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NONEXISTENCE OF LEVI FLAT HYPERSURFACES WITH POSITIVE NORMAL BUNDLE IN COMPACT KÄHLER MANIFOLDS OF DIMENSION \( \geq 3 \)

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In memory of Gennadi M. Henkin

Abstract. Let \( X \) be a compact connected Kähler manifold of dimension \( \geq 3 \) and \( L \) a \( C^\infty \) Levi flat hypersurface in \( X \). Then the normal bundle to the Levi foliation does not admit a Hermitian metric with positive curvature along the leaves. This represents an answer to a conjecture of Marco Brunella.

1. Introduction

A classical theorem of Poincaré-Bendixson [28], [29], [5] states that every leaf of a foliation of the real projective plane accumulates on a compact leaf or on a singularity of the foliation. As a holomorphic foliation \( \mathcal{F} \) of codimension 1 of \( \mathbb{C}P_n \), \( n \geq 2 \), does not contain any compact leaf and its singular set \( \text{Sing } \mathcal{F} \) is not empty, a major problem in foliation theory is the following: can \( \mathcal{F} \) contain a leaf \( F \) such that \( \mathcal{F} \cap \text{Sing } \mathcal{F} = \emptyset \)? If this is the case, then there exists a nonempty compact set \( K \) called exceptional minimal, invariant by \( \mathcal{F} \) and minimal for the inclusion such that \( K \cap \text{Sing } \mathcal{F} = \emptyset \). The problem of the existence of an exceptional minimal in \( \mathbb{C}P_n \), \( n \geq 2 \) is implicit in [12].

In [13] D. Cerveau proved a dichotomy under the hypothesis of the existence of a holomorphic foliation \( \mathcal{F} \) of codimension 1 of \( \mathbb{C}P_n \) which admits an exceptional minimal \( \mathfrak{M} \): \( \mathfrak{M} \) is a real analytic Levi flat hypersurface in \( \mathbb{C}P_n \) (i.e. \( T(\mathfrak{M}) \cap JT(\mathfrak{M}) \) is integrable, where \( J \) is the complex structure of \( \mathbb{C}P_n \)), or there exists \( p \in \mathfrak{M} \) such that the leaf through \( p \) has a hyperbolic holonomy and the range of the holonomy morphism is a linearisable abelian group. This gave rise to the conjecture of the nonexistence of smooth Levi flat hypersurface in \( \mathbb{C}P_n \), \( n \geq 2 \).

The conjecture was proved for \( n \geq 3 \) by A. Lins Neto [22] for real analytic Levi flat hypersurfaces and by Y.-T. Siu [31] for \( C^{12} \) smooth Levi flat hypersurfaces. The methods of proofs for the real analytic case are very different from the smooth case.

A real hypersurface of class \( C^2 \) in a complex manifold is Levi flat if its Levi form vanishes or equivalently, it admits a foliation by complex hypersurfaces. We say that a (non-necessarily smooth) real hypersurface \( L \) in a complex manifold \( X \) is Levi flat if \( X \setminus L \) is pseudoconvex. An example of (non-smooth) Levi flat hypersurface in
\( \mathbb{CP}_2 \) is \( L = \{ [z_0, z_1, z_2] : |z_1| = |z_2| \} \), where \([z_0, z_1, z_2] \) are homogeneous coordinates in \( \mathbb{CP}_2 \) (see [19]).

In [21] Iordan and Mattthey proved the nonexistence of Lipschitz Levi flat hypersurfaces in \( \mathbb{CP}_n \), \( n \geq 3 \), which are of Sobolev class \( W^s, s > 9/2 \). A principal element of the proof is that the Fubini-Study metric induces a metric of positive curvature on any quotient of the tangent space.

Nonexistence questions for the Levi flat hypersurfaces in compact Kähler manifolds were first discussed by T. Ohsawa in [24], who proved the nonexistence of real-analytic Levi flat hypersurfaces with Stein complement in compact Kähler manifolds of dimension \( \geq 3 \).

In [9], M. Brunella proved that the normal bundle to the Levi foliation of a closed real analytic Levi flat hypersurface in a compact Kähler manifold of dimension \( n \geq 3 \) does not admit any Hermitian metric with leafwise positive curvature. The real analytic hypothesis may be relaxed to the assumption of \( C^{2,\alpha} \), \( 0 < \alpha < 1 \), such that the Levi foliation extends to a holomorphic foliation in a neighborhood of the hypersurface.

The main step in his proof is to show that the existence of a Hermitian metric with leafwise positive curvature on the normal bundle to the Levi foliation of a compact Levi flat hypersurface \( L \) in a Hermitian manifold \( X \), implies that \( X \setminus L \) is strongly pseudoconvex, i.e. there exists on \( X \setminus L \) an exhaustion function which is strongly plurisubharmonic outside a compact set. This was generalized in [10] for invariant compact subsets of a holomorphic foliation of codimension one. Of course, if \( X \) is the complex projective space, then every proper pseudoconvex domain in \( X \) is Stein [34].

Brunella stated also the following conjecture [9]: Let \( X \) be a compact connected Kähler manifold of dimension \( n \geq 3 \) and \( L \) a \( C^\infty \) compact Levi flat hypersurface in \( X \). Then the normal bundle to the Levi foliation does not admit any Hermitian metric with leafwise positive curvature.

The assumption \( n \geq 3 \) is necessary in this conjecture (see Example 4.2 of [9]).

In [11] Brunella and Perrone proved that every leaf of a holomorphic foliation \( F \) of codimension one of a projective manifold \( X \) of dimension at least 3 and such that \( Pic(X) = \mathbb{Z} \) accumulates on the singular set of the foliation. In this case the normal bundle to the foliation is ample.

In [25], T. Ohsawa considered a \( C^\infty \) Levi flat compact hypersurface \( L \) in a compact Kähler manifold \( X \) such that the normal bundle to the Levi foliation admits a fiber metric whose curvature is semipositive of rank \( \geq k \) on the holomorphic tangent space to the leaves and proved that \( X \setminus L \) admits an exhaustion plurisubharmonic function of logarithmic growth which is strictly \((n-k)\)-convex. Then, if \( \dim X \geq 3 \), he proved that there are no Levi flat real analytic hypersurfaces such that the normal bundle to the Levi foliation admits a fiber metric whose curvature is semipositive of rank \( \geq 2 \) on the holomorphic tangent space to \( L \). Some possibilities for generalization in the smooth case are also indicated.

In this paper we solve the above mentioned conjecture of Brunella for compact connected Kähler manifolds of dimension \( n \geq 3 \). The principal ingredient of the proof is a refinement of the proof of Brunella [9] of the strong pseudoconvexity of \( X \setminus L \) : we show that there exist a neighborhood \( U \) of \( L \) and a function \( v \) on \( U \) vanishing on \( L \), such that \( -i\partial\bar{\partial} \ln v \geq \omega \) on \( U \setminus L \), where \( c > 0 \) and \( \omega \) is the \((1,1)\)-form associated to the Kähler metric. Then we use the \( L^2 \) estimates [2], [1], [20],
nonexistence of Levi flat hypersurfaces

[15] for the weighted $\bar{\partial}$-equation on $(n, q)$-forms on $X \setminus L$ endowed with a complete Kähler metric. These estimates together with the lower uniform boundedness of the eigenvalues of the Levi form and a duality method developed in [19], allow us to solve the $\bar{\partial}$-equation with compact support for $(0, q)$-forms, $1 \leq q \leq n - 1$, and this leads in dimensions $\geq 3$ to the solution of Brunella’s conjecture.

2. Preliminaries

Let $X$ be a complex $n$-dimensional manifold, $\omega$ a Kähler metric on $X$, $\Omega$ a domain in $X$ and $\sigma$ a positive function on $\Omega$. For $\alpha \in \mathbb{R}$ denote

$$L^2_{(p, q)}(\Omega, \sigma^\alpha, \omega) = \left\{ f \in L^2_{(p, q), \text{loc}}(\Omega) : \int_\Omega |f|^2 \sigma^{2\alpha} dV_\omega < \infty \right\}$$

endowed with the norm

$$N_{\alpha, \omega, \sigma}(f) = \left( \int_\Omega |f|^2 \sigma^{2\alpha} dV_\omega \right)^{1/2}.$$ 

Let $\Omega$ be a pseudoconvex domain in $\mathbb{CP}_n$ and $\partial_{\Omega}$ the geodesic distance to the boundary for the Fubini-Study metric $\omega_{FS}$. By using the $L^2$ estimates for the $\bar{\partial}$-operator of Hörmander with the weight $e^{-\varphi}$, $\varphi = -\alpha \log \delta_{\partial \Omega}$ which is strongly plurisubharmonic by a theorem of Takeuchi [34], Henkin and Iordan proved in [19] the existence and regularity of the $\bar{\partial}$ equation for $\bar{\partial}$-closed forms in $L^2_{(p, q)}(\Omega, \delta_{\partial \Omega}^\alpha, \omega_{FS})$ verifying the moment condition. This gives the regularity of the $\bar{\partial}$-operator in pseudoconcaive domains with Lipschitz boundary [19] and, by using a method of Siu [31], [32], the nonexistence of smooth Levi flat hypersurfaces in $\mathbb{CP}_n$, $n \geq 3$ follows (see [21]). These techniques will be used in the 4th and the 5th paragraph.

We will use also the following theorem of regularity of $\bar{\partial}$ equation of Brinkshulte [7]:

**Theorem 1.** Let $\Omega$ be a relatively compact domain with Lipschitz boundary in a Kähler manifold $(X, \omega)$ and set $\delta_{\partial \Omega}$ the geodesic distance to the boundary of $\Omega$. Let $f \in L^2_{(p, q)}(\Omega, \delta_{\partial \Omega}^{-k}, \omega) \cap C^0_{(p, q)}(\Omega) \cap C^\infty_{(p, q)}(\Omega)$, $q \geq 1$, $k \in \mathbb{N}$ and $u \in L^2_{(p, q-1)}(\Omega, \delta_{\partial \Omega}^{-k}, \omega)$ such that $\bar{\partial}u = f$ and $\bar{\partial}^{-k}u = 0$, where $\bar{\partial}^{-k}$ is the Hilbert space adjoint of the unbounded operator $\bar{\partial}^{-k} : L^2_{(p, q-1)}(\Omega, \delta_{\partial \Omega}^{-k}, \omega) \to L^2_{(p, q)}(\Omega, \delta_{\partial \Omega}^{-k}, \omega)$. Then for $k$ big enough $u \in C^s_{(p, q-1)}(\Omega)$ where $s(k) \sim \sqrt{k}$.

3. Strong pseudoconvexity of the complement of a Levi flat hypersurface

Let $L$ be a smooth Levi flat hypersurface in a Hermitian manifold $X$. As was mentioned in [9] and in [25], by taking a double covering, we can assume that $L$ is orientable and the complement of $L$ has two connected components in a neighborhood of $L$. This will be always supposed in the sequel and for an open neighborhood $U$ of $L$ we will denote by $U^+$ and $U^-$ the two connected components of $U \setminus L$. We will denote by $\delta_L$ the signed geodesic distance to $L$.

In [9] Brunella proved that the complement of a closed Levi flat hypersurface in a compact Hermitian manifold of class $C^{2, \alpha}$, $0 < \alpha < 1$, having the property that the Levi foliation extends to a holomorphic foliation in a neighborhood of $L$ and the normal bundle to the Levi foliation admits a $C^2$ Hermitian metric with leafwise positive curvature is strongly pseudoconvex, i.e. there exists an exhaustion
function which is strongly plurisubharmonic outside a compact set. The following proposition strengthens this result:

**Proposition 1.** Let $L$ be a compact $C^3$ Levi flat hypersurface in a Hermitian manifold $X$ of dimension $n \geq 2$, such that the normal bundle $N^1_L$ to the Levi foliation admits a $C^2$ Hermitian metric with leafwise positive curvature. Then there exist a neighborhood $U$ of $L$, $c > 0$ and a non-negative function $v \in C^2(U)$, vanishing on $L$ and positive on $U \setminus L$ such that $-i\partial \bar{\partial} \ln v \geq c \omega$ on $U \setminus L$, where $\omega$ is the $(1,1)$-form associated to the metric. Moreover, there exists a nonvanishing continuous function $g$ in a neighborhood of $L$ such that $v = g \delta^2_L$.

**Proof.** Let $z_0 \in L$. There exist holomorphic coordinates $z = (z_1, \ldots, z_{n-1}, z_n) = (z', z_n)$ in a neighborhood of $z_0$ such that the local parametric equations for $L$ are of the form

$$z_j = w_j, \quad j = 1, \ldots, n-1, \quad z_n = \varphi(w', t)$$

where $\varphi$ is of class $C^3$ (see [4]) on a neighborhood of the origin in $\mathbb{C}^{n-1} \times \mathbb{R}$, holomorphic in $w'$ and $\partial \varphi / \partial t(z_0) \in \mathbb{R}^*$. We consider a $C^3$ extension $\psi = (\psi_1, \ldots, \psi_n)$ of $\varphi$ on a neighborhood of the origin in $\mathbb{C}^{n-1} \times \mathbb{C}$, $\psi(w', t + is) = (w', \varphi(w', t) + is)$. Then $\psi$ is a $C^3$ diffeomorphism in a neighborhood of $z_0$ and holomorphic in $w'$. It follows that

$$L = \{(z', z_n) : \rho(z', z_n) = 0\},$$

where $\rho = \text{Im} (\psi^{-1})_n$. We denote $f = (\psi^{-1})_n (z', z_n)$. Since $\overline{\partial}_b f = 0$ on $L$, where $\overline{\partial}_b$ is the tangential Cauchy-Riemann operator on $L$, there exists an extension $\tilde{f}$ of class $C^3$ in a neighborhood of $z_0$ such that $\overline{\partial} \tilde{f}$ vanishes to order greater than 2 on $L$, i.e. $D^l \overline{\partial} \tilde{f} = 0$ for $|l| \leq 2$ on $L$.

So there exists an open finite covering $\left( U_j \right)_{j \in J}$ by holomorphic charts of $L$ such that $\overline{\partial}_{\overline{\partial}} \tilde{f}_j \equiv 0$ on $U_j$ such that $U_j = L \cap \overline{U}_j = \left\{ z \in \overline{U}_j : \text{Im} \tilde{f}_j = 0 \right\}$, where $\overline{\partial}_j \tilde{f}_j$ vanishes to order greater than 2 on $L$ and the Levi foliation is given on $U_j$ by $\left\{ z \in U_j : \tilde{f}_j(z) = c_j \right\}$, $c_j \in \mathbb{R}$. Thus $d \overline{\partial}_j \tilde{f}_j = \overline{\partial} \tilde{f}_j$ is a nonvanishing section of $N^1_L$ on $U_j$ and by shrinking $\overline{U}_j$, we may consider that $d \overline{\partial}_j \tilde{f}_j \neq 0$ on $\overline{U}_j$.

We may suppose that $N^1_L$ is represented by a cocycle $\{g_{jk}\}$ of class $C^2$ subordinated to the covering $(U_j)_{j \in J}$ and there exist closed $(1,0)$-forms $\alpha_j$ of class $C^2$ on $U_j$ holomorphic along the leaves such that $T^{1,0} (U_j) = \ker \alpha_j$ for every $j \in J$ and $\alpha_j = g_{jk} \alpha_k$ on $U_j \cap U_k$. So $\{\alpha_j\}_{j \in J}$ defines a global form $\alpha$ on $L$ with values in $N^1_L$ such that locally on $U_j$ we have $\alpha (z) = \alpha_j (z) \otimes \alpha^*_j (z)$ where $\alpha^*_j$ is the dual frame of $\alpha_j$. In particular we have $\alpha^*_k = g_{jk} \alpha^*_j$.

Let $h$ be a $C^2$ Hermitian metric with positive leafwise curvature $\Theta_h \left( N^1_L \right)$ on $N^1_L$. $h$ is defined on each $U_j$ by a $C^2$ function $h_j = |\alpha^*_j|^2$ such that $h_k = |g_{jk}|^2 h_j$ on $U_j \cap U_k$.

Since $\alpha_j = \eta_j d \overline{\partial}_j \tilde{f}_j$ on $U_j$ for every $j$, where $\eta_j$ are nowhere vanishing functions of class $C^2$ on $U_j$ holomorphic along the leaves and

$$\frac{1}{\eta_k} \left( d \overline{\partial}_k \right)^* = \frac{1}{\eta_j} g_{jk} \left( d \overline{\partial}_j \right)^*$$
on \( U_j \cap U_k \), it follows that
\[
|g_{jk}(z)|^2 = \left| \frac{\eta_j(z)}{\eta_k(z)} \right|^2 \left| \frac{d\tilde{f}_k}{d\tilde{f}_j} \right|^2 = \frac{h_k(z)}{h_j(z)}, \quad z \in U_j \cap U_k.
\]

So
\[
h_j |\eta_j|^2 \left( \text{Im} \tilde{f}_j \right)^2 - h_k |\eta_k|^2 \left( \text{Im} \tilde{f}_k \right)^2
\]
vanishes to order greater than 2 on \( U_j \cap U_k \) and \( \left( h_j |\eta_j|^2 \left( \text{Im} \tilde{f}_j \right)^2 \right) \) defines a jet of order 2 on \( L \). By Whitney extension theorem there exists a \( C^2 \) function \( v \) on \( X \) such that \( v - h_j |\eta_j|^2 \left( \text{Im} \tilde{f}_j \right)^2 \) vanishes to order 2 on \( U_j \) for every \( j \in J \). Let \( \tilde{\eta}_j, \tilde{h}_j \) be \( C^2 \) extensions of \( \eta_j, h_j \) on \( \tilde{U}_j \) and set \( \tilde{\alpha}_j = \tilde{\eta}_j d\tilde{f}_j, \tilde{v} = \tilde{h}_j |\tilde{\eta}_j|^2 \left( \text{Im} \tilde{f}_j \right)^2 \).

For \( z \in \tilde{U}_j \) denote \( E'_z = \{ V' \in T^{1,0}_j(X) : \langle \partial \text{Im} \tilde{f}_j, V' \rangle = 0 \} \) and \( E''_z \) the orthogonal of \( E'_z \) in \( T^{1,0}_j(X) \). Then for every \( V \in T^{1,0}_j(X) \) there exists \( V' \in E'_z, V'' \in E''_z \) such that \( V = V' + V'' \). The curvature form \( \Theta \left( \mathcal{A}_L^{1,0} \right) \) is represented by
\[
-\partial \bar{\partial} \ln \left( h_j |\alpha_j|^2 \right) \text{ on } U_j, \quad \text{by shrinking } \tilde{U}_j \text{ we may suppose that there exists } \\
\beta > 0 \text{ such that } \left( -\partial \bar{\partial} \ln \left( h_j |\alpha_j|^2 \right) \right) (V', V') \geq \beta \omega (V', V') \text{ for every } z \in \tilde{U}_j \text{ and } V \in T^{1,0}_j(X).
\]

On \( \tilde{U}_j \backslash L \) we have
\[
(3.1) \quad -\partial \bar{\partial} \ln \tilde{v} = -\partial \bar{\partial} \ln \left( \tilde{h}_j \left| \tilde{\alpha}_j \right|^2 \right) \left( \text{Im} \tilde{f}_j \right)^2 \]
\[
= -\partial \bar{\partial} \ln \tilde{h}_j \left| \tilde{\alpha}_j \right|^2 + \partial \bar{\partial} \ln \left| d\tilde{f}_j \right|^2 - \partial \bar{\partial} \ln \left( \text{Im} \tilde{f}_j \right)^2.
\]

Let \( z \in \tilde{U}_j \) and \( V \in T^{1,0}_j(X) \). Then \( V = V' + V'', V' \in E'_z \) and \( V'' \in E''_z \) and
\[
-\partial \bar{\partial} \ln \left( h_j |\alpha_j|^2 \right) (V, V) = \left( -\partial \bar{\partial} \ln \left( h_j |\alpha_j|^2 \right) \right) (V', V') \]
\[
+ 2 \text{Re} \left( -\partial \bar{\partial} \ln \left( h_j |\alpha_j|^2 \right) \right) (V', V'') \]
\[
+ \left( -\partial \bar{\partial} \ln \left( h_j |\alpha_j|^2 \right) \right) (V'', V'').
\]

There exists a constant \( C > 0 \) depending on the eigenvalues of \( -\partial \bar{\partial} \ln \left( h_j |\alpha_j|^2 \right) \) with respect to \( \omega \) such that for every \( \varepsilon > 0 \)
\[
2 \left| \text{Re} \left( -\partial \bar{\partial} \ln \left( h_j |\alpha_j|^2 \right) \right) (V', V'') \right| \leq C \left( \varepsilon \omega (V', V') + \frac{1}{\varepsilon} \omega (V'', V'') \right),
\]
so
\[
-\partial \bar{\partial} \ln \left( h_j |\alpha_j|^2 \right) (V, V) \geq \beta \omega (V', V') - C \left( \varepsilon \omega (V', V') - \frac{1}{\varepsilon} \omega (V'', V'') \right)
\]
\[
(3.2) \quad -\left\| -\partial \bar{\partial} \ln \left( h_j |\alpha_j|^2 \right) \right\|_\omega (V', V')
\]
Since $\overline{\partial}f_j$ vanishes to order greater than 2 on $L$, for every $\gamma > 0$ there exists a neighborhood of $L$ such that
\[
\left| i\partial\overline{\partial} \ln d\bar{f}_j \right|^2 (V, \overline{V}) \leq \gamma (V, \overline{V})
\] (3.3) and
\[
\left| i\partial\overline{\partial} \Im \bar{f}_j (V, \overline{V}) \right| \leq \gamma \left( \Im \bar{f}_j \right) \omega (V, \overline{V}).
\] (3.4)

Let $z \in \mathcal{U}_j \setminus L$. By (3.4) it follows that
\[
-i\partial\overline{\partial} \ln \left( \Im \bar{f}_j \right)^2 (V, \overline{V}) = \left( -2 i\partial\overline{\partial} \Im \bar{f}_j + 2i \frac{\partial \Im \bar{f}_j \wedge \overline{\partial} \Im \bar{f}_j}{(\Im \bar{f}_j)^2} \right) (V, \overline{V})
\] (3.5)
\[
\geq -2\gamma \omega (V, \overline{V}) + 2i \frac{\partial \Im \bar{f}_j \wedge \overline{\partial} \Im \bar{f}_j}{(\Im \bar{f}_j)^2} (V'', \overline{V}'')
\]
\[
\geq -2\gamma \omega (V, \overline{V}) + \frac{2\inf_{\mathcal{U}_j} \left| \partial \Im \bar{f}_j \right|^2}{(\Im \bar{f}_j)^2} \omega (V'', \overline{V}'').
\]

By using (3.2), (3.3) and (3.5), from (3.1) we obtain
\[
(-i\partial\overline{\partial} \ln \bar{v}) (V, \overline{V}) \geq (\beta - C\varepsilon) \omega (V', \overline{V'})
\]
\[
+ \left( \frac{2}{(\Im \bar{f}_j)^2} \inf_{\mathcal{U}_j} \left| \partial \Im \bar{f}_j \right|^2 - \frac{C}{\varepsilon} - \left\| -i\partial\overline{\partial} \ln h_j |\alpha_j|^2 \right\|_{\omega} \right) \omega (V'', \overline{V}'')
\]
\[
-2\gamma \omega (V, \overline{V}).
\]

By choosing $0 < C\varepsilon < \beta$ and by shrinking $\mathcal{U}_j$ such that $\frac{2}{(\Im \bar{f}_j)^2}$ is big enough and $\gamma$ small enough, we obtain that there exists $c > 0$ such that $-i\partial\overline{\partial} \ln \bar{v} \geq c\omega$ on $\mathcal{U}_j \setminus L$. Finally, since $v - \bar{v}$ vanishes to order greater than 2 on $L$, it follows that there exists a neighborhood $U'$ of $L$ such that $-\ln v$ is strongly plurisubharmonic on $U' \setminus L$. We can now take $U = \{ z \in U' : v(z) < \mu \}$ for $\mu > 0$ small enough.

$L$ is a $C^3$ manifold, so the signed distance function $\delta_L$ is a defining function of class $C^3$ for $L$. Since $v$ is of class $C^2$ on $U$ and vanishes to order greater than 2 on $L$, we have $v = g\delta_L^2$ with $g$ continuous in a neighborhood of $L$.

Suppose that there exists $x \in L$ such that $g(x) = 0$. Then $v = o\left(\delta_L^2\right)$ in a neighborhood of $x$. But there exists $j$ such that $x \in U_j$ and $v = h_j |\eta_j|^2 \left(\Im \bar{f}_j\right)^2 + o\left(\delta_L^2\right)$. Since $\Im \bar{f}_j = 0$ and $d\Im \bar{f}_j \neq 0$ on $L$ it follows that $|\nabla^2 v|(x) \neq 0$. This contradiction shows that $g(x) \neq 0$ on $L$.}

4. Weighted estimates for the $\overline{\partial}$-equation

Remark 1. Under the hypothesis and conclusions of Proposition 1, we consider a positive extension $\bar{v}$ of the restriction of $v$ on a neighborhood of $L$ to $X \setminus L$. Let $s > 0$ such that $\{ v < e^{-s}\} \subset U$ and let $\varphi$ be a smooth function on $\mathbb{R}$ such that $\varphi = 0$ on $| - \infty, s]$ and $\varphi$ is strictly convex increasing on $|s, \infty[. Then $\psi = \varphi(-\ln \bar{v})$ is a
plurisubharmonic exhaustion function of $X \setminus L$, which is strongly plurisubharmonic outside a compact subset of $X \setminus L$.

In the sequel, $L$ will be a compact $C^\infty$ Levi flat hypersurface in a compact Kähler manifold $X$ of dimension $n \geq 2$, verifying the hypothesis and the conclusions of Proposition 1. We denote $X^\pm$ the connected components of $\{z \in X : v > 0\}$ endowed with a complete Kähler metric $\tilde{\omega}$ which will be defined later and we set

$$D_{(p,q)}(X^\pm) = \left\{ f \in C^\infty(X^\pm) : \text{supp } f \subset X^\pm \right\}$$

and

$$H_{(p,q)}(X^\pm, \tilde{\omega}) = \ker \overline{\partial} \cap \ker \overline{\partial}_\alpha^* \subset L^2_{(p,q)}(X^\pm, \tilde{\omega}^{\alpha}, \tilde{\omega})$$

where $\overline{\partial}_\alpha$ is the Hilbert space adjoint of the operator $\overline{\partial} : L^2_{(p,q)}(X^\pm, \tilde{\omega}^{\alpha}, \tilde{\omega}) \to L^2_{(p,q+1)}(X^\pm, \tilde{\omega}^{\alpha}, \tilde{\omega})$.

**Proposition 2.** For every $\alpha > 0$, there exists a complete Kähler metric $\tilde{\omega}$ on $X \setminus L$, $\omega \leq \tilde{\omega} \leq C \omega$, $C > 0$, such that the range $\mathcal{R}_{(n,q)}\overline{\partial}_\alpha^*$ of the operator $\overline{\partial}_\alpha : L^2_{(n,q-1)}(X^\pm, \tilde{\omega}^{\alpha}, \tilde{\omega}) \to L^2_{(n,q)}(X^\pm, \tilde{\omega}^{\alpha}, \tilde{\omega})$ is closed for $1 \leq q \leq n$.

**Proof.** The proof is based on methods of [16] (see also [14]).

Denote by $\omega$ the Kähler metric of $X$. Since $i\partial\bar{\partial}(-\ln v) \geq c\omega$ on $U \setminus L$, $c > 0$, by a method developed in [27] it follows that there exist a neighborhood $V$ of $L$ and $\eta > 0$ such that $-v^\eta$ is strongly plurisubharmonic on $V \setminus L$. Then for $0 < \beta < \eta$, we have the Donnelly-Fefferman estimate [17]

$$(4.1) \quad i\partial(-\ln v) \wedge \bar{\partial}(-\ln v) \leq i\alpha\partial\bar{\partial}(-\ln v).$$

on $V \setminus L$, with $0 < r = \beta/\eta < 1$. This is equivalent to say that the norm of $\partial(-\ln v)$ measured in the metric $i\partial\bar{\partial}(-\ln v)$ is smaller than $r$ on $V \setminus L$ (see also [6] and [19]).

Let $\alpha > 0$. We consider the trivial line bundle $E$ on $X \setminus L$ endowed with the Hermitian metric $h_\alpha = e^{\alpha \ln \tilde{v}}$. Set

$$\tilde{\omega} = i\Theta(E) + K\omega = i\alpha\partial\bar{\partial}(-\ln \tilde{v}) + K\omega$$

with $K$ a positive constant. Since $-\ln \tilde{v}$ is an exhaustion function on $X \setminus L$, it follows by (4.1) that for $K$ big enough $\tilde{\omega}$ is a complete Kähler metric on $X \setminus L$ such that $\omega \leq \tilde{\omega} \leq C \omega$, $C > 0$.

Denote $\lambda_j$ (respectively $\overline{\lambda}_j$) the eigenvalues of $i\Theta(E)$ with respect to $\omega$ (respectively $\tilde{\omega}$), $1 \leq j \leq n$, in increasing order. By Proposition 1, there exists $c > 0$ such that $i\Theta(E) = i\alpha\partial\bar{\partial}(-\ln \tilde{v}) \geq c\omega$ on $\{\psi > b\}$ for $b$ big enough. So, as in [16] (1.6) we have

$$(4.2) \quad 1 \geq \overline{\lambda}_j = \lambda_j / \lambda_j + K \geq \frac{\alpha c}{\alpha c + K} > 0, \quad 1 \leq j \leq n$$

on $\{\psi > b\}$. By Bochner-Kodaira-Nakano inequality (see for ex. [14]) we have

$$(4.3) \quad N_{\alpha,\tilde{\omega}}(\overline{\partial}u)^2 + N_{\alpha,\tilde{\omega}}(\overline{\partial}_\alpha^* u)^2 \geq \int_{X^\pm} (\mathbf{[i\Theta(E), A_{\tilde{\omega}}]} u, u)_{\alpha,\tilde{\omega}} dV_{\tilde{\omega}}$$

for every $u \in D_{(n,q)}(X \setminus L)$, where $N_{\alpha,\tilde{\omega}} = \int_{X^\pm} |u|_\alpha^2 \tilde{\omega}^\alpha dV_{\tilde{\omega}}$.

Let $\chi$ be a smooth function on $X$ such that $0 \leq \chi \leq 1$, $\chi = 0$ on a neighborhood of $\{\psi < b\}$ and $\chi = 1$ on a neighborhood $\{\psi > b'\}$ of $L$, $b' > b$. By (4.3) and (4.2),

$$\int_{X^\pm} (\mathbf{[i\Theta(E), A_{\tilde{\omega}}]} u, u)_{\alpha,\tilde{\omega}} dV_{\tilde{\omega}} \geq C \int_{X^\pm} |u|_\alpha^2 \tilde{\omega}^\alpha dV_{\tilde{\omega}}$$

where $C > 0$ is a constant. This implies the result.
Finally it follows that there exists a compact subset 
\( \tilde{c} \subseteq X \) with 
\[-\int_{\{\psi > b\}} (\lambda_1 + \cdots + \lambda_n) |\chi|u^2_\tilde{\omega} \tilde{\omega}^\alpha dV_{\tilde{\omega}} \gtrless \frac{\alpha c}{\alpha c + K} \int_{\{\psi > b\}} |u|^2_\tilde{\omega} \tilde{\omega}^\alpha dV_{\tilde{\omega}} \]
so there exists \( C, c' > 0 \) such that
\[ 2N_{\alpha,\tilde{\omega},\tilde{\omega}}(\tilde{\omega}u)^2 + 2N_{\alpha,\tilde{\omega},\tilde{\omega}}(\tilde{\omega}^\alpha u)^2 + C \int_{\text{supp}(\chi')} |u|^2_\tilde{\omega} \tilde{\omega}^\alpha dV_{\tilde{\omega}} \gtrless c' \int_{X \setminus L} |u|^2_\tilde{\omega} \tilde{\omega}^\alpha dV_{\tilde{\omega}} - c' \int_{\{\psi < b\}} |u|^2_\tilde{\omega} \tilde{\omega}^\alpha dV_{\tilde{\omega}}. \]
Finally it follows that there exists a compact subset \( F = \text{supp}(\chi') \cup \{\psi \lessdot b'\} \) of 
\( X^\pm \) such that for every \( u \in D_{(n,q)}(X \setminus L) \)
\[ (4.4) \quad c'N_{\alpha,\tilde{\omega},\tilde{\omega}}(u)^2 \lessdot 2N_{\alpha,\tilde{\omega},\tilde{\omega}}(\tilde{\omega}u)^2 + 2N_{\alpha,\tilde{\omega},\tilde{\omega}}(\tilde{\omega}^\alpha u)^2 + (C + c') \int_F |u|^2_\tilde{\omega} \tilde{\omega}^\alpha dV_{\tilde{\omega}}. \]
Since \( \tilde{\omega} \) is a complete metric on \( X \setminus L \), (4.4) is valid for every \( u \in (\text{Dom} \tilde{\omega}) \cap \big( \text{Dom} \tilde{\omega}^\alpha \big) \). The conclusion of Proposition 2 is now a consequence of Proposition 1.2 of [23].

**Corollary 1.** For every \( \alpha > 0 \) and \( 1 \leq q \leq n \) we have \( H_{(n,q)}(X^\pm, \tilde{\omega}^\alpha, \tilde{\omega}) = \{0\} \).

**Proof.** As \((X^\pm, \tilde{\omega})\) is a connected weakly 1-complete Kähler manifold and the bundle \( E \) defined in the proof of Theorem 2 is a semi-positive line bundle on \( X^\pm \) which is positive outside a compact subset of \( X^\pm \), the Corollary 1 is a consequence of [33], Corollary of the Main Theorem (see also [3], [30] and [26], Corollary 2.10).}

By taking in account Corollary 1, a classical application of Proposition 2 (see for example [18]) is the following:

**Corollary 2.** For every \( \alpha > 0 \) and \( 1 \leq q \leq n \) we have:

1. There exists the \( \tilde{\omega} \)-Neumann operator \( N_{(n,q)}^\alpha : L^2_{(n,q)}(X^\pm, \tilde{\omega}^\alpha, \tilde{\omega}) \to L^2_{(n,q)}(X^\pm, \tilde{\omega}^\alpha, \tilde{\omega}) \) such that for every \( f \in L^2_{(n,q)}(X^\pm, \tilde{\omega}^\alpha, \tilde{\omega}) \) we have the orthogonal decomposition 
   \[ f = \overline{\partial}_{\alpha}^* N_{(n,q)}^\alpha f + \overline{\partial}_{\alpha} \overline{\partial} N_{(n,q)}^\alpha f \quad \text{and} \quad \overline{\partial} N_{(n,q)}^\alpha = N_{(n,q+1)}^\alpha \overline{\partial}, \overline{\partial}^* N_{(n,q)}^\alpha = N_{(n,q-1)}^\alpha \overline{\partial}^*. \]

2. For every \( \tilde{\omega} \)-closed form \( f \in L^2_{(n,q)}(X^\pm, \tilde{\omega}^\alpha, \tilde{\omega}) \), 
   \( \tilde{\omega} \left( \overline{\partial}_{\alpha}^* N_{(n,q)}^\alpha f \right) = f \).

**Lemma 1.** Let \( f \in C^{\infty}_{(0,q)}(X) \), \( 1 \leq q \leq n-1 \), be a \( \tilde{\omega} \)-closed form such that \( f \) vanishes to infinite order on \( L \). Let \( \psi_1, \psi_2 \in \text{Dom} \tilde{\omega} \subset L^2_{(n,n-q)}(X^\pm, \tilde{\omega}^\alpha, \tilde{\omega}) \) such that \( \tilde{\omega} \psi_1 = \tilde{\omega} \psi_2 \). Then
\[ \int_{X^\pm} f \wedge (\psi_1 - \psi_2) = 0. \]
Proof. By Corollary 2, there exists $h \in L^2_{(n,q-1)}(X^\pm, \overline{v}^\alpha, \overline{\omega})$ such that $\psi_1 - \psi_2 = \overline{\partial} h$. Since $f$ vanishes to infinite order on $L$ and $\overline{\omega} \leq \frac{C}{\partial_x} \overline{\omega}$, it follows that
\[
\int_{X^\pm} f \wedge (\psi_1 - \psi_2) = \lim_{\varepsilon \to 0} \int_{\{v > \varepsilon\} \cap X^\pm} f \wedge \overline{\partial} h = \lim_{\varepsilon \to 0} \left( \int_{\{v > \varepsilon\}} \overline{\partial} f \wedge h + \int_{\{v = \varepsilon\} \cap X^\pm} f \wedge h \right) = 0.
\]

Proposition 3. Let $f \in C^\infty_{(0,q)}(X)$, $1 \leq q \leq n-1$, be a $\overline{\partial}$-exact form such that $f$ vanishes to infinite order on $L$. Then for every $\alpha > 0$, there exists $u \in L^2_{(0,q-1)}(X^\pm, \overline{v}^{-\alpha}, \overline{\omega})$ such that $\overline{\partial} u = f$ and $N_{-\alpha, \overline{\omega}, \overline{\partial}}(u) \leq C_n N_{-\alpha, \overline{\omega}, \overline{\partial}}(f)$, with $C_n > 0$ independent of $f$.

Proof. Step 1. Definition by duality of $u \in L^2_{(0,q-1)}(X^\pm, \overline{v}^{-\alpha}, \overline{\omega})$, $1 \leq q \leq n-1$.

The proof of this point is inspired from [19], Proposition 5.3. By Proposition 2, $\mathcal{R}^\alpha_{(n,q)}(X^\pm)$ is closed for every $\alpha > 0$ and by Corollary 2 we can find a bounded operator $T^\alpha_{(n,q)} = \overline{\partial}^\alpha \mathcal{N}^\alpha_{(n,q)} : \mathcal{R}^\alpha_{(n,q)}(X^\pm) \to L^2_{(0,q-1)}(X^\pm, \overline{v}^\alpha, \overline{\omega})$, such that $\overline{\partial} T^\alpha_{(n,q)} \varphi = \varphi$ for every $\varphi \in \mathcal{R}^\alpha_{(n,q)}(X^\pm)$, $1 \leq q \leq n$.

Define now the continuous linear form $\Phi_f$ on $\mathcal{R}^\alpha_{(n,n-q+1)}(X^\pm)$, $1 \leq q \leq n$, by
\[
\Phi_f(\varphi) = \int_{X^\pm} f \wedge T^\alpha_{(n,n-q+1)} \varphi, \varphi \in \mathcal{R}^\alpha_{(n,n-q+1)}(X^\pm).
\]

By the Hahn-Banach theorem, we extend $\Phi_f$ as a linear form $\overline{\Phi}_f$ on $L^2_{(n,n-q+1)}(X^\pm, \overline{v}^\alpha, \overline{\omega})$ such that $\|\overline{\Phi}_f\| = \|\Phi_f\|$. Since $\left(L^2_{(n,n-q+1)}(X^\pm, \overline{v}^\alpha, \overline{\omega})\right)' = L^2_{(0,q-1)}(X^\pm, \overline{v}^{-\alpha}, \overline{\omega})$

by the pairing
\[
(\beta_1, \beta_2) = \int_{X^\pm} \beta_1 \wedge \beta_2, \beta_1 \in L^2_{(0,q-1)}(X^\pm, \overline{v}^{-\alpha}, \overline{\omega}), \beta_2 \in L^2_{(n,n-q+1)}(X^\pm, \overline{v}^\alpha, \overline{\omega}),
\]

there exists $u \in L^2_{(0,q-1)}(X^\pm, \overline{v}^{-\alpha}, \overline{\omega})$ such that
\[
\overline{\Phi}_f(\varphi) = \int_{X^\pm} u \wedge \varphi
\]

for every $\varphi \in \mathcal{R}^\alpha_{(n,n-q+1)}(X^\pm)$.

Step 2. We prove that $\overline{\partial} (-1)^q u = f$, $1 \leq q \leq n-1$.

Let $\varphi = \overline{\partial} \psi \in C^\infty_{(n,n-q+1)}(X^\pm)$ with $\psi \in \mathcal{D}_{(n,n-q)}(X^\pm)$. Set $g_n = \overline{\partial} N^\alpha_{(n,n-q+1)} \overline{\partial} \psi \in L^2_{(n,n-q)}(X^\pm, \overline{v}^\alpha, \overline{\omega})$. By Corollary 2, $\overline{\partial} g_n = \varphi$ and by Lemma 1
\[
(4.5) \quad \int_{X^\pm} f \wedge g_n = \int_{X^\pm} f \wedge \psi.
\]

But by step 1 we have
\[
(4.6) \quad \overline{\Phi}_f(\varphi) = \int_{X^\pm} u \wedge \overline{\partial} \psi = \Phi_f(\varphi) = \int_{X^\pm} f \wedge g_n
\]

and by (4.5) and (4.6) it follows that
\[
\int_{X^\pm} f \wedge \psi = \int_{X^\pm} u \wedge \overline{\partial} \psi
\]

for every $\psi \in \mathcal{D}_{(n,n-q)}(X^\pm)$. Therefore $\overline{\partial} (-1)^q u = f$ and the Proposition is proved. $\blacksquare$
Remark 2. Since $\omega \leq \bar{w} \leq C\omega$, by Lemma VIII.6.3 of [14] it follows that:

a) Let $f$ be a smooth $(n, q)$-form on $X$ such that $f$ vanishes to order $k$ on $L$. Then $f \in L^2(n,q)\left(X^+,\bar{v}^{-k},\bar{w}\right)$

Indeed
\[
\int_{X^\pm} |f|^2 \bar{v}^{-k} dV^X \leq \int_{X^\pm} |f|^2 \bar{v}^{-k} dV^\omega < \infty
\]

b) Let $f \in L^2(n,q)\left(X^+,\bar{v}^{-k},\bar{w}\right), k > 2$. Then $f \in L^2(n,q)\left(X^+,\bar{v}^{-k+2},\omega\right)$.

Indeed
\[
\int_{X^\pm} |f|^2 \bar{v}^{-k+2} dV^X \leq C \int_{X^\pm} |f|^2 \bar{v}^{-k} dV^\omega < \infty, \quad C > 0.
\]

5. Nonexistence of Levi flat hypersurfaces

Proposition 4. Let $L$ be a compact $C^\infty$ Levi flat hypersurface in a Kähler manifold $X$ of dimension $n \geq 3$ such that the normal bundle $\mathcal{N}_L^{(1,0)}$ to the Levi foliation admits a $C^2$ Hermitian metric with leafwise positive curvature. Let $u \in C^\infty_{(0,q)}(L)$, $1 \leq q \leq n-2$, such that $\overline{\partial} u = 0$. Then for every $k \in \mathbb{N}^*$ there exist a $\overline{\partial}$-closed extension $U_k \in C^k_{(0,q)}(X)$ of $u$.

Proof. By Proposition 1 there exist a neighborhood $U$ of $L$, $c > 0$ and a non-negative function $v \in C^2(U)$ vanishing on $L$ such that $v = g\delta_L^2$ and $-i\partial\overline{\partial} \ln v \geq c\omega$ on $U \setminus L$. Let $\bar{u} \in C^\infty_{(0,q)}(X)$ be an extension of $u$ such that $\overline{\partial} \bar{u}$ vanishes to infinite order on $L$. Since $\overline{\partial} \bar{u} \in L^2_{(0,q+1)}(X^+,\delta_L^{2k},\omega), q+1 \leq n-1$ and $L^2_{(0,q)}(X^+,\delta_L^{2k},\omega) = L^2_{(0,q)}(X^+,\bar{v}^{-k},\omega)$ for every $k \in \mathbb{N}$, by Remark 2 a) and Proposition 3 it follows that for every $k \in \mathbb{N}^*$ there exist a Hermitian complete metric $\bar{w}$ on $X \setminus L$, $\omega \leq \bar{w} \leq C\omega$ and $\bar{h}^\pm \in L^2_{(0,q)}(X^+,\delta_L^{2k},\bar{w})$ such that $\overline{\partial} \bar{h}^\pm = \overline{\partial} \bar{u}$ on $X^\pm$. By Remark 2 b) we have $\bar{h}^\pm \in L^2_{(0,q)}(X^+,\delta_L^{2k+4},\omega)$. So by using Theorem 1, for $k$ big enough we can choose $h^\pm \in C^k_{(0,q)}(X^\pm), s(k) \sim \sqrt{k}$. This means that for $k$ big enough, the form $h$ defined as $h^\pm$ on $X^\pm$ is of class $C^k$ on $X$ and vanishes on $L$. So $U_k = \bar{u} - h^\pm$ is a $C^k$-smooth $\overline{\partial}$-closed form on $X$ which is an extension of $u$.

Theorem 2. Let $X$ be a compact connected Kähler manifold of dimension $n \geq 3$ and $L$ a $C^\infty$ compact Levi flat hypersurface. Then the normal bundle to the Levi foliation does not admit any Hermitian metric of class $C^2$ with leafwise positive curvature.

Proof. Suppose that the normal bundle $\mathcal{N}$ to the Levi foliation admits a Hermitian metric of class $C^2$ with leafwise positive curvature. Since $\mathcal{N}$ is topologically trivial, its curvature form $\Theta^\mathcal{N}$ for the Kähler metric of $X$ is $d$-exact. So there exists a 1-form $u$ of class $C^\infty$ on $L$ such that $du = \Theta^\mathcal{N}$; we may suppose that $u$ is real and $u = u^{0,1} + u^{0,T}$, where $u^{0,1}$ is the $(0,1)$ component of $u$. Since $\Theta^\mathcal{N}$ is a $(1,1)$-form, it follows that $\overline{\partial}_b u^{0,1} = 0$, where $\overline{\partial}_b$ is the tangential Cauchy-Riemann operator. By Proposition 4 there exists a $C^k$-extension $U^{0,1}$ of $u^{0,1}$ to $X$, $k \geq 2$, such that $\overline{\partial} U^{0,1} = 0$.

By Hodge symmetry and Dolbeault isomorphism $H^{0,1}(X,\mathbb{C}) \cong H^{1,0}(X,\mathbb{C}) \cong H^0(X,\Omega^1_X)$, where $\Omega^1_X$ is the sheaf of holomorphic 1-forms on $X$. So there exists $\eta \in H^0(X,\Omega^1_X)$ and $\Phi \in C^k(X)$ such that $U^{0,1} = \eta + \overline{\partial}_b \Phi$. It follows that $\Theta^\mathcal{N} = \eta$. 

i∂_b \overline{\partial_b} \text{Im } \Phi \text{ on } L \text{ and this gives a contradiction at the point of } L \text{ where Im } \Phi \text{ reaches its maximum.}

**Remark 3.** A first version of this paper was announced on arXiv in 2014, but there was a gap in the proofs of §4, which is now corrected. Recently, Brinkschulte proved a generalization of Theorem 2 for compact Levi flat hypersurfaces in complex manifolds (see Theorem 1.1. of [8]). She uses crucially the Proposition 4.1 of [8], whose statement and proof are the same as Proposition 1 of this paper and which are unchanged from 2014 in our preprint arXiv:1406.5712. However she refers only to Proposition 1.1 of [25], where the lower positive bound for the eigenvalues of the strongly plurisubharmonic function is not mentioned.

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**References**


