Regulation of renewable resource exploitation

Idris Kharroubi, Thomas Lim, Thibaut Mastrolia

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Abstract

We investigate the impact of a regulation policy imposed on an agent exploiting a possibly renewable natural resource. We adopt a principal-agent model in which the Principal looks for a contract, *i.e.* taxes/compensations, leading the Agent to a certain level of exploitation. For a given contract, we first describe the Agent's optimal harvest using the BSDE theory. Under regularity and boundedness assumptions on the coefficients, we express almost optimal contracts as solutions to HJB equations. We then extend the result to coefficients with less regularity and logistic dynamics for the natural resource. We end by numerical examples to illustrate the impact of the regulation in our model.

Key words: Contract Theory, BSDEs, HJB PDE, Logistic SDE.

1 Introduction

The exploitation of natural resources is fundamental for the survival and development of the growing human population. However, natural resources are limited since they are either non renewable (e.g. minerals, oil, gas and coal) so that the available quantity is limited, or renewable (e.g. food, water and forests) and in this case the natural resource is limited by its ability to renew itself. In particular, an excessive exploitation of such resources might lead to their extinctions and therefore affect the depending economies with, for instance, high increases of prices and higher uncertainty on the future. Thus, the natural resource manager faces a dilemma: either harvesting intensively the resource to increase her incomes, or taking into account the potential externalities induced by an overexploitation of the resource and impacting her future ability to harvest the resource. It has been nevertheless emphasized in [7] that in some cases it is optimal for natural resource manager to harvest until the extinction of the resource. This optimal harvesting strategy thus leads to costs for the global welfare related to the environment degradation.

Therefore, the management and the monitoring of the exploitation of natural resources are a balance between optimal harvest for the natural resource manager and ecological implications for public organizations. This second issue has attracted a lot of interest, especially from governance institutions. For example in its last annual report on sustainable development, the statistical office of the European Union *Eurostat* dedicates a full section to the question of sustainable consumption and production (see [11], Section 12).

The management of natural resources have also attracted a lot of interest from the academic community. Many studies on natural resources exploitation tried to describe the possible effect of economic incentives on the exploitation (see e.g. [5, 14, 27, 15]). These references stress the need of an incentive policy to ensure the sustainability of the resource. However, even if

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the regulator have access to the abundance level of natural resource, the unobservability of the natural resource manager behavior induces moral hazards. Thus, the regulator's issue is to incentivize the resource manager to optimally reduce the cost of the resource degradation, together with ensuring a minimal incomes for the manager, under moral hazard. To the best of our knowledge, this question has been addressed only in the discrete-time framework (see for instance [13]) without considering any randomness in the dynamics of the resource. The aim of this work is to investigate this problem in continuous time with randomness in the system.

To deal with this issue, we consider a principal-agent model under moral hazard. The first elements of contract theory with moral hazard appeared in the 60's with the articles [3, 30] in which the mechanisms of controlled management were investigated. Then, it has been extended and named as *agency problem* (see among others [29, 24]) by considering discrete-time models. Concerning the continuous-time framework, the agency problem with moral hazard has been first studied in [17] by modelling the uncertainty of risky incomes with a Brownian motion.

The agency problem can be roughly described as follows. We associate a moral hazard problem with a Stackelberg game in which the leader (named the Principal) proposes at time 0 a compensation to the follower (named the Agent) given at a maturity $T > 0$ fixed by the contract, to manage the wealth of the leader. Moreover, the Principal has to propose a compensation high enough (called the reservation utility) to ensure a certain level of utility for the Agent. Although the Principal cannot directly observe the action of the Agent, the former can anticipate the best reaction effort of the latter with respect to a fixed compensation. Hence the agency problem remains to design an optimal compensation proposed by the Principal to the Agent given all the constraints mentioned above under moral hazard.

The common approach to solve this problem consists in proceeding in two steps. The first step is to compute the optimal reaction of the Agent given a fixed compensation proposed by the Principal, *i.e.* solving the utility maximization problem of the Agent. In all the papers mentioned above, the shape of considered contracts is fundamental to solve the Agent problem by assuming that the compensation is composed by

- a constant part depending on the reservation utility of the Agent,
- a part indexed by the (risky) incomes of the Principal,
- the certain equivalent gain of utility appearing in the Agent maximization.

Using the theory of Backward Stochastic Differential Equations (BSDE for short), [9] proved that this class of *smooth* contracts, having a relevant economic interpretation, is not restrictive to solve the agency problem. The second step consists in solving the Principal problem. Taking into account this optimal reaction of the Agent, the goal is to compute the optimal compensation. As emphasized in [26] and then in [8, 9], this problem remains to a (classical) stochastic control problem with the wealth of the Principal and the continuation utility value of the Agent as state variables.

In this paper, we identify the natural resource manager as the Agent. The Principal refers to a regulator, which can be a public institution that monitors the resource manager's activities.

The resource manager can either harvest or renew the natural resource. In the first case the production is sold at a given price on the market and in the second case the resource manager pays for each unit of renewed natural resource. To regulate the natural resource exploitation, the Principal imposes a tax/compensation to the Agent depending on the remaining level of resource at the terminal time horizon. We suppose here that the Agent is risk-averse and we model his preference with an exponential utility function[1]. For a given harvesting strategy, the Agent total gain is composed by the cumulated amounts paid/earned by renewing/harvesting the natural resource and the regulation compensation/tax. The Agent's aim is then to maximize the expected utility of his total gain over possible harvesting strategies.

On the other side, given the previous optimal harvest of the Agent, the regulator aims at fixing a tax/compensation policy that incentives the Agent to let a reasonable remaining level

[1] See for instance [4] for more details on this kind of utility function and the economical interpretations of it.

of natural resource. As a public institution, we assume that the regulator is risk-neutral.

The main features to model the dynamic of a renewable natural resource are its birth and death rates and the inter-species competition. Besides, due to random evolution of the population, we consider uncertainty in the available abundance. Following [12, 2, 22] we choose to model the evolution of the natural resource by a stochastic logistic diffusion.

We then focus on the Principal-agent problem. We first characterize the Agent behavior for a fixed regulation policy represented by a random variable ξ. Following the BSDEs approaches to deal with exponential utility maximization, we get a unique optimal harvesting strategy as a function of the Z component of the solution to a quadratic BSDE with terminal condition ξ (see [25, 18]).

We next turn to the regulator problem which consists of maximizing an expected terminal reward depending on the regulation tax ξ and the level of remaining natural resource according to the Agent's optimal response. By writing the explicit form of the resource manager's optimal strategy, we turn the regulator problem into a Markov stochastic control problem of a diffusion with controlled drift. We then look for a regular solution to the related PDE to proceed by verification. However, in our case we face the following three issues.

- By considering the logistic dynamics for the resource abundance population, the HJB PDE related to the Principal problem involves a term of the form $x^2\partial_x v$ where x stands for the resource population abundance and v is the Principals value function. This term, induced by the inter-species competition in the classical logistic case, prevents us from using existence results of regular solutions to PDEs.
- The shape of the optimal harvest of the manager leads to irregular coefficients for the related PDE, which also prevents from getting regular solutions.
- Due to the exponential preferences of the Agent, the Principal's admissible strategies need to satisfy an exponential integrability condition. However, the linear preferences of the Principal leads to an optimal contract that is not necessarily exponential integrable. Therefore, the regulator problem might not have an optimal regulation policy.

To deal with these issues, we first study a model for which the inter-species competition coefficient μ of the population is bounded. Hence, the term $x^2\partial_x v$ is replaced by $x\mu(x)\partial_x v$. We then construct a regular approximation of the Hamiltonian. By considering the related PDE, we derive a regular solution (see Proposition 4.1) together with an almost optimal control satisfying the admissibility condition (see Theorem 4.2). We notice that our approach can be related to that of Fleming and Soner [28], which consists of an approximation of the value function by a sequence of smooth value functions to derive a dynamic programming principle. We next turn to the logistic case *i.e.* $\mu(x) = x$ for which we show that the almost optimal strategy obtained for a truncation of μ remains an almost optimal strategy for a large value of the truncation parameter (see Theorem 4.3).

We finally illustrate our results by numerical experiments. We compute the almost optimal strategies using approximations of solutions to HJB PDEs and show that the regulation has a significant effect on the level of remaining natural resource.

The remainder of the paper is the following. In Section 2 we describe the considered mathematical problem. We then solve in Section 3 the manager's problem for a given regulation policy. In Section 4, we first provide almost optimal strategies in the case where the coefficient μ is bounded and we extend our result to the logisitic dynamics. We end Section 4 by economical insights and numerical experiments.

Notations and spaces

We give in this part all the notations used in this paper. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. We assume that this space is equipped with a standard Brownian motion W and we denote by $\mathbb{F} := (\mathcal{F}_t)_{t\geq 0}$ its right-continuous and complete natural filtration.

Let $p \geq 1$ and a time horizon $T > 0$, we introduce the following spaces

– $\mathcal{P}(\mathbb{R})$ (resp. $\mathcal{P}r(\mathbb{R})$) will denote the σ-algebra of $\mathbb{R}$-progressively mesurable, $\mathbb{F}$-adapted (resp. $\mathbb{F}$-predictable) integrable processes.

– $\mathcal{S}^p_T$ is the set of processes X, $\mathcal{P}(\mathbb{R})$-mesurable and continuous satisfying

$$\mathbb{E}[\sup_{0\leq t\leq T} |X_t|^p] < +\infty \, .$$

– $\mathbb{H}^p_T$ is the set of processes X, $\mathcal{P}r(\mathbb{R})$-mesurable satisfying

$$\mathbb{E}\Big[\Big(\int_0^T |X_t|^2 dt\Big)^{\frac{p}{2}}\Big] < +\infty \, .$$

– For an integer $q \geq 0$, a subset D of $\mathbb{R}^q$ and for any $\nu \in (0,1)$, we denote by $C^{1+\nu}(D)$ the set of continuously differentiable functions $f : \ D \to \mathbb{R}$ such that

$$|f|_1 = \sup_{x\in D}\Big(|f(x)| + \sum_{1\leq i\leq q} |\partial_{x_i} f(x)| + \sup_{x,y\in D} \sum_{1\leq i\leq q} \frac{|\partial_{x_i} f(x) - \partial_{x_i} f(y)|}{|x-y|^\nu}\Big) < \infty \, ,$$

and by $C^{2+\nu}(D)$ the set of twice continuously differentiable functions $f : \ D \to \mathbb{R}$ such that

$$\begin{aligned} |f|_{2+\nu} \quad &= \quad \sup_{x\in D}\Big(|f(x)| + \sum_{1\leq i\leq q} |\partial_{x_i} f(x)| + \sum_{1\leq i,j\leq q} |\partial_{x_i,x_j} f(x)|\Big) \\ &\quad + \sup_{x,y\in D} \sum_{1\leq i,j\leq q} \frac{|\partial_{x_i,x_j} f(x) - \partial_{x_i,x_j} f(y)|}{|x-y|^\nu} \ < \ \infty \, . \end{aligned}$$

2 The model

2.1 The natural resource

We fix a deterministic time horizon $T > 0$ and we suppose that the natural resource abundance X^μ_t at time $t \geq 0$ is given by

$$X^\mu_t = X_0 + \int_0^t X^\mu_s(\lambda - \mu(X^\mu_s))ds + \int_0^t \sigma X^\mu_s dW_s \, , \quad t \in [0,T] \, , \tag{2.1}$$

where X_0, λ and σ are positive constants. The quantities X_0 and λ correspond to the initial natural resource abundance and the growth rate respectively. The map μ represents the competition inside the species considered or more generally an auto-degradation parameter for a natural resource. We assume that the map μ satisfies the following assumption

(H0) μ is a map from $\mathbb{R}_+$ to $\mathbb{R}_+$ such that (2.1) admits a unique strong solution in $\mathcal{S}^2_T$.

Note that Assumption **(H0)** holds for instance if the map $x \longmapsto x\mu(x)$ is Lipschitz continuous. Another important example is the so-called logistic equation where $\mu(x) = x$ on $\mathbb{R}_+$, see for example in [12]. In this last case, SDE (2.1) admits an explicit unique solution that will be denoted in the sequel by X and given by

$$X_t = \frac{X_0 e^{(\lambda - \frac{\sigma^2}{2})t + \sigma W_t}}{1 + X_0 \int_0^t e^{(\lambda-\frac{\sigma^2}{2})s + \sigma W_s} ds} \, , \quad t \in [0,T] \, .$$

The ecological interpretation of this model is the following. At time t, if the coefficient $\mu(X_t^\mu)$ is larger than λ then the drift of the diffusion is negative. Therefore the abundance of the natural resource X_t^μ decreases in mean. Conversely, if $\mu(X_t^\mu)$ is smaller than λ then the drift of the diffusion is positive. Hence, the abundance X_t^μ increases in mean. For more details see for instance [22, Proposition 3.4].

More general models can be used in practice and one of the main challenges, see [22], is to rely branching processes with birth and death intensities to the solutions of continuous SDEs.

2.2 The Agent's problem

We consider an agent who tries to make profit from the natural resource. We suppose that this agent owns facilities to either harvest or renew this resource. We assume that his action happends continuously in time and we denote by α_t his intervention rate at time t, *i.e.* the abundance X_t^μ will decrease of an amount $\alpha_t X_t^\mu$ per unit of time. This means that if the intervention rate α_t is positive (resp. negative), the Agent harvests (resp. renews) the natural resource. We denote by $\mathcal{A}$ the set of $\mathbb{F}$-adapted processes defined on $[0,T]$ and valued in $[-\underline{M},\overline{M}]$ where $\underline{M}$ and $\overline{M}$ are two nonnegative constants. If the Agent is prohibited to renew the resource then $\underline{M}=0$. This set $\mathcal{A}$ is called the set of admissible actions.

To take into account the control α of the Agent on the natural resource abundance, we introduce the probability measure $\mathbb{P}^\alpha$ defined by its density H^α w.r.t. $\mathbb{P}$ given by

$$\frac{d\mathbb{P}^\alpha}{d\mathbb{P}}\Big|_{\mathcal{F}_T} := H_T^\alpha ,$$

where the process H^α is defined by

$$H_t^\alpha := \exp\Big(-\int_0^t \frac{\alpha_s}{\sigma} dW_s - \frac{1}{2}\int_0^s \Big|\frac{\alpha_s}{\sigma}\Big|^2 ds\Big) , \quad t \in [0,T] .$$

In the sequel, we denote by $\mathbb{E}^\alpha$ and $\mathbb{E}_t^\alpha$ the expectation and conditional expectation given $\mathcal{F}_t$ respectively, for any $t\in[0,T]$, under the probability measure $\mathbb{P}^\alpha$.

For $\alpha\in\mathcal{A}$, we get from Girsanov Theorem (see e.g. Theorem 5.1 in [19]) that the process W^α defined by

$$W_t^\alpha := W_t + \int_0^t \frac{\alpha_s}{\sigma} ds , \quad t\in[0,T] ,$$

is a Brownian motion under the probability $\mathbb{P}^\alpha$. Thus, for a given admissible effort $\alpha\in\mathcal{A}$, the dynamics of X can be rewritten under the probability $\mathbb{P}^\alpha$ as

$$X_t^\mu = x + \int_0^t \big(X_s^\mu(\lambda - \mu(X_s^\mu)) - \alpha_s X_s^\mu\big)ds + \int_0^t \sigma X_s^\mu dW_s^\alpha , \quad t\in[0,T] .$$

This new dynamics reflects the evolution of the population with a rate α_t per unit of time. Hence, $\alpha_t X_t^\mu$ has to be seen as the speed of the exploitation of the natural resource at time t.

Remark 2.1. *We choose a harvesting/renewing component of the form αX^μ. This allows to take into account not only the impact of the effort of the agent on the system but also the abundance of the resource itself. We note that this form of strategy is a particular case of the form presented in [16, (2.4)]. From a technical point of view, this particular form is due to the presence of a quadratic cost (see* (2.2)*) and allows to derive optimal efforts under a closed form. Possible extensions which are let for further research are to consider more general forms including finite variations and jumps as in the dynamic proposed in [16].*

We then are given a price function $p:\ \mathbb{R}_+\to\mathbb{R}_+$ and we suppose that the price per unit of the natural resource on the market is given by $p(X_t^\mu)$ at time $t\geq 0$. We make the following assumption on the price function p.

$(\mathbf{H}_p)$ There exists a constant P such that $p(x)x \le P$ for all $x \in [0, +\infty)$.

This price function p allows to take into account the dependence w.r.t. the abundance (the more abundant the resource is, the cheaper it will be and conversely). Such a price dependence has already been used to model liquidity effects on financial market, where empirical studies showed that the impact is of the form $p(x) = Pe^{-\beta_1 x^{\beta_2}}$, $x \in \mathbb{R}_+$, for some positive constants P, β_1 and β_2 (see e.g. [1, 21]). In particular, $(\mathbf{H}_p)$ is satisfied for this type of dependence. Another basic example for which $(\mathbf{H}_p)$ holds is the case $p(x) = Px^{-1}$, $x > 0$. This last example reflects the inability to buy the natural resource once it is extinct.

We assume that the manager sells the harvested resource on the market at price $p(X_t^\mu)$ per unit at time t if α_t is positive, and pays the price $p(X_t^\mu)$ per unit of natural resource at time t if α_t is negative to renew this one. This provides the global amount $\int_0^T p(X_t^\mu)X_t^\mu \alpha_t dt$ over the time horizon $[0, T]$.

We also suppose that giving an effort is costly for the manager and we consider the classical quadratic cost function $k : \mathbb{R} \to \mathbb{R}_+$ given by $k(\alpha) = \frac{|\alpha|^2}{2}$, $\alpha \in \mathbb{R}$. Thus, the Agent is penalized by the instantaneous amount $k(\alpha_t)$ per unit of time for a given effort $\alpha \in \mathcal{A}$. This leads to the global payment $\int_0^T k(\alpha_t)dt$ over the considered time horizon $[0, T]$.

In our investigation, we recall that the activity of the natural resource manager is regulated by an institution (usually an environment administration) who is taking care about the size of the remaining natural resource. To avoid an over-exploitation, the regulator imposes a tax on the Agent depending on the remaining resource. This tax amount is represented by an $\mathcal{F}_T$-measurable random variable ξ and is paid at time T. Note that ξ can be either positive or negative. In this last case, it means that the regulator gives a compensation to the manager.

Throughout the paper we assume that the Agent's preferences are given by the exponential utility function u_A defined by

$$u_A(x) := -\exp\big(-\gamma x\big)\ , \quad x \in \mathbb{R}\ ,$$

where γ is a positive constant corresponding to the risk aversion of the Agent. We define the value function $V_A(\xi)$ of the Agent associated to the taxation policy ξ by

$$V_A(\xi) := \sup_{\alpha \in \mathcal{A}} \mathbb{E}^\alpha \Big[-\exp\Big(-\gamma\big(\int_0^T p(X_s^\mu)X_s^\mu \alpha_s ds - \int_0^T \frac{|\alpha_s|^2}{2} ds - \xi \big)\Big)\Big]\ . \tag{2.2}$$

For a fixed tax ξ, we denote by $\mathcal{A}^*(\xi)$ the set of efforts $\alpha^* \in \mathcal{A}$ satisfying the following equality

$$\mathbb{E}^{\alpha^*} \Big[-\exp\Big(-\gamma\big(\int_0^T p(X_s^\mu)X_s^\mu \alpha_s^* ds - \int_0^T \frac{|\alpha_s^*|^2}{2} ds - \xi \big)\Big)\Big] = V_A(\xi)\ .$$

An effort $\alpha^* \in \mathcal{A}^*(\xi)$ is said to be optimal for the fixed tax ξ.

2.3 The Principal's problem

The aim of the regulator is to stabilize the resource population at a fixed target size at the maturity T. For that, a tax ξ is chosen to incentivize the Agent to manage the natural resource so that the remaining population is close to the targeted size. Hence, the regulator benefits from the tax paid by the Agent and is penalized through a cost function f depending on the size of the resource at maturity T. The expected reward under the action $\alpha \in \mathcal{A}$ of the Agent is then given by

$$\mathbb{E}^\alpha\big[\xi - f(X_T^\mu)\big]\ .$$

Typically, we have in mind $f(x) = c(\beta - x)^+$ meaning that the regulator targets a population size $\beta > 0$ at time T for the sustainability of the resource and pays the cost c per unit if the

natural resource is over-consumed. This function f can be seen as the amount that the regulator must pay to reintroduce the missing resource.
We suppose that the resource manager is rational. Therefore, the Principal anticipates that for a tax ξ, the Agent will choose an effort α in the set $\mathcal{A}^*(\xi)$. Note that this set is not necessarily reduced to a singleton[2], hence, as usual in moral hazard problems (see for instance [17] for the formulation of the moral hazard problem), the regulator solves

$$\sup_{\xi} V^P(\xi) \ , \ \text{with } V^P(\xi) = \sup_{\alpha \in \mathcal{A}^*(\xi)} \mathbb{E}^\alpha \Big[\xi - f(X_T^\mu) \Big], \tag{2.3}$$

where ξ lives in a set of suitable contracts defined in the following section.

2.4 Class of contracts and utility reservation

We now introduce a reserve utility R which is a negative constant. This reserve means that the regulator cannot penalize too strongly the Agent for economical reasons so that the utility $V_A(\xi)$ expected by the Agent has to be greater than R. For instance, we can choose R such that the regulator monitors the Agent by promising the same expected utility as the case without regulation (see Section 4.3.1 for more details). This example reflects a non-punitive taxation policy in which the regulator purely monitors the activities of the Agent. In our model, the sign of the tax ξ is on purpose. This means that the natural resource manager pays the fee to the regulator when ξ is positive and conversely, the regulator compensates the Agent's activity when ξ is negative. Moreover, we need to impose an exponential integrability on the tax ξ to ensure the well-posdness of $V_A(\xi)$. We therefore introduce the class $\mathcal{C}_R^\mu$ of admissible taxes defined as the set of $\mathcal{F}_T$-measurable random variables ξ such that

$$V_A(\xi) \geq R \ , \tag{2.4}$$

and there exists a constant $\gamma' > 2\gamma$ such that

$$\mathbb{E}\big[\exp(\gamma'|\xi|)\big] < +\infty \ . \tag{2.5}$$

This last condition is very convenient since it allows to deal with the problem by using the theory of BSDEs. Moreover, a straightforward application of Cauchy-Schwarz inequality ensures that the optimization problems $V_A(\xi)$ and $V^P(\xi)$ take finite values.

3 Optimal effort of the natural resource's manager

We first solve the optimal problem of the Agent (2.2) under taxation policy $\xi \in \mathcal{C}_R^\mu$. As in [9], the following result shows that solving the Agent problem gives both an optimal effort α^* and a particular representation of the tax ξ with respect to the solution of a BSDE.

Theorem 3.1. *Let $\xi \in \mathcal{C}_R^\mu$ and Assumption ($\boldsymbol{H_p}$) be satisfied. There exists a unique pair $(Y_0, Z) \in (-\infty, \tilde{R}] \times \mathbb{H}_T^2$ with $\tilde{R} := \frac{\log(-R)}{\gamma}$ such that*

(i) the tax has the following decomposition

$$\xi = Y_0 - \int_0^T \big(g(X_t^\mu, Z_t) + \frac{\sigma^2}{2}\gamma |Z_t|^2\big)dt + \int_0^T \sigma Z_t dW_t \ , \tag{3.6}$$

where g is defined for any $(x,z) \in \mathbb{R}_+ \times \mathbb{R}$ by

$$g(x,z) = \frac{|a^*(x,z)|^2}{2} - p(x)xa^*(x,z) - a^*(x,z)z \ ,$$

and

$$a^*(x,z) = \big((p(x)x + z) \vee (-\underline{M})\big) \wedge \overline{M} \ , \tag{3.7}$$

[2] In our investigation, we will show that the set $\mathcal{A}^*(\xi)$ is reduced to a single element.

(ii) the value of the Agent is given by

$$V_A(\xi) = -\exp(\gamma Y_0)\ ,$$

(iii) the process $\alpha^*(\xi)$ defined by $\alpha^*_t(\xi) = a^*(X^\mu_t, Z_t)$ is the unique optimal effort associated with the tax ξ given by (3.6).

Proof. The proof is divided in three steps and is related to the BSDE associated with the value function of the Agent. We first introduce a dynamic extension of the optimization problem (2.2). We denote by $J(t,\xi)$ the dynamic value function of the Agent at time t for a tax ξ which is defined by

$$J(t,\xi) := \operatorname*{ess\,inf}_{\alpha\in\mathcal{A}} \mathbb{E}^\alpha_t\Big[\exp\Big(-\gamma\big(\int_t^T p(X^\mu_s)X^\mu_s\alpha_s ds - \int_t^T k(\alpha_s)ds - \xi\big)\Big)\Big]\ .$$

Note that $V_A(\xi) = -J(0,\xi)$.

Step 1. Dynamic utility of the Agent and BSDE. We characterize $J(\cdot,\xi)$ as the unique solution of a BSDE and we derive the optimal control by using comparison results.

Let $\alpha\in\mathcal{A}$, we introduce the process $J^\alpha(\xi)$ defined by

$$J^\alpha_t(\xi) := \mathbb{E}^{\mathbb{P}^\alpha}_t\Big[\exp\Big(-\gamma\big(\int_t^T p(X^\mu_s)X^\mu_s\alpha_s ds - \int_t^T k(\alpha_s)ds - \xi\big)\Big)\Big]\ ,$$

so that

$$J(t,\xi) := \operatorname*{ess\,inf}_{\alpha\in\mathcal{A}_t} J^\alpha_t(\xi)\ . \tag{3.8}$$

Step 1a. Martingale representation and integrability.
We know that the process $H^\alpha_t(\exp(\gamma\int_0^t\big(k(\alpha_s)-p(X^\mu_s)X^\mu_s\alpha_s\big)ds)J^\alpha_t(\xi))_{0\le t\le T}$ is a $(\mathbb{P},\mathbb{F})$-martingale. In view of the condition (2.5), there exists $\varepsilon>0$ and $q>1$ such that $(2+\varepsilon)q\gamma\le\gamma'$. Hence, for $p>1$ such that $\frac{1}{p}+\frac{1}{q}=1$, since α is bounded and Condition (2.5) is satisfied, we get from Hölder's inequality

$$\mathbb{E}[|\overline{H}^\alpha_T J^\alpha_T(\xi)|^{2+\varepsilon}] \ \le\ \mathbb{E}[|\overline{H}^\alpha_T|^{(2+\varepsilon)p}]^{\frac{1}{p}}\mathbb{E}[|e^{(2+\varepsilon)q\gamma\xi}|]^{\frac{1}{q}} \ <\ +\infty\ ,$$

where $\overline{H}^\alpha_t := H^\alpha_t\exp\Big(\gamma\int_0^t\big(k(\alpha_s)-p(X^\mu_s)X^\mu_s\alpha_s\big)ds\Big)$. Hence, by using Doob's maximal inequality, $\overline{H}^\alpha J^\alpha(\xi)\in\mathcal{S}^{2+\varepsilon}$. So by using the martingale representation theorem, we know there exists a process $\overline{Z}^\alpha\in\mathbb{H}^{2+\varepsilon}_T$ such that

$$\overline{H}^\alpha_t J^\alpha_t(\xi) \ =\ J^\alpha_0 + \int_0^t \sigma\overline{Z}^\alpha_s dW_s\ ,\quad t\in[0,T]\ .$$

Therefore, J^α satisfies

$$dJ^\alpha_t(\xi) \ =\ (\alpha_t\tilde{Z}^\alpha_t - \gamma(k(\alpha_t)-p(X^\mu_t)X^\mu_t\alpha_t)J^\alpha_t(\xi))dt + \sigma\tilde{Z}^\alpha_t dW_t\ ,$$

where $\tilde{Z}^\alpha_t = \frac{\overline{Z}^\alpha_t}{\overline{H}^\alpha_t} + J^\alpha_t\frac{\alpha_t}{\sigma^2}$ for any $t\in[0,T]$, and $J^\alpha_T(\xi) = \exp(\gamma\xi)$.

We now prove that $\tilde{Z}^\alpha\in\mathbb{H}^2_T$. From (2.5), the boundedness of α and Assumption $(\mathbf{H}_p)$, there exists a positive constant $C>0$ such that

$$\begin{aligned}\mathbb{E}\Big[\int_0^T|\frac{\overline{Z}^\alpha_t}{\overline{H}^\alpha_t} + J^\alpha_t\frac{\alpha_t}{\sigma^2}|^2dt\Big] \ &\le\ 2\Big(\mathbb{E}\Big[\int_0^T|\frac{\overline{Z}^\alpha_t}{\overline{H}^\alpha_t}|^2dt\Big] + \mathbb{E}\Big[\int_0^T|J^\alpha_t\frac{\alpha_t}{\sigma^2}|^2dt\Big]\Big)\\ &\le\ C\Big(1+\mathbb{E}\Big[\int_0^T|\frac{\overline{Z}^\alpha_t}{\overline{H}^\alpha_t}|^2dt\Big]\Big)\ .\end{aligned}$$

We set $\tilde{q} := 1 + \frac{\varepsilon}{2}$ and $\tilde{p} > 1$ such that $\frac{1}{\tilde{p}} + \frac{1}{\tilde{q}} = 1$. Using Hölder and BDG Inequalities and since $\overline{Z}^\alpha \in \mathbb{H}^{2+\varepsilon}_T$, we get

$$\begin{aligned}
\mathbb{E}[\int_0^T |\frac{\overline{Z}^\alpha_t}{\overline{H}^\alpha_t}|^2 dt] &\leq \mathbb{E}[\sup_{t\in[0,T]} |(\overline{H}^\alpha_t)^{-1}|^2 \int_0^T |\overline{Z}^\alpha_t|^2 dt] \\
&\leq \mathbb{E}[\sup_{t\in[0,T]} |(\overline{H}^\alpha_t)^{-1}|^{2\tilde{p}}]^{\frac{1}{\tilde{p}}} \mathbb{E}[\big(\int_0^T |\overline{Z}^\alpha_t|^2 dt\big)^{\tilde{q}}]^{\frac{1}{\tilde{q}}} \\
&< +\infty \, .
\end{aligned}$$

Consequently, we get $\tilde{Z}^\alpha \in \mathbb{H}^2_T$.

Step 1b. Comparison of BSDEs and optimal effort. We now turn to the characterization of the solution to (3.8) by a BSDE. We introduce the following BSDE

$$d\underline{J}_t(\xi) = -\inf_{a\in[-\underline{M},\overline{M}]} G(X^\mu_t, \underline{J}_t(\xi), \underline{\tilde{Z}}_t, a)dt + \sigma\underline{\tilde{Z}}_t dW_t \, , \quad \underline{J}_T(\xi) = \exp(\gamma\xi) \, , \qquad (3.9)$$

where

$$G(x, j, \tilde{z}, a) := \gamma(k(a) - p(x)xa)j - a\tilde{z} \, .$$

This BSDE has a Lipschitz generator and square integrable terminal condition from (2.5). Therefore it admits a unique solution in $\mathcal{S}^2 \times \mathbb{H}^2_T$. Moreover, for any $\alpha \in \mathcal{A}$, we notice that $(J^\alpha(\xi), \tilde{Z}^\alpha)$ satisfies the following BSDE

$$dJ^\alpha_t(\xi) = -G(X^\mu_t, J^\alpha_t(\xi), \tilde{Z}^\alpha_t, \alpha_t)dt + \sigma\tilde{Z}^\alpha_t dW_t \, , \quad J^\alpha_T(\xi) = \exp(\gamma\xi) \, .$$

By classical comparison Theorem (see for instance [10, Theorem 2.2]), we have

$$\underline{J}_t(\xi) \leq J(t, \xi) \, , \quad \forall t \in [0, T] \, .$$

Then, we notice that BSDE (3.9) can be rewritten

$$d\underline{J}_t(\xi) = -G\Big(X^\mu_t, \underline{J}_t(\xi), \underline{\tilde{Z}}_t, a^*\big(X^\mu_t, \frac{\underline{\tilde{Z}}_t}{\gamma\underline{J}_t(\xi)}\big)\Big)dt + \sigma\underline{\tilde{Z}}_t dW_t \, , \quad \underline{J}_T(\xi) = \exp(\gamma\xi) \, .$$

In particular, we have $\underline{J}(\xi) = J^{a^*\big(X^\mu, \frac{\underline{\tilde{Z}}}{\gamma\underline{J}(\xi)}\big)}(\xi)$ by uniqueness of the solution to BSDE (3.9). Therefore, we get

$$\underline{J}_t(\xi) = J(t, \xi) \quad \text{and} \quad a^*\big(X^\mu, \frac{\underline{\tilde{Z}}}{\gamma\underline{J}(\xi)}\big) \in \mathcal{A}^*(\xi) \, . \qquad (3.10)$$

We now prove that this optimal effort is unique. Let $\tilde{\alpha} \in \mathcal{A}$ be another optimal effort, then we have

$$J^{\tilde{\alpha}}_0 = J_0^{a^*\big(X^\mu, \frac{\underline{\tilde{Z}}}{\gamma\underline{J}(\xi)}\big)} \, .$$

From strict comparison Theorem (see again [10, Theorem 2.2]) we get $J^{\tilde{\alpha}} = J^{a^*\big(X^\mu, \frac{\underline{\tilde{Z}}}{\gamma\underline{J}(\xi)}\big)}$ and

$$G\Big(X^\mu, \underline{J}(\xi), \underline{\tilde{Z}}, a^*\big(X^\mu, \frac{\underline{\tilde{Z}}}{\gamma\underline{J}(\xi)}\big)\Big) = G\Big(X^\mu, \underline{J}(\xi), \underline{\tilde{Z}}, \tilde{\alpha}\Big) \, , \quad dt \otimes d\mathbb{P} - a.e.$$

By the uniqueness of the minimizer of $G(X^\mu_t, \underline{J}_t(\xi), \underline{\tilde{Z}}_t, \cdot)$, we deduce that $\tilde{\alpha} = a^*\big(X^\mu, \frac{\underline{\tilde{Z}}}{\gamma\underline{J}(\xi)}\big)$.

Step 2. Representation of ξ.

Since, by definition, the process $J^{a^*\left(X^\mu, \frac{\tilde Z}{\gamma \underline{J}(\xi)}\right)}$ is positive, we can define the processes Y and Z by

$$Y \;:=\; \frac{\log\left(J^{a^*\left(X^\mu, \frac{\tilde Z}{\gamma \underline{J}(\xi)}\right)}\right)}{\gamma} \quad \text{and} \quad Z \;:=\; \frac{\tilde Z}{\gamma J^{a^*\left(X^\mu, \frac{\tilde Z}{\gamma \underline{J}(\xi)}\right)}} \, . \tag{3.11}$$

We obtain

$$\begin{aligned} dY_t &= -\big(k(a^*(X^\mu, Z_t)) - p(X^\mu_t)X^\mu_t a^*(X^\mu, Z_t) - a^*(X^\mu, Z_t)Z_t + \frac{\sigma^2}{2}\gamma|Z_t|^2\big)dt + \sigma Z_t dW_t \, , \\ Y_T &= \xi \, . \end{aligned}$$

We first prove $Y \in \mathcal{S}^2$. Note for any $t \in [0, T]$, by using Jensen inequality, we have

$$\frac{1}{\gamma}\log(\underline{J}_t(\xi)) \;\;\geq\;\; \mathbb{E}^{\alpha^*}_t\Big[\int_t^T \big(k(\alpha^*_s) - p(X^\mu_s)X^\mu_s\alpha^*_s\big)ds + \xi\Big] \;\geq\; \mathbb{E}^{\alpha^*}_t[\xi] - TPM \, ,$$

where α^* stands for $a^*\big(X^\mu, \frac{\tilde Z}{\gamma \underline{J}(\xi)}\big)$ and $M = \underline{M} \vee \overline{M}$. We then notice

– if $\underline{J}_t(\xi) \geq 1$ we have

$$0 \;\;\leq\;\; \log(\underline{J}_t(\xi)) \;\;\leq\;\; \underline{J}_t(\xi) \, ,$$

– if $0 \leq \underline{J}_t(\xi) < 1$ we have

$$\Big|\frac{1}{\gamma}\log(\underline{J}_t(\xi))\Big| \;\;\leq\;\; TPM + \mathbb{E}^{\alpha^*}_t[|\xi|] \, .$$

Hence, there exists a constant $C > 0$ such that

$$\Big|\frac{1}{\gamma}\log(\underline{J}_t(\xi))\Big|^2 \;\;\leq\;\; C(1 + \mathbb{E}^{\alpha^*}_t[|\xi|]^2) + \frac{1}{\gamma^2}|\underline{J}_t(\xi)|^2 \, , \quad t \in [0, T] \, .$$

From Young inequality, we get

$$\mathbb{E}\Big[\sup_{t\in[0,T]}\Big|\frac{1}{\gamma}\log(\underline{J}_t(\xi))\Big|^2\Big] \;\;\leq\;\; 2C\Big(1 + \mathbb{E}\Big[\sup_{t\in[0,T]}\Big(\frac{H^{\alpha^*}_T}{H^{\alpha^*}_t}\Big)^4\Big] + \mathbb{E}[|\xi|^4]\Big) + \frac{1}{\gamma^2}\mathbb{E}\Big[\sup_{t\in[0,T]}|\underline{J}_t(\xi)|^2\Big] \, .$$

Since α^* is bounded, we have $\mathbb{E}\Big[\sup_{t\in[0,T]}\Big(\frac{H^{\alpha^*}_T}{H^{\alpha^*}_t}\Big)^4\Big] < +\infty$. Using $\underline{J}(\xi) \in \mathcal{S}^2$, we obtain

$$\mathbb{E}[\sup_{t\in[0,T]}\Big|\frac{1}{\gamma}\log(\underline{J}_t(\xi))\Big|^2] < +\infty \, .$$

Which implies $Y \in \mathcal{S}^2$.

We now check $Z \in \mathbb{H}^2_T$. To this end, we use a localization procedure by introducing the sequence of stopping times $(\tau_n)_{n\geq 1}$ defined by

$$\tau_n \;\;:=\;\; \inf\Big\{t \in [0, T], \; \int_0^t |Z_s|^2 ds \geq n\Big\} \wedge T \, ,$$

for any $n \geq 1$. Similarly to the proof of [6, Theorem 2], we apply Itô's Formula to $\iota(|Y|)$ where $\iota(x) = \frac{1}{\gamma^2}(e^{\gamma x} - \gamma x - 1)$ for $x \in \mathbb{R}$. We obtain

$$\begin{aligned} \iota(|Y_0|) \;\;&=\;\; \iota(|Y_{\tau_n}|) + \int_0^{\tau_n}\Big(\iota'(|Y_s|)\mathrm{sgn}(Y_s)\big(g(X^\mu_s, Z_s) + \frac{\sigma^2\gamma}{2}|Z_s|^2\big) - \frac{1}{2}\iota''(|Y_s|)\sigma^2|Z_s|^2\Big)ds \\ &\quad - \int_0^{\tau_n}\sigma\iota'(|Y_s|)\mathrm{sgn}(Y_s)Z_s dW_s \, . \end{aligned}$$

Since $\iota'' - \gamma\iota' = 1$ and $\iota'(x) \geq 0$ for $x \geq 0$, we get from BDG and Young inequalities

$$\mathbb{E}\Big[\int_0^{\tau_n} |Z_s|^2 ds\Big] \quad \leq \quad C\Big(1 + \mathbb{E}\Big[\sup_{t\in[0,T]} e^{\gamma|Y_t|} + \int_0^T e^{\gamma|Y_t|}\big(1+|Y_s|\big)\Big]\Big) . \tag{3.12}$$

From the definition of Y and since α^* is bounded, there exists a constant C such that

$$2\gamma|Y_t| \quad \leq \quad C + 2\gamma\mathbb{E}_t^{\alpha^*}[|\xi|] .$$

Using Jensen and Hölder inequalities we get another constant C' such that

$$\mathbb{E}\Big[\sup_{t\in[0,T]} e^{2\gamma|Y_t|}\Big] \leq C'\mathbb{E}\Big[\sup_{t\in[0,T]} \Big(\frac{H_T^{\alpha^*}}{H_t^{\alpha^*}}\Big)^{\frac{\gamma'}{\gamma'-\gamma}}\Big]^{\frac{\gamma'-\gamma}{\gamma'}} \mathbb{E}\Big[e^{2\gamma'|\xi|}\Big]^{\frac{\gamma}{\gamma'}} .$$

Since α^* is bounded, we have $\mathbb{E}\Big[\sup_{t\in[0,T]}\Big(\frac{H_T^{\alpha^*}}{H_t^{\alpha^*}}\Big)^{\frac{\gamma'}{\gamma'-\gamma}}\Big] < +\infty$ and we get from (2.5)

$$\mathbb{E}\Big[\sup_{t\in[0,T]} e^{2\gamma|Y_t|}\Big] \quad < \quad +\infty .$$

Sending n to ∞ in (3.12), we get from Fatou's Lemma $Z \in \mathbb{H}^2$.

Step 3. Conclusion. We directly deduce (ii) and (iii) from (3.11) together with (3.10) given that (i) has been proved in Step 2. □

4 The problem of the regulator

In this section, we focus on the regulation policy. In view of (2.3) and Theorem 3.1 the regulator's problem turns to be

$$V_R^P \quad = \quad \sup_{\xi\in\mathcal{C}_R^\mu} \mathbb{E}^{\alpha^*(\xi)}[\xi - f(X_T^\mu)] . \tag{4.13}$$

We first provide almost optimal contracts for a bounded parameter μ by a PDE approach. We then extend the study to the logistic case with $\mu(x) = x$.

4.1 Almost optimal strategies for bounded auto-degradation and cost parameters

We introduce the following class of contracts

$$\begin{aligned} \Xi^\mu \quad := \quad \Big\{ & Y_T^{Y_0,Z,\mu} = Y_0 - \int_0^T \big(h(X_t^\mu, Z_t) + \frac{\sigma^2}{2}\gamma|Z_t|^2\big)dt + \int_0^T \sigma Z_t dW_t , \\ & Y_0 \leq \tilde{R} ,\ Z \in \mathcal{Z}\Big\} , \end{aligned} \tag{4.14}$$

where $\mathcal{Z}$ denotes the subset of predictable processes of $\mathbb{H}_T^2$ such that

$$\mathbb{E}[\exp(\gamma'|Y_T^{Y_0,Z,\mu}|)] \quad < \quad +\infty , \tag{4.15}$$

for some $\gamma' > 2\gamma$ and we recall that $\tilde{R} = \frac{\log(-R)}{\gamma}$. When μ is the identity, we omit the exponent μ in the previous definitions.

From Theorem 3.1, constraint (2.4) and integrability conditions (2.5) and (4.15), the set $\mathcal{C}_R^\mu$ coincides with Ξ^μ so that the regulator's problem (4.13) becomes

$$V_R^P \quad = \quad \sup_{Y_0\leq\tilde{R},\ Z\in\mathcal{Z}} \mathbb{E}^{a^*(X^\mu,Z)}[Y_T^{Y_0,Z,\mu} - f(X_T^\mu)] , \tag{4.16}$$

with

$$Y_t^{Y_0,Z,\mu} \quad = \quad Y_0 - \int_0^t \big(k(\alpha_s^*) - p(X_s^\mu)X_s^\mu a^*(X_s^\mu, Z_s) + \frac{\sigma^2}{2}\gamma|Z_s|^2\big)dt + \int_0^t \sigma Z_s dW_s^* \ , \quad t \in [0,T] \ ,$$

where W^* stands for $W^{a^*(X^\mu,Z)}$. We notice that the function to maximize in V_R^P is non-deacreasing w.r.t. the variable Y_0. Therefore the constraint $Y_0 \leq \tilde{R}$ is saturated and (4.16) can be rewritten under the following form

$$V_R^P \quad = \quad \sup_{Z\in\mathcal{Z}} \mathbb{E}^{a^*(X^\mu,Z)}[Y_T^{\tilde{R},Z,\mu} - f(X_T^\mu)] \ . \tag{4.17}$$

To construct a solution to the problem (4.17), we introduce the related HJB PDE given by

$$\left\{ \begin{array}{l} -\partial_t v - H\Big(x, \partial_x v(t,x), \partial_{xx} v(t,x)\Big) = 0 \ , \quad (t,x) \in [0,T) \times \mathbb{R}_+^* \ , \\ v(T,x) = -f(x) \ , \quad x \in \mathbb{R}_+^* \ , \end{array} \right. \tag{4.18}$$

where the Hamiltonian H is given by

$$\begin{aligned} H(x,\delta_1,\delta_2) \quad = \quad & \sup_{z\in\mathbb{R}} \Big\{ xp(x)a^*(x,z) - k(a^*(x,z)) - \frac{\sigma^2}{2}\gamma z^2 + x(\lambda - \mu(x) - a^*(x,z))\delta_1 \Big\} \\ & + \frac{\sigma^2}{2} x^2 \delta_2 \ , \quad (x,\delta_1,\delta_2) \in \mathbb{R}_+^* \times \mathbb{R} \times \mathbb{R} \ , \end{aligned}$$

and a^* is given by (3.7). We first extend PDE (4.18) to the whole domain $[0,T]\times\mathbb{R}$ by considering the change of variable $w(t,y) := v(t,e^y)$ for any $(t,y) \in [0,T] \times \mathbb{R}$. We get the following PDE

$$\left\{ \begin{array}{l} -\partial_t w - \mathcal{H}\Big(y, \partial_y w(t,y), \partial_{yy} w(t,y)\Big) = 0 \ , \quad (t,y) \in [0,T) \times \mathbb{R} \ , \\ w(T,y) = -f(e^y) \ , \quad y \in \mathbb{R} \ , \end{array} \right. \tag{4.19}$$

where

$$\begin{aligned} \mathcal{H}(y,\delta_1,\delta_2) \quad := \quad & \sup_{z\in\mathbb{R}} \Big\{ e^y p(e^y)a^*(e^y,z) - \frac{a^*(e^y,z)^2}{2} - \frac{\sigma^2}{2}\gamma z^2 + (\lambda - \frac{\sigma^2}{2} - \mu(e^y) - a^*(e^y,z))\delta_1 \Big\} \\ & + \frac{\sigma^2}{2}\delta_2 \ , \quad (y,\delta_1,\delta_2) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} \ . \end{aligned}$$

Our aim is to construct a regular solution to this PDE to proceed by verification. Unfortunately, the coefficients of PDE (4.19) are not smooth enough to do so. To overcome this issue, we provide a smooth approximation $\mathcal{H}_\varepsilon$ of $\mathcal{H}$ for which we get regular solutions.

Moreover, we introduce the following assumption, which ensure that the optimal control derived from the PDE satisfies the admissibility condition, *i.e.* belongs to $\mathcal{Z}$.

(H') There exists $\nu \in (0,1)$ such that

(i) the map $y \mapsto \mu(e^y)$ belongs to $C^{1+\nu}(\mathbb{R})$,

(ii) the map $y \mapsto f(e^y)$ belongs to $C^{2+\nu}(\mathbb{R})$,

(iii) the map $y \mapsto p(e^y)e^y$ belongs to $C^{1+\nu}(\mathbb{R})$.

Proposition 4.1. *Under **(H')**, there exists a family $\{\mathcal{H}_\varepsilon, \ \varepsilon > 0\}$ of functions from $\mathbb{R}^3$ to $\mathbb{R}$ such that the PDE*

$$\left\{ \begin{array}{l} -\partial_t w_\varepsilon - \mathcal{H}_\varepsilon\Big(y, \partial_y w_\varepsilon(t,y), \partial_{yy} w_\varepsilon(t,y)\Big) = 0 \ , \quad (t,y) \in [0,T) \times \mathbb{R} \ , \\ w_\varepsilon(T,y) = -f(e^y) \ , \quad y \in \mathbb{R} \ , \end{array} \right. \tag{4.20}$$

admits a unique solution w_ε in $C^{2+\nu}([0,T]\times\mathbb{R})$ and

$$\sup_{\mathbb{R}^3} \big|\mathcal{H} - \mathcal{H}_\varepsilon\big| \quad \leq \quad \varepsilon \tag{4.21}$$

for any $\varepsilon > 0$.

The proof of Proposition 4.1 consists in an approximation by regularization of the original Hamiltonian $\mathcal{H}$. As it is quite technical we postpone this proof to the appendix.

We are now able to describe almost optimal contracts and related almost optimal efforts using the functions w_ε given by Proposition 4.1.

Theorem 4.2. *Suppose that **(H')** holds. For any $\varepsilon > 0$, the tax policy ξ_ε given by*

$$\xi_\varepsilon = \tilde{R} - \int_0^T \big(g(X_t^\mu, Z_t^\varepsilon) + \frac{1}{2}\sigma^2\gamma|Z_t^\varepsilon|^2 + Z_t^\varepsilon(\lambda - \mu(X_t^\mu))\big)dt + \int_0^T \frac{Z_t^\varepsilon}{X_t^\mu} dX_t^\mu ,$$

where

$$Z_t^\varepsilon \quad = \quad -\frac{\partial_x w_\varepsilon(t, \log(X_t^\mu))}{1+\gamma\sigma^2} , \quad t \in [0,T] , \tag{4.22}$$

is $2T\varepsilon$-optimal for the regulator problem:

$$V_R^P \quad \leq \quad \mathbb{E}^{a^*(X^\mu, Z^\varepsilon)}\big[\xi_\varepsilon - f(X_T^\mu)\big] + 2T\varepsilon .$$

Proof. We fix some control $Z \in \mathcal{Z}$ and we apply Itô's formula to the process $\big(Y_t^{\tilde{R},Z,\mu} + w_\varepsilon(t, \log(X_t^\mu))\big)_{t\in[0,T]}$

$$\begin{aligned}
Y_T^{\tilde{R},Z,\mu} + w_\varepsilon(T, \log(X_T^\mu)) \quad &= \quad \tilde{R} + w_\varepsilon(0, \log(X_0)) \\
&\quad + \int_0^T \Big(\partial_t w_\varepsilon(s, \log(X_s^\mu)) \\
&\quad +(\lambda - \frac{\sigma^2}{2} - \mu(X_s^\mu) - a^*(X_s^\mu, Z_s))\partial_x w_\varepsilon(s, \log(X_s^\mu)) \\
&\quad +p(X_s^\mu)a^*(X_s^\mu, Z_s)X_s^\mu - k(a^*(X_s^\mu, Z_s)) - \frac{\sigma^2}{2}\gamma|Z_s|^2 \\
&\quad +\frac{\sigma^2}{2}\partial_{xx} w_\varepsilon(s, \log(X_s^\mu))\Big)ds \\
&\quad +\sigma\int_0^T (\partial_x w_\varepsilon(s, \log(X_s^\mu)) + Z_s)\, dW_s^* ,
\end{aligned}$$

where W^* stands for $W^{a^*(X^\mu,Z)}$. Since $w_\varepsilon \in C^{2+\nu}([0,T]\times\mathbb{R})$ and $Z \in \mathcal{Z}$ we get

$$\begin{aligned}
\mathbb{E}^{a^*(X^\mu,Z)}\Big[Y_T^{\tilde{R},Z,\mu} + w_\varepsilon(T, \log(X_T^\mu))\Big] \ &\leq \ \tilde{R} + w_\varepsilon(0, \log(X_0)) \\
&\quad + \int_0^T \mathbb{E}^{a^*(X^\mu,Z)}\Big[\Big(\partial_t w_\varepsilon + \mathcal{H}\Big(., \partial_y w_\varepsilon, \partial_{yy} w_\varepsilon\Big)\Big)(s, \log(X_s^\mu))\Big]ds .
\end{aligned}$$

From (4.21) we get

$$\begin{aligned}
\mathbb{E}^{a^*(X^\mu,Z)}\Big[Y_T^{\tilde{R},Z,\mu} + w_\varepsilon(T, \log(X_T^\mu))\Big] \ &\leq \ \tilde{R} + w_\varepsilon(0, \log(X_0)) + T\varepsilon \\
&\quad + \int_0^T \mathbb{E}^{a^*(X^\mu,Z)}\Big[\Big(\partial_t w_\varepsilon + \mathcal{H}_\varepsilon\Big(., \partial_y w_\varepsilon, \partial_{yy} w_\varepsilon\Big)\Big)(s, \log(X_s^\mu))\Big]ds ,
\end{aligned}$$

and since w_ε is solution to (4.20), we get

$$\mathbb{E}^{a^*(X^\mu,Z)}\Big[Y_T^{\tilde{R},Z,\mu} - f(X_T^\mu)\Big] \quad \leq \quad \tilde{R} + w_\varepsilon(0, \log(X_0)) + T\varepsilon .$$

Since Z is arbitrarily chosen in $\mathcal{Z}$ we get

$$V_R^P \quad \leq \quad \tilde{R} + w_\varepsilon(0, \log(X_0)) + T\varepsilon . \tag{4.23}$$

We now take $Z = Z^\varepsilon$ where Z^ε is given by (4.22). We now notice that $Z^\varepsilon \in \mathcal{Z}$ since Z^ε is bounded, and by definition of Z^ε we have

$$\begin{aligned}
& \mathcal{H}\Big(\log(X^\mu), \partial_y w_\varepsilon(., \log(X^\mu)), \partial_{yy} w_\varepsilon(., \log(X^\mu))\Big) \\
= \quad & X^\mu p(X^\mu) a^*(X^\mu, Z^\varepsilon) - \frac{a^*(X^\mu, Z^\varepsilon)^2}{2} - \frac{\sigma^2}{2}\gamma\big|Z^\varepsilon\big|^2 + \frac{\sigma^2}{2}\partial_{yy} w_\varepsilon(., \log(X^\mu)) \\
& + \quad \big(\lambda - \frac{\sigma^2}{2} - \mu(X^\mu) - a^*(X^\mu, Z^\varepsilon)\big)\partial_y w_\varepsilon(., \log(X^\mu))
\end{aligned}$$

for any $[0, T]$. A straightforward application of Itô's formula and Girsanov Theorem give

$$\begin{aligned}
\mathbb{E}^{a^*(X^\mu, Z^\varepsilon)}[Y_T^{\tilde{R}, Z^\varepsilon, \mu} - f(X_T^\mu)] \quad = \quad & \tilde{R} + w_\varepsilon(0, \log(X_0)) \\
& + \int_0^T \mathbb{E}^{a^*(X^\mu, Z^\varepsilon)}\Big[\Big(\partial_t w_\varepsilon + \mathcal{H}\Big(., \partial_y w_\varepsilon, \partial_{yy} w_\varepsilon\Big)\Big)(s, \log(X_s^\mu))\Big] ds\ .
\end{aligned}$$

From Propositions (4.20) and (4.21) we get

$$\mathbb{E}^{a^*(X^\mu, Z^\varepsilon)}[Y_T^{\tilde{R}, Z^\varepsilon, \mu} - f(X_T^\mu)] \quad \geq \quad \tilde{R} + w_\varepsilon(0, \log(X_0)) - T\varepsilon\ .$$

Hence, we get from (4.23)

$$V_R^P \quad \leq \quad \mathbb{E}^{a^*(X^\mu, Z^\varepsilon)}[Y_T^{\tilde{R}, Z^\varepsilon, \mu} - f(X_T^\mu)] + 2T\varepsilon\ .$$

Therefore, we get $\xi_\varepsilon = Y_T^{\tilde{R}, Z^\varepsilon, \mu}$ is a $2T\varepsilon$-optimal policy for the regulator. □

4.2 Extension to the logistic equation and continuous cost

We consider in this section an approximation method to build a sequence of almost optimal taxes in the case the classical logistic dynamic for SDE (2.1), i.e. $\mu(x) = x$. More precisely, we introduce a sequence of approximated models from which we derive almost optimal strategy from the previous section. We show that this sequence remains almost optimal for the logistic model. We also weaken the assumption **(H')** (ii) as follows.
(H$_f$') The function f is bounded and continuous on $\mathbb{R}$.

We introduce the sequence of mollifiers $\rho_n : \mathbb{R} \to \mathbb{R}$, $n \geq 1$, defined by

$$\rho_n(x) \quad := \quad \frac{n\rho(nx)}{\int_{\mathbb{R}} \rho(u)du}\ , \qquad x \in \mathbb{R}\ ,$$

where the function $\rho : \mathbb{R} \to \mathbb{R}$ is defined by

$$\rho(x) := \exp\big(\frac{-1}{1 - |x|^2}\big)\mathbb{1}_{|x]<1}\ .$$

We then define the functions f_n, $n \geq 1$, by

$$f_n(x) \quad := \quad \int_{\mathbb{R}} f(y)\rho_n(x - y)dy\ , \quad x \in \mathbb{R}\ .$$

From classical results, we know that f_n satisfies **(H')**(ii) for all $n \geq 1$ and f_n converges to f as n goes to infinity uniformly on every compact subset of $\mathbb{R}$.
We also define the functions $\mu_n : \mathbb{R} \to \mathbb{R}$, $n \geq 1$, by

$$\mu_n(x) \quad := \quad x\big(\Theta(x + e^n + 1) - \Theta(x - (e^n + 1))\big)\ , \quad x \in \mathbb{R}\ , \tag{4.24}$$

where the function $\Theta : \mathbb{R} \to \mathbb{R}$ is given by

$$\Theta(u) \quad := \quad \frac{\int_{-\infty}^u \rho(r)dr}{\int_{\mathbb{R}} \rho(r)dr}\ , \quad u \in \mathbb{R}\ . \tag{4.25}$$

We then notice that μ_n satisisfes **(H')** (i) and

$$\mu_n(x) \quad = \quad x\,, \quad x \in [-e^n, e^n]\,,$$

for $n \geq 1$. We first have the following preliminary result on the convergence of X^{μ_n} to X.

Lemma 4.1. *There exists a constant C such that*

$$\sup_{t\in[0,T]} \mathbb{E}\Big[\big|X_t^{\mu_n} - X_t\big|^2\Big] \quad \leq \quad C\exp\Big(-\frac{n^2}{4\sigma^2}\Big)\,,$$

for all $n \geq 1$.

Proof. We define the sequence of stopping times $(\tau_n)_{n\geq 1}$ by

$$\tau_n \quad := \quad \inf\{t \in [0,T],\ X_t \geq e^n\}\,, \quad n \geq 1\,. \tag{4.26}$$

We then notice that

$$\begin{aligned} X_{t\wedge\tau_n} \quad &= \quad x + \int_0^{t\wedge\tau_n} X_s(\lambda - X_s)ds + \int_0^{t\wedge\tau_n}\sigma X_s dW_s \\ &= \quad x + \int_0^t \mathbf{1}_{s\leq\tau_n} X_s(\lambda - \mu_n(X_s))ds + \int_0^t \sigma\mathbf{1}_{s\leq\tau_n}X_s dW_s\,, \end{aligned}$$

for all $t \in [0,T]$. Therefore $(X_{t\wedge\tau_n})_{t\in[0,T]}$ and $(X^{\mu_n}_{t\wedge\tau_n})_{t\in[0,T]}$ satisfy the same SDE with random and Lipschitz coefficients. By strong uniqueness, we have

$$X_{t\wedge\tau_n} \quad = \quad X^{\mu_n}_{t\wedge\tau_n}\,, \quad t \in [0,T]\,.$$

Which implies

$$\sup_{t\in[0,T]} \mathbb{E}\Big[\big|X_t^{\mu_n} - X_t\big|^2\Big] = \sup_{t\in[0,T]} \mathbb{E}\Big[\big|X_t^{\mu_n} - X_t\big|^2 \mathbb{1}_{\tau_n\leq t}\Big]\,.$$

Hence, by using Cauchy-Schwarz Inequality, we have

$$\sup_{t\in[0,T]} \mathbb{E}\Big[\big|X_t^{\mu_n} - X_t\big|^2\Big] \quad \leq \quad \sup_{t\in[0,T]} \mathbb{E}\Big[\big|X_t^{\mu_n} - X_t\big|^4\Big]^{\frac{1}{2}} \mathbb{P}\big(\tau_n \leq T\big)^{\frac{1}{2}}\,, \tag{4.27}$$

for all $n \geq 1$.

We then notice that

$$X^{\mu_0} \geq X^{\mu_n} \geq X > 0 \tag{4.28}$$

for all $n \geq 1$. Indeed, by setting $\delta^n := X^{\mu_n} - X$, we have

$$\delta_t^n = \int_0^t b_s\delta_s^n ds + \int_0^t \sigma\delta_s^n dW_s + \int_0^t c_s ds$$

where

$$b := \begin{cases} \frac{X^{\mu_n}(\lambda - \mu_n(X^{\mu_n})) - X(\lambda - \mu_n(X))}{\delta_n} & \text{if} \quad X^{\mu_n} - X \neq 0\,, \\ 0 & \text{if} \quad X^{\mu_n} - X = 0\,, \end{cases}$$

is a bounded process since $x \mapsto x(\lambda - \mu_n(x))$ is Lipschitz continuous, and

$$c \quad = \quad X(\lambda - \mu_n(X)) - X(\lambda - X)$$

is a nonnegative process since $\mu_n(x) \leq x$ for $x \in [0, +\infty)$. A straightforward computation shows that

$$\delta_t^n \quad = \quad R_t\int_0^t \frac{c_s}{R_s}ds\,, \quad t \in [0,T]\,,$$

where $R_t = \exp(\int_0^t (b_s - \sigma^2/2)ds + \sigma W_t)$ for $t \in [0,T]$. Since $c \geq 0$ we get $X^{\mu_n} \geq X$. The same argument applied to $\tilde{\delta}^n = X^{\mu_n} - X^{\mu_{n+1}}$ gives $X^{\mu_n} \geq X^{\mu_{n+1}}$.

From (4.27) and (4.28), there exists a constant C such that

$$\sup_{t\in[0,T]} \mathbb{E}\Big[\big|X_t^{\mu_n} - X_t\big|^2\Big] \quad \leq \quad C\mathbb{P}\big(\tau_n \leq T\big)^{\frac{1}{2}} \ , \quad n \geq 1 \ . \tag{4.29}$$

Still using $X^{\mu_0} \geq X > 0$ and since X^{μ_0} is a geometric drifted brownian motion, we have

$$\mathbb{P}\big(\tau_n \leq T\big) \quad \leq \quad \mathbb{P}\Big(\sup_{t\in[0,T]} W_t \geq \frac{n - (\lambda - \sigma^2/2)T}{\sigma}\Big) \ .$$

Since $\sup_{t\in[0,T]} W_t$ has the same law as $|W_T|$, we get

$$\mathbb{P}\big(\tau_n \leq T\big) \leq 2\int_{\frac{n-(\lambda-\sigma^2/2)T}{\sigma}}^{+\infty} e^{\frac{-y^2}{2}} \frac{dy}{\sqrt{2\pi}} \leq C\exp\big(-\frac{n^2}{2\sigma^2}\big) \ , \quad n \geq 1 \ , \tag{4.30}$$

and we get the result from (4.29). □

We then define the function $V_{R,n}^P$ as the optimal value of the regulator problem in the model with coefficients μ_n and f_n in place of μ and f respectively

$$V_{R,n}^P \quad = \quad \sup_{\xi\in\mathcal{C}_R^{\mu_n}} \mathbb{E}^{a^*(X^{\mu_n},Z)}[\xi - f_n(X_T^{\mu_n})] \ .$$

Since μ_n and f_n satisfy Assumptions **(H')** (i) and **(H')** (ii) respectively for all $n \geq 1$, we get, from Theorem 4.2, a sequence of bounded processes $(Z^{\varepsilon,n})_{n\geq 1}$ such that

$$V_{R,n}^P \quad \leq \quad \mathbb{E}^{a^*(X^{\mu_n},Z^{\varepsilon,n})}[Y_T^{\tilde{R},Z^{\varepsilon,n},\mu_n} - f_n(X_T^{\mu_n})] + 2T\varepsilon \ , \quad n \geq 1 \ .$$

We introduce the control $\tilde{Z}^{\varepsilon,n}$ defined by

$$\tilde{Z}_t^{\varepsilon,n} \quad := \quad Z_t^{\varepsilon,n}\mathbb{1}_{[0,\tau_n]}(t) \ , \quad t \in [0,T] \ ,$$

where the stopping time τ_n is defined in (4.26), and we denote by $\tilde{\xi}_{\varepsilon,n}$ the related contract

$$\tilde{\xi}_{\varepsilon,n} = Y_T^{R,\tilde{Z}^{\varepsilon,n}} \ .$$

We then have the following almost optimality result.

Theorem 4.3. *Suppose that **(H$_f$ ')** and **(H')** (iii) hold. Then $V_R^P < +\infty$ and we have*

$$\limsup_{n\to+\infty}\Big(V_R^P - \mathbb{E}^{\mathbb{P}^{a^*(X,\tilde{Z}^{\varepsilon,n})}}\big[\tilde{\xi}_{\varepsilon,n} - f(X_T)\big]\Big) \leq 2T\varepsilon$$

for any $\varepsilon > 0$.

Proof. We proceed in four steps.

Step 1. The optimal value V_R^P is finite.
From (4.17) and the dynamics of $Y^{\tilde{R},Z}$ we have

$$V_R^P \quad \leq \quad \sup_{Z\in\mathcal{Z}} \mathbb{E}^{a^*(X,Z)}[\tilde{R} + T\underline{M}P - f(X_T)] \quad < \quad +\infty \ .$$

Since f is bounded, we get $V_R^P < +\infty$.

Step 2. Comparison of Ξ^μ and $\Xi^{\mu'}$.

We fix two functions μ, μ' satisfying **(H0)** and we show that $\Xi^\mu = \Xi^{\mu'}$where Ξ^μ and $\Xi^{\mu'}$ are defined by (4.14). Let $\xi = Y_T^{Y_0,Z,\mu} \in \Xi^\mu$. Then, we have by definition

$$\mathbb{E}[\exp(\gamma'|Y_T^{Y_0,Z,\mu}|)] \quad < \quad +\infty \ ,$$

for some $\gamma' > 2\gamma$, with

$$Y_T^{Y_0,Z,\mu} = Y_0 - \int_0^T \big(\frac{|a^*(X_s^\mu, Z_s)|^2}{2} - a^*(X_s^\mu, Z_s)(p(X_s^\mu)X_s^\mu + Z_s) + \frac{\sigma^2}{2}\gamma|Z_s|^2\big)ds + \int_0^T \sigma Z_s dW_s \ .$$

Since the optimal effort a^* is bounded and Assumption **(H')** (iii) holds, there exists a positive constant C such that

$$\begin{aligned}
\mathbb{E}\Big[\exp(\gamma'|Y_T^{Y_0,Z,\mu'}|)\Big] \quad &= \quad \mathbb{E}\Big[e^{\gamma'\left|Y_0 - \int_0^T \left(\frac{|a^*(X_s^{\mu'},Z_s)|^2}{2} - a^*(X_s^{\mu'},Z_s)(p(X_s^{\mu'})X_s^{\mu'}+Z_s) + \frac{\sigma^2}{2}\gamma|Z_s|^2\right)ds + \int_0^T \sigma Z_s dW_s\right|}\Big] \\
&\leq \quad C\mathbb{E}\Big[e^{\gamma'\left(\left|Y_0 - \int_0^T \left(\frac{|a^*(X_s^{\mu},Z_s)|^2}{2} - a^*(X_s^{\mu'},Z_s)Z_s + \frac{\sigma^2}{2}\gamma|Z_s|^2\right)ds + \int_0^T \sigma Z_s dW_s\right|\right)}\Big] \\
&\leq \quad C\mathbb{E}\Big[e^{\gamma'|Y_T^{Y_0,Z,\mu}|}e^{\delta_T^*}\Big] \ ,
\end{aligned}$$

where $\delta_T^* = \gamma' \int_0^T |a^*(X_s^{\mu'}, Z_s) - a^*(X_s^\mu, Z_s)||Z_s|ds$. We then notice that

$$\begin{aligned}
|a^*(X', z) - a^*(X, Z)||z| \quad &= \quad \big|(\overline{M} \wedge (p(x')x' + z) \vee (-\underline{M})) - (\overline{M} \wedge (p(x)x + z) \vee (-\underline{M}))\big||z| \\
&\leq \quad (\overline{M} + \underline{M})(P + \overline{M} + \underline{M})
\end{aligned}$$

for any $z \in \mathbb{R}$. Hence, we get

$$\mathbb{E}[\exp(\gamma'|Y_T^{Y_0,Z,\mu'}|)] \leq Ce^{\gamma' T(\overline{M}+\underline{M})(P+\overline{M}+\underline{M})}\mathbb{E}\big[e^{\gamma'|Y_T^{Y_0,Z,\mu}|}\big] \ .$$

Hence, we get $\xi \in \Xi^\mu$. We then write Ξ for Ξ^μ in the sequel.

Step 3. Convergence of the values for a given Z.
We fix $Z \in \Xi$. We then have

$$\begin{aligned}
\Delta_n(Z) &:= \big|\mathbb{E}^{a^*(X^{\mu_n},Z)}[Y_T^{\tilde{R},Z,\mu_n} - f_n(X_T^{\mu_n})] - \mathbb{E}^{a^*(X,Z)}[Y_T^{\tilde{R},Z} - f(X_T)]\big| \\
&\leq \Delta_1^n(Z) + \Delta_2^n(Z) \ ,
\end{aligned}$$

with

$$\Delta_1^n(Z) \quad = \quad \mathbb{E}\Big[|H_T^{a^*(X^{\mu_n},Z)} - H_T^{a^*(X,Z)}||Y_T^{\tilde{R},Z} - f(X_T)|\Big]$$

and

$$\Delta_2^n(Z) \quad = \quad \mathbb{E}\Big[H_T^{a^*(X^{\mu_n},Z)}|(Y_T^{\tilde{R},Z} - f(X_T)) - (Y_T^{\tilde{R},Z,\mu_n} - f_n(X_T^{\mu_n}))|\Big] \ .$$

We now study the convergence of Δ_n^1 and Δ_n^2.

Substep 3.1. Convergence of Δ_1^n.
By Cauchy-Schwarz inequality we have

$$\Delta_1^n(Z) \quad \leq \quad \mathbb{E}[|H_T^{a^*(X^{\mu_n},Z)} - H_T^{a^*(X,Z)}|^2]^{\frac{1}{2}}\mathbb{E}[|Y_T^{\tilde{R},Z} - f(X_T)|^2]^{\frac{1}{2}} \ .$$

From (4.15) and **(H$_f$')**, $\mathbb{E}[|Y_T^{\tilde{R},Z} - f(X_T)|^2]^{\frac{1}{2}}$ is uniformly bounded w.r.t. n. The convergence of $\Delta_1^n(Z)$ remains to the convergence of

$$\tilde{\Delta}_1^n(Z) \quad = \quad \mathbb{E}[|H_T^{a^*(X^{\mu_n},Z)} - H_T^{a^*(X,Z)}|^2] \ .$$

From the definition of a^*, for all $q \geq 1$, there exists a constant C_q such that

$$\mathbb{E}[|a^*(X,Z) - a^*(X^{\mu_n},Z)|^q] \quad \leq \quad C_q \mathbb{E}[|a^*(X,Z) - a^*(X^{\mu_n},Z)|^2] \;, \quad n \geq 1 \;.$$

Therefore we have

$$\sup_{t\in[0,T]} \mathbb{E}[|a^*(X_t,Z_t) - a^*(X_t^{\mu_n},Z_t)|^q] \quad \xrightarrow[n\to+\infty]{} \quad 0$$

for all $q \geq 1$. From Lemma 4.1 and Theorem 2.8.1 in [20] we get $\tilde{\Delta}_1^n(Z) \xrightarrow[n\to+\infty]{} 0$.

Substep 3.2. Convergence of Δ_2^n.
From Cauchy-Schwarz Inequality, there exists a positive constant $C > 0$ such that

$$\begin{aligned}
\Delta_2^n(Z) \quad & \leq \quad \mathbb{E}[|H_T^{a^*(X^{\mu_n},Z)}|^2]^{\frac{1}{2}} \mathbb{E}[|(Y_T^{\tilde{R},Z} - f(X_T)) - (Y_T^{\tilde{R},Z,\mu_n} - f_n(X_T^{\mu_n}))|^2]^{\frac{1}{2}} \\
& \leq \quad C\, \mathbb{E}[|H_T^{a^*(X^{\mu_n},Z)}|^2]^{\frac{1}{2}} \Big(\mathbb{E}\big[\big|Y_T^{\tilde{R},Z,\mu_n} - Y_T^{\tilde{R},Z}\big|^2\big]^{\frac{1}{2}} + \mathbb{E}\big[\big|f(X_T) - f_n(X_T^{\mu_n})\big|^2\big]^{\frac{1}{2}} \Big) \;.
\end{aligned}$$

First note that

$$\begin{aligned}
\mathbb{E}[|H_T^{a^*(X^{\mu_n},Z)}|^2] \quad & = \quad \mathbb{E}[e^{-2\int_0^T a^*(X_s^{\mu_n},Z_s)\sigma^{-1}dW_s - \int_0^T |a^*(X_s^{\mu_n},Z_s)|^2\sigma^{-2}ds}] \\
& = \quad \mathbb{E}^{\mathbb{Q}}[e^{\int_0^T |a^*(X_s^{\mu_n},Z_s)|^2\sigma^{-2}ds}] \;,
\end{aligned}$$

with $d\mathbb{Q}/d\mathbb{P} = H_T^{2a^*(X^{\mu_n},Z)}$. Since $|a^*(X^{\mu_n},Z)|$ is bounded by $\underline{M} \vee \overline{M}$, we deduce that

$$\mathbb{E}[|H_T^{a^*(X^{\mu_n},Z)}|^2] \leq e^{\frac{T(\underline{M}\vee\overline{M})^2}{\sigma^2}} .$$

We then have

$$\mathbb{E}\big[\big|f(X_T) - f_n(X_T^{\mu_n})\big|^2\big]^{\frac{1}{2}} \leq C\, \mathbb{E}\big[\big|f(X_T) - f_n(X_T)\big|^2\big]^{\frac{1}{2}} + C\, \mathbb{E}\big[\big|f_n(X_T) - f_n(X_T^{\mu_n})\big|^2\big]^{\frac{1}{2}} \;.$$

Since f is continuous and bounded, we get from the dominated convergence Theorem

$$\mathbb{E}\big[\big|f(X_T) - f_n(X_T)\big|^2\big] \xrightarrow[n\to+\infty]{} 0 \;.$$

Then from the definition of f_n there exists a constant L such that f_n is $n \times L$-Lipchitz continuous for all $n \geq 1$. Therefore, we get from Lemma 4.1

$$\mathbb{E}\big[\big|f_n(X_T) - f_n(X_T^{\mu_n})\big|^2\big] \quad \xrightarrow[n\to+\infty]{} \quad 0 \;.$$

Since a^* is continuous and bounded, and $Z \in \mathcal{Z}$, we get from Lemma 4.1 and the definition of $Y^{\tilde{R},Z,\mu_n}$ and $Y^{\tilde{R},Z}$

$$\mathbb{E}\big[\big|Y_T^{\tilde{R},Z,\mu_n} - Y_T^{\tilde{R},Z}\big|^2\big] \quad \xrightarrow[n\to+\infty]{} \quad 0 \;.$$

Hence we get $\lim_{n\to+\infty} \Delta_2^n(Z) = 0$.

Step 4. Almost optimality of $\tilde{Z}^{\varepsilon,n}$.
We fix $\eta > 0$ and $Z^\eta \in \Xi$ such that

$$V_R^P \quad \leq \quad \mathbb{E}^{a^*(X,Z^\eta)}[Y_T^{\tilde{R},Z^\eta} - f(X_T)] + \eta \;.$$

By definition of $Z^{\varepsilon,n}$, we get

$$V_R^P - \mathbb{E}^{\mathbb{P}^{a^*(X^{\mu_n},Z^{\varepsilon,n})}}[\xi_{\varepsilon,n} - f_n(X_T^{\mu^n})] \quad \leq \quad \eta + 4T\varepsilon + \Delta_n(Z^\eta) \;, \quad n \geq 1 \;.$$

Sending n to ∞, we get from Step 2

$$\limsup_{n\to+\infty} V_R^P - \mathbb{E}^{\mathbb{P}^{a^*(X^{\mu_n},Z^{\varepsilon,n})}}[\xi_{\varepsilon,n} - f_n(X_T^{\mu_n})] \quad \leq \quad 4T\varepsilon + \eta$$

for any $\eta > 0$. Which implies

$$\limsup_{n\to+\infty} V_R^P - \mathbb{E}^{\mathbb{P}^{a^*(X^{\mu_n},Z^{\varepsilon,n})}}[\xi_{\varepsilon,n} - f_n(X_T^{\mu_n})] \quad \leq \quad 4T\varepsilon \ . \tag{4.31}$$

Since f and a^* are bounded, there exists a constant C such that

$$\begin{aligned}
&|\mathbb{E}^{\mathbb{P}^{a^*(X^{\mu_n},Z^{\varepsilon,n})}}[f_n(X_T^{\mu_n})] - \mathbb{E}^{\mathbb{P}^{a^*(X,\tilde{Z}^{\varepsilon,n})}}[f(X_T)]| \\
&\qquad \leq C\Big(\mathbb{E}[|H_T^{a^*(X^{\mu_n},Z^{\varepsilon,n})} - H_T^{a^*(X,Z^{\varepsilon,n})}|^2]^{\frac{1}{2}} + \mathbb{P}(\tau_n \leq T) + \mathbb{E}\big[\big|f(X_T) - f_n(X_T^{\mu_n})\big|^2\big]^{\frac{1}{2}}\Big) \ .
\end{aligned}$$

We therefore get from Step 3 and (4.30)

$$\mathbb{E}^{\mathbb{P}^{a^*(X^{\mu_n},Z^{\varepsilon,n})}}[f_n(X_T^{\mu_n})] - \mathbb{E}^{\mathbb{P}^{a^*(X,\tilde{Z}^{\varepsilon,n})}}[f(X_T)] \quad \xrightarrow[n\to+\infty]{} \quad 0 \ . \tag{4.32}$$

From **(H')** (i) and the definition of a^* we have

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}^{a^*(X^{\mu_n},Z^{\varepsilon,n})}}[Y_T^{R,Z^{\varepsilon,n}}] - \mathbb{E}^{\mathbb{P}^{a^*(X,\tilde{Z}^{\varepsilon,n})}}[\tilde{\xi}_{\varepsilon,n}] \\
&\qquad \leq \mathbb{E}\Big[H_T^{a^*(X,\tilde{Z}^{\varepsilon,n})}\int_0^T\Big(\frac{a^*(X_s,\tilde{Z}_s^{\varepsilon,n})^2}{2} + a^*(X_s,\tilde{Z}_s^{\varepsilon,n})p(X_s)X_s\Big)ds\Big] \\
&\qquad\quad - \mathbb{E}\Big[H_T^{a^*(X^{\mu_n},Z^{\varepsilon,n})}\int_0^T\Big(\frac{a^*(X_s^{\mu_n},Z_s^{\varepsilon,n})^2}{2} + a^*(X_s^{\mu_n},Z_s^{\varepsilon,n})p(X_s^{\mu_n})X_s^{\mu_n}\Big)ds\Big] \\
&\qquad\quad + \frac{\gamma\sigma^2}{2}\mathbb{E}\Big[H_T^{a^*(X,\tilde{Z}^{\varepsilon,n})}\int_0^T|\tilde{Z}_s^{\varepsilon,n}|^2ds - H_T^{a^*(X^{\mu_n},Z^{\varepsilon,n})}\int_0^{\tau_n}|Z_s^{\varepsilon,n}|^2ds\Big] \ .
\end{aligned} \tag{4.33}$$

By definition of τ_n we have

$$\mathbb{E}\Big[H_T^{a^*(X,\tilde{Z}^{\varepsilon,n})}\int_0^T|\tilde{Z}_s^{\varepsilon,n}|^2ds - H_T^{a^*(X^{\mu_n},Z^{\varepsilon,n})}\int_0^{\tau_n}|Z_s^{\varepsilon,n}|^2ds\Big] \quad = \quad 0 \ , \tag{4.34}$$

and

$$\begin{aligned}
&\mathbb{E}\Big[H_T^{a^*(X,\tilde{Z}^{\varepsilon,n})}\int_0^{T\wedge\tau_n}\Big(\frac{a^*(X_s,\tilde{Z}_s^{\varepsilon,n})^2}{2} + a^*(X_s,\tilde{Z}_s^{\varepsilon,n})p(X_s)X_s\Big)ds\Big] \\
&-\mathbb{E}\Big[H_T^{a^*(X^{\mu_n},Z^{\varepsilon,n})}\int_0^{T\wedge\tau_n}\Big(\frac{a^*(X_s^{\mu_n},Z_s^{\varepsilon,n})^2}{2} + a^*(X_s^{\mu_n},Z_s^{\varepsilon,n})p(X_s^{\mu_n})X_s^{\mu_n}\Big)ds\Big] \quad = \quad 0 \ .
\end{aligned} \tag{4.35}$$

Since a^* and $x \mapsto p(x)x$ are bounded, we get from (4.30)

$$\begin{aligned}
&\mathbb{E}\Big[H_T^{a^*(X,\tilde{Z}^{\varepsilon,n})}\int_{T\wedge\tau_n}^T\Big(\frac{a^*(X_s,\tilde{Z}_s^{\varepsilon,n})^2}{2} + a^*(X_s,\tilde{Z}_s^{\varepsilon,n})p(X_s)X_s\Big)ds\Big] \\
&-\mathbb{E}\Big[H_T^{a^*(X^{\mu_n},Z^{\varepsilon,n})}\int_{T\wedge\tau_n}^T\Big(\frac{a^*(X_s^{\mu_n},Z_s^{\varepsilon,n})^2}{2} + a^*(X_s^{\mu_n},Z_s^{\varepsilon,n})p(X_s^{\mu_n})X_s^{\mu_n}\Big)ds\Big] \quad \xrightarrow[n\to+\infty]{} \quad 0 \ .
\end{aligned} \tag{4.36}$$

Therefore we get from (4.32), (4.33), (4.34), (4.35) and (4.36)

$$\limsup_{n\to+\infty}\mathbb{E}^{\mathbb{P}^{a^*(X^{\mu_n},Z^{\varepsilon,n})}}[Y_T^{R,Z^{\varepsilon,n}} - f_n(X_T^{\mu_n})] - \mathbb{E}^{\mathbb{P}^{a^*(X,\tilde{Z}^{\varepsilon,n})}}[\tilde{\xi}_{\varepsilon,n} - f(X_T)] \quad \leq \quad 0 \ .$$

This last inequality with (4.31) give the result. □

Remark 4.2. *The previous proof shows that the construction of almost optimal strategies given by Theorem 4.3 can be extended to more general dynamics with a function μ such that **(H0)** holds and there exists an approximating sequence $(\mu_n)_{n\in\mathbb{N}}$ of μ such that*

– *μ_n satisfies **(H')** (i),*

– $\lim\limits_{n\to+\infty} \frac{1}{n^2}\sup_{t\in[0,T]}\mathbb{E}[|X_t^{\mu_n}-X_T^{\mu}|^2]=0$,

– *the sequence of stopping times* $(\tau_n)_n$ *defined by*

$$\tau_n = \inf\{t\geq 0,\ X_t^{\mu_n}\neq X_t^{\mu}\}\wedge T,$$

satisfies $\tau_n \xrightarrow[n\to+\infty]{\mathbb{P}-a.s} T$.

We also notice that the result can be extended to a multi-dimensional system of N*-interacting resources of the form*

$$dX_t^i = X_t^i(\lambda^i-\mu^i(X_t))dt+\sigma^i X_t^i dW_t^i,\ 1\leq i\leq N$$

where $X=(X^1,\dots,X^N)$ *is the vector abundance of species,* $W=(W^1,\dots,W^N)$ *is an* N*-dimensional Brownian motion,* λ^i,σ^i *are respectively the death/birth rate and volatility of the resource* i *and* $\mu=(\mu^1,\dots,\mu^N)$ *is a map from* $\mathbb{R}^N$ *into* $\mathbb{R}^N$ *such that the multi-dimensional version of the HJB equation* (4.18) *admits a smooth solution.*

4.3 Applications and economical interpretations

4.3.1 Non-regulated case and reservation utility

In this part, we provide a way to monitor the activities of the natural resource manager without penalizing him by choosing a relevant reservation utility R. The natural way is to consider the problem of the regulator without regulation policy

$$\overline{V}_A := \sup_{\alpha\in\mathcal{A}}\mathbb{E}^{\alpha}\Big[-\exp\Big(-\gamma\big(\int_0^T p(X_s^{\mu})\alpha_s X_s^{\mu}ds-\int_0^T\frac{|\alpha_s|^2}{2}ds\big)\Big)\Big].$$

This problem can be solved explicitly by direct computations. If the regulator chooses $R=\overline{V}_A$ then any admissible tax ξ will satisfy $V_A(\xi)\geq\overline{V}_A$. In other words, the choice of R ensures a non-punitive regulation policy.

4.3.2 Numerical examples

We now give some numerical results to illustrate our theoretical results. For that we consider μ_n given by (4.24), $p(x)=px^{-1}$ with $p>0$ and $f(x)=(c-\frac{c}{\beta}x)\mathbf{1}_{x<\beta}$ where c is the cost of the resource for the regulator and β is the target size of the population. We use the following parameters: $\gamma=0.1$, $\lambda=1.2$, $\sigma=0.1$, $P=1$, $T=1$, $\beta=0.9$, $c=3$, $\underline{M}=\overline{M}=10$, $n=100$, $\varepsilon=0.01$ with $X_0=1.2$. We use an approximation grid of 2000 points for the space and 5000 points for the time. In our case $R=\overline{V}_A=-\exp(-\gamma P^2T)$.

Figure 1 shows the Agent harvests moderately at the beginning and the rate is increasing w.r.t. the abundance population. On the contrary, at times close to the maturity the strategy depends on the abundance of the resource. Indeed, for an abundance below the target β, *i.e.* $X_t^{\alpha}<0.9$, the Agent renews the population, and for an abundance higher than β, *i.e.* $X_t^{\alpha}\geq 0.9$, the Agent harvests. Moreover, the lowest the abundance is, the most the Agent renews the population, and the highest is the abundance, the most the Agent harvests the population.

Figure 2 presents the graph of the value function w_ε of the Principal. The function w_ε is increasing w.r.t. the population abundance. This property is expected in view of the Principal optimization problem. We also remark that the value function w_ε is decreasing w.r.t. the time to maturity. Indeed, the longer is the time to maturity, the best it is for the Agent and the Principal since the resource has more time to regenerates itself.

Figure 3 shows that the Agent harvests with an important rate at the beginning : $\alpha_t^*(\xi)$ is around 0.6. As he get closer to the matirity, the Agent slows down the harvest and then renew the resource. This can be interpreted as follows. The Agent harvests with a high rate

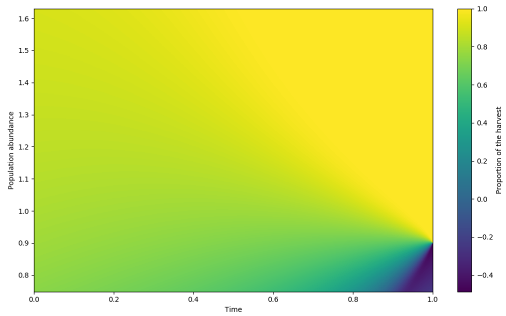

Figure 1: The optimal harvest rate w.r.t. the time t and the population abundance X_t.

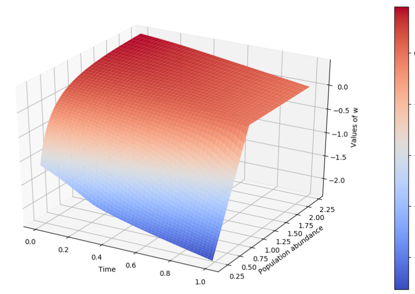

Figure 2: The value function w_ε w.r.t. the time and the population abundance.

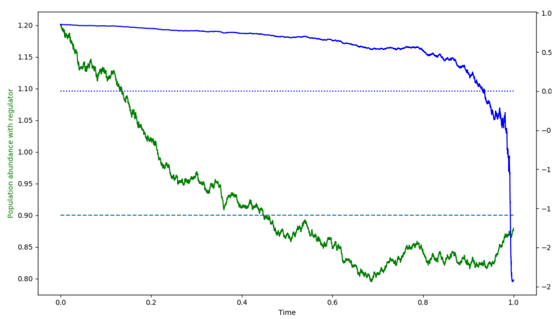

Figure 3: A trajectory of the optimally controlled population abundance (green curve, y-axis on the left) and the associated optimal harvest rate (blue curve, y-axis on the right) w.r.t. the time (x-axis). The dotted line corresponds to $\alpha = 0$, and the dashed line corresponds to $X_t^{\alpha^*} = \beta$.

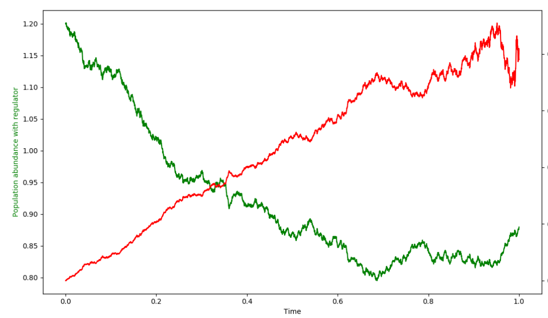

Figure 4: The evolution of the penalty (y-axis on the right, red curve) and the population abundance (y-axis on the left, green curve) w.r.t. the time (x-axis).

and does not care about the tax at maturity since the population has time to regenerate itself. Getting closer to the maturity, the Agent take into account the tax and slows down the harvest. When very close to maturity, $t \approx 0.93$, the Agent renews the population to ensure an abundance close to the target $\beta = 0.9$ to limit the tax. This shows that the incentive policy is efficient.

Figure 4 presents the forecast of the penalty (*i.e.* $Y^{\tilde{R},Z^\varepsilon,n}$) in red, and the abundance population in green. We notice these two quantities evolve in opposite ways: for high values of the abundance, the expected tax is low, and for low abundance the penalty becomes greater. We now study the sensitivity of the incentive policy w.r.t. the target β (see Figure 5) and w.r.t. the renewal cost c (see Figure 6).

Figure 5 presents the evolution in mean of the abundance w.r.t. time for several values of β. The mean is approximated by the empirical mean over 1000 trajectories. We remark that at each time the mean of the population abundance is more important as β is larger. This shows that the choice of β influences the behavior of the resource manager: the most important β is, the least the Agent harvests. We also notice that for each value of β, the mean terminal value reaches the target, which also shows the incentive effect of the parameter β.

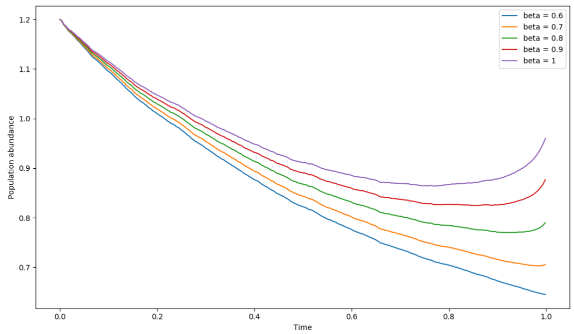

Figure 5: Evolution in mean of the population abundance w.r.t. the time for different values of β.

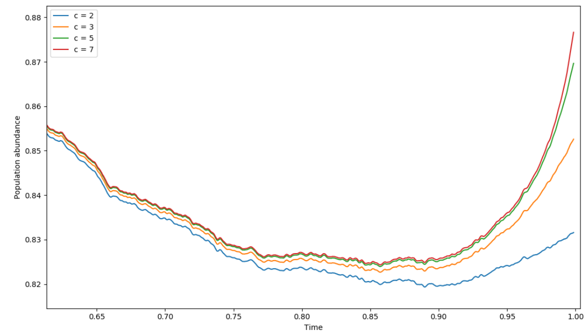

Figure 6: Evolution in mean of the population abundance w.r.t. the time for different values of c.

Figure 6 shows the evolution of the mean of the resource abundance w.r.t. time for several values of the costs parameter c. The mean is approximated by the empirical mean over for 1000 trajectories. We remark that the population abundance is nondecreasing w.r.t. c. In particular, the highest the penalty is, the most the Agent is concerned, through the incentive policy, by the size of the population at the end.

We now compare the situation for which the Agent can renew the population abundance (that is $\underline{M} > 0$) with the situation for which the Agent can only harvest (that is $\underline{M} = 0$). In Figure

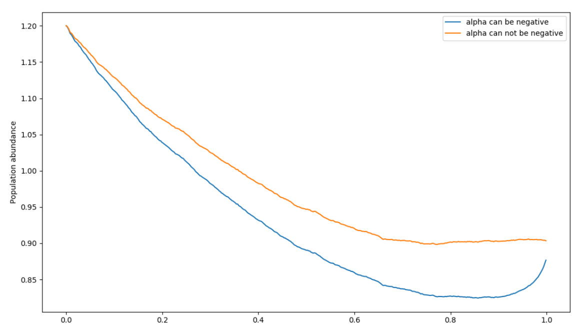

Figure 7: Evolution in mean of the population abundance w.r.t. the time when the Agent can (blue curve) and cannot (orange curve) renew the population abundance.

7 the mean is approximated by the empirical mean over 1000 trajectories. We remark that at each time the population abundance is more important in mean if the Agent cannot renew the resource. Indeed, if the resource is not renewable, the Agent reduces his harvesting rate in prevision of the terminal tax. On the contrary, if the resource can be renewed, the Agent harvests more to generate a higher profit since he can reduce the terminal tax by renewing the resource at the end.

A Proof of Proposition 4.1

We first recall that $\mathcal{H}$ is defined by

$$\begin{aligned}\mathcal{H}(y,\delta_1,\delta_2) &= \sup_{z\in\mathbb{R}}\Big\{e^yp(e^y)a^*(e^y,z)-\frac{a^*(e^y,z)^2}{2}-\frac{\sigma^2}{2}\gamma z^2+(\lambda-\frac{\sigma^2}{2}-\mu(e^y)-a^*(e^y,z))\delta_1\Big\}\\ &\quad +\frac{\sigma^2}{2}\delta_2\,,\quad (y,\delta_1,\delta_2)\in\mathbb{R}\times\mathbb{R}\times\mathbb{R}\,.\end{aligned}$$

From the definition of a^* given in (3.7) we can rewrite $\mathcal{H}$ by considering the different cases $e^yp(e^y)+z<-\underline{M}$, $-\underline{M}\leq e^yp(e^y)+z\leq\overline{M}$ and $e^yp(e^y)+z>\overline{M}$, and making the variable change $z-e^yp(e^y)$ for z under the following form

$$\mathcal{H}(y,\delta_1,\delta_2) = \max\Big\{\mathcal{K}_1(y,\delta_1),\mathcal{K}_2(y,\delta_1),\mathcal{K}_3(y,\delta_1)\Big\}+(\lambda-\frac{\sigma^2}{2}-\mu(e^y))\delta_1+\frac{\sigma^2}{2}\delta_2\,,$$

where

$$\begin{aligned}\mathcal{K}_1(y,\delta_1) &= \sup_{z\leq-\underline{M}}\Big\{\Big(\delta_1-e^yp(e^y)-\frac{\underline{M}}{2}\Big)\underline{M}-\frac{\sigma^2}{2}\gamma\left(z-e^yp(e^y)\right)^2\Big\}\\ &= \Big(\delta_1-e^yp(e^y)-\frac{\underline{M}}{2}\Big)\underline{M}-\frac{\sigma^2}{2}\gamma\left(\underline{M}+e^yp(e^y)\right)^2\,,\end{aligned}$$

$$\begin{aligned}\mathcal{K}_2(y,\delta_1) &= \sup_{z\geq\overline{M}}\Big\{\Big(\delta_1-e^yp(e^y)-\frac{\overline{M}}{2}\Big)\overline{M}-\frac{\sigma^2}{2}\gamma\left(z-e^yp(e^y)\right)^2\Big\}\\ &= \Big(-\delta_1+e^yp(e^y)-\frac{\overline{M}}{2}\Big)\overline{M}-\frac{\sigma^2}{2}\gamma\Big(\big[\overline{M}-e^yp(e^y)\big]_+\Big)^2\,,\end{aligned}$$

and

$$\mathcal{K}_3(y,\delta_1)=\sup_{-\underline{M}\leq z\leq\overline{M}}\Big\{-\frac{1}{2}(1+\gamma\sigma^2)z^2+\big(e^yp(e^y)(1+\gamma^2\sigma)-\delta_1\big)z\Big\}-\frac{\sigma^2}{2}\gamma|e^yp(e^y)|^2$$

for all $y,\delta_1\in\mathbb{R}$.

A straightforward computation gives $\mathcal{K}_3(y,\delta_1)=Q(e^yp(e^y),\delta_1)$ where

$$\begin{aligned}Q(\mathfrak{p},\delta_1) &= \frac{\big(\mathfrak{p}(1+\gamma^2\sigma)-\delta_1\big)^2}{2(1+\gamma\sigma^2)}\mathbb{1}_{\mathfrak{p}-\frac{\delta_1}{1+\gamma\sigma^2}\in[-\underline{M},\overline{M}]}\\ &\quad+\Big(-\frac{1}{2}(1+\gamma\sigma^2)\overline{M}^2+\big(\mathfrak{p}(1+\gamma^2\sigma)-\delta_1\big)\overline{M}\Big)\mathbb{1}_{\mathfrak{p}-\frac{\delta_1}{1+\gamma\sigma^2}\in(\overline{M},+\infty)}\\ &\quad+\Big(-\frac{1}{2}(1+\gamma\sigma^2)\underline{M}^2-\big(\mathfrak{p}(1+\gamma^2\sigma)-\delta_1\big)\underline{M}\Big)\mathbb{1}_{\mathfrak{p}-\frac{\delta_1}{1+\gamma\sigma^2}\in(-\infty,-\underline{M})}\\ &\quad-\frac{\sigma^2}{2}\gamma|\mathfrak{p}|^2\end{aligned}$$

for all $\mathfrak{p},\delta_1\in\mathbb{R}$.

We then introduce the functions $\mathrm{abs}_\varepsilon:\ \mathbb{R}\to\mathbb{R}_+$ and $\max_\varepsilon:\ \mathbb{R}^2\to\mathbb{R}$ defined by

$$\begin{aligned}\mathrm{abs}_\varepsilon(x) &= |x|\Big(\Theta\Big(-\frac{4}{\varepsilon}x-3\Big)+\Theta\Big(\frac{4}{\varepsilon}x-3\Big)\Big)\,,\quad x\in\mathbb{R}\,,\\ \max_\varepsilon(x,y) &= \frac{\mathrm{abs}_\varepsilon(x-y)+x+y}{2}\,,\quad x,y\in\mathbb{R}\,,\end{aligned}$$

for any $\varepsilon>0$ where we recall that the function Θ is defined by (4.25). From the definition of Θ, the function $\max_\varepsilon$ is infinitely differentiable with bounded derivatives and we have

$$\sup_{x,\,y\,\in\,\mathbb{R}}\big|\max(x,y)-\max_\varepsilon(x,y)\big| \leq \frac{\varepsilon}{3} \tag{A.37}$$

for all $\varepsilon > 0$.

Fix the constant $\Gamma := \max\big\{\overline{M}(1+\gamma\sigma^2), (P+\underline{M})(1+\gamma\sigma^2)\big\}$. Since the function Q is continuous on $\mathbb{R}^2$, there exists Q_ε infinitely differentiable on $\mathbb{R}^2$ such that

$$\sup_{(\mathfrak{p},\delta_1)\in[0,P]\times[-(\Gamma+1),\,\Gamma+1]} |Q(\mathfrak{p},\delta_1) - Q_\varepsilon(\mathfrak{p},\delta_1)| \quad\leq\quad \frac{\varepsilon}{3}\,. \tag{A.38}$$

We then define $\mathcal{K}_{3,\varepsilon}$ for any $y,\delta_1\in\mathbb{R}$ by

$$\begin{aligned}\mathcal{K}_{3,\varepsilon}(y,\delta_1) \quad = \quad & Q_\varepsilon(e^y p(e^y),\delta_1)\Big(1-\Theta(2(\delta_1-\Gamma)-1)-\Theta\big(-2(\Gamma+\delta_1)-1)\big)\Big)\\ & + \left(-\frac{1}{2}(1+\gamma\sigma^2)\underline{M}^2 - \big(e^y p(e^y)(1+\gamma^2\sigma)-\delta_1\big)\underline{M}\right)\Theta\big(2(\delta_1-\Gamma)-1\big)\\ & + \left(-\frac{1}{2}(1+\gamma\sigma^2)\overline{M}^2 + \big(e^y p(e^y)(1+\gamma^2\sigma)-\delta_1\big)\overline{M}\right)\Theta\big(-2(\Gamma+\delta_1)-1\big)\,.\end{aligned}$$

From (A.38) we have

$$\sup_{\mathbb{R}^2}|\mathcal{K}_{3,\varepsilon}-\mathcal{K}_3| \leq \frac{\varepsilon}{3}\,.$$

We then define for any $y,\delta_1,\delta_2\in\mathbb{R}$ the approximated Hamiltonian $\mathcal{H}_\varepsilon$ by

$$\mathcal{H}_\varepsilon(y,\delta_1,\delta_2) := \max{}_\varepsilon\Big\{\mathcal{K}_1(y,\delta_1),\mathcal{K}_2(y,\delta_1),\mathcal{K}_{3,\varepsilon}(y,\delta_1)\Big\} + \frac{\sigma^2}{2}\delta_2 + (\lambda - \frac{\sigma^2}{2} - \mu(e^y))\delta_1\,.$$

We therefore get from (A.37)

$$\sup_{\mathbb{R}^3}|\mathcal{H}_\varepsilon - \mathcal{H}| \leq \varepsilon\,.$$

We then turn the PDE driven by $\mathcal{H}_\varepsilon$ that writes

$$\begin{cases} -\partial_t w_\varepsilon - \mathcal{H}_\varepsilon\Big(y,\partial_y w_\varepsilon(t,y),\partial_{yy}w_\varepsilon(t,y)\Big) = 0\,, \quad (t,y)\in[0,T)\times\mathbb{R}\,,\\ w_\varepsilon(T,y) = -f(e^y)\,, \quad y\in\mathbb{R}\,.\end{cases} \tag{A.39}$$

Under Assumption **(H')**, we can write the approximated Hamiltonian $\mathcal{H}_{3,\varepsilon}$ under the form

$$\mathcal{H}_{3,\varepsilon}(y,\delta_1,\delta_2) = \frac{\sigma^2}{2}\delta_2 + b_\varepsilon(u,\delta_1)\delta_1 + c_\varepsilon(y,\delta_1)$$

where the coefficients b_ε and c_ε satisfy Conditions A, B and D of [23]. Then, according to Theorem 14 in [23], PDE (A.39) admits a unique solution $w_\varepsilon\in C^{2+\nu}([0,T]\times\mathbb{R})$.

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