bar{\nu}}\left(\overline{\partial^*_{\alpha}} u\right)^2 + C \int_{\text{supp}(\chi')} |u|^2 \bar{\nu}^n dV_{\bar{\omega}} \\
\geq c' \int_{X \setminus L} |u|^2 \bar{\nu}^n dV_{\bar{\omega}} - c' \int_{\{\psi < b'\}} |u|^2 \bar{\nu}^n dV_{\bar{\omega}}.
\]
Finally it follows that there exists a compact subset \( F = \text{supp}(\chi') \cup \{\psi \leq b'\} \) of \( X^\pm \) such that for every \( u \in \mathcal{D}_{(n,q)}(X \setminus L) \)
\[
(4.4) \quad c' N_{\alpha,\bar{\omega},\bar{\nu}}(u)^2 \leq 2N_{\alpha,\bar{\omega},\bar{\nu}}(\partial u)^2 + 2N_{\alpha,\bar{\omega},\bar{\nu}}\left(\overline{\partial^*_{\alpha}} u\right)^2 + (c + c') \int_F |u|^2 \bar{\nu}^n dV_{\bar{\omega}}.
\]
Since \( \bar{\omega} \) is a complete metric on \( X \setminus L \), (4.4) is valid for every \( u \in (\text{Dom} \overline{\partial}) \cap (\text{Dom} \overline{\partial^*_{\alpha}}) \). The conclusion of Proposition 2 is now a consequence of Proposition 1.2 of [23].

**Corollary 1.** For every \( \alpha > 0 \) and \( 1 \leq q \leq n \) we have
\( \mathcal{H}_{(n,q)}(X^\pm, \bar{\nu}^n, \bar{\omega}) = \{\} \).

**Proof.** As \( (X^\pm, \bar{\omega}) \) is a connected weakly 1-complete Kähler manifold and the bundle \( E \) defined in the proof of Theorem 2 is a semi-positive line bundle on \( X^\pm \) which is positive outside a compact subset of \( X^\pm \), the Corollary 1 is a consequence of [33], Corollary of the Main Theorem (see also [3], [30] and [26], Corollary 2.10).

By taking in account Corollary 1, a classical application of Proposition 2 (see for example [18]) is the following:

**Corollary 2.** For every \( \alpha > 0 \) and \( 1 \leq q \leq n \) we have:

1. There exists the \( \overline{\partial} \)-Neumann operator \( N^\alpha_{(n,q)} : L^2_{(n,q)}(X^\pm, \bar{\nu}^n, \bar{\omega}) \to L^2_{(n,q)}(X^\pm, \bar{\nu}^n, \bar{\omega}) \) such that for every \( f \in L^2_{(n,q)}(X^\pm, \bar{\nu}^n, \bar{\omega}) \) we have the orthogonal decomposition
\[
f = \overline{\partial \alpha_N^\alpha} f + \overline{\partial \alpha_N^\alpha(N_{(n,q)}) f} \text{ and } \overline{\partial N^\alpha_{(n,q)}} = N^\alpha_{(n,q+1)} \overline{\partial} \text{, } \overline{\partial N^\alpha_{(n,q)}} = N^\alpha_{(n,q-1)} \overline{\partial^*}\alpha.
\]

2. For every \( \overline{\partial} \)-closed form \( f \in L^2_{(n,q)}(X^\pm, \bar{\nu}^n, \bar{\omega}) \), \( \overline{\partial} \left( \overline{\partial^*_{\alpha}} N^\alpha_{(n,q)} f \right) = f \).

**Lemma 1.** Let \( f \in C^\infty_{(0,q)}(X) \), \( 1 \leq q \leq n - 1 \), be a \( \overline{\partial} \)-closed form such that \( f \) vanishes to infinite order on \( L \). Let \( \psi_1, \psi_2 \in \text{Dom} \overline{\partial} \subset L^2_{(n,q)}(X^\pm, \bar{\nu}^n, \bar{\omega}) \) such that \( \overline{\partial} \psi_1 = \overline{\partial} \psi_2 \). Then
\[
\int_{X^\pm} f \wedge (\psi_1 - \psi_2) = 0.
\]
Proof. By Corollary 2, there exists \( h \in L^2_{(n,n-q-1)}(X^\pm, \bar{\nu}^\alpha, \bar{\omega}) \) such that \( \psi_1 - \psi_2 = \bar{\partial}h \). Since \( f \) vanishes to infinite order on \( L \) and \( \bar{\omega} \leq \frac{C_\alpha}{\nu} \bar{\omega} \), it follows that

\[
\int_{X^\pm} f \wedge (\psi_1 - \psi_2) = \lim_{\epsilon \to 0} \int_{\{v > \epsilon\} \cap X^\pm} f \wedge \bar{\partial}h = \lim_{\epsilon \to 0} \left( \int_{\{v > \epsilon\}} \bar{\partial}f \wedge h + \int_{\{v = \epsilon\} \cap X^\pm} f \wedge h \right) = 0.
\]

\[\blacksquare\]

**Proposition 3.** Let \( f \in C^\infty(\partial_{(q,0)}) (X) \), \( 1 \leq q \leq n - 1 \), be a \( \bar{\partial} \)-exact form such that \( f \) vanishes to infinite order on \( L \). Then for every \( \alpha > 0 \), there exists \( u \in L^2_{(0,q-1)}(X^\pm, \bar{\nu}^{-\alpha}, \bar{\omega}) \) such that \( \bar{\partial}u = f \) and \( N_{-\alpha,\bar{\nu},\bar{\omega}} (u) \leq C_\alpha N_{-\alpha,\bar{\nu},\bar{\omega}} (f) \), with \( C_\alpha > 0 \) independent of \( f \).

**Proof.** Step 1. Definition by duality of \( u \in L^2_{(0,q-1)}(X^\pm, \bar{\nu}^{-\alpha}, \bar{\omega}) \), \( 1 \leq q \leq n - 1 \).

The proof of this point is inspired from [19], Proposition 5.3. By Proposition 2, \( R^\alpha_{(n,q)}(X^\pm) \) is closed for every \( \alpha > 0 \) and by Corollary 2 we can find a bounded operator \( T^\alpha_{(n,q)} = \bar{\partial}^\alpha N^\alpha_{(n,q)} : R^\alpha_{(n,q)}(X^\pm) \to L^2_{(n,n-q-1)}(X^\pm, \bar{\nu}^\alpha, \bar{\omega}) \), such that \( \bar{\partial}T^\alpha_{(n,q)} \varphi = \varphi \) for every \( \varphi \in R^\alpha_{(n,q)}(X^\pm) \), \( 1 \leq q \leq n \).

Define now the continuous linear form \( \Phi_f \) on \( R^\alpha_{(n,n-q-1)}(X^\pm) \), \( 1 \leq q \leq n \), by

\[
\Phi_f (\varphi) = \int_{X^\pm} f \wedge T^\alpha_{(n,n-q-1)} \varphi, \quad \varphi \in R^\alpha_{(n,n-q-1)}(X^\pm).
\]

By the Hahn-Banach theorem, we extend \( \Phi_f \) as a linear form \( \tilde{\Phi}_f \) on \( L^2_{(n,n-q+1)}(X^\pm, \bar{\nu}^\alpha, \bar{\omega}) \) such that \( \| \tilde{\Phi}_f \| = \| \Phi_f \| \). Since \( \left( L^2_{(n,n-q+1)}(X^\pm, \bar{\nu}^\alpha, \bar{\omega}) \right)' = L^2_{(0,q-1)}(X^\pm, \bar{\nu}^{-\alpha}, \bar{\omega}) \) by the pairing

\[
(\beta_1, \beta_2) = \int_{X^\pm} \beta_1 \wedge \beta_2, \quad \beta_1 \in L^2_{(0,q-1)}(X^\pm, \bar{\nu}^{-\alpha}, \bar{\omega}), \quad \beta_2 \in L^2_{(n,n-q+1)}(X^\pm, \bar{\nu}^\alpha, \bar{\omega}),
\]

there exists \( u \in L^2_{(0,q-1)}(X^\pm, \bar{\nu}^{-\alpha}, \bar{\omega}) \) such that

\[
\tilde{\Phi}_f (\varphi) = \int_{X^\pm} u \wedge \varphi
\]

for every \( \varphi \in R^\alpha_{(n,n-q-1)}(X^\pm) \).

Step 2. We prove that \( \bar{\partial}(-1)^q u = f \), \( 1 \leq q \leq n - 1 \).

Let \( \varphi = \bar{\partial} \psi \in C^\infty_{(n,n-q+1)}(X^\pm) \) with \( \psi \in D_{(n,n-q)}(X^\pm) \). Set \( g_\alpha = \bar{\partial}^\alpha N^\alpha_{(n,n-q+1)} \bar{\partial} \psi \in L^2_{(n,n-q)}(X^\pm, \bar{\nu}^\alpha, \bar{\omega}) \). By Corollary 2, \( \bar{\partial}g_\alpha = \varphi \) and by Lemma 1

\[
\int_{X^\pm} f \wedge g_\alpha = \int_{X^\pm} f \wedge \psi.
\]

But by step 1 we have

\[
\tilde{\Phi}_f (\varphi) = \int_{X^\pm} u \wedge \bar{\partial} \psi = \Phi_f (\varphi) = \int_{X^\pm} f \wedge g_\alpha
\]

and by (4.5) and (4.6) it follows that

\[
\int_{X^\pm} f \wedge \psi = \int_{X^\pm} u \wedge \bar{\partial} \psi
\]

for every \( \psi \in D_{(n,n-q)}(X^\pm) \). Therefore \( \bar{\partial}(-1)^q u = f \) and the Proposition is proved. \( \blacksquare \)
Remark 2. Since $\omega \leq \bar{\omega} \leq \frac{C}{n} \omega$, by Lemma VIII.6.3 of [14] it follows that:

a) Let $f$ be a smooth $(n, q)$-form on $X$ such that $f$ vanishes to order $k$ on $L$. Then $f \in L^2_{(n, q)}(X^\pm, \bar{\omega}^{-k}, \bar{\omega})$

Indeed

$$\int_{X^\pm} |f|^2 \bar{\omega}^{-k} \omega \leq \int_{X^\pm} |f|^2 \bar{\omega}^{-k} \omega < \infty$$

b) Let $f \in L^2_{(n, q)}(X^\pm, \bar{\omega}^{-k}, \bar{\omega})$, $k > 2$. Then $f \in L^2_{(n, q)}(X^\pm, \bar{\omega}^{-k+2}, \omega)$.

Indeed

$$\int_{X^\pm} |f|^2 \bar{\omega}^{-k+2} \omega \leq C \int_{X^\pm} |f|^2 \bar{\omega}^{-k} \omega < \infty, \quad C > 0.$$

5. Nonexistence of Levi flat hypersurfaces

Proposition 4. Let $L$ be a compact $C^\infty$ Levi flat hypersurface in a Kähler manifold $X$ of dimension $n \geq 3$ such that the normal bundle $\mathcal{N}^{(1,0)}_L$ to the Levi foliation admits a $C^2$ Hermitian metric with leafwise positive curvature. Let $u \in C^\infty_{(0,q)}(L)$, $1 \leq q \leq n - 2$, such that $\partial_{\bar{\mathcal{B}}}u = 0$. Then for every $k \in \mathbb{N}^*$ there exist a $\overline{\partial}$-closed extension $U_k \in C^k_{(0,q)}(X)$ of $u$.

Proof. By Proposition 1 there exist a neighborhood $U$ of $L$, $c > 0$ and a non-negative function $v \in C^2(\bar{U})$ vanishing on $L$ such that $v = q \delta L$ and $-i \partial \overline{\partial} \ln v \geq c \omega$ on $U \setminus L$. Let $\bar{u} \in C^\infty_{(0,q)}(X)$ be an extension of $u$ such that $\overline{\partial} \bar{u}$ vanishes to infinite order on $L$. Since $\overline{\partial} \bar{u} \in L^2_{(q+1)}(X^\pm, \delta L^{2k}, \omega)$, $q+1 \leq n-1$ and $L^2_{(0,q)}(X^\pm, \delta L^{2k+2}, \omega) = L^2_{(0,q)}(X^\pm, \bar{\omega}^{-k}, \omega)$ for every $k \in \mathbb{N}$, by Remark 2 a) and Proposition 3 it follows that for every $k \in \mathbb{N}^+$ there exist a Hermitian complete metric $\bar{\omega}$ on $X \setminus L$, $\omega \leq \bar{\omega} \leq \frac{C}{n} \omega$ and $h^\pm \in L^2_{(0,q)}(X^\pm, \delta L^{2k}, \bar{\omega})$ such that $\overline{\partial} h^\pm = \overline{\partial} \bar{u}$ on $X^\pm$. By Remark 2 b) we have $h^\pm \in L^2_{(0,q)}(X^\pm, \delta L^{2k+4}, \omega)$. So by using Theorem 1, for $k$ big enough we can choose $h \in C^k_{(0,q)}(X^\pm)$, $s(k) \sim \sqrt{k}$. This means that for $k$ big enough, the form $h$ defined as $h^\pm$ on $X^\pm$ is of class $C^k$ on $X$ and vanishes on $L$. So $U_k = \bar{u} - h^\pm$ is a $C^k$-smooth $\overline{\partial}$-closed form on $X$ which is an extension of $u$.

Theorem 2. Let $X$ be a compact connected Kähler manifold of dimension $n \geq 3$ and $L$ a $C^\infty$ compact Levi flat hypersurface. Then the normal bundle to the Levi foliation does not admit any Hermitian metric of class $C^2$ with leafwise positive curvature.

Proof. Suppose that the normal bundle $\mathcal{N}$ to the Levi foliation admits a Hermitian metric of class $C^2$ with leafwise positive curvature. Since $\mathcal{N}$ is topologically trivial, its curvature form $\Theta^\mathcal{N}$ for the Kähler metric of $X$ is $d$-exact. So there exists a 1-form $u$ of class $C^\infty$ on $L$ such that $du = \Theta^\mathcal{N}$: we may suppose that $u$ is real and $u = u^{0,1} + u^{0,1}$, where $u^{0,1}$ is the $(0, 1)$ component of $u$. Since $\Theta^\mathcal{N}$ is a $(1, 1)$-form, it follows that $\overline{\partial} u^{0,1} = 0$, where $\overline{\partial}$ is the tangential Cauchy-Riemann operator. By Proposition 4 there exists a $C^k$-extension $U^{0,1}$ of $u^{0,1}$ to $X$, $k \geq 2$, such that $\overline{\partial} U^{0,1} = 0$.

By Hodge symmetry and Dolbeault isomorphism $H^{0,1}(X, \mathbb{C}) \approx H^{1,0}(X, \mathbb{C}) \approx H^0(X, \Omega^{1,0}_X)$, where $\Omega^{1,0}_X$ is the sheaf of holomorphic 1-forms on $X$. So there exists $\eta \in H^{0}(X, \Omega^{1,0}_X)$ and $\Phi \in C^k(X)$ such that $U^{0,1} = \eta + \Phi$. It follows that $\Theta^\mathcal{N} =$
$i\partial_b \overline{\partial}_b \text{Im } \Phi$ on $L$ and this gives a contradiction at the point of $L$ where $\text{Im } \Phi$ reaches its maximum.

Remark 3. A first version of this paper was announced on arXiv in 2014, but there was a gap in the proofs of §4, which is now corrected. Recently, Brinkschulte proved a generalization of Theorem 2 for compact Levi flat hypersurfaces in complex manifolds (see Theorem 1.1. of [8]). She uses crucially the Proposition 4.1 of [8], whose statement and proof are the same as Proposition 1 of this paper and which are unchanged from 2014 in our preprint arXiv:1406.5712. However she refers only to Proposition 1.1 of [25], where the lower positive bound for the eigenvalues of the strongly plurisubharmonic function is not mentioned.

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References


