Homogenized Model with Memory for Two-Phase Compressible Flow in Double-Porosity Media

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A completely averaged model of two-phase flow of compressible fluids in a medium with double porosity is developed. The variational asymptotic two-scale averaging method with splitting the nonlocality and nonlinearity is presented. Several mechanisms of delay are detected, as the nonequilibrium capillary redistribution of phases, pressure field relaxation caused by the compressibility, and the cross effects of fluid extrusion from pores due to the rock com- paction and fluid expansion. A generalized non-equilibrium capillary equation is obtained. All characteristic times of delay are explicitly defined as functions of saturation.

I. INTRODUCTION

A double porosity medium, which is also called fractured-porous medium, is the classical model applied to study the effects of memory or delay occurrence in fluid flow after changing the scale of examination. Such a medium consists of low permeable blocks and highly permeable fractures. A single-phase compressible flow in such a medium is characterized by the delay in pressure behaviour. Indeed, in tight blocks, non-stationary processes of perturbation propagation are much slower than in fractures, therefore, the average pressure in blocks reacts to a perturbation coming from fractures with a delay. As the result, the average pressure in the blocks depends on the all history of variation of the average pressure in the fractures, but not only on its current value, which is the phenomenon of memory. Such a phenomenon is observed in terms of the averaged pressures, i.e. on the macroscopic scale. In the single-phase case, it is possible to obtain completely averaged model and to describe explicitly the kernel of the memory operator. The first mathematical model was suggested in[11], where the concept of a delay between the pressures was introduced. Later, in[9] and[22] it was shown by homogenization that high contrast in the permeability of blocks and fractures leads to a long memory and changes the original diffusion equation into an integro-differential homogenized equation. In some papers a more complete version of compressible single-phase flow in double-porosity medium was homogenized, by introducing elastic deformations of the medium described by Biot's equations,[1],[21].

In the case of two-phase flow, another type of memory arises, which is caused by the capillarity. The capillary forces push the more wetting fluid from fractures into blocks, while displacing the non-wetting fluid from the blocks to the fractures. This phenomenon is known as the counter-current spontaneous imbibition. In particular, this mechanism is essential for oil recovery by water injection in heterogeneous reservoirs,[20]. With respect to the flow in fractures, the spontaneous imbibition in blocks is very slow, which causes the capillary delay effects in terms of the averaged saturation of water (the term "saturation" means the volume fraction).

The capillary imbibition can be represented in terms of the propagation of the wave of water saturation from the block boundary towards the block centre. The dynamics of this wave is nonlinear, proper to two-phase flow in general. In classical two-phase model the counter-current imbibition is described by a nonlinear diffusion equation with nonlinear boundary conditions. We are, thus, faced with a situation where delay/memory (or time nonlocality) at the macroscale is caused by the propagation of a nonlinear wave at the local scale.

In the flow is two-phase and compressible simultaneously, the two types of the memory mentioned above, caused by the capillarity and the compressibility, should interact in some way, which could generate new physical cross phenomena. The detection of such phenomena is the main objective of the present paper. For this, it was necessary to develop the macroscale model of two-phase compressible flow in double-porosity media.

Attempts to develop the model of two-phase flow in double porosity media have been undertaken essentially for incompressible fluids[4],[7],[8],[10],[12],[13],[19],[22],[28]. The appearance of nonlocality in a nonlinear system, mentioned above, is a strong obstacle for obtaining averaged model. This is why the problem remains open for two-phase case, even non-compressible. Two papers devoted to compressible two-phase flow in double-porosity media,[2],[8], in which one phase was an ideal gas while the second one was incompressible, have illustrated even deeper problems of interference between the nonlocality and twice nonlinearity. The attempts to overcome this difficulty by linearizing the equations of capillary imbibition, as, for instance, in[6],[7],[10], mean physically that the flow in blocks becomes single-phase.

As a result of such an imposition of nonlocality and nonlinearity, the models obtained are not completely averaged. Namely, macroscopic equations and the cell problems make up a coupled system of equations, which

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should be solved together. This means that macroscopic equations are not spared from microscopic variables, and cell problems are not spared from macroscopic variables. Despite the importance of such works for mathematics, the significance of such models for physics is disputable. This is one of the main reasons why the results of the homogenization theory were not well received by the physical community.

This is why, the physics of two-phase flow in fractured-porous media continue to be studied mainly experimentally, or numerically. Among the recent experiments, it is important to cite the studies based on transparent glass micromodels:[17],[18],[25], and on real rock samples scanned by nuclear magnetic resonance technique:[20]. The numerical modeling of the first kind is based on Darcy's scale model:[14],[15],[23],[25],[26], in which the main problem is focused on matching two-dimensional schemes for fractures with three-dimensional schemes for blocks. The numerical modeling of the second kind is based on pore-scale models of two-phase flow:[16],[24],[27]. In all these papers, the Navier-Stokes and Cahn-Hilliard equations were used (the diffuse interface method). The results obtained in all these studies converge regarding the role of the capillary imbibition on the oil recovery: the higher the capillary forces, the higher the imbibition effect and the ultimate oil recovery,[17],[18],[24]. In[24] the capillary forces were varied by the surface tension, in[16] by the flow rate, while in[18] by injecting foams. However, the results remain contradictory regarding the dependence of the ultimate recovery on the injection rate. In[16],[18],[24],[25] the ultimate recovery decreases with the injection rate, whilst in[17],[20] the result is inverse. This indicates that the general physical theory of two-flow in double porosity media remains an open problem.

However, the development of such a theory is possible. More exactly, it is possible to develop the completely averaged model of such systems, in which nonlocality and nonlinearity are superimposed. The idea of such a method consists of spreading nonlinearity and nonlocality into different levels of the asymptotic expansion, that is, to weaken their interaction. Such a splitting can be performed in so-called media with moderately contrasting properties.

On the basis of this idea, in the present work, we construct a completely averaged model of two-phase compressible flow, which combines elements of single-phase compressible and two-phase incompressible systems. We show that the interference between the capillarity and compressibility is more complicated than a simple sum of the saturation and pressure delay, but leads to more complicated cross effects:

- the effect of saturation delay caused by asymmetric extrusion of phases from the blocks due to the expansion and compaction of pores;

- the effect of saturation delay caused by nonlinear overlap of compressibility and capillarity.

All characteristic times have been obtained explicitly as the functions of saturation.

We applied the homogenization method based on two-scale asymptotic expansion in variational form, which is technically close to the constructive part of the two-scale convergence method[3]. It enables us to present all the calculations in the most compact form. This method is frequently applied to heterogeneous media. We extended the technique of the method to double porosity media, so that the method enables us to obtain the averaged equations both in fractures and blocks, and to remove macroscopic variables in the cell problem for the delay operator. This is why we prefer to explain the method in the main body of the paper, but not to place it in the appendix.

We underline that the paper does not study "deformable" media, but only "compressible" media, i.e. the media in which the porosity and permeability depend on pressure.

We also note that the model of periodic double porosity medium may be applied even for randomly heterogeneous media. This is shown, for instance, in[27], where a two-phase flow was calculated on the pore scale. It was shown that the width of fingers (i.e. the width of fractures) and the distance between fingers (i.e. the width of blocks) are monotonically decreasing functions of the variance of the permeability field.

II. PROBLEM FORMULATION

A. Equations of two-phase compressible flow

Compressibility means that the density of liquids and the porosity/permeability of the medium depend on pressure. Traditionally, one uses the exponential laws for liquid and tight rocks:

$$\frac{d\rho_w}{dp} = \widetilde{C}_w \rho_w, \quad \frac{d\rho_o}{dp} = \widetilde{C}_o \rho_o, \quad \frac{d\phi}{dp} = \widetilde{C}_\phi \phi \qquad (1)$$

where p is the pressure, $\widetilde{C}_w$, $\widetilde{C}_o$, and $\widetilde{C}_\phi$ are the isothermal compressibility coefficients, which are considered to be constant and positive. Their dimension is Pa^{-1}. The characteristic values of this parameter are discussed further in section VII A. The positive derivative $d\phi/dp$ means that the porosity decreases if the pressure of liquid in the pores drops, which corresponds to the pore compaction under the weight of superimposed rocks.

For single-phase compressible fluids in compressible medium, one uses the classical simplification of flow equations, which consists in the following. Let us substitute the equation of compressibility (1) in the mass conservation law:

$$\partial_t (\rho\phi) + \nabla \cdot (\rho \mathbf{V}) = 0 \qquad (2)$$

where $\mathbf{V}$ is the Darcy velocity. Then we obtain:

$$0 = \frac{d(\rho\phi)}{dp}\, \partial_t p + \rho \nabla \cdot \mathbf{V} + \frac{d\rho}{dp}\, \mathbf{V} \cdot \nabla p \qquad (3)$$

The last term, $\mathbf{V}\cdot\nabla p \sim (\nabla p)^2$ is negligible, since the gradients of pressure are very low in geological porous reservoirs. Then we obtain the classical equation of single-phase compressible flow in compressible porous media:

$$\phi C\,\partial_t p + \nabla\cdot\mathbf{V} = 0, \qquad C \equiv \frac{1}{\rho\phi}\frac{d(\rho\phi)}{dp} = \widetilde{C}_w + \widetilde{C}_\phi \quad (4)$$

Let us apply the same assumption for two-phase fluids. Consider the immiscible phases and call them "water" and "oil". Each phase has its own pressure: p_w and p_o. Note that $\rho_w = \rho_w(p_w)$, $\rho_o = \rho_o(p_o)$, and $\phi = \phi(p_w)$ for the pores occupied by water, or $\phi = \phi(p_o)$ for the pores occupied by oil.

The initial equations of mass conservation of the phases are:

$$\partial_t\left(\rho_w \phi s\right) + \nabla\cdot\left(\rho_w \mathbf{V}_w\right) = 0 \quad (5)$$

and

$$\partial_t\left(\rho_o \phi (1-s)\right) + \nabla\cdot\left(\rho_o \mathbf{V}_o\right) = 0 \quad (6)$$

where s is the volume fraction of water ("the saturation"); ρ is the phase density; ϕ is the medium porosity; p is the pressure; $\mathbf{V}$ is the Darcy velocity vector. Indices w and o mean water and oil. Differentiating by parts, we obtain for water:

$$\rho_w\phi\,\partial_t s + \frac{d(\rho_w\phi)}{dp_w}\,s\,\partial_t p_w + \rho_w \nabla\cdot\mathbf{V}_w + \frac{d\rho_w}{dp_w}\,\mathbf{V}\cdot\nabla p_w = 0 \quad (7)$$

Neglecting the terms of the order $\sim (\nabla p)^2$, we finally obtain the equations of two-pase compressible flow in a porous medium

$$\begin{cases} \phi\,\partial_t s + \phi C_w s\,\partial_t p_w + \nabla\cdot\mathbf{V}_w = 0, \\ -\phi\,\partial_t s + \phi C_o (1-s)\,\partial_t p_o + \nabla\cdot\mathbf{V}_o = 0 \end{cases} \quad (8)$$

where $C_w \equiv \widetilde{C}_w + \widetilde{C}_\phi$, $C_o \equiv \widetilde{C}_o + \widetilde{C}_\phi$.

This system is complemented by the equations of conservation of momentum in the form of Darcys law for each phase and the equation of capillary equilibrium, which relates the phase pressures:

$$\mathbf{V}_\alpha = -K\lambda_\alpha(s)\nabla p_\alpha, \quad \lambda_\alpha(s) \equiv \frac{k_\alpha(s)}{\mu_\alpha}, \quad \alpha = w, o \quad (9a)$$

$$p_o = p_w + p_c(s) \quad (9b)$$

where K is the absolute permeability of the medium; μ is the dynamic viscosity of the phase. The functions of relative phase permeability $k_w(s)$ and $k_o(s)$, and the capillary pressure $p_c(s)$ are given. They have the following properties:

- $k_w(s)$ is continuous monotonically increasing function of water saturation s, such that $k_w \equiv 0$ for $s \in [0, s_*]$, and $k_w(1) = 1$;
- $k_o(s)$ is continuous monotonically decreasing function of s, such that $k_o(0) = 1$ and $k_o \equiv 0$ for $s \in [s^*, 1]$;
- $p_c(s)$ is monotonically decreasing, such that $p_c(1) = 0$ and $p_c \to \infty$ for $s \to s_*$. It is undefined for $s \in [0, s_*]$;

The values s_* and $1 - s_*$ are called the percolation thresholds or residual saturations. They mark the critical water and oil saturation below which the corresponding phase becomes immobile in porous medium. An example of these functions is given in Fig. 2.

III. AVERAGING PROBLEM

Let the flow domain be a bounded, connected region with a periodic microstructure of a characteristic scale ε, which is the ratio of the length of a period of heterogeneity to the length of the entire region. Parameter ε is small: $0 < \varepsilon \ll 1$ (Fig. 1a).

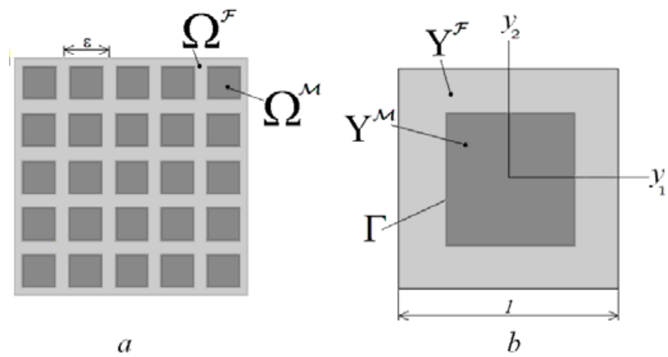

FIG. 1. Medium structure (a) and a single cell (b)

Four equations of mass and momentum conservation (8) and (9) can be reduced to two equations, excluding the Darcy velocities:

$$\phi^\varepsilon\,\partial_t s^\varepsilon_\alpha + \phi^\varepsilon C^\varepsilon_\alpha s^\varepsilon_\alpha\,\partial_t p^\varepsilon_\alpha = \nabla\cdot\left(K^\varepsilon \lambda^\varepsilon_\alpha \nabla p^\varepsilon_\alpha\right) \quad (10)$$

where $s^\varepsilon_o \equiv 1 - s^\varepsilon_w$, $\lambda^\varepsilon_\alpha(x, s)$ is defined in (9a).

The system (10) and (9b) defines three unknown functions: water saturation s^ε_w, pressure in water p^ε_w, and pressure in oil p^ε_o.

At the interface between the media, the normal phase fluxes and the phase pressures are continuous.

The boundary conditions specify the water pressure and saturation. The initial conditions fixes a specified saturation and pressure distribution in the domain: $s^\varepsilon(x,0) = s^0(x)$ and $p^\varepsilon_w(x,0) = p^0_w(x)$. If these initial values do not satisfy the condition of local capillary equilibrium, then the stated problem describes the relaxation of the initially non-equilibrium system to an equilibrium state.

A. Parameters of the problem

Porosity, compressibility factors, phase permeability, and capillary pressure are different in blocks and fractures, but remain of the same order:

$$\phi^\varepsilon(x) = \begin{cases} \phi^{\mathcal{F}} \\ \phi^{\mathcal{M}} \end{cases}, \quad C_\alpha^\varepsilon(x) = \begin{cases} C_\alpha^{\mathcal{F}}, & x \in \Omega^{\mathcal{F}} \\ C_\alpha^{\mathcal{M}}, & x \in \Omega^{\mathcal{M}} \end{cases} \tag{11}$$

$$\lambda_\alpha^\varepsilon(s,x) = \begin{cases} \lambda_\alpha^{\mathcal{F}}(s) \\ \lambda_\alpha^{\mathcal{M}}(s) \end{cases}, \quad p_c^\varepsilon(s,x) = \begin{cases} p_c^{\mathcal{F}}(s), & x \in \Omega^{\mathcal{F}} \\ p_c^{\mathcal{M}}(s), & x \in \Omega^{\mathcal{M}} \end{cases} \tag{12}$$

where $\alpha = w, o$.

Parameters $\phi^{\mathcal{F}}$, $\phi^{\mathcal{M}}$, $C_\alpha^{\mathcal{F}}$, $C_\alpha^{\mathcal{M}}$, $\lambda_\alpha^{\mathcal{F}}(s)$, $\lambda_\alpha^{\mathcal{M}}(s)$, $p_c^{\mathcal{F}}(s)$, $p_c^{\mathcal{M}}(s)$ do not depend on ε. Parameters $\phi^{\mathcal{F}}$, $\phi^{\mathcal{M}}$, $C_\alpha^{\mathcal{F}}$, $C_\alpha^{\mathcal{M}}$ are positive. Functions $\lambda_\alpha^{\mathcal{F}}(s)$, $\lambda_\alpha^{\mathcal{M}}(s)$, $p_c^{\mathcal{F}}(s)$, $p_c^{\mathcal{M}}(s)$ are non-negative. We assume that $p_c^{\mathcal{M}}(s) \geq p_c^{\mathcal{F}}(s)$, for any s, because the capillary pressure is higher when the permeability is lower.

At the same time, the absolute permeability of blocks and fractures can be very different. It is known that the permeability changes by several orders of magnitude when the porosity varies within $0.1 - 0.2$. Therefore, we accept the following basic condition of double porosity:

$$K^\varepsilon(x) = \begin{cases} K^{\mathcal{F}}, & x \in \Omega^{\mathcal{F}} \\ \delta K^{\mathcal{M}}, & x \in \Omega^{\mathcal{M}} \end{cases} \tag{13}$$

Coefficients $K^{\mathcal{F}}$ and $K^{\mathcal{M}}$ are positive and independent of ε.

Parameter δ is the degree of the contrast between the permeabilities of blocks and fractures. If the blocks are much less permeable than fractures, then this causes a delay/memory. The delay rate (or the memory length) can ne measured by another parameter, ω, defined as:

$$\omega = \frac{\varepsilon^2}{\delta} \tag{14}$$

Two types of media are of interest, regarding the contrast degree and the memory length[13]:

- media with long memory (or strong contrast): $\omega \sim 1$ or $\delta \sim \varepsilon^2$. They are also called ε^2-media;

- media with short memory (or moderate contrast): $\varepsilon \ll \omega \ll 1$ or $\varepsilon^2 \ll \delta \ll \varepsilon$.

There are also non-contrast media: $\delta \sim 1$, and media with impermeable blocks: $\delta \ll \varepsilon^2$. In the first case, the exchange process between blocks and fractures is without delay ($\omega \sim \varepsilon^2 \ll 1$), while it is completely absent in the second one due to a very strong delay ($\omega \gg 1$). Both these cases are of no interest for our research purposes.

IV. METHOD OF HOMOGENIZATION WITH SPLITTING NONLOCALITY AND NONLINEARITY

A. The idea of the method

The purpose of this paper is to obtain a completely averaged model. This concept was introduced in[7]. A model is called completely averaged, if its macroscopic equations do not contain microscopic variables, and the cell problem, which determines macroscopic coefficients, does not depend on macroscopic variables and, therefore, is solved only once.

To avoid the superimposition of the nonlinearity and the nonlocality, the idea is to spread them into different levels of asymptotic expansion over the parameter of memory length ω, that is, nonlinearity is preserved in terms of zero order, whilst the nonlocality is allowed only in the first order. Since in asymptotic expansions the problem for the first approximation is usually linear, this enables us to treat the nonlocality completely and explicitly. For incompressible fluids, this was done in[13] for the so-called moderate contrast between blocks and fractures.

This idea can be realized by considering ω as a small parameter and developing the solution of our problem into the asymptotic series over ω. Therefore, we necessarily deal with media of moderate contrast in the permeability of block and fractures. For the sake of simplicity, we assume that

$$\omega \sim \sqrt{\varepsilon}, \quad \text{or} \quad \delta = \varepsilon\sqrt{\varepsilon} \tag{15}$$

which was used in[13]. Then, the total asymptotic expansion over ε and ω reduces to a single expansion over ε but which should include the fractional powers of ε.

B. Two-scale formulation

First of all, let us introduce the new variable y that belongs to the unit cell of the microstructure, $Y = (0,1)^d$, which is shown in Fig. 1b, where d is the space dimension. Domain Y consists of two subdomains: a connected subdomain $Y^{\mathcal{F}}$ (*Fractures*) and $Y^{\mathcal{M}}$ (*Matrix block*). We denote by Γ the boundary between the two subdomains in Y. Let us introduce the solution extension $\widetilde{s}^\varepsilon(x,y,t)$, $\widetilde{p}_\alpha^\varepsilon(x,y,t)$, where $y \in Y$, so that

$$s^\varepsilon(x,t) = \widetilde{s}^\varepsilon(x,y,t)\big|_{y=\frac{x}{\varepsilon}}, \quad p_\alpha^\varepsilon(x,t) = \widetilde{p}_\alpha^\varepsilon(x,y,t)\big|_{y=\frac{x}{\varepsilon}} \tag{16}$$

Then the following is true for the derivatives:

$$\frac{\partial f(x,t)}{\partial x_i} \to \left(\frac{\partial \widetilde{f}(x,y,t)}{\partial x_i} + \frac{1}{\varepsilon}\frac{\partial \widetilde{f}(x,y,t)}{\partial y_i} \right) \tag{17}$$

where f is any function from $s^\varepsilon, p_w^\varepsilon, p_o^\varepsilon$

Since the operations of extension and differentiation commute, we find that the argument y can be considered as an independent variable of x, and we should put it equal to x/ε in final results. This is the main idea of the two-scale homogenization method. The variable x is called "slow", while y is "fast".

Further, we will omit tilde.

The two-scale formulation of the problem (10) has the following form:

$$\begin{cases} \phi^\varepsilon(y)\partial_t s^\varepsilon_\alpha + \phi^\varepsilon(y)\, C^\varepsilon_\alpha(y) s^\varepsilon_\alpha \partial_t p^\varepsilon_\alpha = \\ \qquad \left(\partial_{xi} + \dfrac{1}{\varepsilon}\partial_{yi}\right)\left(K^\varepsilon(y)\,\lambda^\varepsilon_\alpha\left(\partial_{xi}p^\varepsilon_\alpha + \dfrac{1}{\varepsilon}\partial_{yi}p^\varepsilon_\alpha\right)\right), \\ \qquad\qquad x\in\Omega, \quad y\in Y, \quad t\in(0,T) \\ p^\varepsilon_o = p^\varepsilon_w + p^\varepsilon_c, \qquad s^\varepsilon_o = 1 - s^\varepsilon_w \end{cases} \tag{18}$$

with conditions:

$$\begin{cases} s^\varepsilon_w|_{t=0} = s^0(x), \quad p^\varepsilon_w|_{t=0} = p^0_w(x) \\ \left[K^\varepsilon\,\lambda^\varepsilon_\alpha \dfrac{\partial p^\varepsilon_\alpha}{\partial n}\right]_\Gamma = 0, \quad [p^\varepsilon_\alpha]_\Gamma = 0, \quad \alpha = w, o \\ s^\varepsilon_w, p^\varepsilon_w \ \text{ are } y-\text{periodic} \end{cases} \tag{19}$$

where $[\cdot]$ means a jump. Note that the continuity of phase pressures means the continuity of the capillary pressure: $[p^\varepsilon_c] = 0$, which determine a discontinuity of the saturation s^ε.

It can be presented in variational form:

$$\int_\Omega\int_Y \zeta\,\phi^\varepsilon \partial_t s^\varepsilon_\alpha\,dxdy + \int_\Omega\int_Y \zeta\,\phi^\varepsilon C^\varepsilon_\alpha s^\varepsilon_\alpha\,\partial_t p^\varepsilon_\alpha\,dxdy = $$
$$-\frac{1}{\varepsilon^2}\int_\Omega\int_{Y^\mathcal{F}} K^\mathcal{F}\,\lambda^\mathcal{F}_\alpha\left(\partial_{yi}p^\varepsilon_\alpha + \varepsilon\partial_{xi}p^\varepsilon_\alpha\right)\left(\partial_{yi}\zeta + \varepsilon\partial_{xi}\zeta\right)dydx -$$
$$\frac{1}{\sqrt{\varepsilon}}\int_\Omega\int_{Y^\mathcal{M}} K^\mathcal{M}\,\lambda^\mathcal{M}_\alpha\left(\partial_{yi}p^\varepsilon_\alpha + \varepsilon\partial_{xi}p^\varepsilon_\alpha\right)\left(\partial_{yi}\zeta + \varepsilon\partial_{xi}\zeta\right)dydx \tag{20}$$

for any function $\zeta(x,y)$ in $\Omega\times Y$, such that ζ is continuous, $\zeta|_{\partial\Omega} = 0$ and is periodic with respect to y.

The relationship (20) is obtained by multiplying (18) by ζ, integrating by parts, and using the Gauss-Ostrogradsky theorem. The integrals over the boundary $\partial\Omega$ are zero due to the fact that the function ζ is zero on $\partial\Omega$. The integrals over the boundary ∂Y of the period Y are zero due to the periodicity of all functions with respect to y.

C. Asymptotic expansion

Homogenization is performed by the method of two-scale asymptotic expansions with respect to parameter ε, as well as regular asymptotic series with respect to the nonlocality parameter ω, which means, given (15), the appearance of terms containing fractional powers $\varepsilon^{1/2}$. The general structure of the expansion is as follows, for $\alpha = w, o$:

$$p^\varepsilon_\alpha(x,y,t) = $$
$$\begin{cases} p_{\alpha 0}(x,t) + \sqrt{\varepsilon}\,p^\mathcal{M}_{\alpha\,1/2}(x,y,t) + \varepsilon\,p^\mathcal{M}_{\alpha 1}(x,y,t) + ..., \quad y\in Y^\mathcal{M} \\ p_{\alpha 0}(x,t) + \sqrt{\varepsilon}\,p^\mathcal{F}_{\alpha\,1/2}(x,t) + \varepsilon\,p^\mathcal{F}_{\alpha 1}(x,y,t) + ..., \quad y\in Y^\mathcal{F} \end{cases}$$

$$s^\varepsilon_\alpha(x,y,t) = $$
$$\begin{cases} s_{\alpha 0}(x,t) + \sqrt{\varepsilon}\,s^\mathcal{M}_{\alpha\,1/2}(x,y,t) + \varepsilon\,s^\mathcal{M}_{\alpha 1}(x,y,t) + ..., \quad y\in Y^\mathcal{M} \\ s_{\alpha 0}(x,t) + \sqrt{\varepsilon}\,s^\mathcal{F}_{\alpha\,1/2}(x,t) + \varepsilon\,s^\mathcal{F}_{\alpha 1}(x,y,t) + ..., \quad y\in Y^\mathcal{F} \end{cases} \tag{21}$$

In addition, the condition of continuity of phase pressures holds:

$$p^\mathcal{M}_{\alpha\,1/2}(x,y,t)\Big|_{y\in\Gamma} = p^\mathcal{F}_{\alpha\,1/2}(x,t) \tag{22}$$

The independence of the zero terms of y is easy to prove by substituting the asymptotic expansion into (20). For nonlinear functions in (20), the expansion is as follows:

$$\lambda^\varepsilon_\alpha = \begin{cases} \lambda_{\alpha 0}(x,t) + \sqrt{\varepsilon}\,\lambda^\mathcal{M}_{\alpha\,1/2}(x,y,t) + \varepsilon ..., \quad y\in Y^\mathcal{M} \\ \lambda_{\alpha 0}(x,t) + \sqrt{\varepsilon}\,\lambda^\mathcal{F}_{\alpha\,1/2}(x,t) + \varepsilon ..., \quad y\in Y^\mathcal{F} \end{cases} \tag{23}$$

Then the integral identity (20) takes the form:

$$\int_\Omega\int_Y \zeta\,\phi^\varepsilon\left(\partial_t s_{\alpha 0} + \sqrt{\varepsilon}\,\partial_t s_{\alpha\,1/2}\right)dxdy +$$
$$\int_\Omega\int_Y \zeta\,\phi^\varepsilon\,C^\varepsilon_\alpha\left(s_{\alpha 0} + \sqrt{\varepsilon}\,s_{\alpha\,1/2}\right)\left(\partial_t p_{\alpha 0} + \sqrt{\varepsilon}\,\partial_t p_{\alpha\,1/2}\right)dxdy =$$
$$-\frac{1}{\varepsilon}\int_\Omega\int_{Y^\mathcal{F}} K^\mathcal{F}\left(\lambda^\mathcal{F}_{\alpha 0} + \sqrt{\varepsilon}\,\lambda^\mathcal{F}_{\alpha\,1/2}\right)\Big(\partial_{yi}p_{\alpha 1} + \partial_{xi}p_{\alpha 0} +$$
$$\sqrt{\varepsilon}\left(\partial_{yi}p_{\alpha\,3/2} + \partial_{xi}p_{\alpha\,1/2}\right)\Big)\left(\partial_{yi}\zeta + \varepsilon\partial_{xi}\zeta\right)dydx -$$
$$-\int_\Omega\int_{Y^\mathcal{M}} K^\mathcal{M}\,\lambda^\mathcal{M}_{\alpha 0}\left(\partial_{yi}p_{\alpha\,1/2} + \sqrt{\varepsilon}\left(\partial_{yi}p_{\alpha 1} + \partial_{xi}p_{\alpha 0}\right)\right)\times$$
$$\left(\partial_{yi}\zeta + \varepsilon\partial_{xi}\zeta\right)dydx + \mathcal{O}(\varepsilon) \tag{24}$$

The homogenization technique consists of substituting the asymptotic expansion into the integral identity (24) and obtaining closed expressions for subsequent terms of the expansion by selecting various types of test functions ζ.

D. Result of homogenization: macroscopic model

We immediately give the result of homogenization. Its derivation is given in the next section.

Let us define the averaged phase saturations and pressures in the blocks and fractures as follows:

$$\mathcal{P}_\alpha^{\mathcal{F}} \equiv \frac{1}{|Y^{\mathcal{F}}|}\int\limits_{Y^{\mathcal{F}}}\left(p_{\alpha 0}+\sqrt{\varepsilon}\,p_{\alpha\, 1/2}^{\mathcal{F}}\right)dy = p_{\alpha 0}+\sqrt{\varepsilon}\,p_{\alpha\, 1/2}^{\mathcal{F}},$$
$$\mathcal{P}_\alpha^{\mathcal{M}} \equiv \frac{1}{|Y^{\mathcal{M}}|}\int\limits_{Y^{\mathcal{M}}}\left(p_{\alpha 0}+\sqrt{\varepsilon}\,p_{\alpha\, 1/2}^{\mathcal{M}}\right)dy = p_{\alpha 0}+\sqrt{\varepsilon}\left\langle p_{\alpha\, 1/2}^{\mathcal{M}}\right\rangle_{\mathcal{M}},$$
$$\mathcal{S}_\alpha^{\mathcal{F}} \equiv \frac{1}{|Y^{\mathcal{F}}|}\int\limits_{Y^{\mathcal{F}}}\left(s_{\alpha 0}+\sqrt{\varepsilon}\,s_{\alpha\, 1/2}^{\mathcal{F}}\right)dy = s_{\alpha 0}+\sqrt{\varepsilon}s_{\alpha\, 1/2}^{\mathcal{F}},$$
$$\mathcal{S}_\alpha^{\mathcal{M}} \equiv \frac{1}{|Y^{\mathcal{M}}|}\int\limits_{Y^{\mathcal{M}}}\left(s_{\alpha 0}+\sqrt{\varepsilon}\,s_{\alpha\, 1/2}^{\mathcal{M}}\right)dy = s_{\alpha 0}+\sqrt{\varepsilon}\left\langle s_{\alpha\, 1/2}^{\mathcal{M}}\right\rangle_{\mathcal{M}} \tag{25}$$

where $\langle\cdot\rangle_i = \frac{1}{|Y^i|}\int\limits_{Y^i}(\cdot)dy, \quad i=\mathcal{F},\mathcal{M}$

The following links exist between them:

$$\mathcal{S}_o^{\mathcal{F}} = 1-\mathcal{S}_w^{\mathcal{F}}, \quad \mathcal{S}_o^{\mathcal{M}} = 1-\mathcal{S}_w^{\mathcal{F}},$$
$$\mathcal{P}_o^{\mathcal{F}} = \mathcal{P}_w^{\mathcal{F}} + p_c^{\mathcal{F}}\left(\mathcal{S}_w^{\mathcal{F}}\right), \quad \mathcal{P}_o^{\mathcal{M}} = \mathcal{P}_w^{\mathcal{M}} + p_c^{\mathcal{M}}\left(\mathcal{S}_w^{\mathcal{M}}\right) \tag{26}$$

Note that macroscopic capillary pressures in fractures and blocks, $\mathcal{P}_c^{\mathcal{F}}\left(\mathcal{S}_w^{\mathcal{F}}\right)$ and $\mathcal{P}_c^{\mathcal{M}}\left(\mathcal{S}^{\mathcal{M}}\right)$, do not figure in this model, since they are simply equal to the original capillary pressure functions taken from the average saturations, with accuracy $\mathcal{O}(\varepsilon)$:

$$\mathcal{P}_c^{\mathcal{F}}\left(\mathcal{S}_w^{\mathcal{F}}\right) = p_c^{\mathcal{F}}\left(\mathcal{S}_w^{\mathcal{F}}\right), \qquad \mathcal{P}_c^{\mathcal{M}}\left(\mathcal{S}_w^{\mathcal{M}}\right) = p_c^{\mathcal{M}}\left(\mathcal{S}_w^{\mathcal{M}}\right) \tag{27}$$

The macroscopic model has the form of the following system of equations for the average pressures and saturations in the blocks and fractures:

$$\phi^{\mathcal{F}}(1-\theta)\partial_t\mathcal{S}_\alpha^{\mathcal{F}} + \phi^{\mathcal{F}}C_\alpha^{\mathcal{F}}(1-\theta)\mathcal{S}_\alpha^{\mathcal{F}}\partial_t\mathcal{P}_\alpha^{\mathcal{F}} =$$
$$\partial_{xi}\left(\mathbb{K}_{ik}\lambda_\alpha^{\mathcal{F}}\partial_{xi}\mathcal{P}_\alpha^{\mathcal{F}}\right) + \xi_\alpha\left(\mathcal{P}_\alpha^{\mathcal{M}} - \mathcal{P}_\alpha^{\mathcal{F}}\right), \quad \alpha = w,o \tag{28a}$$

$$\mathcal{P}_w^{\mathcal{M}} = \mathcal{P}_w^{\mathcal{F}} - \frac{\tau_w^{\text{com}}}{C_w^{\mathcal{M}}\mathcal{S}_w^{\mathcal{M}}}\left[\partial_t\mathcal{S}_w^{\mathcal{M}} + C_w^{\mathcal{M}}\mathcal{S}_w^{\mathcal{M}}\partial_t\mathcal{P}_w^{\mathcal{M}}\right], \tag{28b}$$

$$p_c^{\mathcal{M}}\left(\mathcal{S}_w^{\mathcal{M}}\right) = p_c^{\mathcal{F}} + \frac{\tau^{\text{cap}}+\tau^{\text{cc}}}{C_o^{\mathcal{M}}}\partial_t\mathcal{S}_w^{\mathcal{M}} + \tag{28c}$$
$$\left(\tau_w^{\text{com}} - \tau_o^{\text{com}}\right)\partial_t\mathcal{P}_w^{\mathcal{M}} \tag{28d}$$

where θ is the volume fractions of the blocks.

The effective permeability is defined as:

$$\mathbb{K}_{ik} \equiv \int\limits_{Y^{\mathcal{F}}} K^{\mathcal{F}}\left(\partial_{yi}\psi_k + \delta_{ik}\right)dy \tag{29}$$

and the cell functions $\psi_k(y)$ are the solution to the first problem in a cell (in the fracture), for $k=1,2,3$:

$$\begin{cases} \partial_{yi}\Big(K^{\mathcal{F}}\left(\partial_{yi}\psi_k+\delta_{ik}\right)\Big) = 0, \quad y\in Y^{\mathcal{F}} \\ K^{\mathcal{F}}\left(\partial_{yi}\psi_k+\delta_{ik}\right)n_i^{\Gamma}\big|_{y\in\Gamma} = 0, \\ \psi_k \text{ is } y-\text{periodic} \\ \langle\psi_k\rangle_{Y^{\mathcal{F}}} = 0 \end{cases} \tag{30}$$

The delay times between blocks and fractures caused by the phase compressibility are defined as:

$$\tau_\alpha^{\text{com}}\left(\mathcal{S}_w^{\mathcal{M}}\right) = \sqrt{\varepsilon}\,\langle\varphi\rangle_{\mathcal{M}}\phi^{\mathcal{M}}\frac{C_\alpha^{\mathcal{M}}\mathcal{S}_\alpha^{\mathcal{M}}}{\lambda_\alpha^{\mathcal{M}}}, \quad \alpha = w,o \tag{31}$$

The delay times caused by capillarity and joint action of capillarity and compressibility are as follows:

$$\tau^{\text{cap}}\left(\mathcal{S}_w^{\mathcal{M}}\right) = \sqrt{\varepsilon}\,\langle\varphi\rangle_{\mathcal{M}}\phi^{\mathcal{M}}C_o^{\mathcal{M}}\frac{\left(\lambda_w^{\mathcal{M}}+\lambda_o^{\mathcal{M}}\right)}{\lambda_w^{\mathcal{M}}\lambda_o^{\mathcal{M}}},$$
$$\tau^{\text{cc}}\left(\mathcal{S}_w^{\mathcal{M}}\right) = -C_o^{\mathcal{M}}\frac{dp_c^{\mathcal{M}}}{d\mathcal{S}^{\mathcal{M}}}\tau_o^{\text{com}} \tag{32}$$

where the function $\varphi(y)$ is the solution of the second cell problem (in the block):

$$\begin{cases} \dfrac{\partial}{\partial y_i}\left(K^{\mathcal{M}}\dfrac{\partial\varphi}{\partial y_i}\right) = -1, \quad y\in Y^{\mathcal{M}} \\ \varphi|_{y\in\Gamma} = 0 \end{cases} \tag{33}$$

The transfer functions ξ_α are defined as:

$$\xi_\alpha \equiv \frac{\theta\lambda_\alpha^{\mathcal{M}}}{\sqrt{\varepsilon}\,\langle\varphi\rangle_{\mathcal{M}}} \tag{34}$$

where θ is the volume fraction of blocks.

The model (28) is the formal asymptotic expansion of the original problem that keeps the terms of order $\mathcal{O}\left(\sqrt{\varepsilon}\right)$. The order of residual terms is, thus, $\mathcal{O}\left(\varepsilon\right)$.

V. DERIVATION OF THE AVERAGED MODEL

A. First step of homogenization: expansion in fractures

Let $\zeta = \zeta(x,y)$ in $\Omega\times Y$, $\zeta|_{\partial\Omega} = 0$, ζ is continuous and periodic with respect to y. Then one obtains from (24) for negatives powers of ε:

$$0 = \frac{1}{\varepsilon}\int\limits_{\Omega}\int\limits_{Y^{\mathcal{F}}} K^{\mathcal{F}}\lambda_{\alpha 0}^{\mathcal{F}}\left(\partial_{yi}p_{\alpha 1}+\partial_{xi}p_{\alpha 0}\right)\partial_{yi}\zeta\, dydx,$$
$$0 = \frac{1}{\sqrt{\varepsilon}}\int\limits_{\Omega}\int\limits_{Y^{\mathcal{F}}} K^{\mathcal{F}}\Big[\lambda_{\alpha 0}^{\mathcal{F}}\left(\partial_{yi}p_{\alpha\, 3/2}+\partial_{xi}p_{\alpha\, 1/2}\right) + \lambda_{\alpha\, 1/2}^{\mathcal{F}}\left(\partial_{yi}p_{\alpha 1}+\partial_{xi}p_{\alpha 0}\right)\Big]\partial_{yi}\zeta\, dydx + \mathcal{O}\left(\varepsilon\right) \tag{35}$$

This yields the following:

$$0 = -\int\limits_{\Omega}\int\limits_{Y^{\mathcal{F}}} \zeta\, \partial_{yi}\left(K^{\mathcal{F}}\lambda^{\mathcal{F}}_{\alpha 0}\left(\partial_{yi}p_{\alpha 1} + \partial_{xi}p_{\alpha 0}\right)\right) dxdy + \int\limits_{\Omega}\int\limits_{\Gamma} \zeta\, K^{\mathcal{F}}\lambda^{\mathcal{F}}_{\alpha 0}\left(\partial_{yi}p_{\alpha 1} + \partial_{xi}p_{\alpha 0}\right) n_i^{\Gamma}\, dxdy \tag{36}$$

$$0 = \int\limits_{\Omega}\int\limits_{Y^{\mathcal{F}}} \zeta\, \partial_{yi}\Big(K^{\mathcal{F}}\lambda^{\mathcal{F}}_{\alpha 0}\left(\partial_{yi}p_{\alpha\, {}^3\!/_2} + \partial_{xi}p_{\alpha\, {}^1\!/_2}\right) + K^{\mathcal{F}}\lambda^{\mathcal{F}}_{\alpha 1}\left(\partial_{yi}p_{\alpha 1} + \partial_{xi}p_{\alpha 0}\right)\Big) dydx + \int\limits_{\Omega}\int\limits_{\Gamma} \zeta\left[K^{\mathcal{F}}\lambda^{\mathcal{F}}_{\alpha 0}\left(\partial_{yi}p_{\alpha\, {}^3\!/_2} + \partial_{xi}p_{\alpha\, {}^1\!/_2}\right) + K^{\mathcal{F}}\lambda^{\mathcal{F}}_{\alpha 1}\left(\partial_{yi}p_{\alpha 1} + \partial_{xi}p_{\alpha 0}\right)\right] n_i^{\Gamma}\, dxdy \tag{37}$$

Taking into account (23), this is equivalent to two problems in classical formulation:

$$\begin{cases} \partial_{yi}\left(K^{\mathcal{F}}\left(\partial_{yi}p^{\mathcal{F}}_{\alpha 1} + \partial_{xi}p_{\alpha 0}\right)\right) = 0, \quad y \in Y^{\mathcal{F}} \\ K^{\mathcal{F}}\left(\partial_{yi}p^{\mathcal{F}}_{\alpha 1} + \partial_{xi}p_{\alpha 0}\right) n_i^{\Gamma}\Big|_{y\in\Gamma} = 0 \end{cases} \tag{38}$$

$$\begin{cases} \partial_{yi}\left(K^{\mathcal{F}}\left(\partial_{yi}p^{\mathcal{F}}_{\alpha\, {}^3\!/_2} + \partial_{xi}p_{\alpha\, {}^1\!/_2}\right)\right) = 0, \quad y \in Y^{\mathcal{F}} \\ K^{\mathcal{F}}\left(\partial_{yi}p^{\mathcal{F}}_{\alpha\, {}^3\!/_2} + \partial_{xi}p_{\alpha\, {}^1\!/_2}\right) n_i^{\Gamma}\Big|_{y\in\Gamma} = 0 \end{cases} \tag{39}$$

This leads to the following representation of $p^{\mathcal{F}}_{\alpha 1}$ and $p^{\mathcal{F}}_{\alpha\, {}^3\!/_2}$ through $p_{\alpha 0}$ and $p^{\mathcal{F}}_{\alpha\, {}^1\!/_2}$:

$$\begin{aligned} p^{\mathcal{F}}_{\alpha 1} &= \psi_k(y)\frac{\partial p_{\alpha 0}}{\partial x_k} + \bar{p}^{\mathcal{F}}_{\alpha 1}(x,t), \\ p^{\mathcal{F}}_{\alpha\, {}^3\!/_2} &= \psi_k(y)\frac{\partial p^{\mathcal{F}}_{\alpha\, {}^1\!/_2}}{\partial x_k} + \bar{p}^{\mathcal{F}}_{\alpha\, {}^3\!/_2}(x,t) \end{aligned} \tag{40}$$

where $\bar{p}^{\mathcal{F}}_{\alpha 1}(x,t)$ and $\bar{p}^{\mathcal{F}}_{\alpha\, {}^3\!/_2}(x,t)$ are some slow functions, which do not enter in the homogenized model. For functions ψ_k, we obtain the first cell problem (30), in which the latter condition is added to ensure the solution uniqueness.

B. Second step: the averaged equation in fractures

Let us select the test functions as $\zeta = \zeta(x)$ in Ω and $\zeta|_{\partial\Omega} = 0$. Then one obtains from (24):

$$\begin{aligned} &\int\limits_{\Omega}\int\limits_{Y} \zeta\, \phi\left(\partial_t s_{\alpha 0} + \sqrt{\varepsilon}\partial_t s_{\alpha\, {}^1\!/_2}\right) dxdy + \\ &\int\limits_{\Omega}\int\limits_{Y} \zeta\, \phi C_\alpha\left(s_{\alpha 0} + \sqrt{\varepsilon}s_{\alpha\, {}^1\!/_2}\right)\left(\partial_t p_{\alpha 0} + \sqrt{\varepsilon}\partial_t p_{\alpha\, {}^1\!/_2}\right) dxdy = \\ &-\int\limits_{\Omega}\int\limits_{Y^{\mathcal{F}}} K^{\mathcal{F}}\left(\lambda^{\mathcal{F}}_{\alpha 0} + \sqrt{\varepsilon}\lambda^{\mathcal{F}}_{\alpha\, {}^1\!/_2}\right)\Big[\partial_{yi}p_{\alpha 1} + \partial_{xi}p_{\alpha 0} + \\ &\sqrt{\varepsilon}\left(\partial_{yi}p_{\alpha\, {}^3\!/_2} + \partial_{xi}p_{\alpha\, {}^1\!/_2}\right)\Big]\partial_{xi}\zeta\, dydx + \mathcal{O}(\varepsilon) \end{aligned} \tag{41}$$

Using (40) we deduce:

$$\begin{aligned} &\int\limits_{\Omega}\int\limits_{Y} \zeta\, \phi\partial_t\left(s_{\alpha 0} + \sqrt{\varepsilon}s_{\alpha\, {}^1\!/_2}\right) dxdy + \\ &\int\limits_{\Omega}\int\limits_{Y} \zeta\, \phi\, C_\alpha\left(s_{\alpha 0} + \sqrt{\varepsilon}s_{\alpha\, {}^1\!/_2}\right)\partial_t\left(p_{\alpha 0} + \sqrt{\varepsilon}p_{\alpha\, {}^1\!/_2}\right) dxdy = \\ &\int\limits_{\Omega}\int\limits_{Y^{\mathcal{F}}} \zeta\, \partial_{xi}\Big(K^{\mathcal{F}}\left(\lambda^{\mathcal{F}}_{\alpha 0} + \sqrt{\varepsilon}\lambda^{\mathcal{F}}_{\alpha\, {}^1\!/_2}\right)\left(\partial_{yi}\psi_k + \delta_{ik}\right)\times \\ &\partial_{xk}\left(p_{\alpha 0} + \sqrt{\varepsilon}p^{\mathcal{F}}_{\alpha\, {}^1\!/_2}\right)\Big) dydx + \mathcal{O}(\varepsilon) \end{aligned} \tag{42}$$

Introducing the averaged pressures and saturations (25), we obtain:

$$\begin{aligned} &\int\limits_{\Omega} \zeta\left[\phi^{\mathcal{F}}(1-\theta)\partial_t\mathcal{S}^{\mathcal{F}}_\alpha + \phi^{\mathcal{M}}\theta\partial_t\mathcal{S}^{\mathcal{M}}_\alpha\right] dx + \\ &\int\limits_{\Omega} \zeta\left[\phi^{\mathcal{F}}C^{\mathcal{F}}_\alpha(1-\theta)\mathcal{S}^{\mathcal{F}}_\alpha\partial_t\mathcal{P}^{\mathcal{F}}_\alpha + \phi^{\mathcal{M}}C^{\mathcal{M}}_\alpha\theta\mathcal{S}^{\mathcal{M}}_\alpha\partial_t\mathcal{P}^{\mathcal{M}}_\alpha\right] dx = \\ &\int\limits_{\Omega} \zeta\, \partial_{xi}\Big(\mathbb{K}_{ik}\lambda^{\mathcal{F}}_\alpha\left(\mathcal{S}_\alpha\mathcal{F}\right)\partial_{xk}\mathcal{P}^{\mathcal{F}}_\alpha\Big) dx \end{aligned} \tag{43}$$

where the effective permeability tensor is (29). This gives the first averaged equation (28a) in the as-yet-incomplete form:

$$\begin{aligned} &\phi^{\mathcal{F}}(1-\theta)\partial_t\mathcal{S}^{\mathcal{F}}_\alpha + \phi^{\mathcal{F}}C^{\mathcal{F}}_\alpha(1-\theta)\mathcal{S}^{\mathcal{F}}_\alpha\partial_t\mathcal{P}^{\mathcal{F}}_\alpha = \\ &\partial_{xi}\Big(\mathbb{K}_{ik}\lambda^{\mathcal{F}}_\alpha\left(\mathcal{S}_\alpha\mathcal{F}\right)\partial_{xk}\mathcal{P}^{\mathcal{F}}_\alpha\Big) - \phi^{\mathcal{M}}\theta\partial_t\mathcal{S}^{\mathcal{M}}_\alpha - \\ &\phi^{\mathcal{M}}C^{\mathcal{M}}_\alpha\theta\mathcal{S}^{\mathcal{M}}_\alpha\partial_t\mathcal{P}^{\mathcal{M}}_\alpha, \quad \alpha = w, o \end{aligned} \tag{44}$$

C. Third step: expansion in blocks

Let $\zeta=\zeta(x,y)$ in $\Omega\times Y$, $\zeta|_{\partial\Omega}=0$, $\zeta\equiv 0$ in $Y^{\mathcal{F}}$, and ζ is continuous. Then the identity (24) gives:

$$\int\limits_{\Omega}\int\limits_{Y^{\mathcal{M}}}\zeta\,\phi\left(\partial_t s_{\alpha 0}+\sqrt{\varepsilon}\,\partial_t s^{\mathcal{M}}_{\alpha\,1/2}\right)dxdy+$$
$$\int\limits_{\Omega}\int\limits_{Y^{\mathcal{M}}}\zeta\,\phi\,C_\alpha\left(s_{\alpha 0}+\sqrt{\varepsilon}s^{\mathcal{M}}_{\alpha\,1/2}\right)\left(\partial_t p_{\alpha 0}+\sqrt{\varepsilon}\partial_t p^{\mathcal{M}}_{\alpha\,1/2}\right)dxdy=$$
$$-\int\limits_{\Omega}\int\limits_{Y^{\mathcal{M}}}K^{\mathcal{M}}\left(\lambda_{\alpha 0}+\sqrt{\varepsilon}\lambda^{\mathcal{M}}_{\alpha\,1/2}\right)\Big[\partial_{yi}p_{\alpha\,1/2}+$$
$$\sqrt{\varepsilon}\left(\partial_{yi}p_{\alpha 1}+\partial_{xi}p_{\alpha 0}\right)\Big]\partial_{xi}\zeta\,dydx+\mathcal{O}\left(\varepsilon\right) \tag{45}$$

The integral over the block boundary is zero, since functions w are zero in the fracture and are continuous. The zero-order terms yield:

$$\int\limits_{\Omega}\int\limits_{Y^{\mathcal{M}}}\zeta\,\phi\left(\partial_t s_{\alpha 0}+C_\alpha s_{\alpha 0}\partial_t p_{\alpha 0}\right)dxdy=$$
$$\int\limits_{\Omega}\int\limits_{Y^{\mathcal{M}}}\zeta\,\partial_{yi}\left(K^{\mathcal{M}}\lambda_{\alpha 0}\partial_{yi}p_{\alpha\,1/2}\right)dydx \tag{46}$$

This produces the following expression for $p^{\mathcal{M}}_{\alpha\,1/2}$ given (22):

$$p^{\mathcal{M}}_{\alpha\,1/2}=p^{\mathcal{F}}_{\alpha\,1/2}-\varphi(y)\frac{\phi^{\mathcal{M}}}{\lambda_{\alpha 0}}\left(\partial_t s_{\alpha 0}+C^{\mathcal{M}}_\alpha s_{\alpha 0}\partial_t p_{\alpha 0}\right) \tag{47}$$

where $\varphi(y)$ is he solution of the second cell problem (33). The boundary condition in (33) results from the continuity of the phase pressures (22).

D. Fourth step: averaged equation in blocks

Formula (47) enables us to obtain an explicit relation between $\mathcal{P}^{\mathcal{M}}_\alpha$ and $\mathcal{P}^{\mathcal{F}}_\alpha$. Indeed, taking the average of (47) over $Y^{\mathcal{M}}$ (and noting that only the function ϕ depends on y in the right-hand side of (47)), multiplying by $\sqrt{\varepsilon}$ and adding $p_{\alpha 0}$, we obtain:

$$p_{\alpha 0}+\sqrt{\varepsilon}\left\langle p^{\mathcal{M}}_{\alpha\,1/2}\right\rangle_{\mathcal{M}}=p_{\alpha 0}+\sqrt{\varepsilon}p^{\mathcal{F}}_{\alpha\,1/2}-$$
$$\sqrt{\varepsilon}\left\langle\varphi\right\rangle_{\mathcal{M}}\frac{\phi^{\mathcal{M}}}{\lambda_{\alpha 0}}\left(\partial_t s_{\alpha 0}+C^{\mathcal{M}}_\alpha s_{\alpha 0}\partial_t p_{\alpha 0}\right) \tag{48}$$

Using the definition of the averaged pressures and saturations (25), we deduce:

$$\mathcal{P}^{\mathcal{M}}_\alpha=\mathcal{P}^{\mathcal{F}}_\alpha-\frac{\tau^{\rm com}_\alpha}{C^{\mathcal{M}}_\alpha\mathcal{S}^{\mathcal{M}}_\alpha}\left(\partial_t\mathcal{S}^{\mathcal{M}}_\alpha+C^{\mathcal{M}}_\alpha\mathcal{S}^{\mathcal{M}}_\alpha\partial_t\mathcal{P}^{\mathcal{M}}_\alpha\right),\quad \alpha=w,o \tag{49}$$

which is identical to (28b). The characteristic times of delay are defined as (31).

In the structure of the delay times, we have taken into account the following circumstances:

$$\partial_t s_{\alpha 0}=\partial_t\left(s_{\alpha 0}+\sqrt{\varepsilon}\left\langle s^{\mathcal{M}}_{\alpha,01}\right\rangle_{\mathcal{M}}\right)-\sqrt{\varepsilon}\,\partial_t\left\langle s^{\mathcal{M}}_{\alpha,01}\right\rangle_{\mathcal{M}}=$$
$$\partial_t\mathcal{S}^{\mathcal{M}}_\alpha+\mathcal{O}\left(\sqrt{\varepsilon}\right), \tag{50}$$

$$\lambda^{\mathcal{M}}_{\alpha 0}=\lambda^{\mathcal{M}}_\alpha\left(s_{\alpha 0}\right)+\sqrt{\varepsilon}\,\frac{d\lambda^{\mathcal{M}}_\alpha}{ds_{\alpha 0}}\left\langle s^{\mathcal{M}}_{\alpha\,1/2}\right\rangle_{\mathcal{M}}-$$
$$\sqrt{\varepsilon}\,\frac{d\lambda^{\mathcal{M}}_\alpha}{ds_{\alpha 0}}\left\langle s^{\mathcal{M}}_{\alpha\,1/2}\right\rangle_{\mathcal{M}}=\lambda^{\mathcal{M}}_\alpha\left(s_{\alpha 0}+\sqrt{\varepsilon}\left\langle s^{\mathcal{M}}_{\alpha\,1/2}\right\rangle_{\mathcal{M}}\right)+$$
$$\mathcal{O}\left(\sqrt{\varepsilon}\right)=\lambda^{\mathcal{M}}_\alpha\left(\mathcal{S}^{\mathcal{M}}_\alpha\right)+\mathcal{O}\left(\sqrt{\varepsilon}\right) \tag{51}$$

Consequently, we deduce the following:

$$\frac{\sqrt{\varepsilon}}{\lambda^{\mathcal{M}}_{\alpha 0}}\,\partial_t s_{\alpha 0}=\frac{\sqrt{\varepsilon}\,\partial_t\mathcal{S}^{\mathcal{M}}_\alpha+\mathcal{O}\left(\varepsilon\right)}{\lambda^{\mathcal{M}}_\alpha\left(\mathcal{S}^{\mathcal{M}}_\alpha\right)+\mathcal{O}\left(\sqrt{\varepsilon}\right)}=\frac{\sqrt{\varepsilon}\,\partial_t\mathcal{S}^{\mathcal{M}}_\alpha}{\lambda^{\mathcal{M}}_\alpha\left(\mathcal{S}^{\mathcal{M}}_\alpha\right)}+\mathcal{O}\left(\varepsilon\right) \tag{52}$$

and similarly for other terms.

Instead of two equations (49) for phase pressures, it is possible to replace one of them by the equation for the capillary pressure. Subtracting one equation (49) from another one, we deduce the relation between the averaged capillary pressures in the blocks and fractures:

$$\mathcal{P}^{\mathcal{M}}_w=\mathcal{P}^{\mathcal{F}}_w+\tau^{\rm com}_w\partial_t\mathcal{P}^{\mathcal{M}}_w-\tau^{\rm com}_o\partial_t\left(\mathcal{P}^{\mathcal{M}}_w+\mathcal{P}^{\mathcal{M}}_c\right)+$$
$$\left(\frac{\tau^{\rm com}_w}{C^{\mathcal{M}}_w\mathcal{S}^{\mathcal{M}}_w}+\frac{\tau^{\rm com}_o}{C^{\mathcal{M}}_o\mathcal{S}^{\mathcal{M}}_o}\right)\partial_t\mathcal{S}^{\mathcal{M}}_w=$$
$$\mathcal{P}^{\mathcal{F}}_w+\left(\frac{\tau^{\rm com}_w}{C^{\mathcal{M}}_w\mathcal{S}^{\mathcal{M}}_w}+\frac{\tau^{\rm com}_o}{C^{\mathcal{M}}_o\mathcal{S}^{\mathcal{M}}_o}-\tau^{\rm com}_w\frac{d\mathcal{P}^{\mathcal{M}}_c}{d\mathcal{S}^{\mathcal{M}}_w}\right)\partial_t\mathcal{S}^{\mathcal{M}}_w+$$
$$\left(\tau^{\rm com}_w-\tau^{\rm com}_o\right)\partial_t\mathcal{P}^{\mathcal{M}}_w \tag{53}$$

which is reduced to (28d) if we prove the link (49) between the average capillary pressures $\mathcal{P}^{\mathcal{M}}_c$ and $\mathcal{P}^{\mathcal{F}}_c$ and the original capillary pressure curves $p^{\mathcal{M}}_c$ and $p^{\mathcal{F}}_c$. This is done in the following way. Let us expand the expression for $p^{\mathcal{M}}_c$ and $p^{\mathcal{F}}_c$ in Taylor series:

$$p^{\mathcal{F}}_c\left(\mathcal{S}^{\mathcal{F}}_w\right)=p^{\mathcal{F}}_c\left(s_{w0}\right)+\sqrt{\varepsilon}\,\frac{dp^{\mathcal{F}}_c}{ds_{w0}}\,s^{\mathcal{F}}_{w\,1/2}+\mathcal{O}\left(\varepsilon\right),$$
$$p^{\mathcal{M}}_c\left(\mathcal{S}^{\mathcal{M}}_w\right)=p^{\mathcal{M}}_c\left(s_{w0}\right)+\sqrt{\varepsilon}\,\frac{dp^{\mathcal{M}}_c}{ds_{w0}}\left\langle s^{\mathcal{M}}_{w\,1/2}\right\rangle_{\mathcal{M}}+\mathcal{O}\left(\varepsilon\right) \tag{54}$$

Since all the functions on the right do not depend on y, then the averaging over the fractures or blocks does not change anything:

$$\mathcal{P}^{\mathcal{F}}_c\left(\mathcal{S}^{\mathcal{F}}_w\right)\equiv\left\langle p^{\mathcal{F}}_c\left(\mathcal{S}^{\mathcal{F}}_w\right)\right\rangle_{\mathcal{F}}=p^{\mathcal{F}}_c\left(\mathcal{S}^{\mathcal{F}}_w\right)+\mathcal{O}\left(\varepsilon\right),$$
$$\mathcal{P}^{\mathcal{M}}_c\left(\mathcal{S}^{\mathcal{M}}_w\right)\equiv\left\langle p^{\mathcal{M}}_c\left(\mathcal{S}^{\mathcal{M}}_w\right)\right\rangle_{\mathcal{M}}=p^{\mathcal{M}}_c\left(\mathcal{S}^{\mathcal{M}}_w\right)+\mathcal{O}\left(\varepsilon\right) \tag{55}$$

which proves (27).

E. Fifth step: final form of Eqs. (28a)

For the expressions in the right-hand side of the macroscopic equations (28a), it is possible to obtain an explicit relation through the pressure difference, using (28b):

$$C_\alpha^{\mathcal{M}} \mathcal{S}_\alpha^{\mathcal{M}} \partial_t \mathcal{P}_\alpha^{\mathcal{M}} = -\frac{C_\alpha^{\mathcal{M}} \mathcal{S}_\alpha^{\mathcal{M}}}{\tau_\alpha^{\mathrm{com}}} \left(\mathcal{P}_\alpha^{\mathcal{M}} - \mathcal{P}_\alpha^{\mathcal{F}} \right) \tag{56}$$

which gives a more traditional form to the terms of exchange between blocks and fractures and leads to the definite form (28a).

VI. PHYSICAL INTERPRETATION OF THE RESULTS OBTAINED

A. Particular case of compressible single-phase flow

In the single-phase case, system (28) takes the form:

$$\begin{cases} \phi^{\mathcal{F}} C^{\mathcal{F}} (1-\theta) \partial_t \mathcal{P}^{\mathcal{F}} - \partial_{xi} \left(\mathbb{K}_{ik} \partial_{xi} \mathcal{P}^{\mathcal{F}} \right) = \xi \left(\mathcal{P}^{\mathcal{M}} - \mathcal{P}^{\mathcal{F}} \right) \\ \mathcal{P}^{\mathcal{M}} = \mathcal{P}^{\mathcal{F}} - \tau^{\mathrm{com}} \partial_t \mathcal{P}^{\mathcal{M}} \end{cases} \tag{57}$$

where $\xi \equiv \dfrac{\theta}{\sqrt{\varepsilon} \, \langle \varphi \rangle_{\mathcal{M}}}$, $\tau^{\mathrm{com}} = \sqrt{\varepsilon} \, \langle \varphi \rangle_{\mathcal{M}} \, \phi^{\mathcal{M}} C^{\mathcal{M}}$, which is the well-known model of single-phase flow in double porosity medium with moderate contrast[11].

B. Particular case of an incompressible two-phase system

In the incompressible case, system (28) takes the following form:

$$\begin{aligned} &\phi^{\mathcal{F}} (1-\theta) \partial_t \mathcal{S}_\alpha^{\mathcal{F}} = \partial_{xi} \left(\mathbb{K}_{ik} \lambda_\alpha^{\mathcal{F}} \partial_{xi} \mathcal{P}_\alpha^{\mathcal{F}} \right) + \\ &\qquad \xi_\alpha \left(\mathcal{P}_\alpha^{\mathcal{M}} - \mathcal{P}_\alpha^{\mathcal{F}} \right), \quad \alpha = w, o \\ &\mathcal{P}_w^{\mathcal{M}} = \mathcal{P}_w^{\mathcal{F}} - \frac{\tau_w^{\mathrm{com}}}{C_w^{\mathcal{M}} \mathcal{S}_w^{\mathcal{M}}} \partial_t \mathcal{S}_w^{\mathcal{M}}, \\ &p_c^{\mathcal{M}} \left(\mathcal{S}_w^{\mathcal{M}} \right) = p_c^{\mathcal{F}} + \frac{\tau^{\mathrm{cap}}}{C_o^{\mathcal{M}}} \partial_t \mathcal{S}_w^{\mathcal{M}} \end{aligned} \tag{58}$$

where parameters $\frac{\tau_w^{\mathrm{com}}}{C_w^{\mathcal{M}} \mathcal{S}_w^{\mathcal{M}}}$ and $\frac{\tau^{\mathrm{cap}}}{C_o^{\mathcal{M}}}$ do not depend on the compressibility coefficients:

$$\begin{aligned} &\frac{\tau_w^{\mathrm{com}}}{C_w^{\mathcal{M}} \mathcal{S}_w^{\mathcal{M}}} = \sqrt{\varepsilon} \, \langle \varphi \rangle_{\mathcal{M}} \frac{\phi^{\mathcal{M}}}{\lambda_w^{\mathcal{M}}}, \\ &\frac{\tau^{\mathrm{cap}}}{C_o^{\mathcal{M}}} = \sqrt{\varepsilon} \, \langle \varphi \rangle_{\mathcal{M}} \, \phi^{\mathcal{M}} \frac{\left(\lambda_w^{\mathcal{M}} + \lambda_o^{\mathcal{M}} \right)}{\lambda_w^{\mathcal{M}}} \end{aligned} \tag{59}$$

As seen from two last equations in (58), only the latter is differential (with respect to $\mathcal{S}_w^{\mathcal{M}}$), while the former is algebraic with respect to $\mathcal{P}_w^{\mathcal{M}}$. This means that the delay effects, which are caused only by the capillarity in this case, concern the capillary pressure and saturation. The difference in phase pressures in blocks and fractures also exists, but only as a consequence of the link of phase pressures with capillary pressure.

C. Capillary delay and delay caused by compressibility

Comparison with the case of incompressible fluids (58) and single-phase flow (57) enables us to better understand the essence of the obtained model (28) and the role of compressibility in two-phase systems. The nonequilibrium behavior of system (28) is determined by the subsystem of two ordinary differential equations (28b) and (28d) for saturation and pressure. This means that the pressure and the saturation of water in blocks are delayed with respect to those in fractures. Such a delay is caused by the following mechanisms on the microscale.

- Non-equilibrium capillary redistribution of phases between the blocks and fractures, caused by the fact that the average saturation in blocks changes more slowly than in fractures, which violates the equality of average capillary pressures (the condition of capillary equilibrium). As a result, a difference in the average capillary pressures arises, which depends on the rate of variation of the saturation in the blocks. In equation (28d), this process is described by the first term:

$$p_c^{\mathcal{M}} - p_c^{\mathcal{F}} = \frac{\tau^{\mathrm{cap}}}{C_o^{\mathcal{M}}} \partial_t \mathcal{S}_w^{\mathcal{M}} \tag{60}$$

  The difference in capillary pressures automatically causes a difference in phase pressures, which is expressed by the first term in (28b):

$$\mathcal{P}_w^{\mathcal{M}} - \mathcal{P}_w^{\mathcal{F}} = \frac{\tau_w^{\mathrm{com}}}{C_w^{\mathcal{M}} \mathcal{S}_w^{\mathcal{M}}} \partial_t \mathcal{S}_w^{\mathcal{M}} \tag{61}$$

  In a medium with strong contrast in permeability, such a process would lead to long memory, but in the case of moderate contrast, we obtain the short memory described by the kinetic relationship (28d). This process does not depend on the compressibility of phases and rocks and is the same as in the incompressible case (58).

- Nonuniform pressure redistribution between the blocks and fractures. The compressibility of the system determines the appearance of pressure waves, whose rate of propagation is equal to $K/(\mu C \phi)$. Thus, it is lower in low permeable blocks than in fractures. This leads to a delay in the behavior of pressures, which is described by the second term in equation (28b):

$$\mathcal{P}_w^{\mathcal{M}} - \mathcal{P}_w^{\mathcal{F}} = -\tau_w^{\mathrm{com}} \partial_t \mathcal{P}_w^{\mathcal{M}} \tag{62}$$

This process does not depend on the two-phase nature of the system and is the same as in the single-phase case, (57).

- Non-equilibrium asymmetric extrusion of the phases (peristalsis) due to their expansion and compaction of pores. Compressibility leads to expansion of liquids and compaction of pores under the weight of overlying rocks, which leads to the extrusion of both phases from the pores. This effect is similar to peristalsis, when a fluid in a channel is driven by the deformation of the channel walls. If such extrusion is symmetrical for both phases, they are extruded as a whole, which does not change their volume fraction. In contrast, an asymmetrical extrusion leads to a redistribution of phases in space, which affects their saturation. In blocks, such a movement caused by extrusion is delayed, which involves an additional non-equilibrium in the saturation behavior, and enhances the capillary disequilibrium. This leads to an additional difference in capillary pressures, which is reflected by the third term in (28d):

$$p_c^{\mathcal{M}} - p_c^{\mathcal{F}} = (\tau_w^{\mathrm{com}} - \tau_o^{\mathrm{com}})\, \partial_t \mathcal{P}_w^{\mathcal{M}} \tag{63}$$

  As seen, this effect is really zeroed if the extrusion of the phases is symmetrical, that is if $\tau_w^{\mathrm{com}} - \tau_o^{\mathrm{com}}$.

- The nonlinear component of extrusion, which is also caused by the imposition of the capillarity, compressibility and nonlinearity of the flow equations. This effect is described by the second term in (28d):

$$p_c^{\mathcal{M}} - p_c^{\mathcal{F}} = \frac{\tau^{\mathrm{cc}}}{C_o^{\mathcal{M}}}\, \partial_t \mathcal{S}_w^{\mathcal{M}} \tag{64}$$

VII. QUANTITATIVE ANALYSIS

To analyse the role of various memory effects, we will consider two examples of the application of this model to an underground reservoir of oil or gas in an aquifer:
- Case I: monotonic depletion of an oil reservoir;
- Case II: oscillatory functioning of an underground gas storage in an aquifer.

In both cases we assume that the saturation and pressure in fractures are constant in space but vary in time as given functions, whose behaviour reflects the physical process we are analyzing. For instance, for the depletion process, the fracture pressure and saturation are monotonic functions of time, whilst for the gas storage they are oscillatory in time. Then it is sufficient to solve only the system of two differential equations (28b) and (28d), which can be presented in the following form:

$$\begin{cases} a_{11}\, \partial_t \mathcal{S}^{\mathcal{M}} + a_{12}\, \partial_t \mathcal{P}^{\mathcal{M}} = -\left(\mathcal{P}^{\mathcal{M}} - \mathcal{P}^{\mathcal{F}}\right) \\ a_{21}\, \partial_t \mathcal{S}^{\mathcal{M}} + a_{22}\, \partial_t \mathcal{P}^{\mathcal{M}} = p_c^{\mathcal{M}} - p_c^{\mathcal{F}} \end{cases} \tag{65}$$

where $\mathcal{S} \equiv \mathcal{S}_w$, $\mathcal{P} \equiv \mathcal{P}_w$, and

$$\begin{aligned} a_{11} &\equiv \frac{\tau_w^{\mathrm{com}}}{C_w^{\mathcal{M}} \mathcal{S}^{\mathcal{M}}}, \qquad a_{12} \equiv \tau_w^{\mathrm{com}} \\ a_{21} &\equiv \frac{(\tau^{\mathrm{cap}} + \tau^{\mathrm{cc}})}{C_o^{\mathcal{M}}}, \qquad a_{22} \equiv \tau_w^{\mathrm{com}} - \tau_o^{\mathrm{com}} \end{aligned} \tag{66}$$

A. Parameters of natural systems

All the variables in the system (28) are assumed to be dimensionless.

The dimensionless value of compressibility parameter corresponds to $C_w p^0$ where p^0 is the initial pressure ((19)). In natural underground reservoirs, $p^0 \sim 20 - 50$ MPa. The compressibility of water and consolidated rocks (silicates) is of order $10^{-9} - 10^{-8}$ Pa^{-1}, which corresponds to dimensionless value of $C_w \sim 0.02 - 0.5$. The compressibility of oil and soft rocks is higher: $\sim 10^{-8} - 10^{-7}$ Pa^{-1}, so that the dimensionless compressibility is $C_o \sim 0.2 - 5$.

The exponential law of compressibility (1) is applied to liquids and solids. But it can also be used for highly compressed gas, whose properties are close to those of liquid. The compressibility of such a dense gas may be much higher than any liquid, so that the dimensionless parameter C is of order $2 - 50$.

Thus, we conclude that:
- $C_\alpha \sim 0.03$: low compressibility, 10^{-9} Pa^{-1};
- $C_\alpha \sim 0.3$: moderate compressibility, 10^{-8} Pa^{-1};
- $C_\alpha \sim 3$: high compressibility, 10^{-7} Pa^{-1};
- $C_\alpha \sim 30$: very high compressibility (of dense gases), 10^{-6} Pa^{-1}.

Other parameters are: $K^{\mathcal{F}} = 1$, $K^{\mathcal{M}} = 1$, $\varepsilon = 0.2$, $\phi^{\mathcal{F}} = 0.2$, $\phi^{\mathcal{M}} = 0.2$, $\theta = 0.8$, $\mu_w/\mu_o = 0.5$. We use the following analytical curves of relative permeability and capillary pressure:

$$\begin{aligned} k_w(s) &= s^2, \qquad k_o(s) = (1-s)^2 \\ p_c^{\mathcal{F}}(s) &= \gamma\sqrt{-\ln s}, \qquad p_c^{\mathcal{M}}(s) = \frac{\gamma}{\eta}\sqrt{-\ln s} \end{aligned} \tag{67}$$

where $\gamma = 0.1$ and $\eta = 0.65$. These functions are shown in Fig. 2. For the sake of simplicity, the relative permeabilities are assumed to be identical in blocks and fractures.

The cell problem (33) represents a Dirichlet problem for Poisson's equation. It was solved numerically in a square block by the finite element method using the Matlab PDE Toolbox, which is the solver of partial differential equations integrated into the package Matlab. The domain was discretized by triangular elements. The basic functions were linear. The solution has the form of a convex surface having the maximum at the block centre and zero at the block boundary. Its average value is $\langle\varphi\rangle_{\mathcal{M}} = 0.025$.

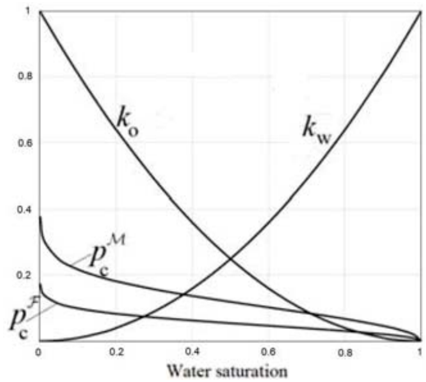

FIG. 2. Original curves of relative permeability and capillary pressure

B. Monotonic depletion of an oil reservoir

Let the reservoir contain initially oil and water. The initial pressure is uniform in the entire reservoir, while the saturations are different in blocks and fractures, being controlled by the capillary equilibrium:

$$\mathcal{P}_w^{\mathcal{F}} = \mathcal{P}_w^{\mathcal{M}} = 1, \quad \mathcal{S}_w^{\mathcal{F}} = 0.2, \quad \text{at} \quad t = 0 \qquad (68)$$

The production of oil leads to the decrease in reservoir pressure and to the invasion of the aquifer water, which leads, in turn, to the increase in water saturation. This can be expressed in terms of the following behaviour of the pressure and saturation in fractures:

$$\mathcal{P}_w^{\mathcal{F}} = 0.7 - qt, \qquad \mathcal{S}_w^{\mathcal{F}} = 0.2 + qt$$

where q is the depletion rate.

This means that $\mathcal{P}_w^{\mathcal{F}}$ instantaneously drops from 1 to 0.7 and then decreases linearly with time.

The behaviour of block pressure and saturation is shown in Fig. 3 for high compressibility: $C_w = C_o = 3$.

A significant delay in pressure propagation is observed only for highly compressible systems ($C_w \gg 1$).

The saturation field is much more affected by the compressibility effects than pressure.

C. Oscillatory regimes of injection-production

Let the pressure and saturation in fractures oscillate in time, which corresponds in practice to functioning of an underground storage of gas in an aquifer. A half of year the gas in injected (the pressure increases and the water saturation decreases). For another half year, the gas is withdrawn, so that the pressure decreases and the water saturation increases:

$$\mathcal{P}^{\mathcal{F}} = 1 + A\cos(\nu t), \qquad \mathcal{S}^{\mathcal{F}} = 0.2 + A\sin(\nu t) \qquad (69)$$

where A and ν are the amplitude and the period of oscillations.

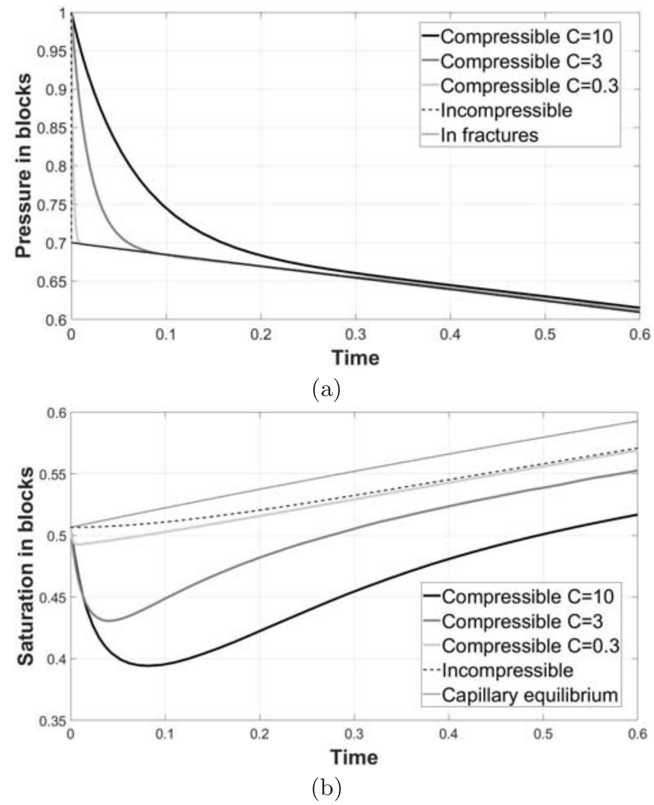

FIG. 3. Variation of pressure (a) and water saturation (b) in blocks, for various $C_w^{\mathcal{M}} = C_o^{\mathcal{M}} = 0.3, 3, 10$

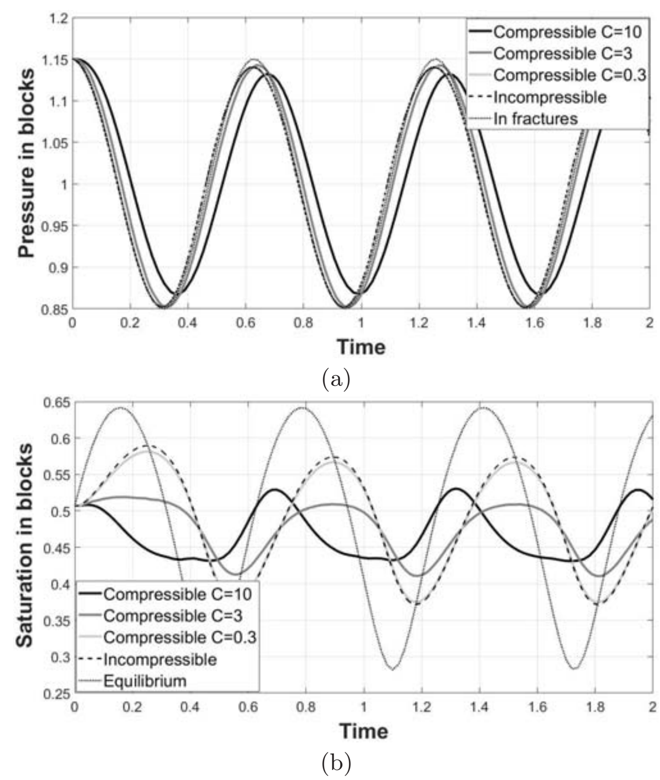

FIG. 4. Variation of pressure (a) and saturation (b) in blocks, for various $C_w^{\mathcal{M}} = C_o^{\mathcal{M}} = 0.3, 3, 10$

The behaviour of pressure and saturation is shown in Fig. 4, for high compressibility ($C \sim 3$).

As in the previous case, the pressure delay is observed only for very high compressibility of fluids or rocks. In contrast, the delay in the saturation is significant even for moderate values of compressibility $C_w, C_o \sim 1-3$. In the case of high compressibility, the saturation in blocks is totally different from the incompressible case.

D. Impact of asymmetrical compressibility of phases

If the compressibility of two phases is very different, it has a significant impact on saturation, even for moderate compressibility. This impact is even greater than that of a strong but identical compressibility for the two phases. This is clearly seen in Fig. 5, which compares two cases: the high compressibility but identical for both phases (the grey curve, $C_w = C_o = 3$), and the case of different compressibility (the black curve, $C_w = 0.3$ and $C_o = 3$).

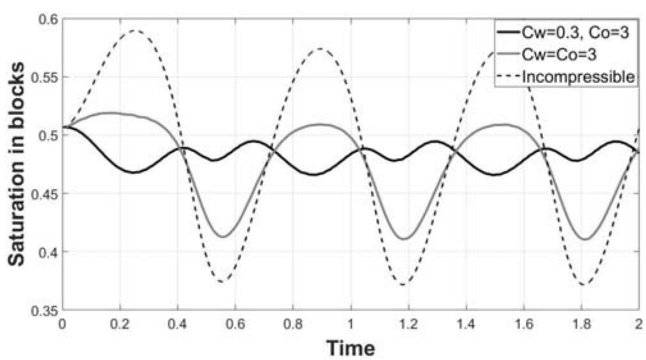

FIG. 5. Variation of the saturation in blocks in the case of asymmetrical compressibility, for $C_w^{\mathcal{M}} = 0.3$ and $C_o^{\mathcal{M}} = 3$

If the compressibility is asymmetrical and high, then the saturation in blocks changes even qualitatively the behaviour with respect to the incompressible case. In Fig. 6 the black curve (high compressibility of both phases) has not only much higher amplitude but also the inverse maximums/minimus comparing to the incompressible case (the dashed curve) or to the case of an identical compressibility (the grey curve).

In all the situations, one sees that the delay effects concern much more the saturation, than the pressure. A moderate and even low but asymmetrical compressibility has significant impact on the saturation.

E. Simplification of the macroscopic model

Given the results of simulations, we can suggest a simplified approximate macroscopic model, in which we neglect the delay in the behaviour of pressure. Then the version of the model (28), in which only the saturation

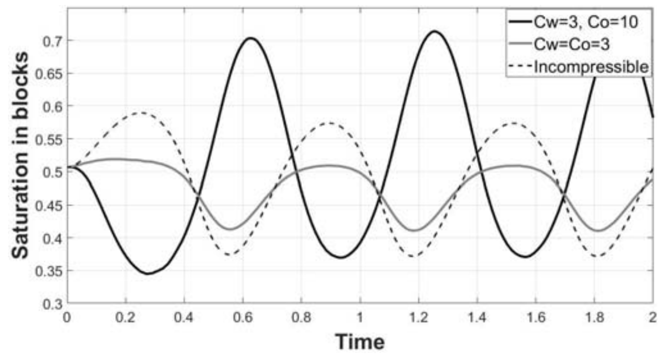

FIG. 6. Variation of the saturation in blocks in the case of asymmetrical compressibility, for $C_w^{\mathcal{M}} = 3$ and $C_o^{\mathcal{M}} = 10$

is in disequilibrium (but not the pressure) takes the following form:

$$\phi^{\mathcal{F}}(1-\theta)\partial_t \mathcal{S}_\alpha^{\mathcal{F}} + \phi^{\mathcal{F}} C_\alpha^{\mathcal{F}}(1-\theta)\mathcal{S}_\alpha^{\mathcal{F}}\partial_t \mathcal{P}_\alpha^{\mathcal{F}} =$$

$$\partial_{xi}\left(\mathbb{K}_{ik}\lambda_\alpha^{\mathcal{F}}\partial_{xi}\mathcal{P}_\alpha^{\mathcal{F}}\right), \quad \alpha = w, o \tag{70a}$$

$$\mathcal{P}_w^{\mathcal{M}} = \mathcal{P}_w^{\mathcal{F}}, \tag{70b}$$

$$\frac{\tau^{\text{cap}} + \tau^{\text{cc}}}{C_o^{\mathcal{M}}}\partial_t \mathcal{S}_w^{\mathcal{M}} = p_c^{\mathcal{M}}\left(\mathcal{S}_w^{\mathcal{M}}\right) - p_c^{\mathcal{F}} - \tag{70c}$$

$$\left(\tau_w^{\text{com}} - \tau_o^{\text{com}}\right)\partial_t \mathcal{P}_w^{\mathcal{M}} \tag{70d}$$

Contrarily to the exact model (28), the first two equations (70a) do not contain anymore the exchange term, due to a fast equalization of pressure in space. Moreover, two kinetic differential equations (28b) and (28d) transform into a single ordinary differential equation (70d) with respect to the saturation $\mathcal{S}_w^{\mathcal{M}}$.

Equation (70d) can be considered as the generalization to compressible systems of the capillary non-equilibrium equation obtained in[13].

This simplified model is valid for $C_w, C_o < 3-4$. These dimensionless values correspond to the compressibility of 10^{-7} Pa^{-1}, which is a highly compressible system. Fig. 7 illustrates this.

This is however not the case of very high compressibility, as illustrated in Fig. 8.

CONCLUSION

First of all, we note that the averaged model (28) is completely homogenized, despite the presence of the nonlinearity and the memory. This means that it does not contain microscopic variables, and the cell problem does not contain macroscopic variables, so that it is solved only once.

This was possible to do due to the method of splitting nonlocality and nonlinearity, proposed earlier[13], which

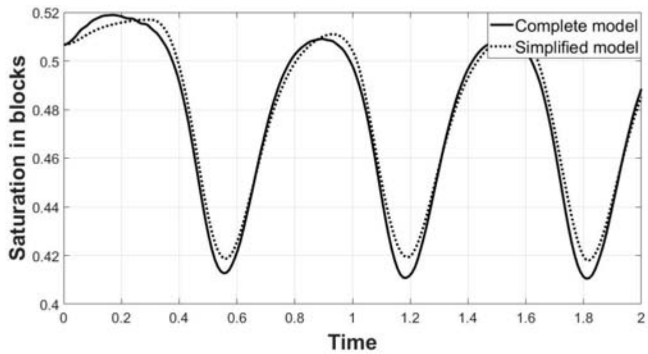

FIG. 7. Comparison of the simplified model (the dotted curve) and the exact model (the solid curve) for $C_w = C_o = 3$

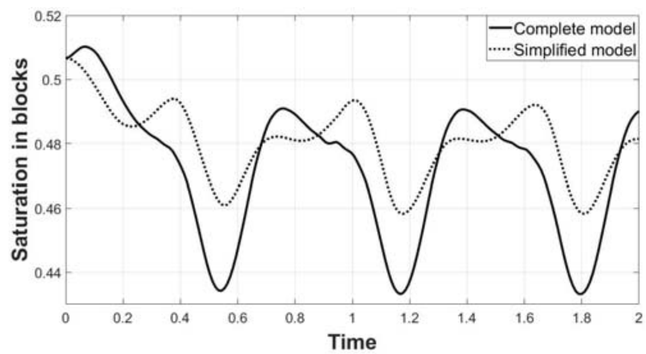

FIG. 8. Comparison of the simplified model (the dotted curve) and the exact model (the solid curve) for a highly compressible system: $C_w = C_o = 5$

showed once again its effectiveness. Averaging by the variational method (which remains non-popular yet in mechanics of porous media), is also highly efficient, since it allows minimizing the calculations.

Secondly, the model predicts and describes several mechanisms causing the delay or memory effects. Along with the pure capillary delay, typical to two-phase systems, and the delay of pressure wave propagation, typical for compressible systems, two additional cross-effects arise, such as peristalsis (or phase extrusion) and nonlinear peristalsis, which, in turn, depend non-trivially on the asymmetry or symmetry of the compression with respect to two phases.

The practical application of this model is necessary in cases of strongly non-stationary processes, when compressibility plays a significant role in the propagation of perturbations.

Various memory effects are particularly pronounced in processes with a variable direction of evolution. For example, the gas injection-production in underground gas storage. The accumulation of a history of pressure and saturation oscillations in such processes can lead to a strong delay and a complete discrepancy in the qualitative behavior of the process compared to the process without memory.

We considered only the case of the exponential compressibility law, which is applicable for liquids and solids. The applicability of this model for a gas-liquid system is acceptable only for high pressures, or for highly compressed gas, whose properties are close to a liquid.

The delay caused by the compressibility affects the saturation (through the asymmetrical peristalsis) even at moderate values of compressibility parameter.

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