On completely multiplicative automatic sequences
Shuo Li

To cite this version:

HAL Id: hal-02995722
https://hal.sorbonne-universite.fr/hal-02995722
Submitted on 9 Nov 2020

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers. L’archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d’enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.
On completely multiplicative automatic sequences

Shuo LI
CNRS, Institut de Mathématiques de Jussieu-PRG
Université Pierre et Marie Curie, Case 247
4 Place Jussieu
F-75252 Paris Cedex 05 (France)
shuo.li@imj-prg.fr

Abstract

In this article we prove that all completely multiplicative automatic sequences \((a_n)_{n \in \mathbb{N}}\) defined on \(\mathbb{C}\), vanishing or not, can be written in the form \(a_n = b_n \chi_n\) where \((b_n)_{n \in \mathbb{N}}\) is an almost constant sequence, and \((\chi_n)_{n \in \mathbb{N}}\) is a Dirichlet character.

1 Introduction

In this article we give a decomposition of completely multiplicative automatic sequences, which will be referred as CMAS. In article [SP11], the author proves that a non-vanishing CMAS is almost periodic (defined in [SP11]). In article [AG18], the authors give a formal expression to all non-vanishing CMAS and also some examples in the vanishing case (named as mock characters). In article [Hu17], the author studies completely multiplicative sequences, which will be referred as CMS, taking values on a general field, which have finitely many prime numbers such that \(a_p \neq 1\), she proves that such CMS have complexity \(p^a(n) = O(n^k)\) where \(k = \# \{p | p \in \mathbb{P}, a_p \neq 1, 0\}\). In this article we prove that all completely multiplicative sequences \((a_n)_{n \in \mathbb{N}}\) defined on \(\mathbb{C}\), vanishing or not, can be written in the form \(a_n = b_n \chi_n\) where \((b_n)_{n \in \mathbb{N}}\) is an almost constant sequence, and \((\chi_n)_{n \in \mathbb{N}}\) is a Dirichlet character.

Let us consider a CMAS \((a_n)_{n \in \mathbb{N}}\) defined on \(\mathbb{C}\). We firstly prove that all CMAS are mock characters (defined in [AG18]) with an exceptional case. Secondly, we study the CMAS satisfying the condition \(C\) :

\[
\sum_{p | a_p \neq 1, p \in \mathbb{P}} \frac{1}{p} < \infty,
\]

where \(\mathbb{P}\) is the set of prime numbers, we prove that in this case, there is at most one prime \(p\) such that \(a_p \neq 1\) or 0. In the third part we prove that all CMAS are either Dirichlet-like sequence or strongly aperiodic sequences. Lastly we conclude by proving that a strongly aperiodic sequence can not be automatic.

2 Definitions, notation and basic propositions

Let us recall the definition of automatic sequences and complete multiplicativity:
Definition Let \((a_n)_{n\in\mathbb{N}}\) be an infinite sequence and \(k \geq 2\) an integer, we say this sequence is \(k\)-automatic if there is a finite set of sequences containing \((a_n)_{n\in\mathbb{N}}\) and closed under the maps
\[ a_n \rightarrow a_{kn+i}, \quad i = 0, 1, \ldots, k-1. \]

There is another definition of a \(k\)-automatic sequence \((a_n)_{n\in\mathbb{N}}\) via an automaton. An automaton is an oriented graph with one state distinguished as initial state, and for each state there are exactly \(k\) edges pointing from this state to others, these edges are labeled as \(0, 1, \ldots, k-1\).

There is an output function \(f\) which maps the set of states to a set \(U\). For an arbitrary \(n \in \mathbb{N}\), the \(n\)-th element of the automatic sequence can be computed as follows: writing the \(k\)-ary expansion of \(n\), starting from the initial state and moving from one state to another by taking the edge read in the \(k\)-ary expansion one by one until stop on some state. The value of \(a_n\) is the evaluation of \(f\) on the stopping state. If we read the expansion from right to left, then we call this automaton a reverse automaton of the sequence, otherwise it is called a direct automaton.

In this article, all automata considered are direct automata.

Definition We define a subword\(^1\) of a sequence as a finite length string of the sequence. We let \(w_l\) denote a subword of length \(l\).

Definition Let \((a_n)_{n\in\mathbb{N}}\) be an infinite sequence, we say this sequence is completely multiplicative if for any \(p, q \in \mathbb{N}\), we have \(a_pa_q = a_{pq}\).

It is easy to see that a CMAS can only take finite many values, either 0 or a \(k\)-th root of unity (see, for example, Lemma 1 [SP11]).

Definition Let \((a_n)_{n\in\mathbb{N}}\) be a CMS, we say that \(a_p\) is a prime factor of \((a_n)_{n\in\mathbb{N}}\) if \(p\) is a prime number and \(a_p \neq 1\). Moreover, we say that \(a_p\) is a non-trivial factor if \(a_p \neq 0\); and respectively, we say that \(a_p\) is a 0-factor if \(a_p = 0\). We say that a sequence \((a_n)_{n\in\mathbb{N}}\) is generated by \(a_{p_1}, a_{p_2}, \ldots\) if and only if \(a_{p_1}, a_{p_2}, \ldots\) are the only prime factors of the sequence.

Definition We say that a sequence is an almost-0-sequence if there is only one non-trivial factor \(a_p\) and \(a_q = 0\) for all primes \(q \neq p\).

Proposition 1 Let \((a_n)_{n\in\mathbb{N}}\) be a \(k\)-CMAS and \(q\) be the number of states of a direct automaton generating \((a_n)_{n\in\mathbb{N}}\) then for any \(m, y \in \mathbb{N}\). We have equality between the sets \(\{a_n|mk^q \leq n < (m+1)k^q\} = \{a_n|mk^{qy} \leq n < (m+1)k^{qy}\}\).

Proof In article [SP11](Lemma 3 and Theorem 1), the author proves that in an automaton, every state which can be reached from a specific state, say \(s\), with \(q!\) steps, can be reached with \(yy!\) steps for every \(y \geq 1\); and conversely, if a state can be reached with \(yy!\) steps for some \(y \geq 1\), then it can already be reached with \(q!\) steps. This proves the proposition.

\(^1\)what we call subword here is also called factor in the literature, but we use factor with a different meaning.
Let us consider \((a_n)_{n \in \mathbb{N}}\) a CMS taking values in a finite Abelian group \(G\), we define

\[
E = \left\{ g \in G, \sum_{a_p = g, p \in \mathbb{P}} \frac{1}{p} = \infty \right\}
\]

and \(G_1\) the subgroup of \(G\) generated by \(E\).

**Definition** We say that an element \(\zeta\) of a sequence \((a_n)_{n \in \mathbb{N}}\) has a natural density if and only if 
\[
\lim_{N \to \infty} \frac{\# \{ n \mid a_n = \zeta, 0 \leq n \leq N \}}{N+1}
\]
exists, and we say that the sequence \((a_n)_{n \in \mathbb{N}}\) has a mean value if and only if 
\[
\lim_{N \to \infty} \frac{\sum_{n=0}^{N} a_n}{N+1}
\]
exists.

**Proposition 2** Let \((a_n)_{n \in \mathbb{N}}\) be a CMS taking values in a finite Abelian group \(G\), then for all elements \(g \in G\), the sequence \(a^{-1}(g) = \{ n : a_n = g \}\) has a non-zero natural density. Furthermore, this density depends only on the coset \(rG_1\) on which the element \(g\) lies. The statement is still true in the case that \(G\) is a semi-group generated by a finite group and 0, under the condition that there are finitely many primes \(p\) such that \(a_p = 0\).

**Proof** When \(G\) is an Abelian group the proposition is proved in Theorem 3.10, [Ruz77], and when \(G\) is a semi-group, the Theorem 7.3, [Ruz77] shows that all elements in \(G\) have a natural density. To conclude the proof, it is enough to consider the following fact: let \(f_0\) be a CMS such that there exists a prime \(p\) with \(a_p = 0\), let \(f_1\) be another CMS such that

\[
f_1(q) = \begin{cases} f_0(q) & \text{if } q \in \mathbb{P}, q \neq p \\ 1 & \text{otherwise,} \end{cases}
\]

if \(d_0(g), d_1(g)\) denote respectively the natural density of \(g\) in the sequence \((f_0(n))_{n \in \mathbb{N}}\) and \((f_1(n))_{n \in \mathbb{N}}\), then we have the equality

\[
d_1(g) = d_0(g) + \frac{1}{p}d_0(g) + \frac{1}{p^2}d_0(g) + \cdots = \frac{p}{p-1}d_0(g).
\]

Doing this regressively until a non-vanishing sequence, we can conclude by the first part of the proposition.

### 3 Finiteness of the numbers of 0-factors

In this section we will prove that a CMAS is either a mock character, which means it has only finitely many 0-factors, or an almost-0-sequence, that is to say \(a_m = 0\) for all \(m\) which is not a power of \(p\) and \(a_{pk} = \delta^k\) for some \(\delta\), where \(\delta\) is a root of unity or 0 and \(p\) is a prime number.

**Proposition 3** Let \((a_n)_{n \in \mathbb{N}}\) be a \(p\)-CMAS, then it is either a mock character or an 0-almost-sequence.

**Proof** If \((a_n)_{n \in \mathbb{N}}\) is not a mock character, then it contains infinitely many 0-factors. Here we prove that, in this case, if there is some \(a_m \neq 0\) then \(m\) must be a power of \(p\) and \(p\) must be a prime number. Let us suppose that there are \(q\) states on automaton generating the sequence. As there are infinitely many 0-factors, it is easy to find a subword of length \(p^{2q}\) such that all its
elements are:
It is equivalent to find some \( m \in \mathbb{N} \) and \( p^{2q} \) 0-factors, say \( a_{p_1}, a_{p_2}, \ldots, a_{p_{r'}} \), such that
\[
\begin{cases}
  m \equiv 0 \pmod{p_1} \\
  m + 1 \equiv 0 \pmod{p_2} \\
  m + 2 \equiv 0 \pmod{p_3} \\
  \ldots
  \end{cases}
\]
\( m + p^{2q} - 1 \equiv 0 \pmod{p_{r'}} \)

If \( m \) is a solution of the above system, then the subword \( a_{m_1}a_{m_1+1} \ldots a_{m_1+p^{2q}m_1} \) is constant to 0. So there exists a \( m' \) such that \( m \leq m'p^q < (m' + 1)p^q \leq m + p^{2q} \). Because of Proposition 1, for any \( y \in \mathbb{N} \), \( a_k = 0 \) for all \( k \) such that \( m'p^q \leq k < (m' + 1)p^q \). Taking an arbitrary prime \( r \), if \( r \) and \( p \) are not multiplicatively dependent, then \( a_r = 0 \) because there exists a power of \( r \) satisfying \( m'p^q \leq r^t < (m' + 1)p^q \), this inequality holds because we can find some integer \( t \) and \( y \) such that:
\[
\log_p m' \leq t \log_p r - yq < \log_p (m' + 1).
\]

The above argument shows that if \( (a_n)_{n \in \mathbb{N}} \) is not a sequence such that \( a_m = 0 \) for all \( m > 1 \), then \( p \) must be a power of a prime number \( p' \). Otherwise, as \( p \) is not multiplicatively dependent with any prime numbers, \( a_m = 0 \) for all \( m > 1 \). Furthermore, the sequence \( (a_n)_{n \in \mathbb{N}} \) can have at most one non-zero prime factor, and if it exists, it should be \( a_{p'} \). Using the automaticity, we can replace \( p' \) with \( p \).

4 CMAS satisfying condition C

From this section, we consider only the CMAS with finitely many 0-factors.

In this section we prove that all CMAS satisfying C can have at most one non-trivial factor, and we do this in several steps.

**Proposition 4** Let \( (a_n)_{n \in \mathbb{N}} \) be a non-vanishing CMS taking values in the set \( G = \{ \zeta^r | r \in \mathbb{N} \} \), where \( \zeta \) is a non-trivial \( k \)-th root of unity, having \( u \) prime factors \( a_{p_1}, a_{p_2}, \ldots, a_{p_u} \), then there exist \( g \in G \) (where \( a_{p_1} = g \)) and a subword \( \overline{w} \) appearing periodically in the sequence \( (a_n)_{n \in \mathbb{N}} \) such that all its letters are different from \( g \). Furthermore, the period does not have any other prime factor than \( p_1, p_2, \ldots, p_u \).

**Proof** We prove it by induction, for \( u = 1 \), the above statement is trivial. It is easy to check that the sequence \( (a_{n^2 + p_1^k + p_1^k + p_1^k})_{n \in \mathbb{N}} \) is a constant sequence of 1, the period is \( p_1^k+1 \) and \( g = a_{p_1} \).

Supposing the statement is true for some \( u = n_0 \), let us consider the case \( u = n_0 + 1 \). We firstly consider the the sequence \( (a'_n)_{n \in \mathbb{N}} \) defined as \( a'_n = a_{n^{p_{n_0+1}+1}} \), a sequence having \( n_0 \) prime factors, where \( v_p(n) \) denotes the largest integer \( r \) such that \( p^r \mid n \). Using the hypothesis of induction we get a subword \( \overline{w}_{n_0} \) satisfying the statement. Let us suppose that the first letter of this subword appears in the sequence \( (a'_{m_{n_0}n + l_{n_0}})_{n \in \mathbb{N}} \). We can extract from this sequence a sequence of the form \( (a'_{m_{n_0}n + l_{n_0}})_{n \in \mathbb{N}} \) such that \( m_{n_0}' = m_{n_0}\prod_{j=1}^{n_0} p_j^{d_j} \) for some \( d_j \in \mathbb{N}^+ \) and \( v_{p_j}(m_{n_0}'n + l_{n_0} + n_0) = v_{p_j}(l_{n_0} + n_0) \) for all \( j \leq n_0 \). In this case the sequence \( (a'_{m_{n_0}n + l_{n_0} + n_0})_{n \in \mathbb{N}} \) is a constant sequence, say all letters equal \( C \).
Here we consider two residue classes $N_1(n), N_2(n)$ satisfying separately the conditions:

\[ m_{n_0}N_1(n) \equiv -l_{n_0} - n_0 \mod p_{n_0+1} \]

and

\[ m_{n_0}N_2(n) \not\equiv -l_{n_0} - n_0 \mod p_{n_0+1} \]

In these two cases we have respectively $a_{m_{n_0}}N_1(n)+l_{n_0}+n_0 = Cp_{n_0+1}$ and $a_{m_{n_0}}N_2(n)+l_{n_0}+n_0 = C^2p_{n_0+1}$ for all $n \in \mathbb{N}$. As $a_{p_{n_0+1}} \neq 1$, there is at least one element of $Cp_{n_0+1}, C^2p_{n_0+1}$ not equal to $g$. If $N_1(n)$ is the associated residue class, then $N_1(n) = p_1^{l+1}n + t$ for all integer $n$ with $t \in \mathbb{N}, i = 1$ or 2.

Now let us choose $m_{n_0+1} = m_{n_0}p_{n_0+1}$ and $l_{n_0} = l_{n_0}+tm_{n_0}$, so that the sequence $(a_{m_{n_0}+1+n_{n_0}})_{n \in \mathbb{N}}$ is a subsequence of $(a_{m_{n_0}n+n_{n_0}})_{n \in \mathbb{N}}$, thus the subword $a_{m_{n_0}+1+n_{n_0}+1}a_{m_{n_0}+n_{n_0}+1}a_{m_{n_0}+n_{n_0}+1}...a_{m_{n_0}+n_{n_0}+1}$ is constant and none of its letters equals $g$ because of the hypothesis of induction. Furthermore, $a_{m_{n_0}+1+n_{n_0}+n_0} = a_{m_{n_0}N_1(n)+l_{n_0}+n_0}$ is independent of $n$ and different from $g$ because of the choice of residue class. The properties saying that the prime number $p_{n_0+1}$ is larger than $n_0 + 1$ and $p_{n_0+1}m_{n_0}N_1(n) + l_{n_0} + n_0$ by construction imply the fact that for all $j$ such that $0 \leq j \leq n_0 - 1$, $p_{n_0+1} \nmid m_{n_0+1}l_{n_0}+j$. So we conclude that for all $n, j \in \mathbb{N}$ such that $0 \leq j \leq n_0 - 1$, $v_{p_{n_0+1}}(m_{n_0+1}l_{n_0}+j) = 0$.

This means that the subword $a_{m_{n_0}+1+n_{n_0}+1}a_{m_{n_0}+n_{n_0}+1}a_{m_{n_0}+n_{n_0}+1}...a_{m_{n_0}+n_{n_0}+1+n_0}$ is a subword of length $n_0 + 1$ independent of $n$ and none of its letters equals $g$, what is more $m_{n_0+1}$ does not have any prime factor other than $p_1, p_2, ... p_{n_0}$.

**Proposition 5** Let $(a_n)_{n \in \mathbb{N}}$ be a non-vanishing CMS defined on a finite set $G$ satisfying condition $C$, and let $(a'_n)_{n \in \mathbb{N}}$ be another CMS generated by the first $r$ prime factors of $(a_n)_{n \in \mathbb{N}}$, say $a_{p_1}, a_{p_2}, ..., a_{p_r}$. If there is a subword $\mathfrak{w}_r$ appearing periodically in $(a'_n)_{n \in \mathbb{N}}$, and the period does not have any other prime factors than $p_1, p_2, ..., p_r$, then this subword appears at least once in $(a_n)_{n \in \mathbb{N}}$.

**Proof** Let us denote by $p_1, p_2, ..., p_{n_0}$ the sequence of prime numbers such that $a_{p_i} \neq 1$. Supposing that the first letter of the subword $\mathfrak{w}_r$ belongs to the sequence $(a_{m_{n_0}+1})_{n \in \mathbb{N}}$ for some $m_r \in \mathbb{N}, r_0 \in \mathbb{N}$, by hypothesis, $m_r$ does not have any other prime factor than $p_1, p_2, ..., p_{r_0}$. So the total number of such words in the sequence $(a_n)_{n \in \mathbb{N}}$ can be bounded by the inequality:

\[
\# \{ a_k|k \leq n, a_{k+1}, a_{k+2}, ..., a_{k+r_0-1} = \mathfrak{w}_r \} \geq \# \{ a_k|k \leq n, k = m_rk' + l_r, k' \in \mathbb{N}; p_i \not\mid k + j, \forall (i,j) \text{ with } 0 \leq j \leq r - 1, i > r \}
\]

Let us consider the sequence defined as $N(t) = \prod_{j=1}^{t} p_{r+j}$, we have

\[
\# \{ a_k|k \leq N(t)m_r + l_r, k = m_rk' + l_r, k' \in \mathbb{N}; p_i \not\mid k + j, \forall (i,j) \text{ with } 0 \leq j \leq r - 1, r < i \leq r + t \} = \prod_{j=1}^{t} (p_{r+j} - r)
\]

This equality holds because of Chinese reminder theorem, and the fact that $p_{r+j} \nmid r$ and $p_{r+j} > r$ for all $j \geq 1$. 

5
So we have

\[
\# \{ a_k | k \leq N(t)m_r + l_r, k = m_rk' + l_r, k' \in \mathbb{N}; p_i \nmid k + j, \forall (i, j) \text{ with } 0 \leq j \leq r - 1, i > r \} \\
\# \{ a_k | k \leq N(t)m_r + l_r, k = m_rk' + l_r, k' \in \mathbb{N}; p_i \nmid k + j, \forall (i, j) \text{ with } 0 \leq j \leq r - 1, r < i \leq r + t \} \\
- \# \{ a_k | k \leq N(t)m_r + l_r, k = m_rk' + l_r, k' \in \mathbb{N}; p_i \nmid k + j, \forall (i, j) \text{ with } 0 \leq j \leq r - 1, i > r + t \} \\
> \sum_{i > r + t} \# \{ a_k | k \leq N(i)m_r + l_r, k = m_rk' + l_r, k' \in \mathbb{N}; p_i \nmid k + j, \forall (i, j) \text{ with } 0 \leq j \leq r - 1, r < i \leq r + t \} \\
> \sum_{i > r + t} \sum_{k > r + t} \frac{N(t)}{p_k} - r\pi(N(t) + r).
\]

(3)

where \([a]\) represents the smallest integer larger than \(a\) and \(\pi\) is the prime counting function.

However,

\[
\prod_{j=1}^{t} (p_{r+j} - r) - r \sum_{i > r + t, p_i < N(t) + r} \frac{N(t)}{p_i} - r\pi(N(t) + r).
\]

(4)

The last formula can be approximated as

\[
\prod_{j=1}^{t} \frac{p_{r+j} - r}{p_{r+j}} = \exp(\sum_{j=1}^{t} \log(\frac{p_{r+j} - r}{p_{r+j}})) = \exp(-\Theta(\sum_{j=1}^{t} \frac{r}{p_{r+j}})),
\]

the last equality holds because \(\log(1 - x) \sim x\) when \(x\) is small. Because of \(C\), the above quantity does not diverge to 0. We conclude that, if \(t\) is large enough, there exists a \(c\) with \(0 < c < 1\) such that

\[
\prod_{j=1}^{t} (p_{r+j} - r) > cN(t).
\]

On the other hand, we remark that for all \(i > r + t, p_i > \prod_{j=1}^{t} p_{r+j} = N(t)\), so \(p_i > N(t)^{\frac{1}{2}}\)

\[
\sum_{i > r + t, p_i < N(t) + r} \frac{N(t)}{p_i} < N(t) \sum_{N(t)^{\frac{1}{2}} < p < N(t) + r} \frac{1}{p}.
\]

(5)

And the term \(N(t)^{\frac{1}{2}}\) can be bounded by

\[
N(t)^{\frac{1}{2}} = \left( \prod_{j=1}^{t} p_{r+j} \right)^{\frac{1}{2}} \geq \frac{t}{\sum_{j=1}^{t} \frac{1}{p_{r+j}}} > \frac{t}{\sum_{j=1}^{t} \frac{1}{q_j}}.
\]

(6)

where \(q_j\) is the \(j\)-th prime number in \(\mathbb{N}\). For any \(x \in \mathbb{N}\), \# \{ \(p_i | p_i \leq x\) \} \sim \frac{x}{\log(x)}\) and \(\sum_{p_i \leq x} \frac{1}{p_i} \sim \log \log(x)\), so \(N(t)^{\frac{1}{2}}\) tends to infinity when \(t\) tends to infinity, because of \(C\), we can conclude that there exists some \(t_0 \in \mathbb{N}\) such that for all \(t > t_0\), \(\sum_{N(t)^{\frac{1}{2}} < p < N(t) + r} \frac{1}{p} < \frac{1}{2\pi}c\).

To conclude, for all \(t > t_0\),

\[
\# \{ a_k | k \leq N(t)m_r + l_r, k = m_rk' + l_r, k' \in \mathbb{N}; p_i \nmid k + j, \forall (i, j) \text{ with } 0 \leq j \leq r - 1, r < i \leq r + t \} \\
> \prod_{j=1}^{t} (p_{r+j} - r) - r \sum_{k > r + t} \frac{N(t)}{p_k} - r\pi(N(t) + r) \\
> cN(t) - \frac{1}{2}cN(t) - r\pi(N(t) + r).
\]

(7)
When $t$ tends to infinity, the set $\{a_k | k \leq n, a_k, a_{k+1}, ..., a_{k+r-1} = w\}$ is not empty.

**Proposition 6** Let $(a_n)_{n \in \mathbb{N}}$ be a $p$-CMAS, vanishing or not, satisfying condition $C$, then there exists at most one prime number $k$ such that $a_k \neq 1$ or 0.

**Proof** Supposing that the sequence $(a_n)_{n \in \mathbb{N}}$ has infinitely many prime factors not equal to 0 or 1. Let us consider firstly the sequence $(a'_n)_{n \in \mathbb{N}}$ defined as follows:

$$a'_n = a_n \prod_{p \in \mathbb{P} \cap \mathbb{N}} p^{v_p(a_n)},$$

where $\mathbb{Z} = \{p|p \in \mathbb{P}, a_p = 0\}$, because of Proposition 3, this set is finite.

Using the Proposition 4 and 5, there exists a subword of length $p^{2q}$, say $w_{p^{2q}}$, appearing in $(a'_n)_{n \in \mathbb{N}}$ such that none of its letters equals $g = a'_{p_1} = a_{p_1}$, where $q$ is the number of states of the automaton generating $(a_n)_{n \in \mathbb{N}}$. Then, by construction, there is a subword of same length, say $w_{p^{2q}}$, appearing at the same position on the sequence $(a_n)_{n \in \mathbb{N}}$ such that none of its letters equals $g$. Extracting a subword $w'_{p^{2q}}$ contained in $w_{p^{2q}}$ of the form $w_{p^{2q}} - a_{(u+1)p^q-1}$ for some $u \in \mathbb{N}$ and using the Proposition 1, we have for every $y$ such that $y \geq 1$ and every $m$ such that $0 \leq m \leq p^{2q} - 1$, $a_{yp^{2q}+m} \neq g$. In particular,

$$\lim_{y \to \infty} \frac{1}{p^{2q}} \# \{a_s = g | yp^{2q} \leq s < (u+1)p^{2q} - 1\} = 0.$$

which contradicts the fact that $g$ has a non-zero natural density proved by Proposition 2.

So we have proved that the sequence $(a_n)_{n \in \mathbb{N}}$ must have finitely many prime factors. However, Corollary 2 of [Hu17] proves that, in this case, the sequence $(a_n)_{n \in \mathbb{N}}$ can have at most one prime $k$ such that $a_k \neq 1$ or 0.

## 5 Classification of CMAS

In this section we will prove that a CMAS is either strongly aperiodic or a Dirichlet-like sequence.

**Definition** A sequence $(a_n)_{n \in \mathbb{N}}$ is said to be aperiodic if and only if for any couple of integers $(s, r)$, we have

$$\lim_{N \to \infty} \frac{\sum_{i=0}^{N} a_{si+r}}{N} = 0.$$

**Definition** Let $\mathcal{M}$ be the set of completely multiplicative functions, let $D : \mathcal{M} \times \mathcal{M} \times \mathbb{N} \to [0, \infty]$ be given by:

$$D(f, g, N)^2 = \sum_{p \in \mathcal{P} \cap [N]} \frac{1 - Re(f(p)g(p))}{p}$$

and $M : \mathcal{M} \times \mathbb{N} \to [0, \infty)$ be given by:

$$M(f, N) = \min_{|t| \leq N} D(f, n^t, N)^2$$

A sequence $(a_n)_{n \in \mathbb{N}}$ is said to be strongly aperiodic if and only if $M(f \chi, N) \to \infty$ as $N \to \infty$ for every Dirichlet character $\chi$. 

7
**Definition** A sequence \((a_n)_{n \in \mathbb{N}}\) is said to be (trivial) Dirichlet-like if and only if there exists a (trivial) Dirichlet character \(X(n)_{n \in \mathbb{N}}\) such that there exists at most one prime number \(p\) satisfying \(a_p \neq X(p)\).

**Proposition 7** Let \((a_n)_{n \in \mathbb{N}}\) be a CMAS, then either there exists a Dirichlet character \((X(n))_{n \in \mathbb{N}}\) such that the sequence \((a_n X(n))_{n \in \mathbb{N}}\) is a trivial Dirichlet-like character, or it is strongly aperiodic.

**Proof** Firstly, it is easy to check that, there is an integer \(r\) such that \(a_p\) is \(r\)-th root of unity for all but finitely many primes \(p\) (see Lemma 1 [SP11]). If \((a_n)_{n \in \mathbb{N}}\) is not strongly aperiodic, then because of Proposition 6.1 in [Fra18], there exists a Dirichlet character \((X(n))_{n \in \mathbb{N}}\) such that

\[
\lim_{N \to \infty} D(a, X, N) < \infty (*).
\]

However, the sequence \((a_nX(n))_{n \in \mathbb{N}}\) is also CMAS and satisfies condition \(C\), the last fact is from (*) . Because of Proposition 6, \((a_nX(n))_{n \in \mathbb{N}}\) is a trivial Dirichlet-like character.

**Proposition 8** Let \((a_n)_{n \in \mathbb{N}}\) be a CMAS and \(X_t(n)_{n \in \mathbb{N}}\) a Dirichlet character \((mod t)\). If the sequence \((a_n X_t(n))_{n \in \mathbb{N}}\) is the trivial Dirichlet-like character \((mod t)\) then \((a_n)_{n \in \mathbb{N}}\) is either a Dirichlet character \((mod t)\), or a Dirichlet-like character \(a_n = \epsilon v_p(n) X(n)^{\frac{n}{p^m p_i}}\), where \(p\) is a prime divisor of \(t\) and \(\epsilon\) is a root of unity.

**Proof** Let \((a_n)_{n \in \mathbb{N}}\) be a CMAS satisfying the above hypothesis, then all possibilities for such \((a_n)_{n \in \mathbb{N}}\) are the sequences of the form:

\[
a_n = \prod_{i=1}^{m} \epsilon_i^{v_{p_i}(n)} X\left(\frac{n}{\prod_{i=1}^{m} p_i^{v_{p_i}(n)}}\right),
\]

for each \(n\), where \(\epsilon_i\) are all non zero complex numbers and \(p_i\) are all prime factors of \(t\).

Let us consider the Dirichlet sequence \(f(s)\) associated with the sequence \((a_n)_{n \in \mathbb{N}}\), which can be written

\[
f(s) = L(s, X_t) \prod^{m}_{i=1} \frac{1 - \frac{n}{p_i^{m_i}}}{1 - \frac{n}{p_i}}.
\]

So all the poles of \(f(s)\) can be found on

\[
s = \frac{\log a_{p_i} + 2im\pi}{\log p_i},
\]

for all \(i\) such that \(1 \leq i \leq m\) and \(n \in \mathbb{N}\).

But on the other hand, if \((a_n)_{n \in \mathbb{N}}\) is a \(k\)-automatic sequence for some integer \(k\), then the poles should be located at points

\[
s = \frac{\log \lambda}{\log k} + \frac{2im\pi}{\log k} - l + 1,
\]

where \(\lambda\) is any eigenvalue of a certain matrix defined from the sequence \((\chi_n)_{n \in \mathbb{N}}\), and \(m \in \mathbb{Z}, l \in \mathbb{N}\), and \(\log\) is a branch of the complex logarithm [AFP00]. By comparing the two sets of possible location of poles for the same function, we can see that there is at most one \(a_{p_i} \neq 0\).
6 Conclusion

In this section we conclude this article by proving that strongly aperiodic CMAS does not exist. To do so, we give a definition of the block complexity of sequences.

**Definition** Let \((a_n)_{n \in \mathbb{N}}\) be a sequence, the the block complexity of \((a_n)_{n \in \mathbb{N}}\) is a sequence, which will be denoted by \((p(k))_{k \in \mathbb{N}}\), such that \(p(k)\) is the number of subwords of length \(k\) that occur (as consecutive values) in \((a_n)_{n \in \mathbb{N}}\).

**Proposition 9** If \((a_n)_{n \in \mathbb{N}}\) is a CMAS then it is not strongly aperiodic.

**Proof** From Theorem 2 in ([FH19]) and the following remark, the block complexity of sequence \((a_n)_{n \in \mathbb{N}}\) should satisfy the property that \(\lim_{n \to \infty} \frac{p(n)}{n} = \infty\), which contradicts to the fact that the block complexity of an automatic sequence is bounded by a linear function [Cob72]. So the non-existence of strongly aperiodic CMAS.

**Theorem 1** Let \((a_n)_{n \in \mathbb{N}}\) be a CMAS, then it can be written in the form:

- either there is at most one prime \(p\) such that \(a_p \neq 0\) and \(a_q = 0\) for all other primes \(q\);

- or \(a_n = e^{\nu(n)} X\left(\frac{n}{p^{\nu(n)}}\right)\), where \((X(n))_{n \in \mathbb{N}}\) is a Dirichlet character.

7 Acknowledgement

In recent papers, we found some results in the literature on similar topics, which have applications to the classification of CMAS. In [LM18], the authors prove that all continuous observables in a substitutional dynamical system \((X_\theta, S)\) are orthogonal to any bounded, aperiodic, multiplicative function, where \(\theta\) represents a primitive uniform substitution and \(S\) is the shift operator. As an application, multiplicative and automatic sequences produced by primitive automata are Weyl rationally almost periodic. REMARKING THAT a sequence \((b_n)_{n \in \mathbb{N}}\) is called Weyl rationally almost periodic if it can be approximated by periodic sequences in same alphabet in the pseudo-metric

\[
d_W(a, b) = \lim_{N \to \infty} \sup_{l \geq 1} \frac{1}{N} \left| \{ l \leq n < l + N : a(n) \neq b(n) \} \right|.
\]

This result is stronger than Proposition 10 in this article by restricting the automatic sequence to the primitive case, however, this result could probably be generated to non-primitive case.

In [KM17], the authors consider general multiplicative functions with the condition \(\liminf_{N \to \infty} |b_{n+1} - b_n| > 0\). They prove that if \((b_n)_{n \in \mathbb{N}}\) is a completely multiplicative sequence, than most primes, at a fixed power, gives the same values as a Dirichlet character.

References


