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EXTENDING REPRESENTATION FORMULAE FOR BOUNDARY VOLTAGE PERTURBATIONS OF LOW VOLUME FRACTION TO VERY CONTRASTED CONDUCTIVITY INHOMOGENEITIES

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Imposing either Dirichlet or Neumann boundary conditions on the boundary of a smooth bounded domain Ω, we study the perturbation incurred by the voltage potential when the conductivity is modified in a set of small measure. We consider $(\gamma_n)_{n\in\mathbb{N}}$, a sequence of perturbed conductivity matrices differing from a smooth γ_0 background conductivity matrix on a measurable set well within the domain, and we assume $(\gamma_n-\gamma_0)\,\gamma_n^{-1}\,(\gamma_n-\gamma_0)\to 0$ in $L^1(\Omega)$. Adapting the limit measure, we show that the general representation formula introduced for bounded contrasts in [4] can be extended to unbounded sequences of matrix valued conductivities.

1. The general framework

Given $d\geq 2$, let $\Omega\subset\mathbb{R}^d$ be an open, bounded Lipschitz domain. We study the following family of solutions of perturbed boundary value problems for the conductivity equation. Given $g\in H^{\frac{1}{2}}(\partial\Omega)$, we consider $(u_n)_{n\in\mathbb{N}}\in H^1(\Omega)^{\mathbb{N}}$, a sequence of perturbations of $u_0\in H^1(\Omega)$ given by

$$\begin{cases}-\operatorname{div}(\gamma_0\nabla u_0) &=0 \quad\text{in}\quad \Omega,\\ u_0 &= g \quad\text{on}\quad \partial\Omega,\end{cases}\quad\text{and}\quad\begin{cases}-\operatorname{div}\left(\gamma_n\nabla u_n\right) &=0 \quad\text{in}\quad \Omega,\\ u_n &= g \quad\text{on}\quad \partial\Omega.\end{cases}\tag{1.1}$$

Alternatively, given $h\in H^{-\frac{1}{2}}(\partial\Omega)$ with $\int_{\partial\Omega}h\mathrm{d}\sigma=0$, we consider $(u_n)_{n\in\mathbb{N}}\in H^1(\Omega)^{\mathbb{N}}$, a sequence of perturbations of $u_0\in H^1(\Omega)$ given by

$$\begin{cases}-\operatorname{div}(\gamma_0\nabla u_0) &=0 \quad\text{in}\quad \Omega,\\ \gamma_0\nabla u_0\cdot n &= h \quad\text{on}\quad \partial\Omega,\\ \int_{\partial\Omega}u_0\mathrm{d}\sigma &=0,\end{cases}\quad\text{and}\quad\begin{cases}-\operatorname{div}\left(\gamma_n\nabla u_n\right) &=0 \quad\text{in}\quad \Omega,\\ \gamma_n\nabla u_n\cdot n &= h \quad\text{on}\quad \partial\Omega,\\ \int_{\partial\Omega}u_n\mathrm{d}\sigma &=0.\end{cases}\tag{1.2}$$

The conductivity coefficients are assumed to be symmetric positive definite matrix-valued functions with $\gamma_0\in W^{2,d}_{\text{loc}}\left(\mathbb{R}^d;\mathbb{R}^{d\times d}\right)$, $\gamma_n\in L^\infty\left(\Omega;\mathbb{R}^{d\times d}\right)$, and they satisfy the ellipticity condition

$$\lambda_0|\zeta|^2\leq\gamma_0\zeta\cdot\zeta\leq\Lambda_0|\zeta|^2\quad\text{and}\quad\lambda_n|\zeta|^2\leq\gamma_n\zeta\cdot\zeta\leq\Lambda_n|\zeta|^2,\quad\forall\zeta\in\mathbb{R}^d,$$

with $0<\lambda_n<\Lambda_n$ for all $n\in\mathbb{N}$.

Definition 1. Given $(A_n)_{n\in\mathbb{N}}$ and $(B_n)_{n\in\mathbb{N}}$, two sequences of measurable subsets of Ω whose Lebesgue measures tend to zero, we define $d_n\in L^\infty\left(\Omega;\mathbb{R}^{d\times d}\right)$, a positive semi-definite matrix valued function by

$$d_n=\left(\gamma_n+\gamma_0\gamma_n^{-1}\gamma_0\right)1_{A_n\cup B_n}.$$

We make the following assumptions on the inclusion sets.

Assumption. *We assume that the following assumptions are satisfied:*

(1) There exists K an open subset of Ω with C^∞ boundary such that $d(\partial K, \partial\Omega) > 0$ and
$$\bigcup_{n\in\mathbb{N}} (A_n \cup B_n) \subset K.$$
(2) The perturbation vanishes asymptotically in $L^1(\Omega)$, that is,
$$\|d_n\|_{L^1(\Omega)} \leq 1 \text{ and } \lim_{n\to\infty} \|d_n\|_{L^1(\Omega)} = 0.$$
(3) There holds, for all $n \geq 1$,
$$\gamma_n = \gamma_0 \text{ in } \Omega \setminus (B_n \cup A_n).$$
The sets A_n and B_n are disjoint and
$$\gamma_n \geq \gamma_0 \text{ a.e. in } A_n, \quad \gamma_n \leq \gamma_0 \text{ a.e. in } B_n$$
these inequalities being understood in the sense of quadratic forms.
(4) If $A_n \neq \emptyset$ for all n, we assume that one of the following integrability properties are satisfied:
   (a) There exists $p > d$ such that
$$\limsup_{n\to\infty} \|d_n\|_{L^p(A_n)} < \infty.$$
   (b) When $d = 2$, for some $p > 2$ there holds
$$\limsup_{n\to\infty} \|d_n\|_{L^p(B_n)} < \infty.$$
   (c) There exists $p > \frac{d}{2}$ such that
$$\limsup_{n\to\infty} \|d_n\|_{L^p(A_n)} < \infty,$$
   and there exists $\tau < \frac{1}{d-1}$ such that for all $n \in \mathbb{N}$,
$$d(A_n, B_n) > \|d_n\|^\tau_{L^1(A_n)}.$$

For $f \in L^p(\Omega)$, $1 \leq p \leq \infty$, $\|f\|_{L^p(\Omega)}$ is the canonical $L^P(\Omega)$ norm. For $U \in L^p(\Omega; \mathbb{R}^d)$ the notation $\|U\|_{L^p(\Omega)} = \||U|_d\|_{L^p(\Omega)}$ where $|\cdot|_d$ denotes the Euclidean norm in $\mathbb{R}^d$. For $A \in L^p(\Omega; \mathbb{R}^{d\times d})$, $\|A\|_{L^p(\Omega)}$ means $\||A|_F\|_{L^p(\Omega)}$ where $|\cdot|_F$ is the Frobenius norm, that is, the Euclidean norm on $\mathbb{R}^{d\times d}$. We remind the reader that $|AU|_d \leq |A|_F |U|_d$ a.e. in Ω, even though the Frobenius norm isn't the subordinate matrix norm associated with the Euclidean distance in $\mathbb{R}^d$. If A and B are non negative symmetric semi-definite matrices such that $A \leq B$ in the sense of quadratic forms, then $|A|_F \leq |B|_F$.

Remark 2. Definition 1 implies that on $A_n \cup B_n$,
$$d_n = (\gamma_n - \gamma_0)\gamma_n^{-1}(\gamma_n - \gamma_0) + 2\gamma_0.$$
Thus
$$d_n \geq 2\gamma_0 \text{ and } d_n > (\gamma_n - \gamma_0)\gamma_n^{-1}(\gamma_n - \gamma_0).$$
For all $x \in B_n$, $d_n > \gamma_0 \geq \gamma_n \geq \gamma_0 - \gamma_n \geq 0$. For all $x \in A_n$, $d_n = \gamma_n + \gamma_0\gamma_n^{-1}\gamma_0 \geq \gamma_n \geq \gamma_n - \gamma_0$. All in all, there holds

$$(1.3)\qquad \begin{cases} |d_n|_F & \geq |\gamma_0|_F \\ |d_n|_F & \geq |\gamma_n|_F \\ |d_n|_F & \geq |\gamma_n - \gamma_0|_F \\ |d_n|_F & \geq \left|(\gamma_n - \gamma_0)\,\gamma_n^{-1}\,(\gamma_n - \gamma_0)\right|_F \end{cases} \quad \text{a.e. on } A_n \cup B_n.$$

We will use these estimates frequently.

Remark 3. Assumption 1 comes from the fact that near the boundary of the domain, the behaviour of the solution is different, as the imposed boundary condition plays an increased role.

Assumption 2 is sufficient and sharp in general. Example 5 illustrates the fact that for some inclusions $u_n \not\to u_0$ when $\|d_n\|_{L^1(\Omega)} \not\to 0$.

Assumption 3 imposes a limitation for anisotropic conductivities since $A_n \cap B_n = \emptyset$: there cannot be an anisotropic inclusion which is very large in one direction and very small in another. In the case of isotropic materials, it is simply means that the inhomogeneities are located in A_n and B_n.

Assumption 4 imposes additional integrability properties for d_n only on highly conductive inclusions, not on insulating ones, in general. If $A_n = \emptyset$, assumption 4 is always satisfied. In dimension two, in the presence of both insulating and conductive inclusions, if they are arbitrarily mixed, an extra integrability of either of the two types of inclusions suffices. Alternatively, if the insulating and conductive inclusions are not too finely intertwined, a weaker integrability condition is required. While any of the conditions listed under assumption 4 is sufficient for our results to hold, it is not clear that an assumption is necessary. As far as the authors are aware, this is the first result allowing both very insulating and very highly conductive inclusions.

For any $y \in \Omega$, the Green function $G(\cdot, y)$ is the weak solution to the boundary value problem given by

$$\begin{aligned} \operatorname{div}\big(\gamma_0 \nabla G(\cdot, y)\big) &= \delta_y \text{ in } \Omega \\ G(\cdot, y) &= 0 \text{ on } \partial\Omega \end{aligned}$$

where δ_y denotes the Dirac measure at the point y, and the Neumann function $N(\cdot, y)$ is the weak solution to the boundary value problem given by

$$\begin{aligned} \operatorname{div}\big(\gamma_0 \nabla N(\cdot, y)\big) &= \delta_y \text{ in } \Omega \\ \gamma_0 \nabla N(\cdot, y) \cdot n &= \frac{1}{|\partial\Omega|} \text{ on } \partial\Omega. \end{aligned}$$

The main result of this article is that the general representation formula introduced in [4] can be extended to this context. This result was presented in a preliminary form in [11].

Theorem 4. *Let d_n be given by definition 1. Suppose that assumptions 1, 2, 3 and 4 hold. Then, there exists a subsequence also denoted by d_n and a matrix valued function $M \in L^2\big(\Omega, \mathbb{R}^{d\times d}; d\mu\big)$, where μ is the Radon measure generated by the sequence $\frac{1}{\|d_n\|_{L^1(\Omega)}} |d_n|_F$, such that for any $y \in \overline{\Omega \setminus K}$,*

- *if u_n and u_0 are solutions to* (1.1) *there holds*

$$u_n(y) - u_0(y) = \|d_n\|_{L^1(\Omega)} \int_\Omega M_{ij}(x) \frac{\partial u_0}{\partial x_i}(x) \frac{\partial G(x, y)}{\partial x_j} d\mu(x) + r_n(y),$$

- *if* u_n *and* u_0 *are solutions to* (1.2) *there holds*

$$u_n(y) - u_0(y) = \|d_n\|_{L^1(\Omega)} \int_\Omega M_{ij}(x) \frac{\partial u_0}{\partial x_i}(x) \frac{\partial N(x,y)}{\partial x_j} d\mu(x) + r_n'(y),$$

in which $r_n \in L^\infty(\overline{\Omega \setminus K})$ *(respectively* $r_n' \in L^\infty(\overline{\Omega \setminus K})$ *) satisfies* $\frac{\|r_n\|_{L^\infty(\Omega\setminus K)}}{\|d_n\|_{L^1(\Omega)}} \to 0$ *(resp.* $\frac{\|r_n'\|_{L^\infty(\Omega\setminus K)}}{\|d_n\|_{L^1(\Omega)}} \to 0$ *) uniformly in* $g \in H^{\frac{1}{2}}(\partial\Omega)$ *(resp.* $h \in H^{-\frac{1}{2}}(\partial\Omega)$ *) with* $\|g\|_{H^{\frac{1}{2}}(\partial\Omega)} \leq 1$ *satisfies (resp.* $\|h\|_{H^{-\frac{1}{2}}(\partial\Omega)} \leq 1$*).*

The matrix valued function $M \in L^2(\Omega, d\mu)$ *is symmetric. The tensor* M *can be written as* $M = D - W$*, where* W *satisfies*

$$0 \leq W\zeta\cdot\zeta \leq \zeta\cdot\zeta \quad \mu \text{ a.e. in } \Omega,$$

and if γ_n *and* γ_0 *are isotropic,*

$$0 \leq W\zeta\cdot\zeta \leq \frac{1}{\sqrt{d}}\zeta\cdot\zeta \quad \mu \text{ a.e. in } \Omega.$$

whereas D *is limit in the sense of measures of* $\|d_n\|_{L^1(\Omega)}^{-1}(\gamma_n - \gamma_1)$.

Definition 11 specifies the matrix valued function $W \in L^2(\Omega, \mathbb{R}^{d\times d}; \mathrm{d}\mu)$. The tensor M is, up to a factor, the polarisation tensor introduced in [4]. Its properties are briefly discussed in section §4, following [6].

The question of large contrast limits has been considered by other authors. In [10], the authors consider the case of diametrically bounded inclusions. In [7], the authors consider thin inhomogeneities. Unlike what is done in these articles, we do not go beyond the perturbation regime. On the other hand, in this work no geometric assumption is made on the shape of the inhomogeneities.

To document the sharpness of assumption 2, the following example shows that it may happen that the asymptotic limit of u_n is different from u for some sequence $(\gamma_n)_{n\in\mathbb{N}}$ when $\|d_n\|_{L^1(\Omega)} \not\to 0$ even though $|A_n \cup B_n| \to 0$.

Example 5. Suppose that $\Omega = B(0,2) \subset \mathbb{R}^d$, choose $A_n = B\left(0, 1+\frac{1}{n}\right) \setminus B\left(0, 1-\frac{1}{n}\right)$, and $g = x_1$. Then for $\gamma_0 = I_d$, the unperturbed solution of (1.1) corresponds to $u = x_1$.

Suppose that γ_n is radial and constant on $(I_i)_{i\leq 1\leq 4}$, where $I_1 = \left(0, 1-\frac{1}{n}\right)$, $I_2 = \left(1-\frac{1}{n}, 1\right)$. $I_3 = \left(1, 1+\frac{1}{n}\right)$, $I_4 = \left(1+\frac{1}{n}, 2\right)$, with values

$$\gamma_n = \chi_{I_1\cup I_4} + n^\alpha \chi_{I_2} + n^\beta \chi_{I_3},$$

where α, β are real parameters. Then,

$$\int_\Omega |d_n|_F \, \mathrm{d}x = \sqrt{d}\left(n^{\alpha-1} + n^{-\alpha-1} + n^{\beta-1} + n^{-\beta-1}\right)$$

and the solution u_n of (1.1) takes the form

$$u_n = \sum_{i=1}^4 A_i^n x_1 \mathbf{1}_{I_i}(|x|) + |x|^{-d} \sum_{i=2}^4 B_i^n x_1 \mathbf{1}_{I_i}(|x|),$$

for some constants $(A_i^n)_{1\leq i\leq 4}$ and $(B_i^n)_{2\leq i\leq 4}$. As $n \to \infty$, then $u_n \to v$ pointwise where $v = (\lim_{n\to\infty} A_1^n)\, x_1$ for $x < 1$ and $v = (\lim_{n\to\infty} A_4^n)\, x_1 + (\lim_{n\to\infty} B_4^n)\, |x|^{-d} x_1$ for $x > \frac{1}{2}$. Computing the value of the constants, we find that $(\lim_{n\to\infty} A_1^n) = (\lim_{n\to\infty} A_4^n) = 1$ and $(\lim_{n\to\infty} B_4^n) = 0$ if and only if $-1 < \alpha < 1$ and $-1 < \beta < 1$. We further note that if we write $\delta = \min(1+\alpha, 1+\beta, 1-\alpha, 1-\beta) > 0$, $u_n - x_1$ is of order $n^{-\delta}$. Written in a slightly different form, there exists a positive constant C depending on α,β and d but independent of n such that for all $n \geq 1$ there holds

$$C^{-1} \int_{\Omega} |d_n|_F \, \mathrm{d}x \leq \|u_n - x\|_{L^1(\Omega)} \text{ and } \|u_n - x\|_{L^\infty(\Omega)} \leq C \int_{\Omega} |d_n|_F \, \mathrm{d}x.$$

In this family of examples, the assumption $\int_\Omega |d_n|_F \, \mathrm{d}x \to 0$ is necessary for the perturbation regime to exist.

Following the steps in [4], the asymptotic formula that we derive makes use of

(1) A limiting Radon measure μ which describes the geometry of the limiting set,
(2) A background fundamental solution $G(x, y)$,
(3) A limit vector $\mathcal{M} \in [L^2(\Omega, \mathrm{d}\mu)]^d$ which describes the variations of the field ∇u_n in the presence of inhomogeneity sets,
(4) A polarisation tensor M, independent of u_n, u_0, the larger domain Ω and the type of boundary condition, such that $\mathcal{M} = M\nabla u_0$ in $L^2(\Omega, \mathrm{d}\mu)$.

This will be particularly familiar to readers acquainted to the subsequent article [6] where an energy-based approach is also used. It turns out that under assumption 1 and assumption 2 only, we can express the first order expansion in terms of $\mathcal{M}$.

Given $u_n, u_0 \in H^1(\Omega)$ given by (1.1) or (1.2), we define $w_n = u_n - u_0 \in X$ where $X = H^1_0(\Omega)$ for the Dirichlet problem and $X = \left\{\phi \in H^1(\Omega) : \int_\Omega \phi \, \mathrm{d}x = 0\right\}$ for the Neumann problem. Here, w_n is the weak solution of

$$\int_\Omega \gamma_n \nabla w_n \cdot \nabla \phi \mathrm{d}x = \int_\Omega (\gamma_0 - \gamma_n) \nabla u_0 \cdot \nabla \phi \mathrm{dx} \text{ for all } \phi \in X. \tag{1.4}$$

Note that if u_0 is the background solution of (1.1) or (1.2), then by classical regularity results [8, theorem 2.1], $u_0 \in H^1(\Omega) \cap C^1(K)$ and $\|u_0\|_{C^1(K)} \leq C(\Omega) \|g\|_{H^{\frac{1}{2}}(\partial\Omega)}$, or $\|u_0\|_{C^1(K)} \leq C(\Omega) \|h\|_{H^{-\frac{1}{2}}(\partial\Omega)}$ respectively.

Lemma 6. *Let $d_n \in L^\infty(\Omega; \mathbb{R}^{d\times d})$ be given by definition 1. Then, the sequence $\frac{|d_n|_F}{\|d_n\|_{L^1(\Omega)}}$ converges up to the possible extraction of a subsequence, in the sense of measures to a positive radon measure μ, that is,*

$$\int_\Omega \frac{1}{\|d_n\|_{L^1(\Omega)}} |d_n|_F \, \phi \, dx \to \int_\Omega \phi \, d\mu \text{ for all } \phi \in C(\overline{\Omega}). \tag{1.5}$$

For each $i, j \in \{1, \dots, d\}^2$, $\frac{1}{\|d_n\|_{L^1(\Omega)}} (\gamma_n - \gamma_0)_{ij}$ converges in the sense of measures to a limit $D_{ij} \in [L^2(\Omega, d\mu)]$

$$\int_\Omega \frac{1}{\|d_n\|_{L^1(\Omega)}} (\gamma_n - \gamma_0)_{ij} \, \phi \, dx \to \int_\Omega D_{ij} \, \phi \, d\mu \text{ for all } \phi \in C(\overline{\Omega}). \tag{1.6}$$

Proof. See appendix A. □

Remark 7. The sequence $\|d_n\|^{-1}_{L^1(\Omega)} |d_n|_F$ only converges to a given measure after extraction of a subsequence a priori. In the case of an isotropic, constant, conductivity in the inclusions, $\|d_n\|^{-1}_{L^1(\Omega)} |d_n|_F = 1_{A_n \cup B_n} |A_n \cup B_n|^{-1}$, and this measure does not depend on the values taken by γ_n or γ_0 on $A_n \cup B_n$.

The quantity d_n appears in the following energy estimate.

Proposition 8. *The weak solution of* (1.4) $w_n \in X$ *satisfies*

$$E(w_n) := \int_\Omega \gamma_n \nabla w_n \cdot \nabla w_n dx \leq \|d_n\|_{L^1(\Omega)} \|\nabla u_0\|^2_{L^\infty(K)}. \tag{1.7}$$

As a consequence, there holds

$$\|(\gamma_n - \gamma_0)\nabla w_n\|_{L^1(\Omega)} \leq \|d_n\|_{L^1(\Omega)} \|\nabla u_0\|_{L^\infty(K)}. \tag{1.8}$$

Furthermore, up to the possible extraction of a subsequence, $\frac{1}{\|d_n\|_{L^1(\Omega)}}(\gamma_0 - \gamma_n)\nabla w_n$ *converges in the sense of measures to a limit*

$$\int_\Omega \frac{1}{\|d_n\|_{L^1(\Omega)}}(\gamma_0 - \gamma_n)\nabla w_n \cdot \Psi\, dx \to \int_\Omega \mathcal{W} \cdot \Psi\, d\mu, \tag{1.9}$$

where $\mathcal{W} \in [L^2(\Omega, d\mu)]^d$ *and* μ *is given by* (1.5).

Remark 9. The upper estimates (1.7) and (1.8) are sharp with respect to the order of dependence on $\|d_n\|_{L^1(\Omega)}$ as shown in example 5.

Proof. The proof of proposition 8 is similar to the moderate contrast case in [4], but with estimates in terms of $\|d_n\|_{L^1(\Omega)}$. It is provided in appendix B. □

Under assumption 3 an improved Aubin–Céa–Nitsche estimate can be derived (lemma 14), which allows to consider extreme contrast and depends on the L^1 norm of d_n only. This allows in particular to show independence with respect to the domain and the prescribed boundary condition, as stated below (see also [6, lemma 1]).

Lemma 10. *Suppose that assumptions 1, 2, and 3 hold. Let* $\tilde{\Omega}$ *be any bounded regular open set such that* $K \subset \tilde{\Omega}$ *with* $dist(K, \tilde{\Omega}) > 0$. *Let* Y *be one of the spaces*

$$H^1_0(\tilde{\Omega}), \quad \tilde{H}^1(\tilde{\Omega}) := \left\{\phi \in H^1(\tilde{\Omega}) : \int_{\tilde{\Omega}\setminus K} \phi\, dx = 0\right\}$$

or

$$H^1_\#(\tilde{\Omega}) := \left\{\phi \in H^1_{loc}(\mathbb{R}^d) : \int_{\tilde{\Omega}\setminus K} \phi\, dx = 0 \text{ and } \phi \quad \tilde{\Omega} - \text{periodic}\right\},$$

the latter if $\tilde{\Omega}$ *is a cube. We write the weak solution of (1.4)* $w_n^X \in X$ *and we set* w_n^Y *to be the unique weak solution to*

$$\int_Q \gamma_n \nabla w_n^Y \cdot \nabla \phi dx = \int_Q (\gamma_0 - \gamma_n)\nabla u_0 \cdot \nabla \phi dx \text{ for all } \phi \in Y, \tag{1.10}$$

then for any $\tau \in \left(0, \frac{1}{2(d-1)}\right)$, *there exists* $C > 0$ *which may depend on* τ, Ω, K, Λ_0, λ_0 *and* $\|\gamma_0\|_{W^{2,d}(\Omega)}$ *only such that*

$$\frac{1}{\|d_n\|_{L^1(\Omega)}}\|(\gamma_n - \gamma_0)\nabla\left(w_n^Y - w_n^X\right)\|_{L^1(\Omega)} \leq C\|d_n\|^\tau_{L^1(\Omega)} \|\nabla u_0\|_{L^\infty(\Omega)}.$$

As a consequence, the measured valued vector $\mathcal{M}^X$ *and* $\mathcal{M}^Y$ *obtained from any two of these variational problems via proposition 8 are equal.*

The proof of this result is provided in section §2. It now suffices to focus on Dirichlet problem to establish theorem 4. To prove polarisability, that is, $\mathcal{M} = M\nabla u_0$, our argument requires one of the additional requirements detailed in assumption 4.

Definition 11. For each $i = 1, \ldots, d$, we define the correctors $w_n^i \in H_0^1(\Omega)$ as the weak solutions of

$$\int_\Omega \gamma_n \nabla w_n^i \cdot \nabla \phi \, \mathrm{d}x = \int_\Omega (\gamma_0 - \gamma_n) \, \mathbf{e}_i \cdot \nabla \phi \, \mathrm{d}x \text{ for all } \phi \in H_0^1(\Omega). \tag{1.11}$$

We call $W_{ij} \in L^2(\Omega, d\mu)$ the scalar weak* limit of $\frac{1}{\|d_n\|_{L^1(\Omega)}} \left(\nabla w_n^i \cdot (\gamma_0 - \gamma_n) \, \mathbf{e}_j \right)$.

Remark. The connection between this tensor and its parent introduced in [4] is discussed in section §4.

Proposition 12. *Suppose assumptions 1, 2, 3 and 4 are satisfied. Given Ω' a smooth open subset of Ω containing K such that $d(\Omega', \partial\Omega) > \frac{1}{3} d(K, \partial\Omega)$ and $d(K, \partial\Omega') > \frac{1}{3} d(K, \partial\Omega)$, there holds*

$$\int_\Omega (\gamma_n - \gamma_0) \nabla w_n \cdot \nabla x_i \, \phi \, dx = \int_\Omega (\gamma_n - \gamma_0) \nabla w_n^i \cdot \nabla u_0 \, \phi \, dx + \int_\Omega r_n \cdot \nabla \phi \, dx$$

with

$$\|r_n\|_{L^1(\Omega)} \leq C \, \|d_n\|_{L^1(\Omega)}^{1+\eta} \left(\|\nabla u_0\|_{L^\infty(K)} + \|u_0\|_{L^\infty(\partial\Omega')} \right),$$

where the positive constants C and η may depend only on τ, Ω, K, $\|\gamma_0\|_{W^{2,d}(\Omega)}$, Λ_0, λ_0, and possibly $\|d_n\|_{L^p(A_n)}$ or $\|d_n\|_{L^p(B_n)}$ for some p depending on which of the alternatives listed in assumption 4 is satisfied.

Proof. The proof of proposition 12 is the purpose of section §3. Depending on whether both insulating and conducting inhomogeneities are present, and whether the dimension is 2 or more, it is the combined conclusion of proposition 18, proposition 24 and proposition 26. ☐

We are now in position to conclude the proof of theorem 4, but for the properties of the polarisation tensor M, left for lemma 29.

End of the proof of theorem 4. Consider the Dirichlet case. Observing that the weak formulation for the solution $w_n = u_n - u_0$ reads

$$\int_\Omega \gamma_0 \nabla w_n \cdot \nabla \phi \mathrm{d}x = \int_\Omega (\gamma_0 - \gamma_n) (\nabla w_n + \nabla u_0) \cdot \nabla \phi \, \mathrm{d}x \tag{1.12}$$

for any $\phi \in H_0^1(\Omega)$, we choose a sequence $\phi_m \in C_c^1(\Omega)$ such that $\phi_m \to G_y$ in $W^{1,1}(\Omega)$ and $\phi_m \to \nabla G_y$ in $C^0(K)$. Using the fact that w_n is smooth away from the set K and the fact that $\gamma_n - \gamma_0$ is supported in K, we may insert ϕ_m into (1.12) and pass to the limit to conclude that

$$\int_\Omega \gamma_0 \nabla w_n \cdot \nabla_x G(x,y) \, \mathrm{d}x = \int_\Omega (\gamma_0 - \gamma_n) (\nabla u_0 + \nabla w_n) \cdot \nabla_x G(x,y) \, \mathrm{d}x.$$

After an integration by parts we obtain

$$\begin{aligned}
(u_n - u_0)(y) &= \int_\Omega (\gamma_n - \gamma_0) (\nabla w_n + \nabla u_0) \cdot \nabla_x G(x,y) \, \mathrm{d}x \\
&= \|d_n\|_{L^1(\Omega)} \int_\Omega \frac{1}{\|d_n\|_{L^1(\Omega)}} (\gamma_n - \gamma_0) \nabla u_0 \cdot \nabla_x G(x,y) \, \mathrm{d}x \\
&- \|d_n\|_{L^1(\Omega)} \int_\Omega \frac{1}{\|d_n\|_{L^1(\Omega)}} (\gamma_0 - \gamma_n) \nabla w_n \cdot \nabla_x G(x,y) \, \mathrm{d}x
\end{aligned}$$

Using the fact that $\forall y \in \overline{\Omega \setminus K}$ and $\forall x \in \cup_{n=1}^\infty (A_n \cup B_n)$, we may find a smooth function $\phi_y \in C^0(\overline{\Omega})$ such that

$$\phi_y(x) = \nabla_x G(x,y) \quad \forall x \in K,$$

and thanks to proposition 12, and lemma 6, we have

$$(u_n - u_0)(y) = \|d_n\|_{L^1(\Omega)} \int_\Omega (D_{ij} - W_{ij}) \frac{\partial u_0}{\partial x_i} \frac{\partial G(x,y)}{\partial x_j} \mathrm{d}\mu(x) + r_n(y),$$

where $W \in L^2\big(\Omega, \mathbb{R}^{d\times d}; \mathrm{d}\mu\big)$ is introduced in definition 11. Note that ϕ_y is uniformly bounded $\forall (x,y) \in K \times \overline{\Omega \setminus K}$. Moreover, the remainder estimate from proposition 12 only depends on $\|g\|_{H^{\frac{1}{2}}(\partial\Omega)}$, therefore $\|r_n\|_{L^\infty(\Omega)}/\|d_n\|_{L^1(\Omega)}$ converges to 0 uniformly in $y \in \overline{\Omega \setminus K}$ and g in the unit ball of the space $H^{\frac{1}{2}}(\partial\Omega)$. The Neumann case is similar. □

The rest of paper is structured as follows. In section §2 we derive a number of a priori estimates, and prove lemma 10. Section §3 is devoted to the proof of proposition 12. In section §4 we briefly discuss some of the properties of the tensor M, and prove lemma 29. Finally in section §5 we show with an example that the a priori bounds for M given in theorem 4 are attained.

2. Proof of lemma 10 and a priori estimates.

Lemma 13. *Given Ω' a smooth domain as defined in proposition 12, there holds*

$$\|u_n\|_{L^\infty(\partial\Omega')} + \|\nabla u_n\|_{L^\infty(\partial\Omega')} \le C\left(\|\nabla u_0\|_{L^\infty(K)} + \|u_0\|_{L^\infty(\partial\Omega')}\right),$$
$$\|w_n\|_{L^\infty(\partial\Omega')} + \|\nabla w_n\|_{L^\infty(\partial\Omega')} \le C\,\|w_n\|_{L^2(\Omega\setminus K)}$$

where $C>0$ depends on Ω', K, Ω, Λ_0, λ_0 and $\|\gamma_0\|_{W^{2,d}(\Omega)}$ only. Furthermore,

$$\|w_n\|_{L^\infty(K)} \le C\left(\|\nabla u_0\|_{L^\infty(K)} + \|u_0\|_{L^\infty(\partial\Omega')}\right). \tag{2.1}$$

Notation. In the sequel, we use the notation $a \lesssim b$ to mean $a \le Cb$, where C is a constant, possibly changing from line to line depending on the parameters announced in the claim we wish to prove.

Proof. Let Ω' and Ω'' be two open domains such that $K \subset \Omega'' \subset \Omega' \subset \Omega$, with $9d(\Omega'',\partial\Omega') > d(K,\partial\Omega)$ and $9d(K,\partial\Omega'') > d(K,\partial\Omega)$. Since

$$-\operatorname{div}(\gamma_0 \nabla w_n) = 0 \quad \text{on} \quad \Omega'' \setminus \Omega'$$

and $\gamma_0 \in W^{2,d}(\Omega)$, classical regularity theory shows that

$$\|w_n\|_{C^1\left(\overline{\Omega'\setminus\Omega''}\right)} \lesssim \|w_n\|_{L^2(\Omega\setminus K)}. \tag{2.2}$$

By Poincaré's inequality (or Poincaré-Wirtinger's inequality depending on X) since w_n vanishes on $\partial\Omega$, there holds

$$\|w_n\|_{L^2(\Omega\setminus K)} \lesssim \|\nabla w_n\|_{L^2(\Omega\setminus K)}.$$

On the other hand, using the fact that $\gamma_n = \gamma_0 \ge \lambda_0 I_d$ on $\Omega \setminus K$, there holds

$$\begin{aligned}\|\nabla w_n\|_{L^2(\Omega\setminus K)} &\le \frac{1}{\sqrt{\lambda_0}} \left(E(w_n)\right)^{\frac{1}{2}} \\ &\lesssim \|d_n\|_{L^1(\Omega)}^{\frac{1}{2}} \|\nabla u_0\|_{L^\infty(K)} \\ &\lesssim \|\nabla u_0\|_{L^\infty(K)},\end{aligned}$$

where we used (1.7) for the penultimate inequality and the fact that the sequence $\|d_n\|_{L^1(\Omega)}$ is bounded on the last line. Therefore on $\Omega \setminus \Omega'$, the function w_n satisfies $\operatorname{div}(\gamma_0 \nabla w_n) = 0$

with $|w_n| \lesssim \|\nabla u_0\|_{L^\infty(K)}$ on $\partial\Omega'$ and satisfies a homogeneous boundary condition on $\partial\Omega$ (or periodicity). By comparison, this implies

$$\|w_n\|_{L^\infty(\Omega\setminus\Omega')} \lesssim \|\nabla u_0\|_{L^\infty(K)}\,.$$

Furthermore, $u_n = w_n + u_0$ satisfies $\|u_n\|_{C^1(\partial\Omega')} \leq \|w_n\|_{C^1(\partial\Omega')} + \|u_0\|_{C^1(\partial\Omega')}$. Finally, since $\operatorname{div}(\gamma_n \nabla u_n) = 0$ on Ω', by comparison $\|u_n\|_{L^\infty(\Omega')} = \|u_n\|_{C(\partial\Omega')}$, and $\|w_n\|_{L^\infty(\Omega)} \leq \|w_n\|_{L^\infty(\Omega')} + \|w_n\|_{L^\infty(\Omega\setminus\Omega')} \lesssim \|\nabla u_0\|_{L^\infty(K)} + \|u_0\|_{L^\infty(\partial\Omega')}$ and the conclusion follows. □

Following the strategy introduced in [4], we now show that the potential tends to zero faster than the gradient via an Aubin–Céa–Nitsche argument. The novelty of this result is that it depends on γ_n only via on $\|d_n\|_{L^1(\Omega)}$.

Lemma 14. *For any $\tau \in \left[1, \frac{d}{d-1}\right)$, and given Ω' a smooth domain as defined in proposition 12, there holds*

$$\|w_n\|_{L^2(\Omega)} \leq C\, \|d_n\|_{L^1(\Omega)}^{\frac{\tau}{2}} \left(\|\nabla u_0\|_{L^\infty(K)} + \|u_0\|_{L^\infty(\partial\Omega')} \right), \tag{2.3}$$

with the constant C may depend on τ, Ω, K, $\|\gamma_0\|_{W^{2,d}(\Omega)}$, and the a priori bounds Λ_0 and λ_0 only.

Proof. Consider the following auxiliary equation

$$\begin{aligned} -\operatorname{div}(\gamma_0 \nabla \psi_n) &= w_n \quad \text{in} \quad \Omega \\ \psi_n &= 0 \quad \text{on} \quad \partial\Omega. \end{aligned} \tag{2.4}$$

Since $\gamma_0 \in W^{2,d}\left(\Omega; \mathbb{R}^{d\times d}\right)$ we infer from elliptic regularity theory (see e.g. [8]) that for any $q \geq 2$, the solution ψ_n satisfies

$$\|\psi_n\|_{W^{2,q}(\Omega)} + \|\psi_n\|_{W^{1,q}(\Omega)} \lesssim \|w_n\|_{L^q(\Omega)}\,. \tag{2.5}$$

Testing (2.4) with w_n, and recalling that $\operatorname{supp}(\gamma_n - \gamma_0) \subset (A_n \cup B_n) \subset K$, an integration by parts shows

$$\begin{aligned} \|w_n\|_{L^2(\Omega)}^2 &= \int_\Omega \gamma_0 \nabla\psi_n \cdot \nabla w_n \,\mathrm{d}x \\ &= \int_\Omega (\gamma_0 - \gamma_n) \nabla w_n \cdot \nabla \psi_n \,\mathrm{d}x + \int_\Omega \gamma_n \nabla\psi_n \cdot \nabla w_n \,\mathrm{d}x \\ &= \int_{A_n\cup B_n} (\gamma_0 - \gamma_n) \nabla w_n \cdot \nabla\psi_n + \int_{A_n\cup B_n} (\gamma_0 - \gamma_n) \nabla u_0 \cdot \nabla \psi_n \end{aligned} \tag{2.6}$$

Using Cauchy–Schwarz, we find

$$\begin{aligned} &\int_{A_n\cup B_n} (\gamma_0 - \gamma_n) \nabla w_n \cdot \nabla\psi_n \,\mathrm{d}x \\ \leq &\left(\int_{A_n\cup B_n} \gamma_n \nabla w_n \cdot \nabla w_n \,\mathrm{d}x \right)^{\frac{1}{2}} \left(\int_\Omega d_n \nabla\psi_n \cdot \nabla\psi_n \,\mathrm{d}x \right)^{\frac{1}{2}}, \end{aligned}$$

and thanks to (1.7),

$$\int_{A_n\cup B_n} (\gamma_0 - \gamma_n) \nabla w_n \cdot \nabla \psi_n \,\mathrm{d}x \leq \|d_n\|_{L^1(\Omega)} \|\nabla u_0\|_{L^\infty(K)} \|\nabla\psi_n\|_{L^\infty(K)}\,.$$

Similarly, using (1.3),

$$\begin{aligned}&\int_{A_n\cup B_n} (\gamma_0-\gamma_n)\nabla u_0\cdot\nabla\psi_n\,\mathrm{d}x\\ \leq&\left(\int_{A_n\cup B_n}\gamma_n\nabla u_0\cdot\nabla u_0\,\mathrm{d}x\right)^{\frac12}\left(\int_\Omega d_n\nabla\psi_n\cdot\nabla\psi_n\,\mathrm{d}x\right)^{\frac12}\\ \leq&\left\|d_n\right\|_{L^1(\Omega)}\left\|\nabla u_0\right\|_{L^\infty(K)}\left\|\nabla\psi_n\right\|_{L^\infty(K)}.\end{aligned}$$

and (2.6) becomes

$$\left\|w_n\right\|^2_{L^2(\Omega)} \leq 2\left\|d_n\right\|_{L^1(\Omega)}\left\|\nabla u_0\right\|_{L^\infty(K)}\left\|\nabla\psi_n\right\|_{L^\infty(K)}. \tag{2.7}$$

On the other hand, choosing $q = d+\epsilon$ in (2.5) there holds

$$\left\|\nabla\psi_n\right\|_{L^\infty(\Omega)} \lesssim \left\|\psi_n\right\|_{W^{2,d+\epsilon}(\Omega)} \lesssim \left\|w_n\right\|_{L^{d+\epsilon}(\Omega)} \tag{2.8}$$

By interpolation, and using the a priori bound (2.1) for w_n given in lemma 13, we find

$$\left\|w_n\right\|_{L^{d+\epsilon}(\Omega)} \leq \left\|w_n\right\|^{\frac{2}{d+\epsilon}}_{L^2(\Omega)}\left\|w_n\right\|^{1-\frac{2}{d+\epsilon}}_{L^\infty(\Omega)} \lesssim \left\|w_n\right\|^{\frac{2}{d+\epsilon}}_{L^2(\Omega)}\Big(\left\|\nabla u_0\right\|_{L^\infty(K)}+\left\|u_0\right\|_{L^\infty(\partial\Omega')}\Big)^{1-\frac{2}{d+\epsilon}}. \tag{2.9}$$

Combining (2.7), 2.8, and (2.9), we have obtained

$$\begin{aligned}\left\|w_n\right\|^{2\left(1-\frac{1}{d+\epsilon}\right)}_{L^2(\Omega)} &\lesssim \left\|d_n\right\|_{L^1(\Omega)}\left\|\nabla u_0\right\|_{L^\infty(K)}\Big(\left\|\nabla u_0\right\|_{L^\infty(K)}+\left\|u_0\right\|_{L^\infty(\partial\Omega')}\Big)^{1-\frac{2}{d+\epsilon}}\\ &\lesssim \left\|d_n\right\|_{L^1(\Omega)}\Big(\left\|\nabla u_0\right\|_{L^\infty(K)}+\left\|u_0\right\|_{L^\infty(\partial\Omega')}\Big)^{2\left(1-\frac{1}{d+\epsilon}\right)},\end{aligned}$$

which is equivalent to (2.3). □

Remark 15. Note that estimate (2.3) improves on previous estimates, even in the case of bounded contrasts (see [4, lemma 1]). It is arbitrarily close to the estimate one obtains for a fixed, scaled shape with constant scalar conductivity [1].

Corollary 16. *For any $q \geq 2$ and any $\tau \in \left[1, \frac{d}{d-1}\right)$, with the same notations as in lemma 14, there holds*

$$\left\|w_n\right\|_{L^q(\Omega)} \leq C\left\|d_n\right\|^{\frac{\tau}{q}}_{L^1(\Omega)}\Big(\left\|\nabla u_0\right\|_{L^\infty(K)}+\left\|u_0\right\|_{L^\infty(\partial\Omega')}\Big). \tag{2.10}$$

Furthermore, w_n solution of (1.4) *satisfies*

$$[\![\nabla w_n]\!]_{L^\infty(\partial\Omega')}+[\![w_n]\!]_{L^\infty(\partial\Omega')} \leq C\left\|d_n\right\|^{\frac{\tau}{2}}_{L^1(\Omega)}\Big(\left\|\nabla u_0\right\|_{L^\infty(K)}+\left\|u_0\right\|_{L^\infty(\partial\Omega')}\Big). \tag{2.11}$$

Proof. We write

$$\left\|w_n\right\|_{L^s(\Omega)} \leq \left\|w_n\right\|^{\frac{2}{q}}_{L^2(\Omega)}\left\|w_n\right\|^{1-\frac{2}{q}}_{L^\infty(\Omega)}$$

and estimate (2.10) follows from (2.3) and (2.1). Estimate (2.11) follows from lemma 13 and lemma 14. □

We now address the independence of the polarisation tensor M from the boundary conditions.

Proof of lemma 10. . Given $\tau = \left(0, \frac{1}{2}\frac{1}{d-1}\right)$, Following the steps of 14 with (1.10) and w_n^Y, we find

$$\left\|w_n^Y\right\|_{L^2\left(\tilde\Omega\right)} \lesssim \left\|d_n\right\|^{\frac{1+2\tau}{2}}_{L^1(\Omega)}\Big(\left\|\nabla u_0\right\|_{L^\infty(K)}+\left\|u_0\right\|_{L^\infty(\partial\Omega')}\Big). \tag{2.12}$$

Now, we choose a smooth cut-off function $\chi \in C_c^\infty\left(\tilde{\Omega}\right)$ such that $\chi = 1$ on K. Noting that $\operatorname{div}\left(\gamma_n \nabla\left(w_n^X - w_n^Y\right)\right) = 0$ on $\hat{\Omega}$, Caccioppoli's inequality writes

$$\int_{\tilde{\Omega}} \gamma_n \nabla\left(\chi\left(w_n^Y - w_n^X\right)\right) \cdot \nabla\left(\chi\left(w_n^Y - w_n^X\right)\right) \mathrm{d}x = \int_{\tilde{\Omega}\setminus K} \left(\gamma_0 \nabla\chi \cdot \nabla\chi\right)\left(w_n^Y - w_n^X\right)^2 \mathrm{d}x,$$

that is,

$$\begin{aligned}\int_{\tilde{\Omega}} \gamma_n \nabla\left(w_n^Y - w_n^X\right) \cdot \nabla\left(w_n^Y - w_n^X\right) \mathrm{d}x &\leq C\left(\tilde{\Omega}, K\right)\left(\left\|w_n^X\right\|_{L^2(\Omega)}^2 + \left\|w_n^Y\right\|_{L^2(\tilde{\Omega})}^2\right), \\ &\lesssim \|d_n\|_{L^1(\Omega)}^{1+2\tau}\left(\|\nabla u_0\|_{L^\infty(K)} + \|u_0\|_{L^\infty(\partial\Omega')}\right)^2\end{aligned}$$

This in turn shows, by Cauchy-Schwarz,

$$\begin{aligned}\|\left(\gamma_n - \gamma_0\right)\nabla\left(w_n^Y - w_n^X\right)\|_{L^1(\Omega)} &\leq \|d_n\|_{L^1(\Omega)}^{\frac{1}{2}}\left(\int_K \gamma_n \nabla\left(w_n^Y - w_n^X\right) \cdot \nabla\left(w_n^Y - w_n^X\right) \mathrm{d}x\right)^{\frac{1}{2}} \\ &\leq C\left(\tilde{\Omega}, K\right)\|d_n\|_{L^1(\Omega)}^{1+\tau}\left(\|\nabla u_0\|_{L^\infty(K)} + \|u_0\|_{L^\infty(\partial\Omega')}\right)\end{aligned}$$

As a result, $\frac{1}{\|d_n\|_{L^1(\Omega)}}\left\|\left(\gamma_n - \gamma_0\right)\nabla\left(w_n^Y - w_n^X\right)\right\|_{L^1(\Omega)} \to 0$, which implies equivalence that the limiting measures resulting from $\frac{1}{\|d_n\|_{L^1(\Omega)}}\left(\gamma_n - \gamma_0\right)\nabla w_n^X$ and $\frac{1}{\|d_n\|_{L^1(\Omega)}}\left(\gamma_n - \gamma_0\right)\nabla w_n^Y$ are equal. □

3. Proof of proposition 12

We use the following corollary to the a priori energy estimate given in proposition 8.

Corollary (Corollary to proposition 8)**.** *For any $p \geq 1$, there holds*

$$\|\gamma_n \nabla w_n\|_{L^{\frac{2p}{p+1}}(A_n)} \leq d^{\frac{1}{4}}\|d_n\|_{L^1(\Omega)}^{\frac{1}{2}}\|d_n\|_{L^p(A_n)}^{\frac{1}{2}}\|\nabla u_0\|_{L^\infty(K)}. \tag{3.1}$$

Proof. Using Hölder's inequality, it holds that for any $p \geq 1$

$$\|\gamma_n \nabla w_n\|_{L^{\frac{2p}{p+1}}(A_n)} \leq \left\|\gamma_n^{\frac{1}{2}}\right\|_{L^{2p}(A_n)}\left(E\left(w_n\right)\right)^{\frac{1}{2}}. \tag{3.2}$$

We have

$$\left\|\gamma_n^{\frac{1}{2}}\right\|_{L^{2p}(A_n)} = \left(\int_{A_n}\left|\gamma_n^{\frac{1}{2}}\right|_F^{2p} \mathrm{d}x\right)^{\frac{1}{2p}},$$

and, using the fact that for $d \times d$ symmetric matrix A, $\left|A^2\right|_F \leq |A|_F^2 \leq \sqrt{d}\left|A^2\right|_F$,we find, using (1.3),

$$\left\|\gamma_n^{\frac{1}{2}}\right\|_{L^{2p}(A_n)} \leq d^{\frac{1}{4}}\left(\int_{A_n}|\gamma_n|_F^p \mathrm{d}x\right)^{\frac{1}{2p}} = d^{\frac{1}{4}}\|\gamma_n\|_{L^p(A_n)}^{\frac{1}{2}} \leq d^{\frac{1}{4}}\|d_n\|_{L^p(A_n)}^{\frac{1}{2}}. \tag{3.3}$$

Putting together (1.7), (3.2) and (3.3) the conclusion follows. □

The following error estimate is a key tool for the proof of proposition 12

Proposition 17. *For any $\phi \in C^1(\overline{\Omega})$, there holds*

$$\int_\Omega \Big(\Big(\gamma_n - \gamma_0\Big)\nabla w_n \cdot \nabla x_i\Big)\,\phi\,dx = \int_\Omega \Big(\Big(\gamma_n - \gamma_0\Big)\nabla w_n^i \cdot \nabla u_0\Big)\,\phi\,dx + \int_\Omega r_n \cdot \nabla\phi\,dx \tag{3.4}$$

with $r_n \in L^1(\Omega)$. Furthermore for any $\tau \in \big[1, \frac{2d-1}{2d-2}\big)$, the following estimate holds

$$\left|\int_\Omega r_n \cdot \nabla\phi\,dx\right| \le C\,\|\nabla\phi\|_{L^\infty(\Omega)}\Big(\|d_n\|^\tau_{L^1(\Omega)}\Big(\|\nabla u_0\|_{L^\infty(K)} + \|u_0\|_{L^\infty(\partial\Omega')}\Big) + \varepsilon_n\Big), \tag{3.5}$$

The constant C may depends on τ, Ω, Ω', K, $\|\gamma_0\|_{W^{2,d}(\Omega)}$, and the a priori bounds Λ_0 and λ_0 only. The remainder term ε_n satisfies the following two a priori estimates

$$\varepsilon_n \le \|d_n\|_{L^1(\Omega)}\Big(\|w_n\|_{L^\infty(A_n)} + \big\|w_n^i\big\|_{L^\infty(A_n)}\|\nabla u_0\|_{L^\infty(K)}\Big) \tag{3.6}$$

and, for $p > d$,

$$\varepsilon_n \le \|d_n\|^{1+\eta}_{L^1(\Omega)}\,\|d_n\|^{\frac12}_{L^p(A_n)}\Big(\|\nabla u_0\|_{L^\infty(K)} + \|u_0\|_{L^\infty(\partial\Omega')}\Big). \tag{3.7}$$

where $\eta > 0$ depends only on p.

Remark. Note that estimates (3.6) and (3.7) imply that $\epsilon_n \le 0$ when $A_n = \emptyset$.

Proof. We write Z as a shorthand for $\|\nabla u_0\|_{L^\infty(K)} + \|u_0\|_{L^\infty(\partial\Omega')}$. A computation shows that

$$\int_\Omega \left(\left(\gamma_n - \gamma_0\right)\nabla w_n \cdot \nabla x_i\right)\phi\,\mathrm{d}x = \int_\Omega \left(\left(\gamma_0 - \gamma_n\right)\nabla w_n^i \cdot \nabla u_0\right)\,\phi\,\mathrm{d}x + \int_\Omega r_n \cdot \nabla\phi\,\mathrm{d}x$$

where the remainder term $r_n \in L^1(\Omega)$ is

$$r_n = \left(\gamma_n - \gamma_0\right)\left(w_n^i \nabla u_0 - w_n \nabla x_i\right) + w_n^i \gamma_n \nabla w_n - w_n \gamma_n \nabla w_n^i.$$

Now, write $T_1 = \mathbf{1}_{\gamma_n \le \gamma_0}\left(r_n \cdot \nabla\phi\right)$ and $T_2 = r_n \cdot \nabla\phi - T_1$.

$$\begin{aligned}
\|T_1\|_{L^1(\Omega)} \le{}& \int_{\Omega\cap\{\gamma_n\le\gamma_0\}} \left|w_n\left(\gamma_n \nabla w_n^i\right)\cdot\nabla\phi\right|\,\mathrm{d}x + \int_{\Omega\cap\{\gamma_n\le\gamma_0\}} \left|w_n^i\left(\gamma_n\nabla w_n\right)\cdot\nabla\phi\right|\,\mathrm{d}x \\
&+ \int_{\Omega\cap\{\gamma_n\le\gamma_0\}} \left|w_n^i\left(\gamma_n - \gamma_0\right)\nabla u_0\cdot\nabla\phi\right|\,\mathrm{d}x + \int_{\Omega\cap\{\gamma_n\le\gamma_0\}} \left|w_n\left(\gamma_n-\gamma_0\right)\nabla x_i\cdot\nabla\phi\right|\,\mathrm{d}x \\
\le{}& \|\nabla\phi\|_{L^\infty(\Omega)}\,\|\gamma_0\|^{\frac12}_{L^\infty(\Omega)}\Big(\|w_n\|_{L^2(\Omega)}\,E\left(w_n^i\right)^{\frac12} + \left\|w_n^i\right\|_{L^2(\Omega)}E\left(w_n\right)^{\frac12} \\
&+ \left\|w_n^i\right\|_{L^2(\Omega)}\|d_n\|^{\frac12}_{L^1(\Omega)}\|\nabla u_0\|_{L^\infty(K)} + \|w_n\|_{L^2(\Omega)}\|d_n\|^{\frac12}_{L^1(\Omega)}\Big)
\end{aligned}$$

Thanks to estimate (1.7) and (2.3) (applied to $u_0 = x_i$ for the corrector terms w_n^i) we find

$$\|T_1\|_{L^1(\Omega)} \lesssim \|d_n\|^{\frac12+\frac12\tau'}_{L^1(\Omega)}\,\|\nabla\phi\|_{L^\infty(\Omega)}\,Z,$$

with $\tau' \in \big[1, \frac{d}{d-1}\big)$, so that $\tau = \frac{1+\tau'}{2} \in \big[1, \frac{2d-1}{2d-2}\big)$. We now turn to the other term. The triangle inequality gives

$$\begin{aligned}
\|T_2\|_{L^1(\Omega)} \le{}& \int_{A_n}\left|w_n\left(\gamma_n\nabla w_n^i\right)\cdot\nabla\phi\right|\,\mathrm{d}x + \int_{A_n}\left|w_n^i\left(\gamma_n\nabla w_n\right)\cdot\nabla\phi\right|\,\mathrm{d}x \\
&+ \int_{A_n}\left|w_n^i\left(\gamma_n-\gamma_0\right)\nabla u_0\cdot\nabla\phi\right|\,\mathrm{d}x + \int_{A_n}\left|w_n\left(\gamma_n-\gamma_0\right)\nabla x_i\cdot\nabla\phi\right|\,\mathrm{d}x.
\end{aligned} \tag{3.8}$$

Recall that thanks to (1.3), $|\gamma_n - \gamma_0|_F < |d_n|_F$. Thus using (3.1) with $p = 1$, and (1.3), we deduce from (3.8) that

$$\|T_2\|_{L^1(\Omega)} \lesssim \|d_n\|_{L^1(\Omega)} \left(\|w_n\|_{L^\infty(A_n)} + \left\|w_n^i\right\|_{L^\infty(A_n)} \|\nabla u_0\|_{L^\infty(K)} \right) \|\nabla \phi\|_{L^\infty(K)},$$

which corresponds to estimate (3.6).

Alternatively, applying Hölder's inequality, then the L^p bound (3.1) and the L^q bound (2.10) with the conjugate exponent, we find for any $p \geq 1$, and any $\theta \in \left[1, \frac{d}{d-1}\right)$,

$$\begin{aligned} &\int_{A_n} \left|w_n \gamma_n \nabla w_n^i \cdot \nabla \phi\right| \,\mathrm{d}x \\ &\leq \left\|\gamma_n \nabla w_n^i\right\|_{L^{\frac{2p}{p+1}}(A_n)} \|w_n\|_{L^{\frac{2p}{p-1}}(A_n)} \|\nabla \phi\|_{L^\infty(K)} \\ &\lesssim \|d_n\|_{L^1(\Omega)}^{\frac12} \|d_n\|_{L^p(A_n)}^{\frac12} \|d_n\|_{L^1(\Omega)}^{\left(\frac12 - \frac{1}{2p}\right)\theta} Z \|\nabla \phi\|_{L^\infty(K)}. \end{aligned}$$

Similarly

$$\int_{A_n} \left|w_n^i \gamma_n \nabla w_n \cdot \nabla \phi\right| \,\mathrm{d}x \lesssim \|d_n\|_{L^1(\Omega)}^{\frac12} \|d_n\|_{L^p(A_n)}^{\frac12} \|d_n\|_{L^1(\Omega)}^{\left(\frac12 - \frac{1}{2p}\right)\theta} Z \|\nabla \phi\|_{L^\infty(K)}.$$

Using (1.3), Hölder's inequality and the L^s bounds (2.10), we write

$$\begin{aligned} \int_{A_n} \left|w_n^i (\gamma_n - \gamma_0) \nabla u_0 \cdot \nabla \phi\right| \mathrm{d}x &\leq \left\|d_n^{\frac12}\right\|_{L^2(A_n)} \left\|d_n^{\frac12} w_n^i\right\|_{L^2(A_n)} \|\nabla u_0\|_{L^\infty(K)} \|\nabla \phi\|_{L^\infty(K)} \\ &\lesssim \|d_n\|_{L^1(A_n)}^{\frac12} \|d_n\|_{L^p(A_n)}^{\frac12} \left\|w_n^i\right\|_{L^{\frac{2p}{p-1}}(A_n)} \|\nabla u_0\|_{L^\infty(K)} \|\nabla \phi\|_{L^\infty(K)} \\ &\lesssim \|d_n\|_{L^1(\Omega)}^{\frac12} \|d_n\|_{L^p(A_n)}^{\frac12} \|d_n\|_{L^1(\Omega)}^{\left(\frac12 - \frac{1}{2p}\right)\theta} Z \|\nabla \phi\|_{L^\infty(K)}, \end{aligned}$$

and by the same argument,

$$\int_{A_n} |w_n (\gamma_n - \gamma_0) \nabla x_i \cdot \nabla \phi| \,\mathrm{d}x \lesssim \|d_n\|_{L^1(\Omega)}^{\frac12} \|d_n\|_{L^p(A_n)}^{\frac12} \|d_n\|_{L^1(\Omega)}^{\left(\frac12 - \frac{1}{2p}\right)\theta} Z \|\nabla \phi\|_{L^\infty(K)}.$$

Altogether, for any $p \geq 1$, and any $\theta \in \left[1, \frac{d}{d-1}\right)$,

$$\|T_2\|_{L^1(\Omega)} \lesssim \|d_n\|_{L^1(\Omega)}^{\frac12} \|d_n\|_{L^p(A_n)}^{\frac12} \|d_n\|_{L^1(\Omega)}^{\left(\frac12 - \frac{1}{2p}\right)\theta} Z \|\nabla \phi\|_{L^\infty(K)}.$$

For any $p > d$, pick $\theta = \frac12 \left(\frac{p}{p-1} + \frac{d}{d-1}\right)$, then

$$\eta = \frac12 \left(\frac{d}{d-1} \frac{p-1}{p} - 1 \right) > 0,$$

and

$$\|T_2\|_{L^1(\Omega)} \leq \|d_n\|_{L^1(\Omega)}^{1+\eta} \|d_n\|_{L^p(A_n)}^{\frac12} Z \|\nabla \phi\|_{L^\infty(K)},$$

which concludes the proof of estimate (3.7). □

Proposition 18. *Suppose assumptions 1, 2, and 3 are satisfied. Additionally assume that either $A_n = \emptyset$, or assumption 4a holds. Given Ω' a smooth domain as defined in proposition 12, there holds*

$$\int_\Omega \left(\left(\gamma_n - \gamma_0\right) \nabla w_n \cdot \nabla x_i \right) \phi \, dx = \int_\Omega \left(\left(\gamma_n - \gamma_0\right) \nabla w_n^i \cdot \nabla u_0 \right) \phi \, dx + \int_\Omega r_n \cdot \nabla \phi \, dx$$

with

$$\|r_n\|_{L^1(\Omega)} \leq C \|d_n\|_{L^1(\Omega)}^{1+\eta} \left(\|\nabla u_0\|_{L^\infty(K)} + \|u_0\|_{L^\infty(\partial\Omega')} \right),$$

where the positive constants C and η may depend only on τ, Ω, K, $\|\gamma_0\|_{W^{2,d}(\Omega)}$, Λ_0 and λ_0 and $\|d_n\|_{L^p(A_n)}$.

Proof. This is an immediate consequence of proposition 17. □

3.1. **The high conductivity inclusion case when $d = 2$.** This section addresses the case when assumption 4b holds. When $d = 2$, as it is well known, there is a direct relation between high and low conductivity problem, by means of stream functions (see e.g. [9]). We use this indirect method to obtain the polarisability result under assumption 4b. We remind the reader of the following classical result.

Lemma 19 ([Lemma I.1 2])**.** *Let Ω be any smooth open set in $\mathbb{R}^2$, not necessarily simply connected, and D be a vector field such that*

$$div(D) = 0 \text{ on } \Omega, \text{ and } \int_{\Gamma_i} D \cdot n d\sigma = 0$$

on each connected component Γ_i of $\partial\Omega$. Then, there exists a function H such that

$$D = (-\partial_{x_2} H, \partial_{x_1} H) \text{ on } \Omega.$$

Let $(\Gamma_i)_{1\leq i\leq N}$ the connected components of $\partial\Omega$ and let Fb_n and Fb_0 the unique solutions of

$$\begin{cases} \operatorname{div}(\gamma_n \nabla Fb_n) = 0 & \text{on } \Omega', \\ \gamma_n \nabla Fb_n \cdot n = \frac{1}{|\Gamma_i|}\int_{\Gamma_i} \gamma_n \nabla u_n \cdot n \mathrm{d}\sigma & \text{on each } \Gamma_i. \\ \int_\Omega Fb_n \mathrm{d}x = 0. \end{cases} \tag{3.9}$$

and

$$\begin{cases} \operatorname{div}(\gamma_0 \nabla Fb_0) = 0 & \text{on } \Omega', \\ \gamma_0 \nabla Fb_0 \cdot n = \frac{1}{|\Gamma_i|}\int_{\Gamma_i} \gamma_0 \nabla u_0 \cdot n \mathrm{d}\sigma & \text{on each } \Gamma_i. \\ \int_\Omega Fb_0 \mathrm{d}x = 0. \end{cases} \tag{3.10}$$

Then applying lemma 19 to $\gamma_n \nabla(u_n - Fb_n)$ and $\gamma_0 \nabla(u_0 - Fb_0)$ there exist stream functions $\psi_n, \psi_0 \in H^1(\Omega')$ such that

$$\gamma_n \nabla(u_n - Fb_n) = J\nabla\psi_n \text{ and } \gamma_0 \nabla(u_0 - Fb_0) = J\nabla\psi_0 \text{ a.e. in } \Omega'. \tag{3.11}$$

where J is the antisymmetric matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. As the stream functions may be chosen uniquely up to an additive constant, we may assume without loss of generality that they satisfy the constraint

$$\int_\Omega \psi_n \mathrm{d}x = 0 = \int_\Omega \psi_0 \mathrm{d}x.$$

Thus, ψ_n and ψ_0 are weak solutions of

$$-\operatorname{div}(\sigma_n \nabla\psi_n) = 0 \text{ in } \Omega'$$
$$-\operatorname{div}(\sigma_0 \nabla\psi_0) = 0 \text{ in } \Omega'$$

where the conductivity matrices σ_n and σ_0 are defined as

$$\sigma_n := J^T\gamma_n^{-1}J \quad \text{and} \quad \sigma_0 := J^T\gamma_0^{-1}J.$$

When then define Σ_n as d_n was with respect to γ_0 and γ_n,that is

Definition 20. We set

$$\Sigma_n = \left(\sigma_n + \sigma_0\sigma_n^{-1}\sigma_0\right) 1_{A_n\cup B_n}.$$

Proposition 21. *Given Ω' a smooth domain as defined in proposition 12, given ψ_n and ψ_0 be the stream functions defined in* (3.11)*. The function $\varphi_n = \psi_n - \psi_0$ satisfies*

$$-\, div\,(\sigma_n \nabla \varphi_n) = div\,((\sigma_n - \sigma_0)\,\nabla \psi_0) \ \ in\ \mathcal{D}'\,(\Omega') \tag{3.12}$$

and for any $\tau \in \left(0, \frac{1}{2}\right)$ there holds

$$\|\sigma_n \nabla \varphi_n \cdot \nu\|_{H^{-\frac{1}{2}}(\partial\Omega')} \leq C\, \|d_n\|_{L^1(\Omega)}^{\frac{1}{2}+\tau}\, \|g\|_{H^{\frac{1}{2}}(\partial\Omega')}\,, \tag{3.13}$$

where the constant C may depend only on τ, Ω, K, $\|\gamma_0\|_{W^{2,d}(\Omega)}$, Λ_0 and λ_0 .

Proof. Thanks to (3.11), since $d\,(\partial\Omega', K) > 0$, on $\partial\Omega'$

$$\begin{aligned}\sigma_n \nabla \varphi_n = \sigma_0 \nabla \varphi_n &= J^T \nabla\,(u_n - F b_n - u_0 - F b_0)\\ &= J^T \nabla\,(w_n + F b_n - F b_0)\,.\end{aligned}$$

Thanks to estimate (2.11) applied to w_n and to Fb_n and Fb_0, there holds

$$\|\sigma_n \nabla \varphi_n\|_{L^\infty(\partial\Omega')} \leq C\, \|d_n\|_{L^1(\Omega)}^{\frac{1}{2}+\tau} \left(\|\nabla u_0\|_{L^\infty(K)} + \|u_0\|_{L^\infty(\partial\Omega')}\right).$$

which implies (3.13). □

We note that the role of B_n and A_n are swapped when considering (3.12) rather than (1.4). The polarisability for φ_n is therefore established from proposition 18 provided $\|d_n\|_{L^p(B_n)} < \infty$ for some $p > 2$.

Corollary 22. *Suppose that Assumptions 1, 2, and 3 are satisfied. Additionally assume that $d = 2$ and for some $p > 2$,*

$$\limsup_n \|\Sigma_n\|_{L^p(B_n)} < \infty.$$

The function $\varphi_n = \psi_n - \psi_0$, the weak solution to (3.12)*, satisfies*

$$\frac{1}{\|\Sigma_n\|_{L^1(\Omega)}} \Big(\sigma_0 - \sigma_n\Big) \nabla \varphi_n \, dx \ \ \overset{*}{\rightharpoonup} \ \ \tilde{N} \nabla \psi_0 \, d\nu \tag{3.14}$$

in the space of bounded Radon measures where $\tilde{N} \in L^2\big(\Omega, \mathbb{R}^{d\times d}; d\nu\big)$, and ν is the Radon measure generated by the sequence $\frac{1}{\|\Sigma_n\|_{L^1(\Omega)}}\Sigma_n$. The convergence is uniform with respect to $g \in H^{1/2}\,(\partial\Omega)$ provided $\|g\|_{H^{\frac{1}{2}}(\partial\Omega)} \leq 1$.

Proof. The proof follows directly from proposition 17 and lemma 10. □

Lemma 23. *The symmetric positive definite matrix Σ_n given by definition 20 satisfies*

$$\Sigma_n = \sigma_n + \sigma_0 \sigma_n^{-1} \sigma_0 = J^T \gamma_0^{-1} d_n \gamma_0^{-1} J.$$

As a consequence, denoting ν and μ to be the Radon measures generated by the sequences $\frac{\Sigma_n}{\|\Sigma_n\|_{L^1(\Omega)}}$ and $\frac{d_n}{\|d_n\|_{L^1(\Omega)}}$ respectively, the Radon-Nikodym derivatives $\frac{d\nu}{d\mu}$ and $\frac{d\mu}{d\nu}$ belongs to $L^\infty\,(\Omega; d\mu)$ and $L^\infty\,(\Omega; d\nu)$ respectively, and the spaces $L^p\,(\Omega; d\mu)$ are equivalent to $L^p\,(\Omega; d\nu)$ for any $p > 1$.

Proof. The formula $\Sigma_n = J^T \gamma_0^{-1} d_n \gamma_0^{-1} J$ is straightforward to verify. It follows that

$$|d_n|_F \left(\min_{\overline{\Omega}} \lambda\left(\gamma_0^{-1}\right)\right)^2 \leq |\Sigma_n|_F \leq |d_n|_F \left(\max_{\overline{\Omega}} \lambda\left(\gamma_0^{-1}\right)\right)^2. \tag{3.15}$$

Since these two quantities are equivalent, the conclusion follows. □

Proposition 24. *Suppose Assumptions 1, 2, and 3 are satisfied. Additionally assume that $d = 2$ and for some $p > 2$,*

$$\limsup_{n} \|d_n\|_{L^p(B_n)} < \infty.$$

Given Ω' a smooth domain as defined in proposition 12, there holds

$$\int_\Omega ((\gamma_n - \gamma_0) \nabla w_n \cdot \nabla x_i) \phi \, dx = \int_\Omega \left((\gamma_n - \gamma_0) \nabla w_n^i \cdot \nabla u_0\right) \phi \, dx + \int_\Omega r_n \cdot \nabla\phi \, dx$$

with

$$\|r_n\|_{L^1(\Omega)} \le C \|d_n\|_{L^1(\Omega)}^{1+\eta} \left(\|\nabla u_0\|_{L^\infty(K)} + \|u_0\|_{L^\infty(\partial\Omega')}\right),$$

where the positive constants C and η may depend only on τ, Ω, K, $\|\gamma_0\|_{W^{2,d}(\Omega)}$, Λ_0, λ_0 and $\|d_n\|_{L^p(A_n)}$.

The proof of this result is given in appendix C.

3.2. **The non finely intertwined case.** The main result of this section is the establishes proposition 12 in the final case, namely when assumption 4c holds. Example 25 is an illustration of such a configuration.

Example 25. Suppose that $\Omega \subset \mathbb{R}^d$ is the ball $B(0, d)$ of radius d centred at the origin. Assume that $\gamma_0 = I_d$. Given $\epsilon > 0$, for $n \ge 2$, we set

$$A_n = \bigcup_{k=1}^{n} \left(\frac{k}{n}, \frac{k}{n} + \frac{1}{n^{d+1+\epsilon}}\right) \times (0,1)^{d-1}, \quad B_n = \bigcup_{k=1}^{n} \left(\frac{k}{n} + \frac{1}{2n}, \frac{k}{n} + \frac{3}{4n}\right) \times (0,1)^{d-1},$$

and

$$\gamma_n = \left(\left(n\frac{i-1}{d-1} + \frac{d-i}{d-1}\right)\delta_{ij}\right)_{1\le i,j\le d} \text{ on } A_n, \quad \gamma_n = \frac{\ln n}{n} I_d \text{ on } B_n.$$

We have $A_n \cup B_n \subset (0,1)^d \subset \Omega$. The insulating and conductive strips are separated by a distance $d(A_n, B_n) \propto \frac{1}{n}$. We have

$$\|d_n\|_{L^1(A_n)} \propto \frac{1}{n^{d-1+\epsilon}}, \quad \|d_n\|_{L^1(B_n)} \propto \frac{1}{\ln n},$$

therefore $\|d_n\|_{L^1(\Omega)} \to 0$. We have $d(A_n, B_n) > \|d_n\|_{L^1(A_n)}^\tau$ for $\tau \in \left(0, \frac{1}{d-1}\right)$ .We compute that $\|d_n\|_{L^p(A_n)} \propto n^{p-(d+\epsilon)}$. In particular for $p = d > \frac{d}{2}$ there holds $\|d_n\|_{L^p(A_n)} \to 0$. Notice that the conductive strips are narrowed to accomodate the extra integrability, whereas the insulating one are just chosen to so that $\|d_n\|_{L^1(\Omega)} \to 0$.

Proposition 26. *Suppose assumptions 1, 2, and 3 are satisfied. Suppose additionally that for some $p > \frac{d}{2}$,*

$$\limsup_{n} \|d_n\|_{L^p(A_n)}^{\frac{1}{2}} < \infty$$

and that there exists a sequence of function $(\chi_n)_{n\in\mathbb{N}} \in \left(W^{1,\infty}(\Omega; [0,1])\right)^{\mathbb{N}}$ such that $\chi_n \equiv 0$ on B_n, $\chi_n = 1$ on A_n and

$$\|d_n\|_{L^1(\Omega)}^\tau \|\nabla\chi_n\|_{L^\infty(\Omega)} < \infty,$$

for some $\tau < \frac{1}{(d-1)}$. Given Ω' a smooth domain as defined in proposition 12, there holds

$$\int_\Omega \left(\left(\gamma_n - \gamma_0\right)\nabla w_n \cdot \nabla x_i\right) \phi \, dx = \int_\Omega \left(\left(\gamma_n - \gamma_0\right)\nabla w_n^i \cdot \nabla u_0\right) \phi \, dx + \int_\Omega r_n \cdot \nabla\phi \, dx$$

with

$$\|r_n\|_{L^1(\Omega)} \leq C \|d_n\|_{L^1(\Omega)}^{1+\eta} \left(\|\nabla u_0\|_{L^\infty(K)} + \|u_0\|_{L^\infty(\partial\Omega')}\right),$$

where the positive constants C and η may depend only on τ, Ω, K, $\|\gamma_0\|_{W^{2,d}(\Omega)}$, Λ_0 and λ_0, p and τ only.

Proof. This a direct consequence of estimate (3.6) in proposition 17 and lemma 27. □

Lemma 27. *If for some $p > \frac{d}{2}$,*

$$\limsup_n \|d_n\|_{L^p(A_n)}^{\frac{1}{2}} < \infty$$

and if there exists a sequence of function $(\chi_n)_{n\in\mathbb{N}} \in \left(W^{1,\infty}(\Omega;[0,1])\right)^{\mathbb{N}}$ such that $\chi_n \equiv 0$ on B_n, $\chi_n = 1$ on A_n and

$$\|d_n\|_{L^1(\Omega)}^{\tau} \|\nabla \chi_n\|_{L^\infty(\Omega)} < \infty,$$

for some $\tau < \frac{1}{(d-1)}$ then there exists $\eta > 0$ depending on p and τ only such that

$$\|w_n\|_{L^\infty(A_n)} \leq C \|d_n\|_{L^1(\Omega)}^{\eta} \left(\|\nabla u_0\|_{L^\infty(K)} + \|u_0\|_{L^\infty(\partial\Omega')}\right),$$

where C depends on K, Ω, Λ_0, λ_0 , $\|\gamma_0\|_{W^{2,d}(\Omega)}$, p and τ only.

Proof. We apply Stampacchia's truncation method [12]. We denote $u \to G_k(u)$ to be the truncation operator, i.e $G_k(u) = \begin{cases} u & |u| \leq k \\ k & u > k \\ -k & u < -k \end{cases}$ with $k > 0$, and we write $m_k = \{x \in \Omega \,:\, |u_n| > k\}$.
We test equation (1.4) against $\chi_n^2 v_n$, with $v_n = w_n - G_k(w_n)$, and obtain

$$\begin{aligned} &\int_\Omega \gamma_n \nabla w_n \cdot \nabla\left(\chi_n^2 v_n\right) \mathrm{d}x \\ =& \int_\Omega \gamma_n \nabla(\chi_n v_n) \cdot \nabla(\chi_n v_n)\,\mathrm{d}x - \int_\Omega \gamma_n \nabla\chi_n \cdot \nabla\chi_n v_n^2 \mathrm{d}x \\ =& \int_\Omega \chi_n(\gamma_0 - \gamma_n)\nabla u_0 \cdot \nabla(\chi_n v_n)\,\mathrm{d}x + \int_\Omega \chi_n v_n (\gamma_0 - \gamma_n)\nabla u_0 \cdot \nabla\chi \mathrm{d}x \end{aligned}$$

Write $\gamma_n^+ = \max(\gamma_n, \gamma_0)$. Since $\chi \equiv 0$ on B_n, and $\nabla\chi$ is supported on $\Omega \setminus (A_n \cup B_n)$ and v_n is supported on m_k, we may simplify the above identity to

$$\int_\Omega \gamma_n^+ \nabla(\chi v_n) \cdot \nabla(\chi v_n)\,\mathrm{d}x = \int_{m_k} \gamma_0 \nabla\chi_n \cdot \nabla\chi_n v_n^2 \mathrm{d}x - \int_{m_k} \left(\gamma_0 - \gamma_n^+\right)\nabla u_0 \cdot \nabla(\chi_n v_n)\,\mathrm{d}x$$

Using Cauchy-Schwarz, we find

$$\left|\int_{m_k} \left(\gamma_0 - \gamma_n^+\right)\nabla u_0 \cdot \nabla(\chi_n v_n)\,\mathrm{d}x\right| \leq \left(\int_{m_k} d_n \nabla u_0 \cdot \nabla u_0 \mathrm{d}x\right)^{\frac{1}{2}} \left(\int_\Omega \gamma_n^+ \nabla\chi_n \cdot \nabla\chi_n v_n^2 \mathrm{d}x\right)^{\frac{1}{2}},$$

which shows that

$$\int_\Omega \lambda_0 |\nabla(\chi v_n)|^2 \,\mathrm{d}x \leq \int_\Omega \gamma_n^+ \nabla(\chi v_n) \cdot \nabla(\chi v_n)\,\mathrm{d}x \leq 2\left(\int_{m_k} d_n^+ \nabla u_0 \cdot \nabla u_0 \mathrm{d}x + \int_{m_k} \Lambda_0 |\nabla\chi_n|^2 v_n^2 \mathrm{d}x\right).$$

For any $p > \frac{d}{2}$ we write using Hölder's inequality and the fact that $|v_n| \leq |w_n|$

$$\int_{m_k} d_n^+ \nabla u_0 \cdot \nabla u_0 \mathrm{d}x + \int_{m_k} \gamma_0 \nabla \chi_n \cdot \nabla \chi_n v_n^2 \mathrm{d}x$$
$$\leq \|d_n\|_{L^p(A_n)} \|\nabla u_0\|^2_{L^\infty(K)} |m_k|^{1-\frac{1}{p}} + \|w_n\|^2_{L^{2p}(\Omega)} \Lambda_0 \|\nabla \chi_n\|^2_{L^\infty(\Omega)} |m_k|^{\frac{p-1}{p}},$$

Whereas for any $h > k$, thanks to the Sobolev embedding $H^1(\Omega) \hookrightarrow L^q(\Omega)$ for $q = \left(\frac{p}{p-1} + \frac{d}{d-2}\right)$ if $d > 2$ and $q = \frac{2p}{p-1} + 1$ if $d = 2$,

$$\lambda_{\grave{a}} C(s,\Omega) |k-h|^2 |m_h|^{\frac{2}{q}} < \lambda_q C(s,\Omega) \|\chi v_n\|^2_{L^{3+\frac{2}{s}}(m_k)} < \int_\Omega \lambda_0 |\nabla(\chi v_n)|^2 \mathrm{d}x.$$

This shows that $m_k = 0$, for k large enough, that is,

$$\|\chi_n w_n\|_{L^\infty(\Omega)} \leq C\left(\|d_n\|^{\frac{1}{2}}_{L^p(A_n)} \|\nabla u_0\|_{L^\infty(K)} + \|w_n\|_{L^{2p}(\Omega)} \|\nabla \chi_n\|_{L^\infty(\Omega)}\right),$$

$C > 0$ depends on s, K, Ω, Λ_0 and λ_0 only. Thanks to estimate (2.10), for any $\zeta \in \left[1, \frac{1}{(d-1)}\right)$ there holds

$$\|w_n\|_{L^{2p}(\Omega)} \leq C \|d_n\|^{\frac{d\zeta}{2p}}_{L^1(\Omega)} \left(\|\nabla u_0\|_{L^\infty(K)} + \|u_0\|_{L^\infty(\partial\Omega')}\right),$$

where C depends on η, Ω', K, Ω, Λ_0 and λ_0 and $\|\gamma_0\|_{W^{2,d}(\Omega)}$. Altogether,

$$\|w_n\|_{L^\infty(A_n)} \leq C\left(\|d_n\|^{\frac{1}{2}}_{L^p(A_n)} + \|d_n\|^{\zeta}_{L^1(\Omega)} \|\nabla \chi_n\|_{L^\infty(\Omega)}\right)\left(\|\nabla u_0\|_{L^\infty(K)} + \|u_0\|_{L^\infty(\partial\Omega')}\right). \tag{3.16}$$

Now, given $\tau < \frac{1}{d-1}$ and $p_0 > \frac{d}{2}$ such that

$$\limsup \|d_n\|^{\tau}_{L^1(\Omega)} \|\nabla \chi_n\|_{L^\infty(\Omega)} + \limsup \|d_n\|_{L^{p_0}(A_n)} < \infty,$$

write

$$\kappa = \sup_n \|d_n\|^{\tau}_{L^1(\Omega)} \|\nabla \chi_n\|_{L^\infty(\Omega)} + \|d_n\|_{L^{p_0}(A_n)},$$

and $p_1 = \frac{1}{2}\min\left(\frac{d}{2}\frac{1}{\tau(d-1)}, p_0\right) + \frac{d}{4}$. By interpolation between $L^1(A_n)$ and $L^{p_0}(A_n)$ we have

$$\|d_n\|^{\frac{1}{2}}_{L^{p_1}(A_n)} \leq \|d_n\|^{\theta_1}_{L^1(A_n)} \kappa^{\frac{1}{2}-\theta_1},$$

with $\theta_1 = \frac{p_0 - p_1}{2p_1(p_0-1)} > 0$ and

$$\|d_n\|^{\frac{d\tau}{2p_1}}_{L^1(\Omega)} \|\nabla \chi_n\|_{L^\infty(\Omega)} \leq \|d_n\|^{\theta_2} \kappa,$$

with

$$\theta_2 = \left(\frac{d}{2p_1} - 1\right)\tau > 0.$$

Estimate (3.16) with $p = p_1$ and $\zeta = \tau$ concludes the proof, with $\eta = \min(\theta_1, \theta_2)$. $\square$

4. Properties of the polarisation tensor M

Thanks to lemma 10, we may consider alternative definitions for the tensor M. The most convenient is the periodic one, namely, embedding Ω in a large cube C, we set

$$H^1_\#(C) := \left\{ \phi \in H^1_{\rm loc}\left(\mathbb{R}^d\right) \,:\, \int_{C\setminus K} \phi \,\mathrm{d}x = 0 \text{ and } \phi \quad C-\text{periodic} \right\},$$

and $M_{ij} = D_{ij} - W_{ij} \in L^2(\Omega, d\mu)$ is the scalar weak* limit of $\frac{1}{\|d_n\|_{L^1(\Omega)}}\left((\nabla w_n^i + \mathbf{e}_i)\cdot(\gamma_n - \gamma_0)\,\mathbf{e}_j\right)$, where w_n^i is be the unique weak solution to

$$\int_Q \gamma_n \nabla w_n^i \cdot \nabla\phi \,\mathrm{d}x = \int_Q (\gamma_0 - \gamma_n)\,\mathbf{e}_j \cdot \nabla \phi\,\mathrm{d}x \text{ for all } \phi \in H^1_\#(C). \tag{4.1}$$

In [4] another version $\mathbb{M}$ of this tensor is introduced, and M a natural extension to this context.

Assuming $\gamma_n = ((\gamma_1 - \gamma_0)\,1_{A_n\cup B_n} + \gamma_0)\,I_d$ for some regular functions γ_1 and γ_0, then the tensor $\mathbb{M}$ introduced in [4] is defined as the weak* limit in $L^2(\Omega, d\mu)$ of

$$\frac{1}{|A_n \cup B_n|}\left(\nabla w_n^i + \mathbf{e}_i\right)\cdot \mathbf{e}_j$$

To compare both formulas, suppose γ_1 and γ_0 are constant. Then

$$\frac{1}{\|d_n\|_{L^1(\Omega)}}\left(\left(\nabla w_n^i + \mathbf{e}_i\right)\cdot(\gamma_n - \gamma_0)\,\mathbf{e}_j\right) = \frac{1}{|A_n\cup B_n|}\frac{1}{\sqrt{d}}\frac{\gamma_1}{\gamma_1^2+\gamma_0^2}(\gamma_1 - \gamma_0)\left(\nabla w_n^i + \mathbf{e}_i\right)\cdot \mathbf{e}_j,$$

thus the two tensors are related by the simple fomula

$$M = \frac{1}{\sqrt{d}}\frac{\gamma_1}{\gamma_1^2+\gamma_0^2}(\gamma_0 - \gamma_1)\,\mathbb{M}, \tag{4.2}$$

and most properties can be read directly from [6] with the appropriate changes.

Lemma 28 ([4, theorem 1])**.** *The entries of the polarisation tensor M satisfies $M_{ij} = M_{ji}$ μ-almost everywhere in Ω.*

Lemma 29 (See [Lemma 4 6])**.** *For every $\phi \in C^1_c(C)$, $\phi \geq 0$, and every $\zeta \in \mathbb{R}^d$, there holds*

$$\begin{aligned}\int_\Omega W\zeta\cdot\zeta\phi d\mu = & \frac{1}{\|d_n\|_{L^1(\Omega)}}\int_\Omega d_n'\zeta\cdot\zeta\phi\,dx \\ & - \frac{1}{\|d_n\|_{L^1(\Omega)}}\min_{u\in H^1_\#(C)^d}\int_\Omega \gamma_n\left(\nabla u - \gamma_n^{-1}(\gamma_n-\gamma_0)\,\zeta\right)\cdot\left(\nabla u - \gamma_n^{-1}(\gamma_n-\gamma_0)\,\zeta\right)\phi dx + o(1),\end{aligned}$$

with

$$d_n' = (\gamma_n - \gamma_0)\,\gamma_n^{-1}(\gamma_n-\gamma_0) = d_n - 2\gamma_0 \geq 0.$$

In particular, the tensor M is positive semi-definite and satisfies

$$0 \leq W \leq I_d \;\mu \text{ a.e. in } \Omega.$$

If γ_n and γ_0 are multiples of the identity matrix, that is, the material is isotropic, then

$$0 \leq W \leq \frac{1}{\sqrt{d}} I_d \;\mu \text{ a.e. in } \Omega.$$

Proof. The derivation of the identity is, mutatis mutandis, done in [6, lemma 4]. Choosing $u = 0$, we find

$$\begin{aligned}&\frac{1}{\|d_n\|_{L^1(\Omega)}} \min_{u\in H^1_\#(C)^d} \int_\Omega \gamma_n \left(\nabla u - \gamma_n^{-1}(\gamma_n - \gamma_0)\zeta\right)\cdot\left(\nabla u - \gamma_n^{-1}(\gamma_n - \gamma_0)\zeta\right)\phi\,\mathrm{d}x\\ \leq&\frac{1}{\|d_n\|_{L^1(\Omega)}} \min_{u\in H^1_\#(C)^d} \int_\Omega d'_n\phi\,\mathrm{d}x,\end{aligned}$$

and therefore

$$\int_\Omega W\zeta\cdot\zeta\phi d\mu \geq 0.$$

Since the second term is negative, we find

$$\int_\Omega W\zeta\cdot\zeta\phi d\mu \leq \lim_{n\to\infty}\frac{1}{\|d_n\|_{L^1(\Omega)}}\int_\Omega \phi d'_n\zeta\cdot\zeta\,\mathrm{d}x.$$

We compute

$$\frac{1}{\|d_n\|_{L^1(\Omega)}}\int_\Omega \phi d'_n\zeta\cdot\zeta\,\mathrm{d}x = \int_\Omega \phi\frac{d'_n\zeta\cdot\zeta}{|d_n|_F}\frac{|d_n|_F}{\|d_n\|_{L^1(\Omega)}}\,\mathrm{d}x \leq \int_\Omega \phi\frac{d'_n\zeta\cdot\zeta}{|d'_n|_F}\frac{|d_n|_F}{\|d_n\|_{L^1(\Omega)}}\,\mathrm{d}x,$$

and if $\lambda_1 \leq \ldots \leq \lambda_d$ are the eigenvalues of d'_n at x,

$$\frac{d'_n\zeta\cdot\zeta}{|d'_n|_F} \leq |\zeta|^2\frac{\lambda_d}{\sqrt{\sum_{i=1}^d\lambda_i^2}} \leq |\zeta|^2\begin{cases}1 & \text{in general,}\\ \frac{1}{\sqrt{d}} & \text{if } \lambda_1 = \ldots = \lambda_d\end{cases}.$$

All eigenvalues are equal when γ_0 and γ_n are isotropic, therefore

$$\int_\Omega W\zeta\cdot\zeta\phi d\mu \leq \lim_{n\to\infty}\frac{1}{\|d_n\|_{L^1(\Omega)}}\int_\Omega \phi d'_n\zeta\cdot\zeta\,\mathrm{d}x \leq C|\zeta|^2\int_\Omega\phi\mathrm{d}\mu,$$

with $C = 1$ in general and $C = d^{-\frac{1}{2}}$ in isotropic media. □

5. An example

We revisit an example already considered in [3, 5], namely, elliptic inclusions. In a domain

$$\Omega = \left\{(x,y)\subset\mathbb{R}^2 : \frac{x^2}{\cosh^2(2)} + \frac{y^2}{\sinh^2(2)} \leq 1\right\},$$

consider heterogeneities in a homogeneous medium located in the set

$$E_n = \left\{(x,y)\subset\mathbb{R}^2 : \frac{x^2}{\cosh^2(n^{-1})} + \frac{y^2}{\sinh^2(n^{-1})} \leq 1\right\},$$

which collapses to the line segment $(-1,1)\times\{0\}$ as $n\to\infty$. Consider an isotropic inhomogeneity, with conductivity

$$\gamma_n(x) = \begin{cases}1 & x\in\Omega\setminus Q_n\\ \lambda_n & x\in Q_n,\end{cases}$$

where $\lambda_n \in (0,1)\cup(1,\infty)$. In this case,

$$d_n = \left(\lambda_n + \lambda_n^{-1}\right)I_2.$$

and $\|d_n\|_{L^1(\Omega)} \to 0$ means $\max(n^{-1}\lambda_n, n^{-1}\lambda_n^{-1}) \to 0$. The solution u_n^i to the equation

$$\begin{aligned} -\nabla \cdot \left(\gamma_n \nabla u_n^i\right) &= 0 \quad \text{in} \quad \Omega \\ u_n^i &= x_i \quad \text{on} \quad \partial\Omega \end{aligned} \tag{5.1}$$

can be computed explicitly in elliptic coordinates. In particular we find that

$$\frac{1}{|d_n|_F}(1-\gamma_n)\,\partial_{x_j} w_n^i = \frac{1}{\sqrt{2}}\frac{\lambda_n}{1+\lambda_n^2}(1-\gamma_n)\,1_{E_n}\left(\partial_{x_j} u_n^i - \delta_{ij}\right) = \delta_{ij}\ell_n^i 1_{E_n},$$

with

$$\ell_n^1 = O\left(\frac{\lambda_n}{n}\right) \text{ and } \ell_n^2 = \frac{1}{\sqrt{2}} + O\left(\frac{\lambda_n}{n}\right) \text{ when } \lambda_n > 1,$$

$$\ell_n^1 = O\left(\frac{1}{n^2}\right) \text{ and } \ell_n^2 = O\left(\frac{1}{n\lambda_n}\right) \text{ when } 0 < \lambda_n < 1,$$

As a consequence, when $n\lambda_n \to 0$ with $\lambda_n \to \infty$

$$W = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix}, \quad D = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \quad M = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 \end{pmatrix},$$

Whereas when $\lambda_n \to 0$,we obtain

$$W = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad D = -\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \quad M = -\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix},$$

and both results corresponds extreme cases with respect to the isotropic pointwise bounds derived in lemma 29.

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Appendix A. Additional proofs

Proof of lemma 6. The convergence (1.5) is a direct consequence of the Banach–Alaoglu's theorem and the continuous embedding between $L^1(\Omega) \hookrightarrow C^0\left(\overline{\Omega}\right)^*$, where we have identified the continuous dual space of $C^0\left(\overline{\Omega}\right)$ as the space of bounded Radon measures on Ω. We know from (1.3) that $\left|(\gamma_0 - \gamma_n)_{ij}\right| \leq |d_n|_F$,therefore $\left\|(\gamma_n - \gamma_0)_{ij}\right\|_{L^1(\Omega)} \leq \|d_n\|_{L^1(\Omega)}$. We may extract a subsequence in which

$$\frac{1}{\|d_n\|_{L^1(\Omega)}} (\gamma_n - \gamma_0)_{ij} \overset{*}{\rightharpoonup} \mathrm{d}\mathcal{D}_{ij}$$

in the space of bounded vector Radon measures.

$$\begin{aligned}
\int_\Omega \phi \, d\mathcal{D}_{ij} &= \lim_{n\to\infty} \int_\Omega \frac{1}{\|d_n\|_{L^1(\Omega)}} (\gamma_0 - \gamma_n)_{ij} \phi \, \mathrm{d}x \\
&\leq \lim_{n\to\infty} \int_\Omega \frac{1}{\|d_n\|_{L^1(\Omega)}} |d_n|_F \, \phi \, \mathrm{d}x \\
&\leq \lim_{n\to\infty} \left(\int_\Omega \frac{1}{\|d_n\|_{L^1(\Omega)}} |d_n|_F \, \phi^2 \, \mathrm{d}x \right)^{\frac{1}{2}} \\
&= \left(\int_\Omega \phi^2 \, \mathrm{d}\mu \right)^{\frac{1}{2}},
\end{aligned}$$

where we used Cauchy-Schwarz in the penultimate line. It follows that the functional

$$\phi \to \int_\Omega \phi \cdot \mathrm{d}\mathcal{D}_{ij}$$

may be extended to a bounded linear functional on $[L^2(\Omega,\mathrm{d}\mu)]^d$. Hence, by Riesz's Representation Theorem, we may identify

$$\mathrm{d}\mathcal{D}_{ij} = D_{ij}\mathrm{d}\mu$$

for some function $D_{ij} \in L^2(\Omega,\mathrm{d}\mu)$, which is our statement. $\square$

Appendix B. Proof of proposition 8

Proof. We write

$$d_n' = (\gamma_n - \gamma_0)\,\gamma_n^{-1}\,(\gamma_n - \gamma_0)\,,$$

and note that $d_n' \leq d_n$. Note that w_n is the unique minimiser over X of the functional

$$J(w) = \int_\Omega \gamma_n \left(\nabla w + \gamma_n^{-1}(\gamma_n - \gamma_0)\nabla u_0\right)\cdot\left(\nabla w + \gamma_n^{-1}(\gamma_n-\gamma_0)\nabla u_0\right)\mathrm{d}x,$$

Clearly, $J(w_n) \geq 0$, thus

$$-\int_\Omega \gamma_n \nabla w_n\cdot\nabla w_n\,\mathrm{d}x + 2\int_\Omega \gamma_n\left(\nabla w_n + \gamma_n^{-1}(\gamma_n-\gamma_0)\nabla u_0\right)\cdot\nabla w_n \mathrm{d}x + \int_\Omega d_n'\nabla u_0\cdot\nabla u_0 \mathrm{d}x \geq 0,$$

which shows

$$\int_\Omega \gamma_n\nabla w_n\cdot\nabla w_n\,\mathrm{d}x \leq \int_\Omega d_n'\nabla u_0\cdot\nabla u_0\,\mathrm{d}x. \tag{B.1}$$

Thus, as $u_0 \in C^1(K)$

$$\int_\Omega \gamma_n(x)\nabla w_n\cdot\nabla w_n\,\mathrm{d}x \leq \|\nabla u_0\|^2_{L^\infty(K)}\int_\Omega |d_n|_F\,\mathrm{d}x.$$

We now turn to the second estimate. Using Cauchy–Schwarz we find

$$\begin{aligned}
&\|(\gamma_n-\gamma_0)\nabla w_n\|_{L^1(\Omega)} \\
&= \int_\Omega \sqrt{\left|(\gamma_n-\gamma_0)\,\gamma_n^{-\frac12}\gamma_n^{\frac12}\nabla w_n\right|^2\mathrm{d}x} \\
&\leq \sqrt{\int_\Omega |(\gamma_n-\gamma_0)\,\gamma_n^{-1}(\gamma_n-\gamma_0)|_F\,\mathrm{d}x}\sqrt{\int_\Omega \gamma_n\nabla w_n\cdot\nabla w_n \mathrm{d}x} \\
&\leq \|d_n\|_{L^1(\Omega)}\,\|\nabla u_0\|_{L^\infty(K)}\,.
\end{aligned} \tag{B.2}$$

Since $\frac{1}{\|d_n\|_{L^1(\Omega)}}(\gamma_n-\gamma_0)\nabla w_n$ is uniformly bounded in $L^1(\Omega)$, we may extract a subsequence in which

$$\frac{1}{\|d_n\|_{L^1(\Omega)}}(\gamma_n-\gamma_0)\nabla w_n \overset{*}{\rightharpoonup} \mathrm{d}\mathscr{M}$$

in the space of bounded vector Radon measures. Moreover, for any $\Psi \in C^0(\overline{\Omega};\mathbb{R}^d)$,

$$\begin{aligned}
\int_\Omega \Psi\cdot\mathrm{d}\mathscr{M} &= \lim_{n\to\infty}\int_\Omega \frac{1}{\|d_n\|_{L^1(\Omega)}}(\gamma_n-\gamma_0)\nabla w_n\cdot\Psi\mathrm{d}x \\
&\leq \lim_n \left(\frac{1}{\|d_n\|_{L^1(\Omega)}}\int_\Omega \gamma_n\nabla w_n\cdot\nabla w_n\mathrm{d}x\right)^{\frac12}\left(\frac{1}{\|d_n\|_{L^1(\Omega)}}\int_\Omega d_n'\Psi\cdot\Psi\mathrm{d}x\right)^{\frac12} \\
&\leq C\left(\int_\Omega |\Psi|^2\,\mathrm{d}\mu\right)^{\frac12}
\end{aligned}$$

thanks to the estimate above. As a consequence of this estimate, it follows that the functional

$$\Psi \to \int_\Omega \Psi \cdot \mathrm{d}\mathscr{M}$$

may be extended to a bounded linear functional on $[L^2(\Omega, \mathrm{d}\mu)]^d$. Hence, by Riesz's Representation Theorem, we may identify

$$\mathrm{d}\mathscr{M} = M\mathrm{d}\mu$$

for some function $\mathcal{M} \in [L^2(\Omega, \mathrm{d}\mu)]^d$, which is our statement. □

Appendix C. Proof of Proposition 24

Remark. Note that if Ω' is simply connected, $Fb_n = Fb_0 = 0$. Remark that

$$\frac{1}{|\Gamma_i|}\int_{\Gamma_i} \gamma_0 \nabla u_0 \cdot n \mathrm{d}\sigma = \int_{\Gamma_i} \gamma_n \nabla u_n \cdot n \mathrm{d}\sigma.$$

Let I_i be the solution of

$$\operatorname{div}(\gamma_0 \nabla I_i) = 0 \text{ on } \Omega' \text{ and } I_i = 1 \text{ on } \Gamma_i.$$

By an integration by parts,

$$\begin{aligned}
\int_{\Gamma_i} \gamma_0 \nabla u_0 \cdot n \mathrm{d}\sigma - \int_{\Gamma_i} \gamma_n \nabla u_n \cdot n \mathrm{d}\sigma &= \int_\Omega \gamma_0 \nabla u_0 \cdot \nabla I_i \mathrm{d}x - \int_\Omega \gamma_n \nabla u_n \cdot \nabla I_i \mathrm{d}x. \\
&= \int_\Omega g I_i \mathrm{d}\sigma - \int_\Omega g I_i \mathrm{d}\sigma \\
&= 0.
\end{aligned}$$

Thus, imposing that $g \in H^{\frac{1}{2}}(\partial\Omega)$ is such that $Fb_0 = 0$, which corresponds to $N-1$ contraints in an infinite dimensional space and therefore is not a loss of generality, this implies that $Fb_n = 0$. We shall make that assumption in the rest of this section.

Proof. By the inequality in (3.15), we have

$$\frac{\|\Sigma_n\|_{L^1(\Omega)}}{\|d_n\|_{L^1(\Omega)}} \leq \left(\max_{\overline{\Omega}} \lambda_d\left(\gamma_0^{-1}\right)\right)^2,$$

thus taking a convergent subsequence of $\frac{\|\Sigma_n\|_{L^1(\Omega)}}{\|d_n\|_{L^1(\Omega)}} \to a_0$ and a possible further extraction of the subsequence $\frac{1}{\|\Sigma_n\|_{L^1(\Omega)}}\Big(\sigma_n - \sigma_0\Big)\nabla\phi_n$, corollary 22 implies that, if $\Xi \in C^0\left(\overline{\Omega}, \mathbb{R}^2\right)$ is an arbitrary vector field,

$$\begin{aligned}
&\lim_{n\to\infty}\int_\Omega \left(\frac{1}{\|d_n\|_{L^1(\Omega)}}\Big(\sigma_0-\sigma_n\Big)\nabla\varphi_n\cdot\Xi\right)\,\mathrm{d}x\\
=\;&\lim_{n\to\infty}\int_\Omega \left(\frac{\|\Sigma_n\|_{L^1(\Omega)}}{\|d_n\|_{L^1(\Omega)}}\frac{1}{\|d_n\|_{L^1(\Omega)}}\Big(\sigma_n-\sigma_0\Big)\nabla\varphi_n\cdot\Xi\right)\,\mathrm{d}x\\
=\;&a_0\lim_{n\to\infty}\int_\Omega \left(\frac{1}{\|d_n\|_{L^1(\Omega)}}\Big(\sigma_0-\sigma_n\Big)\nabla\varphi_n\cdot\Xi\right)\,\mathrm{d}x\\
=\;&a_0\int_\Omega \tilde N\nabla\psi_0\cdot\Xi\,\mathrm{d}\nu\\
=\;&\int_\Omega N\nabla\psi_0\cdot\Xi\,\mathrm{d}\mu.
\end{aligned}$$

Where $N=a_0\frac{\mathrm{d}\nu}{\mathrm{d}\mu}\tilde N$ belongs to $L^2(\Omega;\mathrm{d}\mu)$. Alternatively testing against $\big(J^T\gamma_0\big)\,\Xi$ we find

$$\begin{aligned}
&\int_\Omega \frac{1}{\|d_n\|_{L^1(\Omega)}}\Big(\sigma_0-\sigma_n\Big)\nabla\psi_n\cdot\big(J^T\gamma_0\big)\,\Xi\,\mathrm{d}x\\
=&\int_\Omega \frac{1}{\|d_n\|_{L^1(\Omega)}}\,\Big(J^T\gamma_0^{-1}\Big(\gamma_n-\gamma_0\Big)\gamma_n^{-1}J\Big)\,\big(J^T\gamma_n\nabla u_n\big)\cdot\big(J^T\gamma_0\big)\,\Xi\,\mathrm{d}x\\
=&\int_\Omega \frac{1}{\|d_n\|_{L^1(\Omega)}}J^T\gamma_0^{-1}\Big(\gamma_0-\gamma_n\Big)\nabla u_n\cdot J^T\gamma_0\Xi\,\mathrm{d}x.\\
=&\int_\Omega \frac{1}{\|d_n\|_{L^1(\Omega)}}\Big(\gamma_0-\gamma_n\Big)\nabla u_n\cdot\Xi\,\mathrm{d}x
\end{aligned}$$

whereas

$$\begin{aligned}
&\int_\Omega \frac{1}{\|d_n\|_{L^1(\Omega)}}\Big(\sigma_n-\sigma_0\Big)\nabla\psi_0\cdot\big(J^T\gamma_0\big)\,\Xi\,\mathrm{d}x\\
&\int_\Omega \frac{1}{\|d_n\|_{L^1(\Omega)}}\gamma_0\gamma_n^{-1}\Big(\gamma_0-\gamma_n\Big)\nabla u_0\cdot\Xi\,\mathrm{d}x.\\
&\int_\Omega \frac{1}{\|d_n\|_{L^1(\Omega)}}\,\Big(\Big(\gamma_0-\gamma_n\Big)+d_n\Big)\,\nabla u_0\cdot\Xi\,\mathrm{d}x.
\end{aligned}$$

We write $\mathcal{D}$ as the limit limiting tensor corresponding to $\|d_n\|^{-1}_{L^1(\Omega)}\,d_n$ in $L^2(\Omega,d\mu)^{d\times d}$, that is,

$$\lim_{n\to\infty}\int_\Omega \frac{d_n}{\|d_n\|_{L^1(\Omega)}}\nabla u_0\cdot\Xi\,\mathrm{d}x=\int_\Omega \mathcal{D}\nabla u_0\cdot\Xi\,\mathrm{d}\mu$$

Altogether, we have obtained

$$\begin{aligned}
\lim_{n\to\infty}\int_\Omega \frac{1}{\|d_n\|_{L^1(\Omega)}}\Big(\gamma_0-\gamma_n\Big)\nabla w_n\cdot\Xi\mathrm{dx}&=-\int_\Omega \mathcal{D}\nabla u_0\cdot\Xi\,\mathrm{d}\mu+\int_\Omega N\nabla\psi_0\cdot\big(J^T\gamma_0\big)\,\Xi\,\mathrm{d}\mu\\
&=\int_\Omega \Big((\gamma_0J)\,N\,(\gamma_0J)^T-\mathcal{D}\Big)\,\nabla u_0\cdot\Xi\,\mathrm{d}\mu
\end{aligned}$$

which is concludes our proof. □

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