Nonexistence of Levi flat hypersurfaces with positive normal bundle in compact Kähler manifolds of dimension ≥3

Séverine Biard, Andrei Iordan

NONEXISTENCE OF LEVI FLAT HYPERSURFACES WITH POSITIVE NORMAL BUNDLE IN COMPACT KÄHLER MANIFOLDS OF DIMENSION $\geqslant 3$

SÉVERINE BIARD AND ANDREI IORDAN

In memory of Gennadi M. Henkin

Abstract. Let X be a compact connected Kähler manifold of dimension $\geqslant 3$ and L a C^∞ Levi flat hypersurface in X. Then the normal bundle to the Levi foliation does not admit a Hermitian metric with positive curvature along the leaves. This represents an answer to a conjecture of Marco Brunella.

1. Introduction

A classical theorem of Poincaré-Bendixson [28], [29], [5] states that every leaf of a foliation of the real projective plane accumulates on a compact leaf or on a singularity of the foliation. As a holomorphic foliation $\mathcal{F}$ of codimension 1 of $\mathbb{CP}_n$, $n \geqslant 2$, does not contain any compact leaf and its singular set $Sing\ \mathcal{F}$ is not empty, a major problem in foliation theory is the following: can $\mathcal{F}$ contain a leaf F such that $\overline{F} \cap Sing\ \mathcal{F} = \emptyset$? If this is the case, then there exists a nonempty compact set K called exceptional minimal, invariant by $\mathcal{F}$ and minimal for the inclusion such that $K \cap Sing\ \mathcal{F} = \emptyset$. The problem of the existence of an exceptional minimal in $\mathbb{CP}_n$, $n \geqslant 2$ is implicit in [12].

In [13] D. Cerveau proved a dichotomy under the hypothesis of the existence of a holomorphic foliation $\mathcal{F}$ of codimension 1 of $\mathbb{CP}_n$ which admits an exceptional minimal $\mathfrak{M}$: $\mathfrak{M}$ is a real analytic Levi flat hypersurface in $\mathbb{CP}_n$ (i. e. $T(\mathfrak{M}) \cap JT(\mathfrak{M})$ is integrable, where J is the complex structure of $\mathbb{CP}_n$), or there exists $p \in \mathfrak{M}$ such that the leaf through p has a hyperbolic holonomy and the range of the holonomy morphism is a linearisable abelian group. This gave rise to the conjecture of the nonexistence of smooth Levi flat hypersurface in $\mathbb{CP}_n$, $n \geqslant 2$.

The conjecture was proved for $n \geqslant 3$ by A. Lins Neto [22] for real analytic Levi flat hypersurfaces and by Y.-T. Siu [31] for C^{12} smooth Levi flat hypersurfaces. The methods of proofs for the real analytic case are very different from the smooth case.

A real hypersurface of class C^2 in a complex manifold is Levi flat if its Levi form vanishes or equivalently, it admits a foliation by complex hypersurfaces. We say that a (non-necessarly smooth) real hypersurface L in a complex manifold X is Levi flat if $X \backslash L$ is pseudoconvex. An example of (non-smooth) Levi flat hypersurface in

Date: September, 30, 2019.
1991 *Mathematics Subject Classification.* 32V40, 32F32, 32Q15, 32W05.
Key words and phrases. Levi flat hypersurface, weighted $\overline{\partial}$-equation.

$\mathbb{CP}_2$ is $L = \{[z_0, z_1, z_2] : \ |z_1| = |z_2|\}$, where $[z_0, z_1, z_2]$ are homogeneous coordinates in $\mathbb{CP}_2$ (see [19]).

In [21] Iordan and Matthey proved the nonexistence of Lipschitz Levi flat hypersurfaces in $\mathbb{CP}_n$, $n \geqslant 3$, which are of Sobolev class W^s, $s > 9/2$. A principal element of the proof is that the Fubini-Study metric induces a metric of positive curvature on any quotient of the tangent space.

Nonexistence questions for the Levi flat hypersurfaces in compact Kähler manifolds were first discussed by T. Ohsawa in [24], who proved the nonexistence of real-analytic Levi flat hypersufaces with Stein complement in compact Kähler manifolds of dimension $\geqslant 3$.

In [9], M. Brunella proved that the normal bundle to the Levi foliation of a closed real analytic Levi flat hypersurface in a compact Kähler manifold of dimension $n \geqslant 3$ does not admit any Hermitian metric with leafwise positive curvature. The real analytic hypothesis may be relaxed to the assumption of $C^{2,\alpha}$, $0 < \alpha < 1$, such that the Levi foliation extends to a holomorphic foliation in a neighborhood of the hypersurface.

The main step in his proof is to show that the existence of a Hermitian metric with leafwise positive curvature on the normal bundle to the Levi foliation of a compact Levi flat hypersurface L in a Hermitian manifold X, implies that $X \backslash L$ is strongly pseudoconvex, i.e. there exists on $X \backslash L$ an exhaustion function which is strongly plurisubharmonic outside a compact set. This was generalized in [10] for invariant compact subsets of a holomorphic foliation of codimension one. Of course, if X is the complex projective space, then every proper pseudoconvex domain in X is Stein [34].

Brunella stated also the following conjecture [9]: Let X be a compact connected Kähler manifold of dimension $n \geqslant 3$ and L a C^∞ compact Levi flat hypersurface in X. Then the normal bundle to the Levi foliation does not admit any Hermitian metric with leafwise positive curvature.

The assumption $n \geqslant 3$ is necessary in this conjecture (see Example 4.2 of [9]).

In [11] Brunella and Perrone proved that every leaf of a holomorphic foliation $\mathcal{F}$ of codimension one of a projective manifold X of dimension at least 3 and such that $Pic\,(X) = \mathbb{Z}$ accumulates on the singular set of the foliation. In this case the normal bundle to the foliation is ample.

In [25], T. Ohsawa considered a C^∞ Levi flat compact hypersurface L in a compact Kähler manifold X such that the normal bundle to the Levi foliation admits a fiber metric whose curvature is semipositive of rank$\geqslant k$ on the holomorphic tangent space to the leaves and proved that $X \backslash L$ admits an exhaustion plurisubharmonic function of logarithmic growth which is strictly $(n-k)$-convex. Then, if $\dim X \geqslant 3$, he proved that there are no Levi flat real analytic hypersurfaces such that the normal bundle to the Levi foliation admits a fiber metric whose curvature is semipositive of rank$\geqslant 2$ on the holomorphic tangent space to L. Some possibilities for generalization in the smooth case are also indicated.

In this paper we solve the above mentioned conjecture of Brunella for compact connected Kähler manifolds of dimension $n \geqslant 3$. The principal ingredient of the proof is a refinement of the proof of Brunella [9] of the strong pseudoconvexity of $X \backslash L$: we show that there exist a neighborhood U of L and a function v on U vanishing on L, such that $-i\partial\overline{\partial} \ln v \geqslant c\omega$ on $U \backslash L$, where $c > 0$ and ω is the $(1,1)$-form associated to the Kähler metric. Then we use the L^2 estimates [2], [1], [20],

[15] for the weighted $\overline{\partial}$-equation on (n,q)-forms on $X\backslash L$ endowed with a complete Kähler metric. These estimates together with the lower uniform boundedness of the eigenvalues of the Levi form and a duality method developped in [19], allow us to solve the $\overline{\partial}$-equation with compact support for $(0,q)$-forms, $1 \leqslant q \leqslant n-1$, and this leads in dimensions $\geqslant 3$ to the solution of Brunella's conjecture.

2. Preliminaries

Let X be a complex n-dimensional manifold, ω a Kähler metric on X, Ω a domain in X and σ a positive function on Ω. For $\alpha \in \mathbb{R}$ denote

$$L^2_{(p,q)}(\Omega, \sigma^\alpha, \omega) = \left\{ f \in L^2_{(p,q)loc}(\Omega) : \int_\Omega |f|^2 \sigma^{2\alpha} dV_\omega < \infty \right\}$$

endowed with the norm

$$N_{\alpha,\omega,\sigma}(f) = \left(\int_\Omega |f|^2 \sigma^{2\alpha} dV_\omega \right)^{1/2}.$$

Let Ω be a pseudoconvex domain in $\mathbb{CP}_n$ and $\delta_{\partial\Omega}$ the geodesic distance to the boundary for the Fubini-Study metric ω_{FS}. By using the L^2 estimates for the $\overline{\partial}$-operator of Hörmander with the weight $e^{-\varphi}$, $\varphi = -\alpha \log \delta_{\partial\Omega}$ which is strongly plurisubharmonic by a theorem of Takeuchi [34], Henkin and Iordan proved in [19] the existence and regularity of the $\overline{\partial}$ equation for $\overline{\partial}$-closed forms in $L^2_{(p,q)}(\Omega, \delta_{\partial\Omega}^{-\alpha}, \omega_{FS})$ verifying the moment condition. This gives the regularity of the $\overline{\partial}$-operator in pseudoconcave domains with Lipschitz boundary [19] and, by using a method of Siu [31], [32], the nonexistence of smooth Levi flat hypersurfaces in $\mathbb{CP}_n$, $n \geqslant 3$ follows (see [21]). These techniques will be used in the 4th and the 5th paragraph.

We will use also the following theorem of regularity of $\overline{\partial}$ equation of Brinkschulte [7]:

Theorem 1. *Let Ω be a relatively compact domain with Lipschitz boundary in a Kähler manifold (X,ω) and set $\delta_{\partial\Omega}$ the geodesic distance to the boundary of Ω. Let $f \in L^2_{(p,q)}(\Omega, \delta_{\partial\Omega}^{-k}, \omega) \cap C^k_{(p,q)}\left(\overline{\Omega}\right) \cap C^\infty_{(p,q)}(\Omega)$, $q \geqslant 1$, $k \in \mathbb{N}$ and $u \in L^2_{(p,q-1)}(\Omega, \delta_{\partial\Omega}^{-k}, \omega)$ such that $\overline{\partial} u = f$ and $\overline{\partial}^*_{-k} u = 0$, where $\overline{\partial}^*_{-k}$ is the Hilbert space adjoint of the unbounded operator $\overline{\partial}_{-k} : L^2_{(p,q-1)}(\Omega, \delta_{\partial\Omega}^{-k}, \omega) \to L^2_{(p,q)}(\Omega, \delta_{\partial\Omega}^{-k}, \omega)$. Then for k big enough $u \in C^{s(k)}_{(p,q-1)}\left(\overline{\Omega}\right)$ where $s(k) \underset{k\to\infty}{\sim} \sqrt{k}$.*

3. Strong pseudoconvexity of the complement of a Levi flat hypersurface

Let L be a smooth Levi flat hypersurface in a Hermitian manifold X. As was mentioned in [9] and in [25], by taking a double covering, we can assume that L is orientable and the complement of L has two connected components in a neighborhood of L. This will be always supposed in the sequel and for an open neighborhood U of L we will denote by U^+ and U^- the two connected components of $U\backslash L$. We will denote by δ_L the signed geodesic distance to L.

In [9] Brunella proved that the complement of a closed Levi flat hypersurface in a compact Hermitian manifold of class $C^{2,\alpha}$, $0 < \alpha < 1$, having the property that the Levi foliation extends to a holomorphic foliation in a neighborhood of L and the normal bundle to the Levi foliation admits a C^2 Hermitian metric with leafwise positive curvature is strongly pseudoconvex, i.e. there exists an exhaustion function which is strongly plurisubharmonic outside a compact set. The following proposition strenghtens this result:

Proposition 1. *Let L be a compact C^3 Levi flat hypersurface in a Hermitian manifold X of dimension $n \geqslant 2$, such that the normal bundle $\mathcal{N}_L^{1,0}$ to the Levi foliation admits a C^2 Hermitian metric with leafwise positive curvature. Then there exist a neighborhood U of L, $c>0$ and a non-negative function $v \in C^2(U)$, vanishing on L and positive on $U \backslash L$ such that $-i\partial\overline{\partial}\ln v \geqslant c\omega$ on $U\backslash L$, where ω is the $(1,1)$-form associated to the metric. Moreover, there exists a nonvanishing continuous function g in a neighborhood of L such that $v = g\delta_L^2$.*

Proof. Let $z_0 \in L$. There exist holomorphic coordinates $z = (z_1, \cdots, z_{n-1}, z_n) = (z', z_n)$ in a neighborhood of z_0 such that the local parametric equations for L are of the form

$$z_j = w_j,\ j = 1, ..., n-1,\ z_n = \varphi(w', t)$$

where φ is of class C^3 (see [4]) on a neighborhood of the origin in $\mathbb{C}^{n-1} \times \mathbb{R}$, holomorphic in w' and $\frac{\partial\varphi}{\partial t}(z_0) \in \mathbb{R}^*$. We consider a C^3 extension $\psi = (\psi_1, ..., \psi_n)$ of φ on a neighborhood of the origin in $\mathbb{C}^{n-1}\times\mathbb{C}$, $\psi(w', t+is) = (w', \varphi(w', t) + is)$. Then ψ is a C^3 diffeomorphism in a neighborhood of z_0 and holomorphic in w'. It follows that

$$L = \{(z', z_n) : \rho(z', z_n) = 0\}.$$

where $\rho = \operatorname{Im}\left(\psi^{-1}\right)_n$. We denote $f = \left(\psi^{-1}\right)_n(z', z_n)$. Since $\overline{\partial}_b f = 0$ on L, where $\overline{\partial}_b$ is the tangential Cauchy-Riemann operator on L, there exists an extension $\widetilde{f}$ of class C^3 in a neighborhood of z_0 such that $\overline{\partial}\widetilde{f}$ vanishes to order greater than 2 on L, i.e. $D^l\overline{\partial}\widetilde{f} = 0$ for $|l| \leqslant 2$ on L.

So there exists an open finite covering $\left(\widetilde{U_j}\right)_{j\in J}$ by holomorphic charts of L such that $\widetilde{U_j}\backslash L = \widetilde{U_j^+} \cup \widetilde{U_j^-}$ such that $U_j = L \cap \widetilde{U_j} = \left\{z \in \widetilde{U_j} : \operatorname{Im}\widetilde{f_j} = 0\right\}$, where $\overline{\partial}\widetilde{f_j}$ vanishes to order greater than 2 on L and the Levi foliation is given on U_j by $\left\{z \in U_j :\ \widetilde{f_j}(z) = c_j\right\}$, $c_j \in \mathbb{R}$. Thus $d\widetilde{f_j} = \partial\widetilde{f_j}$ is a nonvanishing section of $\mathcal{N}_L^{1,0}$ on U_j and by shrinking $\widetilde{U_j}$, we may consider that $d\widetilde{f_j} \neq 0$ on $\widetilde{U_j}$.

We may suppose that $\mathcal{N}_L^{1,0}$ is represented by a cocycle $\{g_{jk}\}$ of class C^2 subordinated to the covering $(U_j)_{j\in J}$ and there exist closed $(1,0)$-forms α_j of class C^2 on U_j holomorphic along the leaves such that $T^{1,0}(U_j) = \ker\alpha_j$ for every $j \in J$ and $\alpha_j = g_{jk}\alpha_k$ on $U_j \cap U_k$. So $(\alpha_j)_{j\in J}$ defines a global form α on L with values in $\mathcal{N}_L^{1,0}$ such that locally on U_j we have $\alpha(z) = \alpha_j(z) \otimes \alpha_j^*(z)$ where α_j^* is the dual frame of α_j. In particular we have $\alpha_k^* = g_{jk}\alpha_j^*$.

Let h be a C^2 Hermitian metric with positive leafwise curvature $\Theta_h\left(\mathcal{N}_L^{1,0}\right)$ on $\mathcal{N}_L^{1,0}$. h is defined on each U_j by a C^2 function $h_j = \left|\alpha_j^*\right|^2$ such that $h_k = |g_{jk}|^2 h_j$ on $U_j \cap U_k$.

Since $\alpha_j = \eta_j d\widetilde{f_j}$ on U_j for every j, where η_j are nowhere vanishing functions of class C^2 on U_j holomorphic along the leaves and

$$\frac{1}{\eta_k}\left(d\widetilde{f_k}\right)^* = \frac{1}{\eta_j}g_{jk}\left(d\widetilde{f_j}\right)^*$$

on $U_j \cap U_k$, it follows that

$$\left|g_{jk}(z)\right|^2 = \left|\frac{\eta_j(z)}{\eta_k(z)}\right|^2 \left|\frac{\left(d\widetilde{f}_k\right)^*}{\left(d\widetilde{f}_j\right)^*}\right|^2 = \frac{h_k(z)}{h_j(z)},\ z \in U_j \cap U_k.$$

So

$$h_j \left|\eta_j\right|^2 \left(\operatorname{Im} \widetilde{f}_j\right)^2 - h_k \left|\eta_k\right|^2 \left(\operatorname{Im} \widetilde{f}_k\right)^2$$

vanishes to order greater than 2 on $U_j \cap U_k$ and $\left(h_j \left|\eta_j\right|^2 \left(\operatorname{Im} \widetilde{f}_j\right)^2\right)_{j\in J}$ defines a jet of order 2 on L. By Whitney extension theorem there exists a C^2 function v on X such that $v - h_j \left|\eta_j\right|^2 \left(\operatorname{Im} \widetilde{f}_j\right)^2$ vanishes to order 2 on U_j for every $j \in J$. Let $\widetilde{\eta_j}, \widetilde{h_j}$ be C^2 extensions of η_j, h_j on $\widetilde{U_j}$ and set $\widetilde{\alpha_j} = \widetilde{\eta_j} d\widetilde{f}_j$, $\widetilde{v} = \widetilde{h}_j \left|\widetilde{\eta_j}\right|^2 \left(\operatorname{Im} \widetilde{f}_j\right)^2$.

For $z \in \widetilde{U_j}$ denote $E_z' = \left\{V' \in T_z^{1,0}(X) : \left\langle \partial \operatorname{Im} \widetilde{f}_j, V' \right\rangle = 0\right\}$ and E_z'' the orthogonal of E_z' in $T_z^{1,0}(X)$. Then for every $V \in T_z^{1,0}(X)$ there exists $V' \in E_z', V'' \in E_z''$ such that $V = V' + V''$. The curvature form $\Theta\left(\mathcal{N}_L^{1,0}\right)$ is represented by $-i\partial\overline{\partial} \ln \left(h_j \left|\alpha_j\right|^2\right)$ on U_j, so by shrinking $\widetilde{U_j}$ we may suppose that there exists $\beta > 0$ such that $\left(-i\partial\overline{\partial} \ln \left(\widetilde{h}_j \left|\widetilde{\alpha_j}\right|^2\right)\right)\left(V', \overline{V'}\right) \geqslant \beta\omega\left(V', \overline{V'}\right)$ for every $z \in \widetilde{U_j}$ and $V \in T_z^{1,0}(X)$.

On $\widetilde{U_j} \backslash L$ we have

$$\begin{aligned}
(3.1) \qquad -i\partial\overline{\partial} \ln \widetilde{v} &= -i\partial\overline{\partial} \ln \left(\widetilde{h}_j \left|\frac{\widetilde{\alpha_j}}{d\widetilde{f}_j}\right|^2 \left(\operatorname{Im} \widetilde{f}_j\right)^2\right) \\
&= -i\partial\overline{\partial} \ln \widetilde{h}_j \left|\widetilde{\alpha_j}\right|^2 + i\partial\overline{\partial} \ln \left|d\widetilde{f}_j\right|^2 - i\partial\overline{\partial} \ln \left(\operatorname{Im} \widetilde{f}_j\right)^2.
\end{aligned}$$

Let $z \in \widetilde{U_j}$ and $V \in T_z^{1,0}(X)$. Then $V = V' + V''$, $V' \in E_z'$ and $V" \in E_z''$ and

$$\begin{aligned}
-i\partial\overline{\partial} \ln \widetilde{h}_j \left|\widetilde{\alpha_j}\right|^2 \left(V, \overline{V}\right) &= \left(-i\partial\overline{\partial} \ln \left(\widetilde{h}_j \left|\widetilde{\alpha_j}\right|^2\right)\right)\left(V', \overline{V'}\right) \\
&\quad + 2 \operatorname{Re}\left(-i\partial\overline{\partial} \ln \left(\widetilde{h}_j \left|\widetilde{\alpha_j}\right|^2\right)\left(V', \overline{V''}\right)\right) \\
&\quad + \left(-i\partial\overline{\partial} \ln \left(\widetilde{h}_j \left|\widetilde{\alpha_j}\right|^2\right)\right)\left(V'', \overline{V''}\right)
\end{aligned}$$

There exists a constant $C > 0$ depending on the eigenvalues of $-i\partial\overline{\partial} \ln \left(\widetilde{h}_j \left|\widetilde{\alpha_j}\right|^2\right)$ with respect to ω such that for every $\varepsilon > 0$

$$2\left|\operatorname{Re}\left(-i\partial\overline{\partial} \ln \left(\widetilde{h}_j \left|\widetilde{\alpha_j}\right|^2\right)\left(V', \overline{V''}\right)\right)\right| \leqslant C\left(\varepsilon\omega\left(V', \overline{V'}\right) + \frac{1}{\varepsilon}\omega\left(V'', \overline{V''}\right)\right),$$

so

$$\begin{aligned}
-i\partial\overline{\partial} \ln \widetilde{h}_j \left|\widetilde{\alpha_j}\right|^2 \left(V, \overline{V}\right) &\geqslant \beta\omega\left(V', \overline{V'}\right) - C\left(\varepsilon\omega\left(V', \overline{V'}\right) - \frac{1}{\varepsilon}\omega\left(V'', \overline{V''}\right)\right) \\
(3.2) \qquad &\quad - \left\|-i\partial\overline{\partial} \ln \widetilde{h}_j \left|\widetilde{\alpha_j}\right|^2\right\|_\omega \omega\left(V'', \overline{V''}\right)
\end{aligned}$$

Since $\overline{\partial}\widetilde{f}_j$ vanishes to order greater than 2 on L, for every $\gamma > 0$ there exists a neighborhood of L such that

$$\left| i\partial\overline{\partial}\ln\left|d\widetilde{f}_j\right|^2\left(V,\overline{V}\right)\right| \leqslant \gamma\omega\left(V,\overline{V}\right) \tag{3.3}$$

and

$$\left| i\partial\overline{\partial}\,\mathrm{Im}\,\widetilde{f}_j\left(V,\overline{V}\right)\right| \leqslant \gamma\left(\mathrm{Im}\,\widetilde{f}_j\right)\omega\left(V,\overline{V}\right). \tag{3.4}$$

Let $z \in \widetilde{U_j}\backslash L$. By (3.4) it follows that

$$\begin{aligned} -i\partial\overline{\partial}\ln\left(\mathrm{Im}\,\widetilde{f}_j\right)^2\left(V,\overline{V}\right) &= \left(-2\frac{i\partial\overline{\partial}\,\mathrm{Im}\,\widetilde{f}_j}{\mathrm{Im}\,\widetilde{f}_j} + 2i\frac{\partial\,\mathrm{Im}\,\widetilde{f}_j\wedge\overline{\partial}\,\mathrm{Im}\,\widetilde{f}_j}{\left(\mathrm{Im}\,\widetilde{f}_j\right)^2}\right)\left(V,\overline{V}\right) \\ &\geqslant -2\gamma\omega\left(V,\overline{V}\right) + 2i\frac{\partial\,\mathrm{Im}\,\widetilde{f}_j\wedge\overline{\partial}\,\mathrm{Im}\,\widetilde{f}_j}{\left(\mathrm{Im}\,\widetilde{f}_j\right)^2}\left(V'',\overline{V''}\right) \\ &\geqslant -2\gamma\omega\left(V,\overline{V}\right) + \frac{2\inf\limits_{\widetilde{U_j}}\left\|\partial\,\mathrm{Im}\,\widetilde{f}_j\right\|_\omega^2}{\left(\mathrm{Im}\,\widetilde{f}_j\right)^2}\omega\left(V'',\overline{V''}\right). \end{aligned} \tag{3.5}$$

By using (3.2), (3.3) and (3.5), from (3.1) we obtain

$$\begin{aligned} \left(-i\partial\overline{\partial}\ln\widetilde{v}\right)\left(V,\overline{V}\right) \geqslant\ & (\beta - C\varepsilon)\,\omega\left(V',\overline{V'}\right) \\ & + \left(\frac{2}{\left(\mathrm{Im}\,f_j\right)^2}\inf_{\widetilde{U_j}}\left\|\partial\,\mathrm{Im}\,\widetilde{f}_j\right\|_\omega^2 - \frac{C}{\varepsilon} - \left\|-i\partial\overline{\partial}\ln\widetilde{h}_j\left|\widetilde{\alpha_j}\right|^2\right\|_\omega\right)\omega\left(V'',\overline{V''}\right) \\ & -2\gamma\omega\left(V,\overline{V}\right). \end{aligned}$$

By choosing $0 < C\varepsilon < \beta$ and by shrinking $\widetilde{U_j}$ such that $\frac{2}{(\mathrm{Im}\,f_j)^2}$ is big enough and γ small enough, we obtain that there exists $c > 0$ such that $-i\partial\overline{\partial}\ln\widetilde{v} \geqslant c\omega$ on $\widetilde{U_j}\backslash L$. Finally, since $v - \widetilde{v}$ vanishes to order greater than 2 on L, it follows that there exists a neighborhood U' of L such that $-\ln v$ is strongly plurisubharmonic on $U'\backslash L$. We can now take $U = \{z \in U' :\ v(z) < \mu\}$ for $\mu > 0$ small enough.

L is a C^3 manifold, so the signed distance function δ_L is a defining function of class C^3 for L. Since v is of class C^2 on U and vanishes to order greater than 2 on L, we have $v = g\delta_L^2$ with g continuous in a neighborhood of L.

Suppose that there exists $x \in L$ such that $g(x) = 0$. Then $v = o\left(\delta_L^2\right)$ in a neighborhood of x. But there exists j such that $x \in U_j$ and $v = h_j\left|\eta_j\right|^2\left(\mathrm{Im}\,\widetilde{f}_j\right)^2 + o\left(\delta_L^2\right)$. Since $\mathrm{Im}\,\widetilde{f}_j = 0$ and $d\,\mathrm{Im}\,\widetilde{f}_j \neq 0$ on L it follows that $\left|\nabla^2 v\right|(x) \neq 0$. This contradiction shows that $g(x) \neq 0$ on L. ∎

4. Weighted estimates for the $\overline{\partial}$-equation

Remark 1. *Under the hypothesis and conclusions of Proposition 1, we consider a positive extension $\widetilde{v}$ of the restriction of v on a neighborhood of L to $X\backslash L$. Let $s > 0$ such that $\{v < e^{-s}\} \subset U$ and let φ be a smooth function on $\mathbb{R}$ such that $\varphi = 0$ on $]-\infty, s]$ and φ is strictly convex increasing on $]s, \infty[$. Then $\psi = \varphi(-\ln\widetilde{v})$ is a* *plurisubharmonic exhaustion function of* $X\backslash L$*, which is strongly plurisubharmonic outside a compact subset of* $X\backslash L$*.*

In the sequel, L *will be a compact* C^∞ *Levi flat hypersurface in a compact Kähler manifold* X *of dimension* $n \geqslant 2$*, verifying the hypothesis and the conclusions of Proposition 1. We denote* $X^\pm$ *the connected components of* $\{z \in X :\ v > 0\}$ *endowed with a complete Kähler metric* $\widetilde{\omega}$ *which will be defined later and we set*

$$\mathcal{D}_{(p,q)}(X^\pm) = \left\{ f \in C^\infty_{(p,q)}\left(X^\pm\right) :\ supp\ f \subset\subset X^\pm \right\}$$

and

$$\mathcal{H}_{(p,q)}\left(X^\pm, \widetilde{v}^\alpha, \widetilde{\omega}\right) = \ker \overline{\partial} \cap \ker \overline{\partial}^*_\alpha \subset L^2_{(p,q)}\left(X^\pm, \widetilde{v}^\alpha, \widetilde{\omega}\right)$$

where $\overline{\partial}^*_\alpha$ *is the Hilbert space adjoint of the operator* $\overline{\partial} : L^2_{(p,q)}\left(X^\pm, \widetilde{v}^\alpha, \widetilde{\omega}\right) \to L^2_{(p,q+1)}\left(X^\pm, \widetilde{v}^\alpha, \widetilde{\omega}\right)$*.*

Proposition 2. *For every* $\alpha > 0$*, there exists a complete Kähler metric* $\widetilde{\omega}$ *on* $X\backslash L$*,* $\omega \leqslant \widetilde{\omega} \leqslant \frac{C}{\widetilde{v}^2}\omega$*,* $C > 0$*, such that the range* $\mathcal{R}^\alpha_{(n,q)}\left(X^\pm\right)$ *of the operator* $\overline{\partial}_\alpha : L^2_{(n,q-1)}\left(X^\pm, \widetilde{v}^\alpha, \widetilde{\omega}\right) \to L^2_{(n,q)}\left(X^\pm, \widetilde{v}^\alpha, \widetilde{\omega}\right)$ *is closed for* $1 \leqslant q \leqslant n$*.*

Proof. The proof is based on methods of [16] (see also [14]).

Denote by ω the Kähler metric of X. Since $i\partial\overline{\partial}\left(-\ln v\right) \geqslant c\omega$ on $U\backslash L$, $c > 0$, by a method developped in [27] it follows that there exist a neigborhood V of L and $\eta > 0$ such that $-v^\eta$ is strongly plurisubharmonic on $V\backslash L$. Then for $0 < \beta < \eta$, we have the Donnelly-Fefferman estimate [17]

$$i\partial\left(-\ln v\right) \wedge \overline{\partial}\left(-\ln v\right) \leqslant ir\partial\overline{\partial}\left(-\ln v\right). \tag{4.1}$$

on $V\backslash L$, with $0 < r = \beta/\eta < 1$. This is equivalent to say that the norm of $\partial\left(-\ln v\right)$ measured in the metric $i\partial\overline{\partial}\left(-\ln v\right)$ is smaller than r on $V\backslash L$ (see also [6] and [19]).

Let $\alpha > 0$. We consider the trivial line bundle E on $X\backslash L$ endowed with the Hermitian metric $h_\alpha = e^{\alpha \ln \widetilde{v}}$. Set

$$\widetilde{\omega} = i\Theta\left(E\right) + K\omega = i\alpha\partial\overline{\partial}\left(-\ln \widetilde{v}\right) + K\omega$$

with K a positive constant. Since $-\ln \widetilde{v}$ is an exhaustion function on $X\backslash L$, it follows by (4.1) that for K big enough $\widetilde{\omega}$ is a complete Kähler metric on $X\backslash L$ such that $\omega \leqslant \widetilde{\omega} \leqslant \frac{C}{\widetilde{v}^2}\omega$, $C > 0$.

Denote λ_j (respectively $\widetilde{\lambda}_j$) the eigenvalues of $i\Theta\left(E\right)$ with respect to ω (respectively $\widetilde{\omega}$), $1 \leqslant j \leqslant n$, in increasing order. By Proposition 1, there exists $c > 0$ such that $i\Theta\left(E\right) = i\alpha\partial\overline{\partial}\left(-\ln \widetilde{v}\right) \geqslant \alpha c\omega$ on $\{\psi > b\}$ for b big enough. So, as in [16] (1.6) we have

$$1 \geqslant \widetilde{\lambda}_j = \frac{\lambda_j}{\lambda_j + K} \geqslant \frac{\alpha c}{\alpha c + K} > 0,\ 1 \leqslant j \leqslant n \tag{4.2}$$

on $\{\psi > b\}$. By Bochner-Kodaira-Nakano inequality (see for ex. [14]) we have

$$N_{\alpha,\widetilde{\omega},\widetilde{v}}\left(\overline{\partial}u\right)^2 + N_{\alpha,\widetilde{\omega},\widetilde{v}}\left(\overline{\partial}^*_\alpha u\right)^2 \geqslant \int_{X^\pm} \left\langle \left(\left[i\Theta\left(E\right), \Lambda_{\widetilde{\omega}}\right]\right) u, u\right\rangle_{\alpha,\widetilde{\omega},\widetilde{v}} dV_{\widetilde{\omega}} \tag{4.3}$$

for every $u \in \mathcal{D}_{(n,q)}\left(X\backslash L\right)$, where $N_{\alpha,\widetilde{\omega},\widetilde{v}} = \int_{X^\pm} |u|^2_{\widetilde{\omega}}\, \widetilde{v}^\alpha dV_{\widetilde{\omega}}$.

Let χ be a smooth function on X such that $0 \leqslant \chi \leqslant 1$, $\chi = 0$ on a neighborhood of $\{\psi < b\}$ and $\chi = 1$ on a neighborhood $\{\psi > b'\}$ of L, $b' > b$. By (4.3) and (4.2), for every $u \in \mathcal{D}_{(n,q)}(X\backslash L)$ we have

$$\begin{aligned}
N_{\alpha,\widetilde{\omega},\widetilde{v}}\left(\overline{\partial}(\chi u)\right)^2 + N_{\alpha,\widetilde{\omega},\widetilde{v}}\left(\overline{\partial}^*_\alpha(\chi u)\right)^2 &\geqslant \int_{X^\pm} \langle([i\Theta(E),\Lambda_{\widetilde{\omega}}])\chi u, \chi u\rangle_{\alpha,\widetilde{\omega},\widetilde{v}}\, dV_{\widetilde{\omega}} \\
&\geqslant \int_{\{\psi>b'\}} \langle([i\Theta(E),\Lambda_{\widetilde{\omega}}])\chi u, \chi u\rangle_{\alpha,\widetilde{\omega},\widetilde{v}}\, dV_{\widetilde{\omega}} \\
&\geqslant \int_{\{\psi>b'\}} (\lambda_1+\cdots+\lambda_n)\,|\chi u|^2_{\widetilde{\omega}}\,\widetilde{v}^\alpha dV_{\widetilde{\omega}} \\
&\geqslant \frac{\alpha c}{\alpha c+K}\int_{\{\psi>b'\}} |u|^2_{\widetilde{\omega}}\,\widetilde{v}^\alpha dV_{\widetilde{\omega}}
\end{aligned}$$

so there exists $C, c' > 0$ such that

$$\begin{aligned}
&2N_{\alpha,\widetilde{\omega},\widetilde{v}}\left(\overline{\partial}u\right)^2 + 2N_{\alpha,\widetilde{\omega},\widetilde{v}}\left(\overline{\partial}^*_\alpha u\right)^2 + C\int_{supp(\chi')} |u|^2_{\widetilde{\omega}}\,\widetilde{v}^\alpha dV_{\widetilde{\omega}} \\
\geqslant\ & c'\int_{X\backslash L} |u|^2_{\widetilde{\omega}}\,\widetilde{v}^\alpha dV_{\widetilde{\omega}} - c'\int_{\{\psi<b'\}} |u|^2_{\widetilde{\omega}}\,\widetilde{v}^\alpha dV_{\widetilde{\omega}}.
\end{aligned}$$

Finally it follows that there exists a compact subset $F = supp(\chi') \cup \{\psi \leqslant b'\}$ of $X^\pm$ such that for every $u \in \mathcal{D}_{(n,q)}(X\backslash L)$

$$c' N_{\alpha,\widetilde{\omega},\widetilde{v}}(u)^2 \leqslant 2N_{\alpha,\widetilde{\omega},\widetilde{v}}\left(\overline{\partial}u\right)^2 + 2N_{\alpha,\widetilde{\omega},\widetilde{v}}\left(\overline{\partial}^*_\alpha u\right)^2 + (C+c')\int_F |u|^2_{\widetilde{\omega}}\,\widetilde{v}^\alpha dV_{\widetilde{\omega}}. \tag{4.4}$$

Since $\widetilde{\omega}$ is a complete metric on $X\backslash L$, (4.4) is valid for every $u \in \left(Dom\overline{\partial}\right) \cap \left(Dom\overline{\partial}^*_\alpha\right)$. The conclusion of Proposition 2 is now a consequence of Proposition 1.2 of [23]. ∎

Corollary 1. *For every $\alpha > 0$ and $1 \leqslant q \leqslant n$ we have $\mathcal{H}_{(n,q)}(X^\pm, \widetilde{v}^\alpha, \widetilde{\omega}) = \{0\}$.*

Proof. As $(X^\pm, \widetilde{\omega})$ is a connected weakly 1-complete Kähler manifold and the bundle E defined in the proof of Theorem 2 is a semi-positive line bundle on $X^\pm$ which is positive outside a compact suset of $X^\pm$, the Corollary 1 is a consequence of [33], Corollary of the Main Theorem (see also [3], [30] and [26], Corollary 2.10). ∎

By taking in account Corollary 1, a classical application of Proposition 2 (see for example [18]) is the following:

Corollary 2. *For every $\alpha > 0$ and , $1 \leqslant q \leqslant n$ we have:*

1. *There exists the $\overline{\partial}$-Neumann operator $\mathcal{N}^\alpha_{(n,q)} : L^2_{(n,q)}(X^\pm, \widetilde{v}^\alpha, \widetilde{\omega}) \to L^2_{(n,q)}(X^\pm, \widetilde{v}^\alpha, \widetilde{\omega})$ such that for every $f \in L^2_{(n,q)}(X^\pm, \widetilde{v}^\alpha, \widetilde{\omega})$ we have the orthogonal decomposition $f = \overline{\partial}\overline{\partial}^*_\alpha \mathcal{N}^\alpha_{(n,q)} f + \overline{\partial}^*_\alpha \overline{\partial}\mathcal{N}^\alpha_{(n,q)} f$ and $\overline{\partial}\mathcal{N}^\alpha_{(n,q)} = \mathcal{N}^\alpha_{(n,q+1)}\overline{\partial}$, $\overline{\partial}^*_\alpha \mathcal{N}^\alpha_{(n,q)} = \mathcal{N}^\alpha_{(n,q-1)}\overline{\partial}^*_\alpha$.*
2. *For every $\overline{\partial}$-closed form $f \in L^2_{(n,q)}(X^\pm, \widetilde{v}^\alpha, \widetilde{\omega})$, $\overline{\partial}\left(\overline{\partial}^*_\alpha \mathcal{N}^\alpha_{(n,q)} f\right) = f$.*

Lemma 1. *Let $f \in C^\infty_{(0,q)}(X)$, $1 \leqslant q \leqslant n-1$, be a $\overline{\partial}$-closed form such that f vanishes to infinite order on L. Let $\psi_1, \psi_2 \in Dom\overline{\partial} \subset L^2_{(n,n-q)}(X^\pm, \widetilde{v}^\alpha, \widetilde{\omega})$ such that $\overline{\partial}\psi_1 = \overline{\partial}\psi_2$. Then*

$$\int_{X^\pm} f \wedge (\psi_1 - \psi_2) = 0.$$

Proof. By Corollary 2, there exists $h \in L^2_{(n,n-q-1)}(X^\pm, \widetilde{v}^\alpha, \widetilde{\omega})$ such that $\psi_1 - \psi_2 = \overline{\partial}h$. Since f vanishes to infinite order on L and $\widetilde{\omega} \leqslant \frac{C}{v^2}\omega$, it follows that

$$\int_{X^\pm} f \wedge (\psi_1 - \psi_2) = \lim_{\varepsilon \to 0} \int_{\{v>\varepsilon\}\cap X^\pm} f \wedge \overline{\partial} h = \lim_{\varepsilon \to 0} \left(\int_{\{v>\varepsilon\}} \overline{\partial} f \wedge h + \int_{\{v=\varepsilon\}\cap X^\pm} f \wedge h \right) = 0.$$

∎

Proposition 3. *Let $f \in C^\infty_{(0,q)}(X)$, $1 \leqslant q \leqslant n-1$, be a $\overline{\partial}$-exact form such that f vanishes to infinite order on L. Then for every $\alpha > 0$, there exists $u \in L^2_{(0,q-1)}(X^\pm, \widetilde{v}^{-\alpha}, \widetilde{\omega})$ such that $\overline{\partial} u = f$ and $N_{-\alpha,\widetilde{\omega},\widetilde{v}}(u) \leqslant C_\alpha N_{-\alpha,\widetilde{\omega},\widetilde{v}}(f)$, with $C_\alpha > 0$ independent of f.*

Proof. Step 1. Definition by duality of $u \in L^2_{(0,q-1)}(X^\pm, \widetilde{v}^{-\alpha}, \widetilde{\omega})$, $1 \leqslant q \leqslant n-1$.

The proof of this point is inspired from [19], Proposition 5.3. By Proposition 2, $\mathcal{R}^\alpha_{(n,q)}(X^\pm)$ is closed for every $\alpha > 0$ and by Corollary 2 we can find a bounded operator $T^\alpha_{(n,q)} = \overline{\partial}^*_\alpha \mathcal{N}^\alpha_{(n,q)} : \mathcal{R}^\alpha_{(n,q)}(X^\pm) \to L^2_{(n,q-1)}(X^\pm, \widetilde{v}^\alpha, \widetilde{\omega})$, such that $\overline{\partial} T^\alpha_{(n,q)}\varphi = \varphi$ for every $\varphi \in \mathcal{R}^\alpha_{(n,q)}(X^\pm)$, $1 \leqslant q \leqslant n$.

Define now the continuous linear form Φ_f on $\mathcal{R}^\alpha_{(n,n-q+1)}(X^\pm)$, $1 \leqslant q \leqslant n$, by

$$\Phi_f(\varphi) = \int_{X^\pm} f \wedge T^\alpha_{(n,n-q+1)}\varphi,\ \varphi \in \mathcal{R}^\alpha_{(n,n-q+1)}\left(X^\pm\right).$$

By the Hahn-Banach theorem, we extend Φ_f as a linear form $\widetilde{\Phi_f}$ on $L^2_{(n,n-q+1)}(X^\pm, \widetilde{v}^\alpha, \widetilde{\omega})$ such that $\left\|\widetilde{\Phi_f}\right\| = \|\Phi_f\|$. Since $\left(L^2_{(n,n-q+1)}(X^\pm, \widetilde{v}^\alpha, \widetilde{\omega})\right)' = L^2_{(0,q-1)}(X^\pm, \widetilde{v}^{-\alpha}, \widetilde{\omega})$ by the pairing

$$(\beta_1, \beta_2) = \int_{X^\pm} \beta_1 \wedge \beta_2,\ \beta_1 \in L^2_{(0,q-1)}\left(X^\pm, \widetilde{v}^{-\alpha}, \widetilde{\omega}\right),\ \beta_2 \in L^2_{(n,n-q+1)}\left(X^\pm, \widetilde{v}^\alpha, \widetilde{\omega}\right),$$

there exists $u \in L^2_{(0,q-1)}(X^\pm, \widetilde{v}^{-\alpha}, \widetilde{\omega})$ such that

$$\widetilde{\Phi_f}(\varphi) = \int_{X^\pm} u \wedge \varphi$$

for every $\varphi \in \mathcal{R}^\alpha_{(n,n-q+1)}(X^\pm)$.

Step 2. We prove that $\overline{\partial}(-1)^q u = f$, $1 \leqslant q \leqslant n-1$.

Let $\varphi = \overline{\partial}\psi \in C^\infty_{(n,n-q+1)}(X^\pm)$ with $\psi \in \mathcal{D}_{(n,n-q)}(X^\pm)$. Set $g_\alpha = \overline{\partial}^*_\alpha \mathcal{N}^\alpha_{(n,n-q+1)}\overline{\partial}\psi \in L^2_{(n,n-q)}(X^\pm, \widetilde{v}^\alpha, \widetilde{\omega})$. By Corollary 2, $\overline{\partial} g_\alpha = \varphi$ and by Lemma 1

$$\int_{X^\pm} f \wedge g_\alpha = \int_{X^\pm} f \wedge \psi. \tag{4.5}$$

But by step 1 we have

$$\widetilde{\Phi_f}(\varphi) = \int_{X^\pm} u \wedge \overline{\partial}\psi = \Phi_f(\varphi) = \int_{X^\pm} f \wedge g_\alpha \tag{4.6}$$

and by (4.5) and (4.6) it follows that

$$\int_{X^\pm} f \wedge \psi = \int_{X^\pm} u \wedge \overline{\partial}\psi$$

for every $\psi \in \mathcal{D}_{(n,n-q)}(X^\pm)$. Therefore $\overline{\partial}(-1)^q u = f$ and the Proposition is proved. ∎

Remark 2. *Since $\omega \leqslant \widetilde{\omega} \leqslant \frac{C}{\widetilde{v}^2}\omega$, by Lemma VIII.6.3 of* [14] *it follows that:*

a) Let f be a smooth (n,q)-form on X such that f vanishes to order k on L. Then $f \in L^2_{(n,q)}\left(X^\pm, \widetilde{v}^{-k}, \widetilde{\omega}\right)$

Indeed

$$\int_{X^\pm} |f|^2_{\widetilde{\omega}}\, \widetilde{v}^{-k} dV_{\widetilde{\omega}} \leqslant \int_{X^\pm} |f|^2_{\omega}\, \widetilde{v}^{-k} dV_{\omega} < \infty$$

b) Let $f \in L^2_{(n,q)}\left(X^\pm, \widetilde{v}^{-k}, \widetilde{\omega}\right)$, $k > 2$. Then $f \in L^2_{(n,q)}\left(X^\pm, \widetilde{v}^{-k+2}, \omega\right)$.

Indeed

$$\int_{X^\pm} |f|^2_{\omega}\, \widetilde{v}^{-k+2} dV_{\omega} \leqslant C \int_{X^\pm} |f|^2_{\widetilde{\omega}}\, \widetilde{v}^{-k} dV_{\widetilde{\omega}} < \infty,\ C > 0.$$

5. Nonexistence of Levi flat hypersurfaces

Proposition 4. *Let L be a compact C^∞ Levi flat hypersurface in a Kähler manifold X of dimension $n \geqslant 3$ such that the normal bundle $\mathcal{N}^{1,0}_L$ to the Levi foliation admits a C^2 Hermitian metric with leafwise positive curvature. Let $u \in C^\infty_{(0,q)}(L)$, $1 \leqslant q \leqslant n-2$, such that $\overline{\partial}_b u = 0$. Then for every $k \in \mathbb{N}^*$ there exist a $\overline{\partial}$-closed extension $U_k \in C^k_{(0,q)}(X)$ of u.*

Proof. By Proposition 1 there exist a neighborhood U of L, $c > 0$ and a non-negative function $v \in C^2\left(\overline{U}\right)$ vanishing on L such that $v = g\delta^2_L$ and $-i\partial\overline{\partial}\ln v \geqslant c\omega$ on $U\backslash L$. Let $\widetilde{u} \in C^\infty_{(0,q)}(X)$ be an extension of u such that $\overline{\partial}\widetilde{u}$ vanishes to infinite order on L. Since $\overline{\partial}\widetilde{u} \in L^2_{(0,q+1)}\left(X^\pm, \delta_L^{-2k}, \omega\right)$, $q+1 \leqslant n-1$ and $L^2_{(0,q)}\left(X^\pm, \delta_L^{-2k}, \omega\right) = L^2_{(0,q)}\left(X^\pm, \widetilde{v}^{-k}, \omega\right)$ for every $k \in \mathbb{N}$, by Remark 2 a) and Proposition 3 it follows that for every $k \in \mathbb{N}^*$ there exist a Hermitian complete metric $\widetilde{\omega}$ on $X\backslash L$, $\omega \leqslant \widetilde{\omega} \leqslant \frac{C}{v^2}\omega$ and $h^\pm \in L^2_{(0,q)}\left(X^\pm, \delta_L^{-2k}, \widetilde{\omega}\right)$ such that $\overline{\partial}h^\pm = \overline{\partial}\widetilde{u}$ on $X^\pm$. By Remark 2 b) we have $h^\pm \in L^2_{(0,q)}\left(X^\pm, \delta_L^{-2k+4}, \omega\right)$. So by using Theorem 1, for k big enough we can choose $h^\pm \in C^{s(k)}_{(0,q)}\left(\overline{X^\pm}\right)$, $s(k) \underset{k\to\infty}{\sim} \sqrt{k}$. This means that for k big enough, the form h defined as $h^\pm$ on $\overline{X^\pm}$ is of class C^k on X and vanishes on L. So $U_k = \widetilde{u} - h^\pm$ is a C^k-smooth $\overline{\partial}$-closed form on X which is an extension of u. ∎

Theorem 2. *Let X be a compact connected Kähler manifold of dimension $n \geqslant 3$ and L a C^∞ compact Levi flat hypersurface. Then the normal bundle to the Levi foliation does not admit any Hermitian metric of class C^2 with leafwise positive curvature.*

Proof. Suppose that the normal bundle $\mathcal{N}$ to the Levi foliation admits a Hermitian metric of class C^2 with leafwise positive curvature. Since $\mathcal{N}$ is topologically trivial, its curvature form $\Theta^{\mathcal{N}}$ for the Kähler metric of X is d-exact. So there exists a 1-form u of class C^∞ on L such that $du = \Theta^{\mathcal{N}}$; we may suppose that u is real and $u = u^{0,1} + \overline{u^{0,1}}$, where $u^{0,1}$ is the $(0,1)$ component of u. Since $\Theta^{\mathcal{N}}$ is a $(1,1)$-form, it follows that $\overline{\partial}_b u^{0,1} = 0$, where $\overline{\partial}_b$ is the tangential Cauchy-Riemann operator. By Proposition 4 there exists a C^k-extension $U^{0,1}$ of $u^{0,1}$ to X, $k \geqslant 2$, such that $\overline{\partial}U^{0,1} = 0$.

By Hodge symmetry and Dolbeault isomorphism $H^{0,1}(X,\mathbb{C}) \approx \overline{H^{1,0}(X,\mathbb{C})} \approx \overline{H^0\left(X,\Omega^1_X\right)}$, where Ω^1_X is the sheaf of holomorphic 1-forms on X. So there exists $\eta \in H^0\left(X,\Omega^1_X\right)$ and $\Phi \in C^k(X)$ such that $\widetilde{U^{0,1}} = \overline{\eta} + \overline{\partial}\Phi$. It follows that $\Theta^{\mathcal{N}} =$

$i\partial_b\overline{\partial_b}\,\mathrm{Im}\,\Phi$ on L and this gives a contradiction at the point of L where $\mathrm{Im}\,\Phi$ reaches its maximum. ∎

Remark 3. *A first version of this paper was announced on arXiv in 2014, but there was a gap in the proofs of §4, which is now corrected. Recently, Brinkschulte proved a generalization of Theorem 2 for compact Levi flat hypersurfaces in complex manifolds (see Theorem 1.1. of* [8]*). She uses crucially the Proposition 4.1 of* [8]*, whose statement and proof are the same as Proposition 1 of this paper and which are unchanged from 2014 in our preprint arXiv:1406.5712. However she refers only to Proposition 1.1 of* [25]*, where the lower positive bound for the eigenvalues of the strongly plurisubharmonic function is not mentioned.*

Acknowledgement 1. *We would like to thank M. Adachi and T.-C. Dinh for very useful discussions. We would also thank the referees for their remarks.*

References

[1] A.Andreotti and E. Vesentini, *Carleman estimates for the Laplace-Beltrami equation on complex manifolds*, Publ. Math. IHES **24-25** (1965), 81–150.
[2] A. Andreotti and E. Vesentini, *Sopra un teorema di Kodaira*, Ann. Scuola Norm. Sup. Pisa **15** (1961), no. 4, 283–309.
[3] N. Aronszajn, *A unique continuation theorem for solutions of elliptic partial differential equations or inequalities of second order*, J. Math. Pures Appl. **36** (1957), 235–249.
[4] D. E. Barrett and J. E. Fornaess, *On the smoothness of Levi-foliations*, Publ. Mat. **2** (1988), 171–177.
[5] I. Bendixson, *Sur les courbes définies par une équation différentielle*, Acta Matematica **24** (1901), 1–88.
[6] B. Berndtsson and Ph. Charpentier, *A Sobolev mapping property of the Bergman kernel*, Math. Z. **235** (2000), 1–10.
[7] J. Brinkschulte, *The $\overline{\partial}$-problem with support conditions on some weakly pseudoconvex domains*, Ark. fr Mat. **42** (2004), 259–282.
[8] ———, *On the normal bundle of Levi-flat real hypersurfaces*, Math. Ann. (2018), to appear.
[9] M. Brunella, *On the dynamics of codimension one holomorphic foliations with ample normal bundle*, Indiana Univ. Math. J. **57** (2008), 3101–3113.
[10] ———, *Codimension one foliations on complex tori*, Ann. Fac. Sci. Toulouse Math. **19** (2010), 405–418.
[11] M. Brunella and C. Perrone, *Exceptional singularities of codimension one holomorphic foliations*, Publ. Mat. **55** (2011), 295–312.
[12] C. Camacho, A. Lins Neto and P. Sad, *Minimal sets of foliations in complex projective space*, Publ. Math. de I.H.E.S. **68** (1988), 187–203.
[13] D. Cerveau, *Minimaux des feuilletages algébriques de $\mathbb{CP}^n$*, Ann. Inst. Fourier **43** (1993), 1535–1543.
[14] J.-P. Demailly, *Complex Analytic Geometry and Differential Geometry*, http://www-fourier.ujf-grenoble.fr∽demailly/books.html.
[15] ———, *Estimations L^2 pour l'opérateur $\overline{\partial}$ d'un fibré vectoriel holomorphe semi-positif au-dessus d'une variété kählérienne complète*, Ann. Scient. Ec. Norm. Sup. **15** (1982), 457–511.
[16] ———, *Sur les théorèmes d'annulation et de finitude de T. Ohsawa et O. Abdelkader*, Séminaire P. Lelong - P. Dolbeault - H. Skoda (Analyse) 1985/86, Lecture Notes in Mathematics, no. 1295, Springer-Verlag, 1987, pp. 48–58.
[17] H. Donnelly and C. Fefferman, *L^2-cohomology and index theorem for the Bergman metric*, Ann. of Math. **118** (1983), no. 2, 593–618.
[18] G. B. Folland and J. J. Kohn, *The Neumann problem for the Cauchy-Riemann complex*, no. 75, Princeton Univ. Press, Princeton, N. J., 1972.
[19] G. M. Henkin and A. Iordan, *Regularity of $\overline{\partial}$ on pseudoconcave compacts and applications*, Asian J. Math. **4** (2000), no. 4, 855–884, and Erratum to : Regularity of $\overline{\partial}$ on pseudoconcave compacts and applications by G. M. Henkin and A. Iordan, Asian J. Math., 4, 855-884, 2000.

[20] L. Hörmander, *L^2 estimates and existence theorems for the $\overline{\partial}$ operator*, Acta Math. **113** (1965), 89–152.

[21] A Iordan and F. Matthey, *Régularité de l'opérateur $\overline{\partial}$ et théorème de Siu sur la nonexistence d'hypersurfaces Levi-plates dans l'espace projectif complexe $\mathbb{CP}^n$, $n \geqslant 3$*, C. R. Acad. Sc. Paris **346** (2008), 395–400.

[22] A. Lins Neto, *A note on projective Levi flats and minimal sets of algebraic foliations*, Ann. Inst. Fourier **49** (1999), 1369–1385.

[23] T. Ohsawa, *Isomorphism theorems for cohomology groups of weakly 1-complete manifolds*, Publ. RIMS, Kyoto Univ. **18** (1982), 191–232.

[24] ———, *On the complement of Levi flats in Kähler manifolds of dimension $\geqslant 3$*, Nagoya Math. J. **185** (2007), 161–169.

[25] ———, *Nonexistence of certain Levi flat hypersurfaces in Kähler manifolds from the viewpoint of positive normal bundles*, Publ. RIMS Kyoto Univ. **49** (2013), 229–239.

[26] ———, *L^2 approaches in Several Complex Variables*, Springer, 2015.

[27] T. Ohsawa and N. Sibony, *Bounded P.S.H. functions and pseudoconvexity in a Kähler manifold*, Nagoya Math. J. **149** (1998), 1–8.

[28] H. Poincaré, *Mémoire sur les courbes définies par une équation différentielle*, Journal de Math. Pures et Appl. **7** (1881), 375–422.

[29] ———, *Mémoire sur les courbes définies par une équation différentielle*, Journal de Math. Pures et Appl. **8** (1882), 251–296.

[30] O. Riemenschneider, *Characterizing Moišezon spaces by almost positive coherent analytic sheaves*, Math. Z. **123** (1971), 263–284.

[31] Y.-T. Siu, *Nonexistence of smooth Levi-flat hypersurfaces in complex projective spaces of dimension $\geqslant 3$*, Ann. of Math. **151** (2000), 1217–1243.

[32] ———, *$\overline{\partial}$-regularity for weakly pseudoconvex domains in compact Hermitian symmetric spaces with respect to invariant metrics*, Ann. of Math. **156** (2002), 595–621.

[33] K. Takegoshi, *A generalization of vanishing theorems for weakly 1-complete manifolds*, Publ. RIMS Kyoto Univ **17** (1981), 311–330.

[34] A. Takeuchi, *Domaines pseudoconvexes sur les variétés kählériennes*, J. Math. Kyoto Univ. **6** (1967), 323–357.

Institut de Mathématiques, UMR 7586 du CNRS, case 247, Université Pierre et Marie-Curie, 4 Place Jussieu, 75252 Paris Cedex 05, France, Current address: LAMAV,, Université Polytechnique Hauts-de-France,, Campus du Mont Houy,, 59313 Valenciennes Cedex 9, France

E-mail address: `severine.biard@uphf.fr`

Sorbonne Université, Faculté des Sciences et Ingénierie, Institut de Mathématiques de Jussieu-Paris Rive Gauche, 4 Place Jussieu, 75252 Paris Cedex 05, France

E-mail address: `andrei.iordan@imj-prg.fr`