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# *Comptes Rendus*

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## *Mécanique*

Stéphan Fauve

**Effect of neutral modes on the order of a transition**


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**Part of the Thematic Issue:** Tribute to an exemplary man: Yves Couder

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Tribute to an exemplary man: Yves Couder

Instabilities, patterns / *Instabilities, patterns*

# Effect of neutral modes on the order of a transition

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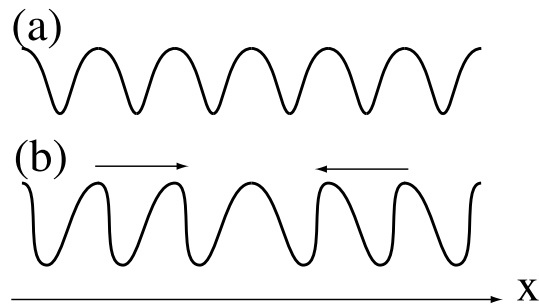
**Abstract.** Neutral modes related to spontaneous broken symmetries at the onset of a pattern-forming instability can strongly modify the nature of secondary instabilities of the pattern. In particular these neutral modes can change the order of the secondary transition making it first order or subcritical in the language of bifurcation theory. We first discuss this phenomenon in the context of the drift bifurcation from stationary to traveling patterns. We then consider patterns that undergo a spatial period-doubling bifurcation like the Peierls transition in solid state physics.

**Résumé.** Je rappelle comment une expérience réalisée par Yves Couder et son groupe a motivé des travaux théoriques qui ont montré que les modes de phase d'une structure cellulaire engendrée par instabilité peuvent affecter la nature des instabilités secondaires de cette structure, à savoir, le caractère propagatif ou non de l'instabilité secondaire et sa sous-criticalité. Je discute ensuite la sous-criticalité résultant du couplage avec les modes de phase sur d'autres exemples tels que la transition de Peierls en physique de la matière condensée.

**Keywords.** Instability, Symmetry, Neutral modes, Drifting patterns, Peierls transition.

## 1. Introduction

Pattern-forming instabilities in fluid mechanics are related to a transition from a spatially homogeneous (often motionless) state, to one varying periodically in space or time. It is a nearly two century old subject. In 1831, Faraday observed that standing surface waves can be generated by vertically vibrating a fluid layer [1]. The Kelvin-Helmholtz instability, i.e. waves amplified by a shear flow at the interface between two fluids of different densities, was also observed long ago in various geophysical flows and understood using linear stability analysis [2]. Thermal convection in a horizontal layer of fluid subjected to a temperature gradient, referred to as Rayleigh-Bénard convection, was first quantitatively studied by Bénard in 1900 [3, 4]. Pattern formation was the subject of renewed interest starting in the 1980s, as part of efforts to understand nonlinear phenomena in out of equilibrium systems. It was found that patterns with quasicrystalline symmetry can be generated from hydrodynamic instabilities [5, 6] and much work was devoted to the study of secondary instabilities of patterns [7] and to the transition to spatiotemporal chaos. Yves Couder and his group played an important role in the last two subjects.



**Figure 1.** (a) A stationary spatially periodic pattern. (b) Above the drift bifurcation, some cells become inclined and thus spontaneously break the reflexion symmetry. The direction of propagation of the inclined cells (shown by arrows) is determined by the sign of the asymmetry.

We will start by describing an experiment they performed on film draining instabilities that motivated theoretical work showing how neutral modes related to a primary instability can modify the nature of secondary instabilities. They studied the stability of an oil–air interface sheared in the opening gap between two cylinders [8]. The two cylinders are horizontal, one inside the other and off-centered so as to be separated by a small gap along a generatrix, say the  $x$ -axis, at the bottom of the set-up. This gap is filled by a small amount of oil that also wets the cylinders since they rotate at rates  $\Omega_1$  and  $\Omega_2$ . When only one cylinder rotates ( $\Omega_2 = 0$ ), the oil–air interface undergoes a stationary instability above a critical value of  $\Omega_1$  and the interface becomes periodically modulated along its axis  $x$  (see Figure 1a). When  $\Omega_1$  is increased further, the pattern becomes strongly anharmonic and its wavelength decreases through transient rearrangements but a stationary state is reached. A time dependent instability is observed when the outer cylinder is set in counter-rotation. The cells become inclined and travel at constant velocity along the  $x$ -axis. At this secondary instability, the reflexion symmetry  $x \rightarrow -x$  is spontaneously broken and the direction of propagation of the cells is determined by the sign of the asymmetry (see Figure 1b). Similar drift instabilities have been observed in many pattern-forming systems, convection in binary fluid mixtures [9], Couette flow between two horizontal cylinders with a partially filled gap [10], directional solidification [11, 12] and the Faraday instability [13, 14].

We will recall how the observations of Couder and coworkers can be understood as a result of the interaction between the primary bifurcating mode at wave number  $k$  and its second harmonic  $2k$ . The order parameter of the secondary bifurcation that spontaneously breaks the reflexion symmetry is  $\Sigma = 2\phi - \theta$  where  $\phi$  (respectively  $\theta$ ) is the phase of the mode at wave number  $k$  (respectively  $2k$ ). Reflexion symmetry is spontaneously broken when  $\Sigma$  bifurcates from zero. One key feature of the drift bifurcation is that the neutral phase mode  $\phi$  that results from broken translational invariance in space at the primary pattern forming bifurcation, is coupled to  $\Sigma$ . This generates the translation at constant velocity of the tilted cells [15]. A similar example has been reported in the time domain and affects the secondary instabilities of limit cycles. The temporal phase of the limit cycle becomes coupled to the order parameter of the secondary bifurcation of the limit cycle that gives rise to intermittency [16]. In all these situations, the effect of the neutral modes is to modify the dynamics observed after the secondary instability onset that becomes of propagative nature. We will first recall this mechanism. Then, we will emphasize another effect of the neutral modes that consists of making the secondary instability subcritical whereas it would have been supercritical in the absence of coupling. This second effect is also observed in the case of other secondary instabilities of periodic patterns, such as the instability that generates

a modulation of the primary pattern at twice its wavelength. Note that throughout the paper, we will use indifferently supercritical (respectively subcritical), or second order (respectively first order) as used for phase transitions.

This paper is organized as follows: we recall in Section 2 how elementary symmetry arguments can be used to determine the form of the equations for the amplitudes of the modes that become unstable above the instability threshold. In particular, we show how the competition between a mode of wave number  $k$  and its second harmonic  $2k$  explains the drift bifurcation observed by the group of Yves Couder. We recall in Section 3 the equations describing the drift bifurcation in general and we show how the coupling with the neutral modes related to the broken symmetry of the primary pattern makes the secondary instability subcritical. We consider in Section 4 the secondary instability of one-dimensional periodic pattern that generates a modulation of the primary pattern at twice its wavelength. A canonical example of this type of instability is the Peierls transition in solid state physics. We show in that case too, that the neutral modes related to broken translational symmetry by the primary pattern can make the secondary transition first order whereas the transition would be of second order in the absence of coupling. We finally conclude in Section 5.

## 2. Elementary considerations using symmetry arguments

### 2.1. Stationary or traveling patterns; the effect of externally broken mirror symmetry

We first consider a hydrodynamic system that undergoes a transition from a homogeneous state to a spatially periodic one with a finite wave number  $k_c$  as displayed in Figure 1a. Slightly above the instability threshold, we write

$$u(x, y, t) = A(t)U_{k_c}(y) \exp ik_c x + \text{c.c.} + \dots, \quad (1)$$

where  $u$  stands for a component of the velocity field for instance, or for the position of the interface between two media,  $k_c$  is the wave number of the most unstable mode, i.e. the one that bifurcates first when some control parameter of the system is varied,  $U_{k_c}$  describes the structure of the mode and  $A(t)$  is its complex amplitude. “c.c.” stands for the complex conjugate of the previous expression.

Close to the instability threshold, the growth rate of the most unstable mode nearly vanishes whereas the other stable modes are linearly damped with a finite time scale. The unstable mode therefore behaves with a much longer time scale than the others. Consequently, all the stable modes adiabatically follow the amplitude of the marginally unstable mode and one expects to be able to express their amplitudes as functions of  $A$  and  $\bar{A}$ . If the leading order non linear terms saturate the growth of the unstable mode,  $A$  remains small in the vicinity of the bifurcation and we expect that adiabatic elimination of stable modes provides an amplitude equation that is well approximated by a power series in  $A$  and  $\bar{A}$ ,

$$\dot{A} = \sum \alpha_{n,p} A^n \bar{A}^p. \quad (2)$$

The form of this equation is constrained by the broken symmetries at the instability threshold [7]. The problem and the base state are spatially homogeneous, i.e. invariant under translation in space,  $x \rightarrow x + x_0$ . This invariance is broken by the solution (1). However, if  $u(x, y, t)$  is a solution, then  $u(x + x_0, y, t)$  is another possible solution. Therefore, if  $A$  is a solution of (2), then  $A \exp ik_c x_0$  should also be a solution whatever the value of  $x_0$ . In other words, the amplitude equation (2) should be invariant by rotation in the complex plane. Therefore, it cannot involve constant or quadratic terms and it is easy to check that the leading order nonlinear term is of cubic order

$$\dot{A} = \alpha_{1,0} A + \alpha_{2,1} A^2 \bar{A}. \quad (3)$$

Further constraints on the coefficients  $\alpha_{i,j}$  exist if the system is mirror symmetric, i.e. invariant under space reflexion  $x \rightarrow -x$ . This transformation amounts to  $A \rightarrow \bar{A}$  in (1). If the system is mirror symmetric, the amplitude equation should be invariant in the transformation  $x \rightarrow -x, A \rightarrow \bar{A}$ . Taking the complex conjugate of the resulting equation and identifying with (3) implies that  $\alpha_{i,j}$  are real coefficients and we get

$$\dot{A} = \mu A - \gamma A^2 \bar{A}. \quad (4)$$

$\mu$  is the linear growth rate of the unstable pattern. It vanishes at the bifurcation threshold and is proportional to the distance from the threshold. The growth rate being real, the amplitude grows in a monotonic way above threshold and the bifurcation is stationary. If  $\gamma > 0$ , it is supercritical. Writing  $A = R \exp i\phi$ , we find that the amplitude  $R_0$  of the stationary solution of (4) increases proportionally to the square root of the distance to threshold,  $R_0 = \sqrt{\mu/\gamma}$ . The phase obeys the equation  $\dot{\phi} = 0$  and is therefore neutral, i.e. it neither grows nor decays. This is also called a collective mode or a Goldstone mode in some other fields of physics. This traces back to the spontaneously broken translational invariance at the instability onset and means that the pattern can be easily shifted along the  $x$ -axis.

We emphasize that a weaker symmetry constraint than mirror symmetry can also lead to an amplitude equation with real coefficients. It is enough to have a system invariant under  $x \rightarrow -x$  combined with another transformation even though not invariant under  $x \rightarrow -x$  alone. For instance in the present problem, invariance under  $x \rightarrow -x, u \rightarrow -u$ , or  $x \rightarrow -x, y \rightarrow y + y_0$ , are both enough to constrain real coefficients in the amplitude equation.

If no such invariance exists, then we generically expect complex coefficients in the amplitude equation

$$\dot{A} = (\mu + i\nu)A - (\gamma + i\delta)A^2 \bar{A}. \quad (5)$$

The instability is oscillatory with frequency  $\nu$  at onset. In other words, we have a Hopf bifurcation for  $\mu = 0$ . In the supercritical case,  $R_0 = \sqrt{(\mu/\gamma)}$  as in the previous situation but  $\phi = (\nu - \mu\delta/\gamma)t$ . Equation (1) therefore describes a pattern that travels along the  $x$ -axis. The two directions along the  $x$ -axis being of different nature, the pattern selects one of them preferentially.

Note that if  $\nu = 0$  and  $\delta \neq 0$ , we get a situation for which linear stability analysis predicts a stationary bifurcation whereas the nonlinearly saturated regime is a wave traveling at a speed proportional to the distance to threshold. Although we expect this situation to be singular, it is observed in the case of instabilities of some parallel flows [17].

## 2.2. *Traveling patterns due to spontaneously broken mirror symmetry*

The film draining experiment performed by Yves Couder and his group has mirror symmetry  $x \rightarrow -x$ , where the  $x$ -axis is along the gap parallel to the axis of the cylinders. We consider the case of counter-rotating cylinders. If one cylinder is slowly rotated, a sequence of transitions are observed when the velocity of the other one is increased in the opposite direction. The flat interface first undergoes a stationary bifurcation and becomes periodically modulated in space. When the velocity is increased further, a secondary bifurcation occurs. Cells become tilted and travel at constant velocity along the  $x$ -axis to the left or the right according to the sign of their tilt (see Figure 1b). This second transition does not occur in a homogeneous way. The first moving cells can appear as localized solitary waves. Domains with cells with opposite tilts therefore traveling to the left and the right can also coexist in the system [18]. These features are characteristic of a subcritical instability.

We proposed to model this secondary bifurcation to drifting cells observed by Couder and coworkers and also in several experiments on directional solidification [11, 12] as a result of the resonant interaction between the bifurcating mode of wave number  $k_c$  and its second

harmonic [19, 20]. This occurs if the second harmonic is weakly damped such that it should be kept in the set of the nearly neutral modes in the vicinity of the bifurcation threshold. We therefore consider

$$u(x, y, t) = A(t)U_{k_c}(y) \exp ik_c x + B(t)U_{2k_c}(y) \exp 2ik_c x + \text{c.c.} + \dots, \quad (6)$$

and write the amplitude equations for  $A$  and  $B$  using symmetry constraints, translational invariance and mirror symmetry. To cubic order, we obtain

$$\dot{A} = \mu A - \bar{A}B - \alpha|A|^2 A - \beta|B|^2 A, \quad (7)$$

$$\dot{B} = -\nu B + \varepsilon A^2 - \gamma|A|^2 B - \delta|B|^2 B. \quad (8)$$

The system of equations (7), (8) arises in many different contexts, thermal convection [21], excitable media [22], von Karman swirling flows [23]. It has been also studied for its own interest [24, 25]. We refer to these last papers for the mathematical aspects and we present here the parameter range of interest for the drift bifurcation.

Slightly above the instability onset, the mode with wave number  $k_c$  is linearly unstable, i.e.  $\mu > 0$ , whereas its second harmonic is linearly damped, i.e.  $\nu > 0$ . The coefficients of the quadratic terms can be taken equal to  $\varepsilon = \pm 1$  by appropriate scaling of the amplitudes; the coefficient of  $\bar{A}B$  can be taken equal to  $-1$ , making the transformation  $A \rightarrow -A, B \rightarrow -B$ , if necessary. Positive values of  $\alpha, \beta, \gamma$  and  $\delta$  ensure global stability.

Writing

$$A = R \exp(i\phi), \quad B = S \exp(i\theta), \quad \Sigma = 2\phi - \theta, \quad (9)$$

we get from (7), (8)

$$\dot{R} = (\mu - \alpha R^2 - \beta S^2)R - RS \cos \Sigma, \quad (10)$$

$$\dot{S} = (-\nu - \gamma R^2 - \delta S^2)S + \varepsilon R^2 \cos \Sigma, \quad (11)$$

$$\dot{\Sigma} = \left(2S - \varepsilon \frac{R^2}{S}\right) \sin \Sigma, \quad (12)$$

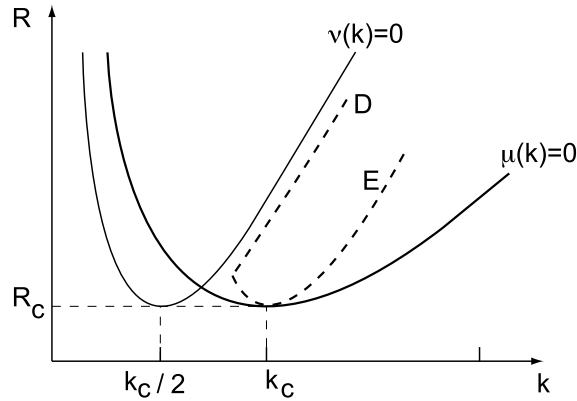
$$\dot{\phi} = S \sin \Sigma. \quad (13)$$

When  $\mu$  becomes positive, the null state bifurcates to a family of stable stationary patterns related to each other by space translation:  $R = R_0 \neq 0, S = S_0 \neq 0, \Sigma = \Sigma_0 = 0$ , and  $\phi$  arbitrary.

Another type of stationary solutions corresponds to  $\Sigma = \Sigma_D \neq 0$ . This implies  $2S - \varepsilon R^2/S = 0$ , and thus  $\varepsilon = 1$ . These patterns are cellular patterns drifting with a constant velocity  $S_D \sin \Sigma_D$  according to (13). Therefore, the coefficients of the quadratic terms must have opposite signs in order to observe the drift instability. Note that this means that the second harmonic does not enhance the stationary instability near onset; indeed, for  $\mu \simeq 0$  and  $\nu > 0$ ,  $B$  follows adiabatically  $A$  ( $B \propto A^2$ ), and the quadratic terms of (7), (8) contribute to saturate the primary instability. Note also that if the coefficients in (7), (8) are such that the system is of potential form ( $\varepsilon = -1/2, \beta = \gamma$ ), the drift bifurcation is not possible as expected. This type of bifurcation cannot occur in systems that minimize some free energy.

The stationary pattern is destabilized when  $2S_0 - R_0^2/S_0$  vanishes as  $\mu$  is increased. This happens if the condition  $1 - \nu(2\gamma + \delta) > 0$  is satisfied, which means that the second harmonic is not strongly damped ( $\nu$  not too large). The system of equations (10)–(12) then undergoes a supercritical pitchfork bifurcation. The two bifurcated stationary states are such that  $R_D^2 = 2S_D^2, \Sigma = \pm \Sigma_D \neq 0$ .

It can be easily checked that  $\Sigma = 2\phi - \theta$  represents the order parameter related to reflexion symmetry. Shifting the origin of the  $x$ -axis by  $-\phi/k_c$ , the pattern (1) becomes of the form  $R \exp ik_c x + S \exp 2ik_c x \exp i(\theta - 2\phi) + \text{c.c.}$  Invariance under the transformation  $x \rightarrow -x$  requires  $\Sigma = 2\phi - \theta = 0[\pi]$ . The bifurcation from the stationary pattern to the traveling one has the following characteristics: Its order parameter,  $\Sigma = 2\phi - \theta$ , undergoes a pitchfork bifurcation that



**Figure 2.** Neutral stability curves  $\mu(k) = 0$  for the mode of wave number  $k$  and  $\nu(k) = 0$  for the mode of wave number  $2k$ . The domain of stable stationary patterns is limited by the Eckhaus instability (E) and the drift bifurcation (D).

spontaneously breaks the basic pattern reflection symmetry. The coupling with the neutral mode  $\phi$  induces the drift motion according to (13), and the direction of propagation is determined by the sign of  $\Sigma$ .

The stability domain of stationary patterns is displayed in Figure 2.  $R$  is the control parameter, for instance the Reynolds number related to the cylinder velocity which is increased. A stationary pattern bifurcates above the neutral stability curve  $R(k)$  because the growth rate of the mode of wave number  $k$ ,  $\mu(k) = R - R(k)$  becomes positive. As usual, the band of stationary patterns above the minimum  $R_c$  of  $R(k)$  is limited by the Eckhaus instability (E) [7]. The growth rate of the mode of wave number  $2k$  is  $\nu(k) = \mu(2k)$ . Therefore the marginal stability curve of the second harmonic,  $\nu(k) = 0$ , has a minimum  $R = R_c$  for  $k = k_c/2$ . When  $R$  is increased above  $R_c$ , the drift bifurcation occurs when the damping rate of the mode of wave number  $2k$  becomes small enough, i.e. slightly below the curve  $\nu(k) = 0$ . This limits the range of stable stationary patterns from above. Note however that the pattern can remain stationary if its wave number is increased such that the damping rate of the second harmonic remains large enough. This is also observed in experiments (see Section 3).

The  $k$ - $2k$  interaction mechanism correctly describes many features of the parity-breaking displayed in the experiment quoted above [18, 26, 27]. As noted in [18], it does not capture a tertiary period doubling bifurcation that occurs for a higher value of the control parameter, but we do not expect that anyway. The  $k$ - $2k$  interaction mechanism has been also proposed in [28] but in a different parameter range that does not describe the results of this experiment.

### 3. The drift bifurcation

#### 3.1. The normal form of the drift bifurcation using symmetry arguments

The structure of the drift bifurcation described above is rather general: a pattern undergoes a secondary instability with a spontaneously broken reflexion symmetry, the order parameter of which becomes coupled with the phase mode that results from broken translational invariance in space. Besides the the  $k$ - $2k$  interaction described above, we have identified another mechanism that involves the same structure in the case of parametrically forced standing waves [13, 20]. The amplitude difference between the two counter-propagating waves plays the role of the order parameter  $V$  in that case. It bifurcates from zero when the amplitude of the standing wave is



increased and breaks reflexion symmetry. The coupling with the spatial phase of the standing wave pattern generates the drift. In all cases, we therefore have at the secondary instability threshold

$$\dot{\phi} = \omega V, \quad (14)$$

$$\dot{V} = \lambda V - \eta V^3. \quad (15)$$

$\eta > 0$  for the  $k$ - $2k$  model, and the bifurcation is supercritical if we take into account only homogeneous perturbations, whereas  $\eta < 0$  in the case of parametrically forced standing waves, and the bifurcation is subcritical. At bifurcation threshold, the instability problem involves a linear operator proportional to

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad (16)$$

that corresponds to a codimension-two bifurcation [29]. This situation is found here by varying only one control parameter because broken translational invariance enforces the existence of a persistent zero eigenvalue [16, 30].

It has been proposed to study the dynamics of the system close to the drift bifurcation by writing coupled equations for  $\phi(X, T)$  and  $V(X, T)$  slowly varying in space and time [15, 31]. To wit, we write

$$u(x, t) = u_0[x + \phi(X, T)] + V(X, T) u_\pi(x) + \dots, \quad (17)$$

where  $V$  is the amplitude of the eigenmode  $u_\pi(x)$  that breaks the reflection symmetry at the secondary instability threshold.  $u_0[x + \phi]$  represents the family of stable stationary patterns related to each other by space translation and responsible for the persistent zero eigenvalue at zero wave number. The form of the equations for  $\phi$  and  $V$  is constrained by symmetry arguments, translational invariance in space ( $\phi \rightarrow \phi + \phi_0$ ), that forbids the existence of terms involving explicitly the phase  $\phi$ , and space reflection symmetry ( $x \rightarrow -x, \phi \rightarrow -\phi, V \rightarrow -V$ ). We obtain to leading orders in the gradient expansion

$$\frac{\partial \phi}{\partial T} = \omega V + \zeta_1 \frac{\partial^2 \phi}{\partial X^2} + \zeta_1 \frac{\partial^2 V}{\partial X^2} + \chi_1 V \frac{\partial \phi}{\partial X} + \rho_1 V \frac{\partial V}{\partial X} + \kappa_1 \frac{\partial \phi}{\partial X} \frac{\partial^2 \phi}{\partial X^2} + \dots, \quad (18)$$

$$\frac{\partial V}{\partial T} = \lambda V - \eta V^3 + \zeta_2 \frac{\partial^2 \phi}{\partial X^2} + \zeta_2 \frac{\partial^2 V}{\partial X^2} + \chi_2 V \frac{\partial \phi}{\partial X} + \rho_2 V \frac{\partial V}{\partial X} + \kappa_2 \frac{\partial \phi}{\partial X} \frac{\partial^2 \phi}{\partial X^2} + \dots. \quad (19)$$

All the terms on the right hand side of (18) except the first one can be removed using a near-identity nonlinear transformation [32]. Keeping the same notations for  $\phi$ ,  $V$  and  $\lambda$  for simplicity, we get equations of the form

$$\frac{\partial \phi}{\partial T} = V + \dots, \quad (20)$$

$$\frac{\partial V}{\partial T} = \lambda V - V^3 + \frac{\partial^2 \phi}{\partial X^2} + \frac{\partial^2 V}{\partial X^2} + \chi V \frac{\partial \phi}{\partial X} + \dots. \quad (21)$$

We have kept the terms of leading orders, chosen appropriate units to scale as many coefficients as possible and assume that the signs of  $\omega$  and of the coefficients of the diffusive terms are such that stability at finite wave number is ensured in the resulting equations. If this is not the case, terms with four  $x$ -derivatives should be kept and the drift bifurcation can be preempted by an oscillatory instability of the pattern [20, 33].

Note that the equations proposed in [15] are (18), (19) with  $\zeta_1 = \chi_1 = \rho_1 = \kappa_1 = \zeta_2 = \rho_2 = \kappa_2 = 0$ . It is not possible to justify this choice using an asymptotic expansion in the vicinity of the bifurcation threshold. Even in the simplest case of (20), (21), it is not possible to obtain all the terms at a given order in the limit of  $\lambda$  small with other coefficients of order one. In addition, a subcritical bifurcation was assumed ( $\eta < 0$ ) in [15] in order to describe traveling tilted domains within the static pattern, as observed in many of the above quoted experiments. We will show that

this is not necessary because the coupling of  $V$  with the phase gradients makes the bifurcation subcritical even for  $\eta > 0$ .

### 3.2. A subcritical bifurcation due to coupling with phase modes

As said above, in the homogeneous case, the  $V = 0$  solution first bifurcates when  $\lambda$  passes through zero and becomes positive. The homogeneous drifting pattern,  $V_0 = \pm\sqrt{\lambda}$ ,  $\phi_0 = V_0 T$ , bifurcates supercritically; however its stability to inhomogeneous disturbances of the form  $\exp(\sigma T + iKX)$ , is governed by the dispersion relation,

$$\sigma^2 + (2\lambda + K^2)\sigma - iK\chi V_0 + K^2 = 0, \tag{22}$$

that shows that the term  $\chi V \partial\phi/\partial X$  destabilizes the homogeneous pattern to long wavelength disturbances independently of the sign of  $\chi$ , since one solution to (22) is a growth rate with a positive real part

$$Re(\sigma) = \left(\frac{\chi}{\lambda}\right)^2 K^2(1 + \mathcal{O}(\lambda)) + o(K^2). \tag{23}$$

This shows that, for a domain with an infinite extension, where disturbances with arbitrarily small  $K$  can develop, the homogeneous drifting pattern is unstable above its bifurcation threshold  $\lambda = 0$  as long as  $\lambda$  remains small enough. This is related to the existence of solutions,  $V = 0$ ,  $\phi = qX$ , of (20), (21) that represent basic patterns with a slightly different wave number. These patterns bifurcate to homogeneous drifting patterns for  $\lambda + \chi q = 0$ , i.e. for a lower threshold in  $\lambda$  if  $\chi q > 0$ . We will show that it is possible to construct traveling solutions of (20), (21), that describe localized drifting regions with tilted cells [34]. These solutions exist below the linear instability threshold of the basic pattern such that we can call them subcritical. In addition, the existence of localized domains of the bifurcated solution that coexist with the basic pattern is characteristic of subcritical bifurcations in extended domains.

We therefore look for traveling solutions of the form  $V = f(X - cT)$ ,  $\phi_X = -f(X - cT)/c$  and get from (20), (21)

$$f'' + \left(c - \frac{1}{c}\right)f' + \lambda f - \frac{\chi}{c}f^2 - f^3 = 0. \tag{24}$$

This equation is that of a particle moving in a potential

$$U(f) = \frac{\lambda}{2}f^2 - \frac{\chi}{3c}f^3 - \frac{f^4}{4}, \tag{25}$$

with a friction coefficient  $c - 1/c$ . In the reference frame moving at velocity  $c$ , a domain boundary separating the basic pattern and the tilted one is stationary provided the dissipation vanishes and  $U$  has two maxima at the same height  $U(f) = 0$  for  $f = 0$  (basic pattern) and  $f = f^* \neq 0$  (tilted pattern), as displayed in Figure 3a. The condition of zero dissipation selects the velocity,  $c^2 = 1$ . The conditions on the potential give

$$\lambda = \lambda^* = -\frac{2}{9}\left(\frac{\chi}{c}\right)^2, \quad f^* = -\frac{2\chi}{3c}, \tag{26}$$

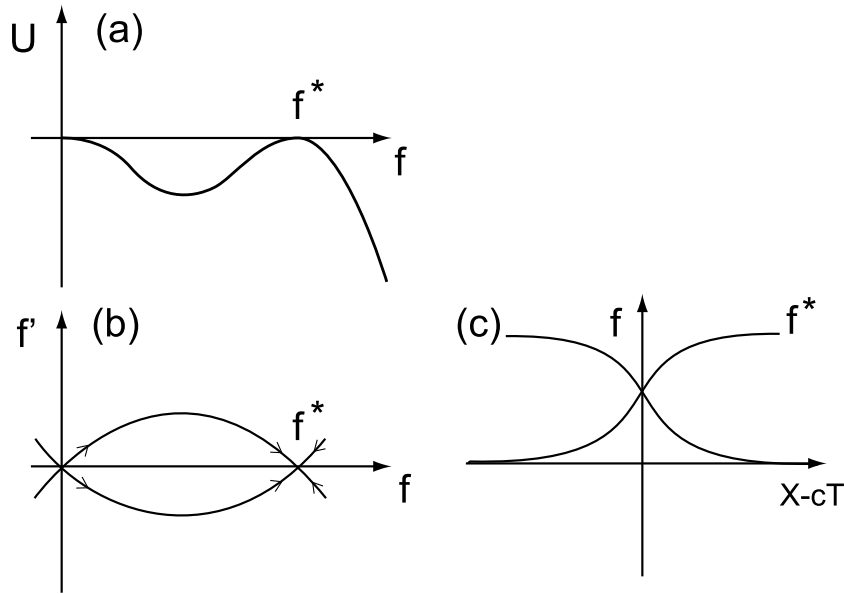
and the potential at the Maxwell plateau  $\lambda = \lambda^*$  is

$$U = U^* = \frac{\lambda^*}{2}f^2 \left(1 - \frac{f}{f^*}\right)^2. \tag{27}$$

For  $\lambda = \lambda^*$ , there are two heteroclinic orbits in the phase space of (24) that connect the tilted domain to the basic pattern (see Figure 3b). They are given by

$$f = \frac{f^*}{1 + \exp[\pm\sqrt{-\lambda^*}(X - cT)]}. \tag{28}$$

These two solutions are related to a domain wall moving at velocity  $c$  between tilted cells and the basic pattern (see Figure 3c). For  $\lambda \approx \lambda^*$ , we can construct solutions describing a tilted domain



**Figure 3.** Kink-type solutions of (24). (a) The shape of the potential  $U(f)$  given by (25) for  $\lambda = \lambda^*$ . (b) Phase space of (24) with the two heteroclinic orbits that connect  $(0, 0)$  and  $(f^*, 0)$ . (c) The corresponding kink-type solutions (28).

within the basic pattern, with two domain walls moving at different velocities such that the tilted domain grows or shrinks [34]. This has been widely observed in experiments close to the drift bifurcation [11, 12, 18, 35]. Inside the tilted domain, the phase gradient is  $\phi_X^* = 2\chi/3c^2$ . If  $\chi < 0$  as for the  $k-2k$  model, the wavelength of the tilted pattern is therefore larger than the one of the basic pattern.

Another experimental observation can be understood in the framework of (20), (21): it is the stationary pattern wavenumber selection often observed as the control parameter is increased. It is observed that a pattern wavenumber modification occurs by nucleation of a transient drifting domain that generates a phase gradient, say  $q$ , and leads to a new periodic pattern with a larger wavenumber if  $q > 0$ . As noted above,  $V = 0$ ,  $\phi = qX$ , is a particular solution of (20), (21) that represents a basic pattern with a slightly different wave number. The damping rate of perturbations in  $V$  becomes  $\lambda + \chi q$ . Consequently the drift bifurcation of this pattern is inhibited for  $q > 0$  if  $\chi < 0$ . Therefore, this pattern remains stationary because of decreased wavelength. Within the framework of the  $k - 2k$  model, this stabilization mechanism is associated to the increase of the second harmonic damping rate when the pattern wavenumber is increased.

#### 4. A spatial period-doubling bifurcation of a one-dimensional periodic pattern

We now consider whether other secondary bifurcations of periodic patterns display similar features as the one reported above for the drift bifurcation: subcriticality induced by coupling with neutral modes. Another secondary bifurcation reported in [11, 18] generates a spatial period doubling of the pattern. It can also be described by (7), (8) when  $\mu < 0$  and  $-\nu > 0$  is increased [21, 28]. Then, the stationary pattern with only wave number  $2k$  can bifurcate to a mixed mode that involves both  $k$  and  $2k$  leading to a pattern with a double fundamental wavelength. In solid state physics, the Peierls transition is a well-known example of spatial period doubling of a one-dimensional lattice [36]. The transition results from a lattice distortion where every other ion

moves closer to one neighbor and further away from the other. The period doubling of the lattice creates a new band gap at smaller wave number. If the band is half-filled by free electrons, the energy saving of the electrons due to the new band gap outweighs the elastic energy cost of the lattice distortion that therefore takes place. The displacement of the ions can of course take place in the two directions, therefore the order parameter  $U$  of this transition is real and breaks the  $x \rightarrow -x$  symmetry (see below). For a phase transition, potential dynamics that minimize some free energy is expected. The Ginzburg–Landau free energy is to leading order

$$F[U(x, t)] = \frac{1}{2} \int \left[ \alpha \left( \frac{\partial U}{\partial x} \right)^2 - \mu U^2 + \frac{\gamma}{2} U^4 \right] dx, \tag{29}$$

that gives

$$\frac{\partial U}{\partial t} = -\frac{\delta F}{\delta U} = \mu U - \gamma U^3 + \alpha \frac{\partial^2 U}{\partial x^2}, \tag{30}$$

where  $\alpha$  and  $\gamma$  are real positive coefficients to ensure stability. A stationary supercritical bifurcation occurs for  $\mu = 0$ . The pattern with double spatial period  $U_0 = \pm \sqrt{\mu/\gamma}$  is stable for  $\mu > 0$ .

However, as explained above, the basic pattern involves neutral modes related to spontaneously broken translational invariance in space, and we therefore expect the spatial phase  $\phi$  to be coupled with  $U$ . The symmetry requirements are  $x \rightarrow x + x_0$  such that only the gradients of  $\phi$  can be involved in the free energy, and  $x \rightarrow -x, U \rightarrow -U, \phi \rightarrow -\phi$ . The leading order nonlinear coupling term is therefore  $U^2 \partial \phi / \partial x$  and we get for the free energy

$$F = \frac{1}{2} \int \left[ \alpha \left( \frac{\partial U}{\partial x} \right)^2 + D \left( \frac{\partial \phi}{\partial x} \right)^2 - \left( \mu + \nu \frac{\partial \phi}{\partial x} \right) U^2 + \frac{\gamma}{2} U^4 \right] dx, \tag{31}$$

where  $D$  is positive and  $\nu$  is a real coefficient. The coupled equations for  $U$  and  $\phi$  are therefore

$$\frac{\partial U}{\partial t} = \left( \mu + \nu \frac{\partial \phi}{\partial x} \right) U - \gamma U^3 + \alpha \frac{\partial^2 U}{\partial x^2}, \tag{32}$$

$$\frac{\partial \phi}{\partial t} = D \frac{\partial^2 \phi}{\partial x^2} - \frac{\nu}{2} \frac{\partial}{\partial x} (U^2). \tag{33}$$

These equations have been written down for the spatial period-doubling instability of a one-dimensional pattern [31]. The only difference for a system without potential dynamics is that the coefficients of the two coupling terms are not necessarily related. The stability of the bifurcated state can be easily studied writing  $U = U_0 + u(x, t)$  and linearizing equations (32), (33). We get

$$\frac{\partial u}{\partial t} = -2\mu u + \alpha \frac{\partial^2 u}{\partial x^2} + \nu U_0 \frac{\partial \phi}{\partial x} + \dots \tag{34}$$

$$\frac{\partial \phi}{\partial t} = D \frac{\partial^2 \phi}{\partial x^2} - \nu U_0 \frac{\partial u}{\partial x} + \dots \tag{35}$$

which leads to the dispersion relation

$$\sigma^2 + (2\mu + (D + \alpha)k^2)\sigma + \mu \left( 2D - \frac{\nu^2}{\gamma} \right) k^2 + \alpha D k^4 = 0. \tag{36}$$

Therefore, the homogeneous bifurcated solution is unstable to long wavelength disturbances if  $\nu^2 > 2\gamma D$ .

The stationary solutions of (33) are

$$\frac{\partial \phi}{\partial x} = \frac{\nu}{2D} U^2 + C, \tag{37}$$

where  $C$  is an integration constant. We get from (32)

$$(\mu + \nu C)U - \left( \gamma - \frac{\nu^2}{2D} \right) U^3 + \alpha \frac{\partial^2 U}{\partial x^2} = 0, \tag{38}$$

which shows that the bifurcation is subcritical for  $\nu^2 > 2\gamma D$ .

In contrast to the drift bifurcation for which coupling with the spatial phase always makes the bifurcation subcritical because of a quadratic nonlinearity, the modification here occurs at cubic order and makes the bifurcation subcritical only if some condition on the coefficients is fulfilled. It has been shown that the same mechanism operates for the zig-zag transition that occurs when repelling particles are confined by a transverse potential in quasi-one-dimensional geometry [37].

Note that when the Fermi wave number is incommensurate with the underlying lattice periodicity, its phase with respect to the lattice should be taken into account such that the order parameter  $Z$  of the Peierls transition is complex. The free energy becomes

$$F = \int \left[ \alpha \left| \frac{\partial Z}{\partial x} \right|^2 + \frac{D}{2} \left( \frac{\partial \phi}{\partial x} \right)^2 - \left( \mu + \nu \frac{\partial \phi}{\partial x} \right) |Z|^2 + \frac{\gamma}{2} |Z|^4 \right] dx. \quad (39)$$

The equations for  $Z$  and  $\phi$  are those that describe the instability of a periodic pattern at a wave number incommensurate with that of the primary pattern [31]. The analysis is similar to that above and shows the same constraint on the coefficients that is required for a subcritical instability, i.e. a first order phase transition. Although many theoretical approaches of the Peierls transition have resulted in second order phase transitions, some experiments have reported a first order transition [38, 39].

## 5. Conclusion

We have shown that neutral modes related to spontaneously broken translation symmetry by a pattern-forming instability can change the nature of secondary instabilities of the pattern by making them subcritical or first order even though they would be supercritical or second order if space dependent fluctuations were discarded. This always occurs when the secondary instability breaks reflexion symmetry. In all the other cases reported in [31], subcriticality results from a condition on the coupling coefficient. The coupling term in the equation for the order parameter  $U$  of the secondary instability is  $\nu U \partial \phi / \partial x$ . It simply traces back to the dependence of the secondary instability threshold on the primary pattern wave number. Indeed, a constant phase gradient  $\partial \phi / \partial x = q$  corresponds to a variation proportional to  $q$  of the primary pattern wave number and shifts the secondary instability growth-rate by an amount  $\nu q$ . Depending on the leading order coupling term in the phase equation, the secondary instability becomes subcritical if this term is linear in  $U$  or possibly subcritical if this term is quadratic in  $U$ . In the latter case, a condition on the amplitude of the coupling coefficients is required. This very general mechanism can provide a simple explanation for the first order Peierls type transitions observed in several experiments.

## Conflicts of interest

The author declares no competing financial interest.

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