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NONEXISTENCE OF LEVI FLAT HYPERSURFACES WITH POSITIVE NORMAL BUNDLE IN COMPACT KÄHLER MANIFOLDS OF DIMENSION ≥ 3

SÉVERINE BIARD AND ANDREI IORDAN

In memory of Gennadi M. Henkin

ABSTRACT. Let X be a compact connected Kähler manifold of dimension ≥ 3 and L a C^∞ Levi flat hypersurface in X . Then the normal bundle to the Levi foliation does not admit a Hermitian metric with positive curvature along the leaves. This represents an answer to a conjecture of Marco Brunella.

1. INTRODUCTION

A classical theorem of Poincaré-Bendixson [28], [29], [5] states that every leaf of a foliation of the real projective plane accumulates on a compact leaf or on a singularity of the foliation. As a holomorphic foliation \mathcal{F} of codimension 1 of $\mathbb{C}\mathbb{P}_n$, $n \geq 2$, does not contain any compact leaf and its singular set $Sing \mathcal{F}$ is not empty, a major problem in foliation theory is the following: can \mathcal{F} contain a leaf F such that $\overline{F} \cap Sing \mathcal{F} = \emptyset$? If this is the case, then there exists a nonempty compact set K called exceptional minimal, invariant by \mathcal{F} and minimal for the inclusion such that $K \cap Sing \mathcal{F} = \emptyset$. The problem of the existence of an exceptional minimal in $\mathbb{C}\mathbb{P}_n$, $n \geq 2$ is implicit in [12].

In [13] D. Cerveau proved a dichotomy under the hypothesis of the existence of a holomorphic foliation \mathcal{F} of codimension 1 of $\mathbb{C}\mathbb{P}_n$ which admits an exceptional minimal \mathfrak{M} : \mathfrak{M} is a real analytic Levi flat hypersurface in $\mathbb{C}\mathbb{P}_n$ (i. e. $T(\mathfrak{M}) \cap JT(\mathfrak{M})$ is integrable, where J is the complex structure of $\mathbb{C}\mathbb{P}_n$), or there exists $p \in \mathfrak{M}$ such that the leaf through p has a hyperbolic holonomy and the range of the holonomy morphism is a linearisable abelian group. This gave rise to the conjecture of the nonexistence of smooth Levi flat hypersurface in $\mathbb{C}\mathbb{P}_n$, $n \geq 2$.

The conjecture was proved for $n \geq 3$ by A. Lins Neto [22] for real analytic Levi flat hypersurfaces and by Y.-T. Siu [31] for C^{12} smooth Levi flat hypersurfaces. The methods of proofs for the real analytic case are very different from the smooth case.

A real hypersurface of class C^2 in a complex manifold is Levi flat if its Levi form vanishes or equivalently, it admits a foliation by complex hypersurfaces. We say that a (non-necessarily smooth) real hypersurface L in a complex manifold X is Levi flat if $X \setminus L$ is pseudoconvex. An example of (non-smooth) Levi flat hypersurface in

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$\mathbb{C}\mathbb{P}_2$ is $L = \{[z_0, z_1, z_2] : |z_1| = |z_2|\}$, where $[z_0, z_1, z_2]$ are homogeneous coordinates in $\mathbb{C}\mathbb{P}_2$ (see [19]).

In [21] Iordan and Matthey proved the nonexistence of Lipschitz Levi flat hypersurfaces in $\mathbb{C}\mathbb{P}_n$, $n \geq 3$, which are of Sobolev class W^s , $s > 9/2$. A principal element of the proof is that the Fubini-Study metric induces a metric of positive curvature on any quotient of the tangent space.

Nonexistence questions for the Levi flat hypersurfaces in compact Kähler manifolds were first discussed by T. Ohsawa in [24], who proved the nonexistence of real-analytic Levi flat hypersurfaces with Stein complement in compact Kähler manifolds of dimension ≥ 3 .

In [9], M. Brunella proved that the normal bundle to the Levi foliation of a closed real analytic Levi flat hypersurface in a compact Kähler manifold of dimension $n \geq 3$ does not admit any Hermitian metric with leafwise positive curvature. The real analytic hypothesis may be relaxed to the assumption of $C^{2,\alpha}$, $0 < \alpha < 1$, such that the Levi foliation extends to a holomorphic foliation in a neighborhood of the hypersurface.

The main step in his proof is to show that the existence of a Hermitian metric with leafwise positive curvature on the normal bundle to the Levi foliation of a compact Levi flat hypersurface L in a Hermitian manifold X , implies that $X \setminus L$ is strongly pseudoconvex, i.e. there exists on $X \setminus L$ an exhaustion function which is strongly plurisubharmonic outside a compact set. This was generalized in [10] for invariant compact subsets of a holomorphic foliation of codimension one. Of course, if X is the complex projective space, then every proper pseudoconvex domain in X is Stein [34].

Brunella stated also the following conjecture [9]: Let X be a compact connected Kähler manifold of dimension $n \geq 3$ and L a C^∞ compact Levi flat hypersurface in X . Then the normal bundle to the Levi foliation does not admit any Hermitian metric with leafwise positive curvature.

The assumption $n \geq 3$ is necessary in this conjecture (see Example 4.2 of [9]).

In [11] Brunella and Perrone proved that every leaf of a holomorphic foliation \mathcal{F} of codimension one of a projective manifold X of dimension at least 3 and such that $\text{Pic}(X) = \mathbb{Z}$ accumulates on the singular set of the foliation. In this case the normal bundle to the foliation is ample.

In [25], T. Ohsawa considered a C^∞ Levi flat compact hypersurface L in a compact Kähler manifold X such that the normal bundle to the Levi foliation admits a fiber metric whose curvature is semipositive of rank $\geq k$ on the holomorphic tangent space to the leaves and proved that $X \setminus L$ admits an exhaustion plurisubharmonic function of logarithmic growth which is strictly $(n - k)$ -convex. Then, if $\dim X \geq 3$, he proved that there are no Levi flat real analytic hypersurfaces such that the normal bundle to the Levi foliation admits a fiber metric whose curvature is semipositive of rank ≥ 2 on the holomorphic tangent space to L . Some possibilities for generalization in the smooth case are also indicated.

In this paper we solve the above mentioned conjecture of Brunella for compact connected Kähler manifolds of dimension $n \geq 3$. The principal ingredient of the proof is a refinement of the proof of Brunella [9] of the strong pseudoconvexity of $X \setminus L$: we show that there exist a neighborhood U of L and a function v on U vanishing on L , such that $-i\partial\bar{\partial} \ln v \geq c\omega$ on $U \setminus L$, where $c > 0$ and ω is the $(1, 1)$ -form associated to the Kähler metric. Then we use the L^2 estimates [2], [1], [20],

[15] for the weighted $\bar{\partial}$ -equation on (n, q) -forms on $X \setminus L$ endowed with a complete Kähler metric. These estimates together with the lower uniform boundedness of the eigenvalues of the Levi form and a duality method developed in [19], allow us to solve the $\bar{\partial}$ -equation with compact support for $(0, q)$ -forms, $1 \leq q \leq n - 1$, and this leads in dimensions ≥ 3 to the solution of Brunella's conjecture.

2. PRELIMINARIES

Let X be a complex n -dimensional manifold, ω a Kähler metric on X , Ω a domain in X and σ a positive function on Ω . For $\alpha \in \mathbb{R}$ denote

$$L^2_{(p,q)}(\Omega, \sigma^\alpha, \omega) = \left\{ f \in L^2_{(p,q)loc}(\Omega) : \int_{\Omega} |f|^2 \sigma^{2\alpha} dV_{\omega} < \infty \right\}$$

endowed with the norm

$$N_{\alpha, \omega, \sigma}(f) = \left(\int_{\Omega} |f|^2 \sigma^{2\alpha} dV_{\omega} \right)^{1/2}.$$

Let Ω be a pseudoconvex domain in $\mathbb{C}\mathbb{P}_n$ and $\delta_{\partial\Omega}$ the geodesic distance to the boundary for the Fubini-Study metric ω_{FS} . By using the L^2 estimates for the $\bar{\partial}$ -operator of Hörmander with the weight $e^{-\varphi}$, $\varphi = -\alpha \log \delta_{\partial\Omega}$ which is strongly plurisubharmonic by a theorem of Takeuchi [34], Henkin and Iordan proved in [19] the existence and regularity of the $\bar{\partial}$ equation for $\bar{\partial}$ -closed forms in $L^2_{(p,q)}(\Omega, \delta_{\partial\Omega}^{-\alpha}, \omega_{FS})$ verifying the moment condition. This gives the regularity of the $\bar{\partial}$ -operator in pseudoconcave domains with Lipschitz boundary [19] and, by using a method of Siu [31], [32], the nonexistence of smooth Levi flat hypersurfaces in $\mathbb{C}\mathbb{P}_n$, $n \geq 3$ follows (see [21]). These techniques will be used in the 4th and the 5th paragraph.

We will use also the following theorem of regularity of $\bar{\partial}$ equation of Brinkschulte [7]:

Theorem 1. *Let Ω be a relatively compact domain with Lipschitz boundary in a Kähler manifold (X, ω) and set $\delta_{\partial\Omega}$ the geodesic distance to the boundary of Ω . Let $f \in L^2_{(p,q)}(\Omega, \delta_{\partial\Omega}^{-k}, \omega) \cap C^k_{(p,q)}(\bar{\Omega}) \cap C^\infty_{(p,q)}(\Omega)$, $q \geq 1$, $k \in \mathbb{N}$ and $u \in L^2_{(p,q-1)}(\Omega, \delta_{\partial\Omega}^{-k}, \omega)$ such that $\bar{\partial}u = f$ and $\bar{\partial}^*_k u = 0$, where $\bar{\partial}^*_k$ is the Hilbert space adjoint of the unbounded operator $\bar{\partial}_{-k} : L^2_{(p,q-1)}(\Omega, \delta_{\partial\Omega}^{-k}, \omega) \rightarrow L^2_{(p,q)}(\Omega, \delta_{\partial\Omega}^{-k}, \omega)$. Then for k big enough $u \in C^{s(k)}_{(p,q-1)}(\bar{\Omega})$ where $s(k) \underset{k \rightarrow \infty}{\sim} \sqrt{k}$.*

3. STRONG PSEUDOCONVEXITY OF THE COMPLEMENT OF A LEVI FLAT HYPERSURFACE

Let L be a smooth Levi flat hypersurface in a Hermitian manifold X . As was mentioned in [9] and in [25], by taking a double covering, we can assume that L is orientable and the complement of L has two connected components in a neighborhood of L . This will be always supposed in the sequel and for an open neighborhood U of L we will denote by U^+ and U^- the two connected components of $U \setminus L$. We will denote by δ_L the signed geodesic distance to L .

In [9] Brunella proved that the complement of a closed Levi flat hypersurface in a compact Hermitian manifold of class $C^{2,\alpha}$, $0 < \alpha < 1$, having the property that the Levi foliation extends to a holomorphic foliation in a neighborhood of L and the normal bundle to the Levi foliation admits a C^2 Hermitian metric with leafwise positive curvature is strongly pseudoconvex, i.e. there exists an exhaustion

function which is strongly plurisubharmonic outside a compact set. The following proposition strenghtens this result:

Proposition 1. *Let L be a compact C^3 Levi flat hypersurface in a Hermitian manifold X of dimension $n \geq 2$, such that the normal bundle $\mathcal{N}_L^{1,0}$ to the Levi foliation admits a C^2 Hermitian metric with leafwise positive curvature. Then there exist a neighborhood U of L , $c > 0$ and a non-negative function $v \in C^2(U)$, vanishing on L and positive on $U \setminus L$ such that $-i\partial\bar{\partial} \ln v \geq c\omega$ on $U \setminus L$, where ω is the $(1,1)$ -form associated to the metric. Moreover, there exists a nonvanishing continuous function g in a neighborhood of L such that $v = g\delta_L^2$.*

Proof. Let $z_0 \in L$. There exist holomorphic coordinates $z = (z_1, \dots, z_{n-1}, z_n) = (z', z_n)$ in a neighborhood of z_0 such that the local parametric equations for L are of the form

$$z_j = w_j, \quad j = 1, \dots, n-1, \quad z_n = \varphi(w', t)$$

where φ is of class C^3 (see [4]) on a neighborhood of the origin in $\mathbb{C}^{n-1} \times \mathbb{R}$, holomorphic in w' and $\frac{\partial \varphi}{\partial t}(z_0) \in \mathbb{R}^*$. We consider a C^3 extension $\psi = (\psi_1, \dots, \psi_n)$ of φ on a neighborhood of the origin in $\mathbb{C}^{n-1} \times \mathbb{C}$, $\psi(w', t + is) = (w', \varphi(w', t) + is)$. Then ψ is a C^3 diffeomorphism in a neighborhood of z_0 and holomorphic in w' . It follows that

$$L = \{(z', z_n) : \rho(z', z_n) = 0\}.$$

where $\rho = \text{Im}(\psi^{-1})_n$. We denote $f = (\psi^{-1})_n(z', z_n)$. Since $\bar{\partial}_b f = 0$ on L , where $\bar{\partial}_b$ is the tangential Cauchy-Riemann operator on L , there exists an extension \tilde{f} of class C^3 in a neighborhood of z_0 such that $\bar{\partial} \tilde{f}$ vanishes to order greater than 2 on L , i.e. $D^l \bar{\partial} \tilde{f} = 0$ for $|l| \leq 2$ on L .

So there exists an open finite covering $(\tilde{U}_j)_{j \in J}$ by holomorphic charts of L such that $\tilde{U}_j \setminus L = \tilde{U}_j^+ \cup \tilde{U}_j^-$ such that $U_j = L \cap \tilde{U}_j = \{z \in \tilde{U}_j : \text{Im} \tilde{f}_j = 0\}$, where $\bar{\partial} \tilde{f}_j$ vanishes to order greater than 2 on L and the Levi foliation is given on U_j by $\{z \in U_j : \tilde{f}_j(z) = c_j\}$, $c_j \in \mathbb{R}$. Thus $d\tilde{f}_j = \partial \tilde{f}_j$ is a nonvanishing section of $\mathcal{N}_L^{1,0}$ on U_j and by shrinking \tilde{U}_j , we may consider that $d\tilde{f}_j \neq 0$ on \tilde{U}_j .

We may suppose that $\mathcal{N}_L^{1,0}$ is represented by a cocycle $\{g_{jk}\}$ of class C^2 subordinated to the covering $(U_j)_{j \in J}$ and there exist closed $(1,0)$ -forms α_j of class C^2 on U_j holomorphic along the leaves such that $T^{1,0}(U_j) = \ker \alpha_j$ for every $j \in J$ and $\alpha_j = g_{jk} \alpha_k$ on $U_j \cap U_k$. So $(\alpha_j)_{j \in J}$ defines a global form α on L with values in $\mathcal{N}_L^{1,0}$ such that locally on U_j we have $\alpha(z) = \alpha_j(z) \otimes \alpha_j^*(z)$ where α_j^* is the dual frame of α_j . In particular we have $\alpha_k^* = g_{jk} \alpha_j^*$.

Let h be a C^2 Hermitian metric with positive leafwise curvature $\Theta_h(\mathcal{N}_L^{1,0})$ on $\mathcal{N}_L^{1,0}$. h is defined on each U_j by a C^2 function $h_j = |\alpha_j^*|^2$ such that $h_k = |g_{jk}|^2 h_j$ on $U_j \cap U_k$.

Since $\alpha_j = \eta_j d\tilde{f}_j$ on U_j for every j , where η_j are nowhere vanishing functions of class C^2 on U_j holomorphic along the leaves and

$$\frac{1}{\eta_k} (d\tilde{f}_k)^* = \frac{1}{\eta_j} g_{jk} (d\tilde{f}_j)^*$$

on $U_j \cap U_k$, it follows that

$$|g_{jk}(z)|^2 = \left| \frac{\eta_j(z)}{\eta_k(z)} \right|^2 \left| \frac{(df_k)^*}{(df_j)^*} \right|^2 = \frac{h_k(z)}{h_j(z)}, \quad z \in U_j \cap U_k.$$

So

$$h_j |\eta_j|^2 (\operatorname{Im} \tilde{f}_j)^2 - h_k |\eta_k|^2 (\operatorname{Im} \tilde{f}_k)^2$$

vanishes to order greater than 2 on $U_j \cap U_k$ and $\left(h_j |\eta_j|^2 (\operatorname{Im} \tilde{f}_j)^2 \right)_{j \in J}$ defines a jet of order 2 on L . By Whitney extension theorem there exists a C^2 function v on X such that $v - h_j |\eta_j|^2 (\operatorname{Im} \tilde{f}_j)^2$ vanishes to order 2 on U_j for every $j \in J$. Let $\tilde{\eta}_j, \tilde{h}_j$ be C^2 extensions of η_j, h_j on \tilde{U}_j and set $\tilde{\alpha}_j = \tilde{\eta}_j d\tilde{f}_j, \tilde{v} = \tilde{h}_j |\tilde{\eta}_j|^2 (\operatorname{Im} \tilde{f}_j)^2$.

For $z \in \tilde{U}_j$ denote $E'_z = \{V' \in T_z^{1,0}(X) : \langle \partial \operatorname{Im} \tilde{f}_j, V' \rangle = 0\}$ and E''_z the orthogonal of E'_z in $T_z^{1,0}(X)$. Then for every $V \in T_z^{1,0}(X)$ there exists $V' \in E'_z, V'' \in E''_z$ such that $V = V' + V''$. The curvature form $\Theta(\mathcal{N}_L^{1,0})$ is represented by $-i\partial\bar{\partial} \ln(h_j |\alpha_j|^2)$ on U_j , so by shrinking \tilde{U}_j we may suppose that there exists $\beta > 0$ such that $(-i\partial\bar{\partial} \ln(\tilde{h}_j |\tilde{\alpha}_j|^2))(V', \bar{V}') \geq \beta \omega(V', \bar{V}')$ for every $z \in \tilde{U}_j$ and $V \in T_z^{1,0}(X)$.

On $\tilde{U}_j \setminus L$ we have

$$\begin{aligned} (3.1) \quad -i\partial\bar{\partial} \ln \tilde{v} &= -i\partial\bar{\partial} \ln \left(\tilde{h}_j \left| \frac{\tilde{\alpha}_j}{d\tilde{f}_j} \right|^2 (\operatorname{Im} \tilde{f}_j)^2 \right) \\ &= -i\partial\bar{\partial} \ln \tilde{h}_j |\tilde{\alpha}_j|^2 + i\partial\bar{\partial} \ln |d\tilde{f}_j|^2 - i\partial\bar{\partial} \ln (\operatorname{Im} \tilde{f}_j)^2. \end{aligned}$$

Let $z \in \tilde{U}_j$ and $V \in T_z^{1,0}(X)$. Then $V = V' + V'', V' \in E'_z$ and $V'' \in E''_z$ and

$$\begin{aligned} -i\partial\bar{\partial} \ln \tilde{h}_j |\tilde{\alpha}_j|^2 (V, \bar{V}) &= \left(-i\partial\bar{\partial} \ln(\tilde{h}_j |\tilde{\alpha}_j|^2) \right) (V', \bar{V}') \\ &\quad + 2 \operatorname{Re} \left(-i\partial\bar{\partial} \ln(\tilde{h}_j |\tilde{\alpha}_j|^2) (V', \bar{V}'') \right) \\ &\quad + \left(-i\partial\bar{\partial} \ln(\tilde{h}_j |\tilde{\alpha}_j|^2) \right) (V'', \bar{V}'') \end{aligned}$$

There exists a constant $C > 0$ depending on the eigenvalues of $-i\partial\bar{\partial} \ln(\tilde{h}_j |\tilde{\alpha}_j|^2)$ with respect to ω such that for every $\varepsilon > 0$

$$2 \left| \operatorname{Re} \left(-i\partial\bar{\partial} \ln(\tilde{h}_j |\tilde{\alpha}_j|^2) (V', \bar{V}'') \right) \right| \leq C \left(\varepsilon \omega(V', \bar{V}') + \frac{1}{\varepsilon} \omega(V'', \bar{V}'') \right),$$

so

$$\begin{aligned} (3.2) \quad -i\partial\bar{\partial} \ln \tilde{h}_j |\tilde{\alpha}_j|^2 (V, \bar{V}) &\geq \beta \omega(V', \bar{V}') - C \left(\varepsilon \omega(V', \bar{V}') + \frac{1}{\varepsilon} \omega(V'', \bar{V}'') \right) \\ &\quad - \left\| -i\partial\bar{\partial} \ln \tilde{h}_j |\tilde{\alpha}_j|^2 \right\|_{\omega} \omega(V'', \bar{V}'') \end{aligned}$$

Since $\bar{\partial}\tilde{f}_j$ vanishes to order greater than 2 on L , for every $\gamma > 0$ there exists a neighborhood of L such that

$$(3.3) \quad \left| i\bar{\partial}\bar{\partial}\ln\left|d\tilde{f}_j\right|^2(V,\bar{V}) \right| \leq \gamma\omega(V,\bar{V})$$

and

$$(3.4) \quad \left| i\bar{\partial}\bar{\partial}\operatorname{Im}\tilde{f}_j(V,\bar{V}) \right| \leq \gamma\left(\operatorname{Im}\tilde{f}_j\right)\omega(V,\bar{V}).$$

Let $z \in \tilde{U}_j \setminus L$. By (3.4) it follows that

$$(3.5) \quad \begin{aligned} -i\bar{\partial}\bar{\partial}\ln\left(\operatorname{Im}\tilde{f}_j\right)^2(V,\bar{V}) &= \left(-2\frac{i\bar{\partial}\bar{\partial}\operatorname{Im}\tilde{f}_j}{\operatorname{Im}\tilde{f}_j} + 2i\frac{\partial\operatorname{Im}\tilde{f}_j \wedge \bar{\partial}\operatorname{Im}\tilde{f}_j}{\left(\operatorname{Im}\tilde{f}_j\right)^2} \right) (V,\bar{V}) \\ &\geq -2\gamma\omega(V,\bar{V}) + 2i\frac{\partial\operatorname{Im}\tilde{f}_j \wedge \bar{\partial}\operatorname{Im}\tilde{f}_j}{\left(\operatorname{Im}\tilde{f}_j\right)^2}(V'',\bar{V}'') \\ &\geq -2\gamma\omega(V,\bar{V}) + \frac{2\inf_{\tilde{U}_j}\left\|\partial\operatorname{Im}\tilde{f}_j\right\|_\omega^2}{\left(\operatorname{Im}\tilde{f}_j\right)^2}\omega(V'',\bar{V}''). \end{aligned}$$

By using (3.2), (3.3) and (3.5), from (3.1) we obtain

$$\begin{aligned} (-i\bar{\partial}\bar{\partial}\ln\tilde{v})(V,\bar{V}) &\geq (\beta - C\varepsilon)\omega(V',\bar{V}') \\ &\quad + \left(\frac{2}{\left(\operatorname{Im}f_j\right)^2}\inf_{\tilde{U}_j}\left\|\partial\operatorname{Im}\tilde{f}_j\right\|_\omega^2 - \frac{C}{\varepsilon} - \left\| -i\bar{\partial}\bar{\partial}\ln\tilde{h}_j|\tilde{\alpha}_j|^2 \right\|_\omega \right) \omega(V'',\bar{V}'') \\ &\quad - 2\gamma\omega(V,\bar{V}). \end{aligned}$$

By choosing $0 < C\varepsilon < \beta$ and by shrinking \tilde{U}_j such that $\frac{2}{\left(\operatorname{Im}f_j\right)^2}$ is big enough and γ small enough, we obtain that there exists $c > 0$ such that $-i\bar{\partial}\bar{\partial}\ln\tilde{v} \geq c\omega$ on $\tilde{U}_j \setminus L$. Finally, since $v - \tilde{v}$ vanishes to order greater than 2 on L , it follows that there exists a neighborhood U' of L such that $-\ln v$ is strongly plurisubharmonic on $U' \setminus L$. We can now take $U = \{z \in U' : v(z) < \mu\}$ for $\mu > 0$ small enough.

L is a C^3 manifold, so the signed distance function δ_L is a defining function of class C^3 for L . Since v is of class C^2 on U and vanishes to order greater than 2 on L , we have $v = g\delta_L^2$ with g continuous in a neighborhood of L .

Suppose that there exists $x \in L$ such that $g(x) = 0$. Then $v = o(\delta_L^2)$ in a neighborhood of x . But there exists j such that $x \in U_j$ and $v = h_j|\eta_j|^2\left(\operatorname{Im}\tilde{f}_j\right)^2 + o(\delta_L^2)$. Since $\operatorname{Im}\tilde{f}_j = 0$ and $d\operatorname{Im}\tilde{f}_j \neq 0$ on L it follows that $|\nabla^2 v|(x) \neq 0$. This contradiction shows that $g(x) \neq 0$ on L . ■

4. WEIGHTED ESTIMATES FOR THE $\bar{\partial}$ -EQUATION

Remark 1. *Under the hypothesis and conclusions of Proposition 1, we consider a positive extension \tilde{v} of the restriction of v on a neighborhood of L to $X \setminus L$. Let $s > 0$ such that $\{v < e^{-s}\} \subset U$ and let φ be a smooth function on \mathbb{R} such that $\varphi = 0$ on $] -\infty, s]$ and φ is strictly convex increasing on $]s, \infty[$. Then $\psi = \varphi(-\ln\tilde{v})$ is a*

plurisubharmonic exhaustion function of $X \setminus L$, which is strongly plurisubharmonic outside a compact subset of $X \setminus L$.

In the sequel, L will be a compact C^∞ Levi flat hypersurface in a compact Kähler manifold X of dimension $n \geq 2$, verifying the hypothesis and the conclusions of Proposition 1. We denote X^\pm the connected components of $\{z \in X : v > 0\}$ endowed with a complete Kähler metric $\tilde{\omega}$ which will be defined later and we set

$$\mathcal{D}_{(p,q)}(X^\pm) = \left\{ f \in C_{(p,q)}^\infty(X^\pm) : \text{supp } f \subset\subset X^\pm \right\}$$

and

$$\mathcal{H}_{(p,q)}(X^\pm, \tilde{v}^\alpha, \tilde{\omega}) = \ker \bar{\partial} \cap \ker \bar{\partial}_\alpha^* \subset L_{(p,q)}^2(X^\pm, \tilde{v}^\alpha, \tilde{\omega})$$

where $\bar{\partial}_\alpha^*$ is the Hilbert space adjoint of the operator $\bar{\partial} : L_{(p,q)}^2(X^\pm, \tilde{v}^\alpha, \tilde{\omega}) \rightarrow L_{(p,q+1)}^2(X^\pm, \tilde{v}^\alpha, \tilde{\omega})$.

Proposition 2. *For every $\alpha > 0$, there exists a complete Kähler metric $\tilde{\omega}$ on $X \setminus L$, $\omega \leq \tilde{\omega} \leq \frac{C}{\alpha^2} \omega$, $C > 0$, such that the range $\mathcal{R}_{(n,q)}^\alpha(X^\pm)$ of the operator $\bar{\partial}_\alpha : L_{(n,q-1)}^2(X^\pm, \tilde{v}^\alpha, \tilde{\omega}) \rightarrow L_{(n,q)}^2(X^\pm, \tilde{v}^\alpha, \tilde{\omega})$ is closed for $1 \leq q \leq n$.*

Proof. The proof is based on methods of [16] (see also [14]).

Denote by ω the Kähler metric of X . Since $i\partial\bar{\partial}(-\ln v) \geq c\omega$ on $U \setminus L$, $c > 0$, by a method developed in [27] it follows that there exist a neighborhood V of L and $\eta > 0$ such that $-v^\eta$ is strongly plurisubharmonic on $V \setminus L$. Then for $0 < \beta < \eta$, we have the Donnelly-Fefferman estimate [17]

$$(4.1) \quad i\partial(-\ln v) \wedge \bar{\partial}(-\ln v) \leq ir\partial\bar{\partial}(-\ln v).$$

on $V \setminus L$, with $0 < r = \beta/\eta < 1$. This is equivalent to say that the norm of $\partial(-\ln v)$ measured in the metric $i\partial\bar{\partial}(-\ln v)$ is smaller than r on $V \setminus L$ (see also [6] and [19]).

Let $\alpha > 0$. We consider the trivial line bundle E on $X \setminus L$ endowed with the Hermitian metric $h_\alpha = e^{\alpha \ln \tilde{v}}$. Set

$$\tilde{\omega} = i\Theta(E) + K\omega = i\alpha\partial\bar{\partial}(-\ln \tilde{v}) + K\omega$$

with K a positive constant. Since $-\ln \tilde{v}$ is an exhaustion function on $X \setminus L$, it follows by (4.1) that for K big enough $\tilde{\omega}$ is a complete Kähler metric on $X \setminus L$ such that $\omega \leq \tilde{\omega} \leq \frac{C}{\alpha^2} \omega$, $C > 0$.

Denote λ_j (respectively $\tilde{\lambda}_j$) the eigenvalues of $i\Theta(E)$ with respect to ω (respectively $\tilde{\omega}$), $1 \leq j \leq n$, in increasing order. By Proposition 1, there exists $c > 0$ such that $i\Theta(E) = i\alpha\partial\bar{\partial}(-\ln \tilde{v}) \geq \alpha c\omega$ on $\{\psi > b\}$ for b big enough. So, as in [16] (1.6) we have

$$(4.2) \quad 1 \geq \tilde{\lambda}_j = \frac{\lambda_j}{\lambda_j + K} \geq \frac{\alpha c}{\alpha c + K} > 0, \quad 1 \leq j \leq n$$

on $\{\psi > b\}$. By Bochner-Kodaira-Nakano inequality (see for ex. [14]) we have

$$(4.3) \quad N_{\alpha, \tilde{\omega}, \tilde{v}}(\bar{\partial}u)^2 + N_{\alpha, \tilde{\omega}, \tilde{v}}(\bar{\partial}_\alpha^*u)^2 \geq \int_{X^\pm} \langle ([i\Theta(E), \Lambda_{\tilde{\omega}}]u, u)_{\alpha, \tilde{\omega}, \tilde{v}} dV_{\tilde{\omega}} \rangle$$

for every $u \in \mathcal{D}_{(n,q)}(X \setminus L)$, where $N_{\alpha, \tilde{\omega}, \tilde{v}} = \int_{X^\pm} |u|_{\tilde{\omega}}^2 \tilde{v}^\alpha dV_{\tilde{\omega}}$.

Let χ be a smooth function on X such that $0 \leq \chi \leq 1$, $\chi = 0$ on a neighborhood of $\{\psi < b\}$ and $\chi = 1$ on a neighborhood $\{\psi > b'\}$ of L , $b' > b$. By (4.3) and (4.2),

for every $u \in \mathcal{D}_{(n,q)}(X \setminus L)$ we have

$$\begin{aligned}
N_{\alpha, \tilde{\omega}, \tilde{v}}(\bar{\partial}(\chi u))^2 + N_{\alpha, \tilde{\omega}, \tilde{v}}(\bar{\partial}_\alpha^*(\chi u))^2 &\geq \int_{X^\pm} \langle ([i\Theta(E), \Lambda_{\tilde{\omega}}] \chi u, \chi u)_{\alpha, \tilde{\omega}, \tilde{v}} \rangle dV_{\tilde{\omega}} \\
&\geq \int_{\{\psi > b'\}} \langle ([i\Theta(E), \Lambda_{\tilde{\omega}}] \chi u, \chi u)_{\alpha, \tilde{\omega}, \tilde{v}} \rangle dV_{\tilde{\omega}} \\
&\geq \int_{\{\psi > b'\}} (\lambda_1 + \dots + \lambda_n) |\chi u|_{\tilde{\omega}}^2 \tilde{v}^\alpha dV_{\tilde{\omega}} \\
&\geq \frac{\alpha c}{\alpha c + K} \int_{\{\psi > b'\}} |u|_{\tilde{\omega}}^2 \tilde{v}^\alpha dV_{\tilde{\omega}}
\end{aligned}$$

so there exists $C, c' > 0$ such that

$$\begin{aligned}
&2N_{\alpha, \tilde{\omega}, \tilde{v}}(\bar{\partial}u)^2 + 2N_{\alpha, \tilde{\omega}, \tilde{v}}(\bar{\partial}_\alpha^*u)^2 + C \int_{\text{supp}(\chi')} |u|_{\tilde{\omega}}^2 \tilde{v}^\alpha dV_{\tilde{\omega}} \\
&\geq c' \int_{X \setminus L} |u|_{\tilde{\omega}}^2 \tilde{v}^\alpha dV_{\tilde{\omega}} - c' \int_{\{\psi < b'\}} |u|_{\tilde{\omega}}^2 \tilde{v}^\alpha dV_{\tilde{\omega}}.
\end{aligned}$$

Finally it follows that there exists a compact subset $F = \text{supp}(\chi') \cup \{\psi \leq b'\}$ of X^\pm such that for every $u \in \mathcal{D}_{(n,q)}(X \setminus L)$

$$(4.4) \quad c' N_{\alpha, \tilde{\omega}, \tilde{v}}(u)^2 \leq 2N_{\alpha, \tilde{\omega}, \tilde{v}}(\bar{\partial}u)^2 + 2N_{\alpha, \tilde{\omega}, \tilde{v}}(\bar{\partial}_\alpha^*u)^2 + (C + c') \int_F |u|_{\tilde{\omega}}^2 \tilde{v}^\alpha dV_{\tilde{\omega}}.$$

Since $\tilde{\omega}$ is a complete metric on $X \setminus L$, (4.4) is valid for every $u \in (\text{Dom} \bar{\partial}) \cap (\text{Dom} \bar{\partial}_\alpha^*)$. The conclusion of Proposition 2 is now a consequence of Proposition 1.2 of [23]. ■

Corollary 1. *For every $\alpha > 0$ and $1 \leq q \leq n$ we have $\mathcal{H}_{(n,q)}(X^\pm, \tilde{v}^\alpha, \tilde{\omega}) = \{0\}$.*

Proof. As $(X^\pm, \tilde{\omega})$ is a connected weakly 1-complete Kähler manifold and the bundle E defined in the proof of Theorem 2 is a semi-positive line bundle on X^\pm which is positive outside a compact subset of X^\pm , the Corollary 1 is a consequence of [33], Corollary of the Main Theorem (see also [3], [30] and [26], Corollary 2.10). ■

By taking in account Corollary 1, a classical application of Proposition 2 (see for example [18]) is the following:

Corollary 2. *For every $\alpha > 0$ and $1 \leq q \leq n$ we have:*

- (1) *There exists the $\bar{\partial}$ -Neumann operator $\mathcal{N}_{(n,q)}^\alpha : L_{(n,q)}^2(X^\pm, \tilde{v}^\alpha, \tilde{\omega}) \rightarrow L_{(n,q)}^2(X^\pm, \tilde{v}^\alpha, \tilde{\omega})$ such that for every $f \in L_{(n,q)}^2(X^\pm, \tilde{v}^\alpha, \tilde{\omega})$ we have the orthogonal decomposition $f = \bar{\partial} \bar{\partial}_\alpha^* \mathcal{N}_{(n,q)}^\alpha f + \bar{\partial}_\alpha^* \bar{\partial} \mathcal{N}_{(n,q)}^\alpha f$ and $\bar{\partial} \mathcal{N}_{(n,q)}^\alpha = \mathcal{N}_{(n,q+1)}^\alpha \bar{\partial}$, $\bar{\partial}_\alpha^* \mathcal{N}_{(n,q)}^\alpha = \mathcal{N}_{(n,q-1)}^\alpha \bar{\partial}_\alpha^*$.*
- (2) *For every $\bar{\partial}$ -closed form $f \in L_{(n,q)}^2(X^\pm, \tilde{v}^\alpha, \tilde{\omega})$, $\bar{\partial}(\bar{\partial}_\alpha^* \mathcal{N}_{(n,q)}^\alpha f) = f$.*

Lemma 1. *Let $f \in C_{(0,q)}^\infty(X)$, $1 \leq q \leq n-1$, be a $\bar{\partial}$ -closed form such that f vanishes to infinite order on L . Let $\psi_1, \psi_2 \in \text{Dom} \bar{\partial} \subset L_{(n,n-q)}^2(X^\pm, \tilde{v}^\alpha, \tilde{\omega})$ such that $\bar{\partial} \psi_1 = \bar{\partial} \psi_2$. Then*

$$\int_{X^\pm} f \wedge (\psi_1 - \psi_2) = 0.$$

Proof. By Corollary 2, there exists $h \in L^2_{(n,n-q-1)}(X^\pm, \tilde{v}^\alpha, \tilde{\omega})$ such that $\psi_1 - \psi_2 = \bar{\partial}h$. Since f vanishes to infinite order on L and $\tilde{\omega} \leq \frac{C}{\tilde{v}^2}\omega$, it follows that

$$\int_{X^\pm} f \wedge (\psi_1 - \psi_2) = \lim_{\varepsilon \rightarrow 0} \int_{\{v > \varepsilon\} \cap X^\pm} f \wedge \bar{\partial}h = \lim_{\varepsilon \rightarrow 0} \left(\int_{\{v > \varepsilon\}} \bar{\partial}f \wedge h + \int_{\{v = \varepsilon\} \cap X^\pm} f \wedge h \right) = 0.$$

■

Proposition 3. *Let $f \in C^\infty_{(0,q)}(X)$, $1 \leq q \leq n-1$, be a $\bar{\partial}$ -exact form such that f vanishes to infinite order on L . Then for every $\alpha > 0$, there exists $u \in L^2_{(0,q-1)}(X^\pm, \tilde{v}^{-\alpha}, \tilde{\omega})$ such that $\bar{\partial}u = f$ and $N_{-\alpha, \tilde{\omega}, \tilde{v}}(u) \leq C_\alpha N_{-\alpha, \tilde{\omega}, \tilde{v}}(f)$, with $C_\alpha > 0$ independent of f .*

Proof. Step 1. Definition by duality of $u \in L^2_{(0,q-1)}(X^\pm, \tilde{v}^{-\alpha}, \tilde{\omega})$, $1 \leq q \leq n-1$.

The proof of this point is inspired from [19], Proposition 5.3. By Proposition 2, $\mathcal{R}^\alpha_{(n,q)}(X^\pm)$ is closed for every $\alpha > 0$ and by Corollary 2 we can find a bounded operator $T^\alpha_{(n,q)} = \bar{\partial}^*_\alpha \mathcal{N}^\alpha_{(n,q)} : \mathcal{R}^\alpha_{(n,q)}(X^\pm) \rightarrow L^2_{(n,q-1)}(X^\pm, \tilde{v}^\alpha, \tilde{\omega})$, such that $\bar{\partial}T^\alpha_{(n,q)}\varphi = \varphi$ for every $\varphi \in \mathcal{R}^\alpha_{(n,q)}(X^\pm)$, $1 \leq q \leq n$.

Define now the continuous linear form Φ_f on $\mathcal{R}^\alpha_{(n,n-q+1)}(X^\pm)$, $1 \leq q \leq n$, by

$$\Phi_f(\varphi) = \int_{X^\pm} f \wedge T^\alpha_{(n,n-q+1)}\varphi, \quad \varphi \in \mathcal{R}^\alpha_{(n,n-q+1)}(X^\pm).$$

By the Hahn-Banach theorem, we extend Φ_f as a linear form $\widetilde{\Phi}_f$ on $L^2_{(n,n-q+1)}(X^\pm, \tilde{v}^\alpha, \tilde{\omega})$ such that $\|\widetilde{\Phi}_f\| = \|\Phi_f\|$. Since $\left(L^2_{(n,n-q+1)}(X^\pm, \tilde{v}^\alpha, \tilde{\omega})\right)' = L^2_{(0,q-1)}(X^\pm, \tilde{v}^{-\alpha}, \tilde{\omega})$ by the pairing

$$(\beta_1, \beta_2) = \int_{X^\pm} \beta_1 \wedge \beta_2, \quad \beta_1 \in L^2_{(0,q-1)}(X^\pm, \tilde{v}^{-\alpha}, \tilde{\omega}), \quad \beta_2 \in L^2_{(n,n-q+1)}(X^\pm, \tilde{v}^\alpha, \tilde{\omega}),$$

there exists $u \in L^2_{(0,q-1)}(X^\pm, \tilde{v}^{-\alpha}, \tilde{\omega})$ such that

$$\widetilde{\Phi}_f(\varphi) = \int_{X^\pm} u \wedge \varphi$$

for every $\varphi \in \mathcal{R}^\alpha_{(n,n-q+1)}(X^\pm)$.

Step 2. We prove that $\bar{\partial}(-1)^q u = f$, $1 \leq q \leq n-1$.

Let $\varphi = \bar{\partial}\psi \in C^\infty_{(n,n-q+1)}(X^\pm)$ with $\psi \in \mathcal{D}_{(n,n-q)}(X^\pm)$. Set $g_\alpha = \bar{\partial}^*_\alpha \mathcal{N}^\alpha_{(n,n-q+1)}\bar{\partial}\psi \in L^2_{(n,n-q)}(X^\pm, \tilde{v}^\alpha, \tilde{\omega})$. By Corollary 2, $\bar{\partial}g_\alpha = \varphi$ and by Lemma 1

$$(4.5) \quad \int_{X^\pm} f \wedge g_\alpha = \int_{X^\pm} f \wedge \psi.$$

But by step 1 we have

$$(4.6) \quad \widetilde{\Phi}_f(\varphi) = \int_{X^\pm} u \wedge \bar{\partial}\psi = \Phi_f(\varphi) = \int_{X^\pm} f \wedge g_\alpha$$

and by (4.5) and (4.6) it follows that

$$\int_{X^\pm} f \wedge \psi = \int_{X^\pm} u \wedge \bar{\partial}\psi$$

for every $\psi \in \mathcal{D}_{(n,n-q)}(X^\pm)$. Therefore $\bar{\partial}(-1)^q u = f$ and the Proposition is proved. ■

Remark 2. Since $\omega \leq \tilde{\omega} \leq \frac{C}{v^2}\omega$, by Lemma VIII.6.3 of [14] it follows that:

a) Let f be a smooth (n, q) -form on X such that f vanishes to order k on L . Then $f \in L^2_{(n, q)}(X^\pm, \tilde{v}^{-k}, \tilde{\omega})$

Indeed

$$\int_{X^\pm} |f|_{\tilde{\omega}}^2 \tilde{v}^{-k} dV_{\tilde{\omega}} \leq \int_{X^\pm} |f|_{\omega}^2 \tilde{v}^{-k} dV_{\omega} < \infty$$

b) Let $f \in L^2_{(n, q)}(X^\pm, \tilde{v}^{-k}, \tilde{\omega})$, $k > 2$. Then $f \in L^2_{(n, q)}(X^\pm, \tilde{v}^{-k+2}, \omega)$.

Indeed

$$\int_{X^\pm} |f|_{\omega}^2 \tilde{v}^{-k+2} dV_{\omega} \leq C \int_{X^\pm} |f|_{\tilde{\omega}}^2 \tilde{v}^{-k} dV_{\tilde{\omega}} < \infty, \quad C > 0.$$

5. NONEXISTENCE OF LEVI FLAT HYPERSURFACES

Proposition 4. Let L be a compact C^∞ Levi flat hypersurface in a Kähler manifold X of dimension $n \geq 3$ such that the normal bundle $\mathcal{N}_L^{1,0}$ to the Levi foliation admits a C^2 Hermitian metric with leafwise positive curvature. Let $u \in C^\infty_{(0, q)}(L)$, $1 \leq q \leq n-2$, such that $\bar{\partial}_b u = 0$. Then for every $k \in \mathbb{N}^*$ there exist a $\bar{\partial}$ -closed extension $U_k \in C^k_{(0, q)}(X)$ of u .

Proof. By Proposition 1 there exist a neighborhood U of L , $c > 0$ and a non-negative function $v \in C^2(\bar{U})$ vanishing on L such that $v = g\delta_L^2$ and $-i\partial\bar{\partial} \ln v \geq c\omega$ on $U \setminus L$. Let $\tilde{u} \in C^\infty_{(0, q)}(X)$ be an extension of u such that $\bar{\partial}\tilde{u}$ vanishes to infinite order on L . Since $\bar{\partial}\tilde{u} \in L^2_{(0, q+1)}(X^\pm, \delta_L^{-2k}, \omega)$, $q+1 \leq n-1$ and $L^2_{(0, q)}(X^\pm, \delta_L^{-2k}, \omega) = L^2_{(0, q)}(X^\pm, \tilde{v}^{-k}, \omega)$ for every $k \in \mathbb{N}$, by Remark 2 a) and Proposition 3 it follows that for every $k \in \mathbb{N}^*$ there exist a Hermitian complete metric $\tilde{\omega}$ on $X \setminus L$, $\omega \leq \tilde{\omega} \leq \frac{C}{v^2}\omega$ and $h^\pm \in L^2_{(0, q)}(X^\pm, \delta_L^{-2k}, \tilde{\omega})$ such that $\bar{\partial}h^\pm = \bar{\partial}\tilde{u}$ on X^\pm . By Remark 2 b) we have $h^\pm \in L^2_{(0, q)}(X^\pm, \delta_L^{-2k+4}, \omega)$. So by using Theorem 1, for k big enough we can choose $h^\pm \in C^{s(k)}_{(0, q)}(\bar{X}^\pm)$, $s(k) \underset{k \rightarrow \infty}{\sim} \sqrt{k}$. This means that for k big enough, the form h defined as h^\pm on \bar{X}^\pm is of class C^k on X and vanishes on L . So $U_k = \tilde{u} - h^\pm$ is a C^k -smooth $\bar{\partial}$ -closed form on X which is an extension of u . ■

Theorem 2. Let X be a compact connected Kähler manifold of dimension $n \geq 3$ and L a C^∞ compact Levi flat hypersurface. Then the normal bundle to the Levi foliation does not admit any Hermitian metric of class C^2 with leafwise positive curvature.

Proof. Suppose that the normal bundle \mathcal{N} to the Levi foliation admits a Hermitian metric of class C^2 with leafwise positive curvature. Since \mathcal{N} is topologically trivial, its curvature form $\Theta^\mathcal{N}$ for the Kähler metric of X is d -exact. So there exists a 1-form u of class C^∞ on L such that $du = \Theta^\mathcal{N}$; we may suppose that u is real and $u = u^{0,1} + \bar{u}^{0,1}$, where $u^{0,1}$ is the $(0, 1)$ component of u . Since $\Theta^\mathcal{N}$ is a $(1, 1)$ -form, it follows that $\bar{\partial}_b u^{0,1} = 0$, where $\bar{\partial}_b$ is the tangential Cauchy-Riemann operator. By Proposition 4 there exists a C^k -extension $U^{0,1}$ of $u^{0,1}$ to X , $k \geq 2$, such that $\bar{\partial}U^{0,1} = 0$.

By Hodge symmetry and Dolbeault isomorphism $H^{0,1}(X, \mathbb{C}) \approx \overline{H^{1,0}(X, \mathbb{C})} \approx H^0(X, \Omega^1_X)$, where Ω^1_X is the sheaf of holomorphic 1-forms on X . So there exists $\eta \in H^0(X, \Omega^1_X)$ and $\Phi \in C^k(X)$ such that $\widetilde{U^{0,1}} = \bar{\eta} + \bar{\partial}\Phi$. It follows that $\Theta^\mathcal{N} =$

$i\partial_b\bar{\partial}_b \operatorname{Im} \Phi$ on L and this gives a contradiction at the point of L where $\operatorname{Im} \Phi$ reaches its maximum. ■

Remark 3. *A first version of this paper was announced on arXiv in 2014, but there was a gap in the proofs of §4, which is now corrected. Recently, Brinkschulte proved a generalization of Theorem 2 for compact Levi flat hypersurfaces in complex manifolds (see Theorem 1.1. of [8]). She uses crucially the Proposition 4.1 of [8], whose statement and proof are the same as Proposition 1 of this paper and which are unchanged from 2014 in our preprint arXiv:1406.5712. However she refers only to Proposition 1.1 of [25], where the lower positive bound for the eigenvalues of the strongly plurisubharmonic function is not mentioned.*

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