Digit frequencies of beta-expansions

Y.-Q. Li

DIGIT FREQUENCIES OF BETA-EXPANSIONS

YAO-QIANG LI

ABSTRACT. Let $\beta > 1$ be a non-integer. First we show that Lebesgue almost every number has a β-expansion of a given frequency if and only if Lebesgue almost every number has infinitely many β-expansions of the same given frequency. Then we deduce that Lebesgue almost every number has infinitely many balanced β-expansions, where an infinite sequence on the finite alphabet $\{0, 1, \cdots, m\}$ is called balanced if the frequency of the digit k is equal to the frequency of the digit $m - k$ for all $k \in \{0, 1, \cdots, m\}$. Finally we consider variable frequency and prove that for every pseudo-golden ratio $\beta \in (1, 2)$, there exists a constant $c = c(\beta) > 0$ such that for any $p \in [\frac{1}{2} - c, \frac{1}{2} + c]$, Lebesgue almost every x has infinitely many β-expansions with frequency of zeros equal to p.

1. INTRODUCTION

To represent real numbers, the most common way is to use expansions in integer bases, especially in base 2 or 10. As a natural generalization, expansions in non-integer bases were introduced by Rényi [26] in 1957, and then attracted a lot of attention until now (see for examples [1, 2, 8, 11, 17, 18, 23, 24, 25, 27, 28]). They are known as beta-expansions nowadays.

Let $\mathbb{N} = \{1, 2, 3, \cdots\}$ be the set of positive integers and $\mathbb{R}$ be the set of real numbers. For $\beta > 1$, we define the alphabet by

$$\mathcal{A}_\beta = \{0, 1, \cdots, \lceil \beta \rceil - 1\}.$$

where $\lceil \beta \rceil$ denotes the smallest integer no less than β, and similarly we use $\lfloor \beta \rfloor$ to denote the greatest integer no larger than β throughout this paper. Let $x \in \mathbb{R}$. A sequence $(\varepsilon_i)_{i \geq 1} \in \mathcal{A}_\beta^{\mathbb{N}}$ is called a β-*expansion* of x if

$$x = \sum_{i=1}^{\infty} \frac{\varepsilon_i}{\beta^i}.$$

For $\beta > 1$, let I_β be the interval $[0, \frac{\lceil \beta \rceil - 1}{\beta - 1}]$, and let I_β^o be the interior of I_β (i.e. $I_\beta^o = (0, \frac{\lceil \beta \rceil - 1}{\beta - 1})$). It is straightforward to check that x has a β-expansion if and only if $x \in I_\beta$. An interesting phenomenon is that an x may have many β-expansions. For examples, [14, Theorem 3] shows that if $\beta \in (1, \frac{1+\sqrt{5}}{2})$, every $x \in I_\beta^o$ has a continuum of different β-expansions, and [29, Theorem 1] shows that if $\beta \in (1, 2)$, Lebesgue almost every $x \in I_\beta$ has a continuum of different β-expansions. For more on the cardinality of β-expansions, we refer the reader to [7, 15, 19].

In this paper we focus on the digit frequencies of β-expansions, which is a classical research topic. For examples, Borel's normal number theorem [9] says that for any integer $\beta > 1$, Lebesgue almost every $x \in [0, 1]$ has a β-expansion in which every finite word on $\mathcal{A}_\beta$ with length k occurs with frequency β^{-k}; Eggleston [13] proved that for each $p \in [0, 1]$, the Hausdorff dimension (see [16] for definition) of the set, consisting of those $x \in [0, 1]$

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having a binary expansion with frequency of zeros equal to p, is equal to $(-p\log p-(1-p)\log(1-p))/(\log 2)$. Let $\beta_T \approx 1.80194$ be the unique zero in $(1,2]$ of the polynomial x^3-x^2-2x+1. Recently, on the one hand, Baker and Kong [6] proved that if $\beta\in(1,\beta_T]$, then every $x\in I_\beta^o$ has a *simply normal* β-expansion (i.e., the frequency of each digit is the same), and on the other hand, Jordan, Shmerkin and Solomyak [20] proved that if $\beta\in(\beta_T,2]$, then there exists $x\in I_\beta^o$ which does not have any simply normal β-expansions. In the recent paper [5], Baker studied the set of frequencies of β-expansions for a general $\beta>1$. (See also [10] for the study of the set of frequencies of greedy β-expansions.)

Let $m\in\mathbb{N}$. For any sequence $(\varepsilon_i)_{i\ge1}\in\{0,1,\cdots,m\}^{\mathbb{N}}$, we define the *upper-frequency*, *lower-frequency* and *frequency* of the digit k by

$$\overline{\text{Freq}}_k(\varepsilon_i) := \varlimsup_{n\to\infty}\frac{\sharp\{1\le i\le n:\varepsilon_i=k\}}{n},$$

$$\underline{\text{Freq}}_k(\varepsilon_i) := \varliminf_{n\to\infty}\frac{\sharp\{1\le i\le n:\varepsilon_i=k\}}{n}$$

and

$$\text{Freq}_k(\varepsilon_i) := \lim_{n\to\infty}\frac{\sharp\{1\le i\le n:\varepsilon_i=k\}}{n}$$

(assuming the limit exists) respectively, where $\sharp$ denotes the cardinality. If $\overline{p}=(\overline{p}_0,\cdots,\overline{p}_m)$, $\underline{p}=(\underline{p}_0,\cdots,\underline{p}_m)\in[0,1]^{m+1}$ satisfy

$$\overline{\text{Freq}}_k(\varepsilon_i)=\overline{p}_k \quad\text{and}\quad \underline{\text{Freq}}_k(\varepsilon_i)=\underline{p}_k \quad \text{for all } k\in\{0,1,\cdots,m\},$$

we say that $(\varepsilon_i)_{i\ge1}$ is of frequency $(\overline{p},\underline{p})$.

The following theorem is the first main result in this paper.

Theorem 1.1. *For all $\beta\in(1,+\infty)\setminus\mathbb{N}$ and $\overline{p},\underline{p}\in[0,1]^{\lceil\beta\rceil}$, Lebesgue almost every $x\in I_\beta$ has a β-expansion of frequency $(\overline{p},\underline{p})$ if and only if Lebesgue almost every $x\in I_\beta$ has infinitely many β-expansions of frequency $(\overline{p},\underline{p})$.*

As the second main result, the next theorem focuses on a special kind of frequency. Let $m\in\mathbb{N}$. A sequence $(\varepsilon_i)_{i\ge1}\in\{0,1,\cdots,m\}^{\mathbb{N}}$ is called *balanced* if $\text{Freq}_k(\varepsilon_i)=\text{Freq}_{m-k}(\varepsilon_i)$ for all $k\in\{0,1,\cdots,m\}$.

Theorem 1.2. *For all $\beta\in(1,+\infty)\setminus\mathbb{N}$, Lebesgue almost every $x\in I_\beta$ has infinitely many balanced β-expansions.*

In the following, we consider variable frequency. Recently, Baker proved in [4] that for any $\beta\in(1,\frac{1+\sqrt5}{2})$, there exists $c=c(\beta)>0$ such that for any $p\in[\frac12-c,\frac12+c]$ and $x\in I_\beta^o$, there exists a β-expansion of x with frequency of zeros equal to p. This result is sharp, since for any $\beta\in[\frac{1+\sqrt5}{2},2)$, there exists an $x\in I_\beta^o$ such that for any β-expansion of x its frequency of zeros exists and is equal to either 0 or $\frac12$ (see the statements between Theorem 1.1 and Theorem 1.2 in [6]). It is natural to ask for which $\beta\in[\frac{1+\sqrt5}{2},2)$, the result can be true for almost every $x\in I_\beta^o$. We give a class of such β in Theorem 1.3 as the third main result in this paper. They are the *pseudo-golden ratios*, i.e., the $\beta\in(1,2)$ such that $\beta^m-\beta^{m-1}-\cdots-\beta-1=0$ for some integer $m\ge2$. Note that the smallest pseudo-golden ratio is the golden ratio $\frac{1+\sqrt5}{2}$.

Theorem 1.3. *Let $\beta\in(1,2)$ such that $\beta^m-\beta^{m-1}-\cdots-\beta-1=0$ for some integer $m\ge2$ and let $c=\frac{(m-1)(2-\beta)}{2(m\beta+\beta-2m)}$ (>0). Then for any $p\in[\frac12-c,\frac12+c]$, Lebesgue almost every $x\in I_\beta$ has infinitely many β-expansions with frequency of zeros equal to p.*

We give some notation and preliminaries in the next section, prove the main results in Section 3 and end this paper with further questions in the last section.

2. NOTATION AND PRELIMINARIES

Let $\beta > 1$. We define the maps $T_k(x) := \beta x - k$ for $x \in \mathbb{R}$ and $k \in \mathbb{N} \cup \{0\}$. Given $x \in I_\beta$, let

$$\Sigma_\beta(x) := \Big\{ (\varepsilon_i)_{i\geq 1} \in \mathcal{A}_\beta^{\mathbb{N}} : \sum_{i=1}^{\infty} \frac{\varepsilon_i}{\beta^i} = x \Big\}$$

and

$$\Omega_\beta(x) := \Big\{ (a_i)_{i\geq 1} \in \{T_k, k \in \mathcal{A}_\beta\}^{\mathbb{N}} : (a_n \circ \cdots \circ a_1)(x) \in I_\beta \text{ for all } n \in \mathbb{N} \Big\}.$$

The following lemma given by Baker is a dynamical interpretation of β-expansions.

Lemma 2.1 ([3, 4]). *For any $x \in I_\beta$, we have $\sharp\Sigma_\beta(x) = \sharp\Omega_\beta(x)$. Moreover, the map which sends $(\varepsilon_i)_{i\geq 1}$ to $(T_{\varepsilon_i})_{i\geq 1}$ is a bijection between $\Sigma_\beta(x)$ and $\Omega_\beta(x)$.*

We need the following concepts and the well known Birkhoff's Ergodic Theorem in the proof of our main results.

Definition 2.2 (Absolute continuity and equivalence)**.** Let μ and ν be measures on a measurable space $(X, \mathcal{F})$. We say that μ is *absolutely continuous* with respect to ν and denote it by $\mu \ll \nu$ if, for any $A \in \mathcal{F}$, $\nu(A) = 0$ implies $\mu(A) = 0$. Moreover, if $\mu \ll \nu$ and $\nu \ll \mu$ we say that μ and ν are *equivalent* and denote this property by $\mu \sim \nu$.

Theorem 2.3 ([30] Birkhoff's Ergodic Theorem)**.** *Let $(X, \mathcal{F}, \mu, T)$ be a measure-preserving dynamical system where the probability measure μ is ergodic with respect to T. Then for any real-valued integrable function $f : X \to \mathbb{R}$, we have*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k x) = \int f d\mu$$

for μ-a.e. (almost every) $x \in X$.

3. PROOF OF THE MAIN RESULTS

Proof of Theorem 1.1. The "if" part is obvious. We only need to prove the "only if" part. Let $\mathcal{L}$ be the Lebesgue measure. Suppose that $\mathcal{L}$-a.e. $x \in I_\beta$ has a β-expansion of frequency $(\overline{p}, \underline{p})$. Let

$$\mathcal{U}_\beta := \Big\{ x \in I_\beta : x \text{ has a unique } \beta\text{-expansion} \Big\}$$

and

$$\mathcal{N}_\beta^{\overline{p},\underline{p}} := \Big\{ x \in I_\beta : x \text{ has no } \beta\text{-expansions of frequency } (\overline{p}, \underline{p}) \Big\}.$$

On the one hand, it is well known that $\mathcal{L}(\mathcal{U}_\beta) = 0$ (see for examples [12, 21]). On the other hand, by condition we know $\mathcal{L}(\mathcal{N}_\beta^{\overline{p},\underline{p}}) = 0$. Let

$$\Psi := \Big(\mathcal{U}_\beta \cup \mathcal{N}_\beta^{\overline{p},\underline{p}}\Big) \cup \bigcup_{n=1}^{\infty} \bigcup_{\varepsilon_1,\cdots,\varepsilon_n \in \mathcal{A}_\beta} T_{\varepsilon_n}^{-1} \circ \cdots \circ T_{\varepsilon_1}^{-1} \Big(\mathcal{U}_\beta \cup \mathcal{N}_\beta^{\overline{p},\underline{p}}\Big).$$

Then $\mathcal{L}(\Psi) = 0$. Let $x \in I_\beta \setminus \Psi$. It suffices to prove that x has infinitely many different β-expansions of frequency $(\overline{p}, \underline{p})$.

Let $(\varepsilon_i)_{i\ge1}$ be a β-expansions of x. Since $x\notin\Psi$ implies $x\notin\mathcal{U}_\beta$, x has another β-expansion $(w_i^{(1)})_{i\ge1}$. There exists $n_1\in\mathbb{N}$ such that $w_1^{(1)}\cdots w_{n_1-1}^{(1)}=\varepsilon_1\cdots\varepsilon_{n_1-1}$ and $w_{n_1}^{(1)}\neq\varepsilon_{n_1}$. By

$$T_{w_{n_1}^{(1)}}\circ T_{\varepsilon_{n_1-1}}\circ\cdots\circ T_{\varepsilon_1}x=T_{w_{n_1}^{(1)}}\circ\cdots\circ T_{w_1^{(1)}}x=\sum_{i=1}^{\infty}\frac{w_{n_1+i}^{(1)}}{\beta^i},$$

we know that $(w_{n_1+i}^{(1)})_{i\ge1}$ is a β-expansion of $T_{w_{n_1}^{(1)}}\circ T_{\varepsilon_{n_1-1}}\circ\cdots\circ T_{\varepsilon_1}x$. Since $x\notin\Psi$ implies $T_{w_{n_1}^{(1)}}\circ T_{\varepsilon_{n_1-1}}\circ\cdots\circ T_{\varepsilon_1}x\notin\mathcal{N}_\beta^{\overline{p},\underline{p}}$, $T_{w_{n_1}^{(1)}}\circ T_{\varepsilon_{n_1-1}}\circ\cdots\circ T_{\varepsilon_1}x$ has a β-expansion $(\varepsilon_{n_1+i}^{(1)})_{i\ge1}$ of frequency $(\overline{p},\underline{p})$. Let $\varepsilon_1^{(1)}\cdots\varepsilon_{n_1-1}^{(1)}\varepsilon_{n_1}^{(1)}:=\varepsilon_1\cdots\varepsilon_{n_1-1}w_{n_1}^{(1)}$. Then $(\varepsilon_i^{(1)})_{i\ge1}$ is a β-expansion of x of frequency $(\overline{p},\underline{p})$ with $\varepsilon_{n_1}^{(1)}\neq\varepsilon_{n_1}$, which implies that $(\varepsilon_i)_{i\ge1}$ and $(\varepsilon_i^{(1)})_{i\ge1}$ are different.

Note that $(\varepsilon_{n_1+i})_{i\ge1}$ is a β-expansion of $T_{\varepsilon_{n_1}}\circ\cdots\circ T_{\varepsilon_1}x$. Since $x\notin\Psi$ implies $T_{\varepsilon_{n_1}}\circ\cdots\circ T_{\varepsilon_1}x\notin\mathcal{U}_\beta$, $T_{\varepsilon_{n_1}}\circ\cdots\circ T_{\varepsilon_1}x$ has another β-expansion $(w_{n_1+i}^{(2)})_{i\ge1}$. There exists $n_2>n_1$ such that $w_{n_1+1}^{(2)}\cdots w_{n_2-1}^{(2)}=\varepsilon_{n_1+1}\cdots\varepsilon_{n_2-1}$ and $w_{n_2}^{(2)}\neq\varepsilon_{n_2}$. By

$$T_{w_{n_2}^{(2)}}\circ T_{\varepsilon_{n_2-1}}\circ\cdots\circ T_{\varepsilon_1}x=T_{w_{n_2}^{(2)}}\circ\cdots\circ T_{w_{n_1+1}^{(2)}}\circ(T_{\varepsilon_{n_1}}\circ\cdots\circ T_{\varepsilon_1}x)=\sum_{i=1}^{\infty}\frac{w_{n_2+i}^{(2)}}{\beta^i},$$

we know that $(w_{n_2+i}^{(2)})_{i\ge1}$ is a β-expansion of $T_{w_{n_2}^{(2)}}\circ T_{\varepsilon_{n_2-1}}\circ\cdots\circ T_{\varepsilon_1}x$. Since $x\notin\Psi$ implies $T_{w_{n_2}^{(2)}}\circ T_{\varepsilon_{n_2-1}}\circ\cdots\circ T_{\varepsilon_1}x\notin\mathcal{N}_\beta^{\overline{p},\underline{p}}$, $T_{w_{n_2}^{(2)}}\circ T_{\varepsilon_{n_2-1}}\circ\cdots\circ T_{\varepsilon_1}x$ has a β-expansion $(\varepsilon_{n_2+i}^{(2)})_{i\ge1}$ of frequency $(\overline{p},\underline{p})$. Let $\varepsilon_1^{(2)}\cdots\varepsilon_{n_2-1}^{(2)}\varepsilon_{n_2}^{(2)}:=\varepsilon_1\cdots\varepsilon_{n_2-1}w_{n_2}^{(2)}$. Then $(\varepsilon_i^{(2)})_{i\ge1}$ is a β-expansion of x of frequency $(\overline{p},\underline{p})$ with $\varepsilon_{n_1}^{(2)}=\varepsilon_{n_1}$ and $\varepsilon_{n_2}^{(2)}\neq\varepsilon_{n_2}$, which implies that $(\varepsilon_i)_{i\ge1}$, $(\varepsilon_i^{(1)})_{i\ge1}$ and $(\varepsilon_i^{(2)})_{i\ge1}$ are all different.

$\cdots$

Generally, suppose that for some $j\in\mathbb{N}$ we have already constructed $(\varepsilon_i^{(1)})_{i\ge1}$, $(\varepsilon_i^{(2)})_{i\ge1}$, $\cdots$, $(\varepsilon_i^{(j)})_{i\ge1}$, which are all β-expansions of x of frequency $(\overline{p},\underline{p})$ such that

$$\begin{cases}\varepsilon_{n_1}^{(1)}\neq\varepsilon_{n_1},\\ \varepsilon_{n_1}^{(2)}=\varepsilon_{n_1},\varepsilon_{n_2}^{(2)}\neq\varepsilon_{n_2},\\ \varepsilon_{n_1}^{(3)}=\varepsilon_{n_1},\varepsilon_{n_2}^{(3)}=\varepsilon_{n_2},\varepsilon_{n_3}^{(3)}\neq\varepsilon_{n_3},\\ \cdots\\ \varepsilon_{n_1}^{(j)}=\varepsilon_{n_1},\varepsilon_{n_2}^{(j)}=\varepsilon_{n_2},\cdots,\varepsilon_{n_{j-1}}^{(j)}\neq\varepsilon_{n_{j-1}},\varepsilon_{n_j}^{(j)}\neq\varepsilon_{n_j}.\end{cases}$$

Note that $(\varepsilon_{n_j+i})_{i\ge1}$ is a β-expansion of $T_{\varepsilon_{n_j}}\circ\cdots\circ T_{\varepsilon_1}x$. Since $x\notin\Psi$ implies $T_{\varepsilon_{n_j}}\circ\cdots\circ T_{\varepsilon_1}x\notin\mathcal{U}_\beta$, $T_{\varepsilon_{n_j}}\circ\cdots\circ T_{\varepsilon_1}x$ has another β-expansion $(w_{n_j+i}^{(j+1)})_{i\ge1}$. There exists $n_{j+1}>n_j$ such that $w_{n_j+1}^{(j+1)}\cdots w_{n_{j+1}-1}^{(j+1)}=\varepsilon_{n_j+1}\cdots\varepsilon_{n_{j+1}-1}$ and $w_{n_{j+1}}^{(j+1)}\neq\varepsilon_{n_{j+1}}$. By

$$T_{w_{n_{j+1}}^{(j+1)}}\circ T_{\varepsilon_{n_{j+1}-1}}\circ\cdots\circ T_{\varepsilon_1}x=T_{w_{n_{j+1}}^{(j+1)}}\circ\cdots\circ T_{w_{n_j+1}^{(j+1)}}\circ(T_{\varepsilon_{n_j}}\circ\cdots\circ T_{\varepsilon_1}x)=\sum_{i=1}^{\infty}\frac{w_{n_{j+1}+i}^{(j+1)}}{\beta^i},$$

we know that $(w_{n_{j+1}+i}^{(j+1)})_{i\ge1}$ is a β-expansion of $T_{w_{n_{j+1}}^{(j+1)}}\circ T_{\varepsilon_{n_{j+1}-1}}\circ\cdots\circ T_{\varepsilon_1}x$. Since $x\notin\Psi$ implies $T_{w_{n_{j+1}}^{(j+1)}}\circ T_{\varepsilon_{n_{j+1}-1}}\circ\cdots\circ T_{\varepsilon_1}x\notin\mathcal{N}_\beta^{\overline{p},\underline{p}}$, $T_{w_{n_{j+1}}^{(j+1)}}\circ T_{\varepsilon_{n_{j+1}-1}}\circ\cdots\circ T_{\varepsilon_1}x$ has a β-expansion $(\varepsilon_{n_{j+1}+i}^{(j+1)})_{i\ge1}$ of frequency $(\overline{p},\underline{p})$. Let $\varepsilon_1^{(j+1)}\cdots\varepsilon_{n_{j+1}-1}^{(j+1)}\varepsilon_{n_{j+1}}^{(j+1)}:=\varepsilon_1\cdots\varepsilon_{n_{j+1}-1}w_{n_{j+1}}^{(j+1)}$. Then

$(\varepsilon_i^{(j+1)})_{i\geq 1}$ is a β-expansion of x of frequency $(\overline{p}, \underline{p})$ with $\varepsilon_{n_1}^{(j+1)} = \varepsilon_{n_1}, \cdots, \varepsilon_{n_j}^{(j+1)} = \varepsilon_{n_j}$ and $\varepsilon_{n_{j+1}}^{(j+1)} \neq \varepsilon_{n_{j+1}}$, which implies that $(\varepsilon_i)_{i\geq 1}$, $(\varepsilon_i^{(1)})_{i\geq 1}$, $\cdots$, $(\varepsilon_i^{(j+1)})_{i\geq 1}$ are all different.

$\cdots$

It follows from repeating the above process that x has infinitely many different β-expansions of frequency $(\overline{p}, \underline{p})$. □

Theorem 1.2 follows immediately from Theorem 1.1 and the following lemma.

Lemma 3.1. *For all $\beta > 1$, Lebesgue almost every $x \in I_\beta$ has a balanced β-expansion.*

Proof. The conclusion follows from the well known Borel's Normal Number Theorem [9] if $\beta \in \mathbb{N}$ and follows from [6, Theorem 4.1] if $\beta \in (1, 2)$. Thus we only need to consider $\beta > 2$ with $\beta \notin \mathbb{N}$ in the following. Let

$$z_1 := \frac{1}{2}\Big(\frac{\lfloor\beta\rfloor}{\beta-1} - \frac{\lfloor\beta\rfloor - 1}{\beta}\Big) \quad \text{and} \quad z_{k+1} := z_k + \frac{1}{\beta} \quad \text{for all } k \in \{1, 2, \cdots, \lfloor\beta\rfloor - 1\}.$$

Define $T : I_\beta \to I_\beta$ by

$$T(x) := \begin{cases} T_0(x) = \beta x & \text{for } x \in [0, z_1), \\ T_k(x) = \beta x - k & \text{for } x \in [z_k, z_{k+1}) \text{ and } k \in \{1, 2, \cdots, \lfloor\beta\rfloor - 1\}, \\ T_{\lfloor\beta\rfloor}(x) = \beta x - \lfloor\beta\rfloor & \text{for } x \in [z_{\lfloor\beta\rfloor}, \frac{\lfloor\beta\rfloor}{\beta-1}]. \end{cases}$$

Let

$$z_0 := \frac{\lfloor\beta\rfloor}{2(\beta-1)} - \frac{1}{2} \quad \text{and} \quad z_{\lceil\beta\rceil} := z_0 + 1 = \frac{\lfloor\beta\rfloor}{2(\beta-1)} + \frac{1}{2}.$$

Then $T_1(z_1) = T_2(z_2) = \cdots = T_{\lfloor\beta\rfloor}(z_{\lfloor\beta\rfloor}) = z_0$ and $T_0(z_1) = T_1(z_2) = \cdots = T_{\lfloor\beta\rfloor-1}(z_{\lfloor\beta\rfloor}) = z_{\lceil\beta\rceil}$.

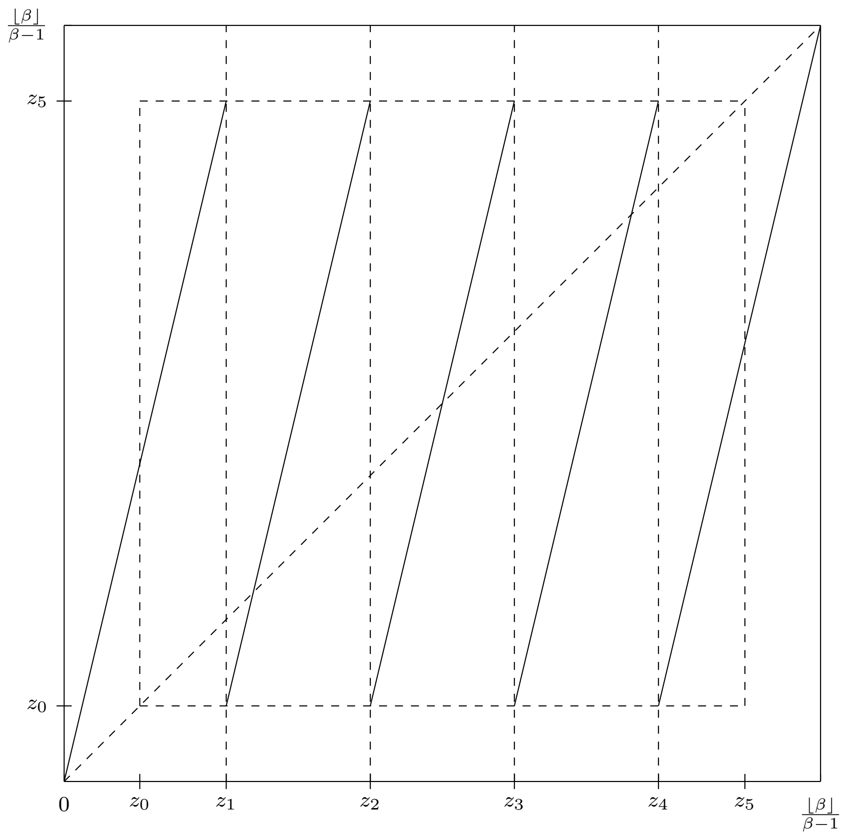

FIGURE 1. The graph of T for some $\beta \in (4,5)$.

We consider the restriction $T|_{[z_0,z_{\lceil\beta\rceil})} : [z_0, z_{\lceil\beta\rceil}) \to [z_0, z_{\lceil\beta\rceil})$. By Theorem 5.2 in [31], there exists a $T|_{[z_0,z_{\lceil\beta\rceil})}$-invariant ergodic Borel probability measure μ on $[z_0, z_{\lceil\beta\rceil})$ equivalent to the Lebesgue measure $\mathcal{L}$. For any $x \in [z_0, z_{\lceil\beta\rceil})$ which is not a preimage of a discontinuity point of $T|_{[z_0,z_{\lceil\beta\rceil})}$, by symmetry, we know that for any $k \in \{0, 1, \cdots, \lfloor\beta\rfloor\}$ and $i \in \{0, 1, 2, \cdots\}$,

$$T^i(x) \in (z_k, z_{k+1}) \Leftrightarrow T^i\Big(\frac{\lfloor\beta\rfloor}{\beta-1} - x\Big) \in (z_{\lfloor\beta\rfloor-k}, z_{\lceil\beta\rceil-k}).$$

For all $k \in \{0, 1, \cdots, \lfloor\beta\rfloor\}$, it follows from Birkhoff's Ergodic Theorem that for $\mathcal{L}$-a.e. $x \in [z_0, z_{\lceil\beta\rceil})$,

$$\mu((z_k, z_{k+1})) = \int_{z_0}^{z_{\lceil\beta\rceil}} \mathbb{1}_{(z_k,z_{k+1})} d\mu = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \mathbb{1}_{(z_k,z_{k+1})}\Big(T^i(x)\Big) \tag{3.1}$$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \mathbb{1}_{(z_{\lfloor\beta\rfloor-k},z_{\lceil\beta\rceil-k})}\Big(T^i\big(\frac{\lfloor\beta\rfloor}{\beta-1} - x\big)\Big) \tag{3.2}$$

and for $\mathcal{L}$-a.e. $y \in [z_0, z_{\lceil\beta\rceil})$,

$$\mu((z_{\lfloor\beta\rfloor-k}, z_{\lceil\beta\rceil-k})) = \int_{z_0}^{z_{\lceil\beta\rceil}} \mathbb{1}_{(z_{\lfloor\beta\rfloor-k},z_{\lceil\beta\rceil-k})} d\mu = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \mathbb{1}_{(z_{\lfloor\beta\rfloor-k},z_{\lceil\beta\rceil-k})}\Big(T^i(y)\Big),$$

which implies that for $\mathcal{L}$-a.e. $(\frac{\lfloor\beta\rfloor}{\beta-1}-x)\in(z_0,z_{\lceil\beta\rceil})$,

$$\mu((z_{\lfloor\beta\rfloor-k},z_{\lceil\beta\rceil-k}))=\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\mathbb{1}_{(z_{\lfloor\beta\rfloor-k},z_{\lceil\beta\rceil-k})}\Big(T^i\big(\frac{\lfloor\beta\rfloor}{\beta-1}-x\big)\Big).$$

So this is also true for $\mathcal{L}$-a.e $x\in(z_0,z_{\lceil\beta\rceil})$. Recall (3.2), we get

$$\mu((z_k,z_{k+1}))=\mu((z_{\lfloor\beta\rfloor-k},z_{\lceil\beta\rceil-k}))\quad\text{for } k\in\{0,1,\cdots,\lfloor\beta\rfloor\}. \tag{3.3}$$

For every $x\in I_\beta$, define a sequence $(\varepsilon_i(x))_{i\geq1}\in\{0,1,\cdots,\lfloor\beta\rfloor\}^{\mathbb{N}}$ by

$$\varepsilon_i(x):=\begin{cases}0 & \text{if } T^{i-1}x\in[0,z_1),\\ k & \text{if } T^{i-1}x\in[z_k,z_{k+1}) \text{ for some } k\in\{1,2,\cdots,\lfloor\beta\rfloor-1\},\\ \lfloor\beta\rfloor & \text{if } T^{i-1}x\in[z_{\lfloor\beta\rfloor},\frac{\lfloor\beta\rfloor}{\beta-1}].\end{cases}$$

Then for all $k\in\{0,1,\cdots,\lfloor\beta\rfloor\}$, $i\in\{0,1,2,\cdots\}$ and $x\in[z_0,z_{\lceil\beta\rceil})$,

$$\mathbb{1}_{[z_k,z_{k+1})}(T^ix)=1\Leftrightarrow T^ix\in[z_k,z_{k+1})\Leftrightarrow\varepsilon_{i+1}(x)=k.$$

By (3.1), we know that for all $k\in\{0,1,\cdots,\lfloor\beta\rfloor\}$ and $\mathcal{L}$-a.e. $x\in[z_0,z_{\lceil\beta\rceil})$,

$$\text{Freq}_k(\varepsilon_i(x))=\lim_{n\to\infty}\frac{\sharp\{1\leq i\leq n:\varepsilon_i(x)=k\}}{n}=\mu((z_k,z_{k+1})).$$

It follows from (3.3) that for all $k\in\{0,1,\cdots,\lfloor\beta\rfloor\}$ and $\mathcal{L}$-a.e. $x\in[z_0,z_{\lceil\beta\rceil})$,

$$\text{Freq}_k(\varepsilon_i(x))=\text{Freq}_{\lfloor\beta\rfloor-k}(\varepsilon_i(x)). \tag{3.4}$$

(1) For any $x\in I_\beta$, we prove that $(\varepsilon_i(x))_{i\geq1}$ is a β-expansion of x, i.e., $\sum_{i=1}^\infty\frac{\varepsilon_i(x)}{\beta^i}=x$. In fact, by Lemma 2.1, it suffices to show $T_{\varepsilon_n(x)}\circ\cdots\circ T_{\varepsilon_1(x)}(x)\in I_\beta$ for all $n\in\mathbb{N}$. We only need to prove $T_{\varepsilon_n(x)}\circ\cdots\circ T_{\varepsilon_1(x)}(x)=T^n(x)$ by induction as follows. Let $n=1$.
 ① If $x\in[0,z_1)$, then $\varepsilon_1(x)=0$ and $T_{\varepsilon_1(x)}(x)=T_0(x)=T(x)$.
 ② If $x\in[z_k,z_{k+1})$ for some $k\in\{1,2,\cdots,\lfloor\beta\rfloor-1\}$, then $\varepsilon_1(x)=k$ and $T_{\varepsilon_1(x)}(x)=T_k(x)=T(x)$.
 ③ If $x\in[z_{\lfloor\beta\rfloor},\frac{\lfloor\beta\rfloor}{\beta-1}]$, then $\varepsilon_1(x)=\lfloor\beta\rfloor$ and $T_{\varepsilon_1(x)}(x)=T_{\lfloor\beta\rfloor}(x)=T(x)$.
Assumes that for some $n\in\mathbb{N}$ we have $T_{\varepsilon_n(x)}\circ\cdots\circ T_{\varepsilon_1(x)}(x)=T^n(x)$.
 ① If $T^n(x)\in[0,z_1)$, then $\varepsilon_{n+1}(x)=0$ and

$$T_{\varepsilon_{n+1}(x)}\circ T_{\varepsilon_n(x)}\circ\cdots\circ T_{\varepsilon_1(x)}(x)=T_0\circ T^n(x)=T^{n+1}(x).$$

 ② If $T^n(x)\in[z_k,z_{k+1})$ for some $k\in\{1,2,\cdots,\lfloor\beta\rfloor-1\}$, then $\varepsilon_{n+1}(x)=k$ and

$$T_{\varepsilon_{n+1}(x)}\circ T_{\varepsilon_n(x)}\circ\cdots\circ T_{\varepsilon_1(x)}(x)=T_k\circ T^n(x)=T^{n+1}(x).$$

 ③ If $T^n(x)\in[z_{\lfloor\beta\rfloor},\frac{\lfloor\beta\rfloor}{\beta-1}]$, then $\varepsilon_{n+1}(x)=\lfloor\beta\rfloor$ and

$$T_{\varepsilon_{n+1}(x)}\circ T_{\varepsilon_n(x)}\circ\cdots\circ T_{\varepsilon_1(x)}(x)=T_{\lfloor\beta\rfloor}\circ T^n(x)=T^{n+1}(x).$$

Combining (1) and (3.4), we know that $\mathcal{L}$-a.e. $x\in[z_0,z_{\lceil\beta\rceil}]$ has a balanced β-expansion. Let

$$N:=\big\{x\in I_\beta: x \text{ has no balanced } \beta\text{-expansions}\big\}.$$

We have already proved $\mathcal{L}(N\cap[z_0,z_{\lceil\beta\rceil}])=0$. To end the proof of this lemma, we need to show $\mathcal{L}(N)=0$. In fact, it suffices to prove $\mathcal{L}(N\cap(0,z_0))=\mathcal{L}(N\cap(z_{\lceil\beta\rceil},\frac{\lfloor\beta\rfloor}{\beta-1}))=0$.

i) Prove $\mathcal{L}(N\cap(0,z_0))=0$.
By $\mathcal{L}(N\cap[z_0,z_{\lceil\beta\rceil}])=0$, we know that for any $n\in\mathbb{N}$, $\mathcal{L}(T_0^{-n}(N\cap[z_0,z_{\lceil\beta\rceil}]))=0$. It suffices to prove $N\cap(0,z_0)\subset\bigcup_{n=1}^{\infty}T_0^{-n}(N\cap[z_0,z_{\lceil\beta\rceil}])$.
(By contradiction) Let $x\in N\cap(0,z_0)$ and assume $x\notin\bigcup_{n=1}^{\infty}T_0^{-n}(N\cap[z_0,z_{\lceil\beta\rceil}])$. By $x\in(0,z_0)$, one can verify that there exists $k\ge 1$ such that $T_0^k x\in[z_0,z_{\lceil\beta\rceil}]$. Since $x\notin T_0^{-k}(N\cap[z_0,z_{\lceil\beta\rceil}])$, we must have $T_0^k x\notin N$. This means that there exists a balanced sequence $(w_i)_{i\ge1}\in\mathcal{A}_\beta^{\mathbb{N}}$ such that $T_0^k x=\sum_{i=1}^{\infty}\frac{w_i}{\beta^i}$, and then

$$x=\frac{0}{\beta}+\frac{0}{\beta^2}+\cdots+\frac{0}{\beta^k}+\sum_{i=1}^{\infty}\frac{w_i}{\beta^{k+i}}=:\sum_{i=1}^{\infty}\frac{\varepsilon_i}{\beta^i}$$

where $\varepsilon_1=\cdots=\varepsilon_k:=0$ and $\varepsilon_{k+i}:=w_i$ for $i\ge1$. It follows that $(\varepsilon_i)_{i\ge1}$ is a balanced β-expansion of x, which contradicts $x\in N$.

ii) The fact $\mathcal{L}(N\cap(z_{\lceil\beta\rceil},\frac{\lfloor\beta\rfloor}{\beta-1}))=0$ follows in a similar way as i) by applying $T_{\lfloor\beta\rfloor}$ instead of T_0.

□

Proof of Theorem 1.3. Let $\beta\in(1,2)$ such that $\beta^m-\beta^{m-1}-\cdots-\beta-1=0$ for some integer $m\ge2$ and let $c=\frac{(m-1)(2-\beta)}{2(m\beta+\beta-2m)}$. We have $c>0$ since $m-1>0$, $2-\beta>0$ and $m\beta+\beta-2m>0$, which is a consequence of

$$m+1<2m<2(\beta^{m-1}+\cdots+\beta+1)=2\beta^m=\frac{2}{2-\beta},$$

where the equalities follows from

$$\beta^m=\beta^{m-1}+\cdots+\beta+1=\frac{\beta^m-1}{\beta-1}.$$

For any $x\in[0,\frac{1}{\beta-1}-1]$, define

$$f(x):=\frac{(\beta-1)(1-(m-1)x)}{m\beta+\beta-2m}.$$

Then

$$f(0)=\frac{\beta-1}{m\beta+\beta-2m}=\frac{1}{2}+c\quad\text{and}\quad f(\frac{1}{\beta-1}-1)=\frac{m\beta+1-2m}{m\beta+\beta-2m}=\frac{1}{2}-c,$$

i.e., $[f(\frac{1}{\beta-1}-1),f(0)]=[\frac12-c,\frac12+c]$. Since f is continuous, for any $p\in[\frac12-c,\frac12+c]$, there exists $b\in[0,\frac{1}{\beta-1}-1]$ such that $f(b)=p$. We only consider $b\in[0,\frac{1}{\beta-1}-1)$ in the following, since the proof for the case $b\in(0,\frac{1}{\beta-1}-1]$ is similar. Define $T:I_\beta\to I_\beta$ by

$$T(x):=\begin{cases}T_0(x)=\beta x & \text{for } x\in[0,\frac{b+1}{\beta}),\\ T_1(x)=\beta x-1 & \text{for } x\in[\frac{b+1}{\beta},\frac{1}{\beta-1}].\end{cases}$$

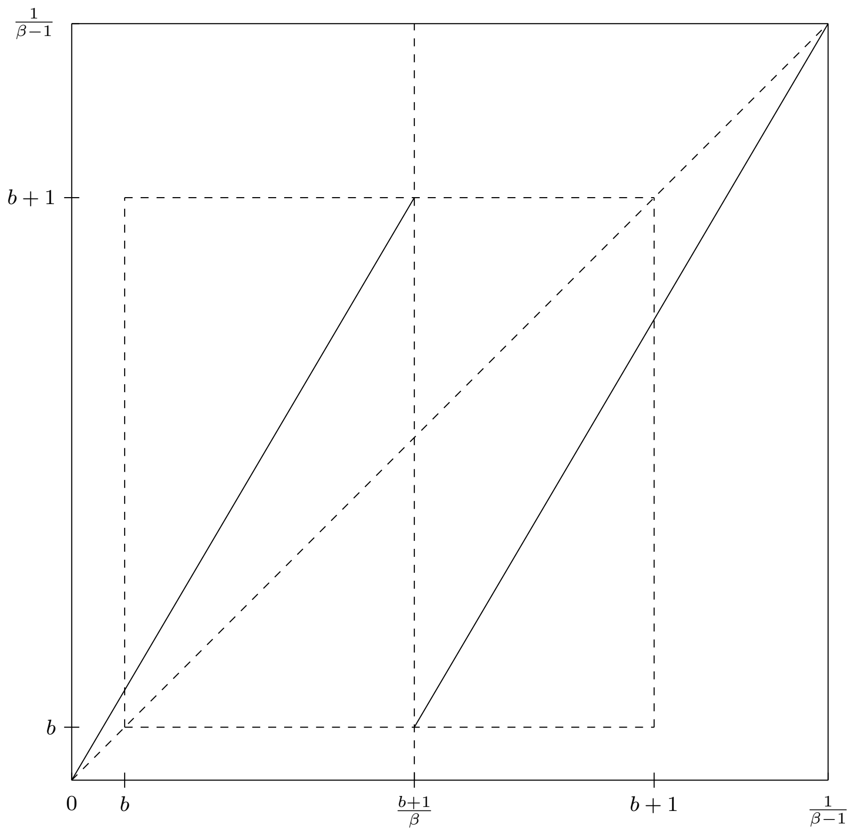

FIGURE 2. The graph of T.

Noting that $T_0(\frac{b+1}{\beta}) = b+1$ and $T_1(\frac{b+1}{\beta}) = b$, by Section 3 in [22], there exists a T-invariant ergodic measure $\mu \ll \mathcal{L}$ (Lebesgue measure) on I_β such that for $\mathcal{L}$-a.e. $x \in I_\beta$,

$$\frac{d\mu}{d\mathcal{L}}(x) = \sum_{n=0}^{\infty} \frac{\mathbb{1}_{[0,T^n(b+1)]}(x)}{\beta^n} - \sum_{n=0}^{\infty} \frac{\mathbb{1}_{[0,T^n(b)]}(x)}{\beta^n} \tag{3.5}$$

and $\nu := \frac{1}{\mu(I_\beta)} \cdot \mu$ is a T-invariant ergodic probability measure on I_β.

(1) For $1 \le n \le m-1$, prove $T^n(b) = \beta^n b < \frac{b+1}{\beta} \le \beta^n b + \beta^n - \beta^{n-1} - \cdots - \beta - 1 = T^n(b+1)$. Note that $\beta^m = \beta^{m-1} + \cdots + \beta + 1 = \frac{\beta^m - 1}{\beta - 1}$.
  ① By $b < \frac{1}{\beta-1} - 1 = \frac{1}{\beta^m - 1} \le \frac{1}{\beta^{n+1}-1}$, we get $\beta^n b < \frac{b+1}{\beta}$.
  ② By $\frac{1}{\beta} + \cdots + \frac{1}{\beta^{n+1}} \le \frac{1}{\beta} + \cdots + \frac{1}{\beta^m} = 1$, we get $\beta^n + \cdots + \beta + 1 \le \beta^{n+1}$ and then $\beta^n + \cdots + \beta + 1 + b \le \beta^{n+1} + \beta^{n+1} b$ which implies $\frac{b+1}{\beta} \le \beta^n b + \beta^n - \beta^{n-1} - \cdots - \beta - 1$.

(2) For $n \ge m$, prove $T^n(b) = T^n(b+1)$.
It suffices to prove $T^m(b) = T^m(b+1)$. In fact, this follows from (1) and $\beta^m b = \beta^m b + \beta^m - \beta^{m-1} - \cdots - \beta - 1$.

Combining (3.5) and (2), we know that for $\mathcal{L}$-a.e. $x \in I_\beta$,

$$\frac{d\mu}{d\mathcal{L}}(x) = \sum_{n=0}^{m-1} \frac{\mathbb{1}_{[0,T^n(b+1)]}(x) - \mathbb{1}_{[0,T^n(b)]}(x)}{\beta^n}. \tag{3.6}$$

Thus

$$\begin{aligned}
\mu[0,\frac{b+1}{\beta}) &= \int_0^{\frac{b+1}{\beta}} \frac{d\mu}{d\mathcal{L}}(x)dx \\
&= \sum_{n=0}^{m-1} \frac{\min\{T^n(b+1),\frac{b+1}{\beta}\}-\min\{T^n(b),\frac{b+1}{\beta}\}}{\beta^n} \\
&\overset{\text{by (1)}}{=\!=} \sum_{n=0}^{m-1} \frac{\frac{b+1}{\beta}-\beta^n b}{\beta^n} \\
&= 1-(m-1)b
\end{aligned}$$

where the last equality follows from $\frac{1}{\beta}+\cdots+\frac{1}{\beta^m}=1$. By

$$\begin{aligned}
\mu(I_\beta) &= \int_0^{\frac{1}{\beta-1}} \frac{d\mu}{d\mathcal{L}}(x)dx \\
&= \sum_{n=0}^{m-1} \frac{T^n(b+1)-T^n(b)}{\beta^n} \\
&\overset{\text{by (1)}}{=\!=} 1+\sum_{n=1}^{m-1} \frac{\beta^n-\beta^{n-1}-\cdots-\beta-1}{\beta^n} \\
&= 1+\sum_{n=1}^{m-1}(1-\frac{1}{\beta}-\cdots-\frac{1}{\beta^n}) \\
&= m-\frac{m-1}{\beta}-\frac{m-2}{\beta^2}-\cdots-\frac{1}{\beta^{m-1}},
\end{aligned}$$

we get

$$\frac{1}{\beta}\cdot\mu(I_\beta)=\frac{m}{\beta}-\frac{m-1}{\beta^2}-\frac{m-2}{\beta^3}-\cdots-\frac{1}{\beta^m}.$$

It follows from the subtraction of the above two equalities that $\mu(I_\beta)=\frac{m\beta+\beta-2m}{\beta-1}$. Therefore $\nu=\frac{\beta-1}{m\beta+\beta-2m}\cdot\mu$ and

$$\nu[0,\frac{b+1}{\beta})=\frac{(\beta-1)(1-(m-1)b)}{m\beta+\beta-2m}=f(b)=p.$$

Since $T:I_\beta\to I_\beta$ is ergodic with respect to ν, it follows from Birkhoff's Ergodic Theorem that for ν-a.e. $x\in I_\beta$ we have

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\mathbb{1}_{[0,\frac{b+1}{\beta})}T^k(x)=\int_0^{\frac{1}{\beta-1}}\mathbb{1}_{[0,\frac{b+1}{\beta})}d\nu=\nu[0,\frac{b+1}{\beta})=p,$$

which implies that for ν-a.e. $x\in[b,b+1]$,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\mathbb{1}_{[0,\frac{b+1}{\beta})}T^k(x)=p.$$

By (3.6) and (1), we know that for $\mathcal{L}$-a.e. $x \in [b, b+1]$, $\frac{d\mu}{d\mathcal{L}}(x) \geq 1$. This implies $\mathcal{L} \ll \mu(\sim \nu)$ on $[b, b+1]$, and then for $\mathcal{L}$-a.e. $x \in [b, b+1]$, we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \mathbb{1}_{[0,\frac{b+1}{\beta})} T^k(x) = p.$$

For every $x \in I_\beta$, define a sequence $(\varepsilon_i(x))_{i\geq 1} \in \{0,1\}^{\mathbb{N}}$ by

$$\varepsilon_i(x) := \begin{cases} 0 & \text{if } T^{i-1}x \in [0, \frac{b+1}{\beta}) \\ 1 & \text{if } T^{i-1}x \in [\frac{b+1}{\beta}, \frac{1}{\beta-1}] \end{cases} \quad \text{for all } i \geq 1.$$

Then by

$$\mathbb{1}_{[0,\frac{b+1}{\beta})}(T^k x) = 1 \Leftrightarrow T^k x \in [0, \frac{b+1}{\beta}) \Leftrightarrow \varepsilon_{k+1}(x) = 0,$$

we know that for $\mathcal{L}$-a.e. $x \in [b, b+1]$,

$$\lim_{n\to\infty} \frac{\sharp\{1 \leq i \leq n : \varepsilon_i(x) = 0\}}{n} = p, \quad \text{i.e.,} \quad \mathrm{Freq}_0(\varepsilon_i(x)) = p. \tag{3.7}$$

By the same way as in the proof of Lemma 3.1, we know that for every $x \in I_\beta$, the $(\varepsilon_i(x))_{i\geq 1}$ defined above is a β-expansion of x, and Lebesgue almost every $x \in I_\beta$ has a β-expansion with frequency of zeros equal to p. Then we finish the proof by applying Theorem 1.1. □

4. FURTHER QUESTIONS

First we wonder whether Theorem 1.1 can be generalized.

Question 4.1. Let $\beta \in (1, +\infty) \setminus \mathbb{N}$ and $\overline{p}, \underline{p} \in [0,1]^{\lceil \beta \rceil}$. Is it true that Lebesgue almost every $x \in I_\beta$ has a β-expansion of frequency $(\overline{p}, \underline{p})$ if and only if Lebesgue almost every $x \in I_\beta$ has a continuum of β-expansions of frequency $(\overline{p}, \underline{p})$?

If a positive answer is given to this question, by Theorem 1.1 and 1.2, there is also a positive answer to the following question.

Question 4.2. Let $\beta \in (2, +\infty) \setminus \mathbb{N}$. Is it true that Lebesgue almost every $x \in I_\beta$ has a continuum of balanced β-expansions?

Even if a negative answer is given to Question 4.1, there may be a positive answer to Question 4.2. An intuitive reason is that, when $\beta > 2$, we have $\sharp\mathcal{A}_\beta \geq 3$ and balanced β-expansions are much more flexible than simply normal β-expansions.

The last question we want to ask is on the variability of the frequency related to Theorem 1.3. Let $\beta > 1$. If there exists $c = c(\beta) > 0$ such that for any $p_0, p_1, \cdots, p_{\lceil\beta\rceil-1} \in [\frac{1}{\lceil\beta\rceil} - c, \frac{1}{\lceil\beta\rceil} + c]$ with $p_0 + p_1 + \cdots + p_{\lceil\beta\rceil-1} = 1$, every $x \in I_\beta^o$ has a β-expansion $(\varepsilon_i)_{i\geq 1}$ with

$$\mathrm{Freq}_0(\varepsilon_i) = p_0,\ \mathrm{Freq}_1(\varepsilon_i) = p_1, \cdots,\ \mathrm{Freq}_{\lceil\beta\rceil-1}(\varepsilon_i) = p_{\lceil\beta\rceil-1},$$

we say that β is a *variational frequency* base. Similarly, if there exists $c = c(\beta) > 0$ such that for any $p_0, p_1, \cdots, p_{\lceil\beta\rceil-1} \in [\frac{1}{\lceil\beta\rceil} - c, \frac{1}{\lceil\beta\rceil} + c]$ with $p_0 + p_1 + \cdots + p_{\lceil\beta\rceil-1} = 1$, Lebesgue almost every $x \in I_\beta$ has a β-expansion $(\varepsilon_i)_{i\geq 1}$ with

$$\mathrm{Freq}_0(\varepsilon_i) = p_0,\ \mathrm{Freq}_1(\varepsilon_i) = p_1, \cdots,\ \mathrm{Freq}_{\lceil\beta\rceil-1}(\varepsilon_i) = p_{\lceil\beta\rceil-1},$$

we say that β is an *almost variational frequency* base.

Obviously, all variational frequency bases are almost variational frequency bases. Baker's results (see the statements between Theorem 1.2 and Theorem 1.3) say that all numbers in

$(1, \frac{1+\sqrt{5}}{2})$ are variational frequency bases and all numbers in $[\frac{1+\sqrt{5}}{2}, 2)$ are not variational frequency bases. Fortunately, Theorem 1.3 says that pseudo-golden ratios (which are all in $[\frac{1+\sqrt{5}}{2}, 2)$) are almost variational frequency bases. We wonder whether all numbers in $[\frac{1+\sqrt{5}}{2}, 2)$ are almost variational frequency bases.

For all integers $\beta > 1$, we know that Lebesgue almost every $x \in [0, 1]$ has a unique β-expansion $(\varepsilon_i)_{i\geq 1}$, and this expansion satisfies

$$\mathrm{Freq}_0(\varepsilon_i) = \mathrm{Freq}_1(\varepsilon_i) = \cdots = \mathrm{Freq}_{\beta-1}(\varepsilon_i) = \frac{1}{\beta}$$

by Borel's normal number theorem. Therefore all integers are not almost variational frequency bases. It is natural to ask the following question.

Question 4.3. Is it true that all non-integers greater than 1 are almost variational frequency bases?

A positive answer is expected.

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INSTITUT DE MATHÉMATIQUES DE JUSSIEU - PARIS RIVE GAUCHE, SORBONNE UNIVERSITÉ - CAMPUS PIERRE ET MARIE CURIE, PARIS, 75005, FRANCE

E-mail address: yaoqiang.li@imj-prg.fr yaoqiang.li@etu.upmc.fr

SCHOOL OF MATHEMATICS, SOUTH CHINA UNIVERSITY OF TECHNOLOGY, GUANGZHOU, 510641, P.R. CHINA

E-mail address: scutyaoqiangli@gmail.com scutyaoqiangli@qq.com