DIFFERENTIAL K-THEORY AND LOCALIZATION FORMULA FOR η-INVARIANTS

Bo Liu, Xiaonan Ma

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Abstract. In this paper we obtain a localization formula in differential K-theory for S^1-actions. We establish a localization formula for equivariant η-invariants by combining this result with our extension of Goette's result on the comparison of two types of equivariant η-invariants. An important step in our approach is to construct a pre-λ-ring structure in differential K-theory.

Contents

0. Introduction

The famous Atiyah-Singer index theorem [7] states that for an elliptic differential operator on a compact manifold, the analytical index (related to the dimension of the space of solutions) is equal to the topological index, computed in terms of characteristic classes. We can view the index as a primitive spectral invariant of an elliptic operator, whereas global spectral invariants such as the η-invariant of Atiyah-Patodi-Singer and the analytic torsion

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of Ray-Singer as the secondary spectral invariants of an elliptic operator. In [6, Proposition 2.10], Atiyah and Segal established a localization formula for the equivariant index using topological K-theory, which computes the equivariant index via the contribution of the fixed point set of the group action. Thus it is natural to ask whether the localization property holds for these secondary spectral invariants. Note that they are neither computable from local data, nor topological invariants as the index.

Note that the Ray-Singer holomorphic analytic torsion [52] (and its families version, Bismut-Köhler torsion form [21]) is the analytic counterpart of the direct image in Arakelov geometry [56]. Bismut-Lebeau's embedding formula [22] for the analytic torsion and Bismut's family extension [15] are the essential analytic ingredients of the arithmetic Riemann-Roch-Grothendieck theorem [34, 36].

Köhler-Roessler established in their proof of the equivariant arithmetic Riemann-Roch theorem, a Lefschetz type fixed point formula [39, Theorem 4.4] in the equivariant arithmetic K-theory. In the arithmetic context, their result gives a relation between the equivariant holomorphic torsion and the contribution of the fixed point set corresponding to the n-th roots of unity. In [40, Lemma 2.3], they discussed in detail this problem and made a conjecture for projective complex manifolds [40, Conjecture, p82]. Köhler-Roessler [40] did not use the comparison formula of Bismut-Goette [19], but used instead their equivariant arithmetic Riemann-Roch theorem. For more applications of the equivariant arithmetic Riemann-Roch theorem, cf. Maillot-Roessler [49] and later works.

Atiyah-Patodi-Singer [4] developed an index theory for the Dirac operator on compact manifolds with boundary. Their index formula involves a contribution of the boundary, called η-invariant. Formally, it is equal to the number of positive eigenvalues of the Dirac operator minus the number of its negative eigenvalues. Cheeger-Simons [27] gave a formula in $\mathbb{R}/\mathbb{Q}$ for the η-invariant by using their differential characters, cf. also the work of Zhang [61].

The η-invariant (and its families version, Bismut-Cheeger η-form [17]) is the analytic counterpart of the direct image in differential K-theory (see e.g., [11, 25, 32]). In particular, the embedding formula of Bismut-Zhang [24, Theorem 2.2] for the η-invariants plays an important role in the proof of Freed-Lott's index theorem in differential K-theory [32, Theorem 7.35]. Various extensions of Bismut-Zhang's embedding formula have been recently established by B. Liu [43] and later work.

In this paper we will establish a localization formula in differential K-theory. Our result is formally similar to [39, Theorem 4.4], but we employ here totally different arguments. For S^1-actions we get a pointwise identification between the equivariant η-invariant and the fixed point set contribution to the η-invariant, modulo a rational function with integral coefficients. The definition of the fixed point set contribution is actually an important part of the localization formula. By combining this identification with our recent extension [45] of Goette's comparison formula for two kinds of equivariant η-invariants [37] we finally conclude our main result: the difference of the equivariant η-invariant and its fixed point set contribution, as a function on the complement of a finite subset of S^1, is the restriction of a rational function on S^1 with integral coefficients. It seems that our result is the first geometric application of differential K-theory.

Let us recall first the Atiyah-Segal localization formula for the equivariant index. Let Y be an S^1-equivariant compact Spin^c manifold. It induces an S^1-equivariant complex line bundle L such that $\omega_2(TY) = c_1(L) \bmod (2)$, where ω_2 is the second Stiefel-Whitney class and c_1 is the first Chern class [41, Appendix D]. Let E be an S^1-equivariant complex vector bundle over Y. Let $D^Y \otimes E$ be the spin^c Dirac operator on $\mathcal{S}(TY, L) \otimes E$, where $\mathcal{S}(TY, L)$ is the spinor associated with this spin^c structure (cf. (1.13)).

For any complex vector bundle F over a manifold X, we use the notation

$$\mathrm{Sym}_t(F) = 1 + \sum_{k>0} \mathrm{Sym}^k(F) t^k, \quad \lambda_t(F) = 1 + \sum_{k>0} \Lambda^k(F) t^k \tag{0.1}$$

for the symmetric and exterior powers of $K^0(X)[[t]]$ respectively and denote by $\mathrm{Sym}(F) := \mathrm{Sym}_1(F)$. Here in (0.1), 1 is understood as the trivial complex line bundle over X in $K^0(X)$, the K-group of X.

Let Y^{S^1} be the fixed point set of the circle action on Y, then each connected component $Y^{S^1}_\alpha$, $\alpha \in \mathfrak{B}$, of Y^{S^1}, is a compact manifold. Let N_α be the normal bundle of $Y^{S^1}_\alpha$ in Y and we can choose a complex structure on N_α through the circle action. For any $\alpha \in \mathfrak{B}$, $Y^{S^1}_\alpha$ also has an equivariant spin^c structure with associated equivariant line bundle $L_\alpha = L|_{Y^{S^1}_\alpha} \otimes (\det N_\alpha)^{-1}$ (see (1.33) for instance).

Let $K^0_{S^1}(Y^{S^1}_\alpha)_{I(g)}$ be the localization of the equivariant K-group $K^0_{S^1}(Y^{S^1}_\alpha)$ at the prime ideal $I(g)$, which consists of all characters of S^1 vanishing at g.

Assume temporarily that Y is even dimensional. Then the spinor is naturally $\mathbb{Z}_2$-graded: $\mathcal{S}(TY, L) = \mathcal{S}^+(TY, L) \oplus \mathcal{S}^-(TY, L)$. Let $D^Y_\pm \otimes E$ be the restrictions of $D^Y \otimes E$ to the space $\mathscr{C}^\infty(Y, \mathcal{S}^\pm(TY, L) \otimes E)$ of smooth sections of $\mathcal{S}^\pm(TY, L) \otimes E$ on Y. Then the kernels $\mathrm{Ker}(D^Y_\pm \otimes E)$ of $D^Y_\pm \otimes E$ are finite dimensional S^1-complex vector spaces. Let

$$\mathrm{Ind}_g(D^Y \otimes E) = \mathrm{Tr}\,|_{\mathrm{Ker}(D^Y_+ \otimes E)}[g] - \mathrm{Tr}\,|_{\mathrm{Ker}(D^Y_- \otimes E)}[g] \tag{0.2}$$

be the equivariant index of $D^Y \otimes E$ corresponding to $g \in S^1$.

For $g \in S^1$ fixed, let χ be a character of S^1 such that $\chi(g) \neq 0$. For $w = (F^+ - F^-)/\chi \in K^0_{S^1}(Y^{S^1}_\alpha)_{I(g)}$ we define the equivariant index of $D^{Y^{S^1}_\alpha} \otimes w$ by

$$\mathrm{Ind}_g(D^{Y^{S^1}_\alpha} \otimes w) := \chi(g)^{-1} \Big(\mathrm{Ind}_g(D^{Y^{S^1}_\alpha} \otimes F^+) - \mathrm{Ind}_g(D^{Y^{S^1}_\alpha} \otimes F^-) \Big). \tag{0.3}$$

This does not depend on the choices of $F^+, F^- \in K^0_{S^1}(Y^{S^1}_\alpha)$ and χ. Here $D^{Y^{S^1}_\alpha} \otimes F^\pm$ are spin^c Dirac operators on $Y^{S^1}_\alpha$ defined in a similar way as $D^Y_\pm \otimes E$.

Theorem 0.1. [6, Lemma 2.7 and Proposition 2.10] *If Y^{S^1} is the fixed point set of $g \in S^1$, then there exists an inverse $\lambda_{-1}(N^*_\alpha)^{-1}$ in $K^0_{S^1}(Y^{S^1}_\alpha)_{I(g)}$. Moreover,*

$$\mathrm{Ind}_g(D^Y \otimes E) = \sum_\alpha \mathrm{Ind}_g \left(D^{Y^{S^1}_\alpha} \otimes \lambda_{-1}(N^*_\alpha)^{-1} \otimes E|_{Y^{S^1}_\alpha} \right). \tag{0.4}$$

For simplicity, we fix a complex structure on N_α such that the weights of the S^1-action on N_α are all positive. Then by [47, (1.15) and (1.17)], we can reformulate Theorem 0.1 in the following way:

$$\mathrm{Ind}_g(D^Y \otimes E) = \sum_\alpha \mathrm{Ind}_g \left(D^{Y^{S^1}_\alpha} \otimes \mathrm{Sym}(N^*_\alpha) \otimes E|_{Y^{S^1}_\alpha} \right) \quad \text{as distributions on } S^1. \tag{0.5}$$

Notice that $\mathrm{Ind}(D^Y\otimes E)=\mathrm{Ker}(D^Y_+\otimes E)-\mathrm{Ker}(D^Y_-\otimes E)$ is a finite dimensional virtual representation of S^1, thus is an element of the representation ring $R(S^1)$ of S^1. For each α,

$$\mathrm{Ind}\left(D^{Y^{S^1}_\alpha}\otimes \mathrm{Sym}(N^*_\alpha)\otimes E|_{Y^{S^1}_\alpha}\right)\in R[S^1] \tag{0.6}$$

is a formal representation of S^1, i.e., each component of weight k of (0.6), denoted by

$$\mathrm{Ind}\left(D^{Y^{S^1}_\alpha}\otimes \mathrm{Sym}(N^*_\alpha)\otimes E|_{Y^{S^1}_\alpha}\right)_k$$

is a finite dimensional virtual vector space. As a consequence of (0.5), we have for any $|k|\gg 1$,

$$\sum_\alpha \mathrm{Ind}\left(D^{Y^{S^1}_\alpha}\otimes \mathrm{Sym}(N^*_\alpha)\otimes E|_{Y^{S^1}_\alpha}\right)_k=0. \tag{0.7}$$

From now on we assume that Y is odd dimensional. Let g^{TY} be an S^1-invariant Riemannian metric on TY, and ∇^{TY} be the Levi-Civita connection on (Y,g^{TY}). Let h^L and h^E be S^1-invariant metrics. Let ∇^L and ∇^E be S^1-invariant Hermitian connections on (L,h^L) and (E,h^E). Put

$$\underline{TY}=(TY,g^{TY},\nabla^{TY}),\quad \underline{L}=(L,h^L,\nabla^L),\quad \underline{E}=(E,h^E,\nabla^E). \tag{0.8}$$

We call them equivariant geometric triples.

For $g\in S^1$ let $\bar\eta_g(\underline{TY},\underline{L},\underline{E})$ be the associated equivariant APS reduced η-invariant (cf. Definition 1.2).

In the rest of this paper we always assume that $Y^{S^1}\neq\emptyset$ except in Section 3.5.

Let $\underline{L_\alpha}$, $\underline{N^*_\alpha}$, $\underline{\mathrm{Sym}(N^*_\alpha)}$ and $\underline{\lambda_{-1}(N^*_\alpha)}$ be the induced geometric triples on $Y^{S^1}_\alpha$. In view of (0.5), it is natural to ask whether we can define $\bar\eta_g\left(\underline{TY^{S^1}_\alpha},\underline{L_\alpha},\underline{\mathrm{Sym}(N^*_\alpha)}\otimes\underline{E}|_{Y^{S^1}_\alpha}\right)$ as a distribution on S^1 for each α and how to compute the difference

$$\bar\eta_g(\underline{TY},\underline{L},\underline{E})-\sum_\alpha\bar\eta_g\left(\underline{TY^{S^1}_\alpha},\underline{L_\alpha},\underline{\mathrm{Sym}(N^*_\alpha)}\otimes\underline{E}|_{Y^{S^1}_\alpha}\right) \tag{0.9}$$

as a distribution on S^1 by using geometric data on Y.

In this paper we give a realization of $\underline{\lambda_{-1}(N^*_\alpha)}^{-1}$ in the localization of equivariant differential K-theory, such that

$$\sum_\alpha\bar\eta_g\left(\underline{TY^{S^1}_\alpha},\underline{L_\alpha},\underline{\lambda_{-1}(N^*_\alpha)}^{-1}\otimes\underline{E}|_{Y^{S^1}_\alpha}\right) \tag{0.10}$$

is well-defined, and then we identify it to $\bar\eta_g(\underline{TY},\underline{L},\underline{E})$ up to a rational function on S^1 with integral coefficients. The remaining challenging problem is to compute precisely this rational function on S^1 in a geometric way.

For $g\in S^1$ let $\widehat{K}^0_g(Y)$ be the g-equivariant differential K-group in Definition 2.14, which is the Grothendieck group of equivalence classes $[\underline{E},\phi]$ (see (2.87) for the equivalence relation) of cycles $(\underline{E},\phi)$, where $\underline{E}$ is an equivariant geometric triple and $\phi\in\Omega^{\mathrm{odd}}(Y^g,\mathbb{C})/d\Omega^{\mathrm{even}}(Y^g,\mathbb{C})$, the space of odd degree complex valued differential forms on the fixed point set Y^g of g, modulo exact forms. Let $\widehat{K}^0_g(Y)_{I(g)}$ be its localization at the prime ideal $I(g)$. Then as explained in (2.93), an element of $\widehat{K}^0_g(Y)_{I(g)}$ can be written as $([\underline{E},\phi]-[\underline{E'},\phi'])/\chi$, where χ is a character of S^1 such that $\chi(g)\neq 0$.

With respect to the S^1-action, we have the decomposition of complex vector bundles $N_\alpha = \bigoplus_{v>0} N_{\alpha,v}$ such that $g \in S^1$ acts on $N_{\alpha,v}$ by multiplication by g^v.

Using the pre-λ-ring structure of the differential K-theory constructed in Theorem 2.6, we obtain the differential K-theory version of the first part of Theorem 0.1:

Theorem 0.2 (See Theorem 2.17)**.** *There exists a finite subset $A \subset S^1$ (cf. Proposition 1.1), such that for $g \in S^1 \backslash A$, $\left[\underline{\lambda_{-1}(N_\alpha^*)}, 0\right]$ is invertible in $\widehat{K}_g^0(Y_\alpha^{S^1})_{I(g)}$ and there exists $\mathcal{N}_0 > 0$, which does not depend on $g \in S^1\backslash A$, such that for any $\mathcal{N} \in \mathbb{N}, \mathcal{N} > \mathcal{N}_0$, we have*

$$\left[\underline{\lambda_{-1}(N_\alpha^*)}, 0\right]^{-1} = \left[\underline{\lambda_{-1}(N_\alpha^*)_{\mathcal{N}}^{-1}}, 0\right] \in \widehat{K}_g^0(Y_\alpha^{S^1})_{I(g)}. \tag{0.11}$$

Here $\underline{\lambda_{-1}(N_\alpha^*)_{\mathcal{N}}^{-1}}$ is defined by truncation up to degree $\mathcal{N} > \mathcal{N}_0$ in the formal expansion of $\underline{\lambda_{-1}(N_\alpha^*)}^{-1}$ given by the γ-filtration (see the precise definition in (2.63)*,* (2.65)*,* (2.66) *and* (2.100)*).*

Remark 0.3. By (2.100), $\underline{\lambda_{-1}(N_\alpha^*)_{\mathcal{N}}^{-1}}$ is a sum of virtual vector bundles on $Y_\alpha^{S^1}$ with coefficients in

$$F(x)/\prod_{v: N_{\alpha,v}\neq 0} (x^v - 1)^{\mathrm{rk}\, N_{\alpha,v} + \mathcal{N}} \in R(S^1)_{I(g)} \tag{0.12}$$

with $F(x) \in \mathbb{Z}[x]$, where $\mathbb{Z}[x]$ means the ring of polynomials in x with integral coefficients and $\mathrm{rk}\, N_\bullet$ is the rank of the complex vector bundle $N_\bullet$.

For $g \in S^1$ set

$$\mathbb{Q}_g := \{P(g)/Q(g) \in \mathbb{C} : P, Q \in \mathbb{Z}[x], Q(g) \neq 0\} \subset \mathbb{C}. \tag{0.13}$$

Let $\iota : Y^{S^1} \to Y$ be the canonical embedding. Let $\hat{\iota}^* : \widehat{K}_g^0(Y)_{I(g)} \to \widehat{K}_g^0(Y^{S^1})_{I(g)}$ be the induced homomorphism.

Theorem 0.4. *For $g \in S^1$, the direct image map $\widehat{f_{Y!}} : \widehat{K}_g^0(Y)_{I(g)} \to \mathbb{C}/\mathbb{Q}_g$,*

$$[\underline{E}, \phi]/\chi \mapsto \chi(g)^{-1}\left(-\int_{Y^g} \mathrm{Td}_g(\nabla^{TY}, \nabla^L) \wedge \phi + \bar{\eta}_g(\underline{TY}, \underline{L}, \underline{E})\right) \tag{0.14}$$

is well-defined.

For any $g \in S^1\backslash A$, $\hat{\iota}^*$ is an isomorphism and the following diagram commutes

$$\begin{tikzcd} \widehat{K}_g^0(Y^{S^1})_{I(g)} \arrow[dr, "\widehat{f_{Y^{S^1}!}}"'] & & \widehat{K}_g^0(Y)_{I(g)} \arrow[ll, "\left[\underline{\lambda_{-1}(N^*)}, 0\right]^{-1} \cup \hat{\iota}^*"'] \arrow[dl, "\widehat{f_{Y!}}"] \\ & \mathbb{C}/\mathbb{Q}_g, & \end{tikzcd} \tag{0.15}$$

where the product $\cup$ is defined in (2.88)*. In particular, taking into account* (0.14)*, we have for any $\mathcal{N} \in \mathbb{N}$ with $\mathcal{N} > \mathcal{N}_0$, where $\mathcal{N}_0$ is as in Theorem 0.2,*

$$\bar{\eta}_g\left(\underline{TY}, \underline{L}, \underline{E}\right) - \sum_\alpha \bar{\eta}_g\left(\underline{TY_\alpha^{S^1}}, \underline{L_\alpha}, \underline{\lambda_{-1}(N_\alpha^*)_{\mathcal{N}}^{-1}} \otimes \underline{E}|_{Y_\alpha^{S^1}}\right) \in \mathbb{Q}_g. \tag{0.16}$$

The final main result of our paper is as follows:

Theorem 0.5. *Let $A \subset S^1$ and $\mathcal{N}_0 \in \mathbb{N}$ be as in Theorem 0.2. Then for any $\mathcal{N} \in \mathbb{N}$, $\mathcal{N} > \mathcal{N}_0$, for any equivariant geometric triple $\underline{E}$ on Y, the function on $g \in S^1 \backslash A$,*

$$\bar{\eta}_g\left(\underline{TY}, \underline{L}, \underline{E}\right) - \sum_{\alpha} \bar{\eta}_g\left(\underline{TY^{S^1}_{\alpha}}, \underline{L_{\alpha}}, \underline{\lambda_{-1}(N^*_{\alpha})^{-1}_{\mathcal{N}}} \otimes \underline{E}|_{Y^{S^1}_{\alpha}}\right) \tag{0.17}$$

is the restriction on $S^1 \backslash A$ of a rational function on S^1 with integral coefficients that does not have poles on $S^1 \backslash A$.

In the last part of this paper (see Section 3.5) we discuss the case when $Y^{S^1} = \emptyset$.

Theorem 0.6 (See Theorem 3.11). *If $Y^{S^1} = \emptyset$ and $A = \{g \in S^1 : Y^g \neq \emptyset\}$, then $\bar{\eta}_g(\underline{TY}, \underline{L}, \underline{E})$ regarded as a function on $S^1 \backslash A$, is the restriction of a rational function on S^1 with integral coefficients and without poles in $S^1 \backslash A$.*

Let us explain why we do not work directly with all elements $g \in S^1 \backslash A$ in Theorems 0.2 and 0.4. Note on one hand that the equivalence relation defining $\widehat{K}^0_g(Y)$ (cf. (2.87) in Definition 2.14) really depends on $g \in S^1$ even for $g \in S^1 \backslash A$. Thus we can neither define the differential K-group $\widehat{K}^0_{S^1}(Y)$, nor localize uniformly on $g \in S^1 \backslash A$. On the other hand, even in the classical situation from Theorem 0.1, we only localize at each element. To get Theorem 0.5, a certain uniform version of Theorem 0.4 on $g \in S^1 \backslash A$, we need to use Theorems 1.9 and 1.10, i.e., our extension of Goette's result, which is roughly saying that the equivariant η-invariant is a meromorphic function in $g \in S^1$ with possible poles in A and whose singularity is locally computable on Y^g.

We give at the end of the introduction a proof of Theorem 0.6 and a formal computation for (0.9) in a special case.

We suppose that there exists an oriented even dimensional S^1-equivariant Spin^c Riemannian compact manifold X with boundary $Y = \partial X$ and associated S^1-equivariant Hermitian line bundle $\underline{\mathcal{L}} = (\mathcal{L}, h^{\mathcal{L}}, \nabla^{\mathcal{L}})$, and an S^1-equivariant Hermitian vector bundle $(\mathcal{E}, h^{\mathcal{E}})$ with S^1-invariant Hermitian connection $\nabla^{\mathcal{E}}$ such that (X, g^{TX}), $\underline{\mathcal{E}}$ and $\underline{\mathcal{L}}$ are of product structure near the boundary and

$$\mathcal{S}(TX, \mathcal{L}) = \mathcal{S}^+(TX, \mathcal{L}) \oplus \mathcal{S}^-(TX, \mathcal{L}), \quad \mathcal{S}^+(TX, \mathcal{L})|_Y = \mathcal{S}(TY, L). \tag{0.18}$$

Then the index $\mathrm{Ind}_{\mathrm{APS}}(D^X \otimes \mathcal{E})$ of the Dirac operator $D^X \otimes \mathcal{E}$ with respect to the APS boundary condition, is a finite dimensional virtual S^1-representation. By [29, Theorem 1.2], we have for any $g \in S^1 \backslash A_1$,

$$\mathrm{Ind}_{\mathrm{APS},g}(D^X \otimes \mathcal{E}) = \int_{X^{S^1}} \mathrm{Td}_g(\underline{TX}, \underline{\mathcal{L}})\, \mathrm{ch}_g(\underline{\mathcal{E}}) - \bar{\eta}_g(\underline{TY}, \underline{L}, \underline{E}), \tag{0.19}$$

where $A_1 = \{h \in S^1 : X^{S^1} \neq X^h\}$.

If $Y^{S^1} = \emptyset$, then X^{S^1} is a manifold without boundary. Thus (0.19) implies that $\bar{\eta}_g(\underline{TY}, \underline{L}, \underline{E})$ is the restriction to $S^1 \backslash A_1$ of a rational function on S^1 with integral coefficients.

Assume now $Y^{S^1} \neq \emptyset$. We denote by $\mathrm{Ind}_{\mathrm{APS}}(D^X \otimes \mathcal{E}, k)$ the multiplicity of weight k part of S^1-representation in $\mathrm{Ind}_{\mathrm{APS}}(D^X \otimes \mathcal{E})$. Then

$$\mathrm{Ind}_{\mathrm{APS},g}(D^X \otimes \mathcal{E}) = \sum_k \mathrm{Ind}_{\mathrm{APS}}(D^X \otimes \mathcal{E}, k) \cdot g^k, \quad \text{for any } g \in S^1. \tag{0.20}$$

Introduce the notation (cf. (1.18) and (3.40))

$$R(q)=q^{-\frac{1}{2}\sum_v v\,\mathrm{rk}\,N_v^X+\frac{1}{2}l}\bigotimes_{v>0}\mathrm{Sym}_{q^{-v}}(\overline{N_v^X})\otimes\Big(\sum_v E_v q^v\Big) = \sum_k R_k q^k\in K(X^{S^1})[[q,q^{-1}]], \tag{0.21}$$

where we denote by $\oplus_{v>0}N_v^X$ the normal bundle of X^{S^1} in X as in (1.5). Now in view of (0.5), we apply the usual APS-index theorem [4] for the operator $D^{X^{S^1}}\otimes R_k$ taking into account that X^{S^1} is a manifold with boundary $Y^{S^1}=\partial X^{S^1}$, and we obtain

$$\mathrm{Ind}_{\mathrm{APS}}(D^{X^{S^1}}\otimes R_k)=\int_{X^{S^1}}\mathrm{Td}(\underline{TX^{S^1}},\underline{\mathcal{L}'})\,\mathrm{ch}(\underline{R_k})-\sum_\alpha\bar{\eta}(\underline{TY_\alpha^{S^1}},\underline{L_\alpha},\underline{R_k}), \tag{0.22}$$

where $\underline{\mathcal{L}'}$ is the associated line bundle over X^{S^1} defined as in (1.33). By the general analytic localization technique in index theory by Bismut-Lebeau [22], we can expect that the arguments in [28, Theorem 1.2], [47, §1.2] extend to the APS-index case, i.e., we can expect that

$$\mathrm{Ind}_{\mathrm{APS}}(D^X\otimes\mathcal{E},k)=\mathrm{Ind}_{\mathrm{APS}}(D^{X^{S^1}}\otimes R_k)+\mathrm{sf}(Y,k),\quad\text{for any }k\in\mathbb{Z}, \tag{0.23}$$

where $\mathrm{sf}(Y,k)\in\mathbb{Z}$ is the spectral flow of a family of deformed $D^Y\otimes E$ operators via the vector field generated by the S^1-action. As in the case of manifolds without boundary we write *formally*

$$\sum_{k\in\mathbb{Z}}\mathrm{Td}(\underline{TX^{S^1}},\underline{\mathcal{L}'})\,\mathrm{ch}(\underline{R_k})\cdot g^k=\mathrm{Td}_g(\underline{TX},\underline{\mathcal{L}})\,\mathrm{ch}_g(\underline{\mathcal{E}}). \tag{0.24}$$

Thus, we have at least formally for any $g\in S^1\backslash A_1$,

$$\bar{\eta}_g(\underline{TY},\underline{L},\underline{E})=\sum_{k\in\mathbb{Z}}\left(\sum_\alpha\bar{\eta}(\underline{TY_\alpha^{S^1}},\underline{L_\alpha},\underline{R_k})-\mathrm{sf}(Y,k)\right)\cdot g^k. \tag{0.25}$$

This heuristic discussion hints to the possibility of computing (0.9) geometrically.

However, the authors are not aware of any result stating that for S^1-equivariant geometric triples $(\underline{TY},\underline{L},\underline{E})$ as above there exist $k>0$ and S^1-equivariant X, $\underline{\mathcal{L}}$, $\underline{\mathcal{E}}$ such that ∂X consists of k properly oriented copies of Y and by restriction to ∂X we get $\underline{TY},\underline{L},\underline{E}$. Another difficulty is how to make a proper sense of the right-hand side of (0.25). This is why our intrinsic formulation of Theorem 0.5 does not rely on the existence of such an X. Also, it shows the usefulness of the γ-filtration that we introduce in differential K-theory.

Finally, it is natural to ask whether there is a similar localization formula (0.17) for the real analytic torsion [23, 51]. However, unlike the case of the holomorphic torsion and the η-invariant, a suitable K-theory where the real analytic torsion is the analytic ingredient of a Riemann-Roch type theorem (cf. [20]) is still lacking.

The main result of this paper is announced in [44].

This paper is organized as follows. In Section 1, we introduce the main object of our paper, the equivariant η-invariant, and we review some of its analytic properties, which we will use in this paper, such as the variation formula, the embedding formula and the comparison of equivariant η-invariants. In Section 2, we prove that the differential K-ring is a pre-λ-ring and construct the inverse of $[\underline{\lambda_{-1}(N_\alpha^*)}, 0]$ in $\widehat{K}_g^0(Y_\alpha^{S^1})_{I(g)}$ explicitly. In Section 3, we prove Theorems 0.4 and 0.5 and study the case $Y^{S^1} = \emptyset$. We also compute in detail the equivariant η-invariant in the case $Y = S^1$.

Notation: For any vector space V and $B \in \mathrm{End}(V)$, we denote by $\mathrm{Tr}[B]$ the trace of B on V. We denote by $\dim_{\mathbb{R}}$ or $\dim_{\mathbb{C}}$ the real or complex dimension of a vector space, and we skip the subscript if it is clear from the context. For a complex vector bundle E, we will denote by $\mathrm{rk}\, E$ its rank as a complex vector bundle, and $E^{\mathbb{R}}$ the underlying real vector bundle.

For $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, we denote by $\Omega^\bullet(X, \mathbb{K})$ the space of smooth $\mathbb{K}$-valued differential forms on a manifold X, and its subspaces of even/odd degree forms by $\Omega^{\mathrm{even/odd}}(X, \mathbb{K})$. Let d be the exterior differential, then the image of d is the space of exact forms, $\mathrm{Im}\, d$.

Let $R(S^1)$ be the representation ring of the circle group S^1. For any finite dimensional virtual S^1-representation $V = M - M' \in R(S^1)$ and $h \in S^1$, its character

$$\chi_V(h) = \mathrm{Tr}\,|_M[h] - \mathrm{Tr}\,|_{M'}[h] \in \mathbb{Z}[h, h^{-1}], \tag{0.26}$$

a polynomial in h and h^{-1} with integral coefficients. Conversely, for any $f \in \mathbb{Z}[h, h^{-1}]$, there exists a finite dimensional virtual S^1-representation $V_f \in R(S^1)$ such that $f = \chi_{V_f}$ on S^1. So in this paper, we will not distinguish the finite dimensional virtual S^1-representation and $f \in \mathbb{Z}[h, h^{-1}]$ as an element of $R(S^1)$.

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1. Equivariant η-invariants

The heat kernel approach to the Atiyah-Singer index theorem, introduced by Mckean-Singer, Gilkey, Atiyah-Bott-Patodi, ..., establishes the local index theorem for Dirac operators. It finds immediately many applications, in particular the discovery of the Atiyah-Patodi-Singer index theorem for manifolds with boundary and of the η-invariant as the boundary contribution in this index formula. Bismut, along with his various collaborators, has made groundbreaking contributions in this direction, in particular by developing various ideas and techniques to study the global spectral invariants such as the η-invariant and the analytic torsion.

In this section we review some facts about the equivariant η-invariants. These results have largely been influenced both philosophically and technically by the analytic localization technique in index theory developed by Bismut-Lebeau. The variation formula which computes the difference of the equivariant η-invariants associated with different metrics and connections, is a direct consequence of Donnelly's equivariant APS index theorem for manifolds with boundary. It is used in Theorem 3.5 to show that the direct image in g-equivariant differential K-theory is well-defined. The embedding formula guarantees that the direct image in g-equivariant differential K-theory is compatible with the embedding. Note that the embedding of the fixed point set into the total manifold appears naturally in our problem (0.9).

To conclude Theorem 0.5 from Theorem 0.4, we need to understand the analyticity of the equivariant η-invariant as a function of $g \in S^1$. In the same way as fixed-point formulas have two equivariant versions, the Lefschetz fixed-point formula and Kirillov-like formulas of Berline-Vergne, also equivariant η-invariants have two versions. In Theorems 1.9, 1.10, we show that the difference of these two equivariant η-invariants is given by an explicit local formula, involving natural Chern-Simons currents. Moreover, the Kirillov-like equivariant η-invariant is analytic near $0 \in \mathrm{Lie}(S^1)$.

This section is organized as follows. In Section 1.1, we study the equivariant decomposition of TY, in particular, we define the finite subset $A \subset S^1$ in Theorem 0.2. In Section 1.2, we define the equivariant η-invariant. In Section 1.3, we first introduce some characteristic classes and Chern-Simons classes which appear in various situations in the whole paper, then we recall the variation formula. In Section 1.4, we review the geometric construction of the direct image for an embedding in topological K-theory, in particular the natural metrics and connections on the direct image constructed by Bismut-Zhang. Finally, we explain the embedding formula. In Section 1.5, we compare the equivariant η-invariant with the equivariant infinitesimal η-invariant.

1.1. **Circle action.** Let Y be a smooth compact manifold with a smooth circle action. For $g \in S^1$, set

$$\begin{aligned} Y^g &= \{y \in Y : gy = y\}, \\ Y^{S^1} &= \{y \in Y : hy = y \text{ for any } h \in S^1\}. \end{aligned} \tag{1.1}$$

Then Y^g is the fixed point set of g-action on Y and Y^{S^1} is the fixed point set of the circle action on Y with connected components $\{Y_\alpha^{S^1}\}_{\alpha\in\mathfrak{B}}$. Since Y is compact, the index set $\mathfrak{B}$ is a finite set. Certainly, for any $g \in S^1$, $Y^{S^1} \subseteq Y^g$.

If $g \in S^1$ is a generator of S^1, that is, $g = e^{2\pi i t}$ with $t \in \mathbb{R}$ irrational, then Y^{S^1} is the fixed point set Y^g of g. We have the decomposition of real vector bundles over $Y_\alpha^{S^1}$

$$TY|_{Y_\alpha^{S^1}} = TY_\alpha^{S^1} \oplus \bigoplus_{v\neq 0} N_{\alpha,v}^{\mathbb{R}}, \tag{1.2}$$

where $N_{\alpha,v}^{\mathbb{R}}$ is the underlying real vector bundle of a complex vector bundle $N_{\alpha,v}$ over $Y_\alpha^{S^1}$ such that g acts on $N_{\alpha,v}$ by multiplication by g^v. Let N be the normal bundle of Y^{S^1} in Y. Then (1.2) induces the canonical identification $N_\alpha := N|_{Y_\alpha^{S^1}} = \oplus_{v\neq 0} N_{\alpha,v}^{\mathbb{R}}$. We will regard N_α as a complex vector bundle. The complex conjugation provides a $\mathbb{C}$-anti-linear isomorphism between the complex vector bundles $N_{\alpha,v}$ and $\overline{N}_{\alpha,-v}$. Since we can choose either $N_{\alpha,v}$ or $\overline{N}_{\alpha,v}$ as the complex vector bundle for $N_{\alpha,v}^{\mathbb{R}}$, in what follows, we may and we will assume that

$$TY|_{Y_\alpha^{S^1}} = TY_\alpha^{S^1} \oplus \bigoplus_{v>0} N_{\alpha,v}^{\mathbb{R}}, \quad N_\alpha = \bigoplus_{v>0} N_{\alpha,v}. \tag{1.3}$$

Since the dimension of TY is finite, there are only finitely many v such that $\operatorname{rk} N_{\alpha,v} \neq 0$. Set

$$q = \max\{v : \text{there exists } \alpha \in \mathfrak{B} \text{ such that } \operatorname{rk} N_{\alpha,v} \neq 0\}. \tag{1.4}$$

Then we have the splittings

$$TY|_{Y^{S^1}_\alpha} = TY^{S^1}_\alpha \oplus \bigoplus_{v=1}^q N^{\mathbb{R}}_{\alpha,v}, \quad N_\alpha = \bigoplus_{v=1}^q N_{\alpha,v}. \tag{1.5}$$

Proposition 1.1. *The set*

$$A = \{g \in S^1 : Y^{S^1} \neq Y^g\} \tag{1.6}$$

is finite.

Proof. Let g^{TY} be an S^1-invariant metric on Y. Then there exists $\varepsilon > 0$ such that the exponential map

$$(y, Z) \in \mathcal{U}_\varepsilon = \{(y, Z) \in N_y : y \in Y^{S^1}, |Z| < \varepsilon\} \to \exp_y(Z) \tag{1.7}$$

is a diffeomorphism from $\mathcal{U}_\varepsilon$ into the tubular neighborhood $\mathcal{V}_\varepsilon$ of Y^{S^1} in Y. Then for any $g_t = e^{2\pi i t} \in S^1$, $(y, Z) \in \mathcal{U}_\varepsilon$ with $Z = (z_{\alpha,v})_{v=1}^q \in N_{\alpha,y}$, we have

$$g_t(y, Z) = (y, (e^{2\pi i t v} z_{\alpha,v})_{v=1}^q). \tag{1.8}$$

Thus for $t \in (0, 1)$,

$$\mathcal{U}_\varepsilon^{g_t} = Y^{S^1}, \quad \text{if } t \notin \mathcal{F}_q := \left\{ \frac{k}{p} : k, p \text{ coprime}, 0 \leq k < p \leq q \right\}. \tag{1.9}$$

Now S^1 acts locally freely on $Y_1 := Y \backslash \mathcal{U}_{\varepsilon/2}$. Thus for any $x \in Y_1$, the stabilizer S^1_x of x is a finite group $\{e^{2\pi i j/k} : 0 \leq j < k\} \simeq \mathbb{Z}_k$ for certain $k \in \mathbb{N}^*$. Set $N_x = T_x Y_1 / T_x(S^1 \cdot x)$, which is a linear representation of $S^1_x \simeq \mathbb{Z}_k$. By the slice theorem, there exists an S^1-equivariant diffeomorphism from an equivariant open neighborhood U_x of the zero section in $S^1 \times_{S^1_x} N_x$ to an open neighborhood of $S^1 \cdot x$ in Y_1, which sends the zero section S^1/S^1_x onto the orbit $S^1 \cdot x$ by the map $g \in S^1 \to g \cdot x$. Now in this neighborhood U_x, for any $g \in S^1 \backslash S^1_x$, $U_x^g = \emptyset$. By using the compactness of Y_1 there is a finite set $A_0 \subset S^1$ such that for any $g \in S^1 \backslash A_0$, $Y_1^g = \emptyset$. Combining with (1.9), we know A in (1.6) is finite.

The proof of Proposition 1.1 is completed. □

1.2. **Equivariant η-invariants.** In the remainder of this section, let Y be an odd dimensional compact oriented manifold with circle action. Then the circle action automatically preserves the orientation of Y. Let g^{TY} be an S^1-invariant metric on TY.

Assume that Y has an S^1-equivariant spin^c structure, i.e., the S^1-action on Y lifts naturally to the associated Spin^c principal bundle, in particular, it induces an S^1-equivariant complex line bundle L such that $\omega_2(TY) = c_1(L) \bmod (2)$, where ω_2 is the second Stiefel-Whitney class and c_1 is the first Chern class [41, Appendix D]. Let $\mathcal{S}(TY, L)$ be the fundamental complex spinor bundle associated with this spin^c structure. It is an S^1-equivariant complex vector bundle in a canonical way, and formally

$$\mathcal{S}(TY, L) = \mathcal{S}_0(TY) \otimes L^{1/2}, \tag{1.10}$$

where $\mathcal{S}_0(TY)$ is the fundamental spinor bundle for the (possibly non-existent) spin structure on TY and $L^{1/2}$ is the (possibly non-existent) square root of L.

Let E be an S^1-equivariant complex vector bundle over Y. Then S^1 acts on $\mathscr{C}^\infty(Y, \mathcal{S}(TY,L)\otimes E)$ by

$$(g.s)(x) = g(s(g^{-1}x)), \quad \text{for } g \in S^1. \tag{1.11}$$

Let h^L and h^E be S^1-invariant Hermitian metrics on L and E respectively. Let $h^{\mathcal{S}_Y}$ be the S^1-invariant Hermitian metric on $\mathcal{S}(TY,L)$ induced by g^{TY} and h^L.

Let ∇^{TY} be the Levi-Civita connection on (TY, g^{TY}). Let ∇^L and ∇^E be S^1-invariant Hermitian connections on (L,h^L) and (E,h^E) respectively. Let $\nabla^{\mathcal{S}_Y}$ be the connection on $\mathcal{S}(TY,L)$ induced by ∇^{TY} and ∇^L [41, Appendix D]. Let $\nabla^{\mathcal{S}_Y\otimes E}$ be the connection on $\mathcal{S}(TY,L)\otimes E$ induced by $\nabla^{\mathcal{S}_Y}$ and ∇^E,

$$\nabla^{\mathcal{S}_Y\otimes E} = \nabla^{\mathcal{S}_Y}\otimes 1 + 1\otimes \nabla^E. \tag{1.12}$$

Let $\{e_j\}$ be a locally orthonormal frame of (TY, g^{TY}). We denote by $c(\cdot)$ the Clifford action of TY on $\mathcal{S}(TY,L)$. Let $D^Y\otimes E$ be the spin^c Dirac operator on Y defined by

$$D^Y\otimes E = \sum_j c(e_j)\nabla^{\mathcal{S}_Y\otimes E}_{e_j} : \mathscr{C}^\infty(Y, \mathcal{S}(TY,L)\otimes E) \to \mathscr{C}^\infty(Y, \mathcal{S}(TY,L)\otimes E). \tag{1.13}$$

Then $D^Y\otimes E$ is an S^1-equivariant first order self-adjoint elliptic differential operator on Y and its kernel $\mathrm{Ker}(D^Y\otimes E)$ is a finite dimensional S^1-complex vector space.

Let $\exp(-u(D^Y\otimes E)^2)$, $u>0$, be the heat semi-group of $(D^Y\otimes E)^2$.

We denote by $\underline{TY}, \underline{L}, \underline{E}$ the equivariant geometric data

$$\underline{TY} = (TY, g^{TY}, \nabla^{TY}), \quad \underline{L} = (L, h^L, \nabla^L), \quad \underline{E} = (E, h^E, \nabla^E). \tag{1.14}$$

We also call $\underline{TY}, \underline{L}, \underline{E}$ equivariant geometric triples over Y.

Definition 1.2. For $g\in S^1$, the equivariant (reduced) η-invariant associated with $\underline{TY}, \underline{L}, \underline{E}$ is defined by

$$\begin{aligned}\bar{\eta}_g(\underline{TY}, \underline{L}, \underline{E}) = \int_0^{+\infty} \mathrm{Tr}\left[g(D^Y\otimes E)\exp(-u(D^Y\otimes E)^2)\right]\frac{du}{2\sqrt{\pi u}} \\ + \frac{1}{2}\mathrm{Tr}|_{\mathrm{Ker}(D^Y\otimes E)}[g] \in \mathbb{C}.\end{aligned} \tag{1.15}$$

The convergence of the integral at $u=0$ in (1.15) is nontrivial (see e.g., [18, Theorem 2.6], [29], [59, Theorem 2.1]).

1.3. **Variation formula.** Since g^{TY} is S^1-invariant, the fixed point set Y^g is an odd dimensional totally geodesic submanifold of Y for any $g\in S^1$. Let $N^{\mathbb{R}}$ be the normal bundle of Y^g in Y, which we identify to the orthogonal complement of TY^g in TY.

Since the S^1-action preserves the spin^c structure, we see that Y^g is canonically oriented (cf. [9, Proposition 6.14], [46, Lemma 4.1]).

Assume first $g = e^{2\pi i t}\in S^1\backslash A$ (cf. (1.6)), then $Y^g = Y^{S^1}$ and by (1.3), we have the decomposition of real vector bundles over Y^g,

$$TY|_{Y^g} = TY^g \oplus \bigoplus_{v>0} N_v^{\mathbb{R}}, \quad N^{\mathbb{R}} = \bigoplus_{v>0} N_v^{\mathbb{R}}, \tag{1.16}$$

where $N_v^{\mathbb{R}}$ is the underlying real vector bundle of the complex vector bundle N_v such that $h \in S^1$ acts by multiplication by h^v. We will fix the orientation on $Y^g = Y^{S^1}$ induced by the canonical orientation on N_v as complex vector bundles and the orientation on TY.

Since g^{TY} is S^1-invariant, the decomposition (1.16) is orthogonal and the restriction of ∇^{TY} on Y^g is split under the decomposition. Let g^{TY^g}, $g^{N^{\mathbb{R}}}$ and $g^{N_v^{\mathbb{R}}}$ be the metrics induced by g^{TY} on TY^g, $N^{\mathbb{R}}$ and $N_v^{\mathbb{R}}$. Let ∇^{TY^g}, $\nabla^{N^{\mathbb{R}}}$ and $\nabla^{N_v^{\mathbb{R}}}$ be the corresponding induced connections on TY^g, $N^{\mathbb{R}}$ and $N_v^{\mathbb{R}}$, with curvatures R^{TY^g}, $R^{N^{\mathbb{R}}}$ and $R^{N_v^{\mathbb{R}}}$. Then under the decomposition (1.16),

$$g^{TY} = g^{TY^g} \oplus g^{N^{\mathbb{R}}}, \quad g^{N^{\mathbb{R}}} = \bigoplus_v g^{N_v^{\mathbb{R}}}, \quad \nabla^{TY}|_{Y^g} = \nabla^{TY^g} \oplus \bigoplus_v \nabla^{N_v^{\mathbb{R}}}. \tag{1.17}$$

Similar to (1.16), we have the orthogonal decomposition of complex vector bundles with connections on Y^g

$$E|_{Y^g} = \bigoplus_v E_v, \quad \nabla^E|_{Y^g} = \bigoplus_v \nabla^{E_v}. \tag{1.18}$$

Here $h \in S^1$ acts by multiplication by h^v on E_v and the connection ∇^{E_v} on E_v is induced by ∇^E. Let R^E, R^{E_v} be the curvatures of ∇^E, ∇^{E_v}.

Definition 1.3. For $g = e^{2\pi it} \in S^1 \backslash A$, set

$$\begin{aligned}
\widehat{\mathrm{A}}(TY^g, \nabla^{TY^g}) : &= \det{}^{1/2}\left(\frac{\frac{i}{4\pi}R^{TY^g}}{\sinh\left(\frac{i}{4\pi}R^{TY^g}\right)}\right),\\
\widehat{\mathrm{A}}_g(N^{\mathbb{R}}, \nabla^{N^{\mathbb{R}}}) : &= \left(i^{\frac{1}{2}\dim N^{\mathbb{R}}}\det{}^{1/2}|_{N^{\mathbb{R}}}\left(1 - g\cdot\exp\left(\frac{i}{2\pi}R^{N^{\mathbb{R}}}\right)\right)\right)^{-1}\\
&= \prod_{v>0}\left(i^{\frac{1}{2}\dim N_v^{\mathbb{R}}}\det{}^{1/2}|_{N_v^{\mathbb{R}}}\left(1 - g\cdot\exp\left(\frac{i}{2\pi}R^{N_v^{\mathbb{R}}}\right)\right)\right)^{-1},\\
\widehat{\mathrm{A}}_g(TY, \nabla^{TY}) : &= \widehat{\mathrm{A}}(TY^g, \nabla^{TY^g}) \cdot \widehat{\mathrm{A}}_g(N^{\mathbb{R}}, \nabla^{N^{\mathbb{R}}}) \in \Omega^\bullet(Y^g, \mathbb{C}),\\
\mathrm{ch}_g(\underline{E}) : &= \mathrm{Tr}\left[g\exp\left(\frac{i}{2\pi}R^E\right)\right]\\
&= \sum_v \mathrm{Tr}\left[\exp\left(\frac{i}{2\pi}R^{E_v} + 2i\pi vt\right)\right] \in \Omega^\bullet(Y^g, \mathbb{C}).
\end{aligned} \tag{1.19}$$

The sign convention in $\widehat{\mathrm{A}}_g(N^{\mathbb{R}}, \nabla^{N^{\mathbb{R}}})$ is that the degree 0 part is given by $\prod_{v>0}(2i\sin(\pi vt))^{-\frac{1}{2}\dim N_v^{\mathbb{R}}}$.

The forms in (1.19) are closed forms on Y^g and their cohomology class does not depend on the S^1-invariant metrics g^{TY}, h^E and connection ∇^E. We denote by $\widehat{\mathrm{A}}(TY^g)$, $\widehat{\mathrm{A}}_g(TY)$, $\mathrm{ch}_g(E)$ their cohomology classes, two of which appear in the equivariant index theorem [9, Chapter 6].

Comparing with (1.19), if $h \in S^1$ acts on $L|_{Y^{S^1}}$ by multiplication by h^l, we write

$$\mathrm{ch}_g(\underline{L^{1/2}}) := \exp\left(\frac{i}{4\pi}R^L|_{Y^g} + i\pi lt\right) \in \Omega^\bullet(Y^g, \mathbb{C}). \tag{1.20}$$

We denote by

$$\mathrm{Td}_g(\nabla^{TY}, \nabla^L) := \widehat{\mathrm{A}}_g(TY, \nabla^{TY})\,\mathrm{ch}_g(\underline{L^{1/2}}). \tag{1.21}$$

Note that the natural lift of $g = e^{2\pi i t}$ on $\mathcal{S}(TY, L)$ over Y^{S^1} is given by

$$\prod_v \prod_j \left(\cos(\pi v t) + \sin(\pi v t)\, c\left(e^v_{2j-1}\right) c\left(e^v_{2j}\right)\right) \cdot e^{i\pi l t}, \tag{1.22}$$

where $\left\{e^v_j\right\}_j$ is an oriented orthonormal frame of N_v. This explains the sign convention in (1.21).

If $g \in A$, we have the decomposition of real vector bundles over Y^g,

$$TY|_{Y^g} = TY^g \oplus \bigoplus_{0<\theta\leq\pi} N(\theta), \tag{1.23}$$

where $N(\theta)$ is a real vector bundle over Y^g which has a complex structure such that g acts by multiplication by $e^{i\theta}$ if $\theta \neq \pi$; or an even dimensional oriented real vector bundle on which g acts by multiplication by -1 if $\theta = \pi$. We fix the orientation on Y^g induced by the orientations on Y and on $N(\theta)$. Then we can still define $\mathrm{ch}_g(\underline{E})$ as in (1.19). Let g act on L_{Y^g} by multiplication by $e^{i\theta'}$, $0 \leq \theta' < 2\pi$. Now the lift of the g-action on $\mathcal{S}(TY, L)$ over Y^g is given by

$$\epsilon \prod_{0<\theta\leq\pi} \prod_j \left(\cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right) c\left(e^\theta_{2j-1}\right) c\left(e^\theta_{2j}\right)\right) \cdot e^{i\theta'/2} \quad \text{and} \quad \epsilon = 1 \text{ or } -1, \tag{1.24}$$

where $\left\{e^\theta_j\right\}_j$ is an oriented orthonormal frame of $N(\theta)$. From (1.24), the sign convention of $\mathrm{Td}_g(\nabla^{TY}, \nabla^L)$ in (1.21) is that its degree 0 part is given by $\epsilon \prod_{0<\theta\leq\pi} (2i\sin(\theta/2))^{-\frac{1}{2}\dim N(\theta)} e^{i\theta'/2}$. This situation is only used in Sections 1.3, 1.5 and 3.2.

We explain now the construction of Chern-Simons classes. Let

$$\underline{TY_j} = (TY, g^{TY}_j, \nabla^{TY}_j),\ \underline{L_j} = (L, h^L_j, \nabla^L_j),\ \text{and}\ \underline{E_j} = (E, h^E_j, \nabla^E_j) \quad \text{for } j = 0, 1 \tag{1.25}$$

be equivariant geometric triples over Y as in (1.14).

Let $\pi : (y, s) \in Y \times \mathbb{R} \to y \in Y$ be the obvious projection. Then the S^1-action lifts naturally on $Y \times \mathbb{R}$, by acting only on the factor Y. Let g^{π^*TY}, h^{π^*L} and h^{π^*E} be S^1-invariant metrics on π^*TY, π^*L and π^*E over $Y \times \mathbb{R}$ such that for $j = 0, 1$,

$$g^{\pi^*TY}|_{Y\times\{j\}} = g^{TY}_j, \quad h^{\pi^*L}|_{Y\times\{j\}} = h^L_j, \quad h^{\pi^*E}|_{Y\times\{j\}} = h^E_j. \tag{1.26}$$

Let ∇^{π^*TY}, ∇^{π^*L} and ∇^{π^*E} be S^1-invariant Hermitian connections on (π^*TY, g^{π^*TY}), (π^*L, h^{π^*L}) and (π^*E, h^{π^*E}) such that for $j = 0, 1$,

$$\nabla^{\pi^*TY}|_{Y\times\{j\}} = \nabla^{TY}_j, \quad \nabla^{\pi^*L}|_{Y\times\{j\}} = \nabla^L_j, \quad \nabla^{\pi^*E}|_{Y\times\{j\}} = \nabla^E_j. \tag{1.27}$$

Let $\underline{\pi^*E} = (\pi^*E, h^{\pi^*E}, \nabla^{\pi^*E})$ be the associated geometric triple on $Y \times \mathbb{R}$.

If $\alpha = \alpha_0 + ds \wedge \alpha_1$ with $\alpha_0, \alpha_1 \in \Lambda^\bullet(T^*Y)$, put

$$\{\alpha\}^{ds} := \alpha_1. \tag{1.28}$$

For $g \in S^1$, the equivariant Chern-Simons classes $\widetilde{\mathrm{ch}}_g(\underline{E_0}, \underline{E_1}), \widetilde{\mathrm{Td}}_g(\nabla^{TY}_0, \nabla^L_0, \nabla^{TY}_1, \nabla^L_1) \in \Omega^{\mathrm{odd}}(Y^g, \mathbb{C})/\mathrm{Im}\, d$ are defined by

$$\begin{aligned} &\widetilde{\mathrm{ch}}_g(\underline{E_0}, \underline{E_1}) = \int_0^1 \{\mathrm{ch}_g(\underline{\pi^*E})\}^{ds} ds \in \Omega^{\mathrm{odd}}(Y^g, \mathbb{C})/\mathrm{Im}\, d, \\ &\widetilde{\mathrm{Td}}_g(\nabla^{TY}_0, \nabla^L_0, \nabla^{TY}_1, \nabla^L_1) = \int_0^1 \{\mathrm{Td}_g(\nabla^{\pi^*TY}, \nabla^{\pi^*L})\}^{ds} ds \in \Omega^{\mathrm{odd}}(Y^g, \mathbb{C})/\mathrm{Im}\, d. \end{aligned} \tag{1.29}$$

Moreover, we have

$$
\begin{aligned}
&d\,\widetilde{\mathrm{ch}}_g(\underline{E_0},\underline{E_1})=\mathrm{ch}_g(\underline{E_1})-\mathrm{ch}_g(\underline{E_0}),\\
&d\,\widetilde{\mathrm{Td}}_g(\nabla_0^{TY},\nabla_0^L,\nabla_1^{TY},\nabla_1^L)=\mathrm{Td}_g(\nabla_1^{TY},\nabla_1^L)-\mathrm{Td}_g(\nabla_0^{TY},\nabla_0^L).
\end{aligned}
\tag{1.30}
$$

Note that the Chern-Simons classes depend only on ∇_j^{TY}, ∇_j^L and ∇_j^E for $j=0,1$ (see [48, Theorem B.5.4]).

Let $\bar{\eta}_g(\underline{TY_j},\underline{L_j},\underline{E_j})$ for $j=0,1$ be the equivariant reduced η-invariants associated with $(\underline{TY_j},\ \underline{L_j},\ \underline{E_j})$. The following variation formula is proved in [42, Proposition 2.14] (see also [43, Theorem 2.6]), which extends the usual well-known non-equivariant variation formula for η-invariants (cf. [5, p95] or [18, Theorem 2.11]).

Recall that for a finite dimensional virtual S^1-representation V, we denote its character by χ_V (cf. (0.26)).

Theorem 1.4. *There exists $V\in R(S^1)$ such that for any $g\in S^1$,*

$$
\begin{aligned}
\bar{\eta}_g(\underline{TY_1},\underline{L_1},\underline{E_1})-\bar{\eta}_g(\underline{TY_0},\underline{L_0},\underline{E_0})=&\int_{Y^g}\widetilde{\mathrm{Td}}_g(\nabla_0^{TY},\nabla_0^L,\nabla_1^{TY},\nabla_1^L)\,\mathrm{ch}_g(\underline{E_1})\\
&+\int_{Y^g}\mathrm{Td}_g(\nabla_0^{TY},\nabla_0^L)\widetilde{\mathrm{ch}}_g(\underline{E_0},\underline{E_1})+\chi_V(g).
\end{aligned}
\tag{1.31}
$$

1.4. **Embedding formula for equivariant η-invariants.** Recall that $\{Y_\alpha^{S^1}\}_{\alpha\in\mathfrak{B}}$ is the set of the connected components of Y^{S^1} and N_α is the normal bundle of $Y_\alpha^{S^1}$ in Y. We consider N_α as a complex vector bundle and denote by $N_\alpha^{\mathbb{R}}$ the underlying real vector bundle of N_α. Then $N_\alpha^{\mathbb{R}}\otimes_{\mathbb{R}}\mathbb{C}=N_\alpha\oplus\overline{N}_\alpha$. Let h^N be the Hermitian metric on N_α induced by g^{N_α}.

Let $C(N_\alpha^{\mathbb{R}})$ be the Clifford algebra bundle of $(N_\alpha^{\mathbb{R}},g^{N^{\mathbb{R}}})$. Then $\Lambda(\overline{N}_\alpha^*)$ is a $C(N_\alpha^{\mathbb{R}})$-Clifford module. Namely, if $u\in N_\alpha$, let $u^*\in\overline{N}_\alpha^*$ be the metric dual of u. The Clifford action on $\Lambda(\overline{N}_\alpha^*)$ is defined by

$$
c(u)=\sqrt{2}u^*\wedge,\quad c(\overline{u})=-\sqrt{2}i_{\overline{u}}\quad\text{for any }u\in N_\alpha.
\tag{1.32}
$$

Here $\wedge, i.$ are the exterior and interior products on forms.

Set

$$
L_\alpha=L|_{Y_\alpha^{S^1}}\otimes(\det N_\alpha)^{-1}.
\tag{1.33}
$$

Then $TY_\alpha^{S^1}$ has an equivariant spin^c structure as $\omega_2(TY_\alpha^{S^1})=c_1(L_\alpha)\in H^2_{S^1}(Y_\alpha^{S^1},\mathbb{Z})\bmod (2)$ (cf. [47, (1.47)]). Let $\mathcal{S}(TY_\alpha^{S^1},L_\alpha)$ be the associated fundamental spinor bundle for $TY_\alpha^{S^1}$ such that

$$
\mathcal{S}(TY,L)|_{Y_\alpha^{S^1}}=\mathcal{S}(TY_\alpha^{S^1},L_\alpha)\otimes\Lambda^\bullet(\overline{N}_\alpha^*).
\tag{1.34}
$$

As in (1.10), formally, we have

$$
\begin{aligned}
\mathcal{S}(TY_\alpha^{S^1},L_\alpha)&=\mathcal{S}_0(TY_\alpha^{S^1})\otimes L^{1/2}|_{Y_\alpha^{S^1}}\otimes(\det N_\alpha)^{-1/2},\\
\Lambda^\bullet(\overline{N}_\alpha^*)&=\mathcal{S}_0(N_\alpha^{\mathbb{R}})\otimes(\det N_\alpha)^{1/2}.
\end{aligned}
\tag{1.35}
$$

Let ∇^N be the Hermitian connection on (N_α,h^N) induced by $\nabla^{N^{\mathbb{R}}}$ in (1.17). Note that the equivariant geometric triple $\underline{N_\alpha}=(N_\alpha,h^N,\nabla^N)$ induces equivariant geometric triples

$\underline{\Lambda^{\mathrm{even}}(N_\alpha^*)}$, $\underline{\Lambda^{\mathrm{odd}}(N_\alpha^*)}$ and $\underline{\det N_\alpha}$. Denote by

$$\begin{aligned}\lambda_{-1}(N_\alpha^*) &= \Lambda^{\mathrm{even}}(N_\alpha^*) - \Lambda^{\mathrm{odd}}(N_\alpha^*),\\ \underline{\lambda_{-1}(N_\alpha^*)} &= \underline{\Lambda^{\mathrm{even}}(N_\alpha^*)} - \underline{\Lambda^{\mathrm{odd}}(N_\alpha^*)}.\end{aligned} \tag{1.36}$$

Let $\underline{L_\alpha}$ be the equivariant geometric triple induced from (1.33).

From [14, (6.26)], we have

$$\mathrm{ch}_g\left(\underline{\lambda_{-1}(N_\alpha^*)}\right) = \widehat{\mathrm{A}}_g(N^{\mathbb{R}}, \nabla^{N^{\mathbb{R}}})^{-1} \cdot \mathrm{ch}_g\left(\underline{(\det N_\alpha)^{-1/2}}\right). \tag{1.37}$$

From (1.19)-(1.21), (1.33) and (1.37), on $Y_\alpha^{S^1}$, we have

$$\mathrm{Td}_g(\nabla^{TY}, \nabla^L)\,\mathrm{ch}_g\left(\underline{\lambda_{-1}(N_\alpha^*)}\right) = \widehat{\mathrm{A}}(TY_\alpha^{S^1}, \nabla^{TY_\alpha^{S^1}})\,\mathrm{ch}_g\left(\underline{L_\alpha^{1/2}}\right) = \mathrm{Td}_g\left(\nabla^{TY_\alpha^{S^1}}, \nabla^{L_\alpha}\right). \tag{1.38}$$

We call F a trivial S^1-equivariant vector bundle over Y if there is a finite dimensional S^1-representation M such that $F = Y \times M$ with the S^1-action on F by $g(y,u) = (gy, gu)$.

Let (μ, h^μ) be an S^1-equivariant Hermitian vector bundle over Y^{S^1} with an S^1-invariant Hermitian connection ∇^μ. Let $\iota : Y^{S^1} \to Y$ be the obvious embedding. In the following, we describe the geometric construction of Atiyah-Hirzebruch's direct image $\iota_! \mu \in K^0(Y)$ of μ for the embedding in K-theory [2], [24, §1b]. It will be clear from its construction that it is compatible with the group action.

For any $\delta > 0$ set $\mathcal{U}_{\alpha,\delta} := \{Z \in N_\alpha^{\mathbb{R}} : |Z| < \delta\}$. Then there exists $\varepsilon_0 > 0$ such that the exponential map $(y, Z) \in N_\alpha^{\mathbb{R}} \to \exp_y^Y(Z)$ is a diffeomorphism between $\mathcal{U}_{\alpha,2\varepsilon_0}$ and an open S^1-equivariant tubular neighbourhood of $Y_\alpha^{S^1}$ in Y for any α. Without confusion we will also regard $\mathcal{U}_{\alpha,2\varepsilon_0}$ as this neighbourhood of $Y_\alpha^{S^1}$ in Y via this identification. We choose $\varepsilon_0 > 0$ small enough such that for any $\alpha \neq \beta \in \mathfrak{B}$, $\overline{\mathcal{U}_{\alpha,2\varepsilon_0}} \cap \overline{\mathcal{U}_{\beta,2\varepsilon_0}} = \emptyset$.

Let $\pi_\alpha : N_\alpha \to Y_\alpha^{S^1}$ denote the projection of the normal bundle N_α over $Y_\alpha^{S^1}$. For $Z \in N_\alpha^{\mathbb{R}}$, let $\tilde{c}(Z) \in \mathrm{End}(\Lambda^\bullet(N_\alpha^*))$ be the transpose of the canonical Clifford action $c(Z)$ on $\Lambda^\bullet(\overline{N}_\alpha^*)$ in (1.32). In particular, for $u \in N_\alpha$, let $\bar{u}^* \in N_\alpha^*$ be the metric dual of $\bar{u} \in \overline{N}_\alpha$, then

$$\tilde{c}(u) = \sqrt{2}\, i_u, \quad \tilde{c}(\bar{u}) = -\sqrt{2}\, \bar{u}^* \wedge . \tag{1.39}$$

Let $\pi_\alpha^*(\Lambda^\bullet(N_\alpha^*))$ be the pull back bundle of $\Lambda^\bullet(N_\alpha^*)$ over N_α. For any $Z \in N_\alpha^{\mathbb{R}}$ with $Z \neq 0$, let $\tilde{c}(Z) : \pi_\alpha^*(\Lambda^{\mathrm{even/odd}}(N_\alpha^*))|_Z \to \pi_\alpha^*(\Lambda^{\mathrm{odd/even}}(N_\alpha^*))|_Z$ denote the corresponding pull back isomorphism at Z.

As S^1 acts trivially on $Y_\alpha^{S^1}$, we can just apply [41, Chapter I, Corollary 9.9] for each weight part to see that (cf. also [55, Proposition 2.4]) there exists an S^1-equivariant vector bundle F_α over $Y_\alpha^{S^1}$ such that $(\Lambda^{\mathrm{even}}(N_\alpha^*) \otimes \mu_\alpha) \oplus F_\alpha$ with $\mu_\alpha = \mu|_{Y_\alpha^{S^1}}$, is a trivializable S^1-equivariant complex vector bundle over $Y_\alpha^{S^1}$ with the S^1-equivariant trivialization map $\varphi_\alpha : Y_\alpha^{S^1} \times M_\alpha \to \Lambda^{\mathrm{even}}(N_\alpha^*) \otimes \mu_\alpha \oplus F_\alpha$. Then

$$\sqrt{-1}\tilde{c}(Z) \oplus \pi_\alpha^* \mathrm{Id}_{F_\alpha} : \pi_\alpha^*(\Lambda^{\mathrm{even}}(N_\alpha^*) \otimes \mu_\alpha \oplus F_\alpha)|_Z \to \pi_\alpha^*(\Lambda^{\mathrm{odd}}(N_\alpha^*) \otimes \mu_\alpha \oplus F_\alpha)|_Z \tag{1.40}$$

induces an S^1-equivariant isomorphism between two S^1-equivariant vector bundles over $\overline{\mathcal{U}_{\alpha,2\varepsilon_0}} \backslash Y_\alpha^{S^1}$. By adding trivial S^1-equivariant vector bundles for the part F_α, we can also assume that $M_\alpha = M_\beta = M$ for any $\alpha \neq \beta \in \mathfrak{B}$. Now the identification $\pi_\alpha^*(\Lambda^{\mathrm{even}}(N_\alpha^*) \otimes \mu_\alpha \oplus F_\alpha)$ (resp. $\pi_\alpha^*(\Lambda^{\mathrm{odd}}(N_\alpha^*) \otimes \mu_\alpha \oplus F_\alpha)$) with $(Y \setminus \cup_\alpha \mathcal{U}_{\alpha,\varepsilon_0}) \times M$ on $\mathcal{U}_{\alpha,2\varepsilon_0} \backslash \mathcal{U}_{\alpha,\varepsilon_0}$ via the map φ_α (resp. $(\sqrt{-1}\tilde{c}(Z) \oplus \pi_\alpha^* \mathrm{Id}_{F_\alpha}) \circ \varphi_\alpha$) defines an S^1-equivariant vector bundle ξ_+ (resp. ξ_-) over Y. Moreover the identity map of the above trivializations of ξ_+ and ξ_- over $Y \setminus \cup_\alpha \mathcal{U}_{\alpha,\varepsilon_0}$

extends smoothly the map (1.40) to a map $v:\xi_+\to\xi_-$. Thus for each $\alpha\in\mathfrak{B}$, there exists an S^1-equivariant vector bundle F_α over $Y_\alpha^{S^1}$ such that

$$\begin{aligned}\xi_\pm|_{\mathcal{U}_{\alpha,2\varepsilon_0}}&=\pi_\alpha^*(\Lambda^{\mathrm{even/odd}}(N_\alpha^*)\otimes\mu_\alpha\oplus F_\alpha)|_{\mathcal{U}_{\alpha,2\varepsilon_0}},\\ v|_{\mathcal{U}_{\alpha,2\varepsilon_0}}&=\sqrt{-1}\tilde{c}(Z)\oplus\pi_\alpha^*\mathrm{Id}_{F_\alpha},\end{aligned}\tag{1.41}$$

and the restriction of v to $Y\backslash\cup_\alpha\mathcal{U}_{\alpha,2\varepsilon_0}$ is invertible. Then the direct image of μ by ι is given by

$$\iota_!\mu=\xi_+-\xi_-\in K^0(Y).\tag{1.42}$$

By a partition of unity argument, we get a metric $h^\xi=h^{\xi_+}\oplus h^{\xi_-}$ over Y such that

$$h^{\xi_\pm}|_{\mathcal{U}_{\alpha,\varepsilon_0}}=\pi_\alpha^*\left(h^{\Lambda^{\mathrm{even/odd}}(N_\alpha^*)\otimes\mu_\alpha}\oplus h^{F_\alpha}\right)\Big|_{\mathcal{U}_{\alpha,\varepsilon_0}},\tag{1.43}$$

where $h^{\Lambda^{\mathrm{even/odd}}(N_\alpha^*)\otimes\mu_\alpha}$ is the S^1-invariant Hermitian metric on $\Lambda^{\mathrm{even/odd}}(N_\alpha^*)\otimes\mu_\alpha$ induced by h^N and h^μ. Again by a partition of unity argument, we get an S^1-invariant $\mathbb{Z}_2$-graded Hermitian connection $\nabla^\xi=\nabla^{\xi_+}\oplus\nabla^{\xi_-}$ on $\xi=\xi_+\oplus\xi_-$ over Y such that

$$\nabla^{\xi_\pm}|_{\mathcal{U}_{\alpha,\varepsilon_0}}=\pi_\alpha^*\left(\nabla^{\Lambda^{\mathrm{even/odd}}(N_\alpha^*)\otimes\mu_\alpha}\oplus\nabla^{F_\alpha}\right)\Big|_{\mathcal{U}_{\alpha,\varepsilon_0}},\tag{1.44}$$

where $\nabla^{\Lambda^{\mathrm{even/odd}}(N_\alpha^*)\otimes\mu_\alpha}$ is the Hermitian connection on $\Lambda^{\mathrm{even/odd}}(N_\alpha^*)\otimes\mu_\alpha$ induced by ∇^N and ∇^μ. We denote now

$$\underline{\xi_\pm}=(\xi_\pm,h^{\xi_\pm},\nabla^{\xi_\pm})\text{ over }Y.\tag{1.45}$$

An equivariant extension of Bismut-Zhang embedding formula [24, Theorem 2.2] (cf. also [28, Theorem 4.1] or [30, Theorem 2.1]) for η-invariants was proved in [43, Corollaries 3.8, 3.9]. The following result follows from [43, Corollaries 3.8, 3.9] applied for $G=S^1$, $g\in S^1\backslash A$.

Theorem 1.5. *There exists $V'\in R(S^1)$, such that for any $g\in S^1\backslash A$,*

$$\overline{\eta}_g(\underline{TY},\underline{L},\underline{\xi_+})-\overline{\eta}_g(\underline{TY},\underline{L},\underline{\xi_-})=\sum_\alpha\overline{\eta}_g(\underline{TY_\alpha^{S^1}},\underline{L_\alpha},\underline{\mu})+\chi_{V'}(g).\tag{1.46}$$

Remark that in [43, Theorem 3.7] applied in the case when the base space is a point, V' is an equivariant spectral flow of a family of deformed Dirac operators on Y with a pseudodifferential operator perturbation obtained from the corresponding perturbation of the Dirac operator on Y^g. Since for any $g\in S^1\backslash A$, $Y^g=Y^{S^1}$ does not change, thus V' does not depend on $g\in S^1\backslash A$.

Remark 1.6. Note that in the general setting of [43, Theorem 3.7] for the embedding $i:Y\to X$, there is an additional term, the equivariant Bismut-Zhang current. Note that the equivariant Bismut-Zhang current is defined for the normal bundle of Y^g in X^g. In our case, since $(Y^g)^g=Y^g$, this term is zero.

1.5. **Comparison of equivariant η-invariants.** In this subsection, we review the comparison formula for equivariant η-invariants in [45], which is an extension of the result of [37] and the analogue of the comparison formulas for the holomorphic torsions [19] and for the de Rham torsions [20].

For $K \in \mathrm{Lie}(S^1)$, let $K^Y(x) = \frac{\partial}{\partial t}\big|_{t=0} e^{tK}\cdot x$ be the induced vector field on Y, and $\mathcal{L}_K$ be the corresponding Lie derivative given by $\mathcal{L}_K s = \frac{\partial}{\partial t}\big|_{t=0}\left(e^{-tK}.\, s\right)$ for $s \in \mathscr{C}^\infty(Y,E)$ (cf. (1.11)). The associated moment maps are defined by [9, Definition 7.5],

$$\begin{aligned} m^E(K) &:= \nabla^E_{K^Y} - \mathcal{L}_K|_E \in \mathscr{C}^\infty(Y,\mathrm{End}(E)),\\ m^{TY}(K) &:= \nabla^{TY}_{K^Y} - \mathcal{L}_K|_{TY} = \nabla^{TY}_{\cdot} K^Y \in \mathscr{C}^\infty(Y,\mathrm{End}(TY)), \end{aligned} \tag{1.47}$$

where the last equation holds since the Levi-Civita connection ∇^{TY} is torsion free.

Let R^E_K and R^{TY}_K be the equivariant curvatures of E and TY defined in [9, §7.1]:

$$R^E_K = R^E - 2i\pi m^E(K), \quad R^{TY}_K = R^{TY} - 2i\pi m^{TY}(K). \tag{1.48}$$

Observe that $m^{TY}(K)|_{Y^g}$ commutes with the circle action for any $g \in S^1$. Then it preserves the decompositions (1.16) and (1.23). Let $m^{TY^g}(K)$, $m^N(K)$ and $m^{N_v}(K)$ be the restrictions of $m^{TY}(K)|_{Y^g}$ to TY^g, $N^{\mathbb{R}}$ and $N^{\mathbb{R}}_v$. Similarly, $m^E(K)|_{Y^g}$ preserves the decomposition (1.18). We define the corresponding equivariant curvatures $R^{TY^g}_K$ and $R^{N^{\mathbb{R}}}_K$ as in (1.48). The following definition is an analogue of Definition 1.3 and (1.20).

Definition 1.7. For $g = e^{2\pi i t} \in S^1$, $K \in \mathrm{Lie}(S^1)$, $|K|$ small enough, set

$$\begin{aligned} \widehat{\mathrm{A}}_{g,K}(TY,\nabla^{TY}) : &= \det{}^{1/2}\left(\frac{\frac{i}{4\pi}R^{TY^g}_K}{\sinh\left(\frac{i}{4\pi}R^{TY^g}_K\right)}\right)\\ &\qquad \times \left(i^{\frac{1}{2}\dim N^{\mathbb{R}}}\det{}^{1/2}\left(1 - g\cdot\exp\left(\frac{i}{2\pi}R^{N^{\mathbb{R}}}_K\right)\right)\right)^{-1} \in \Omega^\bullet(Y^g,\mathbb{C}),\\ \mathrm{ch}_{g,K}(\underline{E}) : &= \mathrm{Tr}\left[g\exp\left(\frac{i}{2\pi}R^E_K\right)\right] \in \Omega^\bullet(Y^g,\mathbb{C}). \end{aligned} \tag{1.49}$$

Let R^L_K be the corresponding equivariant curvature of L. For $g \in S^1\backslash A$, as in (1.20), we define

$$\mathrm{ch}_{g,K}(\underline{L^{1/2}}) : = \exp\left(\frac{i}{4\pi}R^L_K|_{Y^g} + i\pi l t\right). \tag{1.50}$$

For $g \in A$, as we discussed after (1.23), we replace $i\pi l t$ by $\frac{i}{2}\theta'$ in (1.50). As in (1.21), we denote by

$$\mathrm{Td}_{g,K}(\nabla^{TY},\nabla^L) := \widehat{\mathrm{A}}_{g,K}(TY,\nabla^{TY})\,\mathrm{ch}_{g,K}(\underline{L^{1/2}}). \tag{1.51}$$

Certainly for $K=0$, $\widehat{\mathrm{A}}_{g,K}(\cdot) = \widehat{\mathrm{A}}_g(\cdot)$ and $\mathrm{ch}_{g,K}(\cdot) = \mathrm{ch}_g(\cdot)$.

For $K \in \mathrm{Lie}(S^1)$ set

$$d_K = d - 2i\pi i_{K^Y}. \tag{1.52}$$

Then by [9, Theorem 7.7], $\widehat{\mathrm{A}}_{g,K}(TY,\nabla^{TY})$, $\mathrm{ch}_{g,K}(\underline{E})$ and $\mathrm{ch}_{g,K}(\underline{L^{1/2}})$ are d_K-closed.

For $K \in \mathrm{Lie}(S^1)$ let $\vartheta_K \in T^*Y$ be the 1-form which is dual to K^Y by the metric g^{TY}, i.e.,

$$\vartheta_K(X) = \langle K^Y, X\rangle \quad \text{for } X \in TY. \tag{1.53}$$

For $g\in S^1$, $K\in \mathrm{Lie}(S^1)$, $|K|$ small enough, set (cf. [16, Definition 1.7])

$$\mathcal{M}_{g,K}(\underline{TY},\underline{L},\underline{E})=-\int_0^{\infty}\left\{\int_{Y^g}\frac{\vartheta_K}{2i\pi}\exp\left(\frac{v\,d_K\vartheta_K}{2i\pi}\right)\mathrm{Td}_{g,K}(\nabla^{TY},\nabla^L)\,\mathrm{ch}_{g,K}(\underline{E})\right\}dv. \tag{1.54}$$

Note that if $g\in S^1\setminus A$ we have $Y^g=Y^{S^1}$ from (1.6), thus

$$\vartheta_K=0 \text{ on } Y^g \text{ and } \mathcal{M}_{g,K}(\underline{TY},\underline{L},\underline{E})=0 \text{ for } g\in S^1\setminus A. \tag{1.55}$$

By the argument of [38, Proposition 2.2], $\mathcal{M}_{g,K}(\underline{TY},\underline{L},\underline{E})$ is well-defined for $|K|$ small enough. Moreover, for $K_0\in \mathrm{Lie}(S^1)$, $t\in\mathbb{R}$ and $|t|$ small enough, $\mathcal{M}_{g,tK_0}(\underline{TY},\underline{L},\underline{E})$ is smooth at $t\neq 0$ for $g\in A$ and there exist $c_j(K_0)\in\mathbb{C}$ $(j\in\mathbb{N}^*)$ such that as $t\to 0$, we have

$$\mathcal{M}_{g,tK_0}(\underline{TY},\underline{L},\underline{E})=\sum_{j=1}^{(\dim Y^g+1)/2}c_j(K_0)t^{-j}+\mathcal{O}(t^0). \tag{1.56}$$

In the following definition of the equivariant infinitesimal η-invariant, the operator $\sqrt{t}D^Y\otimes E+\frac{c(K^Y)}{4\sqrt{t}}$ was introduced by Bismut [12] in his heat kernel proof of the Kirillov formula for the equivariant index. As observed by Bismut [13, §1d)] (see also [9, §10.7]), its square plus $\mathcal{L}_{K^Y}$ is the square of the Bismut superconnection for a fibration with compact structure group, by replacing K^Y by the curvature of the fibration, thus $\bar{\eta}_{g,K}$ in (1.57) should be understood as certain universal η-forms of Bismut-Cheeger [17, Definition 4.33].

Definition 1.8. [45, Definition 2.3] For $g\in S^1$, $K\in\mathrm{Lie}(S^1)$ and $|K|$ small enough, the equivariant infinitesimal (reduced) η-invariant is defined by

$$\begin{aligned}\bar{\eta}_{g,K}(\underline{TY},\underline{L},\underline{E})=\int_0^{+\infty}\frac{1}{2\sqrt{\pi t}}\,\mathrm{Tr}\Bigg[g\left(D^Y\otimes E-\frac{c(K^Y)}{4t}\right)\\ \cdot\exp\left(-t\left(D^Y\otimes E+\frac{c(K^Y)}{4t}\right)^2-\mathcal{L}_{K^Y}\right)\Bigg]dt+\frac{1}{2}\,\mathrm{Tr}|_{\mathrm{Ker}(D^Y\otimes E)}[ge^K]\in\mathbb{C}.\end{aligned} \tag{1.57}$$

From (1.15), (1.54) and (1.57), we know that $\bar{\eta}_{g,0}(\cdot)=\bar{\eta}_g(\cdot)$, $\mathcal{M}_{g,0}(\cdot)=0$.

The following two theorems are special cases of [45, Theorems 0.1, 0.2], which extend Goette's result [37, Theorem 0.5] as an equality of formal Laurent series in t at $t=0$ when $g=1$ and K_0^Y does not vanish on Y. Here the equivariant η-forms are just equivariant η-invariants and the compact Lie group is S^1.

Theorem 1.9. *Fix $K_0\in\mathrm{Lie}(S^1)$, $g\in S^1$. There exists $\beta>0$ such that for $t\in\mathbb{R}$ and $|t|<\beta$, the equivariant infinitesimal η-invariant $\bar{\eta}_{g,tK_0}(\underline{TY},\underline{L},\underline{E})$ is well-defined and is an analytic function of t. Furthermore, as a function of t near 0, $t^{(\dim Y^g+1)/2}\mathcal{M}_{g,tK_0}(\underline{TY},\underline{L},\underline{E})$ is real analytic.*

Theorem 1.10. *Fix $0\neq K_0\in\mathrm{Lie}(S^1)$. For any $g\in S^1$, there exists $\beta>0$ such that for $|t|<\beta, t\neq 0$, we have*

$$\bar{\eta}_{g,tK_0}(\underline{TY},\underline{L},\underline{E})=\bar{\eta}_{ge^{tK_0}}(\underline{TY},\underline{L},\underline{E})+\mathcal{M}_{g,tK_0}(\underline{TY},\underline{L},\underline{E}). \tag{1.58}$$

Since $\bar{\eta}_{g,tK_0}(\underline{TY},\underline{L},\underline{E})$ is an analytic function of t, when $t\to 0$, the singularity of $\bar{\eta}_{ge^{tK_0}}(\underline{TY},\underline{L},\underline{E})$ is the same as that of $-\mathcal{M}_{g,tK_0}(\underline{TY},\underline{L},\underline{E})$ in (1.56). Thus from Theorem 1.10, we know $\bar{\eta}_g(\underline{TY},\underline{L},\underline{E})$ as a function of $g\in S^1$, is analytic on $S^1\backslash A$, moreover, at $g\in A$, $\bar{\eta}_{ge^{tK_0}}(\underline{TY},\underline{L},\underline{E})+\mathcal{M}_{g,tK_0}(\underline{TY},\underline{L},\underline{E})$ on $0<t<|\beta|$ can be extended as an analytic function on $|t|<|\beta|$.

2. Differential K-theory

K-theory and the λ-ring structure were first introduced by Grothendieck in 1957. The arithmetic K-theory in Arakelov geometry was introduced by Gillet-Soulé in [35]. It extends Grothendieck's K-theory by adding Hermitian metrics on holomorphic vector bundles and differential forms of type (p,p) modulo $\mathrm{Im}\,\partial + \mathrm{Im}\,\bar{\partial}$. In the same way as the topological K-theory of Atiyah and Hirzebruch is the $\mathscr{C}^\infty$-version of Grothendieck's K-theory, also the differential K-theory introduced by Freed-Hopkins [31] and developed further by Hopkins-Singer, Simons-Sullivan, Bunke-Schick, Freed-Lott, is a $\mathscr{C}^\infty$-version of the arithmetic K-theory. It extends the topological K-theory by adding Hermitian metrics and connections on vector bundles and differential forms modulo exact forms.

Note that the λ-ring structure on the arithmetic K-theory was introduced by Gillet-Soulé [35, Theorem 7.3.4] and exploited in detail by Roessler [53, 54], who studied also the associated γ-filtration.

In this section, we start to exploit the λ-ring structure on the vector space of even degree real closed forms on Y and on its direct sum with the space of odd degree real forms modulo exact forms. Then we study its compatibility with the Chern forms and Chern-Simons classes of geometric triples. With this preparation, we can equip the differential K-theory with a pre-λ-ring structure. An important result is that the associated γ-filtration is locally nilpotent.

We consider the circle action on Y now. Recall that N is the normal bundle of Y^{S^1}, the fixed point set of the circle action, in Y. When we apply the above results to our g-equivariant differential K-theory $\widehat{K}^0_g(Y^{S^1})$, it implies that $\underline{\lambda_{-1}(N^*)} := \sum_{i\geq 0}(-1)^i\underline{\Lambda^i(N^*)}$, the exterior algebra bundle of N^* with corresponding metric and connection, is invertible in $\widehat{K}^0_g(Y^{S^1})_{I(g)}$, the localization of $\widehat{K}^0_g(Y^{S^1})$ at the prime ideal $I(g)$ of $R(S^1)$. This result allows us to define the counterpart of the η-invariant on the fixed point set.

This section is organized as follows. In Section 2.1, we define the pre-λ-ring structure and study some examples. In Section 2.2, we construct the pre-λ-ring structure in differential K-theory. In Section 2.3, we study the locally nilpotent property of the γ-filtration in differential K-theory. In Section 2.4, we define the g-equivariant differential K-theory and explicitly construct the inverse of $\underline{\lambda_{-1}(N^*)}$ at differential K-theory level.

2.1. Pre-λ-ring structure.

Definition 2.1. [10, (1.1)-(1.3)] For a commutative ring R with identity, a pre-λ-ring structure is defined by a countable set of maps $\lambda^n : R \to R$ with $n \in \mathbb{N}$ such that for all $x, y \in R$,

a) $\lambda^0(x) = 1$;

b) $\lambda^1(x) = x$;

c) $\lambda^n(x+y) = \sum_{j=0}^n \lambda^j(x)\lambda^{n-j}(y)$.

If R has a pre-λ-ring structure, we call it a pre-λ-ring.

Remark that in [8, §1] the pre-λ-ring here is called the λ-ring.

If t is an indeterminate, we define for $x \in R$,

$$\lambda_t(x) = \sum_{n\geq 0} \lambda^n(x)t^n. \tag{2.1}$$

Then the relations a), c) show that λ_t is a homomorphism from the additive group of R into the multiplicative group $1+R[[t]]^+$, of formal power series in t with constant term 1, i.e.,

$$\lambda_t(x+y)=\lambda_t(x)\lambda_t(y) \quad \text{for any } x,y\in R. \tag{2.2}$$

Now we study some pre-λ-rings which we will use later.

Let Y be a manifold. Let $Z^{\mathrm{even}}(Y,\mathbb{R})$ be the vector space of even degree real closed forms on Y. We define the Adams operation $\Psi^k : Z^{\mathrm{even}}(Y,\mathbb{R})\to Z^{\mathrm{even}}(Y,\mathbb{R})$ for $k\in\mathbb{N}$ by

$$\Psi^k(x)=k^l x \quad \text{for} \quad x\in Z^{2l}(Y,\mathbb{R}). \tag{2.3}$$

For $x\in Z^{\mathrm{even}}(Y,\mathbb{R})$, we define

$$\lambda_t(x)=\sum_{n\geq 0}\lambda^n(x)t^n := \exp\left(\sum_{k=1}^{\infty}\frac{(-1)^{k-1}\Psi^k(x)t^k}{k}\right). \tag{2.4}$$

From the Taylor expansion of the exponential function, we have $\lambda^0(x)=1$ and $\lambda^1(x)=x$. Since

$$\begin{aligned}\lambda_t(x+y)&=\exp\left(\sum_{k=1}^{\infty}\frac{(-1)^{k-1}\Psi^k(x+y)t^k}{k}\right)\\ &=\exp\left(\sum_{k=1}^{\infty}\frac{(-1)^{k-1}\Psi^k(x)t^k}{k}+\sum_{k=1}^{\infty}\frac{(-1)^{k-1}\Psi^k(y)t^k}{k}\right)=\lambda_t(x)\lambda_t(y),\end{aligned} \tag{2.5}$$

we have

$$\lambda^n(x+y)=\sum_{j=0}^{n}\lambda^j(x)\lambda^{n-j}(y). \tag{2.6}$$

Thus (2.4) gives a pre-λ-ring structure on $Z^{\mathrm{even}}(Y,\mathbb{R})$.

Consider the vector space (comparing with [35, §7.3.1])

$$\Gamma(Y) := Z^{\mathrm{even}}(Y,\mathbb{R})\oplus\left(\Omega^{\mathrm{odd}}(Y,\mathbb{R})/\operatorname{Im} d\right). \tag{2.7}$$

We give degree $l\geq 0$ to $Z^{2l}(Y,\mathbb{R})\oplus\left(\Omega^{2l-1}(Y,\mathbb{R})/\operatorname{Im} d\right)$ with $\Omega^{-1}(\cdot)=\{0\}$. We define a pairing on $\Gamma(Y)$ by the formula

$$(\omega_1,\phi_1)*(\omega_2,\phi_2) := (\omega_1\wedge\omega_2, \omega_1\wedge\phi_2+\phi_1\wedge\omega_2-d\phi_1\wedge\phi_2). \tag{2.8}$$

It is easy to verify that this pairing is commutative and associative. Since $*$ is clearly bilinear and $(1,0)$ is a unit, the pairing $*$ defines a graded associative, commutative and unitary $\mathbb{R}$-algebra structure on $\Gamma(Y)$ (comparing with [35, Theorem 7.3.2]). We define the Adams operation $\Psi^k:\Gamma(Y)\to\Gamma(Y)$ for $k\in\mathbb{N}$ (cf. [35, §7.3.1], [54]) by

$$\Psi^k(\alpha,\beta)=(k^l\alpha,k^l\beta) \quad \text{for} \quad (\alpha,\beta)\in Z^{2l}(Y,\mathbb{R})\oplus(\Omega^{2l-1}(Y,\mathbb{R})/\operatorname{Im} d). \tag{2.9}$$

By using the pairing $*$ to replace the multiplicity in (2.4), similarly as $Z^{\mathrm{even}}(Y,\mathbb{R})$, we obtain a pre-$\lambda$-ring structure on $\Gamma(Y)$.

Let p be the projection from $\Gamma(Y)$ to its component $Z^{\mathrm{even}}(Y,\mathbb{R})$ and $\jmath$ be the following injection:

$$\begin{aligned}p&:\Gamma(Y)\to Z^{\mathrm{even}}(Y,\mathbb{R}), \quad (\omega,\varphi)\mapsto\omega,\\ \jmath&:Z^{\mathrm{even}}(Y,\mathbb{R})\to\Gamma(Y), \quad \omega\mapsto(\omega,0).\end{aligned} \tag{2.10}$$

By (2.8), $p, \jmath$ are homomorphisms of pre-λ rings, in particular,

$$\lambda^k(\omega, 0) = (\lambda^k \omega, 0). \tag{2.11}$$

Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. A polynomial $\varphi : \mathfrak{g} \to \mathbb{C}$ is called a G-invariant polynomial if

$$\varphi(\mathrm{Ad}(g^{-1})A) = \varphi(A), \quad \text{for any } g \in G,\ A \in \mathfrak{g}. \tag{2.12}$$

The set of all G-invariant polynomials is denoted by $\mathbb{C}[\mathfrak{g}]^G$.

Let $U(r)$ be the unitary group with Lie algebra $\mathfrak{u}(r)$. For $A \in \mathfrak{u}(r)$, the characteristic polynomial of $-A$ is

$$\det(\lambda I + A) = \lambda^r + c_1(A)\lambda^{r-1} + \cdots + c_r(A). \tag{2.13}$$

So $c_j \in \mathbb{C}[\mathfrak{u}(r)]^{U(r)}$ for $1 \leq j \leq r$. It is well-known that $\mathbb{C}[\mathfrak{u}(r)]^{U(r)}$ is generated by $c_1, \cdots, c_r$ as a polynomial ring:

$$\mathbb{C}[\mathfrak{u}(r)]^{U(r)} = \mathbb{C}[c_1, \cdots, c_r]. \tag{2.14}$$

Let $T^r = \{(e^{it_1}, \cdots, e^{it_r}) : t_1, \cdots, t_r \in \mathbb{R}\}$ be a maximal torus of $U(r)$ with Lie algebra $\mathfrak{t}^r$. Then

$$\mathbb{C}[\mathfrak{t}^r]^{T^r} = \mathbb{C}[u_1, \cdots, u_r], \tag{2.15}$$

where $u_j(x) = x_j$, for any $x = \sum_{j=1}^r x_j \left.\frac{\partial}{\partial t_j}\right|_{t=0} \in T_e(T^r) = \mathfrak{t}^r$. Let

$$\theta : T^r \to U(r), \quad (e^{it_1}, \cdots, e^{it_r}) \mapsto \mathrm{diag}(e^{it_1}, \cdots, e^{it_r}) \tag{2.16}$$

be the diagonal injection, here $\mathrm{diag}(\cdots)$ is the diagonal matrix. It is well-known that

$$\theta^* : \mathbb{C}[\mathfrak{u}(r)]^{U(r)} \to \mathbb{C}[\mathfrak{t}^r]^{T^r} \tag{2.17}$$

is an injective homomorphism and

$$\theta^*(c_j) = \sigma_j(u_1, \cdots, u_r), \quad \theta^*\left(\mathbb{C}[\mathfrak{u}(r)]^{U(r)}\right) = \mathbb{C}[\sigma_1, \cdots, \sigma_r], \tag{2.18}$$

where σ_j is the j-th elementary symmetric polynomial. We define the Adams operations for any $k \in \mathbb{N}$

$$\begin{aligned} &\Psi^k : \mathbb{C}[\mathfrak{u}(r)]^{U(r)} \to \mathbb{C}[\mathfrak{u}(r)]^{U(r)}, \quad && \Psi^k(c_j) = k^j c_j; \\ &\Psi^k : \mathbb{C}[\mathfrak{t}^r]^{T^r} \to \mathbb{C}[\mathfrak{t}^r]^{T^r}, && \Psi^k(u_j) = k u_j. \end{aligned} \tag{2.19}$$

By constructing λ^n as in (2.4), $\mathbb{C}[\mathfrak{u}(r)]^{U(r)}$ and $\mathbb{C}[\mathfrak{t}^r]^{T^r}$ are equipped now pre-λ-ring structures.

Let E be a complex vector bundle over Y of rank r. Let h^E be a Hermitian metric on E. Let ∇^E be a Hermitian connection on (E, h^E). We also denote by $\underline{E} = (E, h^E, \nabla^E)$ the *geometric triple* for this non-equivariant setting. For $\varphi \in \mathbb{C}[\mathfrak{u}(r)]^{U(r)}$, we define the characteristic form $\varphi(\underline{E})$ by

$$\varphi(\underline{E}) = \psi_Y \varphi\left(-R^E\right) \in \Omega^{\mathrm{even}}(Y, \mathbb{C}), \tag{2.20}$$

where $\psi_Y : \Omega^{\mathrm{even}}(Y, \mathbb{C}) \to \Omega^{\mathrm{even}}(Y, \mathbb{C})$ is defined by

$$\psi_Y \omega = (2i\pi)^{-j} \omega \quad \text{for } \omega \in \Omega^{2j}(Y, \mathbb{C}). \tag{2.21}$$

Then by the Chern-Weil theory (cf. [48, Appendix D]), $\varphi(\underline{E})$ is closed. Moreover, $\varphi(\underline{E}) \in \Omega^{\mathrm{even}}(Y, \mathbb{R})$ if $\varphi \in \mathbb{R}[\mathfrak{u}(r)]^{U(r)}$.

The triple $\underline{E}$ induces a homomorphism of rings

$$f_{\underline{E}} : \mathbb{C}[\mathfrak{u}(r)]^{U(r)} \to Z^{\mathrm{even}}(Y, \mathbb{C}), \quad \varphi \mapsto \varphi(\underline{E}). \tag{2.22}$$

Let $\underline{\pi^*E} = (\pi^*E, h^{\pi^*E}, \nabla^{\pi^*E})$ be the triple defined in Section 1.2 without the group action. As in (1.29), the Chern-Simons class $\widetilde{\varphi}(\underline{E_0}, \underline{E_1}) \in \Omega^{\mathrm{odd}}(Y, \mathbb{C})/\mathrm{Im}\, d$ is defined by (cf. [48, Definition B.5.3])

$$\widetilde{\varphi}(\underline{E_0}, \underline{E_1}) := \int_0^1 \{\varphi(\underline{\pi^*E})\}^{ds} ds \in \Omega^{\mathrm{odd}}(Y, \mathbb{C})/\mathrm{Im}\, d. \tag{2.23}$$

Then by [48, Theorem B.5.4],

$$d\widetilde{\varphi}(\underline{E_0}, \underline{E_1}) = \varphi(\underline{E_1}) - \varphi(\underline{E_0}), \tag{2.24}$$

and the Chern-Simons class depends only on ∇^{E_0} and ∇^{E_1}.

Lemma 2.2. *Let $\underline{E_j} = (E, h_j^E, \nabla_j^E)$ for $j = 0, 1, 2$. Let $\varphi, \varphi' \in \mathbb{C}[\mathfrak{u}(r)]^{U(r)}$. Then we have*

(a) $\widetilde{\varphi}(\underline{E_0}, \underline{E_2}) = \widetilde{\varphi}(\underline{E_0}, \underline{E_1}) + \widetilde{\varphi}(\underline{E_1}, \underline{E_2})$;

(b) $\widetilde{\varphi + \varphi'}(\underline{E_0}, \underline{E_1}) = \widetilde{\varphi}(\underline{E_0}, \underline{E_1}) + \widetilde{\varphi'}(\underline{E_0}, \underline{E_1})$;

(c) $\widetilde{\varphi\varphi'}(\underline{E_0}, \underline{E_1}) = \widetilde{\varphi}(\underline{E_0}, \underline{E_1})\varphi'(\underline{E_1}) + \varphi(\underline{E_0})\widetilde{\varphi'}(\underline{E_0}, \underline{E_1})$;

(d) $(\varphi(\underline{E_1}), \widetilde{\varphi}(\underline{E_0}, \underline{E_1})) * (\varphi'(\underline{E_1}), \widetilde{\varphi'}(\underline{E_0}, \underline{E_1})) = (\varphi\varphi'(\underline{E_1}), \widetilde{\varphi\varphi'}(\underline{E_0}, \underline{E_1}))$.

Proof. By (2.23), (a) and (b) are obvious. Let $\pi^*\underline{E_1}$ be the pull-back of the triple $\underline{E_1}$ on $Y \times \mathbb{R}$. Then $\varphi'(\pi^*\underline{E_1}) = \pi^*\varphi'(\underline{E_1})$. Thus by (2.23),

$$\begin{aligned}
\widetilde{\varphi\varphi'}(\underline{E_0}, \underline{E_1}) - \widetilde{\varphi}(\underline{E_0}, \underline{E_1})\varphi'(\underline{E_1}) &= \int_0^1 \Big\{\varphi(\underline{\pi^*E}) \left(\varphi'(\underline{\pi^*E}) - \varphi'(\pi^*\underline{E_1})\right) \Big\}^{ds} ds \\
&= \int_0^1 \Big\{d^{Y\times\mathbb{R}} \Big[\varphi(\underline{\pi^*E})\widetilde{\varphi'}(\pi^*\underline{E_1}, \underline{\pi^*E})\Big]\Big\}^{ds} ds \\
&= -d^Y \int_0^1 \Big\{\varphi(\underline{\pi^*E})\widetilde{\varphi'}(\pi^*\underline{E_1}, \underline{\pi^*E})\Big\}^{ds} ds + \varphi(\underline{\pi^*E})\widetilde{\varphi'}(\pi^*\underline{E_1}, \underline{\pi^*E})\Big|_{Y\times\{t\}}\Big|_{t=0}^{1}.
\end{aligned} \tag{2.25}$$

From (2.25), we get (c).

We establish (d) now. From (2.8), (2.24) and (c), we have

$$\begin{aligned}
&(\varphi(\underline{E_1}), \widetilde{\varphi}(\underline{E_0}, \underline{E_1})) * (\varphi'(\underline{E_1}), \widetilde{\varphi'}(\underline{E_0}, \underline{E_1})) \\
&= \Big(\varphi(\underline{E_1})\varphi'(\underline{E_1}), \varphi(\underline{E_1})\widetilde{\varphi'}(\underline{E_0}, \underline{E_1}) + \widetilde{\varphi}(\underline{E_0}, \underline{E_1})\varphi'(\underline{E_1}) - (\varphi(\underline{E_1}) - \varphi(\underline{E_0}))\widetilde{\varphi'}(\underline{E_0}, \underline{E_1})\Big) \\
&= (\varphi\varphi'(\underline{E_1}), \widetilde{\varphi\varphi'}(\underline{E_0}, \underline{E_1})).
\end{aligned} \tag{2.26}$$

The proof of Lemma 2.2 is completed. □

As in (2.22), the triple $\underline{\pi^*E}$ induces a map

$$\widetilde{f}_{\underline{E}} : \mathbb{C}[\mathfrak{u}(r)]^{U(r)} \to \Gamma(Y), \quad \varphi \mapsto (\varphi(\underline{E_1}), \widetilde{\varphi}(\underline{E_0}, \underline{E_1})). \tag{2.27}$$

It is a ring homomorphism by Lemma 2.2.

Lemma 2.3. *The ring homomorphisms p, θ^*, $f_{\underline{E}}$ and $\widetilde{f}_{\underline{E}}$ in the following diagram are all homomorphisms of pre-λ-rings,*

$$\begin{tikzcd}
\Gamma(Y) \arrow[d, "p"'] & \mathbb{C}[\mathfrak{u}(r)]^{U(r)} \arrow[l, "\tilde{f}_{\underline{E}}"'] \arrow[r, "\theta^*"] \arrow[dl, "f_{\underline{E_1}}"] & \mathbb{C}[\mathfrak{t}^r]^{T^r} \\
Z^{\mathrm{even}}(Y, \mathbb{R}). & &
\end{tikzcd}$$

Moreover, we have

$$f_{\underline{E_1}} = p \circ \tilde{f}_{\underline{E}}. \tag{2.28}$$

Proof. From (2.3), (2.9), (2.10), (2.17), (2.19), (2.22) and (2.27), we see that all homomorphisms here commute with the corresponding Adams operations. So by (2.4), they are all homomorphisms of pre-λ-rings.

The relation (2.28) follows directly from (2.22) and (2.27).

The proof of Lemma 2.3 is completed. □

By (2.18), $(\theta^*)^{-1}(H) \in \mathbb{C}[\mathfrak{u}(r)]^{U(r)}$ is well-defined for any homogeneous symmetric polynomial H in $u_1, \cdots, u_r$. We define the Chern character to be the formal power series

$$\mathrm{ch} = \sum_{k=0}^{\infty} (\theta^*)^{-1}\Big(\frac{1}{k!}\sum_{j=1}^{r} u_j^k\Big). \tag{2.29}$$

It is easy to see that

$$f_{\underline{E}}(\mathrm{ch}) := \sum_{k=0}^{\infty} f_{\underline{E}} \circ (\theta^*)^{-1}\Big(\frac{1}{k!}\sum_{j=1}^{r} u_j^k\Big) \tag{2.30}$$

is the same as the canonical Chern character $\mathrm{ch}(\underline{E})$ in Definition 1.3 for $g = 1$.

Since the manifold is finite dimensional, the right-hand side of (2.30) is a finite sum. In this paper, we only care about the characteristic forms. We could apply θ^* and λ^i on ch formally and obtain the rigorous equality of the characteristic forms after taking the map $f_{\underline{E}}$.

From this point of view we write

$$\mathrm{ch} = (\theta^*)^{-1}\Big(\sum_{j=1}^{r} \exp(u_j)\Big). \tag{2.31}$$

From Lemma 2.3, (2.4), (2.19) and (2.31), we have

$$\begin{aligned}
\theta^* \circ \lambda_t(\mathrm{ch}) &= \lambda_t\Big(\sum_{j=1}^{r} \exp(u_j)\Big) = \exp\left(\sum_{k=1}^{\infty} \frac{1}{k}(-1)^{k-1} t^k \Psi^k\Big(\sum_{j=1}^{r} \exp(u_j)\Big)\right) \\
&= \exp\Big(\sum_{j=1}^{r}\sum_{k=1}^{\infty} \frac{1}{k}(-1)^{k-1} t^k \exp(k u_j)\Big) = \exp\Big(\sum_{j=1}^{r} \log(1 + t\exp(u_j))\Big) \\
&= \prod_{j=1}^{r} (1 + t\exp(u_j)).
\end{aligned} \tag{2.32}$$

From (2.32), we get the following equality (comparing with [35, Lemma 7.3.3]),

$$\lambda^k(\mathrm{ch})(\underline{E}) = \mathrm{ch}(\Lambda^k(\underline{E})) \in Z^{\mathrm{even}}(Y, \mathbb{R}). \tag{2.33}$$

Lemma 2.4. *The following identity holds,*

$$\lambda^k\Big(\,\mathrm{ch}(\underline{E_1}),\widetilde{\mathrm{ch}}(\underline{E_0},\underline{E_1})\Big)=\Big(\,\mathrm{ch}(\Lambda^k(\underline{E_1})),\widetilde{\mathrm{ch}}(\Lambda^k(\underline{E_0}),\Lambda^k(\underline{E_1}))\Big)\in\Gamma(Y).\tag{2.34}$$

Proof. From (2.23) and (2.33), we have modulo exact forms,

$$\begin{aligned}\widetilde{\lambda^k(\mathrm{ch})}(\underline{E_0},\underline{E_1})&=\int_0^1\{\lambda^k(\mathrm{ch})(\underline{\pi^*E})\}^{ds}ds\\&=\int_0^1\{\mathrm{ch}(\Lambda^k(\underline{\pi^*E}))\}^{ds}ds=\widetilde{\mathrm{ch}}(\Lambda^k(\underline{E_0}),\Lambda^k(\underline{E_1})).\end{aligned}\tag{2.35}$$

So from Lemma 2.3, (2.27), (2.33) and (2.35), we get

$$\begin{aligned}\lambda^k\big(\,\mathrm{ch}(\underline{E_1}),\widetilde{\mathrm{ch}}(\underline{E_0},\underline{E_1})\big)&=\lambda^k\big(\tilde f_{\underline{E}}(\mathrm{ch})\big)=\tilde f_{\underline{E}}(\lambda^k(\mathrm{ch}))\\&=\Big(\lambda^k(\mathrm{ch})(\underline{E_1}),\widetilde{\lambda^k(\mathrm{ch})}(\underline{E_0},\underline{E_1})\Big)=\Big(\,\mathrm{ch}(\Lambda^k(\underline{E_1})),\widetilde{\mathrm{ch}}(\Lambda^k(\underline{E_0}),\Lambda^k(\underline{E_1}))\Big).\end{aligned}\tag{2.36}$$

The proof of Lemma 2.4 is completed. □

2.2. **Pre-λ-ring structure in differential K-theory.** We introduce now a pre-λ-ring structure for differential K-ring. It can be understood as the differential K-theory version of the pre-λ-ring structure for arithmetic K-theory in [35, Theorem 7.3.4].

Let Y be a compact manifold.

Definition 2.5. [32, Definition 2.16] A cycle for differential K-theory of Y is a pair $(\underline{E},\phi)$ where $\underline{E}$ is a geometric triple and ϕ is an element in $\Omega^{\mathrm{odd}}(Y,\mathbb{R})/\mathrm{Im}\,d$. Two cycles $(\underline{E_1},\phi_1)$ and $(\underline{E_2},\phi_2)$ are equivalent if there exist a geometric triple $\underline{E_3}$ and a vector bundle isomorphism

$$\Phi: E_1\oplus E_3\to E_2\oplus E_3\tag{2.37}$$

such that

$$\widetilde{\mathrm{ch}}\left(\underline{E_1}\oplus\underline{E_3},\Phi^*\left(\underline{E_2}\oplus\underline{E_3}\right)\right)=\phi_2-\phi_1.\tag{2.38}$$

We define the sum in the obvious way by

$$(\underline{E},\phi)+(\underline{F},\psi)=(\underline{E}\oplus\underline{F},\phi+\psi).\tag{2.39}$$

The differential K-group $\widehat{K}^0(Y)$ is defined as the Grothendieck group of equivalence classes of cycles.

We denote by $[\underline{E},\phi]$ the equivalence class of a cycle $(\underline{E},\phi)$. Then

$$\widehat{K}^0(Y)=\big\{[\underline{E}-\underline{E_1},\phi-\phi_1]:(\underline{E},\phi),(\underline{E_1},\phi_1)\text{ are cycles as above}\big\}\tag{2.40}$$

and $\widehat{K}^0(Y)$ is an abelian group. For $[\underline{E},\phi],[\underline{F},\psi]\in\widehat{K}^0(Y)$, set

$$[\underline{E},\phi]\cup[\underline{F},\psi]=\Big[\underline{E}\otimes\underline{F},[(\mathrm{ch}(\underline{E}),\phi)*(\mathrm{ch}(\underline{F}),\psi)]_{\mathrm{odd}}\Big],\tag{2.41}$$

where $[\cdot]_{\mathrm{odd}}$ is the component of $\Gamma(Y)$ in $\Omega^{\mathrm{odd}}(Y,\mathbb{R})/\mathrm{Im}\,d$. It is easy to check that this product (2.41) is well-defined, commutative and associative (cf. also Lemma 2.15). Let $\underline{\mathbb{C}}$ be the trivial complex line bundle over Y with the trivial metric and connection. Then the element

$$1:=[\underline{\mathbb{C}},0]\tag{2.42}$$

is a unit for the product $\cup$. Thus $(\widehat{K}^0(Y), +, \cup)$ is a commutative ring with unit 1.

From (1.30), (2.37) and (2.38), we see that if $[\underline{E}, 0] = [\underline{F}, 0] \in \widehat{K}^0(Y)$, we have

$$\mathrm{ch}(\underline{E}) = \mathrm{ch}(\underline{F}) \in \Omega^\bullet(Y, \mathbb{R}). \tag{2.43}$$

Theorem 2.6. *There exists a pre-λ-ring structure on $\widehat{K}^0(Y)$.*

Proof. Let $(\underline{E}, \phi)$ be a cycle. Observe that $(\mathrm{ch}(\underline{E}), \phi) \in \Gamma(Y)$. Since $\Gamma(Y)$ is a pre-λ-ring, we define

$$\lambda^k(\underline{E}, \phi) := (\Lambda^k(\underline{E}), [\lambda^k(\mathrm{ch}(\underline{E}), \phi)]_{\mathrm{odd}}). \tag{2.44}$$

It is clear that

$$\lambda^0(\underline{E}, \phi) = 1, \quad \lambda^1(\underline{E}, \phi) = (\underline{E}, \phi). \tag{2.45}$$

By (2.33), (2.39), (2.41) and (2.44), we have for any cycles $(\underline{E}, \phi)$ and $(\underline{F}, \psi)$,

$$\begin{aligned}
\lambda^k((\underline{E}, \phi) + (\underline{F}, \psi)) &= \lambda^k(\underline{E} \oplus \underline{F}, \phi + \psi) \\
&= \left(\Lambda^k(\underline{E} \oplus \underline{F}), [\lambda^k(\mathrm{ch}(\underline{E} \oplus \underline{F}), \phi + \psi)]_{\mathrm{odd}}\right) \\
&= \left(\sum_{j=0}^k \Lambda^j(\underline{E}) \otimes \Lambda^{k-j}(\underline{F}), \left[\sum_{i=0}^k \lambda^i(\mathrm{ch}(\underline{E}), \phi) * \lambda^{k-i}(\mathrm{ch}(\underline{F}), \psi)\right]_{\mathrm{odd}}\right) \\
&= \sum_{j=0}^k \Big(\Lambda^j\left(\underline{E}\right) \otimes \Lambda^{k-j}\left(\underline{F}\right), \Big[\Big(\mathrm{ch}\left(\Lambda^j\left(\underline{E}\right)\right), \left[\lambda^j\left(\mathrm{ch}\left(\underline{E}\right), \phi\right)\right]_{\mathrm{odd}}\Big) \\
&\qquad * \Big(\mathrm{ch}\left(\Lambda^{k-j}\left(\underline{F}\right)\right), \left[\lambda^{k-j}\left(\mathrm{ch}\left(\underline{F}\right), \psi\right)\right]_{\mathrm{odd}}\Big)\Big]_{\mathrm{odd}}\Big) \\
&= \sum_{j=0}^k \Big(\Lambda^j(\underline{E}), [\lambda^j(\mathrm{ch}(\underline{E}), \phi)]_{\mathrm{odd}}\Big) \cup \Big(\Lambda^{k-j}(\underline{F}), [\lambda^{k-j}(\mathrm{ch}(\underline{F}), \psi)]_{\mathrm{odd}}\Big) \\
&= \sum_{j=0}^k \lambda^j(\underline{E}, \phi) \cup \lambda^{k-j}(\underline{F}, \psi),
\end{aligned} \tag{2.46}$$

where the second equality is implied by (2.44), the third equality follows from Definition 2.1c) and the pre-λ-ring structure on $\Gamma(Y)$, the fourth equality is a consequence of the fact that p in (2.10) is a homomorphism of pre-λ rings,

$$\lambda^j(\mathrm{ch}(\underline{E}), \phi) = \Big(\lambda^j\left(\mathrm{ch}\left(\underline{E}\right)\right), \left[\lambda^j\left(\mathrm{ch}\left(\underline{E}\right), \phi\right)\right]_{\mathrm{odd}}\Big) \tag{2.47}$$

and (2.33), the last two equalities follows from (2.41) and (2.44). So we only need to prove that λ^k is well-defined on $\widehat{K}^0(Y)$.

If $(\underline{E_1}, \phi_1) \sim (\underline{E_2}, \phi_2)$, there exist $\underline{E_3}$ and isomorphism $\Phi : E_1 \oplus E_3 \to E_2 \oplus E_3$ such that

$$\widetilde{\mathrm{ch}}\left((\Phi^{-1})^*(\underline{E_1} \oplus \underline{E_3}), \underline{E_2} \oplus \underline{E_3}\right) = \widetilde{\mathrm{ch}}\left(\underline{E_1} \oplus \underline{E_3}, \Phi^*\left(\underline{E_2} \oplus \underline{E_3}\right)\right) = \phi_2 - \phi_1. \tag{2.48}$$

From (2.34), (2.44) and (2.48), we have

$$\begin{aligned}(2.49)\quad & \lambda^k(\underline{E_2}\oplus\underline{E_3}, \widetilde{\mathrm{ch}}\left(\underline{E_1}\oplus\underline{E_3}, \Phi^*(\underline{E_2}\oplus\underline{E_3})\right)) \\ & = \Big(\Lambda^k\left(\underline{E_2}\oplus\underline{E_3}\right), \Big[\lambda^k\Big(\mathrm{ch}(\underline{E_2}\oplus\underline{E_3}), \widetilde{\mathrm{ch}}\left((\Phi^{-1})^*(\underline{E_1}\oplus\underline{E_3}), \underline{E_2}\oplus\underline{E_3}\right)\Big)\Big]_{\mathrm{odd}}\Big) \\ & \qquad = \Big(\Lambda^k\left(\underline{E_2}\oplus\underline{E_3}\right), \widetilde{\mathrm{ch}}\left((\Phi^{-1})^*\Lambda^k(\underline{E_1}\oplus\underline{E_3}), \Lambda^k(\underline{E_2}\oplus\underline{E_3})\right)\Big).\end{aligned}$$

By Definition 2.5, (2.11) and (2.44), we get

$$\begin{aligned}(2.50)\quad & \lambda^k(\underline{E_1}\oplus\underline{E_3}, 0) = \left(\Lambda^k\left(\underline{E_1}\oplus\underline{E_3}\right), \left[\lambda^k\left(\mathrm{ch}(\underline{E_1}\oplus\underline{E_3}), 0\right)\right]_{\mathrm{odd}}\right) \\ & = \left(\Lambda^k\left(\underline{E_1}\oplus\underline{E_3}\right), 0\right) \sim \Big(\Lambda^k\left(\underline{E_2}\oplus\underline{E_3}\right), \widetilde{\mathrm{ch}}\left((\Phi^{-1})^*\Lambda^k(\underline{E_1}\oplus\underline{E_3}), \Lambda^k\left(\underline{E_2}\oplus\underline{E_3}\right)\right)\Big).\end{aligned}$$

From (2.49) and (2.50), we get

$$\lambda^k(\underline{E_1}\oplus\underline{E_3}, 0) \sim \lambda^k\Big(\underline{E_2}\oplus\underline{E_3}, \widetilde{\mathrm{ch}}\left(\underline{E_1}\oplus\underline{E_3}, \Phi^*(\underline{E_2}\oplus\underline{E_3})\right)\Big). \tag{2.51}$$

Since for $j=1,2$,

$$\lambda_t(\underline{E_j}\oplus\underline{E_3}, \phi_j) = \lambda_t((\underline{E_j}\oplus\underline{E_3}, 0) + (0, \phi_j)) = \lambda_t(\underline{E_j}\oplus\underline{E_3}, 0) \cup \lambda_t(0, \phi_j), \tag{2.52}$$

by (2.48) and (2.51), we have

$$\begin{aligned}(2.53)\quad \lambda^k(\underline{E_1}\oplus\underline{E_3}, \phi_1) &= \sum_{i=0}^k \lambda^i(\underline{E_1}\oplus\underline{E_3}, 0) \cup \lambda^{k-i}(0, \phi_1) \\ & \sim \sum_{i=0}^k \lambda^i(\underline{E_2}\oplus\underline{E_3}, \phi_2-\phi_1) \cup \lambda^{k-i}(0, \phi_1) = \lambda^k(\underline{E_2}\oplus\underline{E_3}, \phi_2).\end{aligned}$$

By (2.46), for any $k\geq 1$, $j=1,2$, we have

$$\lambda^k(\underline{E_j}\oplus\underline{E_3}, \phi_j) = \sum_{i=0}^k \lambda^i(\underline{E_j}, \phi_j) \cup \lambda^{k-i}(\underline{E_3}, 0). \tag{2.54}$$

Note that $\lambda^1(\underline{E_1}, \phi_1) = (\underline{E_1}, \phi_1) \sim (\underline{E_2}, \phi_2) = \lambda^1(\underline{E_2}, \phi_2)$. We assume that $\lambda^i(\underline{E_1}, \phi_1) \sim \lambda^i(\underline{E_2}, \phi_2)$ holds for all $1\leq i\leq k-1$. Then

$$\sum_{i=0}^{k-1} \lambda^i(\underline{E_1}, \phi_1) \cup \lambda^{k-i}(\underline{E_3}, 0) \sim \sum_{i=0}^{k-1} \lambda^i(\underline{E_2}, \phi_2) \cup \lambda^{k-i}(\underline{E_3}, 0). \tag{2.55}$$

From (2.53)-(2.55), since $\lambda^0(x)=1$, $\lambda^k(\underline{E_1}, \phi_1) \sim \lambda^k(\underline{E_2}, \phi_2)$. So by induction, for any $k\geq 1$, we have $\lambda^k(\underline{E_1}, \phi_1) \sim \lambda^k(\underline{E_2}, \phi_2)$.

The proof of Theorem 2.6 is completed. □

Lemma 2.7. *If $\sigma: X\to Y$ is a $\mathscr{C}^\infty$ map of compact manifolds, then its pull-back maps*

$$\begin{aligned}&\sigma^*: \Gamma(Y)\to\Gamma(X), && \sigma^*(\omega, \phi) = (\sigma^*\omega, \sigma^*\phi); \\ &\hat{\sigma}^*: \widehat{K}^0(Y)\to\widehat{K}^0(X), && \hat{\sigma}^*[\underline{E}, \phi] = [\sigma^*\underline{E}, \sigma^*\phi],\end{aligned} \tag{2.56}$$

are morphisms of pre-λ-rings.

Proof. From (2.8) and (2.9), σ^* commutes with the operators $*$ and Ψ^k, thus $\sigma^* : \Gamma(Y) \to \Gamma(X)$ is a morphism of pre-λ-rings. Now from (2.22) and (2.23), we have $\sigma^* f_{\underline{E}} = f_{\sigma^*\underline{E}}\sigma^*$, $\widetilde{\varphi}\sigma^* = \sigma^*\widetilde{\varphi}$. From (2.44) and $\sigma^*\Lambda^k(\underline{E}) = \Lambda^k(\sigma^*\underline{E})$, we get that the second map of (2.56) is well-defined and a morphism of pre-λ-rings. □

Remark 2.8. In [10, Chapter V], the λ-ring is well studied, and needs two additional conditions compared to Definition 2.1. It is well-known that the topological K-group $K^0(Y)$ is a λ-ring (cf. [8, Theorem 1.5]). In [54], the author proves that the arithmetic K-group is also a λ-ring. It is natural to ask whether the differential K-group $\widehat{K}^0(Y)$ is also a λ-ring, and we will come back to this question later. However, for our application here, the pre-λ-ring structure for differential K-theory is enough.

2.3. γ-filtration.

Definition 2.9. [10, (1.25)] Let R be any pre-λ-ring with an augmentation homomorphism $\mathrm{rk} : R \to \mathbb{Z}$. The γ-operations are defined by

$$\gamma_t(x) = \sum_{j\geq 0} \gamma^j(x)t^j := \lambda_{\frac{t}{1-t}}(x). \tag{2.57}$$

By Definition 2.1, we have

$$\gamma^0 = 1, \quad \gamma^1(x) = x, \quad \gamma_t(x+y) = \gamma_t(x)\gamma_t(y), \quad \text{for any } x, y \in R. \tag{2.58}$$

Definition 2.10. Set $F^nR := R$ for $n \leq 0$ and F^1R the kernel of $\mathrm{rk} : R \to \mathbb{Z}$. Let F^nR be the additive subgroup generated by $\gamma^{r_1}(x_1)\cdots\gamma^{r_k}(x_k)$, where $x_1, \cdots, x_k \in F^1R$ and $\sum_{j=1}^k r_j \geq n$. The filtration

$$F^1R \supseteq F^2R \supseteq F^3R \supseteq \cdots \tag{2.59}$$

is called the γ-filtration of R. The γ-filtration is said to be **locally nilpotent** at $x \in F^1R$, if there exists $M(x) \in \mathbb{N}$, such that $\gamma^{r_1}(x)\cdots\gamma^{r_k}(x) = 0$ for any $\sum_{j=1}^k r_j > M(x)$.

Let Y be a compact connected manifold.

It is well-known that the classical γ-filtration of $K^0(Y)$ is locally nilpotent for any $x \in F^1K^0(Y)$ [1, Proposition 3.1.5] with the augmentation homomorphism $\mathrm{rk} : E \to \mathrm{rk}\, E$, the rank of the complex vector bundle E. Since $\widehat{K}^0(Y)$ is a pre-λ-ring, the augmentation homomorphism $\mathrm{rk}(\underline{E}, \phi) := \mathrm{rk}\, E$ defines a γ-filtration of $\widehat{K}^0(Y)$.

We identify a geometric triple $\underline{E}$ to the cycle $(\underline{E}, 0)$. By (2.1), (2.11) and (2.44), we have

$$\lambda_t(\underline{E}) = \sum_{j\geq 0} \Lambda^j(\underline{E})t^j. \tag{2.60}$$

So $\gamma^i(\underline{E})$, defined as in (2.57), is a finite dimensional virtual Hermitian vector bundle with induced metric and connection. We also denote it by $\underline{\gamma^i(E)}$.

Let $\underline{k}$ be the k-dimensional trivial complex vector bundle with trivial metric and connection. Then $F^1\widehat{K}^0(Y)$ is generalized by cycles $(\underline{E} - \underline{\mathrm{rk}\, E}, \phi)$ for $\phi \in \Omega^{\mathrm{odd}}(Y, \mathbb{R})/\mathrm{Im}\, d$. By (2.42), (2.57) and (2.60),

$$\lambda_t(\underline{\mathbb{C}}) = 1 + t, \quad \gamma_t(\underline{\mathbb{C}}) = 1 + \frac{t}{1-t} \cdot 1 = \frac{1}{1-t}. \tag{2.61}$$

From (2.57), (2.58), (2.60) and (2.61), letting $r = \operatorname{rk} E$, we have

$$\begin{aligned}(2.62)\quad \gamma_t(\underline{E} - \underline{\operatorname{rk} E}) &= \gamma_t(\underline{E})\gamma_t(\underline{\mathbb{C}})^{-r} = \lambda_{\frac{t}{1-t}}(\underline{E})(1-t)^r = \sum_{i=0}^{r} \Lambda^i(\underline{E}) t^i (1-t)^{r-i} \\ &= \sum_{i=0}^{r}\sum_{j=0}^{r-i}(-1)^j \begin{pmatrix} r-i \\ j \end{pmatrix} \Lambda^i(\underline{E}) t^{i+j} = \sum_{k=0}^{r}\left(\sum_{i=0}^{k}(-1)^{k-i}\begin{pmatrix} r-i \\ k-i \end{pmatrix}\Lambda^i(\underline{E})\right) t^k.\end{aligned}$$

So

$$(2.63)\qquad \gamma^k(\underline{E} - \underline{\operatorname{rk} E}) = \begin{cases} \sum_{i=0}^{k}(-1)^{k-i}\begin{pmatrix} r-i \\ k-i \end{pmatrix}\Lambda^i(\underline{E}) & \text{if } 0 \le k \le r = \operatorname{rk} E; \\ 0 & \text{if } k > r. \end{cases}$$

Since $\lambda_t(x) = \gamma_{t/(1+t)}(x)$, from (2.63), we have

$$\begin{aligned}(2.64)\quad \lambda_t(\underline{E}) &= \lambda_t(\underline{E} - \underline{\operatorname{rk} E}) \cdot \lambda_t(\underline{\operatorname{rk} E}) = \gamma_{\frac{t}{1+t}}(\underline{E} - \underline{\operatorname{rk} E}) \cdot (1+t)^r \\ &= (1+t)^r \Big(1 + \sum_{i=1}^{r} \gamma^i(\underline{E} - \underline{\operatorname{rk} E}) t^i (1+t)^{-i}\Big).\end{aligned}$$

Formally, we have

$$\begin{aligned}(2.65)\quad \lambda_t(\underline{E})^{-1} &= (1+t)^{-r}\Big(1 + \sum_{i=1}^{r}\gamma^i(\underline{E} - \underline{\operatorname{rk} E}) t^i (1+t)^{-i}\Big)^{-1} \\ &= (1+t)^{-r}\left(1 + \sum_{j=1}^{\infty}(-1)^j\Big(\sum_{i=1}^{r}\gamma^i(\underline{E} - \underline{\operatorname{rk} E}) t^i(1+t)^{-i}\Big)^j\right) \\ &= (1+t)^{-r}\left(1 + \sum_{k=1}^{\infty} t^k (1+t)^{-k} \sum_{\substack{(n_1,\cdots,n_r)\in\mathbb{N}^r, \\ \sum_{i=1}^{r} i\cdot n_i = k}} (-1)^{\sum_{i=1}^{r} n_i}\frac{(\sum_{i=1}^{r} n_i)!}{\prod_{i=1}^{r} n_i!}\prod_{i=1}^{r}\big(\gamma^i(\underline{E} - \underline{\operatorname{rk} E})\big)^{n_i}\right).\end{aligned}$$

To simplify the notations, we denote by

$$(2.66)\qquad \lambda_t(\underline{E})^{-1} = (1+t)^{-r}\left(1 + \sum_{k=1}^{\infty} t^k (1+t)^{-k}\big(P_{k,+}(\underline{E}) - P_{k,-}(\underline{E})\big)\right),$$

by using (2.63) in (2.65). Remark that $P_{k,\pm}(\underline{E})$ here are finite dimensional Hermitian vector bundles with induced metrics and connections obtained from $\underline{E}$. We denote by $P_{k,+}(\underline{E}) = P_{k,-}(\underline{E}) = 0$ for $k < 0$ and $P_{0,+}(\underline{E}) = \underline{\mathbb{C}}$, $P_{0,-}(\underline{E}) = 0$. From (2.64) and (2.66), we know that for any $l \in \mathbb{N}^*$,

$$(2.67)\qquad \sum_{i=0}^{r}\gamma^i(\underline{E} - \underline{\operatorname{rk} E})\Big(P_{l-i,+}(\underline{E}) - P_{l-i,-}(\underline{E})\Big) = 0,$$

as geometric triples.

The following theorem is the differential K-theory version of the locally nilpotent property in topological K-theory [1, Propositions 3.1.5, 3.1.10]. The corresponding arithmetic K-theory version was proved by Roessler [53, Proposition 4.5].

Theorem 2.11. *The γ-filtration of $\widehat{K}^0(Y)$ is locally nilpotent at $[\underline{E} - \underline{\operatorname{rk} E}, 0]$. Explicitly, there exists $\mathcal{N}_{r,m} > 0$ (depending only on r, m) such that for any geometric triple $\underline{E}$ on Y with $r = \operatorname{rk} E$, $m = \dim Y$, and $(n_1, \cdots, n_r) \in \mathbb{N}^r$ such that*

$$\sum_{i=1}^{r} i \cdot n_i > \mathcal{N}_{r,m}, \tag{2.68}$$

we have

$$\prod_{i=1}^{r} \left(\gamma^i([\underline{E} - \underline{\operatorname{rk} E}, 0])\right)^{n_i} = \left[\prod_{i=1}^{r} \left(\gamma^i(\underline{E} - \underline{\operatorname{rk} E})\right)^{n_i}, 0\right] = 0 \in \widehat{K}^0(Y). \tag{2.69}$$

Proof. For any complex vector bundle F over Y, we get from (2.63) the γ^k-operation on $K^0(Y)$ by forgetting the metric and connection,

$$\gamma^k(F - \operatorname{rk} F) = \begin{cases} \sum_{i=0}^{k} (-1)^{k-i} \binom{\operatorname{rk} F - i}{k - i} \Lambda^i(F) & \text{if } 0 \le k \le \operatorname{rk} F; \\ 0 & \text{if } k > \operatorname{rk} F. \end{cases} \tag{2.70}$$

From the classical property of the γ-operation in K-theory [1, Propositions 3.1.5, 3.1.10], we know that there exists $a_Y > 0$ (depending only on Y) such that

$$\gamma^{i_1}(x_1)\gamma^{i_2}(x_2)\cdots\gamma^{i_k}(x_k) = 0 \in K^0(Y) \quad \text{if } x_j \in F^1K^0(Y), i_j \in \mathbb{N}, \sum_{j=1}^{k} i_j \ge a_Y. \tag{2.71}$$

Let $\underline{E}$ be a geometric triple on Y with $r = \operatorname{rk} E$, $m = \dim Y$. Let $\mathrm{U}(E)$ be the $\mathrm{U}(r)$-principal bundle of unitary frames of the Hermitian vector bundle (E, h^E). Then $\mathrm{U}(E) \times_{\mathrm{U}(r)} \mathbb{C}^r \simeq E$ and h^E coincides with the Hermitian metric induced by the canonical Hermitian inner product on $\mathbb{C}^r$. The Hermitian connection ∇^E corresponds uniquely to a connection ω on $\mathrm{U}(E)$.

Let $\mathrm{Gr}(r, \mathbb{C}^p)$ be the Grassmannian, which is the space parameterizing all complex linear subspaces of $\mathbb{C}^p$ of given dimension r. Let $\mathrm{V}(p, r)$ be the canonical $\mathrm{U}(r)$-principal bundle over $\mathrm{Gr}(r, \mathbb{C}^p)$ with the canonical connection ω_0, which is induced by the Maurer-Cartan form on $\mathrm{U}(p)$ on the $\mathrm{U}(r)$-principal bundle $\mathrm{V}(p, r) = \mathrm{U}(p)/(\mathrm{I}_r \times \mathrm{U}(p-r)) \to \mathrm{U}(p)/(\mathrm{U}(r) \times \mathrm{U}(p-r)) = \mathrm{Gr}(r, \mathbb{C}^p)$ via the canonical matrix decomposition of $\mathfrak{u}(p)$. Let $H = \mathrm{V}(p, r) \times_{\mathrm{U}(r)} \mathbb{C}^r$. Let h^H be the Hermitian metric on H induced by the canonical inner product on $\mathbb{C}^r$. Let ∇^H be the Hermitian connection on H induced by ω_0. Let $\underline{\operatorname{rk} H}$ be the r-dimensional trivial vector bundle over $\mathrm{Gr}(r, \mathbb{C}^p)$ with trivial metric and connection.

By a theorem of Narasimhan and Ramanan [50, Theorem 1], for

$$p = (m + 1)(2m + 1)r^3, \tag{2.72}$$

there exists a map $f : Y \to \mathrm{Gr}(r, \mathbb{C}^p)$ such that $f^*\mathrm{V}(p, r) = \mathrm{U}(E)$ and $f^*\omega_0 = \omega$. Thus $f^*(H, h^H, \nabla^H) = (E, h^E, \nabla^E)$. Let $\hat{f}^* : \widehat{K}^0(\mathrm{Gr}(r, \mathbb{C}^p)) \to \widehat{K}^0(Y)$ be the pull-back map. From

(2.44), (2.57) and (2.63), we have

$$\prod_{i=1}^{r}\big(\gamma^i([\underline{E}-\underline{\mathrm{rk}\,E},0])\big)^{n_i}=\left[\prod_{i=1}^{r}\big(\gamma^i(\underline{E}-\underline{\mathrm{rk}\,E})\big)^{n_i},0\right]=\hat{f}^*\left\{\left[\prod_{i=1}^{r}\big(\gamma^i(\underline{H}-\underline{\mathrm{rk}\,H})\big)^{n_i},0\right]\right\}\in\widehat{K}^0(Y).\tag{2.73}$$

By (2.71) and (2.72), there exists $a_{r,m}>0$ (depending only on r,m) such that

$$\prod_{i=1}^{r}\big(\gamma^i(H-\mathrm{rk}\,H)\big)^{n_i}=0\in K^0(\mathrm{Gr}(r,\mathbb{C}^p))\quad\text{if}\ \sum_{i=1}^{r}i\cdot n_i>a_{r,m}.\tag{2.74}$$

From Definition 2.5 and (2.74), if $\sum_{i=1}^r i\cdot n_i>a_{r,m}$, there exists $\alpha\in\Omega^{\mathrm{odd}}(\mathrm{Gr}(r,\mathbb{C}^p),\mathbb{R})$ such that

$$\left[\prod_{i=1}^{r}\big(\gamma^i(\underline{H}-\underline{\mathrm{rk}\,H})\big)^{n_i},0\right]=[0,\alpha]\in\widehat{K}^0(\mathrm{Gr}(r,\mathbb{C}^p))\tag{2.75}$$

and

$$-\,d\alpha=\mathrm{ch}\Big(\prod_{i=1}^{r}\big(\gamma^i(\underline{H}-\underline{\mathrm{rk}\,H})\big)^{n_i}\Big)=\prod_{i=1}^{r}\Big(\mathrm{ch}\big(\gamma^i(\underline{H}-\underline{\mathrm{rk}\,H})\big)\Big)^{n_i}\in\Omega^{\mathrm{even}}(\mathrm{Gr}(r,\mathbb{C}^p),\mathbb{R}).\tag{2.76}$$

From (2.22), (2.33) and (2.62), we have

$$\begin{aligned}\mathrm{ch}(\gamma_t(\underline{H}-\underline{\mathrm{rk}\,H}))&=\sum_{i=0}^{r}\mathrm{ch}(\Lambda^i(\underline{H}))t^i(1-t)^{r-i}\\&=\sum_{i=0}^{r}\lambda^i(\mathrm{ch})(\underline{H})t^i(1-t)^{r-i}=\sum_{i=0}^{r}f_{\underline{H}}\big(\lambda^i(\mathrm{ch})\big)t^i(1-t)^{r-i},\end{aligned}\tag{2.77}$$

where $f_{\underline{H}}$ is the map (2.22) with respect to $\underline{H}$. From (2.32), we have

$$\begin{aligned}\sum_{i=0}^{r}\theta^*\lambda^i(\mathrm{ch})t^i(1-t)^{r-i}&=\sum_{i=0}^{r}\sigma_i(e^{u_1},\cdots,e^{u_r})t^i(1-t)^{r-i}\\&=\prod_{j=1}^{r}\Big((1-t)+te^{u_j}\Big)=\prod_{j=1}^{r}\Big(1+t(e^{u_j}-1)\Big)=\sum_{i=0}^{r}\sigma_i(e^{u_1}-1,\cdots,e^{u_r}-1)t^i.\end{aligned}\tag{2.78}$$

Since θ^* is injective and $\sigma_i(e^{u_1}-1,\cdots,e^{u_r}-1)$ is symmetric with respect to u_i's, from (2.77) and (2.78), we have

$$\mathrm{ch}(\gamma^i(\underline{H}-\underline{\mathrm{rk}\,H}))=f_{\underline{H}}\circ(\theta^*)^{-1}\big(\sigma_i(e^{u_1}-1,\cdots,e^{u_r}-1)\big)\in Z^{\mathrm{even}}(\mathrm{Gr}(r,\mathbb{C}^p),\mathbb{R}).\tag{2.79}$$

From Lemma 2.3, we see that the Adams operation Ψ^k commutes with $f_{\underline{H}}\circ(\theta^*)^{-1}$. Since

$$\Psi^k\big(\sigma_i(e^{u_1}-1,\cdots,e^{u_r}-1)\big)=\sigma_i(e^{ku_1}-1,\cdots,e^{ku_r}-1)\tag{2.80}$$

is a power series with respect to k such that the coefficients of $1, k, \cdots, k^{i-1}$ vanish, so is $\Psi^k(\mathrm{ch}(\gamma^i(\underline{H} - \underline{\mathrm{rk}\, H})))$, too. Since $\Psi^k\beta = k^l\beta$ for $\beta \in Z^{2l}(\mathrm{Gr}(r, \mathbb{C}^p), \mathbb{R})$, we have

$$\mathrm{ch}(\gamma^i(\underline{H} - \underline{\mathrm{rk}\, H})) \in \Omega^{\geq 2i}(\mathrm{Gr}(r, \mathbb{C}^p), \mathbb{R}). \tag{2.81}$$

Since $\dim_{\mathbb{R}} \mathrm{Gr}(r, \mathbb{C}^p) = 2r(p-r) = 2r^2((m+1)(2m+1)r^2 - 1)$, for any $(n_1, \cdots, n_r) \in \mathbb{N}^r$ such that $\sum_{i=1}^r i \cdot n_i > r^2((m+1)(2m+1)r^2 - 1)$, we have

$$\mathrm{ch}\left(\prod_{i=1}^r (\gamma^i(\underline{H} - \underline{\mathrm{rk}\, H}))^{n_i}\right) = 0 \in \Omega^{\mathrm{even}}(\mathrm{Gr}(r, \mathbb{C}^p), \mathbb{R}). \tag{2.82}$$

By (2.76) and (2.82), we have $d\alpha = 0$ if $\sum_{i=1}^r i \cdot n_i > \mathcal{N}_{r,m}$, with

$$\mathcal{N}_{r,m} = \sup\{r^2((m+1)(2m+1)r^2 - 1), a_{r,m}\}. \tag{2.83}$$

Since the cohomology of the Grassmannian vanishes in odd degrees, the differential form α is exact. From (2.75), $[\prod_{i=1}^r (\gamma^i(\underline{H} - \underline{\mathrm{rk}\, H}))^{n_i}, 0] = 0 \in \widehat{K}^0(\mathrm{Gr}(r, \mathbb{C}^p))$. By (2.73), for any $(n_1, \cdots, n_r) \in \mathbb{N}^r$ such that $\sum_{i=1}^r i \cdot n_i > \mathcal{N}_{r,m}$, we have

$$\left[\prod_{i=1}^r (\gamma^i(\underline{E} - \underline{\mathrm{rk}\, E}))^{n_i}, 0\right] = 0 \in \widehat{K}^0(Y). \tag{2.84}$$

The proof of Theorem 2.11 is completed. □

Remark 2.12. a). Comparing with (2.69) and (2.71), a natural question about the γ-filtration of $\widehat{K}^0(Y)$ is whether we can replace $\mathcal{N}_{r,m}$ in (2.68) by a constant depending only on Y such that (2.69) still holds. More generally, whether the analogue of (2.71) holds in $\widehat{K}^0(Y)$.

b). Let $\{U_i\}$ be an open covering of $\mathrm{Gr}(r, \mathbb{C}^p)$ such that U_i's are diffeomorphic to open balls, and the covers can be divided into $\dim_{\mathbb{R}} \mathrm{Gr}(r, \mathbb{C}^p) + 1 = 2r(p-r)+1$ classes $\mathcal{U}_1, \cdots, \mathcal{U}_{2r(p-r)+1}$ such that no two U_i's of the same class intersect, from a result of Nash (cf. Ann. of Math. 63 (1956), p.61). Let $V_j = \bigcup_{U_i \in \mathcal{U}_j} U_i$, $j = 1, \cdots, 2r(p-r)+1$. Then the union is a disjoint union with the trivialization $h_j : H|_{V_j} \to V_j \times \mathbb{C}^r$. Let $\{\varphi_j\}$ be a partition of unity of $\{V_j\}$. Let $\pi_1 : H \to \mathrm{Gr}(r, \mathbb{C}^p)$ and $\pi_2 : V_j \times \mathbb{C}^r \to \mathbb{C}^r$ be the canonical projections. Then the map

$$v \in H \mapsto \Big(\pi_1(v), \varphi_1(\pi_1(v)) \cdot \pi_2(h_1(v)), \cdots, \varphi_{2r(p-r)+1}(\pi_1(v)) \cdot \pi_2(h_{2r(p-r)+1}(v))\Big)$$

induces a bundle map $H \to \mathrm{Gr}(r, \mathbb{C}^p) \times \mathbb{C}^{r(2r(p-r)+1)}$ such that it is injective at each fiber. Then we obtain a complex vector bundle F over $\mathrm{Gr}(r, \mathbb{C}^p)$ with rank $2r^2(p-r)$ as the complement of H in $\mathrm{Gr}(r, \mathbb{C}^p) \times \mathbb{C}^{r(2r(p-r)+1)}$. Thus we have

$$\gamma_t(\mathrm{rk}\, H - H) = \gamma_t((\mathrm{rk}\, H + F) - (H + F)) = \gamma_t(F - \mathrm{rk}\, F).$$

From (2.70) and the above equation, $\gamma_t(\mathrm{rk}\, H - H)$ is a polynomial in t with degree $\leq \mathrm{rk}\, F$. By applying the remark before [1, Proposition 3.1.5] to the polynomials $\gamma_t(H - \mathrm{rk}\, H)$, $\gamma_t(\mathrm{rk}\, H - H)$, we can take

$$a_{r,m} = \mathrm{rk}\, H \cdot \mathrm{rk}\, F = 2r^3(p-r) = 2r^4((m+1)(2m+1)r^2 - 1).$$

Thus we can take $\mathcal{N}_{r,m} = 2r^4((m+1)(2m+1)r^2 - 1)$.

From Theorem 2.11, (2.65) and (2.66), we have the following corollary.

Corollary 2.13. *If $k > \mathcal{N}_{r,m}$ with $\mathcal{N}_{r,m}$ as in* (2.83), *we have*

$$[P_{k,+}(\underline{E}), 0] = [P_{k,-}(\underline{E}), 0] \in \widehat{K}^0(Y). \tag{2.85}$$

2.4. **The g-equivariant differential K-theory. Proof of Theorem 0.2.** Let Y be a compact manifold with S^1-action. In the following, we adapt the notation in Section 1.

The following definition is an extension of Definition 2.5.

Definition 2.14. For $g \in S^1$, a cycle for the g-equivariant differential K-theory of Y is a pair $(\underline{E}, \phi)$ where $\underline{E} = (E, h^E, \nabla^E)$ is an S^1-equivariant geometric triple on Y and ϕ is an element in $\Omega^{\mathrm{odd}}(Y^g, \mathbb{C})/\mathrm{Im}\, d$. Two cycles $(\underline{E_1}, \phi_1)$ and $(\underline{E_2}, \phi_2)$ are equivalent if there exist an S^1-equivariant geometric triple $\underline{E_3} = (E_3, h^{E_3}, \nabla^{E_3})$ and an S^1-equivariant vector bundle isomorphism over Y

$$\Phi : E_1 \oplus E_3 \to E_2 \oplus E_3 \tag{2.86}$$

such that

$$\widetilde{\mathrm{ch}}_g \left(\underline{E_1} \oplus \underline{E_3}, \Phi^* \left(\underline{E_2} \oplus \underline{E_3}\right)\right) = \phi_2 - \phi_1. \tag{2.87}$$

We define the sum in the same way as in (2.39). We define the g-equivariant differential K-group $\widehat{K}^0_g(Y)$ to be the Grothendieck group of equivalence classes of cycles.

We denote by $[\underline{E}, \phi] \in \widehat{K}^0_g(Y)$ the equivalence class of a cycle $(\underline{E}, \phi)$. For $[\underline{E}, \phi], [\underline{F}, \psi] \in \widehat{K}^0_g(Y)$, set

$$[\underline{E}, \phi] \cup [\underline{F}, \psi] = [\underline{E} \otimes \underline{F}, \mathrm{ch}_g(\underline{E}) \wedge \psi + \phi \wedge \mathrm{ch}_g(\underline{F}) - d\phi \wedge \psi]. \tag{2.88}$$

From (2.88), we deduce that the element $1 := [\underline{\mathbb{C}}, 0]$ is a unit: the circle action on the total space $Y \times \mathbb{C}$ is defined by $g(y, c) = (gy, c)$ for any $(y, c) \in Y \times \mathbb{C}$, the trivial metric and connection are obviously S^1-invariant.

Lemma 2.15. *The product (2.88) on $\widehat{K}^0_g(Y)$ is well-defined, associative and commutative.*

Proof. If two cycles $(\underline{E_1}, \phi_1)$ and $(\underline{E_2}, \phi_2)$ are equivalent, then by (1.29) and (2.87), we have

$$\widetilde{\mathrm{ch}}_g \left(\left(\underline{E_1} \oplus \underline{E_3}\right) \otimes \underline{F}, \Phi^* \left(\underline{E_2} \oplus \underline{E_3}\right) \otimes \underline{F}\right) = (\phi_2 - \phi_1) \wedge \mathrm{ch}_g \left(\underline{F}\right),$$

and

$$d(\phi_2 - \phi_1) = \mathrm{ch}_g \left(\underline{E_2} \oplus \underline{E_3}\right) - \mathrm{ch}_g \left(\underline{E_1} \oplus \underline{E_3}\right) = \mathrm{ch}_g \left(\underline{E_2}\right) - \mathrm{ch}_g \left(\underline{E_1}\right).$$

Thus $(\underline{E_1} \otimes \underline{F}, \mathrm{ch}_g(\underline{E_1}) \wedge \psi + \phi_1 \wedge \mathrm{ch}_g(\underline{F}) - d\phi_1 \wedge \psi)$ and $(\underline{E_2} \otimes \underline{F}, \mathrm{ch}_g(\underline{E_2}) \wedge \psi + \phi_2 \wedge \mathrm{ch}_g(\underline{F}) - d\phi_2 \wedge \psi)$ are equivalent. The product (2.88) on $\widehat{K}^0_g(Y)$ is well-defined.

The commutativity of (2.88) follows from the facts that $\phi, \psi \in \Omega^{\mathrm{odd}}(Y^g, \mathbb{C})/\mathrm{Im}\, d$ and $d\phi \wedge \psi = d\psi \wedge \phi + d(\phi \wedge \psi)$.

We verify now the associativity of the product. From (2.88), we have

$$\begin{aligned} [\underline{E_1}, \phi_1] \cup \Big([\underline{F_1}, \psi_1] \cup [\underline{F}, \psi]\Big) = \Big[&\underline{E_1} \otimes \underline{F_1} \otimes \underline{F}, \\ &\left(\mathrm{ch}_g(\underline{E_1}) - d\phi_1\right) \wedge \left(\mathrm{ch}_g(\underline{F_1}) \wedge \psi + \psi_1 \wedge \mathrm{ch}_g(\underline{F}) - d\psi_1 \wedge \psi\right) + \phi_1 \wedge \mathrm{ch}_g(\underline{F_1} \otimes \underline{F})\Big], \end{aligned} \tag{2.89}$$

and

$$\begin{aligned}(2.90)\quad \Big([\underline{E_1},\phi_1]\cup[\underline{F_1},\psi_1]\Big)\cup[\underline{F},\psi] = \Big[&\underline{E_1}\otimes\underline{F_1}\otimes\underline{F},\\ &\mathrm{ch}_g(\underline{E_1}\otimes\underline{F_1})\wedge\psi - d\big(\mathrm{ch}_g(\underline{E_1})\wedge\psi_1+\phi_1\wedge\mathrm{ch}_g(\underline{F_1})-d\phi_1\wedge\psi_1\big)\wedge\psi\\ &+\big(\mathrm{ch}_g(\underline{E_1})\wedge\psi_1+\phi_1\wedge\mathrm{ch}_g(\underline{F_1})-d\phi_1\wedge\psi_1\big)\wedge\mathrm{ch}_g(\underline{F})\Big].\end{aligned}$$

Since $\mathrm{ch}_g(\cdot)$ is a closed even form and ϕ_1,ψ_1,ψ are odd forms, from (2.89) and (2.90), we have

$$(2.91)\qquad [\underline{E_1},\phi_1]\cup\big([\underline{F_1},\psi_1]\cup[\underline{F},\psi]\big)=\big([\underline{E_1},\phi_1]\cup[\underline{F_1},\psi_1]\big)\cup[\underline{F},\psi].$$

The proof of Lemma 2.15 is completed. □

Thus $(\widehat{K}_g^0(Y),+,\cup)$ is a commutative ring with unit 1.

Remark 2.16. Certainly, we can replace S^1 by any compact Lie group in Definition 2.14.

As in (2.43), if $[\underline{E},0]=[\underline{F},0]\in\widehat{K}_g^0(Y)$, from (1.30) and (2.87), we have

$$(2.92)\qquad \mathrm{ch}_g(\underline{E})=\mathrm{ch}_g(\underline{F})\in\Omega^{\bullet}(Y^g,\mathbb{C}).$$

Note that $g\in S^1$ defines a prime ideal $I(g)$ in $R(S^1)$, the representation ring of S^1, namely all characters of S^1 which vanish at g. For any $R(S^1)$-module $\mathcal{M}$, we denote by $\mathcal{M}_{I(g)}$ the module obtained from $\mathcal{M}$ by localizing at this prime ideal. An element of $R(S^1)_{I(g)}$ is a "fraction" u/v with $u,v\in R(S^1)$ and $\chi_v(g)\neq 0$, and two fractions u/v and u'/v' represent the same element of $R(S^1)_{I(g)}$ if there exists $w\in R(S^1)$ with $\chi_w(g)\neq 0$ and $wuv'=wu'v\in R(S^1)$. Elements of $\mathcal{M}_{I(g)}$ are "fractions" u/v ($u\in\mathcal{M},v\in R(S^1),\chi_v(g)\neq 0$) with a similar equivalence relation. Thus $\mathcal{M}_{I(g)}$ is a module over the local ring $R(S^1)_{I(g)}$. Since we do not distinguish the finite dimensional virtual representations and the characters of elements in $R(S^1)$, we usually write an element of $\mathcal{M}_{I(g)}$ by

$$(2.93)\qquad u/\chi \quad \text{with } u\in\mathcal{M},\chi\in\mathbb{Z}[h,h^{-1}]\quad \text{for } h\in S^1,\chi(g)\neq 0.$$

For a finite dimensional S^1-representation M, we consider the S^1-action on $Y\times M$ given by

$$(2.94)\qquad g(y,u)=(gy,gu).$$

Thus $Y\times M\to Y$ is an equivariant vector bundle over Y. We denote this equivariant vector bundle by E_M. By construction, the trivial metric h^M and the trivial connection d on E_M are naturally S^1-invariant. Let $\underline{M}=(E_M,h^M,d)$. Note that $E\mapsto E_M\otimes E$ endows the S^1-equivariant K-group $K^0_{S^1}(Y)$ of Y with the structure of an $R(S^1)$-module. From (2.87), since $\mathrm{ch}_g(\underline{M})=\chi_M(g)$, constant on Y^g, $(\underline{E},\phi)\mapsto(\underline{M}\otimes\underline{E},\chi_M(g)\cdot\phi)$ makes $\widehat{K}_g^0(Y)$ an $R(S^1)$-module.

In the following, we will denote by $\underline{'E}$ the corresponding geometric triple when forgetting the group action.

Recall that $Y^{S^1}=\{Y_\alpha^{S^1}\}_{\alpha\in\mathfrak{B}}$ is the fixed point set of the circle action and N_α is the normal bundle of $Y_\alpha^{S^1}$ in Y. We consider N_α as a complex vector bundle. By [55, Proposition 2.2],

$$(2.95)\qquad K^0_{S^1}(Y_\alpha^{S^1})\simeq R(S^1)\otimes K^0(Y_\alpha^{S^1}).$$

By (1.5), we write in the sense of (2.95),

$$\underline{\lambda_{-1}(N_\alpha^*)} \simeq \bigotimes_{v=1}^{q} \lambda_{-h^{-v}}\left(\underline{'N_{\alpha,v}^*}\right) = \bigotimes_{v=1}^{q}\left(1+\sum_{k=1}^{\operatorname{rk} N_{\alpha,v}}(-h^{-v})^k \cdot \Lambda^k\left(\underline{'N_{\alpha,v}^*}\right)\right). \tag{2.96}$$

Set

$$r_{\alpha,v} = \operatorname{rk} N_{\alpha,v}, \quad m_\alpha = \dim Y_\alpha^{S^1}. \tag{2.97}$$

By (2.65) and (2.66), we have formally,

$$\begin{aligned}\lambda_{-h^{-v}}\left(\underline{'N_{\alpha,v}^*}\right)^{-1} & \\ = (1-h^{-v})^{-r_{\alpha,v}} & \left(1+\sum_{k=1}^{\infty}(-h^{-v})^k(1-h^{-v})^{-k}\left(P_{k,+}\left(\underline{'N_{\alpha,v}^*}\right)-P_{k,-}\left(\underline{'N_{\alpha,v}^*}\right)\right)\right)\\ &= \frac{h^{vr_{\alpha,v}}}{(h^v-1)^{r_{\alpha,v}}}\left(1+\sum_{k=1}^{\infty}\frac{(-1)^k}{(h^v-1)^k}\left(P_{k,+}\left(\underline{'N_{\alpha,v}^*}\right)-P_{k,-}\left(\underline{'N_{\alpha,v}^*}\right)\right)\right).\end{aligned} \tag{2.98}$$

By Corollary 2.13, we know that for any $k > \mathcal{N}_{r_{\alpha,v},m_\alpha}$,

$$\left[P_{k,+}\left(\underline{'N_{\alpha,v}^*}\right) - P_{k,-}\left(\underline{'N_{\alpha,v}^*}\right), 0\right] = 0 \in \widehat{K}^0(Y_\alpha^{S^1}). \tag{2.99}$$

We define

$$\begin{aligned}\lambda_{-h^{-v}}\left(\underline{'N_{\alpha,v}^*}\right)_{\mathcal{N}}^{-1} &:= \frac{h^{vr_{\alpha,v}}}{(h^v-1)^{r_{\alpha,v}}}\left(1+\sum_{k=1}^{\mathcal{N}}\frac{(-1)^k}{(h^v-1)^k}\left(P_{k,+}(\underline{'N_{\alpha,v}^*})-P_{k,-}(\underline{'N_{\alpha,v}^*})\right)\right),\\ \underline{\lambda_{-1}(N_\alpha^*)_{\mathcal{N}}^{-1}} &:= \bigotimes_{v:\, r_{\alpha,v}\neq 0} \lambda_{-h^{-v}}\left(\underline{'N_{\alpha,v}^*}\right)_{\mathcal{N}}^{-1}.\end{aligned} \tag{2.100}$$

It follows from (1.6) that for $g \in S^1\backslash A$ we have $Y^g = Y^{S^1}$, thus $g^v - 1 \neq 0$ if $\operatorname{rk} N_{\alpha,v} \neq 0$.

By (2.83) and (2.99), we see that for any $\mathcal{N}, \mathcal{N}' > \sup_{\alpha,v}\mathcal{N}_{r_{\alpha,v},m_\alpha}$,

$$\left[\underline{\lambda_{-1}(N_\alpha^*)_{\mathcal{N}}^{-1}}, 0\right] = \left[\underline{\lambda_{-1}(N_\alpha^*)_{\mathcal{N}'}^{-1}}, 0\right] \in \widehat{K}_g^0(Y_\alpha^{S^1})_{I(g)}. \tag{2.101}$$

Then from (2.67), (2.96)-(2.101), for any $\mathcal{N} > \sup_{\alpha,v}\mathcal{N}_{r_{\alpha,v},m_\alpha}$, we have

$$\left[\underline{\lambda_{-1}(N_\alpha^*)}, 0\right] \cup \left[\underline{\lambda_{-1}(N_\alpha^*)_{\mathcal{N}}^{-1}}, 0\right] = 1 \in \widehat{K}_g^0(Y_\alpha^{S^1})_{I(g)}. \tag{2.102}$$

Summarizing, we obtain the following precise version of Theorem 0.2. A version for arithmetic K-group was obtained in [39, Lemma 4.5].

Theorem 2.17. *For $g \in S^1\backslash A$, $\left[\underline{\lambda_{-1}(N_\alpha^*)}, 0\right]$ is invertible in $\widehat{K}_g^0(Y_\alpha^{S^1})_{I(g)}$ and for any $\mathcal{N} > \sup_{\alpha,v}\mathcal{N}_{r_{\alpha,v},m_\alpha}$ in* (2.83)*, we have*

$$\left[\underline{\lambda_{-1}(N_\alpha^*)}, 0\right]^{-1} = \left[\underline{\lambda_{-1}(N_\alpha^*)_{\mathcal{N}}^{-1}}, 0\right] \in \widehat{K}_g^0(Y_\alpha^{S^1})_{I(g)}. \tag{2.103}$$

Remark that the lower bound $\sup_{\alpha,v}\mathcal{N}_{r_{\alpha,v},m_\alpha}$ does not depend on $g \in S^1\backslash A$.

From (2.43) and (2.99), for any $k > \mathcal{N}_{r_{\alpha,v},m_\alpha}$,

$$\operatorname{ch}\left(P_{k,+}\left(\underline{'N_{\alpha,v}^*}\right)\right) = \operatorname{ch}\left(P_{k,-}\left(\underline{'N_{\alpha,v}^*}\right)\right) \in \Omega^\bullet(Y_\alpha^{S^1}, \mathbb{R}). \tag{2.104}$$

From (2.100), we have

$$\text{(2.105)}\quad \mathrm{ch}_g\left(\underline{\lambda_{-1}(N_\alpha^*)_{\mathcal{N}}^{-1}}\right)$$
$$= \prod_{v:\, r_{\alpha,v}\neq 0} \frac{g^{v r_{\alpha,v}}}{(g^v-1)^{r_{\alpha,v}}}\left(1+\sum_{k=1}^{\mathcal{N}}\frac{(-1)^k}{(g^v-1)^k}\left(\mathrm{ch}\left(P_{k,+}\left(\underline{{}'N_{\alpha,v}^*}\right)\right)-\mathrm{ch}\left(P_{k,-}\left(\underline{{}'N_{\alpha,v}^*}\right)\right)\right)\right).$$

The following corollary follows directly from (2.92), (2.102), (2.104) and (2.105).

Corollary 2.18. *For any $\mathcal{N} > \sup_{\alpha,v}\mathcal{N}_{r_{\alpha,v},m_\alpha}$, $g\in S^1\backslash A$,*

$$\text{(2.106)}\qquad \mathrm{ch}_g\left(\underline{\lambda_{-1}(N_\alpha^*)}\right)\cdot \mathrm{ch}_g\left(\underline{\lambda_{-1}(N_\alpha^*)_{\mathcal{N}}^{-1}}\right) = 1 \in \Omega^\bullet(Y_\alpha^{S^1},\mathbb{C}).$$

3. Localization formula for equivariant η-invariants

In this section, we establish Theorems 0.4 and 0.5 by combining the analytic results on the η-invariant in Section 1 and the algebraic framework of g-equivariant differential K-theory in Section 2.4. We fix $g\in S^1\backslash A$ now. It is relatively easy to verify, by using the exact sequence on g-equivariant differential K-theory and equivariant K-theory, that the localization at $g\in S^1\backslash A$ of the restriction of g-equivariant differential K-theory from the total manifold Y to the fixed point set Y^{S^1} is an isomorphism. We can describe the inverse of this map by using the direct image for the embedding $Y^{S^1}\hookrightarrow Y$ constructed in Section 1.4 and the inverse of $\lambda_{-1}(\underline{N^*})$ constructed in Section 2.4. This result, combined with the embedding formula of η-invariants Theorem 1.5, implies that the difference of the equivariant η-invariant and its contribution on the fixed point set is the value at this element of a rational function with integral coefficients. Note that the coefficients of these rational functions depend a priori on the element $g\in S^1\backslash A$, but thanks to Theorems 1.9, 1.10, we can finally conclude that these rational functions are the same for any $g\in S^1\backslash A$. This ends the proof of our main result, Theorem 0.5.

This section is organized as follows. In Section 3.1, we establish the localization formula in g-equivariant differential K-theory. In Section 3.2, we define a direct image map in g-equivariant differential K-theory. In Section 3.3, we state Theorem 3.7, which is a precise formulation of our main result, Theorem 0.5. In Section 3.4, we establish first Theorem 0.4 by applying the embedding formula of η-invariants, Theorem 1.5, the localization and direct image map in g-equivariant differential K-theory. By using Theorems 0.4, 1.9 and 1.10, we get finally Theorem 3.7. In Section 3.5, we study the case when $Y^{S^1}=\emptyset$ and compute explicitly the equivariant reduced η-invariant when the manifold Y is the circle.

3.1. Localization in g-equivariant differential K-theory.

Let Y be a compact manifold with S^1-action. We explain first the S^1-equivariant K^1-theory on Y, and the equivariant odd Chern character for an element in the S^1-equivariant K^1-group.

Let $K^1_{S^1}(Y)$ be the S^1-equivariant K^1-group of Y. By [55, Definitions 2.7 and 2.8], we have the exact sequence

$$\text{(3.1)}\qquad 0\to K^1_{S^1}(Y)\xrightarrow{\varsigma} K^0_{S^1}(Y\times\widehat{S^1})\xrightarrow{i^*} K^0_{S^1}(Y)\to 0,$$

where $\widehat{S^1}$ is a copy of S^1 with trivial S^1-action and there exists $b \in \widehat{S^1}$ such that the map i is given by $i : Y \ni y \to (y, b) \in Y \times \widehat{S^1}$. Note that $Y \times \widehat{S^1} = Y \times \mathbb{R}/\mathbb{Z}$. We will take $b = \frac{1}{2}$ thus $i(Y) = Y \times \{\frac{1}{2}\}$.

By (3.1), an element y of $K^1_{S^1}(Y)$ can be represented as an element $x = \varsigma(y) \in K^0_{S^1}(Y \times \widehat{S^1})$ such that $i^*(x) = 0 \in K^0_{S^1}(Y)$. We write $x = W - U$, where W and U are equivariant complex vector bundles over $Y \times \widehat{S^1}$. By [55, Proposition 2.4], we may and we will assume that U is a trivial vector bundle associated with a finite dimensional S^1-representation as in (2.94). Since $i^*(x) = W|_{Y\times\{1/2\}} - U|_{Y\times\{1/2\}} = 0 \in K^0_{S^1}(Y)$, by adding on U a trivial vector bundle associated with a finite dimensional S^1-representation, we may and we will assume that $W|_{Y\times\{1/2\}}$ is a trivial vector bundle over $Y \times \{1/2\}$ associated with a finite dimensional S^1-representation M as in (2.94) and

$$U = \left(Y \times \widehat{S^1}\right) \times M. \tag{3.2}$$

Since $[0, 1]$ is contractible, there exists an S^1-equivariant morphism $F \in \mathscr{C}^\infty(Y, \mathrm{Aut}(M))$ such that

$$W = \big(Y \times [0, 1]\big) \times M/\sim_F, \tag{3.3}$$

where $\sim_F$ is the gluing map: $(y, 1, m) \sim_F (y, 0, F(y)m)$ for $y \in Y$, $m \in M$. Then it induces an equivariant vector bundle isomorphism $F : E_M \to E_M$, where the S^1-equivariant vector bundle E_M on Y is defined as in (2.94) by

$$E_M := Y \times M. \tag{3.4}$$

When we restrict the above construction on Y^{S^1}, as S^1 acts trivially on Y^{S^1}, we have

$$\begin{gathered} U|_{Y^{S^1}\times\widehat{S^1}} = \left(Y^{S^1} \times \widehat{S^1}\right) \times M, \quad W|_{Y^{S^1}\times\widehat{S^1}} = \left(Y^{S^1} \times [0, 1]\right) \times M/\sim_F, \\ E_M|_{Y^{S^1}} = Y^{S^1} \times M, \end{gathered} \tag{3.5}$$

where S^1 acts only on the factor M.

Let ∇ be an S^1-invariant connection on E_M. Then $F^*\nabla_\cdot\cdot = F^{-1}\nabla_\cdot(F\cdot)$ is also an S^1-invariant connection on E_M. From (3.3),

$$\nabla^W = dt \wedge \frac{\partial}{\partial t} + (1 - t)\nabla + tF^*\nabla = dt \wedge \frac{\partial}{\partial t} + \nabla + tF^{-1}\nabla F \tag{3.6}$$

is a well-defined S^1-invariant connection on W over $Y \times \widehat{S^1}$.

Recall that the equivariant Chern character form $\mathrm{ch}_g(\underline{E})$ and the equivariant Chern-Simons class $\widetilde{\mathrm{ch}}_g(\underline{E_0}, \underline{E_1})$ defined in (1.19) and (1.29) depend only on the connections, not on the metrics. We often denote the equivariant Chern character form by $\mathrm{ch}_g(E, \nabla^E)$ and the equivariant Chern-Simons class by $\widetilde{\mathrm{ch}}_g(E, \nabla^{E_0}, \nabla^{E_1})$.

Let ∇^U be the trivial connection on U. It is naturally S^1-invariant. For $g \in S^1\backslash A$, the odd equivariant Chern character for $y \in K^1_{S^1}(Y)$ as above, is defined by

$$\begin{aligned} \mathrm{ch}_g(y) &:= \left[\int_{\widehat{S^1}} \left(\mathrm{ch}_g(W, \nabla^W) - \mathrm{ch}_g(U, \nabla^U)\right)\right] \\ &= \left[\int_{\widehat{S^1}} \mathrm{ch}_g(W, \nabla^W)\right] \in H^{\mathrm{odd}}(Y^{S^1}, \mathbb{C}) \subset \Omega^{\mathrm{odd}}(Y^{S^1}, \mathbb{C})/\mathrm{Im}\, d, \end{aligned} \tag{3.7}$$

where the fiberwise integral $\int_{\widehat{S^1}}$ is normalized such that $\int_{\widehat{S^1}} \tilde{\pi}^*\alpha\wedge\beta = \alpha\wedge\int_{\widehat{S^1}}\beta$ for the obvious projection $\tilde{\pi} : Y^{S^1} \times \widehat{S^1} \to Y^{S^1}$ and $\alpha \in \Omega^\bullet(Y^{S^1})$, $\beta \in \Omega^\bullet(Y^{S^1} \times \widehat{S^1})$. As $[\mathrm{ch}_g(W, \nabla^W)] \in H^\bullet(Y^{S^1} \times \widehat{S^1}, \mathbb{C})$ does not depend on the choice of ∇, thus $\mathrm{ch}_g(y)$ also does not depend on ∇. From (1.19), (1.28), (3.6) and (3.7), we have

$$\begin{aligned}(3.8)\quad \mathrm{ch}_g(y) = -\left[\int_{[0,1]} \left\{\mathrm{ch}_g\left([0,1]\times E_M, dt\wedge\frac{\partial}{\partial t} + (1-t)\nabla + tF^*\nabla\right)\right\}^{dt} dt\right] \\ = -\widetilde{\mathrm{ch}}_g(E_M, \nabla, F^*\nabla) \in \Omega^{\mathrm{odd}}(Y^{S^1}, \mathbb{C})/\mathrm{Im}\, d.\end{aligned}$$

If we choose ∇ as the trivial connection d on E_M, by (3.6), the curvature R^W of ∇^W is given by

$$R^W = \left(\nabla^W\right)^2 = dt\wedge(F^{-1}dF) - t(1-t)(F^{-1}dF)^2. \tag{3.9}$$

From (1.19), (1.28), (3.8) and (3.9), we calculate that

$$\mathrm{ch}_g(y) = \sum_{n\geq 0} \frac{1}{(2i\pi)^{n+1}}\frac{n!}{(2n+1)!}\left[\mathrm{Tr}\left[g(F^{-1}dF)^{2n+1}\right]\right]. \tag{3.10}$$

This is just the equivariant version of the odd Chern character in [33] and [60, (1.50)].

From (3.1), $K^1_{S^1}(Y)$ is an $R(S^1)$-module. Moreover, $\phi \mapsto \chi_M(g)\cdot\phi$ makes $\Omega^{\mathrm{odd}}(Y^{S^1}, \mathbb{C})/\mathrm{Im}\, d$ an $R(S^1)$-module.

The following Proposition is the g-equivariant extension of the corresponding results in [25, Proposition 2.20], [26, Proposition 2.24] and [32, (2.21)], which are analogues of Gillet-Soulé's result [35, Theorem 6.2] in arithmetic K-theory.

Proposition 3.1. *If $g \in S^1\backslash A$, we have the exact sequence of $R(S^1)$-modules,*

$$K^1_{S^1}(Y) \xrightarrow{\mathrm{ch}_g} \Omega^{\mathrm{odd}}(Y^{S^1}, \mathbb{C})/\mathrm{Im}\, d \xrightarrow{a} \widehat{K}^0_g(Y) \xrightarrow{\tau} K^0_{S^1}(Y) \longrightarrow 0, \tag{3.11}$$

where

$$a(\phi) = [0, \phi], \quad \tau([\underline{E}, \phi]) = [E]. \tag{3.12}$$

Proof. It is obvious from Definition 2.14 that τ is surjective and $\tau\circ a = 0$.

If $x \in \mathrm{Ker}\,\tau$, it is easy to see from Definition 2.14 that $x \in \mathrm{Im}(a)$.

Now we prove $a\circ\mathrm{ch}_g = 0$. For $y \in K^1_{S^1}(Y)$, we can construct equivariant vector bundle E_M over Y as in (3.4). Let h^M be the metric on E_M induced by an S^1-invariant metric on M via (3.4) and ∇ be an S^1-invariant Hermitian connection on (E_M, h^M). By (3.5) and (3.8), we have

$$\begin{aligned}(3.13)\quad a(\mathrm{ch}_g(y)) = \left[0, -\widetilde{\mathrm{ch}}_g\left(E_M, \nabla|_{Y^{S^1}}, F^*\nabla|_{Y^{S^1}}\right)\right] = [0, -\widetilde{\mathrm{ch}}_g(E_M, \nabla, F^*\nabla)] \\ = [(E_M, h^M, \nabla), 0] - [(E_M, h^M, \nabla), \widetilde{\mathrm{ch}}_g((E_M, h^M, \nabla), (E_M, F^*h^M, F^*\nabla))].\end{aligned}$$

By Definition 2.14, $((E_M, h^M, \nabla), 0)$ and $((E_M, h^M, \nabla), \widetilde{\mathrm{ch}}_g((E_M, h^M, \nabla), (E_M, F^*h^M, F^*\nabla)))$ are equivalent under the equivariant vector bundle isomorphism F over Y. That is, $a(\mathrm{ch}_g(y)) = 0 \in \widehat{K}^0_g(Y)$.

At last, we prove $\operatorname{Ker} a \subseteq \operatorname{Im} \mathrm{ch}_g$. For $\phi' \in \operatorname{Ker} a$, i.e., $[0, \phi'] = 0 \in \widehat{K}_g^0(Y)$. By Definition 2.14, there exists an equivariant geometric triple $\underline{E'} = (E', h^{E'}, \nabla^{E'})$ and an equivariant vector bundle isomorphism over Y:

$$\Phi' : E' \to E' \quad \text{such that} \quad \phi' = -\widetilde{\mathrm{ch}}_g\left(\underline{E'}, \Phi'^* \underline{E'}\right). \tag{3.14}$$

By [55, Proposition 2.4], there exists an S^1-vector bundle E on Y such that $E \oplus E' = E_M$ where M is a finite dimensional S^1-representation. Set

$$\Phi : E \oplus E' \to E \oplus E', \quad (u, v) \mapsto (u, \Phi'(v)). \tag{3.15}$$

Let h^E be an S^1-invariant Hermitian metric on E and ∇^E be an S^1-invariant Hermitian connection on (E, h^E). Then $\widetilde{\mathrm{ch}}_g\left(\underline{E} \oplus \underline{E'}, \Phi^*\left(\underline{E} \oplus \underline{E'}\right)\right) = \widetilde{\mathrm{ch}}_g\left(\underline{E'}, \Phi'^* \underline{E'}\right)$. As in (3.6),

$$\nabla^W = dt \wedge \frac{\partial}{\partial t} + (1-t)(\nabla^E \oplus \nabla^{E'}) + t\Phi^*(\nabla^E \oplus \nabla^{E'}) \tag{3.16}$$

is an S^1-invariant connection on $W = Y \times [0,1] \times M / \sim_\Phi$. Therefore, by (1.29), (3.6) and (3.8), modulo exact forms, we have

$$\begin{aligned} -\widetilde{\mathrm{ch}}_g\left(\underline{E} \oplus \underline{E'}, \Phi^*\left(\underline{E} \oplus \underline{E'}\right)\right) &= -\widetilde{\mathrm{ch}}_g\Big(E \oplus E', (\nabla^E \oplus \nabla^{E'})|_{Y^{S^1}}, \Phi^*(\nabla^E \oplus \nabla^{E'})|_{Y^{S^1}}\Big) \\ &= \int_{\widehat{S^1}} \mathrm{ch}_g(W, \nabla^W). \end{aligned} \tag{3.17}$$

From (3.2), (3.7), (3.14), (3.15) and (3.17), Φ defines an element $y \in K^1_{S^1}(Y)$ by $\varsigma(y) = W - U$ and $\phi' = \mathrm{ch}_g(y)$.

It is obvious that the $R(S^1)$-action commutes with ch_g, a and τ.

The proof of Proposition 3.1 is completed. □

Let $\iota : Y^{S^1} \to Y$ be the canonical embedding. Let

$$\hat{\iota}^* : \widehat{K}_g^0(Y)_{I(g)} \to \widehat{K}_g^0(Y^{S^1})_{I(g)}, \quad [\underline{E}, \phi]/\chi \to [\underline{E}|_{Y^{S^1}}, \phi]/\chi, \tag{3.18}$$

be the induced homomorphism by restriction.

The following localization theorem, which is the differential K-theory version of the Atiyah-Segal localization theorem in topological K-theory [6, Theorem 1.1], is inspired by [26, Theorem 3.27] and [57, Theorem 5.5].

Theorem 3.2 (Localization Theorem). *For $g \in S^1 \backslash A$, the restriction map $\hat{\iota}^* : \widehat{K}_g^0(Y)_{I(g)} \to \widehat{K}_g^0(Y^{S^1})_{I(g)}$ in (3.18) is an $R(S^1)_{I(g)}$-module isomorphism.*

Proof. Since localization preserves exact sequences [3, Proposition 3.3], from Proposition 3.1, we get an exact sequence of $R(S^1)_{I(g)}$-modules

$$K^1_{S^1}(Y)_{I(g)} \xrightarrow{\mathrm{ch}_g} \left(\Omega^{\mathrm{odd}}(Y^{S^1}, \mathbb{C})/\operatorname{Im} d\right)_{I(g)} \xrightarrow{a} \widehat{K}_g^0(Y)_{I(g)} \xrightarrow{\tau} K^0_{S^1}(Y)_{I(g)} \longrightarrow 0. \tag{3.19}$$

Replacing Y by Y^{S^1}, since $(Y^{S^1})^{S^1} = Y^{S^1}$, we get an exact sequence of $R(S^1)_{I(g)}$-modules

$$K^1_{S^1}(Y^{S^1})_{I(g)} \xrightarrow{\mathrm{ch}_g} \left(\Omega^{\mathrm{odd}}(Y^{S^1}, \mathbb{C})/\operatorname{Im} d\right)_{I(g)} \xrightarrow{a} \widehat{K}_g^0(Y^{S^1})_{I(g)} \xrightarrow{\tau} K^0_{S^1}(Y^{S^1})_{I(g)} \longrightarrow 0. \tag{3.20}$$

Furthermore, we have the commutative diagram

$$
\begin{tikzcd}
K^1_{S^1}(Y)_{I(g)} \arrow[r, "\mathrm{ch}_g"] \arrow[d, "\iota^*"] & \left(\Omega^{\mathrm{odd}}(Y^{S^1},\mathbb{C})/\mathrm{Im}\, d\right)_{I(g)} \arrow[r, "a"] \arrow[d, "\mathrm{Id}"] & \widehat{K}^0_g(Y)_{I(g)} \arrow[r, "\tau"] \arrow[d, "\hat{\iota}^*"] & K^0_{S^1}(Y)_{I(g)} \arrow[r] \arrow[d, "\iota^*"] & 0 \\
K^1_{S^1}(Y^{S^1})_{I(g)} \arrow[r, "\mathrm{ch}_g"] & \left(\Omega^{\mathrm{odd}}(Y^{S^1},\mathbb{C})/\mathrm{Im}\, d\right)_{I(g)} \arrow[r, "a"] & \widehat{K}^0_g(Y^{S^1})_{I(g)} \arrow[r, "\tau"] & K^0_{S^1}(Y^{S^1})_{I(g)} \arrow[r] & 0.
\end{tikzcd}
\tag{3.21}
$$

Here $\iota^* : K^*_{S^1}(Y)_{I(g)} \to K^*_{S^1}(Y^{S^1})_{I(g)}$ is the $R(S^1)_{I(g)}$-module map induced by ι. From (3.1) for Y and Y^{S^1}, we have the commutative diagram

$$
\begin{tikzcd}
0 \arrow[r] & K^1_{S^1}(Y)_{I(g)} \arrow[r, "\varsigma"] \arrow[d, "\iota^*"] & K^0_{S^1}(Y\times \widehat{S^1})_{I(g)} \arrow[r, "i^*"] \arrow[d, "\iota^*"] & K^0_{S^1}(Y)_{I(g)} \arrow[r] \arrow[d, "\iota^*"] & 0 \\
0 \arrow[r] & K^1_{S^1}(Y^{S^1})_{I(g)} \arrow[r, "\varsigma"] & K^0_{S^1}(Y^{S^1}\times \widehat{S^1})_{I(g)} \arrow[r, "i^*"] & K^0_{S^1}(Y^{S^1})_{I(g)} \arrow[r] & 0.
\end{tikzcd}
\tag{3.22}
$$

Using localization in topological K-theory [6, Theorem 1.1], ι^* is an isomorphism on $K^0_{S^1}(\cdot)_{I(g)}$. By the five lemma on (3.22), ι^* is an isomorphism on $K^1_{S^1}(\cdot)_{I(g)}$. Then by the five lemma on (3.21), $\hat{\iota}^*$ in (3.21) is an isomorphism.

The proof of Theorem 3.2 is completed. □

As the restriction map $\hat{\iota}^*$ in (3.18) is an isomorphism, it is a natural question to find explicitly its inverse. We solve this problem by combining the construction of the geometric direct image for embeddings in Section 1.4 and the invertibility of the element $[\lambda_{-1}(\underline{N^*}),0]$ in $\widehat{K}^0_g(Y^{S^1})_{I(g)}$ obtained in Theorem 2.17.

In the following definition we adopt the notation in Section 1.4.

Definition 3.3. For $g\in S^1\backslash A$, the direct image map

$$
\hat{\iota}_! : \widehat{K}^0_g(Y^{S^1})_{I(g)} \to \widehat{K}^0_g(Y)_{I(g)}
\tag{3.23}
$$

is defined by

$$
\hat{\iota}_!\left(\left[\underline{\mu},\phi\right]/\chi\right) = \left[\underline{\xi_+}, \mathrm{ch}_g\left(\underline{\Lambda^{\mathrm{even}}(N^*)}\right)\wedge\phi\right]/\chi - \left[\underline{\xi_-}, \mathrm{ch}_g\left(\underline{\Lambda^{\mathrm{odd}}(N^*)}\right)\wedge\phi\right]/\chi.
\tag{3.24}
$$

Theorem 3.4. *The direct image map $\hat{\iota}_!$ is a well-defined isomorphism and*

$$
\hat{\iota}^*\circ\hat{\iota}_! = \left[\underline{\lambda_{-1}(N^*)},0\right]\cup : \widehat{K}^0_g(Y^{S^1})_{I(g)} \xrightarrow{\sim} \widehat{K}^0_g(Y^{S^1})_{I(g)}.
\tag{3.25}
$$

Thus the inverse map of $\hat{\iota}^* : \widehat{K}^0_g(Y)_{I(g)} \to \widehat{K}^0_g(Y^{S^1})_{I(g)}$ in (3.18) is given by $\hat{\iota}_!\circ\left[\underline{\lambda_{-1}(N^*)},0\right]^{-1}\cup$.

Proof. From the construction of $\underline{\xi_\pm}$ in (1.41)-(1.44), we have

$$\begin{aligned}(3.26)\quad &\hat{\imath}^*\left\{\left[\underline{\xi_+}, \mathrm{ch}_g\left(\underline{\Lambda^{\mathrm{even}}(N^*)}\right)\wedge\phi\right]/\chi - \left[\underline{\xi_-}, \mathrm{ch}_g\left(\underline{\Lambda^{\mathrm{odd}}(N^*)}\right)\wedge\phi\right]/\chi\right\}\\
&= \sum_{\alpha\in\mathfrak{B}}\left[\left(\underline{\Lambda^{\mathrm{even}}(N^*_\alpha)}\otimes\underline{\mu_\alpha}\right)\oplus\underline{F_\alpha}, \mathrm{ch}_g\left(\underline{\Lambda^{\mathrm{even}}(N^*_\alpha)}\right)\wedge\phi\right]/\chi\\
&\quad - \sum_{\alpha\in\mathfrak{B}}\left[\left(\underline{\Lambda^{\mathrm{odd}}(N^*_\alpha)}\otimes\underline{\mu_\alpha}\right)\oplus\underline{F_\alpha}, \mathrm{ch}_g\left(\underline{\Lambda^{\mathrm{odd}}(N^*_\alpha)}\right)\wedge\phi\right]/\chi\\
&= \left[\underline{\Lambda^{\mathrm{even}}(N^*)}\otimes\underline{\mu}, \mathrm{ch}_g\left(\underline{\Lambda^{\mathrm{even}}(N^*)}\right)\wedge\phi\right]/\chi\\
&\quad - \left[\underline{\Lambda^{\mathrm{odd}}(N^*)}\otimes\underline{\mu}, \mathrm{ch}_g\left(\underline{\Lambda^{\mathrm{odd}}(N^*)}\right)\wedge\phi\right]/\chi.\end{aligned}$$

Since $\underline{\lambda_{-1}(N^*)} = \underline{\Lambda^{\mathrm{even}}(N^*)} - \underline{\Lambda^{\mathrm{odd}}(N^*)}$, by (2.88) and (3.26), we have

$$\begin{aligned}(3.27)\quad &\left[\underline{\lambda_{-1}(N^*)}, 0\right]\cup[\underline{\mu},\phi]/\chi\\
&= \hat{\imath}^*\left\{\left[\underline{\xi_+}, \mathrm{ch}_g\left(\underline{\Lambda^{\mathrm{even}}(N^*)}\right)\wedge\phi\right]/\chi - \left[\underline{\xi_-}, \mathrm{ch}_g\left(\underline{\Lambda^{\mathrm{odd}}(N^*)}\right)\wedge\phi\right]/\chi\right\}.\end{aligned}$$

Therefore, we have

$$(3.28)\qquad \hat{\imath}^*\circ\hat{\imath}_! = \left[\underline{\lambda_{-1}(N^*)}, 0\right]\cup .$$

By Theorem 3.2, $\hat{\imath}^* : \widehat{K}^0_g(Y)_{I(g)} \xrightarrow{\sim} \widehat{K}^0_g(Y^{S^1})_{I(g)}$ is an $R(S^1)_{I(g)}$-module isomorphism. From Theorem 2.17,

$$(3.29)\qquad (\hat{\imath}^*)^{-1}\circ\left[\underline{\lambda_{-1}(N^*)}, 0\right]\cup : \widehat{K}^0_g(Y^{S^1})_{I(g)} \xrightarrow{\sim} \widehat{K}^0_g(Y)_{I(g)}$$

is a well-defined isomorphism. Equations (3.28) and (3.29) imply that $\hat{\imath}_!$ in (3.24) is a well-defined isomorphism and (3.25) holds. □

3.2. **Direct image in g-equivariant differential K-theory.** In the remainder of this section, Y is an odd dimensional compact oriented manifold and has an S^1-equivariant spin^c structure.

Note that for $g\in S^1$, $\mathbb{Q}_g\subset\mathbb{C}$ was defined in (0.13) and $\mathrm{ch}_g(R(S^1)_{I(g)}) = \mathbb{Q}_g$.

Definition and Theorem 3.5. *Let $g\in S^1$ be fixed. For an equivariant geometric triple $\underline{E}$, $\phi\in\Omega^{\mathrm{odd}}(Y^g,\mathbb{C})/\mathrm{Im}\, d$, $\chi\in R(S^1)$ such that $\chi(g)\neq 0$, the map*

$$(3.30)\qquad \widehat{f_{Y!}}\big((\underline{E},\phi)/\chi\big) := \chi(g)^{-1}\left(-\int_{Y^g}\mathrm{Td}_g(\nabla^{TY},\nabla^L)\wedge\phi + \bar{\eta}_g(\underline{TY},\underline{L},\underline{E})\right)$$

defines a direct image map $\widehat{f_{Y!}} : \widehat{K}^0_g(Y)_{I(g)} \to \mathbb{C}/\mathbb{Q}_g$.

Note that for $g=1$, the family version of (3.30) is [32, Definition 3.12]. In [39, Proposition 4.3] Köhler-Roessler defined an arithmetic K-theory version of (3.30).

Proof. For an S^1-equivariant vector bundle isomorphism $\Phi : E\to E$ over Y, we have by Definition 1.2,

$$(3.31)\qquad \bar{\eta}_g(\underline{TY},\underline{L},\Phi^*\underline{E}) = \bar{\eta}_g(\underline{TY},\underline{L},\underline{E}).$$

For any finite dimensional S^1-representation M and triples $\underline{E}$, $\underline{E_1}$, $\underline{E_2}$, we have from Definition 1.2,

$$\begin{aligned}\bar{\eta}_g(\underline{TY}, \underline{L}, \underline{M} \otimes \underline{E}) &= \chi_M(g) \cdot \bar{\eta}_g(\underline{TY}, \underline{L}, \underline{E}),\\ \bar{\eta}_g(\underline{TY}, \underline{L}, \underline{E_1} \oplus \underline{E_2}) &= \bar{\eta}_g(\underline{TY}, \underline{L}, \underline{E_1}) + \bar{\eta}_g(\underline{TY}, \underline{L}, \underline{E_2}).\end{aligned} \tag{3.32}$$

For cycles $(\underline{E_1}, \phi_1)/\chi_1$ and $(\underline{E_2}, \phi_2)/\chi_2$ of $\widehat{K}^0_g(Y)_{I(g)}$ we have from (3.30) and (3.32),

$$\widehat{f_{Y!}}\big((\underline{E_1}, \phi_1)/\chi_1 + (\underline{E_2}, \phi_2)/\chi_2\big) = \widehat{f_{Y!}}\big((\underline{E_1}, \phi_1)/\chi_1\big) + \widehat{f_{Y!}}\big((\underline{E_2}, \phi_2)/\chi_2\big). \tag{3.33}$$

If $[\underline{E_2} - \underline{E_1}, \phi]/\chi = 0 \in \widehat{K}^0_g(Y)_{I(g)}$, there exists a finite dimensional S^1-representation M such that $[\underline{M} \otimes (\underline{E_2} - \underline{E_1}), \chi_M(g)\phi] = 0 \in \widehat{K}^0_g(Y)$ and $\chi_M(g) \neq 0$. Thus from Definition 2.14, there exist $\underline{E_3}$ and an equivariant vector bundle isomorphism $\Phi : (M \otimes E_1) \oplus E_3 \to (M \otimes E_2) \oplus E_3$ such that

$$\phi = \chi_M(g)^{-1}\widetilde{\mathrm{ch}}_g((\underline{M} \otimes \underline{E_1}) \oplus \underline{E_3}, \Phi^*((\underline{M} \otimes \underline{E_2}) \oplus \underline{E_3})). \tag{3.34}$$

From the variation formula (1.31), (3.31) and (3.32), there exists $\alpha_g \in \mathbb{Z}[g, g^{-1}] := \{f(g) \in \mathbb{C} : f \in \mathbb{Z}[x, x^{-1}]\}$ such that

$$\begin{aligned}&\bar{\eta}_g(\underline{TY}, \underline{L}, \underline{M} \otimes \underline{E_2}) - \bar{\eta}_g(\underline{TY}, \underline{L}, \underline{M} \otimes \underline{E_1})\\ &\quad= \bar{\eta}_g(\underline{TY}, \underline{L}, (\underline{M} \otimes \underline{E_2}) \oplus \underline{E_3}) - \bar{\eta}_g(\underline{TY}, \underline{L}, (\underline{M} \otimes \underline{E_1}) \oplus \underline{E_3})\\ &\quad= \bar{\eta}_g(\underline{TY}, \underline{L}, \Phi^*((\underline{M} \otimes \underline{E_2}) \oplus \underline{E_3})) - \bar{\eta}_g(\underline{TY}, \underline{L}, (\underline{M} \otimes \underline{E_1}) \oplus \underline{E_3})\\ &\quad= \int_{Y^g} \mathrm{Td}_g(\nabla^{TY}, \nabla^L)\widetilde{\mathrm{ch}}_g\Big((\underline{M} \otimes \underline{E_1}) \oplus \underline{E_3}, \Phi^*((\underline{M} \otimes \underline{E_2}) \oplus \underline{E_3})\Big) + \alpha_g.\end{aligned} \tag{3.35}$$

From (3.30), (3.32), (3.34) and (3.35), we have

$$\begin{aligned}&\widehat{f_{Y!}}\big((\underline{E_2} - \underline{E_1}, \phi)/\chi\big)\\ &\quad= \chi(g)^{-1}\left[\bar{\eta}_g(\underline{TY}, \underline{L}, \underline{E_2}) - \bar{\eta}_g(\underline{TY}, \underline{L}, \underline{E_1}) - \int_{Y^g} \mathrm{Td}_g(\nabla^{TY}, \nabla^L)\phi\right]\\ &\quad= \chi(g)^{-1}\chi_M(g)^{-1}\Big[\bar{\eta}_g(\underline{TY}, \underline{L}, \underline{M} \otimes \underline{E_2}) - \bar{\eta}_g(\underline{TY}, \underline{L}, \underline{M} \otimes \underline{E_1})\\ &\quad\quad - \int_{Y^g} \mathrm{Td}_g(\nabla^{TY}, \nabla^L)\widetilde{\mathrm{ch}}_g\Big((\underline{M} \otimes \underline{E_1}) \oplus \underline{E_3}, \Phi^*((\underline{M} \otimes \underline{E_2}) \oplus \underline{E_3})\Big)\Big]\\ &\quad= \chi(g)^{-1}\chi_M(g)^{-1} \cdot \alpha_g \in \mathbb{Q}_g.\end{aligned} \tag{3.36}$$

The proof of Theorem 3.5 is completed. □

3.3. **Main result: Theorem 0.5.** Recall that the orientation of $Y^{S^1}_\alpha$ is given in Section 1.3. From (2.100), there exist equivariant geometric triples $\underline{\mu_{\alpha,\mathcal{N},+}}$ and $\underline{\mu_{\alpha,\mathcal{N},-}}$ such that

$$\underline{\lambda_{-1}(N^*_\alpha)^{-1}_{\mathcal{N}}} = \prod_{v:\, r_{\alpha,v}\neq 0} (h^v - 1)^{-r_{\alpha,v}-\mathcal{N}}\Big(\underline{\mu_{\alpha,\mathcal{N},+}} - \underline{\mu_{\alpha,\mathcal{N},-}}\Big). \tag{3.37}$$

In (3.37), we identify $f(h)\cdot\underline{F}$ with $\underline{M_f\otimes F}$ for triple $\underline{F}$, $f\in\mathbb{Z}[x]$ and virtual S^1-representation M_f associated with f. For $g\in S^1\backslash A$, we define

$$\begin{aligned}(3.38)\quad \bar\eta_g\left(\underline{TY^{S^1}_\alpha},\underline{L_\alpha},\underline{\lambda_{-1}(N^*_\alpha)^{-1}_{\mathcal{N}}}\otimes\underline{E}|_{Y^{S^1}_\alpha}\right)&\\ =\prod_{v:N_{\alpha,v}\neq 0}(g^v-1)^{-r_{\alpha,v}-\mathcal{N}}\Big[\bar\eta_g\left(\underline{TY^{S^1}_\alpha},\underline{L_\alpha},\underline{\mu_{\alpha,\mathcal{N},+}}\otimes\underline{E}|_{Y^{S^1}_\alpha}\right)&\\ -\bar\eta_g\left(\underline{TY^{S^1}_\alpha},\underline{L_\alpha},\underline{\mu_{\alpha,\mathcal{N},-}}\otimes\underline{E}|_{Y^{S^1}_\alpha}\right)\Big]&.\end{aligned}$$

Remark 3.6. Note that from (2.100) and (3.37),

$$\mu_{\alpha,\mathcal{N},\pm}=\bigoplus_{k\geq 0}\xi_{\alpha,k,\pm}\in K^0_{S^1}(Y^{S^1}_\alpha) \tag{3.39}$$

and S^1 acts on $\xi_{\alpha,k}$ with weight k. If S^1 acts on L by sending $g\in S^1$ to g^{l_α} ($l_\alpha\in\mathbb{Z}$) on $Y^{S^1}_\alpha$, then by [47, p139] and (2.97),

$$\sum_v v\,r_{\alpha,v}+l_\alpha=0 \mod(2). \tag{3.40}$$

Now by (1.18), (1.35), (3.39) and (3.40), for $g\in S^1$, we have

$$\begin{aligned}(3.41)\quad \bar\eta_g\left(\underline{TY^{S^1}_\alpha},\underline{L_\alpha},\underline{\mu_{\alpha,\mathcal{N},+}}\otimes\underline{E}|_{Y^{S^1}_\alpha}\right)-\bar\eta_g\left(\underline{TY^{S^1}_\alpha},\underline{L_\alpha},\underline{\mu_{\alpha,\mathcal{N},-}}\otimes\underline{E}|_{Y^{S^1}_\alpha}\right)&\\ =g^{-\frac12\sum_v v\,r_{\alpha,v}+\frac12 l_\alpha}\sum_{k\geq 0,v}g^{k+v}\Big[\bar\eta\left(\underline{TY^{S^1}_\alpha},\underline{L_\alpha},\underline{\xi_{\alpha,k,+}}\otimes\underline{E}_v\right)&\\ -\bar\eta\left(\underline{TY^{S^1}_\alpha},\underline{L_\alpha},\underline{\xi_{\alpha,k,-}}\otimes\underline{E}_v\right)\Big]&.\end{aligned}$$

From (2.83) and (2.103), set

$$\mathcal{N}_0=\sup_{\alpha,v}\mathcal{N}_{r_{\alpha,v},m_\alpha}. \tag{3.42}$$

Now we state our main result of this paper, which is a precise formulation of Theorem 0.5.

Theorem 3.7. *For any $\mathcal{N},\mathcal{N}'\in\mathbb{N}$ and $\mathcal{N}'>\mathcal{N}>\mathcal{N}_0$, for any equivariant geometric triple $\underline{E}$ on Y, the functions on $S^1\backslash A$,*

$$\begin{aligned}(3.43)\quad P_{\mathcal{N},\mathcal{N}'}(g):=\bar\eta_g\left(\underline{TY^{S^1}_\alpha},\underline{L_\alpha},\underline{\lambda_{-1}(N^*_\alpha)^{-1}_{\mathcal{N}'}}\otimes\underline{E}|_{Y^{S^1}_\alpha}\right)&\\ -\bar\eta_g\left(\underline{TY^{S^1}_\alpha},\underline{L_\alpha},\underline{\lambda_{-1}(N^*_\alpha)^{-1}_{\mathcal{N}}}\otimes\underline{E}|_{Y^{S^1}_\alpha}\right)&\end{aligned}$$

and

$$Q_{\mathcal{N}}(g):=\bar\eta_g\left(\underline{TY},\underline{L},\underline{E}\right)-\sum_\alpha\bar\eta_g\left(\underline{TY^{S^1}_\alpha},\underline{L_\alpha},\underline{\lambda_{-1}(N^*_\alpha)^{-1}_{\mathcal{N}}}\otimes\underline{E}|_{Y^{S^1}_\alpha}\right) \tag{3.44}$$

are restrictions of rational functions on S^1 with integral coefficients which do not have poles on $S^1\backslash A$.

3.4. **A proof of Theorems 0.4 and 3.7.** Let $\widehat{f_{Y^{S^1}!}}$ be the direct image map $\widehat{f_{Y^{S^1}!}}$: $\widehat{K}_g^0(Y^{S^1})_{I(g)} \to \mathbb{C}/\mathbb{Q}_g$ defined in Definition 3.5. Explicitly, for any $[\underline{E}, \phi]/\chi \in \widehat{K}_g^0(Y^{S^1})_{I(g)}$,

$$\widehat{f_{Y^{S^1}!}}\big([\underline{E}, \phi]/\chi\big) := \chi(g)^{-1} \sum_{\alpha} \Big[- \int_{Y_\alpha^{S^1}} \mathrm{Td}_g(\nabla^{TY_\alpha^{S^1}}, \nabla^{L_\alpha}) \wedge \phi + \bar{\eta}_g\big(\underline{TY_\alpha^{S^1}}, \underline{L_\alpha}, \underline{E}\big)\Big] \mod \mathbb{Q}_g. \tag{3.45}$$

Lemma 3.8. *For any $\mathcal{N}, \mathcal{N}' \in \mathbb{N}$ and $\mathcal{N} > \mathcal{N}' > \mathcal{N}_0$, $g \in S^1 \backslash A$, we have $P_{\mathcal{N},\mathcal{N}'}(g) \in \mathbb{Q}_g$.*

Proof. From (2.101) and (3.42), for any $\mathcal{N}, \mathcal{N}' > \mathcal{N}_0$, $g \in S^1 \backslash A$,

$$\Big[\underline{\lambda_{-1}(N_\alpha^*)_{\mathcal{N}}^{-1}} \otimes \underline{E}|_{Y_\alpha^{S^1}}, 0\Big] = \Big[\underline{\lambda_{-1}(N_\alpha^*)_{\mathcal{N}'}^{-1}} \otimes \underline{E}|_{Y_\alpha^{S^1}}, 0\Big] \in \widehat{K}_g^0(Y_\alpha^{S^1})_{I(g)}. \tag{3.46}$$

Thus Lemma 3.8 follows directly from Definition and Theorem 3.5. □

Observe that

$$\widehat{K}_g^0(Y^{S^1})_{I(g)} = \bigoplus_{\alpha \in \mathfrak{B}} \widehat{K}_g^0(Y_\alpha^{S^1})_{I(g)}. \tag{3.47}$$

From Theorem 2.17, for $g \in S^1 \backslash A$, $\Big[\underline{\lambda_{-1}(N_\alpha^*)}, 0\Big]$ is invertible in $\widehat{K}_g^0(Y_\alpha^{S^1})_{I(g)}$. We denote by

$$\Big[\underline{\lambda_{-1}(N^*)}, 0\Big]^{-1} = \bigoplus_{\alpha} \Big[\underline{\lambda_{-1}(N_\alpha^*)}, 0\Big]^{-1} \in \widehat{K}_g^0(Y^{S^1})_{I(g)}. \tag{3.48}$$

Proof of Theorem 0.4. The first part of Theorem 0.4 is Theorem 3.5. From Theorem 3.2, for $g \in S^1 \backslash A$, the restriction map $\hat{\imath}^*$ in (3.18) is an $R(S^1)_{I(g)}$-module isomorphism.

For $g \in S^1 \backslash A$, by Theorem 1.5, (1.38), (3.24), (3.30) and (3.45), for any $[\underline{\mu}, \phi]/\chi \in \widehat{K}_g^0(Y^{S^1})_{I(g)}$, we have

$$\begin{aligned}
&\widehat{f_{Y!}} \circ \hat{\imath}_!([\underline{\mu}, \phi]/\chi) \\
&\quad = \widehat{f_{Y!}} \Big\{ \Big[\underline{\xi_+}, \mathrm{ch}_g\Big(\underline{\Lambda^{\mathrm{even}}(N^*)}\Big) \wedge \phi\Big]/\chi - \Big[\underline{\xi_-}, \mathrm{ch}_g\Big(\underline{\Lambda^{\mathrm{odd}}(N^*)}\Big) \wedge \phi\Big]/\chi \Big\} \\
&\quad = \chi(g)^{-1} \Big\{ - \int_{Y^{S^1}} \mathrm{Td}_g(\nabla^{TY}, \nabla^L)\, \mathrm{ch}_g\Big(\underline{\lambda_{-1}(N^*)}\Big) \wedge \phi + \bar{\eta}_g(\underline{TY}, \underline{L}, \underline{\xi_+}) - \bar{\eta}_g(\underline{TY}, \underline{L}, \underline{\xi_-}) \Big\} \\
&\quad = \chi(g)^{-1} \sum_{\alpha} \Big(- \int_{Y_\alpha^{S^1}} \mathrm{Td}_g(\nabla^{TY_\alpha^{S^1}}, \nabla^{L_\alpha}) \wedge \phi + \bar{\eta}_g(\underline{TY_\alpha^{S^1}}, \underline{L_\alpha}, \underline{\mu})\Big) \mod \mathbb{Q}_g \\
&\quad = \widehat{f_{Y^{S^1}!}}([\underline{\mu}, \phi]/\chi).
\end{aligned} \tag{3.49}$$

It means that

$$\widehat{f_{Y!}} \circ \hat{\imath}_! = \widehat{f_{Y^{S^1}!}} : \widehat{K}_g^0(Y^{S^1})_{I(g)} \to \mathbb{C}/\mathbb{Q}_g. \tag{3.50}$$

From Theorems 2.17, 3.4 and (3.50), we have

$$\widehat{f_{Y!}} = \widehat{f_{Y^{S^1}!}} \circ [\underline{\lambda_{-1}(N^*)}, 0]^{-1} \cup \hat{\imath}^* : \widehat{K}_g^0(Y)_{I(g)} \to \mathbb{C}/\mathbb{Q}_g. \tag{3.51}$$

Thus the diagram in (0.15) commutes.

By Definition 3.5, we have

$$\widehat{f_{Y!}}([\underline{E}, 0]) = \bar{\eta}_g(\underline{TY}, \underline{L}, \underline{E}) \mod \mathbb{Q}_g. \tag{3.52}$$

By Theorems 2.17, 3.5, (3.42), (3.45) and (3.48), for any $\mathcal{N} > \mathcal{N}_0$, $g \in S^1\backslash A$, we get

$$\begin{aligned}(3.53)\quad \widehat{f_{Y^{S^1}}}_! \left([\underline{\lambda_{-1}(N^*)}, 0]^{-1} \cup \hat{\imath}^*([\underline{E}, 0])\right) &= \widehat{f_{Y^{S^1}}}_! \left(\bigoplus_\alpha [\underline{\lambda_{-1}(N^*_\alpha)^{-1}_{\mathcal{N}}}, 0] \cup \hat{\imath}^*([\underline{E}, 0])\right) \\ &= \sum_\alpha \bar{\eta}_g \left(\underline{TY^{S^1}_\alpha}, \underline{L_\alpha}, \underline{\lambda_{-1}(N^*_\alpha)^{-1}_{\mathcal{N}}} \otimes \underline{E}|_{Y^{S^1}_\alpha}\right) \bmod \mathbb{Q}_g.\end{aligned}$$

Thus by (3.44), (3.51)-(3.53), for $\mathcal{N} > \mathcal{N}_0$ and $g \in S^1\backslash A$, we have $Q_{\mathcal{N}}(g) \in \mathbb{Q}_g$, i.e., (0.16) holds. The proof of Theorem 0.4 is completed. □

Let $K \in \mathrm{Lie}(S^1)$ be fixed.

Lemma 3.9. *For $g \in S^1\backslash A$, there exists $\beta > 0$ such that for any $t \in \mathbb{R}$, $|t| \leq \beta$, $\mathcal{N}' > \mathcal{N} > \mathcal{N}_0$, $P_{\mathcal{N},\mathcal{N}'}(ge^{tK})$ and $Q_{\mathcal{N}}(ge^{tK})$ are real analytic in t.*

Proof. Recall that $r_{\alpha,v} = \mathrm{rk}\, N_{\alpha,v}$. By (3.37), we have

$$(3.54)\qquad F_{\alpha,\mathcal{N}}(h) \cdot \underline{\lambda_{-1}(N^*_\alpha)^{-1}_{\mathcal{N}}} = \underline{\mu_{\alpha,\mathcal{N},+}} - \underline{\mu_{\alpha,\mathcal{N},-}} \text{ with } F_{\alpha,\mathcal{N}}(h) = \prod_{v:\, r_{\alpha,v} \neq 0} (h^v - 1)^{r_{\alpha,v} + \mathcal{N}}.$$

Set

$$(3.55)\qquad F_{\mathcal{N}}(x) = \prod_{v:\, \max_\alpha r_{\alpha,v} \neq 0} (x^v - 1)^{\max_\alpha r_{\alpha,v} + \mathcal{N}} \in \mathbb{Z}[x].$$

By Theorem 1.10 and (1.55), for $g \in S^1\backslash A$, there exists $\beta > 0$ such that for $|t| < \beta$, we have

$$(3.56)\qquad \bar{\eta}_{ge^{tK}}(\underline{TY}, \underline{L}, \underline{E}) = \bar{\eta}_{g,tK}(\underline{TY}, \underline{L}, \underline{E}).$$

From (3.37), (3.43), (3.44) and (3.56), for $g \in S^1\backslash A$, there exists $\beta > 0$ such that for $|t| < \beta$, $\mathcal{N}' > \mathcal{N} > \mathcal{N}_0$, we have

$$\begin{aligned}(3.57)\quad & F_{\mathcal{N}'}(ge^{tK}) \cdot P_{\mathcal{N},\mathcal{N}'}(ge^{tK}) \\ &= \sum_\alpha \bar{\eta}_{ge^{tK}} \left(\underline{TY^{S^1}_\alpha}, \underline{L_\alpha}, (\underline{\mu_{\alpha,\mathcal{N}',+}} - \underline{\mu_{\alpha,\mathcal{N}',-}}) \otimes \underline{E}|_{Y^{S^1}_\alpha}\right) \cdot \frac{F_{\mathcal{N}'}(ge^{tK})}{F_{\alpha,\mathcal{N}'}(ge^{tK})} \\ &\quad - \sum_\alpha \bar{\eta}_{ge^{tK}} \left(\underline{TY^{S^1}_\alpha}, \underline{L_\alpha}, (\underline{\mu_{\alpha,\mathcal{N},+}} - \underline{\mu_{\alpha,\mathcal{N},-}}) \otimes \underline{E}|_{Y^{S^1}_\alpha}\right) \cdot \frac{F_{\mathcal{N}'}(ge^{tK})}{F_{\alpha,\mathcal{N}}(ge^{tK})},\end{aligned}$$

and

$$\begin{aligned}(3.58)\quad F_{\mathcal{N}}(ge^{tK}) \cdot Q_{\mathcal{N}}(ge^{tK}) &= F_{\mathcal{N}}(ge^{tK}) \bar{\eta}_{g,tK}(\underline{TY}, \underline{L}, \underline{E}) \\ &\quad - \sum_\alpha \bar{\eta}_{ge^{tK}} \left(\underline{TY^{S^1}_\alpha}, \underline{L_\alpha}, (\underline{\mu_{\alpha,\mathcal{N},+}} - \underline{\mu_{\alpha,\mathcal{N},-}}) \otimes \underline{E}|_{Y^{S^1}_\alpha}\right) \cdot \frac{F_{\mathcal{N}}(ge^{tK})}{F_{\alpha,\mathcal{N}}(ge^{tK})}.\end{aligned}$$

Recall that for $g \in S^1\backslash A$, $g^v - 1 \neq 0$ if $r_{\alpha,v} \neq 0$. So there exists $\beta > 0$ such that for $|t| \leq \beta$, $F_{\mathcal{N}}(ge^{tK})^{-1}$ is real analytic in t for any $\mathcal{N}$. By (3.41), $\bar{\eta}_{ge^{tK}}(\underline{TY^{S^1}_\alpha}, \cdots)$ in (3.57) and (3.58) are polynomials on ge^{tK} and $(ge^{tK})^{-1}$. Thus by Theorem 1.9 and (3.41), for $g \in S^1\backslash A$, there exists $\beta > 0$ such that for $|t| \leq \beta$, $\mathcal{N}' > \mathcal{N} > \mathcal{N}_0$, $P_{\mathcal{N},\mathcal{N}'}(ge^{tK})$ and $Q_{\mathcal{N}}(ge^{tK})$ are real analytic in t. □

Proposition 3.10. *For any $\mathcal{N}' > \mathcal{N} > \mathcal{N}_0$, the functions $P_{\mathcal{N},\mathcal{N}'}$ and $Q_{\mathcal{N}}$ on $S^1\backslash A$ are restrictions on $S^1\backslash A$ of rational functions on S^1 with integral coefficients.*

Proof. We prove first this property for $Q_{\mathcal{N}}$.

For $g = e^{2i\pi t} \in S^1\backslash A$ and $\mathcal{N} > \mathcal{N}_0$, we have $Q_{\mathcal{N}}(g) \in \mathbb{Q}_g$ by (0.16) and (3.44). By (0.13), we can write

$$Q_{\mathcal{N}}(g) = \frac{\sum_{k=0}^{N(g)} a_k(g) g^k}{\sum_{k=0}^{M(g)} b_k(g) g^k}, \tag{3.59}$$

where $a_k(g), b_k(g) \in \mathbb{Z}$, $N(g), M(g) \in \mathbb{N}$, the polynomials $\sum_{k=0}^{N(g)} a_k(g)x^k$ and $\sum_{k=0}^{M(g)} b_k(g)x^k$ are relatively prime, and $\sum_{k=0}^{M(g)} b_k(g)g^k \neq 0$.

Let $T_{M,N} = \{g \in S^1\backslash A : M(g) \leq M, N(g) \leq N\}$. Then

$$\cup_{M,N=1}^{\infty} T_{M,N} = S^1\backslash A. \tag{3.60}$$

Fix $g_0 \in S^1\backslash A$. Let U be a connected open neighbourhood of g_0 in $S^1\backslash A$ such that $Q_{\mathcal{N}}$ is real analytic on U by Lemma 3.9. We have $\cup_{M,N=1}^{\infty}(T_{M,N} \cap U) = U$. Since U is an uncountable set, there exist $M_0, N_0 \in \mathbb{N}$ such that $T_{M_0,N_0} \cap U$ is an uncountable set. We define the map $\Psi : T_{M_0,N_0} \cap U \to \mathbb{Z}^{M_0+N_0+2}$ such that

$$\Psi(g) = (a_0(g), \cdots, a_{N_0}(g), b_0(g), \cdots, b_{M_0}(g)). \tag{3.61}$$

Since $\mathbb{Z}^{M_0+N_0+2}$ is a countable set, there exists $I = (a_0, \cdots, a_{N_0}, b_0, \cdots, b_{M_0}) \in \mathrm{Im}(\Psi)$ such that $\Psi^{-1}(I)$ is a uncountable set. Set $h(x) = \frac{\sum_{k=0}^{N_0} a_k x^k}{\sum_{k=0}^{M_0} b_k x^k}$. Then there is an open subset $U' \subset U$ such that h is real analytic on U' and $Q_{\mathcal{N}} = h$ on a uncountable subset of U'. Moreover, since h is a meromorphic function on $\mathbb{C}$, $Q_{\mathcal{N}}$ can be extended as a holomorphic function on an open connected neighborhood $U_0 \subset \mathbb{C}$ of U, we have $h \equiv Q_{\mathcal{N}}$ on U_0, in particular, $h = Q_{\mathcal{N}}$ on U. So for any $g_0 \in S^1\backslash A$, there is an open neighborhood U of g_0 in $S^1\backslash A$ such that $Q_{\mathcal{N}}$ is a rational function on U with integral coefficients. It means that $Q_{\mathcal{N}}$ is a rational function on each connected component of $S^1\backslash A$ with integral coefficients.

For $g \in A$ and for small $t \neq 0$ it follows from Theorem 1.10, similarly to (3.58), that

$$\begin{aligned} Q_{\mathcal{N}}(ge^{tK}) = \bar{\eta}_{g,tK}(\underline{TY}, \underline{L}, \underline{E}) - \mathcal{M}_{g,tK}\left(\underline{TY}, \underline{L}, \underline{E}\right) \\ - F_{\mathcal{N}}(ge^{tK})^{-1} \cdot \sum_{\alpha} \bar{\eta}_{ge^{tK}}\left(\underline{TY_{\alpha}^{S^1}}, \underline{L_{\alpha}}, (\underline{\mu_{\alpha,\mathcal{N},+}} - \underline{\mu_{\alpha,\mathcal{N},-}}) \otimes \underline{E}|_{Y_{\alpha}^{S^1}}\right) \cdot \frac{F_{\mathcal{N}}(ge^{tK})}{F_{\alpha,\mathcal{N}}(ge^{tK})}. \end{aligned} \tag{3.62}$$

From Theorem 1.9, (3.41), (3.55) and (3.62), $Q_{\mathcal{N}}(ge^{tK})$ is a meromorphic function in t near 0. But from the argument before (3.62), we know that for $t > 0$ small

$$Q_{\mathcal{N}}(ge^{tK}) = \frac{P_+(ge^{tK})}{Q_+(ge^{tK})} \tag{3.63}$$

is a rational function in ge^{tK}. As $\frac{P_+(ge^{tK})}{Q_+(ge^{tK})}$ is a meromorphic function in t near 0, this implies (3.63) holds for t near 0. In particular, (3.63) holds for $t < 0$ small. So $Q_{\mathcal{N}}$ as a function on $S^1\backslash A$ is the restriction on $S^1\backslash A$ of a rational function on S^1 with integral coefficients.

By the argument after (3.58), in particular by (3.38) and (3.41), we get that for $\mathcal{N}' > \mathcal{N} > \mathcal{N}_0$, $P_{\mathcal{N},\mathcal{N}'}$ is the restriction on $S^1\backslash A$ of a rational function on S^1 with coefficients in $\mathbb{R}$. To show that that the coefficients are actually in $\mathbb{Z}$ we only need to apply the above argument again.

The proof of Proposition 3.10 is completed. $\square$

By Lemma 3.9 and Proposition 3.10, the proof of Theorem 3.7 is completed.

3.5. **The case when $Y^{S^1} = \emptyset$.** In the remainder of this paper, we discuss the case when $Y^{S^1} = \emptyset$.

As $Y^{S^1} = \emptyset$, from Proposition 1.1, $A = \{g \in S^1 : Y^g \neq \emptyset\}$ is a finite set. From the variation formula (1.31), for $g \in S^1 \backslash A$, up to $\mathbb{Q}_g$, $\bar{\eta}_g(\underline{TY}, \underline{L}, \underline{E})$ does not depend on the geometric data g^{TY}, (h^L, ∇^L), (h^E, ∇^E). Thus similarly as in Definition 3.5, the map $f_! : K^0_{S^1}(Y)_{I(g)} \to \mathbb{C}/\mathbb{Q}_g$ for $g \in S^1 \backslash A$, defined by

$$f_!([E]/\chi) = \chi(g)^{-1} \bar{\eta}_g(\underline{TY}, \underline{L}, \underline{E}) \mod \mathbb{Q}_g \tag{3.64}$$

is well-defined. By [6, Proposition 1.5], $K^0_{S^1}(Y)_{I(g)} = 0$ for $g \in S^1 \backslash A$. So $\bar{\eta}_g(\underline{TY}, \underline{L}, \underline{E}) \in \mathbb{Q}_g$. Since Theorems 1.4 and 1.10 still hold for $Y^{S^1} = \emptyset$, following the same process as in Lemma 3.9 and Proposition 3.10 (note that for the last part of the proof of Proposition 3.10, we only use Theorem 1.10), we obtain:

Theorem 3.11. *If $Y^{S^1} = \emptyset$, $A = \{g \in S^1 : Y^g \neq \emptyset\}$, then $\bar{\eta}_g(\underline{TY}, \underline{L}, \underline{E})$ as a function on $S^1 \backslash A$ is the restriction of a rational function on S^1 with integral coefficients that does not have poles on $S^1 \backslash A$.*

Example 3.12. For $k \in \mathbb{N}^*$, we consider the circle action on $Y = \widehat{S^1}$ with

$$g.e^{i\theta} = e^{2\pi i k t + i\theta}, \quad \text{for } g = e^{2\pi i t} \in S^1. \tag{3.65}$$

Here $\widehat{S^1}$ is a copy of S^1. For $x = e^{i\theta} \in \widehat{S^1}$, if $g.x = x$, we have $kt \in \mathbb{Z}$, which means that $g^k = 1$. So $Y^{S^1} = \emptyset$ and $A = \{g \in S^1 : g^k = 1\}$.

We identify $[0, 2\pi)$ with $\widehat{S^1}$ by sending θ to $e^{i\theta}$. Then the canonical metric on $\widehat{S^1}$ is defined by $|\frac{\partial}{\partial\theta}| = 1$, the spinor of $\widehat{S^1}$ is $\mathcal{S}(\widehat{S^1}) = \mathbb{C}$, and the Clifford action is defined by $c\left(\frac{\partial}{\partial\theta}\right) = -i \in \mathrm{End}(\mathcal{S}(\widehat{S^1}))$. Thus the untwisted Dirac operator on $\widehat{S^1}$ is

$$D = -i\frac{\partial}{\partial\theta}. \tag{3.66}$$

From (3.65) and (3.66), we see that the circle action commutes with D. From (3.66), the eigenvalues of D are $\lambda_n = n$, $n \in \mathbb{Z}$, with eigenspaces $E_{\lambda_n} = \mathbb{C}\{e^{in\theta}\}$. For $g = e^{2\pi i t} \in S^1$, $s \in \mathbb{C}$ and $\mathrm{Re}(s) > 1$, we see that

$$\eta_g(s) := \sum_{n=1}^{+\infty} \frac{\mathrm{Tr}\,|_{E_{\lambda_n}}[g]}{n^s} - \sum_{n=1}^{+\infty} \frac{\mathrm{Tr}\,|_{E_{\lambda_{-n}}}[g]}{n^s} = \sum_{n=1}^{+\infty} \frac{e^{2\pi i n k t}}{n^s} - \sum_{n=1}^{+\infty} \frac{e^{-2\pi i n k t}}{n^s} \tag{3.67}$$

is well-defined.

For $x, y \in \mathbb{R}$, $s \in \mathbb{C}$, let $S_1(x, y, s)$ be the Kronecker zeta function [58, p53],

$$S_1(x, y, s) = \sum_{n\in\mathbb{Z}}{}' (x+n)|x+n|^{-2s} e^{-2\pi i n y}, \tag{3.68}$$

where $\sum'_{n\in\mathbb{Z}}$ is a sum over $n \in \mathbb{Z}$, $n \neq -x$. The series in (3.68) converges absolutely for $\mathrm{Re}(s) > 1$, and defines a holomorphic function of s. Moreover, $s \mapsto S_1(x, y, s)$ has a holomorphic continuation to $\mathbb{C}$ [58, p57]. By (3.67) and (3.68), we have

$$\eta_g(s) = -S_1\left(0, kt, \frac{s+1}{2}\right). \tag{3.69}$$

Thus $\eta_g(s)$ has a holomorphic continuation to $\mathbb{C}$. Also by [58, p57], we have the functional equation for $S_1(x,y,s)$,

$$\Gamma(s)S_1(x,y,s) = -i\pi^{2s-3/2}e^{2\pi ixy}\Gamma\left(\frac{3}{2}-s\right)S_1\left(y,-x,\frac{3}{2}-s\right). \tag{3.70}$$

From (3.69) and (3.70), we have

$$\eta_g(0) = \frac{i}{\pi}S_1(kt,0,1). \tag{3.71}$$

By [58, p57], for $x \notin \mathbb{Z}$, we have

$$S_1(x,0,1) = \pi\cot(\pi x). \tag{3.72}$$

If $g \in S^1\backslash A$, then $kt \notin \mathbb{Z}$. Thus by (3.65), (3.71) and (3.72), for $g \in S^1\backslash A$, the equivairant eta invariant $\eta_g(\underline{\widehat{S^1}})$ of S^1 with the canonical metric, is given by

$$\eta_g(\underline{\widehat{S^1}}) = \eta_g(0) = i\cot(\pi kt) = -\frac{g^{k/2}+g^{-k/2}}{g^{k/2}-g^{-k/2}} = \frac{2}{1-g^k} - 1. \tag{3.73}$$

Since $\mathrm{Ker}(D)$ is the space of the complex valued constant functions on $\widehat{S^1}$, we have $\mathrm{Tr}\,|_{\mathrm{Ker}(D)}[g] = 1$. Thus from (3.73), for $g \in S^1\backslash A$, the equivairant reduced eta invariant

$$\bar{\eta}_g(\underline{\widehat{S^1}}) = \frac{\eta_g(\underline{\widehat{S^1}}) + \mathrm{Tr}\,|_{\mathrm{Ker}(D)}[g]}{2} = \frac{1}{1-g^k}. \tag{3.74}$$

It is a rational function on S^1 with integral coefficients and poles in A.

For $g \in A$, then from (3.67), we get $\eta_g(s) = 0$, for any $s \in \mathbb{C}$. Since $\mathrm{Tr}\,|_{\mathrm{Ker}(D)}[g] = 1$,

$$\bar{\eta}_g(\underline{\widehat{S^1}}) = \bar{\eta}_1(\underline{\widehat{S^1}}) = \bar{\eta}(\underline{\widehat{S^1}}) = \frac{1}{2}, \quad \text{for } g \in A. \tag{3.75}$$

Remark that this reduced eta invariant could also be computed from the equivariant APS index theorem (0.19). Let B be the unit disc in $\mathbb{R}^2$ and $\partial B = \widehat{S^1}$. In polar coordinates, the circle action on B is defined by

$$g.re^{i\theta} = re^{2\pi ikt+i\theta}, \quad \text{for } g = e^{2\pi it},\ k \in \mathbb{N}^*. \tag{3.76}$$

It induces the circle action on $\partial B = \widehat{S^1}$ in (3.65). If $g \in S^1\backslash A$, then $g^k \neq 1$ and the only fixed point set of g on B is $B^g = \{0\}$. Let N be the normal bundle of $\{0\}$ in B. Then the g-action on N is as multiplication by g^k.

We take the metric on B such that it has product structure near the boundary, induces the canonical metric on $\widehat{S^1}$ and commutes with the circle action. We denote by $D^B_{+,\geq 0}$ the Fredholm operator with respect to the Dirac operator D^B_+ on B and the APS boundary condition. The equivariant APS index is defined by

$$\mathrm{Ind}_{\mathrm{APS},g}(D^B_+) = \mathrm{Tr}\,|_{\mathrm{Ker}\left(D^B_{+,\geq 0}\right)}[g] - \mathrm{Tr}\,|_{\mathrm{Coker}\left(D^B_{+,\geq 0}\right)}[g]. \tag{3.77}$$

From the equivariant APS index theorem [29], we have for $g \in S^1\backslash A$,

$$\bar{\eta}_g(\underline{\widehat{S^1}}) = \int_{B^g}\frac{1}{\det(1-g|_N)} - \mathrm{Ind}_{\mathrm{APS},g}(D^B_+) = \frac{1}{1-g^k} - \mathrm{Ind}_{\mathrm{APS},g}(D^B_+). \tag{3.78}$$

Note that the equivariant APS index is invariant when the metric varies near the boundary, without changing the metrics on the boundary. We only need to compute $\mathrm{Ind}_{\mathrm{APS},g}(D^B_+)$ using the canonical metric on B, which is not of product structure near the boundary. In this case, with the coordinate $z = x + iy$ in $\mathbb{R}^2 \simeq \mathbb{C}$, the spinor of B is $\mathcal{S}(B) = \mathbb{C} \oplus \mathbb{C}(d\overline{z}/\sqrt{2}) \simeq \mathbb{C} \oplus \mathbb{C}$ and the Clifford action is given by

$$c\left(\frac{\partial}{\partial x}\right) := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad c\left(\frac{\partial}{\partial y}\right) := \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}. \tag{3.79}$$

Thus $D^B = c(\frac{\partial}{\partial x})\frac{\partial}{\partial x} + c(\frac{\partial}{\partial y})\frac{\partial}{\partial y}$ has the form

$$D^B = \begin{pmatrix} 0 & D^B_- \\ D^B_+ & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\frac{\partial}{\partial x} + i\frac{\partial}{\partial y} \\ \frac{\partial}{\partial x} + i\frac{\partial}{\partial y} & 0 \end{pmatrix} = \begin{pmatrix} 0 & -2\frac{\partial}{\partial z} \\ 2\frac{\partial}{\partial \overline{z}} & 0 \end{pmatrix}, \tag{3.80}$$

In polar coordinates,

$$D^B_+ = -e^{i\theta}\left(-\frac{\partial}{\partial r} - \frac{i}{r}\frac{\partial}{\partial \theta}\right). \tag{3.81}$$

Note that $-\frac{\partial}{\partial r}$ here is the inward normal vector. Let $P_{\geq 0}$ be the orthogonal projection onto the direct sum of the eigenspaces associated with the nonnegative eigenvalues of $\mathcal{A} := -\frac{i}{r}\frac{\partial}{\partial \theta}|_{\partial B}$. Recall that the eigenvalues of $\mathcal{A}$ are $\lambda_n = n$, $n \in \mathbb{Z}$, with eigenspaces $\mathbb{C}\{e^{in\theta}\}$. Thus the APS boundary condition reads in complex coordinates for $f \in \mathscr{C}^\infty(B, \mathbb{C})$,

$$P_{\geq 0} f|_{\partial B} = 0 \Leftrightarrow f|_{\partial B} = \sum_{n<0, n\in\mathbb{Z}} a_n z^n. \tag{3.82}$$

If $D^B_+ f = 0$, f is holomorphic on B. Thus

$$\operatorname{Ker}(D^B_{+,\geq 0}) = \{f \in \mathscr{C}^\infty(B, \mathbb{C}) : D^B_+ f = 0, P_{\geq 0}(f|_{\partial B}) = 0\} = 0. \tag{3.83}$$

By (3.81), the adjoint of $D^B_{+,\geq 0}$ is $D^B_{-,<0}$, i.e., D^B_- with boundary condition

$$P_{<0}(e^{-i\theta} f)|_{\partial B} = 0, \ \text{ with } \ P_{\geq 0} + P_{<0} = \mathrm{Id} \ \text{ on } \mathscr{C}^\infty(\partial B, \mathbb{C}). \tag{3.84}$$

By (3.80) and (3.84), if $D^B_{-,<0} f = 0$, then $\bar{f}$ is holomorphic on B and $f|_{\partial B} = \sum_{n>0, n\in\mathbb{Z}} a_n z^n$. Thus $\operatorname{Coker}(D^B_{+,\geq 0}) = \operatorname{Ker}(D^B_{-,<0}) = 0$.

From (3.77), (3.78) and (3.83), we have $\bar{\eta}_g(\underline{\widehat{S^1}}) = (1 - g^k)^{-1}$ for $g \in S^1 \backslash A$, which is the same as (3.74).

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School of Mathematical Sciences, Shanghai Key Laboratory of PMMP, East China Normal University, Shanghai, P.R. China
E-mail address: bliu@math.ecnu.edu.cn

Université de Paris, CNRS, Institut de Mathématiques de Jussieu-Paris Rive Gauche, F-75013 Paris, France, and School of Mathematical Sciences, University of Science and Technology of China, Herfei, P.R. China
E-mail address: xiaonan.ma@imj-prg.fr