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Abstract

In the sequel, we extend the Strichartz average approach, for the Laplacian on Sierpiński Simplices, which uses average values of a function over basic sets, following the seminal work of S. Kusuoka and X. Y. Zhou, rather than using pointwise ones as classically done in the literature. Until now, the implementation of the related finite volume method, in the case of Sierpiński Simplices, had not been done.

Keywords: Laplacian - Heat equation - Self-similar sets - Finite Volume Method - Convergence.

AMS Classification: 37F20- 28A80-05C63.

1 Introduction

Why should one focus on fractals as, for instance, Sierpiński simplices ? Let us go back to the study of Rammal Rammal, the condensed matter physicist [RT83], who layed the emphasis upon the fact that fractals could "fill the gap between well-ordered crystalline structures and those that are disordered". Then, the work of S. Havlin and D. Ben Avraham [HBA87] showed that diffusion in those disordered systems did not follow classical laws, and thus required specifically designed tools.

In the years that followed, people began to take an interest in the applications of fractals, for numerous reasons.To begin with, one may note that fractal structures, because of the very large size of the length or surface they cover, show isolant or repellent properties. One may refer to the silicium airgel, with a branched microstructure of the fractal type, which is a remarkable insulator, in so far as it almost completely stops the propagation of heat. In chemistry, to go on, scientists started to design dentritic polymers with an arborescent structure (as it is the case of arborol), where patterns such as the Sierpiński Gasket, the Koch curve, or the Menger sponge, are respectively used as model for linear polymer chains, or porous media [Liu86], [HBA87]. In recent years, particularly, one has seen the design of polymers of fractal type (see, for instance, [NWM+06], or [SKM+85], where the authors describe terpyridine-based architecture which simulates a first-generation Sierpiński triangle, as it can

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be observed on Figures 1 and 2).

Dentritic polymers are at stake in diffusion phenomena ; for instance, they are now more and more used in pharmacology, since their very specific architecture enable one to control (prevent or facilitate) drug delivery.

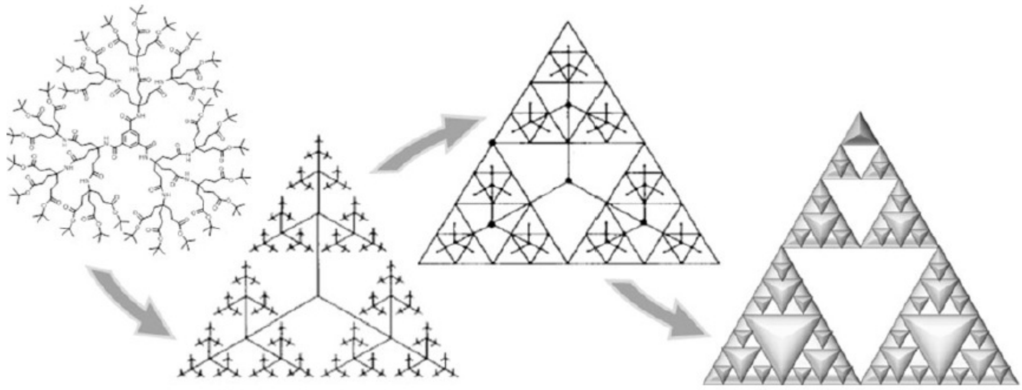

Figure 1 – Conceptual progression of a 1 → 3 dendritic branching pattern and its geometric relationship to the classical Sierpiński triangle [SKM+85]. Figure reprinted with permission from the authors and publisher.

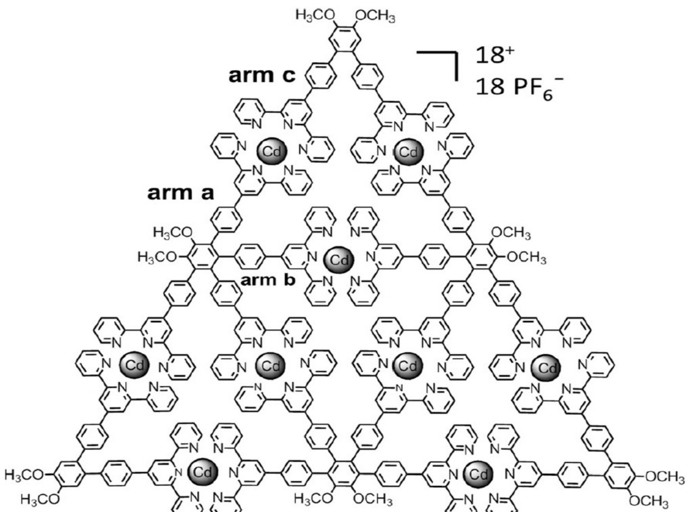

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Figure 2 – Terpyridine-based Sierpiński triangle [SKM+85]. Figure reprinted with permission from the authors and publisher.

One thus require specifically fitted numerical tools. In our work [RD19], following the seminal work of J. Kigami and R. S. Strichartz for Laplacians on Sierpiński simplices, we built the related finite difference scheme as it had not been done before. For instance, if, in the case of he Sierpiński gasket $\mathfrak{SG}$, K. Dalrymple, R. S. Strichartz, and J. Vinson [DSV99] gave an equivalent method for the finite difference approximation, using the spectral shape of the solution (heat kernel), it involved eigenvalues and eigenvectors, calling for an approximation of those quantities.

As for us, the novelty of our contribution layed in defining the discretization of the considered

PDE's (heat and wave equation), by taking into account the recursive construction of the matrix related to a sequence of graph Laplacians. We thus did not call for approximations of the eigenvalues. This enabled us not only to compute the consistency error, but, also, to set stability conditions of Courant-Friedrichs-Lewy type, and, then, to prove the convergence of the scheme.

The extension to the finite volume method (FVM) of those specifically devoted to fractals differential operators was a natural step, especially in the light of the paper of R. S. Strichartz [Str01], where he shows how the symmetric Laplacian on $\mathfrak{S}\mathfrak{G}$ can be defined entirely in terms of average values of a function over basic sets, following the seminal work of S. Kusuoka and X. Y. Zhou [KZ92] for the Sierpiński Carpet, rather than using pointwise ones as classically done in the literature. This approach is interesting, in so far as the Sierpiński Carpet, contrary to Sierpiński simplices, is not what may be called a post-critically finite fractal, i.e. a set which can become totally disconnected if we remove a finite number of points, and therefore requires a non-pointwise valued approach, as in the case of the FVM method. The advantages of dealing with the FVM method consist, first, of its physical approach, which adapts to any geometry, and, in our case, to a fractal one, while being, in the same time, a conservative approach, and the basis of several general numerical codes.

We hereafter present our results, in the case of Sierpiński simplices (Gasket and Tetrahedron), in the case of the heat equation: we give, first, the numerical implementation, then, estimates of the scheme error, and a comparison with the finite-difference method.

2 Framework of the study

Notation. We will denote by $\mathbb{N}$ the set of natural integers. and set:

$$\mathbb{N}^\star = \mathbb{N} \setminus \{0\} \cdot$$

In the following, given $d \in \mathbb{N}^\star$, we place ourselves in the Euclidean space of dimension $d-1$, referred to a direct orthonormal frame. The usual Cartesian coordinates will be denoted by $(x_1, x_2, \ldots, x_{d-1})$.

2.1 Graph approximation

Notation. In the following, $\{f_1, \ldots, f_d\}$ is a set of contractive maps, of ratio $\frac{1}{2}$, where, for any integer i of $\{1, \ldots, d\}$, $P_i \in \mathbb{R}^d$ the fixed point of f_i. For any X of $\mathbb{R}^{d-1}$:

$$f_i(X) = \frac{1}{2}(X + P_i)$$

Theorem 2.1. ***Gluing Lemma [BD85]***

Given a complete metric space (E, δ), and a set $\{f_i\}_{1 \leqslant i \leqslant d}$ of contractions on E with respect to the metric δ, there exists a unique non-empty compact subset $K \subset E$ such that:

$$K = \bigcup_{i=1}^{d} f_i(K) \cdot \tag{1}$$

The set K is said ***self-similar*** *with respect to the family $\{f_1, \ldots, f_d\}$, and called attractor of the iterated function system (IFS) $\{f_1, \ldots, f_d\}$.*

Corollary 2.2. *There exists a unique subset* $\mathfrak{S}\mathfrak{S} \subset \mathbb{R}^{d-1}$ *such that:*

$$\mathfrak{S}\mathfrak{S} = \bigcup_{i=1}^{d} f_i(\mathfrak{S}\mathfrak{S}) \tag{2}$$

which will be called the Sierpiński Simplex.

Definition 2.1. Boundary (or initial) graph

We will denote by V_0 the ordered set of the (boundary) points $\{P_1, \ldots, P_d\}$, where P_i is the fixed point of the contraction f_i. The set V_0, where, for any i of $\{2, \ldots, d-1\}$, the point P_i is respectively connected to P_{i-1} and P_{i+1} by means of a line segment, constitutes a complete graph, that we will denote by $\mathfrak{S}\mathfrak{S}_0$.

V_0 is called the set of vertices of the graph $\mathfrak{S}\mathfrak{S}_0$.

Definition 2.2. m^{th} **order graph,** $m \in \mathbb{N}^\star$

For any strictly positive integer m, we set:

$$V_m = \bigcup_{i=1}^{d} f_i\left(V_{m-1}\right) \cdot \tag{3}$$

The set of points V_m, where the points of the m^{th} cells are linked in the same way as $\mathfrak{S}\mathfrak{S}_0$, is an oriented graph, which we will denote by $\mathfrak{S}\mathfrak{S}_m$. V_m is called the set of vertices of the graph $\mathfrak{S}\mathfrak{S}_m$.
By extension, we will write:

$$\mathfrak{S}\mathfrak{S}_m = \bigcup_{i=1}^{d} f_i\left(\mathfrak{S}\mathfrak{S}_{m-1}\right) \cdot \tag{4}$$

Property 2.3. *For any natural integer* m*:*

$$V_m \subset V_{m+1} \cdot \tag{5}$$

Definition 2.3. Word

Given a strictly positive integer m, we will call **number-letter** any integer $\mathcal{W}_i$ of $\{1, \ldots, d\}$, and **word of length** $|\mathcal{W}| = m$, on the graph $\mathfrak{S}\mathfrak{S}_m$, any set of number-letters of the form:

$$\mathcal{W} = (\mathcal{W}_1, \ldots, \mathcal{W}_m) \cdot \tag{6}$$

We set:

$$f_{\mathcal{W}} = f_{\mathcal{W}_1} \circ \ldots \circ f_{\mathcal{W}_m} \cdot \tag{7}$$

Definition 2.4. Vertex

A point X of $\mathfrak{S}\mathfrak{G}$ will be called ***vertex*** of the graph $\mathfrak{S}\mathfrak{G}$ if there exists a natural integer m such that:

$$X \in V_m \cdot \tag{8}$$

Definition 2.5. Consecutive (or neighbor) vertices

Two points X and Y of $\mathfrak{S}\mathfrak{G}$ will be called ***consecutive vertices***, or ***neighbor vertices***, if there exists a natural integer m, and an integer j of $\{1, \ldots, d\}$, such that:

$$X = (f_{i_1} \circ \ldots \circ f_{i_m})(P_j) \quad \text{and} \quad Y = (f_{i_1} \circ \ldots \circ f_{i_m})(P_{j+1}) \qquad \{i_1, \ldots, i_m\} \in \{1, \ldots, d\}^m \tag{9}$$

or:

$$X = (f_{i_1} \circ \ldots \circ f_{i_m})(P_d) \quad \text{and} \quad Y = (f_{i_1+1} \circ \ldots \circ f_{i_m})(P_1) \cdot \tag{10}$$

Definition 2.6. Edge relation

For any $m \in \mathbb{N}$, two points X and Y of $\mathfrak{S}\mathfrak{G}_m$ will be called ***adjacent*** if and only if X and Y are neighbors in $\mathfrak{S}\mathfrak{G}_m$. We set:

$$X \underset{m}{\sim} Y \cdot \tag{11}$$

This edge relation ensures the existence of a word $\mathcal{W} = (\mathcal{W}_1, \ldots, \mathcal{W}_m)$ of length m, such that X and Y both belong to the iterate:

$$f_{\mathcal{W}} V_0 = (f_{\mathcal{W}_1} \circ \ldots \circ f_{\mathcal{W}_m}) \, V_0 \cdot \tag{12}$$

Given two points X and Y of $\mathfrak{S}\mathfrak{G}$, we will say that X and Y are ***adjacent*** if and only if there exists a natural integer m such that:

$$X \underset{m}{\sim} Y \cdot \tag{13}$$

Definition 2.7. Adresses

For any $m \in \mathbb{N}$, and any vertex X of $\mathfrak{S}\mathfrak{G}_m$, we will call address of the vertex X an expression of the form

$$X = f_{\mathcal{W}}(P_i) \tag{14}$$

where $\mathcal{W}$ is a word of length m, and i a natural integer in $\{1, \ldots, d\}$.

Proposition 2.4. *For any $m \in \mathbb{N}$, we will denote by $\mathcal{N}_m$ the number of vertices of the graph $\mathfrak{S}\mathfrak{G}_m$. We have:*

$$\mathcal{N}_0 = d \quad , \quad \forall\, m \in \mathbb{N} : \quad \mathcal{N}_m = d\,\mathcal{N}_{m-1} - \frac{d\,(d-1)}{2} = \frac{d^{m+1}+d}{2} \cdot \tag{15}$$

Proof. The graph $\mathfrak{S}\mathfrak{G}_m$ is the union of d copies of the graph $\mathfrak{S}\mathfrak{G}_{m-1}$. Each copy shares a vertex with the other ones. So, one may consider the copies as the vertices of a complete graph K_d, the number of edges is equal to $\frac{d\,(d-1)}{2}$, which leads to $\frac{d\,(d-1)}{2}$ vertices to take into account. □

Definition 2.8. For any $m \in \mathbb{N}$, we consider the graph $\mathcal{S}\mathcal{S}_m$, built from $\mathfrak{S}\mathfrak{G}_m$ in the following way:

i. a cell in $\mathfrak{S}\mathfrak{G}_m$ becomes a vertex in $\mathcal{S}\mathcal{S}_m$;

ii. two vertices are linked in $\mathcal{S}\mathcal{S}_m$ if the corresponding cells in $\mathfrak{S}\mathfrak{G}_m$ have a vertex in common.

iii. The vertices number of $\mathcal{S}\mathcal{S}_m$ is d^m.

Remark 2.1. **For the 2-Simplex (Triangle):**

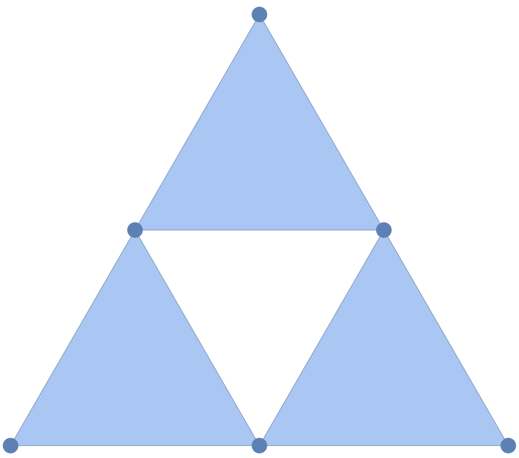

Figure 3 – $\mathfrak{S}\mathfrak{G}_1$

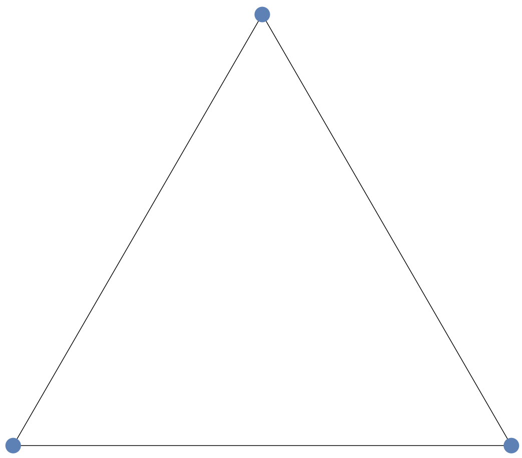

Figure 4 – $\mathcal{S}\mathcal{S}_1$

2.2 Self-similar measures on Sierpiński Simplices

Definition 2.9. Self-similar measure, on the Sierpiński simplex $\mathfrak{S}\mathfrak{G}$

A measure μ on $\mathbb{R}^d$ will be said to be **self-similar** on the Sierpiński simplex $\mathfrak{S}\mathfrak{G}$, if there exists a family of strictly positive weights $(\mu_i)_{1\leqslant i\leqslant d}$ such that:

$$\mu = \sum_{i=1}^{d} \mu_i\, \mu \circ f_i^{-1} \quad , \quad \sum_{i=1}^{d} \mu_i = 1 \cdot \tag{16}$$

For further precisions on self-similar measures, we refer to the works of J. E. Hutchinson (see [Hut81]).

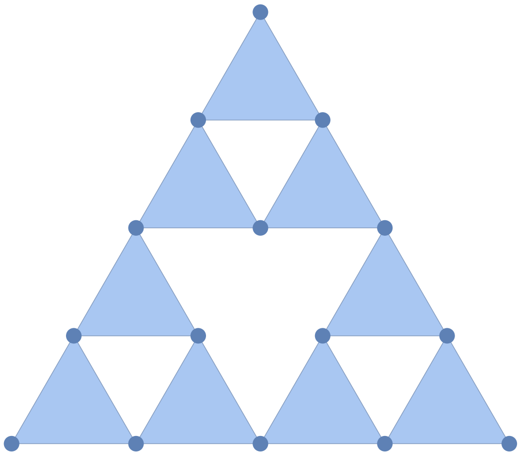

Figure 5 – $\mathfrak{SS}_2$

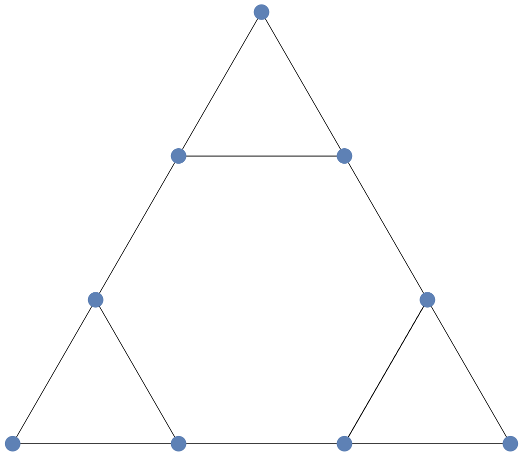

Figure 6 – $\mathcal{SS}_2$

Property 2.5. ***Building of a self-similar measure, for the Sierpiński simplex $\mathfrak{SS}$***

The aforementioned Dirichlet forms require a positive Radon measure with full support. We set, for any integer i belonging to $\{1,\ldots,d\}$:

$$\mu_i = R_i^{D_H(\mathfrak{SS})} \tag{17}$$

where $D_H(\mathfrak{SS})$ is the Hausdorff dimension of the Sierpiński simplex $\mathfrak{SS}$. Since:

$$\sum_{i=1}^{d} R_i^{D_H(K)} = 1 \cdot \tag{18}$$

one may define a self-similar measure μ on $\mathfrak{SS}$ through:

$$\mu = \sum_{i=1}^{d} \mu_i \mu \circ f_i^{-1} \tag{19}$$

which simply yields the standard measure:

$$\mu = \frac{1}{d} \sum_{i=1}^{d} \mu \circ f_i^{-1} \cdot \tag{20}$$

2.3 Laplacians, on Sierpiński Simplices (we refer to [Str06])

Definition 2.10. Energy, on the graph $\mathfrak{SS}_m$, $m \in \mathbb{N}$, of a pair of functions

For any $m \in \mathbb{N}$, and two real valued functions u and v, defined on the set V_m of the vertices of $\mathfrak{SS}_m$, we introduce **the energy, on the graph $\mathfrak{SS}_m$, of the pair of functions** (u,v), as:

$$\mathcal{E}_{\mathfrak{SS}_m}(u,v) = \sum_{X \underset{m}{\sim} Y} \left(u(X)-u(Y)\right)\,\left(v(X)-v(Y)\right) \tag{21}$$

where the adjacency relation $\sim$ has been defined in Definition 2.6.

Definition 2.11. Dirichlet form, on a finite set (see [Kig03])

Let V denote a finite set V, equipped with the usual inner product which, to any pair (u,v) of functions defined on V, associates:

$$(u,v) = \sum_{p\in V} u(p)\,v(p)\cdot \tag{22}$$

A ***Dirichlet form*** on V is a symmetric bilinear form $\mathcal{E}$, such that:

1. For any real valued function u defined on V: $\mathcal{E}(u,u) \geqslant 0$.
2. $\mathcal{E}(u,u) = 0$ if and only if u is constant on V.
3. For any real-valued function u defined on V, if: $u_\star = \min\left(\max(u,0),1\right)$, i.e.:

$$\forall\, p \,\in\, V \;:\quad u_\star(p) = \begin{cases} 1 & \text{if} \quad u(p) \geqslant 1 \\ u(p) & \text{if} \quad 0 < u(p) < 1 \\ 0 & \text{if} \quad u(p) \leqslant 0 \end{cases} \tag{23}$$

then: $\mathcal{E}(u_\star,u_\star) \leqslant \mathcal{E}(u,u)$ (Markov property).

Property 2.6. *Given a natural integer m, the usual inner product of a pair (u,v) of real-valued, continuous functions defined on V_m, through:*

$$\mathcal{E}_{\mathfrak{S}\mathfrak{S}_m}(u,v) = \sum_{X \underset{m}{\sim} Y} \left(u(X)-u(Y)\right)\,\left(v(X)-v(Y)\right) \tag{24}$$

is a Dirichlet form on $\mathfrak{S}\mathfrak{S}_m$.
Moreover:

$$\mathcal{E}_{\mathfrak{S}\mathfrak{S}_m}(u,u) = 0 \Leftrightarrow u \textit{ is constant}\cdot \tag{25}$$

Proposition 2.7. *For any strictly positive integer m, if u is a real-valued function defined on V_{m-1}, its* ***harmonic extension****, denoted by $\tilde{u}$, is obtained as the extension of u to V_m which minimizes the energy:*

$$\mathcal{E}_{\mathfrak{S}\mathfrak{S}_m}(\tilde{u},\tilde{u}) = \sum_{X \underset{m}{\sim} Y} \left(\tilde{u}(X)-\tilde{u}(Y)\right)^2\cdot \tag{26}$$

Remark 2.2. **More explicitly:**

The link between $\mathcal{E}_{\mathfrak{S}\mathfrak{S}_m}$ and $\mathcal{E}_{\mathfrak{S}\mathfrak{S}_{m-1}}$ is obtained through the introduction of two strictly positive constants r_m and r_{m-1} such that:

$$r_m \sum_{X \underset{m}{\sim} Y} \left(\tilde{u}(X)-\tilde{u}(Y)\right)^2 = r_{m-1} \sum_{X \underset{m-1}{\sim} Y} \left(u(X)-u(Y)\right)^2\cdot \tag{27}$$

In particular:

$$r_1 \sum_{X \underset{1}{\sim} Y} (\tilde{u}(X) - \tilde{u}(Y))^2 = r_0 \sum_{X \underset{0}{\sim} Y} (u(X) - u(Y))^2 \cdot \tag{28}$$

We set: $r_0 = 1$. Thus:

$$\mathcal{E}_{\mathfrak{S}\mathfrak{S}_1}(\tilde{u}, \tilde{u}) = \frac{1}{r_1} \mathcal{E}_{\mathfrak{S}\mathfrak{S}_0}(\tilde{u}, \tilde{u}), \cdot \tag{29}$$

Let us introduce:

$$r = \frac{1}{r_1} \tag{30}$$

and:

$$\mathcal{E}_m(u) = r_m \sum_{X \underset{m}{\sim} Y} (\tilde{u}(X) - \tilde{u}(Y))^2 \cdot \tag{31}$$

Since the determination of the harmonic extension of a function appears to be a local problem, on the graph $\mathfrak{S}\mathfrak{S}_{m-1}$, which is linked to the graph $\mathfrak{S}\mathfrak{S}_m$ by a similar process as the one that links $\mathfrak{S}\mathfrak{S}_1$ to $\mathfrak{S}\mathfrak{S}_0$, one deduces, for any strictly positive integer m:

$$\mathcal{E}_{\mathfrak{S}\mathfrak{S}_m}(\tilde{u}, \tilde{u}) = \frac{1}{r_1} \mathcal{E}_{\mathfrak{S}\mathfrak{S}_{m-1}}(\tilde{u}, \tilde{u}) \cdot \tag{32}$$

By induction, one gets:

$$r_m = r_1^m = \frac{1}{r^m} \cdot \tag{33}$$

If v is a real-valued function, defined on V_{m-1}, of harmonic extension $\tilde{v}$, we set:

$$\mathcal{E}_m(u, v) = \frac{1}{r^m} \sum_{X \underset{m}{\sim} Y} (\tilde{u}(X) - \tilde{u}(Y)) (\tilde{v}(X) - \tilde{v}(Y)) \cdot \tag{34}$$

For further details on the construction and existence of harmonic extensions, we refer to *[Sab97]*.

Definition 2.12. Renormalized energy, for a continuous function u, defined on $\mathfrak{S}\mathfrak{S}_m$, $m \in \mathbb{N}$

Given a natural integer m, one defines the **normalized energy**, for a continuous function u, defined on $\mathfrak{S}\mathfrak{S}_m$, by:

$$\mathcal{E}_m(u) = \sum_{X \underset{m}{\sim} Y} \frac{1}{r^m} (u(X) - u(Y))^2 \cdot \tag{35}$$

Definition 2.13. Normalized energy, for a continuous function u, defined on $\mathfrak{S}\mathfrak{S}$

Given a function u defined on $V_\star = \bigcup_{i \in \mathbb{N}} V_i$, one defines the **normalized energy**:

$$\mathcal{E}(u) = \lim_{m \to +\infty} \mathcal{E}_m(u) \tag{36}$$

Definition 2.14. Dirichlet form, for a pair of continuous functions defined on $\mathfrak{S}\mathfrak{G}$

We define the Dirichlet form $\mathcal{E}$ which, to any pair of real-valued, continuous functions (u,v) defined on the graph $\mathfrak{S}\mathfrak{G}$, associates, when it exists:

$$\mathcal{E}(u,v) = \lim_{m\to+\infty} \sum_{X \underset{m}{\sim} Y} \frac{1}{r^m} \left(u_{|V_m}(X) - u_{|V_m}(Y)\right) \left(v_{|V_m}(X) - v_{|V_m}(Y)\right) \cdot \tag{37}$$

Notation. We will denote by:

i. $\operatorname{dom}\mathcal{E}$ the subspace of continuous functions defined on $\mathfrak{S}\mathfrak{G}$, such that: $\mathcal{E}(u) < +\infty$.

ii. $\operatorname{dom}_0\mathcal{E}$ the subspace of continuous functions defined on $\mathfrak{S}\mathfrak{G}$, which take the value zero on V_0, and such that: $\mathcal{E}(u) < +\infty$.

Lemma 2.8. *The map:*

$$\begin{array}{ccc} dom\,\mathcal{E}\big/ Constants \times dom\,\mathcal{E}\big/ Constants & \to & \mathbb{R} \\ (u,v) & \mapsto & \mathcal{E}(u,v) \end{array} \tag{38}$$

defines an inner product on $dom\,\mathcal{E}\big/ Constants$.

Theorem 2.9. *[Str06]*

$\left(dom\,\mathcal{E}\big/ Constants, \mathcal{E}(\cdot,\cdot)\right)$ *is a complete Hilbert space.*

Definition 2.15. Graph Laplacian of order $m \in \mathbb{N}^\star$

For any strictly positive integer m, and any real-valued function u, defined on the set V_m of the vertices of the graph $\mathfrak{S}\mathfrak{G}_m$, we introduce the graph Laplacian of order m, $\Delta_m(u)$, by:

$$\forall\, X \in V_m \setminus V_0 \quad \Delta_m u(X) = \sum_{Y\in V_m,\, Y \underset{m}{\sim} X} \left(u(Y) - u(X)\right) \cdot \tag{39}$$

Definition 2.16. Harmonic functions

A real-valued function u, defined on $V_\star = \displaystyle\bigcup_{i\in\mathbb{N}} V_i$, will be said to be **harmonic** if, for any natural integer m, its restriction $u_{|V_m}$ is harmonic:

$$\forall\, m \in \mathbb{N}, \forall\, X \in V_m \setminus V_0 : \quad \Delta_m u_{|V_m}(X) = 0 \cdot \tag{40}$$

Notation. We will denote by $\operatorname{dom}\mathcal{E}$ the subspace of continuous functions u defined on $\mathfrak{S}\mathfrak{G}$, such that:

$$\mathcal{E}(u) = \lim_{m\to+\infty} \sum_{X \underset{m}{\sim} Y} r^{-m} \left(u_{|V_m}(X) - u_{|V_m}(Y)\right)^2 < +\infty \cdot \tag{41}$$

Definition 2.17. We will denote by $\operatorname{dom}\Delta$ the existence domain of the Laplacian, on $\mathfrak{S}\mathfrak{G}$, as the set of functions u of $\operatorname{dom}\mathcal{E}$ such that there exists a continuous function on $\mathfrak{S}\mathfrak{G}$, denoted by $\Delta\, u$, that we will call **Laplacian of** u, such that, for any $v \in \operatorname{dom}\mathcal{E}$, $v_{|\mathfrak{S}\mathfrak{G}_0} = 0$:

$$\begin{array}{rcl} \mathcal{E}(u,v) & = & \displaystyle\lim_{m\to+\infty} \sum_{X \underset{m}{\sim} Y} r^{-m} \left(u_{|V_m}(X) - u_{|V_m}(Y)\right) \left(v_{|V_m}(X) - v_{|V_m}(Y)\right) \\ & = & \displaystyle -\int_{\mathfrak{S}\mathfrak{G}} v\, \Delta u\, d\mu \end{array} \cdot \tag{42}$$

Theorem 2.10. *[Str06]*

$$(u \in \operatorname{dom}\Delta \quad \text{and} \quad \Delta\, u = 0) \quad \underline{\text{if and only if}}\; u \text{ is harmonic} \cdot \tag{43}$$

Notation. *i.* Given a natural integer m, $\mathcal{S}(\mathcal{H}_0, V_m)$ denotes the space of spline functions "of level m", u, defined on $\mathfrak{S}\mathfrak{G}$, continuous, such that, for any word $\mathcal{W}$ of length m, $u \circ f_{\mathcal{W}}$ is harmonic, i.e.:

$$\Delta_m\, (u \circ f_{\mathcal{W}}) = 0 \cdot \tag{44}$$

ii. $\mathcal{H}_0 \subset \operatorname{dom}\Delta$ denotes the space of harmonic functions, i.e. the space of functions $u \in \operatorname{dom}\Delta$ such that: $\Delta\, u = 0$.

Property 2.11. *For any natural integer m: $\mathcal{S}(\mathcal{H}_0, V_m) \subset \operatorname{dom}\mathcal{E}$.*

Theorem 2.12. ***Pointwise formula***

Let m be a strictly positive integer, $X \in V_\star \setminus V_0$, and $\psi_X^m \in \mathcal{S}(\mathcal{H}_0, V_m)$ a spline function such that:

$$\psi_X^m(Y) = \begin{cases} \delta_{XY} & \forall\; Y \in V_m \\ 0 & \forall\; Y \notin V_m \end{cases} \quad , \quad \text{where} \quad \delta_{XY} = \begin{cases} 1 & \text{if} \quad X = Y \\ 0 & \text{else} \end{cases} \cdot \tag{45}$$

i. For any function u of $\operatorname{dom}\Delta$, such that its Laplacian exists, the sequence

$$\left(r^{-m} \left\{ \int_{\mathfrak{S}\mathfrak{G}} \psi_X^m\, d\mu \right\}^{-1} \Delta_m u(X) \right)_{m\in\mathbb{N}}$$

converges uniformly towards

$$\Delta u(X)$$

ii. Conversely, given a continuous function u on $\mathfrak{S}\mathfrak{G}$ such that the sequence

$$\left(r^{-m} \left\{ \int_{\mathfrak{S}\mathfrak{G}} \psi_X^m \, d\mu \right\}^{-1} \Delta_m u(X) \right)_{m\in\mathbb{N}}$$

converges uniformly towards a continuous function on $V_\star \setminus V_0$, one has:

$$u \in dom\,\Delta \quad and \quad \Delta\, u(X) = \lim_{m\to+\infty} r^{-m} \left\{ \int_{\mathfrak{S}\mathfrak{G}} \psi_X^m \, d\mu \right\}^{-1} \Delta_m u(X) \cdot \tag{46}$$

Definition 2.18. Normal derivative

Given a boundary point $X = f_{\mathcal{W}}(P_i)$ of a cell $f_{\mathcal{W}}(\mathfrak{S}\mathfrak{G})$, $1 \leqslant i \leqslant d$, $\mathcal{W} \in \{1,\ldots,d\}^\ell$, and a continuous function u on $\mathfrak{S}\mathfrak{G}$, we will say that the normal derivative $\partial_n u$ exists if the limit

$$\partial_n u(P_i) = \lim_{m\to+\infty} \frac{1}{r^m} \sum_{\substack{Y \underset{m}{\sim} X \\ Y \in f_{\mathcal{W}}(\mathfrak{S}\mathfrak{G})}} (u(X) - u(Y))$$

exists. The local normal derivative satisfies:

$$\partial_n u(X) = r^{-\ell}\, \partial_n (u \circ f_{\mathcal{W}})(P_i) \cdot \tag{47}$$

Theorem 2.13. ***Green-Gauss formula***

Given $u \in dom_\Delta$ for a measure μ, $\partial_n u$ exists for all $X \in V_0$, and:

$$\mathcal{E}(u,v) = -\int_{\mathfrak{S}\mathfrak{G}} \Delta_\mu u\, v\, d\mu + \sum_{V_0} \partial_n u(X)\, v \tag{48}$$

holds for all $v \in dom\,\mathcal{E}$.

Corollary 2.14.

Given $u \in dom_\Delta$ for a measure μ:

$$\int_{\mathfrak{S}\mathfrak{G}} \Delta_\mu u\, v\, d\mu - \int_{\mathfrak{S}\mathfrak{G}} u\, \Delta_\mu v\, d\mu = \sum_{V_0} (\partial_n u(X)\, v - u\, \partial_n v(X)) \tag{49}$$

holds for all $v \in dom\,\mathcal{E}$.

Theorem 2.15. ***Matching condition***

Given $u \in dom_\Delta$, at each junction point

$$X = f_{\mathcal{W}}(P_i) = f_{\mathcal{W}'}(P_j) \quad , \quad (i,j), \in \{0,\ldots,d-1\}^2 \quad , \quad (\mathcal{W},\mathcal{W}') \in \{1,\ldots,d\}^m \times \{1,\ldots,d\}^m$$

the local normal derivative exists, and

$$\partial_n u(f_{\mathcal{W}}(P_i)) + \partial_n u(f_{\mathcal{W}'}(P_j)) = 0 \tag{50}$$

holds for all $v \in dom\, \mathcal{E}$.

3 The Finite Volume Method

Notation. In the following, T is a strictly positive real number, while N is a non-zero natural integer. Let us introduce:

$$h = \frac{T}{N} \quad , \quad t_n = n \times h \quad , \quad n = 0, 1, \ldots, N-1 \cdot \tag{51}$$

3.1 The heat equation

3.1.1 Formulation of the problem

We hereafter consider a solution u of the problem:

$$\left\{ \begin{array}{ccccc} \dfrac{\partial u}{\partial t}(t,X) - \Delta u(t,X) & = & 0 & \forall (t,X) \in \,]0,T[\times \mathfrak{S}\mathfrak{S} \\ u(t,X) & = & 0 & \forall \, (X,t) \in \partial \mathfrak{S}\mathfrak{S} \times [0,T[\\ u(0,X) & = & g(X) & \forall \, X \in \mathfrak{S}\mathfrak{S} \end{array} \right. \cdot \tag{52}$$

Definition 3.1. Let $m \in \mathbb{N}$. We introduce the m^{th}**-control volume** as the m^{th}-order cell

$$C_m^j = f_{\mathcal{W}^j}(\mathfrak{S}\mathfrak{S}) \quad , \quad \mathcal{W}^j \in \{1, \ldots, d\}^m \tag{53}$$

where $\mathcal{W}^j$ is some word of length m, and whose m^{th}-order cells neighbors are

$$C_m^l = f_{\mathcal{W}^l}(\mathfrak{S}\mathfrak{S}) \quad , \quad \mathcal{W}^l \in \{1, \ldots, d\}^m \quad , \quad l = 1, \ldots, d-1 \cdot \tag{54}$$

Remark 3.1. One may check that:

$$\bigcup_{j=1}^{d^m} C_m^j = \mathfrak{S}\mathfrak{S} \cdot \tag{55}$$

We define then, for any integer j in $\{1, \ldots, d^m\}$:

$$u_j^0 = \frac{1}{\mu(C_m^j)} \int_{C_m^j} g(x) d\mu(x) \tag{56}$$

$$u_j^n = \frac{1}{\mu(C_m^j)} \int_{C_m^j} u(t_n, x) \, d\mu(x) \tag{57}$$

$$\tag{58}$$

and, for any t in $[0,T[$:

$$u_j^t = \frac{1}{d} \sum_{Y \in \partial C_m^j} u(t,Y) \approx \frac{1}{\mu(C_m^j)} \int_{C_m^j} u(t,x)\, d\mu(x) \cdot \tag{59}$$

$$\tag{60}$$

The local Gauss-Green formula enables one to write:

$$\int_{C_m^j} \Delta_\mu u \, d\mu = \sum_{x \in \partial C_m^j} \partial_n u(x) \tag{61}$$

and, given a natural integer n, to integrate the heat equation over $C_m^j \times]t_n, t_{n+1}[$:

$$\int_{C_m^j} u(t_{n+1},X) - u(t_n,X)\, d\mu = \int_{t_n}^{t_{n+1}} \sum_{x \in \partial C_m^j} \partial_n u(t,X)\, dt \cdot \tag{62}$$

We recall that $C_m^j = f_{W^j}(\mathfrak{S}\mathfrak{S})$. The boundary points are given, by:

$$\text{for some } (i,k,\ell) \in \{1,\ldots,d\} \times \{1,\ldots,d\} \times \{1,\ldots,d\} : \quad f_{\mathcal{W}^j}(P_i) = f_{\mathcal{W}^l}(P_k) \cdot \tag{63}$$

One may use the approximation:

$$\begin{aligned}
\partial_n u(t,X) &\approx \frac{1}{r^m} \sum_{\substack{Y \underset{m}{\sim} X \\ Y \in C_m^j}} (u(t,X) - u(t,Y)) &(64)\\
&= \frac{1}{r^m} \left((d-1) u(t,X) - \sum_{\substack{Y \underset{m}{\sim} X \\ Y \in C_m^j}} u(t,Y) \right) &(65)\\
&= \frac{1}{r^m} \left(d\, u(t,X) - u(t,X) - \sum_{\substack{Y \underset{m}{\sim} X \\ Y \in C_m^j}} u(t,Y) \right) &(66)\\
&\approx \frac{1}{r^m}\, d \left(u(t,X) - u_j^t \right) &(67)
\end{aligned}$$

We then introduce the matching condition:

$$\partial_n u(t, f_{\mathcal{W}^j}(P_i)) = -\partial_n u(t, f_{\mathcal{W}^l}(P_k)) \tag{68}$$

i.e.

$$\frac{1}{r^m}\, d \left(u(t,X) - u_j^t \right) = -\frac{1}{r^m}\, d \left(u(t,X) - u_l^t \right) \cdot \tag{69}$$

This implies:

$$u(t,X) = \frac{\left(u_j^t + u_l^t \right)}{2} \cdot \tag{70}$$

The normal derivative is then:

$$\partial_n u(t,X) = \frac{1}{r^m} d \left(\frac{\left(u_j^t + u_l^t\right)}{2} - u_j^t \right) \tag{71}$$

$$= r^{-m} \frac{d}{2} \left(u_l^t - u_j^t\right). \tag{72}$$

Back to the equation (60)

$$u_j^{n+1} = u_j^n + \frac{h}{\mu(C_m^j)} \sum_{X \in \partial C_m^j} \partial_n u(t_n, X) \tag{73}$$

one may now build the finite volume scheme:

$$u_j^{n+1} = u_j^n + \frac{h}{\mu(C_m^j)} \frac{1}{r^m} \frac{d}{2} \sum_{j \underset{m}{\sim} l} \left(u_l^n - u_j^n\right) \tag{74}$$

where $j \underset{m}{\sim} l$ means that the cell $f_{\mathcal{W}^j}(\mathfrak{S}\mathfrak{G})$ and $f_{\mathcal{W}^j}(\mathfrak{S}\mathfrak{G})$ are neighbors.

Remark 3.2.

i. One may note that we retrieve the finite difference scheme, from a totally independent approach.

ii. We can also define the backward scheme:

$$u_j^n = u_j^{n-1} + \frac{h}{\mu(C_m^j)} \frac{1}{r^m} \frac{d}{2} \sum_{j \underset{m}{\sim} l} \left(u_l^t - u_j^t\right) \tag{75}$$

Given $m \in \mathbb{N}$, and denote any $X \in V_m \setminus V_0$ as $X_{\mathcal{W},P_i}$, where $\mathcal{W} \in \{1,\ldots,d\}^m$ denotes a word of length m, and where P_i, $1 \leqslant i \leqslant d$ belongs to V_0.

Let us consider:

$$\forall\, n \in \{0,\ldots,N-1\} : \quad U(n) = \begin{pmatrix} u_1^n \\ \vdots \\ u_{d_m}^n \end{pmatrix} \tag{76}$$

(one has to bear in mind that there are d^m m^{th}-order cells).
One has:

$$\forall\, n \in \{0,\ldots,N-1\} : \quad U(n+1) = A\, U(n)$$

where:

$$A = I_{d^m} - h \frac{d}{2} \tilde{\Delta}_m \tag{77}$$

and where I_{d^m} denotes the $d^m \times d^m$ identity matrix, and $\tilde{\Delta}_m$ the $d^m \times d^m$ Laplacian normalized matrix.

3.1.2 Consistency, stability and convergence

3.1.2.1 Theoretical study of the error

Let us consider a real-valued continuous function u defined on $\mathfrak{SG}$. One has:

$$\forall\,(n,X)\,\in\,\{0,\ldots,N-1\}\times\mathfrak{SG}\;:\quad\int_{t_n}^{t_{n+1}} u(t,X)\,dt = h\,u(t_n,X)+\mathcal{O}(h^2)\cdot \tag{78}$$

On the other hand, given a strictly positive integer m, $X\in V_m\setminus V_0$, and a harmonic function $\psi_X^{(m)}$ on the m^{th}-order cell, taking the value 1 on $X=F_{\mathcal{W}^j}(P_i)=F_{\mathcal{W}^l}(P_k)$ and 0 on the others vertices (see [Str99]), and using the corollary of the Gauss-Green formula:

$$\int_{f_{\mathcal{W}^j}(\mathfrak{SG})} \Delta_\mu u\,\psi_X^{(m)}\,d\mu = \partial_n u(X) - r^{-m}\sum_{\substack{Y\underset{m}{\sim}X\\ Y\in f_{\mathcal{W}^j}(\mathfrak{SG})}} \left(u(t,X)-u(t,Y)\right) \tag{79}$$

By considering the equivalent relation on the neighbor cells $f_{\mathcal{W}^l}(\mathfrak{SG})$, while, at the same time, using the matching condition, one gets:

$$\int_{\mathfrak{SG}} \Delta_\mu u\,\psi_X^{(m)}\,d\mu = \frac{1}{r^m}\,\Delta_m u(X) \tag{80}$$

$$= \mathcal{O}\left(\int_{\mathfrak{SG}} \psi_X^{(m)}\,d\mu\right) \tag{81}$$

We have thus proved that:

$$\partial_n u(X) - \frac{1}{r^m}\sum_{\substack{Y\underset{m}{\sim}X\\ Y\in f_{\mathcal{W}^j}(\mathfrak{SG})}} \left(u(t,X)-u(t,Y)\right) = \mathcal{O}\left(\int_{\mathfrak{SG}} \psi_X^{(m)}d\mu\right) \tag{82}$$

Finally, for the discrete average, on a m^{th}-order cell $f_{\mathcal{W}^j}(\mathfrak{SG})$:

$$\frac{1}{\mu(f_{\mathcal{W}^j}(\mathfrak{SG}))}\int_{F_{\mathcal{W}^j}(\mathfrak{SG})} u(t,X)\,d\mu(X) - \frac{1}{d}\sum_{Y\in\partial f_{\mathcal{W}^j}(\mathfrak{SG})} u(t,Y) = \frac{1}{\mu(f_{\mathcal{W}^j}(\mathfrak{SG}))}\int_{f_{\mathcal{W}^j}(\mathfrak{SG})} u(t,X) - \frac{1}{d}\sum_{Y\in\partial f_{\mathcal{W}^j}(\mathfrak{SG})} u(t,y)\,d\mu(X) \tag{83}$$

$$= \frac{1}{\mu(f_{\mathcal{W}^j}(\mathfrak{SG}))}\int_{f_{\mathcal{W}^j}(\mathfrak{SG})} \left(\frac{1}{d}\sum_{Y\in\partial f_{\mathcal{W}^j}(\mathfrak{SG})} u(t,X)-u(t,Y)\right) d\mu(X) \tag{84}$$

$$\leqslant \max_{Y\in\partial f_{\mathcal{W}^j}(\mathfrak{SG})} \parallel u(t,X)-u(t,Y)\parallel_\infty \tag{85}$$

$$= \delta_u(2^{-m}) \tag{86}$$

where $\delta_u(\cdot)$ is the continuity modulus of u (which is $\mathcal{O}(2^{-\alpha m})$ if u is α-Hölderian, $\alpha>0$).

3.1.2.2 Consistency

Definition 3.2. The scheme is said to be **consistent** if the consistency error tends towards zero when $h\to 0$ and $m\to+\infty$, for a given norm.

For $0\leqslant n\leqslant N-1$, $1\leqslant i\leqslant d^m$, the scheme error is obtained through:

$$\varepsilon_{n,i}^m = \mathcal{O}(h^2) + \mathcal{O}\left(\int_{\mathfrak{S}\mathfrak{G}} \psi_X^{(m)} d\mu\right) + \frac{\delta}{2^m} \tag{87}$$

$$= \mathcal{O}(h^2) + \mathcal{O}\left(\frac{1}{d^m}\right) + \frac{\delta}{2^m} \tag{88}$$

$$= \mathcal{O}(h^2) + \mathcal{O}\left(\frac{1}{2^{\alpha m}}\right) \qquad \text{if} \quad u \in C^{0,\alpha}(\mathfrak{S}\mathfrak{G}) \tag{89}$$

One has:

$$\lim_{h \to 0^+,\, m \to +\infty} \varepsilon_{n,i}^m = 0 \tag{90}$$

which yields the consistency of the scheme.

3.1.2.3 Stability

Notation. Spectral radius
Given a square matrix A, we will denote by $\lambda_{\max}$ its spectral radius.

Notation. Spectral norm
Given a square matrix A, we will denote by $\rho(A)$ its spectral norm, obtained through:

$$\rho(A) = \left(\lambda_{\max}\left(A^T A\right)\right)^{\frac{1}{2}} \tag{91}$$

Proposition 3.1. *We introduce the real valued function Φ, defined on $\mathbb{R}^\star$, by:*

$$\forall\, t \in \mathbb{R}^\star : \quad \Phi(t) = (d + 2 - t)\, t \cdot$$

For any strictly positive integer m, and any λ_m belonging to the spectrum of the Laplacian, one has:

$$\lambda_{m-1} = \Phi\left(\lambda_m\right) \cdot \tag{92}$$

Proof. Let us consider the sequence of graphs $(\mathcal{SS}_m)_{m \geqslant 1}$ related to the sequences of vertices $(\tilde{V}_m)_{m \geqslant 1}$. The initial graph $\mathcal{SS}_1$ is just a d-simplex, and one may construct the next graph as the union of d copies which are linked in the same manner as $\mathcal{SS}_1$, and so on ...

We now fix $m \in \mathbb{N}$, and choose a vertex X_1 of $\mathcal{SS}_m$, of neighbors $X_2, \ldots, X_d, Y$, such that Y belongs to another m-simplex (we can remark that the graph $\mathcal{SS}_m$ is composed by d m-simplices).

We denote by u the eigenfunction related to the eigenvalue λ_m. One has:

$$\{(d-1) - \lambda_m\}\; u(X) = \sum_{i=1}^{d-1} u(X_i) + u(Y) \cdot \tag{93}$$

On the other hand, we have the same idea in the graph $\mathcal{SS}_{m+1}$, if we take the vertex a_1^k and his neighbors $a_2^k, \ldots, a_d^k, a_h^l$ of the graph $\mathcal{SS}_{m+1}$, where a_h^l belongs to another m-simplex, we have, for every interior (non-boundary) vertex:

$$\{(d-1)-\lambda_{m+1}\}\; u(a_i^k) = \sum_{j\neq i} u(a_j^k) + u(a_h^l) \tag{94}$$

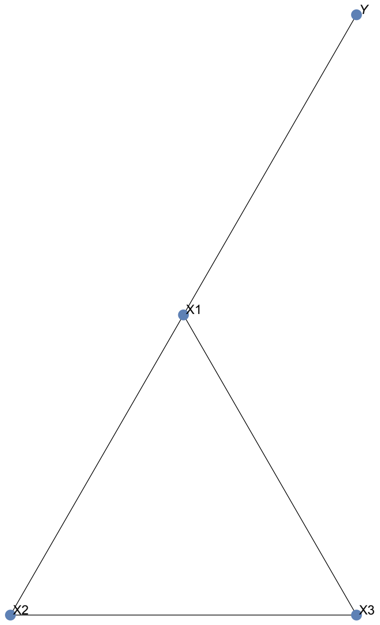

Figure 7 – $\mathcal{SS}_m$ for the Sierpiński triangle.

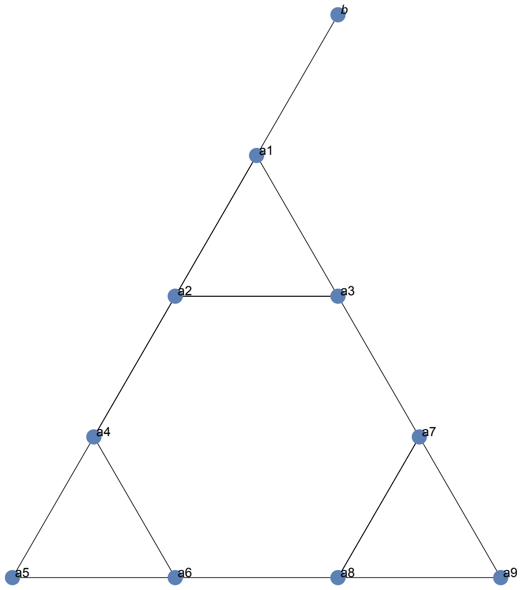

Figure 8 – $\mathcal{SS}_{m+1}$ for the Sierpiński triangle.

Using the mean property:

$$u(X_k) = \frac{1}{d}\sum_{i=1}^{d} u(a_i^k) \tag{95}$$

one obtains, by adding a_i^k to both sides of the eigenfunction relation:

$$(d-\lambda_{m+1})u(a_i^k) = d\,u(X_k) + u(a_h^l) \quad , \quad (d-\lambda_{m+1})u(a_h^l) = d\,u(X_h) + u(a_i^k) \tag{96}$$

which leads to:

$$u(a_i^k) = d\,\frac{((d+1)-\lambda_{m+1})u(X_k) + u(X_h)}{(d+2-\lambda_{m+1})(d-\lambda_{m+1})}\,\cdot \tag{97}$$

Now, given a boundary vertex c_i:

$$((d-1)-\lambda_{m+1})\, u(c_i) = \sum_{j\neq i} u(c_j) \tag{98}$$

$$(d-\lambda_{m+1})\, u(c_i) = d\, u(X_l) \tag{99}$$

$$u(c_i) = \frac{d\, u(X_l)}{(d-\lambda_{m+1})} \tag{100}$$

Finally, by summation over all the $u(a_i^k)$, we obtain:

$$\lambda_m = \lambda_{m+1}\;(d+2-\lambda_{m+1})\;\cdot \tag{101}$$

□

Thus:

$$\forall m \in \mathbb{N}^\star : \quad \begin{cases} \lambda_m^- & = & \dfrac{(d+2) - \left((d+2)^2 - 4\,\lambda_{m-1}\right)^{\frac{1}{2}}}{2} \\ \lambda_m^+ & = & \dfrac{(d+2) - \left((d+2)^2 - 4\,\lambda_{m-1}\right)^{\frac{1}{2}}}{2} \end{cases} . \tag{102}$$

It is then natural to consider the maps ψ^- and ψ^+ defined as follows:

$$\forall x \in \left]-\infty, \frac{(d+2)^2}{4}\right] : \quad \begin{cases} \psi^-(x) & = & \dfrac{(d+2) - \left((d+2)^2 - 4\,x\right)^{\frac{1}{2}}}{2} \\ \psi^+(x) & = & \dfrac{(d+2) + \left((d+2)^2 - 4\,x\right)^{\frac{1}{2}}}{2} \end{cases} . \tag{103}$$

One has:

$$\begin{cases} \psi^-(0) & = & 0 \\ \psi^-\left(\dfrac{(d+2)^2}{4}\right) & = & \dfrac{d+2}{2} \end{cases} , \quad \begin{cases} \psi^+(0) & = & d+2 \\ \psi^+\left(\dfrac{(d+2)^2}{4}\right) & = & \dfrac{d+2}{2} \end{cases} \tag{104}$$

The maps ψ^- and ψ^+ are respectively non decreasing, and non increasing, with respective fixed points:

$$\begin{cases} x^{-,\star} & = & 0 \\ x^{+,\star} & = & (d+2) - 1 \end{cases} \tag{105}$$

Another interesting feature of those two maps is their contractive property, due to:

$$\begin{cases} \left|\dfrac{d}{dx}\psi^-(0)\right| & = & \dfrac{1}{\sqrt{(d+2)^2}} & = & \dfrac{1}{d+2} & < & 1 \\ \left|\dfrac{d}{d\,x}\psi^+\left((d+2)-1\right)\right| & = & \dfrac{1}{\sqrt{(d+2)^2 - 4\,(d+2) + 4}} & = & \dfrac{1}{d} & < & 1 \end{cases} \tag{106}$$

Since V_1 is a complete graph, it has eigenvalues -1 with multiplicity 1, and 2 with multiplicity 2.

The whole Dirichlet spectrum, for $m \geqslant 2$, is generated by the recurrent stable maps ψ^- and ψ^+.

Thus:

$$\forall m \in \mathbb{N} : \quad \lambda_m \in [0, 2\,d] \tag{107}$$

Notation. Matrix norm
In the sequel, we will work with the matrix norm $\|\cdot\|$ defined, for any square matrix A, by:

$$\|A\| = \| A \|_2$$

Definition 3.3.

We will say that:

i. The scheme is *unconditionally stable* if there exists a real constant $C \in]0,1[$ such that:

$$\forall n \in \{1, \ldots, N\} : \quad \rho(A^n) \leqslant C \cdot \tag{108}$$

ii. The scheme is *conditionally stable* if there exist three strictly positive constants α, C_1 and C_2 such that:

$$C_2 < 1 \quad \text{and} \quad \forall n \in \{1, \ldots, N\} : \quad h \leqslant \frac{C_1}{(d+2)^{\alpha m}} \Longrightarrow \rho(A^n) \leqslant C_2 \cdot \tag{109}$$

Proposition 3.2.

If $Sp\,(A) = \{\gamma_1 \ldots, \gamma_{d^m}\}$ is the spectrum of the matrix A, one has:

$$\forall j \in \{1, \ldots, d^m\} : \quad h\,(d+2)^m \leqslant \frac{2}{d^2} \Longrightarrow |\gamma_j| \leqslant 1 \cdot \tag{110}$$

Proof. Since:

$$\forall n \in \{1, \ldots, N\} : \quad U(n+1) = A\,U(n) \qquad \forall n \in \{1, \ldots, N\} \tag{111}$$

where:

$$A = I_{d^m} - h\,\tilde{\Delta}_m. \tag{112}$$

one has:

$$\forall n \in \{1, \ldots, N\} : \quad U(n) = A^n\,U(0) \tag{113}$$

Thus:

$$\forall i \in \{1, \ldots, d^m\} : \quad \gamma_i = 1 - h\,\frac{d}{2}(d+2)^m\,\lambda_i \tag{114}$$

and:

$$\forall i \in \{1, \ldots, d^m\} : \quad 1 - \frac{h\,d}{2}(d+2)^m \times 2\,d \leqslant \gamma_i \leqslant 1 \cdot \tag{115}$$

Consequently, if the condition:

$$h\,(d+2)^m \leqslant \frac{2}{d^2} \tag{116}$$

is satisfied, one obtains:

$$\forall i \in \{1, \ldots, d^m\} : \quad |\gamma_i| \leqslant 1 \cdot \tag{117}$$

□

3.1.2.4 Convergence

Definition 3.4.

i. The scheme will be said to be *convergent* if:

$$\lim_{h \to 0^+,\, m \to +\infty} \left\| \left(u_j^n - \frac{1}{\mu(C_m^j)} \int_{C_m^j} g(x)\,d\mu(x) \right)_{0 \leqslant n \leqslant N,\, 1 \leqslant j \leqslant d^m} \right\| = 0 \cdot \tag{118}$$

ii. The scheme will be said to be *conditionally convergent* if there exist two strictly positive constants α and C such that :

$$\lim_{h \leqslant \frac{C}{(d+2)^{\alpha\, m}},\, m\to+\infty} \left\| \left(u_j^n - \frac{1}{\mu(C_m^j)} \int_{C_m^j} g(x) d\mu(x) \right)_{0\leqslant n\leqslant N,\, 1\leqslant j\leqslant d^m} \right\| = 0\cdot \tag{119}$$

Theorem 3.3.

If the scheme is stable and consistent, then it is also convergent for the norm $\|\cdot\|_{2,\infty}$*, such that:*

$$\left\| \left(u_j^n\right)_{0\leqslant n\leqslant N, 1\leqslant j\leqslant d^m} \right\|_{2,\infty} = \max_{0\leqslant n\leqslant N} \left(\frac{1}{d^m} \sum_{1\leqslant i\leqslant d^m} |u_i^n|^2 \right)^{\frac{1}{2}} \tag{120}$$

Proof. Let us set:

$$w_i^n = u_j^n - \frac{1}{\mu(C_m^j)} \int_{C_m^j} g(x)\, d\mu(x), \quad 0 \leqslant n \leqslant N,\, 1 \leqslant j \leqslant d^m \tag{121}$$

and:

$$\forall\, n \in \{0,\ldots,N\} : \quad W^n = \begin{pmatrix} w_1^n \\ \vdots \\ w_{d^m}^n \end{pmatrix} \quad , \quad E^n = \begin{pmatrix} \varepsilon_{n,1}^m \\ \vdots \\ \varepsilon_{n,d^m}^m \end{pmatrix} \tag{122}$$

Thus:

$$W^0 = 0 \tag{123}$$

and:

$$\forall\, n \in \{0,\ldots,N-1\} : \quad W^{n+1} = A\, W^n + h\, E^n \tag{124}$$

By induction, one gets:

$$\forall\, n \in \{0,\ldots,N-1\} : \quad W^{n+1} = A^n W^0 + h \sum_{j=0}^{n-1} A^j\, E^{n-j-1} = h \sum_{j=0}^{n-1} A^j\, E^{n-j-1} \tag{125}$$

Due to the symmetry of the matrix A, the Courant Fredrichs Lewy stability condition

$$h\,(d+2)^m \leqslant \frac{2}{d^2}$$

yields:

$\forall\, n \in \{0,\ldots,N-1\}$:

$$
\begin{aligned}
|W^n| &\leqslant h \left(\sum_{i=0}^{n-1} \| A \|^i \right) \max_{0 \leqslant n \leqslant i-1} |E^n| \\
&\leqslant h\, n \max_{0 \leqslant n \leqslant i-1} |E^n| \\
&\leqslant h\, N \max_{0 \leqslant n \leqslant i-1} |E^n| \\
&\leqslant T \max_{0 \leqslant n \leqslant i-1} \left(\sum_{j=1}^{d^m} |\varepsilon^m_{n,j}|^2 \right)^{\frac{1}{2}}
\end{aligned}
\tag{126}
$$

and thus:

$$
\max_{0 \leqslant n \leqslant N-1} \left(\frac{1}{d^m} \sum_{i=1}^{d^m} |w^n_i|^2 \right)^{\frac{1}{2}} = \frac{1}{d^{\frac{m}{2}}} \max_{1 \leqslant n \leqslant N-1} |W^n| \tag{127}
$$

$$
\leqslant \frac{1}{d^{\frac{m}{2}}} T \left(\max_{0 \leqslant n \leqslant N-1} \left(\sum_{i=1}^{d^m} |\varepsilon^m_{n,i}|^2 \right)^{\frac{1}{2}} \right) \tag{128}
$$

$$
\leqslant \frac{1}{d^{\frac{m}{2}}} T \left(d^{\frac{m}{2}} \max_{0 \leqslant n \leqslant N-1,\, 1 \leqslant i \leqslant d^m} |\varepsilon^m_{n,i}| \right) \tag{129}
$$

$$
= T \max_{0 \leqslant n \leqslant N-1,\, 1 \leqslant i \leqslant d^m} |\varepsilon^m_{n,i}| \tag{130}
$$

$$
= \mathcal{O}(h^2) + \mathcal{O}(d^{-m}) + \frac{\delta}{2^m} \tag{131}
$$

$$
= \mathcal{O}\left(\frac{1}{(d+2)^{2m}} \right) + \mathcal{O}\left(\frac{1}{d^m} \right) + \frac{\delta}{2^m} \tag{132}
$$

$$
= \mathcal{O}\left(\frac{1}{2^{\alpha\, m}} \right) \cdot \tag{133}
$$

$$
\tag{134}
$$

By assuming the function u to be Hölder-continuous, one obtains the expected result.

□

Remark 3.3. One has to bear in mind that, for piecewise constant functions u on the m^{th}-order cells:

$$
\left\| \left(u^n_j \right) \right\|_2 = \left(\frac{1}{d^m} \sum_{1 \leqslant i \leqslant d^m} |u^n_i|^2 \right)^{\frac{1}{2}} = \left\| \left(u^n_j \right) \right\|_{L^2(\mathfrak{SG})} \cdot \tag{135}
$$

3.1.3 The implicit Euler Method

Let consider the implicit Euler scheme, for any integer k belonging to $\{0, \ldots, N-1\}$:

$$
u^n_j = u^{n-1}_j + \frac{h}{\mu(C^j_m)} \frac{1}{r^m} \frac{d}{2} \sum_{l=1}^{d-1} \left(u^n_l - u^n_j \right) \cdot \tag{136}
$$

It satisfies the recurrence relation:

$$
\tilde{A}\, U(n) = U(n-1) \tag{137}
$$

where:

$$\tilde{A} = I_{d^m} + h \times \tilde{\Delta}_m \tag{138}$$

and where I_{d^m} denotes the $d^m \times d^m$ identity matrix, and $\tilde{\Delta}_m$ the $d^m \times d^m$ normalized Laplacian matrix.

3.1.3.1 Consistency, stability and convergence

i. Consistency

The consistency error of the implicit Euler scheme is given by :
For $0 \leqslant n \leqslant N-1$, $1 \leqslant i \leqslant d^m$, the consistency error of our scheme is given by :

$$\begin{aligned}
\varepsilon_{n,i}^m &= \mathcal{O}(h^2) + \mathcal{O}\left(\frac{1}{d^m}\right) + \frac{\delta}{2^m} && (139)\\
&= \mathcal{O}(h^2) + \mathcal{O}\left(\frac{1}{2^{\alpha\, m}}\right) \qquad \text{if} \quad u \in C^{0,\alpha}(\mathfrak{S}\mathfrak{G}) && (140)
\end{aligned}$$

One has:

$$\lim_{h \to 0^+,\, m \to +\infty} \varepsilon_{n,i}^m = 0$$

which yields the consistency of the scheme.

ii. Stability

Definition 3.5. We will say that:

i. The scheme is *unconditionally stable* for the norm $\|\cdot\|_\infty$ if there exists a constant $C > 0$ independent of h and m such that :

$$\| U_h^m(n) \|_\infty \leqslant C \, \| U_h^m(0) \|_\infty \quad \forall\, n \,\in\, \{1, \ldots, N\} \cdot \tag{141}$$

ii. The scheme is *conditionally stable* if there exist three constants $\alpha > 0$, $C_1 > 0$ and $C_2 < 1$ such that :

$$h \leqslant \frac{C_1}{(d+2)^{m\,\alpha}} \Longrightarrow \| U_h^m(n) \|_\infty \leqslant C_2 \, \| U_h^m(0) \|_\infty \quad \forall\, n \,\in\, \{1, \ldots, N\} \cdot \tag{142}$$

We set:

$$\tilde{A} = I_{\mathcal{N}_m - d} + h\, \tilde{\Delta}_m \cdot \tag{143}$$

One has then:

$$\tilde{A}\, U(n) = U(n-1) \tag{144}$$

We have:

$$\| \tilde{A}^{-1} \|_\infty \leqslant 1 \quad \Longrightarrow \quad \| \tilde{A}^{-n} \|_\infty \leqslant 1 \tag{145}$$

The scheme is thus unconditionally stable :

$$U(n) \leqslant U(0) \cdot \tag{146}$$

iii. Convergence

Theorem 3.4. *The implicit Euler scheme is convergent for the norm* $\| \cdot \|_{2,\infty}$.

Proof. Let:

$$w_i^n = u_j^n - \frac{1}{\mu(C_m^j)} \int_{C_m^j} g(x)d\mu(x), \quad 0 \leqslant n \leqslant N,\, 1 \leqslant j \leqslant d^m \cdot \tag{147}$$

We set:

$$W^n = \begin{pmatrix} w_1^n \\ \vdots \\ w_{d^m}^n \end{pmatrix} \quad , \quad E^n = \begin{pmatrix} \varepsilon_{n,1}^m \\ \vdots \\ \varepsilon_{n,d^m}^m \end{pmatrix} \tag{148}$$

Thus, $W^0 = 0$, and, for $0 \leqslant n \leqslant N-1$:

$$W^{n+1} = \tilde{A}^{-1} W^n + h\, E^n \quad 0 \leqslant n \leqslant N-1 \tag{149}$$

$$\tag{150}$$

We find, by induction, for $0 \leqslant n \leqslant N-1$:

$$W^{n+1} = \tilde{A}^{-n}\, W^0 + h \sum_{j=0}^{n-1} \tilde{A}^{-j}\, E^{n-j-1} \tag{151}$$

$$= h \sum_{j=0}^{n-1} \tilde{A}^{-j} E^{n-j-1} \tag{152}$$

Due to the stability of the scheme, we have, for $n = 0, \ldots, N$:

$$|W^n| \leqslant h \left(\sum_{j=0}^{n-1} \| \tilde{A}^{-1} \|^j \right) \left(\max_{0 \leqslant n \leqslant j-1} |E^n| \right) \tag{153}$$

$$\leqslant h\, n \max_{0 \leqslant n \leqslant j-1} |E^n| \tag{154}$$

$$\leqslant h\, N \max_{0 \leqslant n \leqslant j-1} |E^n|) \tag{155}$$

$$\leqslant T \max_{0 \leqslant n \leqslant j-1} \left(\sum_{i=1}^{d^m} |\varepsilon_{n,i}^m|^2 \right)^{\frac{1}{2}} \tag{156}$$

$$\tag{157}$$

Thus:

$$\max_{0\leqslant n\leqslant N}\left(\frac{1}{d^m}\sum_{i=1}^{d^m}|w_i^n|^2)\right)^{\frac{1}{2}} = \frac{1}{d^{\frac{m}{2}}}\max_{0\leqslant n\leqslant N}|W^n| \tag{158}$$

$$\leqslant \frac{1}{d^{\frac{m}{2}}}\,T\left(\max_{0\leqslant n\leqslant N-1}\left(\sum_{i=1}^{d^m}|\varepsilon_{n,i}^m|^2\right)^{\frac{1}{2}}\right) \tag{159}$$

$$\leqslant \frac{1}{d^{\frac{m}{2}}}\,T\,d^{\frac{m}{2}}\max_{0\leqslant n\leqslant N-1,\,1\leqslant i\leqslant d^m}|\varepsilon_{n,i}^m| \tag{160}$$

$$= T\max_{0\leqslant n\leqslant N-1,\,1\leqslant i\leqslant d^m}|\varepsilon_{n,i}^m| \tag{161}$$

$$= \mathcal{O}(h^2)+\mathcal{O}(d^{-m})+\frac{\delta}{2^m} \tag{162}$$

$$= \mathcal{O}\left(\frac{\delta}{(d+2)^{2\,m}}\right)+\mathcal{O}\left(\frac{1}{d^m}\right)+\frac{\delta}{2^m} \tag{163}$$

$$= \mathcal{O}\left(\frac{1}{2^{\alpha\,m}}\right)\cdot \tag{164}$$

$$\tag{165}$$

The last equality holds if one assumes that u is Hölder-continuous. The scheme is thus convergent. □

3.1.4 Numerical application

3.1.4.1 The graph Laplacian sequence

***i.* The 2-Simplex (Triangle).**

For any $m\in\mathbb{N}^\star$, we will denote by $\text{Corner}_{k,m}$, $1\leqslant k\leqslant 3$, the three corners of a m^{th}-order cell (see figure 3.1.4.1).

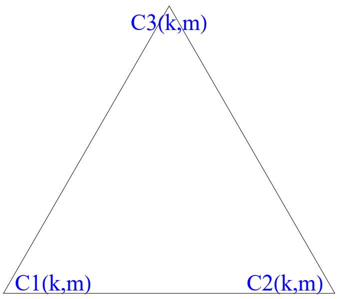

Figure 9 – m^{th}-order cell.

the $(m+1)^{th}$-order triangle is then constructed by connecting three m^{th}-order cells $T(k)$, with $k=1,\,2,\,3$.

The initial triangle is labelled such that $\text{Corner}_1\sim 1$, $\text{Corner}_2\sim 2$ and $\text{Corner}_3\sim 3$ (see figure 1).

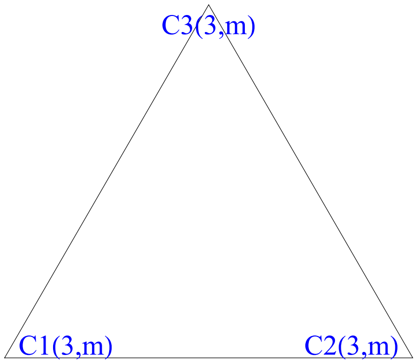

Figure 10 – The third copy T_3

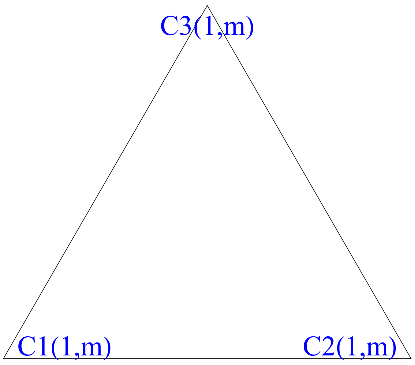

Figure 11 – The first copy T_1

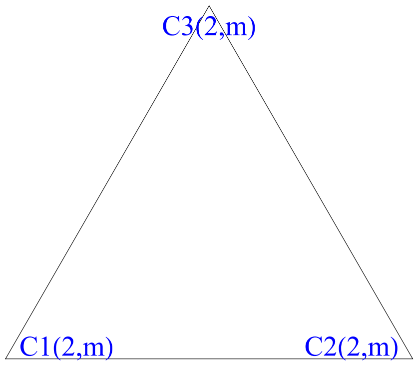

Figure 12 – The second copy T_2

Notation. We set:

$$I_2(1) = 2 \tag{166}$$

and:

$$\forall\, m \,\in\, \mathbb{N}^\star \,:\quad I_2(m) = I_2(m-1) + 3^{m-2} \tag{167}$$

Iterative process:

i. The starting point is the set of vertices V_0 of the initial simplex. The corresponding matrix is given by:

$$A_0 = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \tag{168}$$

ii. At a given order $m \,\in\, \mathbb{N}^\star$, the fusion requires the following connections:

$$\left\{ \begin{array}{lcl} \text{Corner}_2(1,m) & \sim & \text{Corner}_1(2,m) \\ \text{Corner}_3(1,m) & \sim & \text{Corner}_1(3,m) \\ \text{Corner}_3(2,m) & \sim & \text{Corner}_2(3,m) \end{array} \right. \cdot \tag{169}$$

Vertex corners are such that:

$$\left\{ \begin{array}{lcc} \text{Corner}_1(k,m) & = & 1+(k-1)\,3^{m-1} \\ \text{Corner}_2(k,m) & = & I_2(m)+(k-1)\,3^{m-1} \\ \text{Corner}_3(k,m) & = & k\,3^{m-1} \end{array} \right. . \tag{170}$$

where k is the number of copies.

iii. The connection matrix (we refer to [FL04]) is obtained through:

$$\forall\, m \in \mathbb{N}^\star : \quad C_m = \begin{pmatrix} \text{Corner}_2(1,m) & \text{Corner}_3(1,m) & \text{Corner}_3(2,m) \\ \text{Corner}_1(2,m) & \text{Corner}_1(3,m) & \text{Corner}_2(3,m) \end{pmatrix} \tag{171}$$

where the following compatibility conditions are to be satisfied:

$$\left\{ \begin{array}{ccccc} A_{\text{Corner}_m(2,j),C_m(1,j)} & = & A_{\text{Corner}_m(1,j),\text{Corner}_m(2,j)} & = & -1 \\ A_{\text{Corner}_m(2,j),\text{Corner}_m(2,j)} & = & A_{\text{Corner}_m(1,j),\text{Corner}_m(1,j)} & = & 3 \end{array} \right. . \tag{172}$$

***ii.* The $3-$*Simplex* (Tetrahedron).**

For any $m \in \mathbb{N}^\star$, we will denote by $\text{Corner}_{k,m}$, $1 \leqslant k \leqslant 4$, the four corners of a m^{th}-order cell (see figure 3.1.4.1).

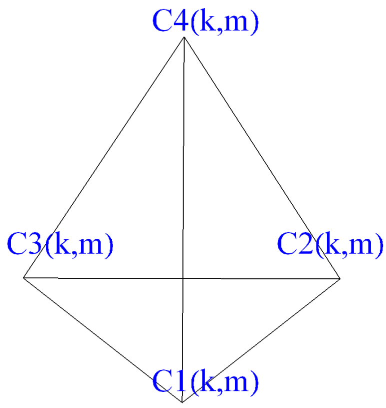

Figure 13 – m^{th}-order cell.

For the fusion, one has to make the following connections:

$$\left\{ \begin{array}{lcl} \text{Corner}_2(1,m) & \sim & \text{Corner}_1(2,m) \\ \text{Corner}_3(1,m) & \sim & \text{Corner}_1(3,m) \\ \text{Corner}_4(1,m) & \sim & \text{Corner}_1(4,m) \\ \text{Corner}_3(2,m) & \sim & \text{Corner}_2(3,m) \\ \text{Corner}_4(2,m) & \sim & \text{Corner}_2(4,m) \\ \text{Corner}_4(3,m) & \sim & \text{Corner}_3(4,m) \end{array} \right. . \tag{173}$$

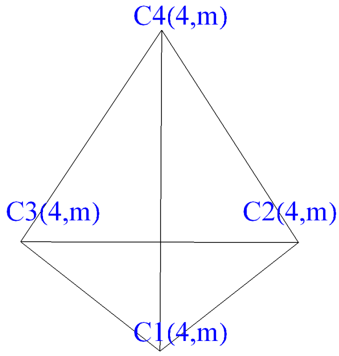

Figure 14 – The fourth copy T_4.

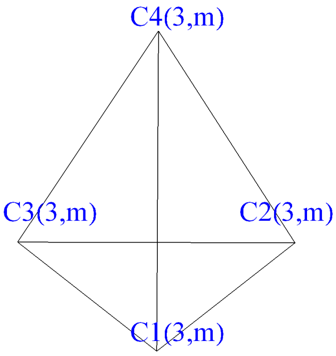

Figure 15 – The third copy T_3.

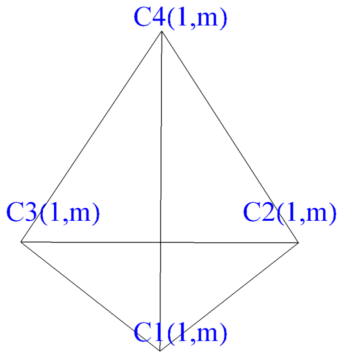

Figure 16 – The first copy T_1.

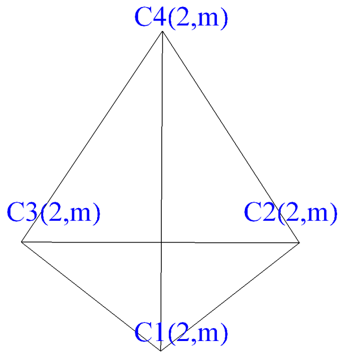

Figure 17 – The second copy T_2.

Notation. We set:

$$I_2(1) = 2 \quad , \quad I_3(1) = 3 \tag{174}$$

and:

$$\forall\, m \,\in\, \mathbb{N}^\star \,:\quad \left\{ \begin{array}{rcl} I_2(m) & = & I_2(m-1) + 4^{m-2} \\ I_3(m) & = & I_3(m-1) + 2 \times 4^{m-2} \end{array} \right. \tag{175}$$

Property 3.5. ***Number of corners of a*** m^{th}***-order cell,*** $m \in \mathbb{N}^\star$

To obtain the number of corners, one writes:

$$\forall\, m \,\in\, \mathbb{N}^\star \,:\quad \left\{ \begin{array}{rcl} Corner_1(k,m) & = & 1 + (k-1)\, 4^{m-1} \\ Corner_2(k,m) & = & I_2(m) + (k-1)\, 4^{m-1} \\ Corner_3(k,m) & = & I_3(m) + (k-1)\, 4^{m-1} \\ Corner_4(k,m) & = & k\, 4^{m-1} \end{array} \right. . \tag{176}$$

Iterative process:

i. The starting point is the set of vertices V_0 of the initial simplex. One has:

$$A_0 = \begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{pmatrix} \cdot \tag{177}$$

ii. The connection matrix is obtained through:
$\forall\, m \,\in\, \mathbb{N}^\star$:

$$C_m = \begin{pmatrix} \text{Corner}_2(1,m) & \text{Corner}_3(1,m) & \text{Corner}_3(2,m) & \text{Corner}_4(1,m) & \text{Corner}_4(2,m) & \text{Corner}_4(3,m) \\ \text{Corner}_1(2,m) & \text{Corner}_1(3,m) & \text{Corner}_2(3,m) & \text{Corner}_1(4,m) & \text{Corner}_2(4,m) & \text{Corner}_3(4,m) \end{pmatrix}$$

where the following compatibility conditions are to be satisfied:

$$\forall\, m \,\in\, \mathbb{N} : \begin{cases} A_{\text{Corner}_m(2,j),\text{Corner}_m(1,j)} & = & A_{\text{Corner}_m(1,j),\text{Corner}_m(2,j)} & = & -1 \\ A_{\text{Corner}_m(1,j),\text{Corner}_m(1,j)} & = & A_{\text{Corner}_m(2,j),\text{Corner}_m(2,j)} & = & 4 \end{cases} \cdot$$

3.1.4.2 Numerical results

Our heat transfer simulation consists in a propagation scenario, where the initial condition is a harmonic spline g, the support of which being an m-cell of $\mathfrak{S}\mathfrak{G}$, such that it takes the value 1 on a vertex X, and 0 otherwise. This implies that g is everywhere null except the cell containing X, which is a vertex of the graph $\mathcal{SS}_m$.

Each point represents a m-cell as before. The color function is related to the gradient of temperature, high values ranging from red to blue.

_i._ The 2-Simplex (Triangle)

In the following (see figures 16 to 19), we give numerical results of (52) in the case where:

$$m = 6 \quad , \quad T = 1 \quad , \quad N = 70.5 \times 10^3 \cdot$$

Each point stands for a m^{th}-order cell of the Simplex.

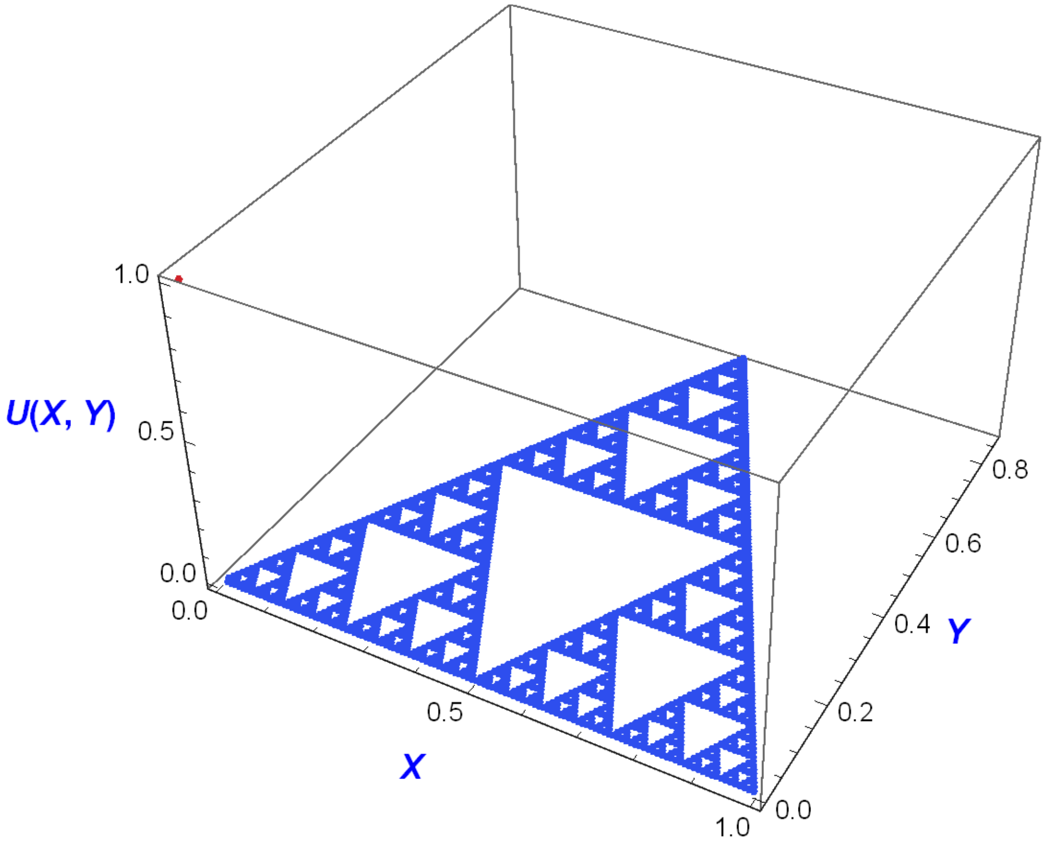

Figure 18 – The numerical solution for the initial condition.

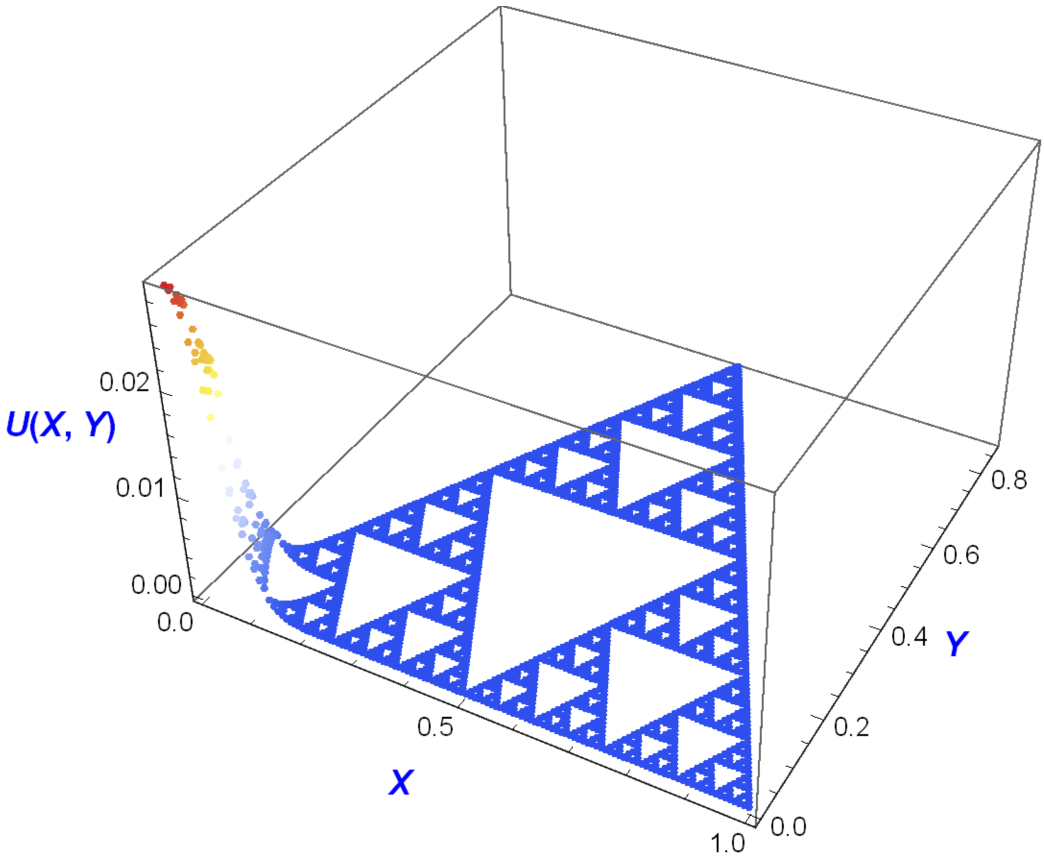

Figure 19 – The numerical solution, for $n = 100$.

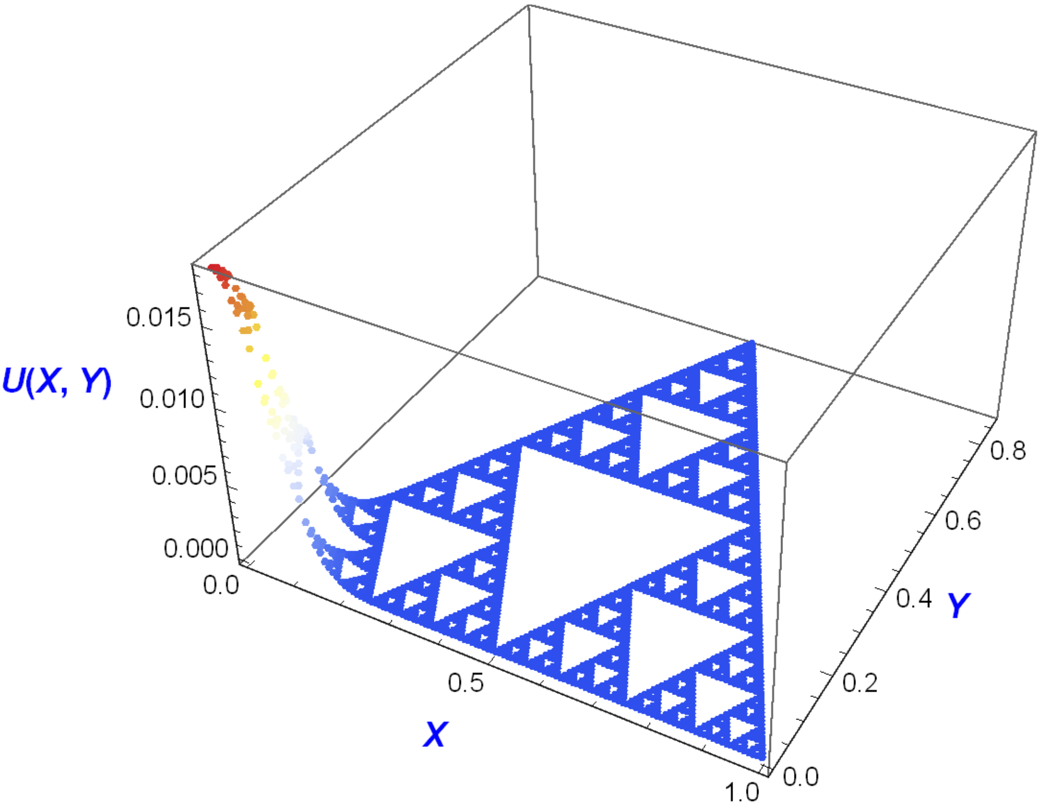

Figure 20 – The numerical solution, for $n = 200$.

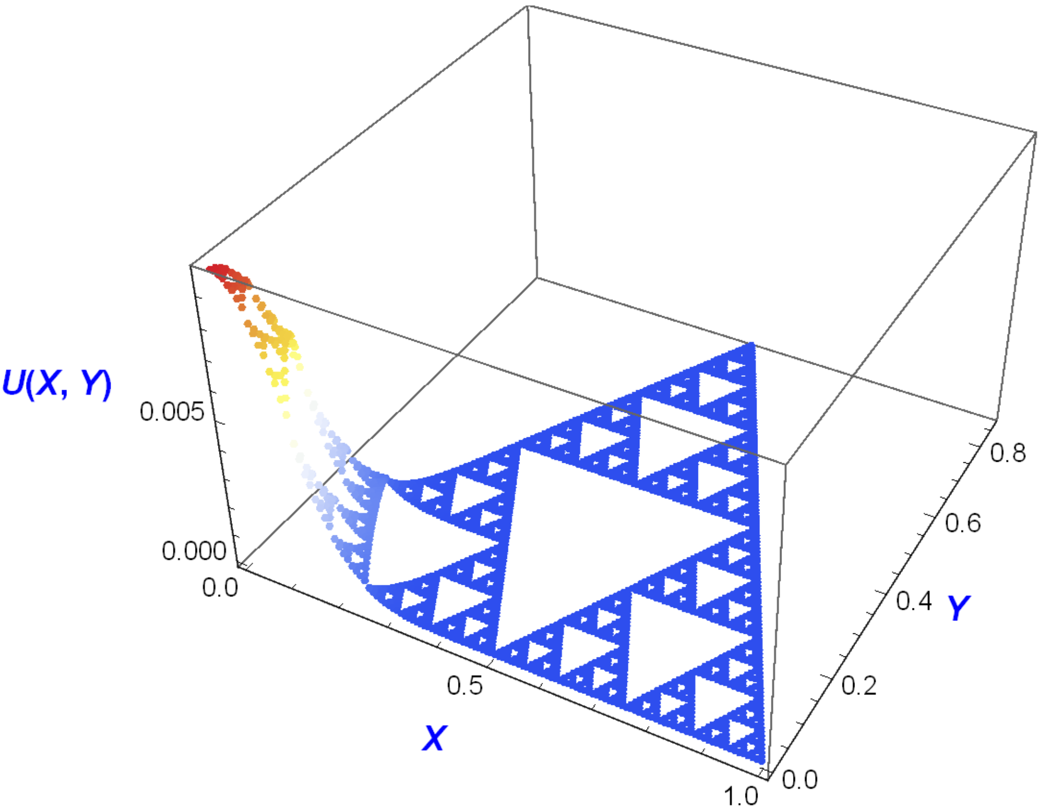

Figure 21 – The numerical solution, for $n = 500$.

ii. The 3-Simplex (Tetrahedron)

In the following (see figures 20, 21, 22, 23, 24), we give a four dimensional representation of the numerical results in the case where:

$$m = 5 \quad , \quad T = 1 \quad , \quad N = 60 \times 10^3 \cdot$$

The color function is related to the gradient of temperature, high values ranging from red to blue.

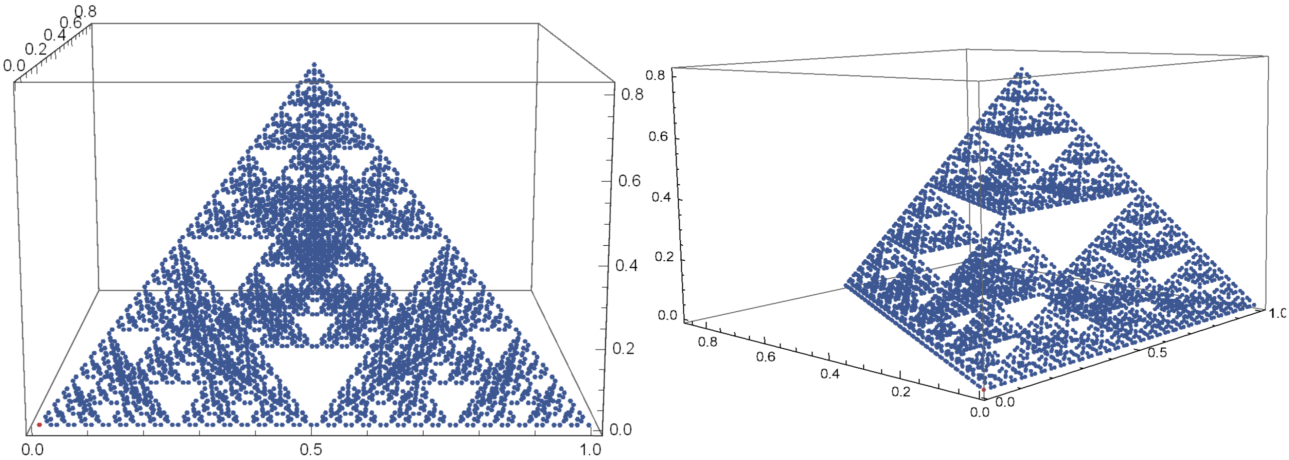

Figure 22 – The numerical solution for the initial condition (front and rotated view).

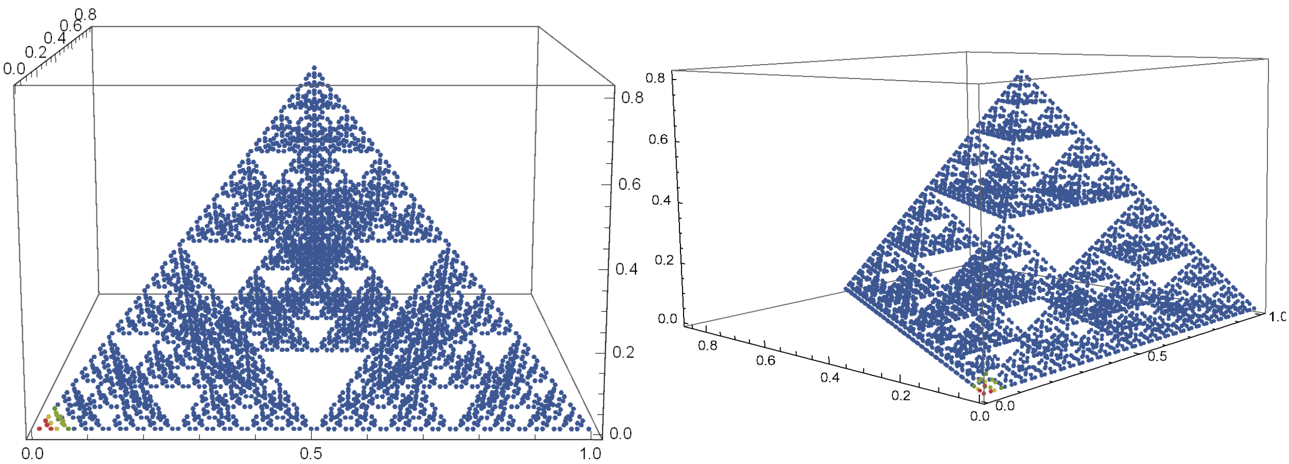

Figure 23 – The numerical solution, for $n = 10$.

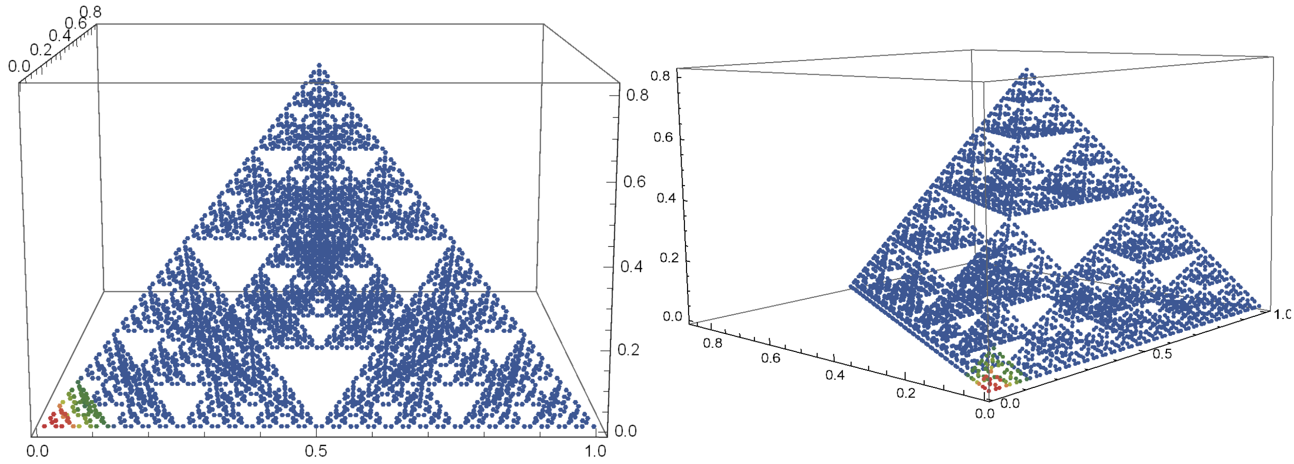

Figure 24 – The numerical solution, for $n = 50$.

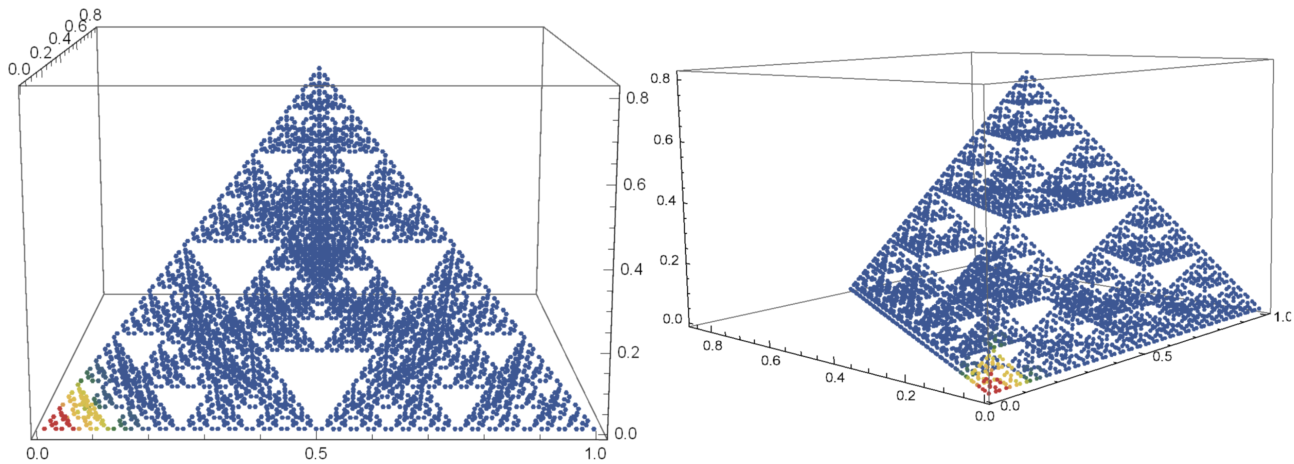

Figure 25 – The numerical solution, for $n = 100$.

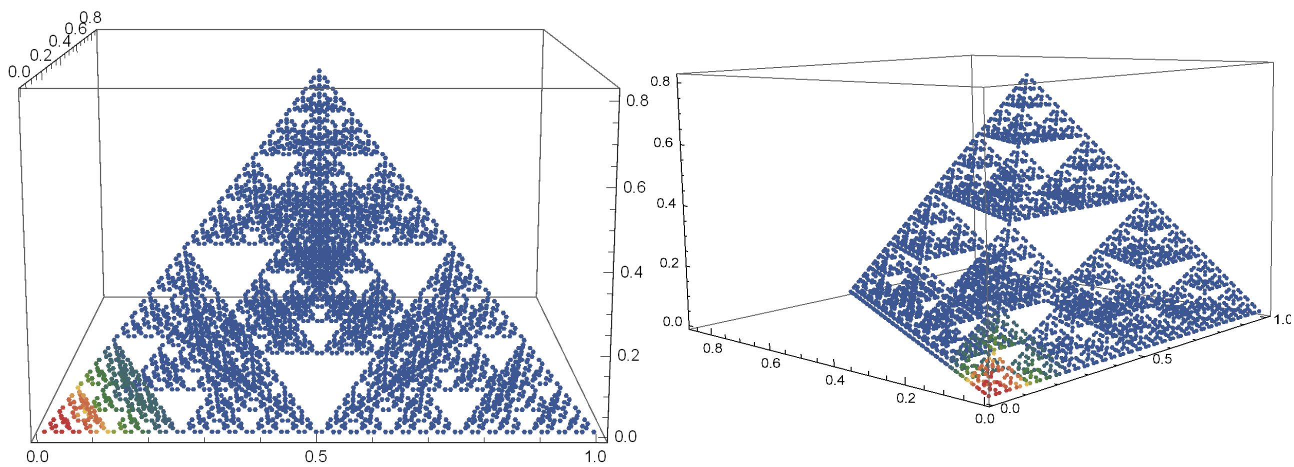

Figure 26 – The numerical solution, for $n = 200$.

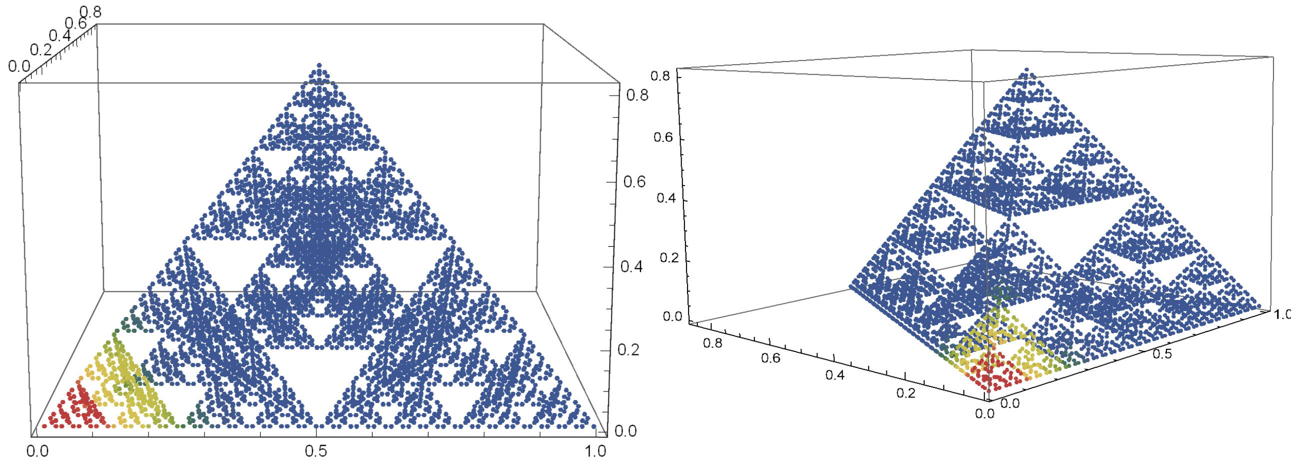

Figure 27 – The numerical solution, for $n = 500$.

iii. Discussion

By plotting the energy

$$E(n) = \int_{\mathfrak{SG}} u(t_n, X)^2 \, d\mu(X)$$

as a function of the iteration step n, one may note the exponential decreasing behaviour (see figure 26).

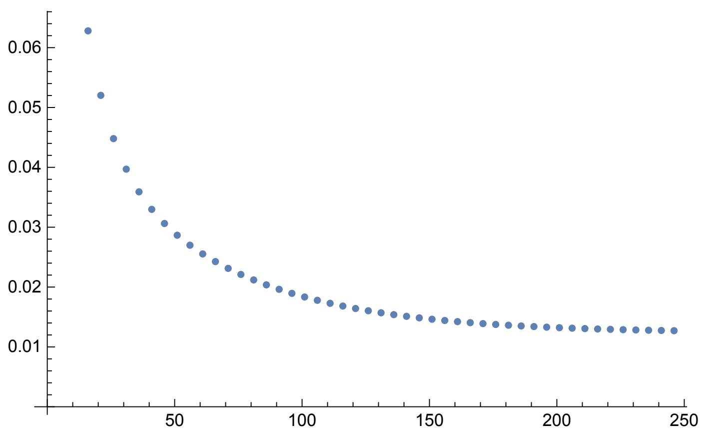

Figure 28 – The energy E on the Sierpiński gasket as a function of the iteration step, for $m = 3$.

More interestingly, as in our previous work [RD19], the loglog plot of the energy (see Figure 29) shows that, numerically, the temperature follows a law of the form:

$$\ln (u(t, x_0)) \approx -1.01275 - 1.44972 \ln t$$

where the slope is close to the spectral dimension $d_S = \frac{2 \ln 3}{\ln 5}$. This yields a power law of the form:

$$u(t, x_0) \approx C \, t^{d_S}$$

where C is a strictly positive real constant. This results holds for different values of m.

This suggests that the spectral dimension belongs to the spectrum of the Laplacian, which is in accordance with theoretical results (see section 3, and [FS92]).

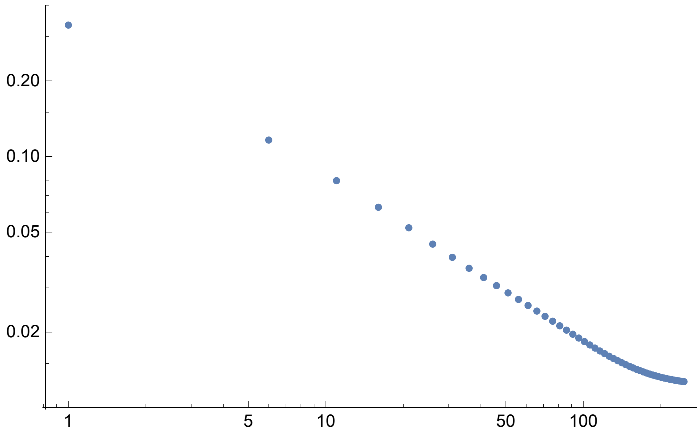

Figure 29 – The loglog plot of the energy E on the Sierpiński gasket, for $m = 3$.

By comparing the explicit finite difference method (FDM) and the finite volume method (FVM), one may deduce from the theoretical results that there are some similarities:

i. The FDM is based on the sequence of graphs $(\mathfrak{S}\mathfrak{S}_m)_{m\in\mathbb{N}}$, and the FVM is based on the sequence of graphs $(\mathcal{S}\mathcal{S}_m)_{m\in\mathbb{N}}$, but both generate the same spectral decimation function.

ii. The theoretical errors of both methods are the same for Hölder continuous functions.

iii. The time theoretical error is of order h in the case of the FDM, and of order h^2 in the case of the FVM (the convergence is faster).

iv. The stability conditions are the same.

v. Finally, the numerical simulation shows the same behavior in both approaches.

Conclusion

So far, there are very few studies on the resolution of partial differential equations on Sierpiński simplices, when it comes to implement numerical methods in accordance to the recent developments in analysis on fractals.

One of the underlying difficulties comes from the fact that there is not any analytical solution, and thus, a missing source of comparison. The only thing one can do is to check that the results are in accordance with the ones related to the fractal dimension of the studied object, which is the case. Then, it satisfactorily appears that the results are also in accordance with our previous numerical study [RD19].

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