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QUADRATIC DOUBLE CENTERS AND THEIR PERTURBATIONS

JEAN–PIERRE FRANÇOISE AND PEIXING YANG

Abstract. This article begins with a full description of the quadratic planar vector fields which display two centers. We follow the method proposed by Chengzhi Li and provide more detailed analysis of the different types of double centers using the classification: Hamiltonian, reversible, Lotka-Volterra, Q_4, currently used for centers of quadratic planar vector fields. We also describe completely the different possible phase portraits and their Poincaré compactification. We show that the double center set is a semi-algebraic set for which we give an explicit stratification (see figure 2). Then we initiate a study of the perturbations within quadratic planar vector fields of the most degenerated case which is the double Lotka-Volterra case. The perturbative analysis is made with the method of successive derivatives of return mappings. As usual, this involves relative cohomology of the first integral which is in that case a rational function. In this case, we have to deal with a kind of "relative logarithmic cohomology" already known in singularity theory. We succeed to compute the first bifurcation function by residue techniques around each centers and they differ from one center to the other.

1. Introduction

We first recall Dulac's theorem on the classification of regular centers of quadratic vector fields. After the important contributions of H. Zoladek ([28]) the different cases are conveniently called: Hamiltonian, Reversible, Lotka-Volterra and Q_4. The center set is an algebraic set, union of strata of different codimensions. It was proved by N.N. Bautin ([1, 2]), that a local perturbation of a linear focus by a quadratic vector field can yield at most three limit cycles. An interesting generalization of the local Hilbert's 16th problem (restricted to quadratic planar vector fields) was proposed by several authors: Try to bound uniformly the number of limit cycle to bifurcate by small perturbations (inside quadratic vector fields) around a center in any fixed component of the center set? After many contributions on that subject (see [5, 6, 7, 8, 18, 26]), there are still some cases missing due to difficulty to investigate the successive bifurcation functions in the reversible case (see [22] for the Hamiltonian stratum of double centers). Indeed, almost nothing is known about the perturbations of the reversible quadratic systems, including the double centers. Although, the articles [25, 21] focuses on examples of reversible double centers

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whose associated invariant curves are elliptic (see also [14]). In the article [21], a situation is uncovered where the perturbation theory displays a configuration $(1, 3)$ (see also [24]).

The set of double centers in quadratic planar vector field was determined in an article by Chengzhi Li ([20]). The list of systems with a double center can be found also in the paper of Zoladek ([28]). We find useful for the subject to include this proof in this article. We follow very closely the method of Chengzhi Li and give more details about the type of centers using the terminology: Hamiltonian, reversible and Lotka-Volterra. Very quickly, we check that Q_4 cannot exist in a double center. We also give a careful analysis of the Poincaré compactification. This part ends with a complete description of the double center set, which is a semi-algebraic set, and of its stratification (see figure 2).

We focus in this article in the perturbations of the highest codimension case that we call the double Lotka-Volterra case "LV+LV". This case has been studied using several techniques like Bautin ideal approach (in [3]), the essential perturbations method of Iliev (see [17, 12]), or the averaging techniques (see [23]). It does not seem possible to extend averaging techniques to reversible systems if they are not isochronous. This is why we develop here the method of successive derivatives introduced in ([11, 15]). In this case, the notion of "relative logarithmic cohomology" (see [19]) appears and this is new for the subject. Actually the first-order perturbation theory reduces to simple residues computation.

2. Quadratic centers

A quadratic vector field near a center is conveniently written in complex notations $z = x + \mathrm{i}y$, see [28]:

$$\dot{z} = (\mathrm{i} + \lambda)z + Az^2 + B \mid z \mid^2 + C\bar{z}^2. \tag{1}$$

with $\lambda, x, y \in \mathbb{R}, (A, B, C) \in \mathbb{C}^3$. The underlying real parameters of the planar vector field are $\lambda, a, a', b, b', c, c'$:

$$\begin{aligned} \dot{x} &= \lambda x - y + ax^2 + bxy + cy^2, \\ \dot{y} &= x + \lambda y + a'x^2 + b'xy + c'y^2, \end{aligned} \tag{2}$$

with the linear relations:

$$\begin{aligned} a + \mathrm{i}a' &= A + B + C \\ b + \mathrm{i}b' &= 2\mathrm{i}(A - C) \\ c + \mathrm{i}c' &= -A + B - C, \\ A &= \frac{1}{4}[a - c + b' + \mathrm{i}(a' - c' - b)] \\ B &= \frac{1}{2}[a + c + \mathrm{i}(a' + c')] \\ C &= \frac{1}{4}[a - c - b' + \mathrm{i}(a' - c' + b)]. \end{aligned}$$

With these variables the Bautin ideal is generated by the four polynomials (with real coefficients):

$$
\begin{aligned}
v_1 &= \lambda \\
v_2 &= \mathrm{Im}(AB) \\
v_3 &= \mathrm{Im}[(2A+\overline{B})(A-2\overline{B})\overline{B}C] \\
v_4 &= \mathrm{Im}[(\mid B\mid^2 - \mid C\mid^2)(2A+\overline{B})\overline{B}^2 C].
\end{aligned}
$$

The components of the center set are then given by:

$$
\begin{aligned}
LV &: \lambda = B = 0 \\
R &: \lambda = \mathrm{Im}(AB) = \mathrm{Im}(\overline{B}^3 C) = \mathrm{Im}(A^3 C) = 0 \\
H &: \lambda = 2A + \overline{B} = 0 \\
Q_4 &: \lambda = (A - 2\overline{B}) = (\mid B\mid - \mid C\mid) = 0.
\end{aligned} \tag{3}
$$

The above computation goes back essentially to Dulac and Kapteyn, see [9, 27, 28]. The usual terminology in the real case is, according to (3) : Hamiltonian H, reversible (or symmetric) R, Lotka-Volterra LV, and co-dimension four Q_4 component of the center set, respectively. Note that in the reversible case, the first two conditions are sufficient.

If we assume $B \neq 0$, performing a suitable rotation and scaling of coordinates, we can suppose $B = 2$. Similarly if $B = 0$ but $A \neq 0$, we take $A = 1$ (LV), and when $A = B = 0$, we take $C = 1$ (Hamiltonian triangle). In the case where $B = 2$, there is a center if and only if the following conditions hold: (i) $A = -1$ (H), (ii) $A = a$ and $C = b$ are real (R), (iii) $A = 4$, $\mid C \mid= 2$ (Codimension 4).

The list of generic quadratic centers looks hence as follows:

- $\dot{z} = -\mathrm{i}z - z^2 + 2\mid z\mid^2 + (b+\mathrm{i}c)\overline{z}^2$, Hamiltonian (H)
- $\dot{z} = -\mathrm{i}z + az^2 + 2\mid z\mid^2 + b\overline{z}^2$, Reversible (R)
- $\dot{z} = -\mathrm{i}z + z^2 + (b+\mathrm{i}c)\overline{z}^2$, Lotka-Volterra (LV)
- $\dot{z} = -\mathrm{i}z + 4z^2 + 2\mid z\mid^2 + (b+\mathrm{i}c)\overline{z}^2, \mid b+\mathrm{i}c\mid = 2$, Codimension 4 ($Q_4$)

3. Double centers

We revisit the list of all double centers for quadratic vector fields. This has been done some years ago by Li Chengzhi (cf.[20]). Following Chengzhi Li's method, we precise his classification by appending the type of different centers, either Lotka-Volterra, Reversible, Hamiltonian or Q_4. One of the key ideas is to fix a system of parameters $(a, b, c; a', b', c')$ of a quadratic planar vector field:

$$
\begin{aligned}
\dot{x} &= -y + ax^2 + bxy + cy^2, \\
\dot{y} &= x + a'x^2 + b'xy + c'y^2,
\end{aligned} \tag{4}
$$

which displays a center at $(0,0)$. We first look for necessary conditions to have a double center. We can thus assume that the vector field displays somewhere another center. We use the invariance of the vector field performing a suitable rotation of coordinates so that the other center is also on the y-axis. We get accordingly another system of parameters $(l, m, n; p, q)$ so that the vector fields write in the following normal form:

$$\begin{aligned} \dot{x} &= -y + lx^2 + mxy + ny^2, \\ \dot{y} &= x + px^2 + qxy. \end{aligned} \tag{5}$$

We should note that it is not always possible to proceed with an affine transformation depending regularly on the parameters. This is a problem, when studying the cyclicity, as in the case of the Kapteyn normal form. It is important to note that once we have proceeded with this rotation, we can no longer use the Kapteyn normal form. The nice aspect about this other choice of rotation is that it brings the other center at the point $(0, \frac{1}{n})$. We are going to write explicitely the conditions for double centers in the parameter set $(l, m, n; p, q)$ of the normal form. Explicit writing of these conditions in the initial parameters looks rather difficult. Also, it should be noticed that the normal form of a quadratic system with a double center is not unique (for instance there are at least two different ways to determine which center is $(0, 0)$).

3.1. **Restriction of the center conditions at** $(0, 0)$**.**

- The Lotka-Volterra conditions for a center are: $B = 0$ or equivalently $l = -n, p = 0$.
- The Hamiltonian conditions are: $2A + \overline{B} = 0$ or equivalently $m = 0, 2l + q = 0$ and the corresponding Hamiltonian is $H = \frac{1}{2}(x^2 + y^2) - lx^2y - \frac{n}{3}y^3 + \frac{p}{3}x^3$.
- The Reversible conditions are:
  (i) $\mathrm{Im}(AB) = 0$ equivalent to
  $$(p - m)(l + n) + p(l - n + q) = 0.$$
  (ii) $\mathrm{Im}(\overline{B}^3 C) = 0$ equivalent to
  $$(p + m)[(l + n)^3 - 3(l + n)p^2] + (l - n - q)[-3p(l + n)^2 + p^3] = 0.$$
- The Q_4 component is given by: $(A - 2\overline{B}) = (\mid B \mid - \mid C \mid) = 0$ which is equivalent to:
  $$5p = m, 5n = -3l + q, 4[(l + n)^2 + p^2] = [(l - n - q)^2 + (p + m)^2].$$

3.2. **Center conditions at** $(0, \frac{1}{n})$**.** Following Chengzhi Li's method, we translate the origin to $(0, \frac{1}{n})$: $x = \xi, y = \eta + \frac{1}{n}$. The vector field displays:

$$\begin{aligned} \dot{\xi} &= \eta + \frac{m}{n}\xi + l\xi^2 + m\xi\eta + n\eta^2, \\ \dot{\eta} &= (1 + \frac{q}{n})\xi + p\xi^2 + q\xi\eta. \end{aligned} \tag{6}$$

The origin must be a linear center and this implies $m = 0$ and $\frac{n+q}{n} < 0$. We set $\frac{n+q}{n} = -\omega^2$. We get accordingly:

$$\begin{aligned} \dot{\xi} &= \eta + l\xi^2 + n\eta^2, \\ \dot{\eta} &= -\omega^2\xi + p\xi^2 + q\xi\eta. \end{aligned} \tag{7}$$

Changing successively $(\xi, \eta) \mapsto (-\xi, -\eta), t \mapsto -t$, we get

$$\begin{aligned} \dot{\xi} &= -\eta + l\xi^2 + n\eta^2, \\ \dot{\eta} &= \omega^2\xi + p\xi^2 + q\xi\eta. \end{aligned} \tag{8}$$

Then we change finaly $\xi \mapsto \frac{1}{\omega}\xi$ and $t \mapsto t/\omega$. This displays:

$$\begin{aligned} \dot{\xi} &= -\eta + \frac{l}{\omega^2}\xi^2 + n\eta^2, \\ \dot{\eta} &= \xi + \frac{p}{\omega^3}\xi^2 + \frac{q}{\omega^2}\xi\eta. \end{aligned} \tag{9}$$

We have thus obtained an expression of the vector field which is analogous to the equation centered at $(0,0)$ where we change the parameters (l,n,p,q) into $(\frac{l}{\omega^2}, n, \frac{p}{\omega^3}, \frac{q}{\omega^2})$. We are thus ready to apply the center equations to this data.

3.3. **Center conditions of $(0,0)$ under the conditions $m = 0, n(n+q) < 0$.**

Proposition 1. *It is impossible that $(0,0)$ be a Q_4-center*

Proof. Consider first the case where $(0,0)$ would be a center of type Q_4. Replacing $m = 0$ into the equations of the Q_4 component yields: $p = m = 0, q = 5n + 3l$ and $4(l+n)^2 = (l-n-q)^2 = (2l+6n)^2$ which gives $l = -2n$, and then $q = -n$ which is obviously in contradiction with the other condition $\frac{n+q}{n} < 0$. So it is impossible that $(0,0)$ would be a Q_4 center. $\square$

Consider next the reversible case for $(0,0)$.

Proposition 2. *The generic component of the reversible case for $(0,0)$ is given by $m = p = 0$, $n(n+q) < 0$.*

Proof. The first condition (i) for $m = 0$ becomes:

$$p(2l + q) = 0.$$

If $2l + q = 0$ and $m = 0$, we recover the Hamiltonian case. We postpone the discussion to the degenerated case. The second possible case is $p = 0$. The two other conditions (ii) and (iii) are then satisfied. The generic reversible component is then given by $m = p = 0$. $\square$

Consider now the Hamiltonian component.

Proposition 3. *The point $(0,0)$ is a Hamiltonian center if and only if $m = 0, 2l + q = 0, n(n+q) < 0$.*

Proof. Indeed the extra condition $m = 0$ is already contained in the Hamiltonian conditions. $\square$

Consider now the Lotka-Volterra case for $(0,0)$.

Proposition 4. *The Lotka-Volterra case for $(0,0)$ is given by $l = -n, p = 0, m = 0, n(n+q) < 0$. It is thus contained in the reversible component.*

Proof. This is obvious from the equations of the Lotka-Volterra case. $\square$

We consider now the degenerated cases. The intersection of LV and R has just been considered above as $LV \subset R$. The only case pending is the case of a reversible Hamiltonian.

Proposition 5. *The intersection of the Hamiltonian and Reversible strata, noted RH is given by $p = 0, m = 0, 2l + q = 0, n(n+q) < 0$; It is a two-dimensional family of Hamiltonian system $H = \frac{1}{2}(x^2 + y^2) - lx^2y - \frac{n}{3}y^3$.*

Proof. In the condition (ii), we set $m = 0$ and this yields

$$p[(l+n)^3 - 3(l+n)p^2] + (l-n-q)[-3p(l+n)^2 + p^3] = 0.$$

If we assume $p \neq 0$ and write with $q = -2l$:

$$[(l+n)^3 - 3(l+n)p^2] + (3l-n)[-3(l+n)^2 + p^2] = 0,$$

which displays:

$$-np^2 + (l+n)^2(-2l+n) = 0,$$
$$np^2 = (l+n)^2(-2l+n),$$

which is impossible because $n(n-2l) < 0$. □

3.4. **Possible configurations of double centers.**

Theorem 6. *The list of quadratic double centers is given by:*

- *The double generic Hamiltonian center denoted "H+H" depending of three parameters, $H = \frac{1}{2}(x^2+y^2) - lx^2y - \frac{n}{3}y^3 + \frac{p}{3}x^3, n(n-2l) < 0$.*
- *The double generic reversible center denoted "R+R", $(m = p = 0, (n+q)n < 0$, depending of three parameters.*
- *The double Lotka-Volterra-reversible "LV+R", $l = -n, p = 0, m = 0, n(n+q) < 0$, which is two-dimensional.*
- *The double reversible-Lotka-Volterra "R+LV", $l = n+q$, $p = m = 0$, $nl < 0$, which is two-dimensional.*
- *The double reversible Hamiltonian center "HR+HR", $p = 0 = m, 2l+q = 0, n(n+q) < 0$, represented by the Hamiltonian family $H = \frac{1}{2}(x^2+y^2) - lx^2y - \frac{n}{3}y^3$.*
- *The double Lotka-Volterra "LV+LV", $l = -n, p = 0, m = 0, 2n+q = 0, n(n+q) < 0$, which is of dimension one.*

Proof. Assume that $(0,0)$ is a generic Hamiltonian center, $m = 0, q = -2l, n(n+q) < 0$, then $\frac{q}{\omega^2} = -2\frac{l}{\omega^2}$ and thus the Hamiltonian displays another center around $(0, \frac{1}{n})$. So this is an example of double center that we denote "H+H". It is a 3 parameter family of double centers:

$$\begin{aligned} \dot{x} &= -\frac{\partial H}{\partial y}, \\ \dot{y} &= \frac{\partial H}{\partial x}, \\ H &= \frac{1}{2}(x^2+y^2) - lx^2y - \frac{n}{3}y^3 + \frac{p}{3}x^3, \\ &\frac{n-2l}{n} < 0. \end{aligned} \tag{10}$$

Consider now that $(0,0)$ is a generic reversible center $m = p = 0, \frac{n+q}{n} < 0$, then it is easy to check that $(0, \frac{1}{n})$ is also a center of reversible type. This is also a 3-dimensional family that we denote "R+R":

$$\begin{aligned} \dot{x} &= -y + lx^2 + ny^2, \\ \dot{y} &= x + qxy. \end{aligned} \tag{11}$$

Consider now that $(0,0)$ is a Lotka-Volterra $l=-n, p=0, m=0, n(n+q)<0$. Then we know that $(0,\frac{1}{n})$ is a reversible center. This is a 2-dimensional component that we denote “LV+R”

$$\begin{aligned} \dot{x} &= -y + l(x^2 - y^2), \\ \dot{y} &= x + qxy. \end{aligned} \tag{12}$$

If we assume $(0,\,\frac{1}{n})$ is a Lotka-Volterra, we should have $\frac{l}{\omega^2}=-n$, $p=0$, $m=0$, $n(n+q)<0$, which is $l=n+q$. It is easy to check $(0,\,0)$ is a center of reversible type. This is a $2-$dimensional component that we denote “R+LV”

$$\begin{aligned} \dot{x} &= -y + lx^2 + (l-q)y^2, \\ \dot{y} &= x + qxy. \end{aligned} \tag{13}$$

We should check if it would be possible that $(0,\,0)$ and $(0,\,\frac{1}{n})$ are both Lotka-Volterra. We should have $l=-n$, $\frac{l}{\omega^2}=-n$, hence $\omega=1$ and thus $2n+q=0$. This is a component of dimension 1 that we note “LV+LV”

$$\begin{aligned} \dot{x} &= -y + l(x^2 - y^2), \\ \dot{y} &= x + 2lxy. \end{aligned} \tag{14}$$

Note that this example can be reduced to a single one by change of scaling so that: $l=1$.

Finally, if we assume that $(0,0)$ is a reversible Hamiltonian center, $p=0=m, 2l+q=0, n(n+q)<0$, it is immediate that $(0,0)$ is also a reversible Hamitonian center. This case of double centers, denoted “HR+HR” is represented by the two-dimensional family of Hamiltonian systems $H=\frac{1}{2}(x^2+y^2)-lx^2y-\frac{n}{3}y^3$.

The dimension mentioned here refers to the above normal form and it should be distinguished from the dimension in the initial moduli space of quadratic vector fields.

□

3.5. **Stationary points in the Poincaré compactification.**

Theorem 7. *The list of global phase portrait for quadratic double centers is given by*

- *For the double center “LV+LV”, $l=-n$, $p=m=0$, $q=-2n$, it has two centers in real domain, and two saddles at infinity. It has an invariant line $y=\frac{1}{2n}$, see figure (a).*
- *For the double center “LV+R”, $l=-n$, $p=m=0$, $n(n+q)<0$, it has two centers in real domain, and two saddles at infinity. It has an invariant line $y=-\frac{1}{q}$, see figure (a).*
- *For the double center “R+LV”, $l=n+q$, $p=m=0$, $nl<0$, it has two centers in real domain, and two saddles at infinity. It has an invariant line $y=-\frac{1}{q}$, see figure (a).*
- *For the double generic center “R+R”, $p=m=0$,*

  i) If $n(n+q)<0$, $l(n+q)>0$, $n(q-l)<0$, it has two centers in real domain, two saddles at infinity. It has an invariant line $y=-\frac{1}{q}$, see figure (a).

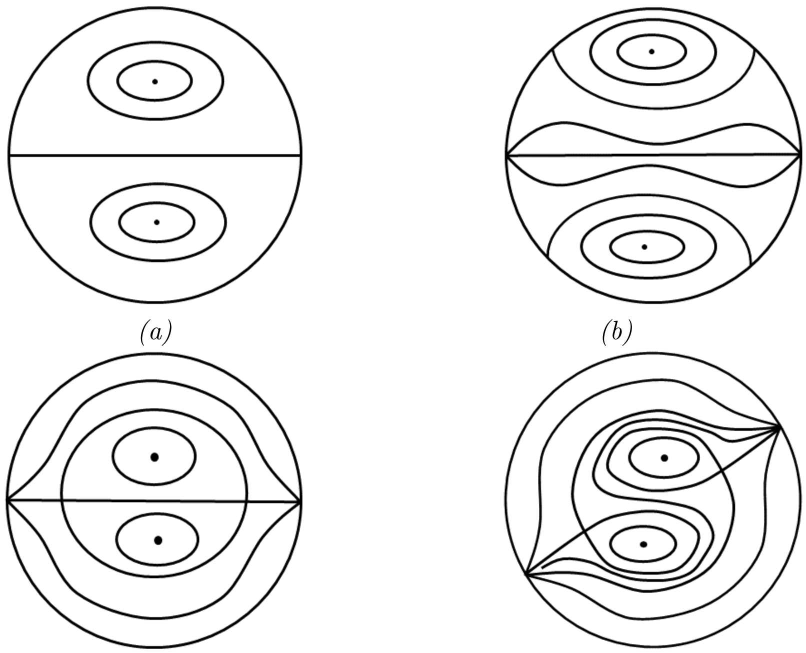

FIGURE 1. Global phase portrait for quadratic double centers. A similar figure appeared in [20].

ii) If $n(n+q) < 0$, $l(n+q) > 0$, $n(q-l) > 0$, it has two centers in real domain, two nodes and four saddles at infinity. It has an invariant line $y = -\frac{1}{q}$, see figure (b).

iii) If $n(n+q) < 0$, $l(n+q) < 0$, it has two centers and two saddles in real domain, two nodes at infinity. It has an invariant line $y = -\frac{1}{q}$, see figure (c).

- *For the double generic Hamiltonian center "H+H", $m = 2l + q = 0$, $n(n+q) < 0$, it has two centers and two saddles in real domain, and two nodes at infinity, see figure (d).*
- *For the double Hamiltonian and reversible center "HR+HR", $p = m = 2l + q = 0$, $n(n+q) < 0$, it has two centers and two saddles in real domain, and two nodes at infinity, It has an invariant line $y = \frac{1}{2l}$, see figure (c).*

Before the proof, for system (5), we make Poincaré transformation

$$x = \frac{1}{z}, \quad y = \frac{u}{z}, \quad dt = zd\tau.$$

Then

$$\begin{aligned} \frac{du}{d\tau} &= p + (q-l)u + z - mu^2 + zu^2 - nu^3, \\ \frac{dz}{d\tau} &= z(-l - mu + zu - nu^2). \end{aligned} \tag{15}$$

With the transformation of $x = \frac{v}{z}, \quad y = \frac{1}{z}, \quad dt = zd\tau$, We obtain

$$\begin{aligned} \frac{dv}{d\tau} &= n - z + mv + (l-q)v^2 - v^2 z - pv^3, \\ \frac{dz}{d\tau} &= -vz^2 - pv^2 z - qvz. \end{aligned}$$

Here because $n \neq 0$, so $(0, 0)$ is not a singular point, in the following discussion, we just consider system (15) at infinity.

Proposition 8. *The double Lotka-Volterra center "LV+LV," has two centers in real domain, and two saddles at infinity. It displays also an invariant line* $y = \frac{1}{2n}$.

Proof. Under this case, we have $l = -n$, $p = m = 0$, $q = -2n$, which is

$$\begin{aligned} \dot{x} &= -y + l(x^2 - y^2), \\ \dot{y} &= x + 2lxy. \end{aligned} \tag{16}$$

It has no other singularity in real domain except two centers $(0, 0)$ and $(0, \frac{1}{n})$.

At infinity, substituting $l = -n$, $p = m = 0$, $q = -2n$ into (15), we have

$$\begin{aligned} \frac{du}{d\tau} &= -nu + z + zu^2 - nu^3, \\ \frac{dz}{d\tau} &= z(n + zu - nu^2). \end{aligned}$$

It has an unique real root $(0, 0)$. And the Jacobian determinant is $-n^2$, so it is a saddle.

So in this case, it has two centers in real domain, and two saddles at infinity, by the way, it has an invariant line $y = \frac{1}{2n}$. See figure (a). $\square$

Proposition 9. *For the Lotka-Volterra-reversible center "LV+R," there are two centers in real domain, and two saddles at infinity. It also displays an invariant line* $y = \frac{1}{2n}$.

Proof. Under this case, we have $l = -n$, $p = m = 0$, $n(n+q) < 0$ which is

$$\begin{aligned} \dot{x} &= -y - n(x^2 - y^2), \\ \dot{y} &= x + qxy. \end{aligned} \tag{17}$$

If $y = -\frac{1}{q}$, we can obtain $x^2 = \frac{n+q}{nq^2}$ from the first equation, combining with $n(n+q) < 0$, it has no other singularity in real domain except two centers $(0, 0)$ and $(0, \frac{1}{n})$.

At infinity, substituting $l = -n$, $p = m = 0$ into (15), we have

$$\begin{aligned} \frac{du}{d\tau} &= (q+n)u + z + zu^2 - nu^3, \\ \frac{dz}{d\tau} &= z(n + zu - nu^2). \end{aligned}$$

It has an unique real root $(0, 0)$, because $q + n = nu^2$ is a contradiction with $n(n+q) < 0$. And the Jacobian determinant at $(0, 0)$ is $n(n+q) < 0$, so it is a saddle.

Hence in this case, it has two centers in real domain, and two saddles at infinity, by the way, it has an invariant line $y = -\frac{1}{q}$. See figure (a). $\square$

Proposition 10. *For the reversible-Lotka-Volterra center "R+LV," it has two centers in real domain, and two saddles at infinity, it has an invariant line* $y = -\frac{1}{q}$.

Proof. Under this case, we have $l = n + q$, $p = m = 0$, $n(n+q) < 0$ which is

$$\begin{aligned} \dot{x} &= -y + (n+q)x^2 + ny^2, \\ \dot{y} &= x + qxy. \end{aligned}$$

If $y = -\frac{1}{q}$, we can obtain $x^2 = -\frac{1}{q^2}$ from the first equation, so it has no other singularity in real domain except two centers (0, 0) and (0, $\frac{1}{n}$).

At infinity, substituting $l = n + q$, $p = m = 0$ into (15), we have

$$\begin{aligned} \frac{du}{d\tau} &= -nu + z + zu^2 - nu^3, \\ \frac{dz}{d\tau} &= z(-n - q + zu - nu^2). \end{aligned}$$

It has an unique real root (0, 0). And the Jacobian determinant at (0, 0) is $n(n+q) < 0$, so it is a saddle.

Hence in this case, it has two centers in real domain, and two saddles at infinity, by the way, it has an invariant line $y = -\frac{1}{q}$. See figure (a). □

Proposition 11. *For the double generic reversible center "R+R," there are three cases as follows,*

- *If* $p = m = 0$, $n(n+q) < 0$, $l(n+q) > 0$, $n(q-l) < 0$, *there are two centers in real domain, two saddles at infinity and an invariant line* $y = -\frac{1}{q}$, *see figure (a).*
- *If* $p = m = 0$, $n(n+q) < 0$, $l(n+q) > 0$, $n(q-l) > 0$, *there are two centers in real domain, two nodes and four saddles at infinity and an invariant line* $y = -\frac{1}{q}$, *see figure (b).*
- *If* $p = m = 0$, $n(n+q) < 0$, $l(n+q) < 0$, *there are two centers and two saddles in real domain, two nodes at infinity and an invariant line* $y = -\frac{1}{q}$, *see figure (c).*

Proof. Here we substitute $p = m = 0$ into system(5), which is

$$\begin{aligned} \dot{x} &= -y + lx^2 + ny^2, \\ \dot{y} &= x + qxy. \end{aligned}$$

If $y = -\frac{1}{q}$, we can obtain $x^2 = -\frac{n+q}{lq^2}$ from the first equation.

If $l(n+q) > 0$, there are no other singularity in real domain except two centers (0, 0) and (0, $\frac{1}{n}$).

If $l(n+q) < 0$, there are four singular points (0, 0), (0, $\frac{1}{n}$), $(\sqrt{-\frac{n+q}{lq^2}}, -\frac{1}{q})$ and $(-\sqrt{-\frac{n+q}{lq^2}}, -\frac{1}{q})$. Here the Jacobian determinant at $(\sqrt{-\frac{n+q}{lq^2}}, -\frac{1}{q})$ and $(-\sqrt{-\frac{n+q}{lq^2}}, -\frac{1}{q})$ are both $-\frac{2(n+q)}{q}$, from $n(n+q) < 0$ we have $nq < 0$ and $q(n+q) > 0$, which implies the other two singular points are saddles.

At infinity, substituting $p = m = 0$ into (15), we have

$$\begin{aligned} \frac{du}{d\tau} &= (q-l)u + z + zu^2 - nu^3, \\ \frac{dz}{d\tau} &= z(-l + zu - nu^2). \end{aligned}$$

When $z = 0$, $u = 0$ or $u^2 = \frac{q-l}{n}$.

If $n(n+q) < 0$, $l(n+q) > 0$, $n(q-l) > 0$, which implies system (5) only has two centers in real domain, and three roots at infinity. The Jacobian determinant at (0, 0) is $-l(q-l)$, the condition $n(n+q) < 0$, $l(n+q) > 0$, $n(q-l) > 0$ gives to $l(q-l) < 0$, so The Jacobian determinant at (0, 0) is positive, the discriminant at (0, 0) is equal to q^2, so (0, 0) is a node. The Jacobian determinant at $(\sqrt{\frac{q-l}{n}}, 0)$ and $(-\sqrt{\frac{q-l}{n}}, 0)$ are both $2q(q-l)$, the condition $n(n+q) < 0$, $n(q-l) > 0$ gives rise to $q(q-l) < 0$, we can obtain $(\sqrt{\frac{q-l}{n}}, 0)$ and $(-\sqrt{\frac{q-l}{n}}, 0)$ are saddles.

Hence under the condition $n(n+q) < 0$, $l(n+q) > 0$, $n(q-l) > 0$, it has two centers in real domain, two nodes and four saddles at infinity, by the way, it has an invariant line $y = -\frac{1}{q}$. See figure (b).

If $n(n+q) < 0$, $l(n+q) > 0$, $n(q-l) < 0$, which implies system (5) only has two centers in real domain, and a unique root (0, 0) at infinity. The Jacobian determinant at (0, 0) is $-l(q-l)$, the condition $n(n+q) < 0$, $l(n+q) > 0$, $n(q-l) < 0$ gives to $l(q-l) > 0$, so the Jacobian determinant at (0, 0) is negative, so (0, 0) is a saddle.

Hence under the condition $n(n+q) < 0$, $l(n+q) > 0$, $n(q-l) < 0$, it has two centers in real domain, two saddles at infinity, by the way, it has an invariant line $y = -\frac{1}{q}$. See figure (a).

If $n(n+q) < 0$, $l(n+q) < 0$, which implies system (5) has two centers and two saddles in real domain, with this condition, we can obtain $n(q-l) < 0$, which implies it has a unique root (0, 0) at infinity. The Jacobian determinant at (0, 0) is $-l(q-l)$, the condition $n(n+q) < 0$, $l(n+q) < 0$, $n(q-l) < 0$ gives to $-l(q-l) > 0$, and the discriminant is equal to q^2, so (0, 0) is a node.

Hence under the condition $n(n+q) < 0$, $l(n+q) < 0$, it has two centers and two saddles in real domain, two nodes at infinity, by the way, it has an invariant line $y = -\frac{1}{q}$. See figure (c). Here we complete the proof. □

Proposition 12. *For the double generic Hamiltonian center "H+H," there are two centers and two saddles in real domain, and two nodes at infinity, and no invariant line.*

Proof. Under this condition, $m = 2l + q = 0$, $n(n+q) < 0$, we have

$$\begin{aligned}\dot{x} &= -y + lx^2 + ny^2,\\ \dot{y} &= x + px^2 - 2lxy.\end{aligned}$$

If $p \neq 0$, we can obtain $x = \frac{2ly-1}{p}$ from the second equation. Substituting the equation into the first equation and simplifying, we can get

$$(4l^3 + np^2)y^2 - (4l^2 + p^2)y + l = 0.$$

The discriminant is $p^2(p^2+4l(2l-n))$, from $n(n+q) < 0$ and $2l+q = 0$, we can derive $nl > 0$, $n(2l-n) > 0$, which implies $l(2l-n) > 0$, so the discriminant is positive. So it must have four singular points (0, 0), $(0, \frac{1}{n})$, $A_1(\frac{2l(4l^2+p^2+\sqrt{p^2(p^2+4l(2l-n))})}{2p(4l^3+np^2)} - \frac{1}{p}, \frac{4l^2+p^2+\sqrt{p^2(p^2+4l(2l-n))}}{2(4l^3+np^2)})$ and $A_2(\frac{2l(4l^2+p^2-\sqrt{p^2(p^2+4l(2l-n))})}{2p(4l^3+np^2)} - \frac{1}{p}, \frac{4l^2+p^2-\sqrt{p^2(p^2+4l(2l-n))}}{2(4l^3+np^2)})$ in the real domain.

The Jacobian determinant at A_1 is

$$
\begin{aligned}
D_1 &= -\frac{\sqrt{p^2(p^2+8l^2-4ln)(lp^2-np^2+l\sqrt{p^2(8l^2-4ln+p^2)})}}{p^2(4l^3+np^2)} \\
&= -\frac{\sqrt{p^2(p^2+8l^2-4ln)((2l-n)p^2+l\sqrt{p^2(8l^2-4ln+p^2)})-lp^2}}{p^2(4l^3+np^2)} \qquad (18)\\
&= -\frac{\sqrt{p^2(p^2+8l^2-4ln)((2l-n)p^2+l(\sqrt{p^2(8l^2-4ln+p^2)})-\sqrt{p^4})}}{p^2(4l^3+np^2)}.
\end{aligned}
$$

Here we assume $2l-n>0$, then we have $n>0$ and $l>0$, which is obtained from $n(n+p)<0$, $p+2l=0$. So under the assumption, we can derive (18) is negative. if we assume $2l-n<0$, we have $n<0$ and $l<0$, and we can get $D_1<0$. Hence A_1 is a saddle.

The Jacobian determinant at A_2 is

$$
\begin{aligned}
D_2 &= -\frac{\sqrt{p^2(p^2+8l^2-4ln)(-lp^2+np^2+l\sqrt{p^2(8l^2-4ln+p^2)})}}{p^2(4l^3+np^2)} \\
&= -\frac{\sqrt{p^2(p^2+8l^2-4ln)(np^2+l\sqrt{p^2(8l^2-4ln+p^2)}-\sqrt{p^4})}}{p^2(4l^3+np^2)}.
\end{aligned}
$$

As the same derivation, we can get D_2 is negative. So A_2 is also a saddle.

At infinity, substituting $m=2l+q=0$ into (15), we have

$$
\begin{aligned}
\frac{du}{d\tau} &= p+(q-l)u+z+zu^2-nu^3, \\
\frac{dz}{d\tau} &= z(-l+zu-nu^2).
\end{aligned}
$$

If $z=0$, we have

$$p+(q-l)u-nu^3=0, \qquad (19)$$

if (19) has a unique zero, if and only if

$$-27p^2n^2+4n(-3l)^3<0.$$

Here from $n(n+q<0)$ and $q+2l=0$ we can obtain $nl>0$. So $-27p^2n^2+4n(-3l)^3<0$ holds, and (19) has a unique root $(u_0,0)$. The Jacobian determinant at $(u_0,0)$ is $3(l+nu_0^2)^2$, and the discriminant is $4(l+nu_0^2)^2$, so $(u_0,0)$ is a node.

If $p=0$, we recover the double Hamiltonian-reversible center.

Hence, for double generic Hamiltonian center, it has two centers and two saddles in real domain, and two nodes at infinity, and it has no invariant line. □

Proposition 13. *For the double Hamiltonian-reversible center "HR+HR," it has two centers and two saddles in real domain, and two nodes at infinity, it has an invariant line* $y=\frac{1}{2l}$.

Proof. Under this condition, $p=m=2l+q=0$, $n(n+q)<0$, we have

$$
\begin{aligned}
\dot{x} &= -y+lx^2+ny^2, \\
\dot{y} &= x-2lxy.
\end{aligned}
$$

If $y = \frac{1}{2l}$, we can obtain $x^2 = \frac{2l-n}{4l^3}$ from the first equation. From $n(n+q) < 0$, $q + 2l = 0$, we get $nl > 0$, $l(n-2l) < 0$, so it must have four singular points $(0, 0)$, $(0, \frac{1}{n})$, $A_3(\sqrt{\frac{2l-n}{4l^3}}, \frac{1}{2l})$ and $A_4(\sqrt{-\frac{2l-n}{4l^3}}, \frac{1}{2l})$ in real domain.

The Jacobian determinant at A_3 and A_4 are both $\frac{n-2l}{l}$, it is negative, so the other two singular points are saddles.

At infinity, substituting $p = m = 2l + q = 0$ into (15), we have

$$\begin{aligned} \frac{du}{d\tau} &= -3lu + z + zu^2 - nu^3, \\ \frac{dz}{d\tau} &= z(-l + zu - nu^2). \end{aligned}$$

If $z = 0$, we have $u = 0$ or $u^2 = -\frac{3l}{n}$, here $nl > 0$, so it has a unique root $(0, 0)$. The Jacobian determinant at $(0, 0)$ is $3l^2$, and the discriminant is $4l^2$, so $(0, 0)$ is a node.

Hence, for double Hamiltonian-reversible center, it has two centers and two saddles in real domain, and two nodes at infinity, and it has an invariant line $y = \frac{1}{2l}$. □

3.6. **Darboux integral for the generic reversible case.** We consider again the generic reversible case:

$$\begin{aligned} \dot{x} &= -y + lx^2 + ny^2, \\ \dot{y} &= x + qxy. \end{aligned}$$

By Dulac's theorem (see [9, 12]), we know that there exists a linear polynomial p_1 and a quadratic polynomial p_2 so that the 1-form associated with the vector field:

$$\omega = (-y + lx^2 + ny^2)dy - (x + qxy)dx = \lambda_1 p_2 dp_1 + \lambda_2 p_1 dp_2.$$

We can explicitely compute these scalars (λ_1, λ_2) and the polynomials p_1, p_2.

Proposition 14. *In the case of a generic reversible double center, the vector field displays the following Darboux integral:*

$$H = (1 + qy)(\frac{l}{q}x^2 + ay^2 + by + c)^{-q/2l},$$

where

$$a = \frac{nl}{q(l-q)},\ b = \frac{2l(n-l+q)}{q(2l-q)(l-q)},\ c = \frac{(n-l+q)}{q(2l-q)(l-q)}.$$

Proof. From the special form of the Darboux integral for reversible case, it is clear that the system should have $p_1 = 0$ as invariant line. So we try $p_1 = 1 + qy$. This yields:

$$\omega = q\lambda_1 p_2 dy + \lambda_2(1 + qy)dp_2.$$

This displays:

$$\begin{aligned} \omega = (-y + lx^2 + ny^2)dy - x(1 + qy)dp_2 = \\ q\lambda_1 p_2 dy + \lambda_2(1 + qy)p'_{2x}dx + \lambda_2(1 + qy)p'_{2y}dy. \end{aligned}$$

By equating the two terms in front of dx, we obtain:

$$\lambda_2 p_2' x = -x,$$

$$p_2 = -\frac{1}{2\lambda_2}x^2 + f(y).$$

Replacing this equality in the terms factorizing dy yields:

$$-y + lx^2 + ny^2 = q\lambda_1(-\frac{1}{2\lambda_2}x^2) + q\lambda_1 f(y) + \lambda_2(1+qy)f'(y).$$

Comparing the two terms in x^2, we get $l/q = -\frac{\lambda_1}{2\lambda_2}$. In fact if H is a Darboux integral, H^k is also a Darboux integral so that we can fix arbitrarily $\lambda_1 = 1$. We then deduce that $\lambda_2 = -\frac{q}{2l}$. We know that $f(y)$ should be quadratic, introduce $f(y) = ay^2 + by + c$ in the equation $-y + ny^2 = q\lambda_1 f(y) + \lambda_2(1+qy)f'(y)$. It is then obvious that it determines (a, b, c) as prescribed in the proposition. $\square$

An important corollary is that not only the system leaves invariant the line $p_1 = 1 + qy = 0$ but also the conic $p_2 = lx^2 + ay^2 + by + c = 0$. The two branches of this conic are nothing else than the two heteroclinic connections between the saddles at infinity.

We explain now how the Darboux integral can be used to describe precisely the geometry of the phase portrait in the three generic cases (a), (b) and (c) of reversible cases with double centers. In this part, we include a further simplification $n = 1$. This is easily achieved after a scaling of variables $(x, y) \to \lambda(x, y)$. Note that with this scaling, the two centers are now $(0, 0)$ and $(0, 1)$. The solutions are given by the curves:

$$\frac{l}{q}x^2 + \frac{l}{q(l-q)}y^2 + by + c = h(1+qy)^{2l/q}, \tag{20}$$

where the parameter h varies within some limits which will be separatedly specified. We now discuss each cases.

- The case (a) corresponds to the conditions $1+q < 0$, $l(1+q) > 0$ (equivalent to $l < 0$) and $q - l < 0$. Note that $1 > \frac{l}{q} > 0$. For all fixed h, we look at the asymptotic behaviour when $(x, y) \to \infty$ of

$$\frac{l}{q}x^2 + \frac{l}{q(l-q)}y^2 + by + c - h(1+qy)^{2l/q}. \tag{21}$$

  The leading term of this function is $\frac{l}{q}x^2 + \frac{l}{q(l-q)}y^2$ because $l/q < 1$. We see that $(x, y) \to \infty$ is impossible and so for all h, the curve is bounded. The dynamics imposes that $\frac{l}{q}x^2 + \frac{l}{q(l-q)}y^2 + by + c - h(1+qy)^{2l/q} = 0$ is a periodic orbit. The limit $h \to \infty$ yields the invariant line $1 + qy = 0$. We obtain the phase portrait (a).
- The case (b) corresponds to the conditions $1+q < 0$, $l(1+q) > 0$ (equivalent to $l < 0$) and $q - l > 0$ which implies $\frac{l}{q} > 1$. For $h = 0$, the curve $\frac{l}{q}x^2 + \frac{l}{q(l-q)}y^2 + by + c = 0$ is an hyperbola. It can be easily checked that the two branches of this hyperbola defines two heteroclinic connexions between the four saddles at infinity. Direct computation shows that the values of h at each centers is negative. So, we have necessarily $h < 0$ on all

the two domains defined above and below the two branches of hyperbolas which are separatrices for the dynamics. We now look at the expression:

$$\frac{l}{q}x^2 + \frac{l}{q(l-q)}y^2 + by + c - h(1+qy)^{2l/q}, \tag{22}$$

when $h < 0$, and its asymptotics when $(x, y) \to \infty$. The leading term is now $\frac{l}{q}x^2 - h(1+qy)^{2l/q}$ and it cannot be compensated by the other terms. This impossibility show that the curves with $h < 0$ are bounded and so they are periodic orbits. The two domains defined by $h < 0$ are periodic annuli with external boundaries given by the two heteroclinic connexions defined by the equation of the hyperbola.

For $h > 0$ on the contrary, the leading term $\frac{l}{q}x^2 - h(1+qy)^{2l/q}$ is the sum of contributions of different signs and they can compensate. The asymptotic shows that the curves have the two limit points which are nodes at infinity. The special solution $1 + qy = 0$ is obtained at the limit $h \to +\infty$.

- The case (c) corresponds to $1+q < 0$ and $l(1+q) < 0$ (equivalent to $l > 0$), we have necessarily $\frac{l}{q} < 0$ and $l > q$. The curve $h = 0, \frac{l}{q}x^2 + \frac{l}{q(l-q)}y^2 + by + c$ is an ellipse and it intersects the invariant line in the two saddles in the finite plane. The special values of $h = h_{(0,0)} = \frac{1-l+q}{q(2l-q)(l-q)}$ and $h = h_{(0,1)} = (1+q)^{-q/2l}\frac{(l+1)(q+1)}{q(2l-q)(l-q)}$, corresponding respectively to the two centers are positive. So inside the ellipse, we have $h > 0$. The leading term when $(x, y) \to \infty$ of the expression (22) is $\frac{l}{q}x^2 + \frac{l}{q(l-q)}y^2$ and it cannot be compensated. So all orbits (except the invariant line) are bounded and thus periodic orbits. We obtain two periodic annuli whose external boundary is the invariant line and the ellipse $h = 0$.

  Consider now the curves so that $h < 0$. As $l/q < 0$, the quantity $-h(1+qy)^{2l/q}$ tends to $+\infty$ as $y \to -1/q$. This term can only be compensated by $x \to \pm\infty$. So the orbits are unbounded. Their asymptotics is given by the leading term $\frac{l}{q}x^2 - h(1+qy)^{2l/q}$. We obtain heteroclinic connexions from $(-\infty, -\frac{1}{q})$ and $(-\frac{1}{q}, +\infty)$. This yields the phase portrait of case (c).

3.7. **The Hamiltonian function H in the Hamiltonian-reversible case.** We have seen the Hamiltonian H in the generic case of the Hamiltonian component:

$$H = \frac{1}{2}(x^2 + y^2) - lx^2y - \frac{n}{3}y^3 + \frac{p}{3}x^3.$$

This Hamiltonian on the intersection Hamiltonian-reversible yields:

$$H = \frac{1}{2}(x^2 + y^2) - lx^2y - \frac{n}{3}y^3.$$

But as the system displays the invariant line $1 - 2ly = 0$ in that case, it is more natural to factorize H in such way:

$$\begin{aligned} H = (1-2ly)[\frac{1}{2}x^2 + \frac{n}{6l}y^2 - \frac{1}{2l}(\frac{1}{2} - \frac{n}{6l})y - \frac{1}{4l^2}(\frac{1}{2} - \frac{n}{6l})] \\ + \frac{1}{4l^2}(\frac{1}{2} - \frac{n}{6l}). \end{aligned}$$

The Hamiltonian function associated to a Hamiltonian system is defined up to a constant. So we can consider as well the first integral

$$H - \frac{1}{4l^2}(\frac{1}{2} - \frac{n}{6l}) =$$
$$(1-2ly)[\frac{1}{2}x^2 + \frac{n}{6l}y^2 - \frac{1}{2l}(\frac{1}{2} - \frac{n}{6l})y - \frac{1}{4l^2}(\frac{1}{2} - \frac{n}{6l})].$$

We should compare this first integral with the Darboux first integral obtained in the reversible case when we fix $2l+q=0$. A easy computation shows that if we fix $2l+q=0$, then the Darboux integral becomes polynomial and that it coincides with

$$-(1-2ly)[\frac{1}{2}x^2 + \frac{n}{6l}y^2 - \frac{1}{2l}(\frac{1}{2} - \frac{n}{6l})y - \frac{1}{4l^2}(\frac{1}{2} - \frac{n}{6l})].$$

We further discuss the limit when a Hamiltonian case approach a Reversible ones. We note that the phase portrait of the Hamiltonian case (a) displays three families of heteroclinic connexions between the two nodes at infinity. Each of these families are separated by two special heteroclinic which are also respectively homoclinic to one of the two saddles in the finite plane. A Hamiltonian Reversible belongs necessarily to the case (c) because $2l+q=0$. What happens of this family as the parameter p goes to zero? A consequence of the previous computation is that we can write the levels of the Hamiltonian as follows:

$$H = (1-2ly)[\frac{1}{2}x^2 + \frac{n}{6l}y^2 - \frac{1}{2l}(\frac{1}{2} - \frac{n}{6l})y - \frac{1}{4l^2}(\frac{1}{2} - \frac{n}{6l})] + \frac{p}{3}x^3 = h.$$

We can compute the level of this Hamiltonian, $h = h(A_1)$ and $h = H(A_2)$ which corresponds respectively to the two homoclinic loops around A_1 and A_2 and see what happens as $p \to 0$. We see that these two homoclinic loops tend to the invariant line $1-2ly=0$ and two sets of heteroclinic connexions which connect the two saddles between themselves and the nodes at infinity. The two points A_1 and A_2 tend to the saddles of the finite plane. This is a quite interesting bifurcation of quadratic double centers which connects case (d) and case (c).

3.8. **The Darboux integral in the Lotka-Volterra center.** We first consider the case LV+ Reversible case where we have to impose $l=-n$. In that case, the Darboux integral yields:

$$H = (1+qy)(\frac{-n}{q}x^2 + ay^2 + by + c)^{q/2n},$$

where

$$a = \frac{n^2}{q(n+q)},\ b = \frac{-2n}{q(n+q)},\ c = \frac{1}{q(n+q)}.$$

This is equivalent to

$$H = (1+qy)[-\frac{n}{q}x^2 + \frac{1}{q(n+q)}(ny-1)^2]^{q/2n},$$

and so there are three invariant lines. Note that the sign condition $n(n+q)<0$ implies that two of these lines are complex (conjugated).

Finally, we consider the case of the double LV-LV case where we add the condition $q=-2n$. This displays a rational first integral:

$$H = \frac{1+2ly}{[\frac{1}{2}x^2 + \frac{1}{2}(y+\frac{1}{l})^2]}.$$

Indeed, it can be easily checked that this is an integral of

$$\begin{aligned} \dot{x} &= -y + l(x^2-y^2), \\ \dot{y} &= x + 2lxy. \end{aligned} \tag{23}$$

4. Geometry of the stratified set of quadratic double centers

We can easily eliminate one parameter by scaling $(x,y)\mapsto(\lambda x,\lambda y)$. By convention, we fix the value of n equal to 1. This fix the two centers to $(0,0)$ and $(0,1)$. Then, the set of double centers is semi-algebraic of (generic) codimension one in the space of parameters (l,q,p). It is the union of two components, the generic reversible set and the generic Hamiltonian case described as follows:

$$\begin{aligned} \dot{x} &= -y + lx^2 + y^2, \\ \dot{y} &= x + qxy + px^2, \end{aligned} \tag{24}$$

The set of reversible centers is given by

$$\begin{aligned} \dot{x} &= -y + lx^2 + y^2, \\ \dot{y} &= x + qxy. \end{aligned} \tag{25}$$

Here, $p=0$ with the condition $n(n+q)=1+q<0$, hence $q<-1$. So this is an open half-plane of equations $p=0, q<-1$. Within this set, there are three subdomains:

- The open stratum (dimension two) defined by $p=0, l(1+q)>0$ and $q<l$ where the phase portrait is of type (a), see figure 2.
- The open stratum (dimension two) defined by $p=0, l(1+q)>0$ and $q>l$ where the phase portrait is of type (b), see figure 2.
- The open stratum (dimension two) defined by $p=0, l(1+q)<0$ where the phase portrait is of type (c), see figure 2.

We have also three strata of dimension one:

- The open stratum (dimension one) defined by $p=0, l(1+q)>0$ and $q=l$ which is adherent both to strata (a) and (b) and is represented by generalized Darboux integral. We find:

$$H = -\frac{1}{4}Y^{-2}x^2 + \int Y^{-3}(-y+y^2)dy,$$

  with $Y=1+qy$.

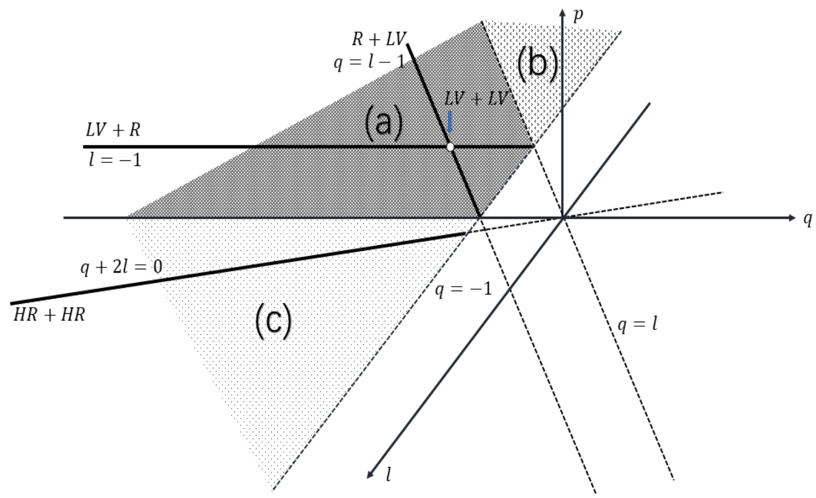

FIGURE 2. Geometry of the stratified set of quadratic double centers

- The open stratum (dimension one) represented by $p = 0, l = 0$ which is in the closure of (a) and (c). These points are also represented by generalized Darboux integrals. In that case, the equation is separable:

$$\frac{dy}{dx} = x\frac{1+qy}{-y+y^2},$$

which yields to the first integral:

$$H = \frac{1}{2q}y^2 - \frac{q+1}{q^2}y - \frac{q+1}{q^3}ln(1+qy) - \frac{1}{2}x^2.$$

- Inside the domain (a), the semi-line $p = 0, l = -1, q < -1$ defines the set LV+R and the semi-line $p = 0$, $q = l - 1$, $l < 0$ defines the set R+LV.

We have one stratum of dimension zero:

- On the semi-line $p = 0, l = -1, q < -1$, the point $(-1, -2, 0)$ corresponds to the LV+LV case.

The set of Hamiltonian centers is given by:

$$\begin{aligned} \dot{x} &= -y + lx^2 + y^2, \\ \dot{y} &= x - 2lxy + px^2. \end{aligned} \tag{26}$$

Hence $q = -2l$, with the condition $l > 1/2$. This is also an half-plane which is transverse to the previous reversible set. The intersection of the two stata reversible and Hamiltonian corresponds to the reversible Hamiltonians. It is one-dimensional, contained in the domain (c) with the equation $q + 2l = 0$, see figure 2.

It should be said that the reversible part of the double-centers set is left globally invariant by the mapping, that we denote $\phi : (x, y) \mapsto (Ax, 1-y), A = \sqrt{-\frac{1}{1+q}}$ which defines an orbital conjugacy:

$$(x, y, t) \mapsto (Ax, 1-y, At) \tag{27}$$

of the vector field:

$$\begin{aligned} \dot{x} &= -y + lx^2 + y^2, \\ \dot{y} &= x + qxy, \end{aligned}$$

into the vector field:

$$\begin{aligned} \dot{x} &= -y + Lx^2 + y^2, \\ \dot{y} &= x + Qxy, \end{aligned}$$

with

$$L = -\frac{l}{1+q}, Q = -\frac{q}{1+q}. \tag{28}$$

This mapping leaves globally invariant each domains of the double center set; It exchanges LV+R into R+LV. Note that (l, q) and (L, Q) are colinear with $(0, 0)$. The fixed point set of the mapping is the line $q = -2$ which contains the point LV+LV. In restriction to this set the mapping is an involution and thus the line $q = -2$ coincides with the set of double centers which are invariant by the involution $(x, y) \mapsto (x, 1-y)$. It determines another special point in the part (c) of figure 2, which is the intersection of this line $q = -2$ with the line HR+HR: $(q, l, n) = (-2, 1, 1)$. This point corresponds to the Hamiltonian:

$$H = (1-2y)[\frac{1}{2}x^2 + \frac{1}{6}(y^2 - y) + \frac{1}{12}], \tag{29}$$

which defines a family of elliptic curves. This case was considered in [22].

To be complete, a full perturbation theory of the double centers within the quadratic planar vector fields should consider perturbations of all the different strata of the stratified set described above. This has not been done yet. We propose here to focus on the perturbation of the highest codimension case, the double LV+LV system.

5. Perturbation theory of the double LV+LV center

In this section, we focus on the perturbation theory, within quadratic planar vector fields, of double "LV+LV"centers. For that purpose, we need to explicit the integrating factor. This is also the opportunity to provide another way to find the first integral. The full perturbation theory of reversible double centers is certainly much harder than what is done here but we can hope to get some insight of the general case by analyzing the simplest case.

5.1. **The integrating factor.** We are concerned here with the 1-form:

$$\omega_0 = (-y + lx^2 + y^2)dy - (x + qxy)dx.$$

With $X = \frac{1}{2}x^2$, we get:

$$\omega_0 = (-y + 2lX + y^2)dy - (1 + qy)dX = (-y + y^2)dy + [2lXdy - (1 + qy)dX].$$

We look for an integrating factor of $[2lXdy-(1+qy)dX]$ of the form $(1+qy)^\alpha, \alpha \in \mathbb{R}$. Choose $\alpha = -\frac{2l}{q} - 1$, so that:

$$[2lX(1 + qy)^\alpha dy - (1 + qy)^{\alpha+1}dX] = d(-(1 + qy)^{\alpha+1}X) = d(-(1 + qy)^{-\frac{2l}{q}}X).$$

This yields:

$$(1+qy)^\alpha\omega_0 = (1+qy)^\alpha(-y+y^2)dy+d(-(1+qy)^{\alpha+1}X) = d[(1+qy)^{\alpha+1}(Ay^2+By+C-X)],$$

where $A = \frac{1}{q(3-\alpha)}$, $B = \frac{\alpha q-3q-2}{q^2(\alpha-2)(\alpha-3)}$, $C = \frac{\alpha q-3q-2}{q^3(\alpha-3)(\alpha-2)(\alpha-1)}$, or in the initial notations:

$$(1 + qy)^{-\frac{2l}{q}-1}\omega_0 = d[(1 + qy)^{-\frac{2l}{q}}(Ay^2 + By + C - \frac{1}{2}x^2)].$$

This yields:

$$A = \frac{1}{2(q-l)}, B = -[\frac{q-l+1}{(q-l)(q-2l)}], C = -[\frac{q-l+1}{2l(q-l)(q-2l)}].$$

In the following, we consider more particularly, the LV+LV case: $(l, q, p) = (-1, -2, 0)$:

$$\begin{aligned} \dot{x} &= -y - x^2 + y^2, \\ \dot{y} &= x - 2xy. \end{aligned}$$

5.2. **The first-order bifurcation function.** This displays:

$$\begin{aligned} (1-2y)^{-2}\omega_0 &= dH, \\ H &= (2y-1)^{-1}[\frac{1}{2}x^2 + \frac{1}{2}y^2 - \frac{1}{4}y + \frac{1}{8}]. \end{aligned} \tag{30}$$

So that the integrating factor is $\psi = (1 - 2y)^{-2}$ and the level sets $H = h$ are circles centered at $(0, \frac{1}{4} + 2h)$ of radius $\rho^2 = (\frac{1}{4} + 2h)^2 - (\frac{1}{4} + 2h)$. The domain of variations of h is defined by $(\frac{1}{4}+2h)^2-(\frac{1}{4}+2h) \geq 0$ or equivalently by the union of the two intervals $h \leq -1/8$ and $h \geq 3/8$ and the circle is reduced to a point $(0, 0)$ for $h = -\frac{1}{8}$ and respectively $(0, 1)$ for respectively $h = \frac{3}{8}$.

We are thus concerned with the first-order Melnikov function $M_1(h)$ (also called the first-order bifurcation function) of system (30) with quadratic polynomial perturbation:

$$\omega_0 + \epsilon\omega_1 \tag{31}$$

with

$$M_1(h) = \oint_{H=h} \frac{1}{(2y-1)^2}\omega_1 \tag{32}$$

where

$$\omega_1 = \sum_{0\le i+j\le 2} (a_{ij}x^i y^j dy + b_{ij}x^i y^j dx).$$

First we denote $\omega_{ij}^k = \frac{x^i y^j}{(2y-1)^k}dx$ and $\delta_{ij}^k = \frac{x^i y^j}{(2y-1)^k}dy$. Then the first-order Melnikov function can be written as

$$\begin{aligned} M_1(h) = \oint_{H=h} & a_{00}\delta_{00}^2 + a_{01}\delta_{01}^2 + a_{02}\delta_{02}^2 + a_{10}\delta_{10}^2 + a_{11}\delta_{11}^2 + a_{20}\delta_{20}^2 \\ & + b_{00}\omega_{00}^2 + b_{01}\omega_{01}^2 + b_{02}\omega_{02}^2 + b_{10}\omega_{10}^2 + b_{11}\omega_{11}^2 + b_{20}\omega_{20}^2. \end{aligned} \tag{33}$$

At this point, it is important to recall the notion of "relative cohomology" which is closely related with the technics of computing bifurcation functions by the algorithm of the successive derivatives (cf. [11, 15]).

Definition 1. *Let ω be a polynomial 1-form. We say that ω is relatively exact (with respect to the function H) on an open set U if there are two analytic functions g and R, defined on the open set U such that $\omega = gdH + dR$. More generally, we say that the two 1-forms ω and ω' are relatively cohomologous with respect to the function H on the domain U, if there are two analytic functions g and R such that $\omega - \omega' = gdH + dR$.*

Lemma 15. *The 1−forms $\delta_{00}^2, \delta_{01}^2, \delta_{02}^2, \delta_{20}^2, \omega_{10}^2, \omega_{11}^2$ are relatively exact (w.r. to H) on the open period annuli defined by H:*

$$\begin{aligned} &\delta_{00}^2 = d(\tfrac{1}{2(1-2y)}),\\ &\delta_{01}^2 = d\Big(\tfrac{1}{4(1-2y)} + \tfrac14 ln|2y-1|\Big),\\ &\delta_{02}^2 = d(\tfrac14 y + \tfrac14 ln|2y-1| - \tfrac{1}{8(2y-1)}),\\ &\delta_{20}^2 = -ln|2y-1|dH + d\Big(Hln|2y-1| - \tfrac14 y - \tfrac18 ln|2y-1| + \tfrac{1}{8(2y-1)}\Big),\\ &\omega_{10}^2 = \tfrac{2}{2y-1}dH + d(-\tfrac{H}{2y-1} - \tfrac14 ln|2y-1| + \tfrac18\tfrac{1}{2y-1}),\\ &\omega_{11}^2 = \Big(\tfrac{y}{2y-1} - \tfrac12 ln|2y-1| + \tfrac{1}{2(2y-1)}\Big)dH\\ &+d\Big(\tfrac12 ln|2y-1|H - \tfrac{H}{2(2y-1)} + \tfrac{1}{16(2y-1)} - \tfrac{3}{16}ln|2y-1| - \tfrac14 y\Big). \end{aligned} \tag{34}$$

Furthermore, the 1-forms $\delta_{11}^2, \omega_{00}^2, \omega_{01}^2, \omega_{02}^2, \omega_{20}^2$ are relatively cohomologous to combinations of $\delta_{10}^1, \delta_{10}^2$, and δ_{10}^3:

$$\begin{aligned} &\delta_{11}^2 = \tfrac12\delta_{10}^2 + \tfrac12\delta_{10}^1\\ &\omega_{00}^2 = d(\tfrac{x}{(2y-1)^2}) + 4\delta_{10}^3,\\ &\omega_{01}^2 = d(\tfrac{xy}{(2y-1)^2}) + \delta_{10}^2 + 2\delta_{10}^3,\\ &\omega_{02}^2 = d(\tfrac{xy^2}{(2y-1)^2}) + \delta_{10}^2 + \delta_{10}^3,\\ &\omega_{20}^2 = \tfrac{x}{2y-1}dH + 2H\delta_{10}^2 - \tfrac14\delta_{10}^2 - \tfrac12\delta_{10}^1. \end{aligned} \tag{35}$$

Proof. Here we omit some decompositions which are obvious. For δ_{ij}^2 and ω_{ij}^2, we expand them at $y = \frac12$, and then derive them as follows:

$$\begin{aligned}
\delta_{11}^2 &= \frac{1}{2}\frac{x}{(2y-1)^2}dy + \frac{1}{2}\frac{x}{2y-1}dy = \frac{1}{2}\delta_{10}^2 + \frac{1}{2}\delta_{10}^1, \\
\delta_{20}^2 &= \frac{2H(2y-1) - y^2 + \frac{1}{2}y - \frac{1}{4}}{(2y-1)^2}dy \\
&= Hd(ln|2y-1|) + \frac{\frac{1}{2}y - y^2 - \frac{1}{4}}{(2y-1)^2}dy \\
&= -ln|2y-1|dH + d(Hln|2y-1| - \frac{1}{4}y - \frac{1}{8}ln|2y-1| + \frac{1}{8(2y-1)}), \\
\omega_{00}^2 &= d(\frac{x}{(2y-1)^2}) - xd(\frac{1}{(2y-1)^2}) = d(\frac{x}{(2y-1)^2}) + 4\delta_{10}^3, \\
\omega_{01}^2 &= d(\frac{xy}{(2y-1)^2}) - xd(\frac{y}{(2y-1)^2}) = d(\frac{xy}{(2y-1)^2}) + \delta_{10}^2 + 2\delta_{10}^3, \\
\omega_{02}^2 &= d(\frac{xy^2}{(2y-1)^2}) - xd(\frac{y^2}{(2y-1)^2}) = d(\frac{xy^2}{(2y-1)^2}) + \delta_{10}^2 + \delta_{10}^3, \\
\omega_{10}^2 &= \frac{1}{(2y-1)^2}d(\frac{1}{2}x^2) \\
&= \frac{1}{(2y-1)^2}d(H(2y-1) - \frac{1}{2}y^2 + \frac{1}{4}y - \frac{1}{8}) \\
&= \frac{1}{2y-1}dH + \frac{2H}{(2y-1)^2}dy - \frac{y}{(2y-1)^2}dy + \frac{1}{4(2y-1)^2}dy \\
&= \frac{2}{2y-1}dH + d(-\frac{H}{2y-1} - \frac{1}{4}ln|2y-1| + \frac{1}{8}\frac{1}{2y-1}), \\
\omega_{20}^2 &= \frac{x}{(2y-1)^2}d(\frac{1}{2}x^2) \\
&= \frac{x}{(2y-1)^2}d(H(2y-1) - \frac{1}{2}y^2 + \frac{1}{4}y - \frac{1}{8}) \\
&= \frac{x}{2y-1}dH + 2H\delta_{10}^2 - \frac{1}{4}\delta_{10}^2 - \frac{1}{2}\delta_{10}^1,
\end{aligned}$$

$$\begin{aligned}
\omega_{11}^2 &= \frac{y}{(2y-1)^2}d(\frac{1}{2}x^2) \\
&= \frac{y}{(2y-1)^2}d(H(2y-1) - \frac{1}{2}y^2 + \frac{1}{4}y - \frac{1}{8}) \\
&= \frac{y}{2y-1}dH + \frac{2Hy}{(2y-1)^2}dy - \frac{y^2}{(2y-1)^2}dy + \frac{y}{4(2y-1)^2}dy \\
&= (\frac{y}{2y-1} + \frac{1}{2(2y-1)} + \frac{1}{2}ln|2y-1|)dH \\
&+ d(\frac{1}{2}ln|2y-1|H - \frac{1}{2}\frac{H}{2y-1} + \frac{1}{16(2y-1)} - \frac{3}{16}ln|2y-1| - \frac{1}{4}y).
\end{aligned}$$

This completes the proof of Lemma 15. □

The presence of the term $ln|2y-1|$ justifies the name "relative logarithmic cohomology".

An easy calculation with Lemma 15 gives the next lemma:

Lemma 16. *The 1-form $\frac{\omega_1}{(2y-1)^2}$ can be decomposed into*

$$\frac{\omega_1}{(2y-1)^2} = g(x,\,y)dH + dR(x,\,y) + N(x,\,y),$$

where $g(x,\,y)$, $R(x,\,y)$ are functions in x and y and $N(x,\,y)$ is a 1-form combination of the three forms $\delta_{10}^i, i = 1, 2, 3$:

$$g(x,\,y) = -(a_{20} + \frac{1}{2}b_{11})ln|2y-1| + 2b_{10}\frac{1}{2y-1} + \frac{1}{2}b_{11}\frac{2y+1}{2y-1} + b_{20}\frac{x}{2y-1},$$

$$\begin{aligned} R(x,\,y) = &\frac{1}{2y-1}(\frac{1}{8}a_{20} + \frac{1}{8}b_{10} + \frac{1}{16}b_{11} - \frac{1}{2}a_{00} - \frac{1}{4}a_{01} - \frac{1}{8}a_{02}) \\ &+ \frac{x}{(2y-1)^2}(b_{00} + b_{01}y + b_{02}y^2) + \frac{1}{4}y(a_{02} - a_{20} - b_{11}) \\ &+ ln|2y-1|(\frac{1}{4}a_{01} + \frac{1}{4}a_{02} - \frac{1}{8}a_{20} - \frac{1}{4}b_{10} - \frac{3}{16}b_{11}) \\ &+ Hln|2y-1|(a_{20} + \frac{1}{2}b_{11}) - \frac{H}{2y-1}(b_{10} + \frac{1}{2}b_{11}) \end{aligned}$$

and

$$\begin{aligned} N(x,\,y) = &\, a_{10}\delta_{10}^2 + a_{11}(\frac{1}{2}\delta_{10}^2 + \frac{1}{2}\delta_{10}^1) + 4b_{00}\delta_{10}^3 + b_{01}(\delta_{10}^2 + 2\delta_{10}^3) \\ &+ b_{02}(\delta_{10}^2 + \delta_{10}^3) + b_{20}(2H\delta_{10}^2 - \frac{1}{4}\delta_{10}^2 - \frac{1}{2}\delta_{10}^1) \\ = &\,(\frac{1}{2}a_{11} - \frac{1}{2}b_{20})\delta_{10}^1 + (a_{10} + \frac{1}{2}a_{11} + b_{01} + b_{02} - \frac{1}{4}b_{20})\delta_{10}^2 \\ &+ 2b_{20}H\delta_{10}^2 + (4b_{00} + 2b_{01} + b_{20})\delta_{10}^3. \end{aligned} \tag{36}$$

From Lemma 15, Lemma 16 and the equation (33), we can simplify $M_1(h)$ into

$$\begin{aligned} M_1(h) = \oint_{H(x,\,y)=h} N = &\,(\frac{1}{2}a_{11} - \frac{1}{2}b_{20})I_1(h) + (a_{10} + \frac{1}{2}a_{11} + b_{01} + b_{02} - \frac{1}{4}b_{20})I_2(h) \\ &+ 2b_{20}hI_2(h) + (4b_{00} + 2b_{01} + b_{20})I_3(h) \end{aligned} \tag{37}$$

where

$$\begin{aligned} I_1(h) &= \oint_{H(x,\,y)=h} \delta_{10}^1, \\ I_2(h) &= \oint_{H(x,\,y)=h} \delta_{10}^2, \\ I_3(h) &= \oint_{H(x\,y=h)} \delta_{10}^3. \end{aligned}$$

Theorem 17. *We can get the generators $I_i(h)(i = 1,\,2,\,3)$ by direct computation of residues as follows,*

$$
\begin{aligned}
I_1(h) &= 2\pi h + \frac{\pi}{4}, \quad h < -\frac{1}{8}\\
I_1(h) &= 2\pi h - \frac{3\pi}{4}, \quad h > \frac{3}{8}\\
I_2(h) &= -2\pi h - \frac{\pi}{4}, \quad h < -\frac{1}{8}\\
I_2(h) &= 2\pi h - \frac{3\pi}{4}, \quad h > \frac{3}{8}\\
I_3(h) &= (4h^2 - h - \frac{3}{16})(-\pi), \quad h < -\frac{1}{8}\\
I_3(h) &= (4h^2 - h - \frac{3}{16})(\pi), \quad h > \frac{3}{8}.
\end{aligned} \tag{38}
$$

Proof. We consider the complex coordinates

$$
\begin{aligned}
z &= x + \mathrm{i}(y - y_0) = R\mathrm{e}^{\mathrm{i}\theta}\\
\overline{z} &= x - \mathrm{i}(y - y_0) = R\mathrm{e}^{-\mathrm{i}\theta},
\end{aligned}
$$

with

$$
\begin{aligned}
R &= \frac{1}{4}\sqrt{(1+8h)(8h-3)},\\
y_0 &= \frac{1}{4} + 2h\\
y_0(y_0 - 1) &= R^2.
\end{aligned}
$$

Along $H = h$, we can assume that $\overline{z} = R^2/z$. We can thus write:

$$
\delta_{10}^1 = \frac{1}{4}\frac{(z^2+R^2)^2}{z^2(z+\mathrm{i}y_0)(z+\mathrm{i}(y_0-1))} = F_1(z)dz.
$$

We fix $h < -\frac{1}{8}$ and consider the circle C_- of radius $R = \frac{1}{4}\sqrt{(1+8h)(8h-3)}$. The rational form $F_1(z)dz$ has 3 poles:

- $z = 0$ or $(x, y) = (0, y_0)$, which is the center of the circle $C_{-1} = \delta D_{-1}$, hence inside the disk D_-,
- $z = -\mathrm{i}y_0$ or $(x, y) = (0, 0)$, which is a stationary point, also inside the disk D_-,
- $z = -\mathrm{i}(y_0 - 1)$ or $(x, y) = (0, 1)$, the other stationary point outside of the disk D_-.

Hence by Cauchy's residue theorem, we get:

$$
I_1(h) = \int_{C_-} F_1(z)dz = 2\pi\mathrm{i}\{[\mathrm{res}(F_1(z), z = 0)] + [\mathrm{res}(F_1(z), z = -\mathrm{i}y_0)]\}.
$$

We find,

$$
2\pi\mathrm{i}[\mathrm{res}(F_1(z), z = 0)] = \frac{\pi}{2}(4h - \frac{1}{2}),
$$

and

$$2\pi\mathrm{i}[\mathrm{res}(F_1(z), z = -\mathrm{i}y_0)] = \frac{\pi}{2},$$

$$I_1(h) = \int_{C_-} F_1(z)dz = 2\pi h + \frac{\pi}{4}, h < -\frac{1}{8}.$$

We now fix $h > \frac{3}{8}$ and consider the circle $C_+ = \delta D_+$ centered at $(0, y_0)$ of radius $R = \frac{1}{4}\sqrt{(1+8h)(8h-3)}$.

- The point $z = 0$ or $(x, y) = (0, y_0)$, which is the center of the circle $C_{-1} = \delta D_{-1}$, is inside the disk D_-,
- $z = -\mathrm{i}y_0$ or $(x, y) = (0, 0)$, which is a stationary point, is outside of the disk D_-,
- $z = -\mathrm{i}(y_0 - 1)$ or $(x, y) = (0, 1)$, the other stationary point is inside of the disk D_-.

Hence by Cauchy's residue theorem, we get:

$$I_1(h) = \int_{C_-} F_1(z)dz = 2\pi\mathrm{i}\{[\mathrm{res}(F_1(z), z = 0)] + [\mathrm{res}(F_1(z), z = -\mathrm{i}(y_0 - 1))]\}.$$

The first residue was computed above. The second one is given by:

$$2\pi\mathrm{i}[\mathrm{res}(F_1(z), z = -\mathrm{i}(y_0 - 1))] = -\frac{\pi}{2},$$

and thus:

$$I_1(h) = \int_{C_-} F_1(z)dz = 2\pi h - \frac{3\pi}{4}, h > \frac{3}{8}.$$

The 1-form $\delta_{10}^2 = \frac{x}{(2y-1)^2}dy$ can be written (along $H = h$) $\delta_{10}^2 = F_2(z)dz$:

$$F_2(z) = \frac{1}{4\mathrm{i}} \frac{z^2(z + \frac{R^2}{z})(1 + \frac{R^2}{z^2})}{[-\mathrm{i}z^2 + (2y_0 - 1)z + \mathrm{i}R^2]^2},$$

$$F_2(z) = \frac{\mathrm{i}}{4} \frac{(z^2 + R^2)^2}{z[(z + \mathrm{i}y_0)(z + \mathrm{i}(y_0 - 1)]^2}.$$

We first assume that $h < -\frac{1}{8}$ and are concerned with the disk D_- inside which $F_2(z)$ displays two poles $z = 0$ and $z = -\mathrm{i}y_0$.

We find easily that

$$\mathrm{Res}(F_2(z), z = 0) = \frac{\mathrm{i}}{4},$$

and then the residue:

$$\mathrm{Res}(F_2(z), z = -\mathrm{i}y_0) = \frac{\mathrm{i}}{4}[\frac{-2(-y_0^2 + R^2)^2}{y_0} + 3y_0^2 - 2R^2 - \frac{R^4}{y_0^2}]$$

$$\mathrm{Res}(F_2(z), z = -\mathrm{i}y_0) = \frac{\mathrm{i}}{4}[(-\frac{1}{2} - 4h) + (8h)]$$

So Cauchy's residue formula yields:

$$I_2(h) = \int_{H=h} F_2(z)dz = -\frac{\pi}{4} - 2\pi h, h < -\frac{1}{8}.$$

Next we fix $h > \frac{3}{8}$ and are concerned with the upper disk D_+ inside which $F_2(z)$ displays the two poles $z = 0$ and $z = -\mathrm{i}(y_0 - 1)$. The first residue for $z = 0$ is the same as above:

$$\mathrm{Res}(F_2(z), z = 0) = \frac{\mathrm{i}}{4}.$$

The second residue is

$$\mathrm{Res}(F_2(z), z = -\mathrm{i}(y_0-1)) = \frac{\mathrm{i}}{4}[\frac{2[(-(y_0-1)^2+R^2]^2}{(y_0-1)} + 3(y_0-1)^2 - 2R^2 - \frac{R^4}{(y_0-1)^2}],$$

$$\mathrm{Res}(F_2(z), z = -\mathrm{i}(y_0 - 1)) = \frac{\mathrm{i}}{4}[\frac{1}{2}(8h-3) + (2-8h)].$$

And Cauchy's residue formula yields:

$$I_2(h) = \int_{H=h} F_2(z)dz = 2\pi h - \frac{3\pi}{4}, h > \frac{3}{8}.$$

The 1-form $\delta_{10}^3 = \frac{x}{(2y-1)^3}dy$ can be written (along $H = h$) $\delta_{10}^3 = F_3(z)dz$:

$$F_3(z) = \frac{1}{4\mathrm{i}} \frac{z^3(z+\frac{R^2}{z})(1+\frac{R^2}{z^2})}{[-\mathrm{i}z^2 + (2y_0-1)z + \mathrm{i}R^2]^3},$$

$$F_3(z) = -\frac{1}{4} \frac{(z^2+R^2)^2}{[(z+\mathrm{i}y_0)(z+\mathrm{i}(y_0-1)]^3}.$$

Consider first the case $h < -\frac{1}{8}$, the rational function $F_3(z)$ has a single pole $z = -\mathrm{i}y_0$ inside the disk D_-. The residue of $F_3(z)$ at the point $z = -\mathrm{i}y_0$ is given by evaluating:

$$-\frac{1}{4}[\frac{(z^2+R^2)^2}{[z+\mathrm{i}(y_0-1)]^3}]'' \mid_{z=-\mathrm{i}y_0} .$$

This yields

$$\mathrm{Res}(F_3(z), z = -\mathrm{i}y_0) = \frac{\mathrm{i}R^2}{2},$$

and hence:

$$I_3(h) = \int_{H=h} F_3(z)dz = -\pi(4h^2 - h - \frac{3}{16}), h < -\frac{1}{8}.$$

It is then obvious that the residue of $F_3(z)dz$ at infinity is zero. So that we obtain:

$$I_3(h) = \int_{H=h} F_3(z)dz = \pi(4h^2 - h - \frac{3}{16}), h > \frac{3}{8}.$$

□

Remark 1. *The 1-forms* $\delta_{10}^i, i = 1, 2, 3$ *are not independent in the relative cohomology. From the computations above it results that:*

$$2\delta_{10}^1 + (-1 + 8h)\delta_{10}^2 + \delta_{10}^3 = gdH + dR,$$

for some functions g, R *analytic on the open period annuli.*

Substituting (38) into (37), we can obtain:

Theorem 18. *The first-order bifurcation function is given by:*

$$M_1(h) = \begin{cases} -\frac{1}{16}\pi(8h+1)(A_1 h + A_0), & h < -\frac{1}{8} \\ \frac{1}{16}\pi(8h-3)(B_1 h + B_0), & h > \frac{3}{8} \end{cases}$$

where

$$\begin{aligned} A_1 &= 32b_{00} + 16b_{01} + 8b_{02} + 8b_{20}, \\ A_0 &= 4a_{10} - 12b_{00} - 2b_{01} + b_{02} + b_{20}, \\ B_1 &= 32b_{00} + 16b_{01} + 8b_{02} + 8b_{20}, \\ B_0 &= 4a_{10} + 4a_{11} + 4b_{00} + 6b_{01} + 5b_{02} - 3b_{20}. \end{aligned}$$

Remark 2. *It is interesting to note that the involution* $\phi : (x, y) \mapsto (x, 1 - y)$ *leaves invariant the vector field LV+LV, and it permutes its two stationnary points* $(0, 0)$ *and* $(0, 1)$*. The pull-back of the first integral* H *by* ϕ *yields* $\phi^*(H) = -H + \frac{1}{4}$ *and it leaves invariant the integrating factor. This involution induces an action on the space of the parameters of the perturbation that, by abuse of notations, we also denote* $\phi^* : (a_{ij}, b_{ij} \mapsto (\overline{a}_{ij}, \overline{b}_{ij})$ *which can be computed:*

$$\begin{aligned} &\overline{a}_{00} = a_{00} + a_{01} + a_{02}, \overline{a}_{10} = a_{10} + a_{11}, \overline{a}_{01} = -a_{01} - 2a_{02}, \\ &\overline{a}_{20} = a_{20}, \overline{a}_{11} = -a_{11}, \overline{a}_{02} = a_{02} \\ &\overline{b}_{00} = -b_{00} - b_{01} - b_{02}, \overline{b}_{10} = -b_{10} - b_{11}, \overline{b}_{01} = b_{01} + 2b_{02}, \\ &\overline{b}_{20} = -b_{20}, \overline{b}_{11} = b_{11}, \overline{b}_{02} = -b_{02}. \end{aligned}$$

The coefficients A_0, A_1, B_1, B_0 *of the function* $M_1(h)$ *are polynomials in the parameters* (a_{ij}, b_{ij}) *of the pertubation. It is easy to check that the pullback display:* $\phi^*(A_1) = -B_1$ *and* $\phi^*(A_0) + \frac{1}{4}\phi^*(A_1) = B_0$*. So that we get:*

$$\phi^*[-\frac{1}{16}\pi(8h+1)(A_1 h + A_0)] = \frac{1}{16}\pi(8h-3)(B_1 h + B_0).$$

From the theorem 18, we can deduce that the maximal configurations of limit cycles which can appear in a first-order one-parameter perturbation theory of the double center "LV+LV" is $(1, 0), (0, 1), (1, 1)$. This was already proved in [3]. Of course, it is necessary to complete with computations of higher-order bifurcation functions. This was done at order two in [17] and [22] but not directly with the approach of successive derivatives developped in [11, 15]. In particular, the averaging theory used in [22] can only be applied to a perturbation of an isochronous center (or double center). The article [3] was based on a first-order approximation of the Bautin ideal. This has been recently generalized in [12] where the set of all bifurcations functions has been characterized geometrically.

6. Conclusion and perspectives

We gave a full description of the stratification of the semi-algebraic set of the double-centers in the quadratic planar vector fields. We considered the first-order perturbation theory of the double Lotka-Volterra system using “relative logarithmic cohomology”. This approach relates with several subjects currently developped in singularity theory (see for instance [19]) and we believe it could be an appropriated setting for the general perturbation theory of reversible double centers. It should be extended to second-order perturbation theory and compared with Iliev’s essential perturbation theory and Bautin ideal approach (see [12, 17]). We plan also to analyze more deeply the prolongation theory of the limit cycles which appear for small perturbations.

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