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# Extending representation formulas for boundary voltage perturbations of low volume fraction to very contrasted conductivity inhomogeneities

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**Abstract.** Imposing either Dirichlet or Neumann boundary conditions on the boundary of a smooth bounded domain $\Omega$, we study the perturbation incurred by the voltage potential when the conductivity is modified in a set of small measure. We consider $(\gamma_n)_{n\in\mathbb{N}}$, a sequence of perturbed conductivity matrices differing from a smooth $\gamma_0$ background conductivity matrix on a measurable set well within the domain, and we assume $(\gamma_n-\gamma_0)\gamma_n^{-1}(\gamma_n-\gamma_0)\to 0$ in $L^1(\Omega)$. Adapting the limit measure, we show that the general representation formula introduced for bounded contrasts in a previous work from 2003 can be extended to unbounded sequences of matrix valued conductivities.

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## 1. The general framework

Given $d\geq 2$, let $\Omega\subset\mathbb{R}^d$ be an open, bounded Lipschitz domain. We study the following family of solutions of perturbed boundary value problems for the conductivity equation. Given $g\in H^{\frac{1}{2}}(\partial\Omega)$, we consider $(u_n)_{n\in\mathbb{N}}\in H^1(\Omega)^{\mathbb{N}}$, a sequence of perturbations of $u_0\in H^1(\Omega)$ given by

$$\begin{cases}-\operatorname{div}\left(\gamma_0\nabla u_0\right) & =0 \quad \text{in} \quad \Omega,\\ u_0 & =g \quad \text{on} \quad \partial\Omega,\end{cases} \quad\text{and}\quad \begin{cases}-\operatorname{div}\left(\gamma_n\nabla u_n\right) & =0 \quad \text{in} \quad \Omega,\\ u_n & =g \quad \text{on} \quad \partial\Omega.\end{cases} \tag{1}$$

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Alternatively, given $h \in H^{-\frac{1}{2}}(\partial\Omega)$ with $\int_{\partial\Omega} h \mathrm{d}\sigma = 0$, we consider $(u_n)_{n\in\mathbb{N}} \in H^1(\Omega)^{\mathbb{N}}$, a sequence of perturbations of $u_0 \in H^1(\Omega)$ given by

$$\begin{cases} -\operatorname{div}(\gamma_0 \nabla u_0) & = 0 \quad \text{in} \quad \Omega, \\ \gamma_0 \nabla u_0 \cdot n & = h \quad \text{on} \quad \partial\Omega, \\ \int_{\partial\Omega} u_0 \mathrm{d}\sigma & = 0, \end{cases} \quad \text{and} \quad \begin{cases} -\operatorname{div}(\gamma_n \nabla u_n) & = 0 \quad \text{in} \quad \Omega, \\ \gamma_n \nabla u_n \cdot n & = h \quad \text{on} \quad \partial\Omega, \\ \int_{\partial\Omega} u_n \mathrm{d}\sigma & = 0. \end{cases} \tag{2}$$

The conductivity coefficients are assumed to be symmetric positive definite matrix-valued functions with $\gamma_0 \in W^{2,d}_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^{d\times d})$, $\gamma_n \in L^\infty(\Omega;\mathbb{R}^{d\times d})$, and they satisfy the ellipticity condition

$$\lambda_0|\zeta|^2 \le \gamma_0\zeta\cdot\zeta \le \Lambda_0|\zeta|^2 \quad \text{and} \quad \lambda_n|\zeta|^2 \le \gamma_n\zeta\cdot\zeta \le \Lambda_n|\zeta|^2, \quad \forall\, \zeta \in \mathbb{R}^d,$$

with $0 < \lambda_n < \Lambda_n$ for all $n \in \mathbb{N}$.

**Definition 1.** *Given $(\omega_n)_{n\in\mathbb{N}}$ a sequence of measurable subsets of $\Omega$ whose Lebesgue measures tend to zero, we define $d_n \in L^\infty(\Omega;\mathbb{R}^{d\times d})$, a positive semi-definite matrix valued function by*

$$d_n = \left(\gamma_n + \gamma_0\gamma_n^{-1}\gamma_0\right)1_{\omega_n}.$$

*We make the following assumptions on the conductivities $\gamma_n$ and the sets $\omega_n$.*

**Assumptions.** *We assume that the following assumptions are satisfied:*

(1) *There exists $K$ an open subset of $\Omega$ with $C^\infty$ boundary such that $d(\partial K,\partial\Omega) > 0$ and*

$$\bigcup_{n\in\mathbb{N}} \omega_n \subset K.$$

*There holds, for all $n \ge 1$,*

$$\gamma_n = \gamma_0 \text{ in } \Omega \setminus \omega_n.$$

(2) *The perturbation vanishes asymptotically in $L^1(\Omega)$, that is,*

$$\left\|d_n\right\|_{L^1(\Omega)} \le 1 \text{ and } \lim_{n\to\infty} \left\|d_n\right\|_{L^1(\Omega)} = 0.$$

(3) *We write*

$$B_n = \left\{x \in \Omega : \gamma_n \le \Lambda_0 I_d\right\}, \quad A_n = \omega_n \setminus B_n$$
$$D_n = \left\{x \in \Omega : \gamma_n \ge \lambda_0 I_d\right\}, \quad C_n = \omega_n \setminus D_n$$

*these inequality being understood in the sense of quadratic forms. One of the following three properties is satisfied:*

(a) *There exists $p > d$ and $B \in \mathbb{R}$ such that*

$$B = \sup_n \left\|d_n\right\|_{L^p(A_n)}.$$

(b) *The dimension is $d = 2$, there exists $p > 2$ and $B \in \mathbb{R}$ such that*

$$B = \sup_n \left\|d_n\right\|_{L^p(C_n)} < \infty.$$

(c) *The exists $p > \frac{d}{2}$, $B \in \mathbb{R}$ and $\tau < \frac{1}{d-1}$ such that*

$$B = \limsup_{n\to\infty} \left\|d_n\right\|_{L^p(A_n)},$$

*and*

$$\inf\left\{\left|x-y\right| : x \in A_n, y \in C_n\right\} \ge \left\|d_n\right\|^\tau_{L^1(A_n)}.$$

*In particular, $A_n \subset D_n$.*

For $f \in L^p(\Omega)$, $1 \le p \le \infty$, $\|f\|_{L^p(\Omega)}$ is the canonical $L^P(\Omega)$ norm. For $U \in L^p(\Omega;\mathbb{R}^d)$ we use the notation $\|U\|_{L^p(\Omega)} = \||U|_d\|_{L^p(\Omega)}$ where $|\cdot|_d$ denotes the Euclidean norm in $\mathbb{R}^d$. For $A \in L^p(\Omega;\mathbb{R}^{d\times d})$, $\|A\|_{L^p(\Omega)}$ means $\||A|_F\|_{L^p(\Omega)}$ where $|\cdot|_F$ is the Frobenius norm, that is, the Euclidean norm on $\mathbb{R}^{d\times d}$.

**Remark 2.** Assumption (1) comes from the fact that near the boundary of the domain, the behaviour of the solution is different, as the imposed boundary condition plays a larger role.

Assumption (2) is sufficient and sharp in general. Example 4 illustrates the fact that for some inclusions $u_n \not\to u_0$ when $\|d_n\|_{L^1(\Omega)} \not\to 0$.

Assumption (3) imposes additional integrability properties for $d_n$ only on highly conductive inclusions. In dimension two, an extra integrability assumption for $d_n$ on the highly insulating inclusion is also sufficient. Alternatively, if very conductive materials and very insulating ones are not too finely intertwined, a weaker integrability condition is required. While any of the conditions listed under (3) is sufficient for our results to hold, it is not clear that an assumption is necessary.

As far as the authors are aware, this is the first result allowing highly contrasted and anisotropic materials in general inclusions. The question of large contrast limits has been considered by other authors. In [18], the authors address the case of diametrically bounded inclusions without (2). Such a general result does not hold for general inclusions, as Example 4 shows. In [11], the authors consider thin inhomogeneities, and provide a uniform representation formula valid beyond the perturbation regime. We only consider the perturbation regime, with no assumption on the shape or diameter of the inhomogeneities.

For any $y \in \Omega$, the Green function $G(\cdot, y)$ is the weak solution to the boundary value problem given by

$$\begin{aligned} \operatorname{div}\left(\gamma_0 \nabla G(\cdot, y)\right) &= \delta_y \quad \text{in} \quad \Omega \\ G(\cdot, y) &= 0 \quad \text{on} \quad \partial\Omega \end{aligned}$$

where $\delta_y$ denotes the Dirac measure at the point $y$, and the Neumann function $N(\cdot, y)$ is the weak solution to the boundary value problem given by

$$\begin{aligned} \operatorname{div}\left(\gamma_0 \nabla N(\cdot, y)\right) &= \delta_y \quad \text{in} \quad \Omega \\ \gamma_0 \nabla N(\cdot, y) \cdot n &= \frac{1}{|\partial\Omega|} \quad \text{on} \quad \partial\Omega. \end{aligned}$$

The main result of this article is that the general representation formula introduced in [8] can be extended to this context. This result was presented in a preliminary form in [19].

**Theorem 3.** *Let $d_n$ be given by Definition 1. Suppose that Assumptions* (1), (2) *and* (3) *hold. Then, there exists a subsequence also denoted by $d_n$ and a matrix valued function $M \in L^2(\Omega, \mathbb{R}^{d\times d}; \mathrm{d}\mu)$, where $\mu$ is the Radon measure generated by the sequence $\frac{1}{\|d_n\|_{L^1(\Omega)}}|d_n|_F$, such that for any $y \in \overline{\Omega \setminus K}$,*

- *if $u_n$ and $u_0$ are solutions to* (1) *there holds*

$$u_n(y) - u_0(y) = \left\|d_n\right\|_{L^1(\Omega)} \int_\Omega M_{ij}(x) \frac{\partial u_0}{\partial x_i}(x) \frac{\partial G(x,y)}{\partial x_j} \mathrm{d}\mu(x) + r_n(y),$$

- *if $u_n$ and $u_0$ are solutions to* (2) *there holds*

$$u_n(y) - u_0(y) = \left\|d_n\right\|_{L^1(\Omega)} \int_\Omega M_{ij}(x) \frac{\partial u_0}{\partial x_i}(x) \frac{\partial N(x,y)}{\partial x_j} \mathrm{d}\mu(x) + r_n'(y),$$

*in which $r_n \in L^\infty(\overline{\Omega \setminus K})$ (respectively $r_n' \in L^\infty(\overline{\Omega \setminus K})$) satisfies*

$$\frac{\|r_n\|_{L^\infty(\Omega\setminus K)}}{\left\|d_n\right\|_{L^1(\Omega)}} \to 0 \quad \left(\textit{resp. } \frac{\left\|r_n'\right\|_{L^\infty(\Omega\setminus K)}}{\left\|d_n\right\|_{L^1(\Omega)}} \to 0\right)$$

*uniformly in*

$$g \in H^{\frac{1}{2}}(\partial\Omega) \left(\textit{resp. } h \in H^{-\frac{1}{2}}(\partial\Omega)\right)$$

*with*

$$\|g\|_{H^{\frac{1}{2}}(\partial\Omega)} \le 1 \quad \textit{satisfies} \left(\textit{resp. } \|h\|_{H^{-\frac{1}{2}}(\partial\Omega)} \le 1\right).$$

*The matrix valued function $M \in L^2(\Omega, d\mu)$ is symmetric. The tensor $M$ can be written as $M = D - W$, where $W$ satisfies*

$$0 \le W\zeta\cdot\zeta \le \zeta\cdot\zeta \quad \mu \textit{ a.e. in } \Omega,$$

*and if $\gamma_n$ and $\gamma_0$ are isotropic,*

$$0 \le W\zeta\cdot\zeta \le \frac{1}{\sqrt{d}}\zeta\cdot\zeta \quad \mu \textit{ a.e. in } \Omega,$$

*whereas $D$ is limit in the sense of measures of*

$$\|d_n\|^{-1}_{L^1(\Omega)}\left(\gamma_n - \gamma_1\right).$$

Definition 10 specifies the matrix valued function $W \in L^2(\Omega, \mathbb{R}^{d\times d}; \mathrm{d}\mu)$. The tensor $M$ is, up to a factor, the polarisation tensor introduced in [8]. Its properties are briefly discussed in Section 4, following [10].

To document the sharpness of (2), the following example shows that it may happen that the asymptotic limit of $u_n$ is different from $u$ for some sequence $(\gamma_n)_{n\in\mathbb{N}}$ when $\|d_n\|_{L^1(\Omega)} \not\to 0$ even though $|\omega_n| \to 0$.

**Example 4.** Suppose that $\Omega = B(0,2) \subset \mathbb{R}^d$, choose $\omega_n = B(0, 1+\frac{1}{n}) \setminus B(0, 1-\frac{1}{n})$, and $g = x_1$. Then for $\gamma_0 = I_d$, the unperturbed solution of (1) corresponds to $u = x_1$.

Suppose that $\gamma_n$ is radial and constant on $(I_i)_{i\le 1\le 4}$, where

$$I_1 = \left(0, 1-\frac{1}{n}\right), \quad I_2 = \left(1-\frac{1}{n}, 1\right).$$
$$I_3 = \left(1, 1+\frac{1}{n}\right), \quad I_4 = \left(1+\frac{1}{n}, 2\right),$$

with values

$$\gamma_n = \chi_{I_1\cup I_4} + n^\alpha \chi_{I_2} + n^\beta \chi_{I_3},$$

where $\alpha, \beta$ are real parameters. Then,

$$\int_\Omega |d_n|_F \mathrm{d}x = \sqrt{d}\left(n^{\alpha-1} + n^{-\alpha-1} + n^{\beta-1} + n^{-\beta-1}\right)$$

and the solution $u_n$ of (1) takes the form

$$u_n = \sum_{i=1}^4 a_i^n x_1 \mathbf{1}_{I_i}(|x|) + |x|^{-d} \sum_{i=2}^4 b_i^n x_1 \mathbf{1}_{I_i}(|x|),$$

for some constants

$$\left(a_i^n\right)_{1\le i\le 4} \quad \text{and} \quad \left(b_i^n\right)_{2\le i\le 4}.$$

As $n\to\infty$, then $u_n \to v$ pointwise where

$$v = \left(\lim_{n\to\infty} b_1^n\right) x_1 \quad \text{for} \quad x < 1$$

and

$$v = \left(\lim_{n\to\infty} a_4^n\right) x_1 + \left(\lim_{n\to\infty} b_4^n\right) |x|^{-d} x_1 \quad \text{for} \quad x > \frac{1}{2}.$$

Computing the value of the constants, we find that $(\lim_{n\to\infty} a_1^n) = (\lim_{n\to\infty} a_4^n) = 1$ and $(\lim_{n\to\infty} b_4^n) = 0$ if and only if $-1 < \alpha < 1$ and $-1 < \beta < 1$. We further note that if we write $\delta = \min(1+\alpha, 1+\beta, 1-\alpha, 1-\beta) > 0$, $u_n - x_1$ is of order $n^{-\delta}$. Written in a slightly different form, there exists a positive constant $C$ depending on $\alpha$,$\beta$ and $d$ but independent of $n$ such that for all $n \geq 1$ there holds

$$C^{-1}\int_\Omega |d_n|_F \mathrm{d}x \leq \|u_n - x\|_{L^1(\Omega)} \quad \text{and} \quad \|u_n - x\|_{L^\infty(\Omega)} \leq C\int_\Omega |d_n|_F \mathrm{d}x.$$

In this family of examples, the assumption $\int_\Omega |d_n|_F \mathrm{d}x \to 0$ is necessary for the perturbation regime to exist.

Following the steps in [8], the asymptotic formula that we derive makes use of:

(1) A limiting Radon measure $\mu$ which describes the geometry of the limiting set,
(2) A background fundamental solution $G(x, y)$,
(3) A limit vector $\mathcal{M} \in [L^2(\Omega, \mathrm{d}\mu)]^d$ which describes the variations of the field $\nabla u_n$ in the presence of inhomogeneity sets,
(4) A polarisation tensor $M$, independent of $u_n$, $u_0$, the larger domain $\Omega$ and the type of boundary condition, such that $\mathcal{M} = M\nabla u_0$ in $L^2(\Omega, \mathrm{d}\mu)$.

This will be particularly familiar to readers acquainted to the subsequent article [10] where an energy-based approach is also used. It turns out that under (1) and (2) only, we can express the first order expansion in terms of $\mathcal{M}$.

Given $u_n, u_0 \in H^1(\Omega)$ given by (1) or (2), we define $w_n = u_n - u_0 \in X$ where $X = H^1_0(\Omega)$ for the Dirichlet problem and $X = \{\phi \in H^1(\Omega) : \int_\Omega \phi \,\mathrm{d}x = 0\}$ for the Neumann problem. Here, $w_n$ is the weak solution of

$$\int_\Omega \gamma_n \nabla w_n \cdot \nabla\phi \mathrm{d}x = \int_\Omega \left(\gamma_0 - \gamma_n\right)\nabla u_0 \cdot \nabla\phi \mathrm{d}x \quad \text{for all } \phi \in X. \tag{3}$$

Note that if $u_0$ is the background solution of (1) or (2), then by classical regularity results [12, Theorem 2.1],

$$u_0 \in H^1(\Omega) \cap C^1(K) \quad \text{and} \quad \|u_0\|_{C^1(K)} \leq C(\Omega)\,\|g\|_{H^{\frac{1}{2}}(\partial\Omega)},$$
$$\text{or } \|u_0\|_{C^1(K)} \leq C(\Omega)\,\|h\|_{H^{-\frac{1}{2}}(\partial\Omega)} \quad \text{respectively.}$$

**Lemma 5.** *Let $d_n \in L^\infty(\Omega; \mathbb{R}^{d\times d})$ be given by Definition 1. Then, the sequence $\frac{|d_n|_F}{\|d_n\|_{L^1(\Omega)}}$ converges up to the possible extraction of a subsequence, in the sense of measures to a positive radon measure $\mu$, that is,*

$$\int_\Omega \frac{1}{\|d_n\|_{L^1(\Omega)}}|d_n|_F \phi \mathrm{d}x \to \int_\Omega \phi \, d\mu \quad \textit{for all } \phi \in C(\overline{\Omega}). \tag{4}$$

*For each*

$$i, j \in \{1, \dots, d\}^2, \ \frac{1}{\|d_n\|_{L^1(\Omega)}}\left(\gamma_n - \gamma_0\right)_{ij}$$

*converges in the sense of measures to a limit $D_{ij} \in [L^2(\Omega, \mathrm{d}\mu)]$*

$$\int_\Omega \frac{1}{\|d_n\|_{L^1(\Omega)}}\left(\gamma_n - \gamma_0\right)_{ij}\phi \mathrm{d}x \to \int_\Omega D_{ij}\phi \, d\mu \quad \textit{for all } \phi \in C(\overline{\Omega}). \tag{5}$$

**Proof.** See Appendix A. □

**Remark 6.** The sequence $\|d_n\|^{-1}_{L^1(\Omega)}|d_n|_F$ only converges to a given measure after extraction of a subsequence in general. In the case of an isotropic, constant, conductivity in the inclusions, $\|d_n\|^{-1}_{L^1(\Omega)}|d_n|_F = 1_{\omega_n}|\omega_n|^{-1}$, and this measure does not depend on the values taken by $\gamma_n$ or $\gamma_0$ on $\omega_n$.

The quantity $d_n$ appears in the following energy estimate.

**Proposition 7.** *The weak solution of* (3) $w_n \in X$ *satisfies*

$$E(w_n) := \int_\Omega \gamma_n \nabla w_n \cdot \nabla w_n \mathrm{d}x \leq \|d_n\|_{L^1(\Omega)} \|\nabla u_0\|^2_{L^\infty(K)}. \tag{6}$$

*As a consequence, there holds*

$$\|(\gamma_n - \gamma_0)\nabla w_n\|_{L^1(\Omega)} \leq \|d_n\|_{L^1(\Omega)} \|\nabla u_0\|_{L^\infty(K)}. \tag{7}$$

*Furthermore, up to the possible extraction of a subsequence,* $\frac{1}{\|d_n\|_{L^1(\Omega)}}(\gamma_0 - \gamma_n)\nabla w_n$ *converges in the sense of measures to a limit*

$$\int_\Omega \frac{1}{\|d_n\|_{L^1(\Omega)}} (\gamma_0 - \gamma_n)\nabla w_n \cdot \Psi \,\mathrm{d}x \to \int_\Omega \mathscr{W} \cdot \Psi \, d\mu, \tag{8}$$

*where* $\mathscr{W} \in [L^2(\Omega, \mathrm{d}\mu)]^d$ *and* $\mu$ *is given by* (4).

**Remark 8.** The upper estimates (6) and (7) are sharp with respect to the order of dependence on $\|d_n\|_{L^1(\Omega)}$ as shown in Example 4.

**Proof.** The proof of Proposition 7 is similar to the moderate contrast case in [8], but with estimates in terms of $\|d_n\|_{L^1(\Omega)}$. It is provided in Appendix B. □

An Aubin–Céa–Nitsche estimate is derived in Lemma 15. It allows extreme contrasts and depends on the $L^1(\Omega)$ norm of $d_n$ only. This implies independence with respect to the domain and the prescribed boundary condition, as stated below (see also [10, Lemma 1]).

**Lemma 9.** *Suppose that Assumptions* (1) *and* (2) *hold. Let* $\widetilde{\Omega}$ *be any bounded regular open set in* $\mathbb{R}^d$ *such that* $K \subset \widetilde{\Omega}$ *with* $\operatorname{dist}(K, \partial\widetilde{\Omega}) > 0$. *Let* $Y$ *be one of the spaces*

$$H^1_0(\widetilde{\Omega}), \quad \widetilde{H}^1(\widetilde{\Omega}) := \left\{\phi \in H^1(\widetilde{\Omega}) : \int_{\widetilde{\Omega}\setminus K} \phi \mathrm{d}x = 0\right\}$$

*or*

$$H^1_\#(\widetilde{\Omega}) := \left\{\phi \in H^1_{\mathrm{loc}}\left(\mathbb{R}^d\right) : \int_{\widetilde{\Omega}\setminus K} \phi \mathrm{d}x = 0 \quad \textit{and } \phi \quad \widetilde{\Omega} - \textit{periodic}\right\},$$

*the latter if* $\widetilde{\Omega}$ *is a cube. We write the weak solution of* (3) $w_n^X \in X$ *and we set* $w_n^Y$ *to be the unique weak solution to*

$$\int_{\widetilde{\Omega}} \gamma_n \nabla w_n^Y \cdot \nabla\phi \mathrm{d}x = \int_{\widetilde{\Omega}} (\gamma_0 - \gamma_n) \nabla u_0 \cdot \nabla \phi \mathrm{d}x \quad \textit{for all } \phi \in Y, \tag{9}$$

*then for any* $\tau \in (0, \frac{1}{2(d-1)})$ *there exists* $C > 0$ *which may depend on* $\tau, \Omega, K, \Lambda_0, \lambda_0$ *and* $\|\gamma_0\|_{W^{2,d}(\Omega)}$ *only such that*

$$\frac{1}{\|d_n\|_{L^1(\Omega)}} \left\|(\gamma_n - \gamma_0)\nabla\left(w_n^Y - w_n^X\right)\right\|_{L^1(\Omega)} \leq C \|d_n\|^\tau_{L^1(\Omega)} \|\nabla u_0\|_{L^\infty(\Omega)}.$$

*As a consequence, the measured valued vector* $\mathscr{M}^X$ *and* $\mathscr{M}^Y$ *obtained from any two of these variational problems via Proposition 7 are equal.*

The proof of this result is provided in Section 2. It now suffices to focus on Dirichlet problem to establish Theorem 3. To prove polarisability, that is, $\mathscr{M} = M\nabla u_0$, our argument requires one of the additional requirements detailed in item 3.

**Definition 10.** *For each* $i = 1, \ldots, d$, *we define the correctors* $w_n^i \in H^1_0(\Omega)$ *as the weak solutions of*

$$\int_\Omega \gamma_n \nabla w_n^i \cdot \nabla \phi \, dx = \int_\Omega (\gamma_0 - \gamma_n) \mathbf{e}_i \cdot \nabla \phi \, dx \quad \textit{for all } \phi \in H^1_0(\Omega). \tag{10}$$

*We call* $W_{ij} \in L^2(\Omega, d\mu)$ *the scalar weak* limit of* $\frac{1}{\|d_n\|_{L^1(\Omega)}}(\nabla w_n^i \cdot (\gamma_0 - \gamma_n)\mathbf{e}_j)$.

**Remark 11.** The connection between this tensor and its parent introduced in [8] is discussed in Section 4.

**Proposition 12.** *Suppose Assumptions* (1), (2) *and* (3) *are satisfied. Given* $\Omega'$ *a smooth open subset of* $\Omega$ *containing* $K$ *such that* $3d(\Omega',\partial\Omega) > d(K,\partial\Omega)$ *and* $3d(K,\partial\Omega') > d(K,\partial\Omega)$, *there holds*

$$\int_\Omega (\gamma_n-\gamma_0)\nabla w_n\cdot\nabla x_i\,\phi\,\mathrm{d}x = \int_\Omega (\gamma_n-\gamma_0)\nabla w_n^i\cdot\nabla u_0\,\phi\,\mathrm{d}x + \int_\Omega r_n\cdot\nabla\phi\,dx$$

*with om*

$$\|r_n\|_{L^1(\Omega)} \le C\|d_n\|_{L^1(\Omega)}^{1+\eta}\left(\|\nabla u_0\|_{L^\infty(K)} + \|u_0\|_{L^\infty(\partial\Omega')}\right),$$

*where the positive constants* $C$ *and* $\eta$ *may depend only on* $\Omega$, $K$, $\|\gamma_0\|_{W^{2,d}(\Omega)}$, $\Lambda_0$, $\lambda_0$, *and* $B$ *and* $p$ *and* $\tau$ *as introduced in Assumption* (3).

**Proof.** The proof of Proposition 12 is the purpose of Section 3. Depending on whether both insulating and conducting inhomogeneities are present, and whether the dimension is 2 or more, it is the combined conclusion of Proposition 19, Proposition 25 and Proposition 27. □

We conclude the proof of Theorem 3, but for the properties of the polarisation tensor $M$, left for Lemma 30.

**End of the proof of Theorem 3.** Consider the Dirichlet case. Observing that the weak formulation for the solution $w_n = u_n - u_0$ reads

$$\int_\Omega \gamma_0\nabla w_n\cdot\nabla\phi\mathrm{d}x = \int_\Omega (\gamma_0-\gamma_n)(\nabla w_n+\nabla u_0)\cdot\nabla\phi\,\mathrm{d}x \tag{11}$$

for any $\phi\in H_0^1(\Omega)$, we choose a sequence $\phi_m\in C_c^1(\Omega)$ such that $\phi_m\to G$ in $W^{1,1}(\Omega)$ and $\nabla\phi_m\to\nabla_x G$ in $C^0(K)$. Using that $w_n$ is smooth away from the set $K$ and that $\gamma_n-\gamma_0$ is supported in $K$, we may insert $\phi_m$ into (11) and pass to the limit to conclude that

$$\int_\Omega \gamma_0\nabla w_n\cdot\nabla_x G(x,y)\,\mathrm{d}x = \int_\Omega (\gamma_0-\gamma_n)(\nabla u_0+\nabla w_n)\cdot\nabla_x G(x,y)\,\mathrm{d}x.$$

After an integration by parts we obtain

$$\begin{aligned}(u_n-u_0)(y) &= \int_\Omega (\gamma_n-\gamma_0)(\nabla w_n+\nabla u_0)\cdot\nabla_x G(x,y)\mathrm{d}x\\ &= \|d_n\|_{L^1(\Omega)}\int_\Omega \frac{1}{\|d_n\|_{L^1(\Omega)}}(\gamma_n-\gamma_0)\nabla u_0\cdot\nabla_x G(x,y)\,\mathrm{d}x\\ &\quad - \|d_n\|_{L^1(\Omega)}\int_\Omega \frac{1}{\|d_n\|_{L^1(\Omega)}}(\gamma_0-\gamma_n)\nabla w_n\cdot\nabla_x G(x,y)\,\mathrm{d}x\end{aligned}$$

Using the fact that

$$\forall\, y\in\overline{\Omega\setminus K}\quad\text{and}\quad\forall\, x\in\bigcup_{n=1}^\infty \omega_n,$$

we may find a smooth function $\phi_y\in C^0(\overline{\Omega})$ such that

$$\phi_y(x) = \nabla_x G(x,y)\quad\forall\, x\in K,$$

and thanks to Proposition 12 and Lemma 5 we have

$$(u_n-u_0)(y) = \|d_n\|_{L^1(\Omega)}\int_\Omega (D_{ij}-W_{ij})\frac{\partial u_0}{\partial x_i}\frac{\partial G(x,y)}{\partial x_j}\mathrm{d}\mu(x) + r_n(y),$$

where $W\in L^2(\Omega,\mathbb{R}^{d\times d};\mathrm{d}\mu)$ is introduced in 10. Note that $\phi_y$ is uniformly bounded $\forall\,(x,y)\in K\times\overline{\Omega\setminus K}$. Moreover, the remainder estimate from Proposition 12 only depends on

$$\|g\|_{H^{\frac12}(\partial\Omega)},\ \text{therefore}\ \|r_n\|_{L^\infty(\Omega)}/\|d_n\|_{L^1(\Omega)}$$

converges to 0 uniformly in $y \in \overline{\Omega \setminus K}$ and $g$ in the unit ball of the space $H^{\frac{1}{2}}(\partial\Omega)$. The Neumann case is similar. □

The rest of paper is structured as follows. In Section 2 we derive a number of a priori estimates, and prove Lemma 9. Section 3 is devoted to the proof of Proposition 12. In Section 4 we briefly discuss some of the properties of the tensor $M$, and prove Lemma 30. Finally in Section 5 we show with an example that the a priori bounds for $M$ given in Theorem 3 are attained.

## 2. Proof of Lemma 9 and a priori estimates

**Notation.** In the sequel, we use the notation $a \lesssim b$ to mean $a \leq Cb$, where $C$ is a constant, possibly changing from line to line depending on the parameters announced in the claim we wish to prove.

**Remark 13.** We remind the reader that $|AU|_d \leq |A|_F |U|_d$ a.e. in $\Omega$, even though the Frobenius norm isn't the subordinate matrix norm associated with the Euclidean distance in $\mathbb{R}^d$. From Definition 1 on $\omega_n$ there holds

$$d_n = \gamma_n + \gamma_0 \gamma_n^{-1} \gamma_0 = \left(\gamma_n - \gamma_0\right) \gamma_n^{-1} \left(\gamma_n - \gamma_0\right) + 2\gamma_0.$$

Thus $d_n$ is symmetric, non-negative, and bounded below by

$$d_n \geq \gamma_n, \quad d_n \geq 2\gamma_0 \quad \text{and} \quad d_n > \left(\gamma_n - \gamma_0\right) \gamma_n^{-1} \left(\gamma_n - \gamma_0\right).$$

In particular, $d_n \geq \gamma_n - \gamma_0$ and $d_n \geq \gamma_0 - \gamma_n$. If $A$ is a non-negative symmetric matrix, $B$ is a symmetric matrix and there holds $A \geq B$ and $A \geq -B$, then $|A_F \geq |B|_F$. As a consequence, there holds,

$$\begin{cases} \left|d_n\right|_F & \geq \left|\gamma_0\right|_F \\ \left|d_n\right|_F & \geq \left|\gamma_n\right|_F \\ \left|d_n\right|_F & \geq \left|\gamma_n - \gamma_0\right|_F \\ \left|d_n\right|_F & \geq \left|\left(\gamma_n - \gamma_0\right) \gamma_n^{-1} \left(\gamma_n - \gamma_0\right)\right|_F \end{cases} \quad \text{a.e. on } \omega_n. \tag{12}$$

We will use these estimates frequently.

**Lemma 14.** *Given $\Omega'$ a smooth open subset of $\Omega$ containing $K$ such that $d(\Omega', \partial\Omega) > \frac{1}{3} d(K, \partial\Omega)$ and $d(K, \partial\Omega') > \frac{1}{3} d(K, \partial\Omega)$, there holds*

$$\|u_n\|_{L^\infty(\partial\Omega')} + \left\|\nabla u_n\right\|_{L^\infty(\partial\Omega')} \leq C\left(\|\nabla u_0\|_{L^\infty(K)} + \|u_0\|_{L^\infty(\partial\Omega')}\right),$$
$$\|w_n\|_{L^\infty(\partial\Omega')} + \left\|\nabla w_n\right\|_{L^\infty(\partial\Omega')} \leq C \|w_n\|_{L^2(\Omega \setminus K)}$$

*where $C > 0$ depends on $\Omega'$, $K, \Omega$, $\Lambda_0$, $\lambda_0$ and $\|\gamma_0\|_{W^{2,d}(\Omega)}$ only. Furthermore,*

$$\|w_n\|_{L^\infty(K)} \leq C\left(\left\|\nabla u_0\right\|_{L^\infty(K)} + \|u_0\|_{L^\infty(\partial\Omega')}\right). \tag{13}$$

This follows from the maximum principle and standard elliptic regularity theory. The proof is given in Appendix A. Following the strategy introduced in [8], we now show that the potential tends to zero faster than the gradient via an Aubin–Céa–Nitsche argument. The novelty of this result is that it depends on $\gamma_n$ only via on $\|d_n\|_{L^1(\Omega)}$.

**Lemma 15.** *For any $\tau \in [1, \frac{d}{d-1})$, and given $\Omega'$ a smooth domain as defined in Proposition 12, there holds*

$$\|w_n\|_{L^2(\Omega)} \leq C\left\|d_n\right\|_{L^1(\Omega)}^{\frac{\tau}{2}} \left(\left\|\nabla u_0\right\|_{L^\infty(K)} + \|u_0\|_{L^\infty(\partial\Omega')}\right), \tag{14}$$

*with the constant $C$ may depend on $\tau$, $\Omega, K$, $\|\gamma_0\|_{W^{2,d}(\Omega)}$, and the a priori bounds $\Lambda_0$ and $\lambda_0$ only.*

**Proof.** Consider the following auxiliary equation

$$\begin{aligned} -\operatorname{div}\left(\gamma_0 \nabla \psi_n\right) = w_n \quad \text{in} \quad \Omega \\ \psi_n = 0 \quad \text{on} \quad \partial\Omega. \end{aligned} \tag{15}$$

Since $\gamma_0 \in W^{2,d}(\Omega;\mathbb{R}^{d\times d})$ we infer from elliptic regularity theory (see e.g. [12]) that for any $q \geq 2$, the solution $\psi_n$ satisfies

$$\left\|\psi_n\right\|_{W^{2,q}(\Omega)} \lesssim \|w_n\|_{L^q(\Omega)}. \tag{16}$$

Testing (15) with $w_n$, and recalling that $\operatorname{supp}(\gamma_n - \gamma_0) \subset \omega_n \subset K$, an integration by parts shows

$$\begin{aligned} \|w_n\|^2_{L^2(\Omega)} &= \int_\Omega \gamma_0 \nabla \psi_n \cdot \nabla w_n \,\mathrm{d}x \\ &= \int_\Omega \left(\gamma_0 - \gamma_n\right) \nabla w_n \cdot \nabla \psi_n \,\mathrm{d}x + \int_\Omega \gamma_n \nabla \psi_n \cdot \nabla w_n \,\mathrm{d}x \\ &= \int_\Omega \left(\gamma_0 - \gamma_n\right) \nabla w_n \cdot \nabla \psi_n + \int_\Omega \left(\gamma_0 - \gamma_n\right) \nabla u_0 \cdot \nabla \psi_n \end{aligned} \tag{17}$$

Using Cauchy–Schwarz, we find

$$\int_\Omega \left(\gamma_0 - \gamma_n\right) \nabla w_n \cdot \nabla \psi_n \,\mathrm{d}x \leq \left(\int_\Omega \gamma_n \nabla w_n \cdot \nabla w_n \,\mathrm{d}x\right)^{\frac{1}{2}} \left(\int_\Omega d_n \nabla \psi_n \cdot \nabla \psi_n \,\mathrm{d}x\right)^{\frac{1}{2}},$$

and thanks to (6),

$$\int_{\omega_n} \left(\gamma_0 - \gamma_n\right) \nabla w_n \cdot \nabla \psi_n \,\mathrm{d}x \leq \left\|d_n\right\|_{L^1(\Omega)} \|\nabla u_0\|_{L^\infty(K)} \left\|\nabla \psi_n\right\|_{L^\infty(K)}.$$

Similarly, using (12),

$$\begin{aligned} \int_\Omega \left(\gamma_0 - \gamma_n\right) \nabla u_0 \cdot \nabla \psi_n \,\mathrm{d}x &\leq \left(\int_\Omega \gamma_n \nabla u_0 \cdot \nabla u_0 \,\mathrm{d}x\right)^{\frac{1}{2}} \left(\int_\Omega d_n \nabla \psi_n \cdot \nabla \psi_n \,\mathrm{d}x\right)^{\frac{1}{2}} \\ &\leq \left\|d_n\right\|_{L^1(\Omega)} \|\nabla u_0\|_{L^\infty(K)} \left\|\nabla \psi_n\right\|_{L^\infty(K)}, \end{aligned}$$

and (17) becomes

$$\|w_n\|^2_{L^2(\Omega)} \leq 2\left\|d_n\right\|_{L^1(\Omega)} \|\nabla u_0\|_{L^\infty(K)} \left\|\nabla \psi_n\right\|_{L^\infty(K)}. \tag{18}$$

On the other hand, choosing $q = d + \epsilon$ in (16) there holds

$$\left\|\nabla \psi_n\right\|_{L^\infty(\Omega)} \lesssim \left\|\psi_n\right\|_{W^{2,d+\epsilon}(\Omega)} \lesssim \|w_n\|_{L^{d+\epsilon}(\Omega)} \tag{19}$$

By interpolation, and using the a priori bound (13) for $w_n$ given in Lemma 14, we find

$$\|w_n\|_{L^{d+\epsilon}(\Omega)} \leq \|w_n\|^{\frac{2}{d+\epsilon}}_{L^2(\Omega)} \|w_n\|^{1-\frac{2}{d+\epsilon}}_{L^\infty(\Omega)} \lesssim \|w_n\|^{\frac{2}{d+\epsilon}}_{L^2(\Omega)} \left(\left\|\nabla u_0\right\|_{L^\infty(K)} + \|u_0\|_{L^\infty(\partial\Omega')}\right)^{1-\frac{2}{d+\epsilon}}. \tag{20}$$

Combining (18), (19), and (20), we have obtained

$$\begin{aligned} \|w_n\|^{2\left(1-\frac{1}{d+\epsilon}\right)}_{L^2(\Omega)} &\lesssim \left\|d_n\right\|_{L^1(\Omega)} \left\|\nabla u_0\right\|_{L^\infty(K)} \left(\left\|\nabla u_0\right\|_{L^\infty(K)} + \|u_0\|_{L^\infty(\partial\Omega')}\right)^{1-\frac{2}{d+\epsilon}} \\ &\lesssim \left\|d_n\right\|_{L^1(\Omega)} \left(\left\|\nabla u_0\right\|_{L^\infty(K)} + \|u_0\|_{L^\infty(\partial\Omega')}\right)^{2\left(1-\frac{1}{d+\epsilon}\right)}, \end{aligned}$$

which is equivalent to (14). □

**Remark 16.** Note that estimate (14) improves on previous estimates, even in the case of bounded contrasts (see [8, lemma 1]). It is arbitrarily close to the estimate one obtains for a fixed, scaled shape with constant scalar conductivity [2].

**Corollary 17.** *For any $q \geq 2$ and any $\tau \in [1, \frac{d}{d-1})$, with the same notations as in Lemma 15, there holds*

$$\|w_n\|_{L^q(\Omega)} \leq C\left\|d_n\right\|^{\frac{\tau}{q}}_{L^1(\Omega)} \left(\left\|\nabla u_0\right\|_{L^\infty(K)} + \|u_0\|_{L^\infty(\partial\Omega')}\right). \tag{21}$$

*Furthermore, $w_n$ solution of* (3) *satisfies*

$$\left\|\nabla w_n\right\|_{L^\infty(\partial\Omega')} + \|w_n\|_{L^\infty(\partial\Omega')} \leq C\left\|d_n\right\|_{L^1(\Omega)}^{\frac{\tau}{2}}\left(\left\|\nabla u_0\right\|_{L^\infty(K)} + \|u_0\|_{L^\infty(\partial\Omega')}\right). \tag{22}$$

**Proof.** We write

$$\|w_n\|_{L^s(\Omega)} \leq \|w_n\|_{L^2(\Omega)}^{\frac{2}{q}}\|w_n\|_{L^\infty(\Omega)}^{1-\frac{2}{q}}$$

and estimate (21) follows from (14) and (13). Estimate (22) follows from Lemmas 14 and 15. □

We now address the independence of the polarisation tensor $M$ from the boundary conditions.

**Proof of Lemma 9.** Given $\tau = (0, \frac{1}{2}\frac{1}{d-1})$, Following the steps of Lemma 15 starting from (9) and $w_n^Y$, we find

$$\left\|w_n^Y\right\|_{L^2(\tilde\Omega)} \lesssim \|d_n\|_{L^1(\Omega)}^{\frac{1+2\tau}{2}}\left(\left\|\nabla u_0\right\|_{L^\infty(K)} + \|u_0\|_{L^\infty(\partial\Omega')}\right). \tag{23}$$

Choose a smooth cut-off function $\chi \in C_c^\infty(\widetilde\Omega)$ such that $\chi = 1$ on $K$. Noting that

$$\operatorname{div}\left(\gamma_n\nabla\left(w_n^X - w_n^Y\right)\right) = 0 \text{ on } \widehat\Omega,$$

Caccioppoli's inequality writes

$$\int_{\tilde\Omega}\gamma_n\nabla\left(\chi\left(w_n^Y - w_n^X\right)\right)\cdot\nabla\left(\chi\left(w_n^Y - w_n^X\right)\right)\mathrm{d}x = \int_{\tilde\Omega\setminus K}\left(\gamma_0\nabla\chi\cdot\nabla\chi\right)\left(w_n^Y - w_n^X\right)^2\mathrm{d}x,$$

that is,

$$\begin{aligned}\int_{\tilde\Omega}\gamma_n\nabla\left(w_n^Y - w_n^X\right)\cdot\nabla\left(w_n^Y - w_n^X\right)\mathrm{d}x &\leq C\left(\widetilde\Omega, K\right)\left(\left\|w_n^X\right\|_{L^2(\Omega)}^2 + \left\|w_n^Y\right\|_{L^2(\tilde\Omega)}^2\right),\\ &\lesssim \left\|d_n\right\|_{L^1(\Omega)}^{1+2\tau}\left(\left\|\nabla u_0\right\|_{L^\infty(K)} + \|u_0\|_{L^\infty(\partial\Omega')}\right)^2\end{aligned}$$

This in turn shows, by Cauchy–Schwarz,

$$\begin{aligned}\left\|(\gamma_n - \gamma_0)\nabla\left(w_n^Y - w_n^X\right)\right\|_{L^1(\Omega)} &\leq \left\|d_n\right\|_{L^1(\Omega)}^{\frac{1}{2}}\left(\int_K\gamma_n\nabla\left(w_n^Y - w_n^X\right)\cdot\nabla\left(w_n^Y - w_n^X\right)\mathrm{d}x\right)^{\frac{1}{2}}\\ &\leq C\left(\widetilde\Omega, K\right)\left\|d_n\right\|_{L^1(\Omega)}^{1+\tau}\left(\left\|\nabla u_0\right\|_{L^\infty(K)} + \|u_0\|_{L^\infty(\partial\Omega')}\right)\end{aligned}$$

As a result,

$$\frac{1}{\left\|d_n\right\|_{L^1(\Omega)}}\left\|\left(\gamma_n - \gamma_0\right)\nabla\left(w_n^Y - w_n^X\right)\right\|_{L^1(\Omega)} \to 0,$$

which implies that the limiting measures resulting from

$$\frac{1}{\left\|d_n\right\|_{L^1(\Omega)}}\left(\gamma_n - \gamma_0\right)\nabla w_n^X \quad\text{and}\quad \frac{1}{\left\|d_n\right\|_{L^1(\Omega)}}\left(\gamma_n - \gamma_0\right)\nabla w_n^Y \quad\text{are equal.}$$

□

## 3. Proof of Proposition 12

We use the following corollary to the a priori energy estimate given in Proposition 7.

**Corollary (Corollary to Proposition 7).** *For any $p \geq 1$, there holds*

$$\left\|\gamma_n\nabla w_n\right\|_{L^{\frac{2p}{p+1}}(A_n)} \leq d^{\frac{1}{4}}\left\|d_n\right\|_{L^1(\Omega)}^{\frac{1}{2}}\left\|d_n\right\|_{L^p(A_n)}^{\frac{1}{2}}\|\nabla u_0\|_{L^\infty(K)}. \tag{24}$$

**Proof.** Using Hölder's inequality, it holds that for any $p \geq 1$

$$\left\|\gamma_n\nabla w_n\right\|_{L^{\frac{2p}{p+1}}(A_n)} \leq \left\|\gamma_n^{\frac{1}{2}}\right\|_{L^{2p}(A_n)}\left(E(w_n)\right)^{\frac{1}{2}}. \tag{25}$$

We have

$$\left\|\gamma_n^{\frac{1}{2}}\right\|_{L^{2p}(A_n)} = \left(\int_{A_n}\left|\gamma_n^{\frac{1}{2}}\right|_F^{2p}\mathrm{d}x\right)^{\frac{1}{2p}},$$

and, using the fact that for $d\times d$ symmetric matrix $A$, $|A^2|_F \le |A|_F^2 \le \sqrt{d}|A^2|_F$,we find, using (12),

$$\left\|\gamma_n^{\frac{1}{2}}\right\|_{L^{2p}(A_n)} \le d^{\frac{1}{4}}\left(\int_{A_n}\left|\gamma_n\right|_F^p \mathrm{d}x\right)^{\frac{1}{2p}} = d^{\frac{1}{4}}\left\|\gamma_n\right\|_{L^p(A_n)}^{\frac{1}{2}} \le d^{\frac{1}{4}}\left\|d_n\right\|_{L^p(A_n)}^{\frac{1}{2}}. \tag{26}$$

Putting together (6), (25) and (26) the conclusion follows. □

The following error estimate is a key tool for the proof of Proposition 12.

**Proposition 18.** *For any $\phi\in C^1(\overline{\Omega})$, there holds*

$$\int_\Omega\left(\left(\gamma_n-\gamma_0\right)\nabla w_n\cdot\nabla x_i\right)\phi\,\mathrm{d}x = \int_\Omega\left(\left(\gamma_n-\gamma_0\right)\nabla w_n^i\cdot\nabla u_0\right)\phi\,\mathrm{d}x + \int_\Omega r_n\cdot\nabla\phi\,\mathrm{d}x \tag{27}$$

*with $r_n\in L^1(\Omega)$. Furthermore for any $\tau\in[1,\frac{2d-1}{2d-2})$, the following estimate holds*

$$\left|\int_\Omega r_n\cdot\nabla\phi\,\mathrm{d}x\right| \le C\left\|\nabla\phi\right\|_{L^\infty(\Omega)}\left(\left\|d_n\right\|_{L^1(\Omega)}^\tau\left(\left\|\nabla u_0\right\|_{L^\infty(K)} + \|u_0\|_{L^\infty(\partial\Omega')}\right) + \varepsilon_n\right), \tag{28}$$

*The constant $C$ may depends on $\tau$, $\Omega$, $K$, $\|\gamma_0\|_{W^{2,d}(\Omega)}$, and the a priori bounds $\Lambda_0$ and $\lambda_0$ only. The remainder term $\varepsilon_n$ satisfies the following two a priori estimates*

$$\varepsilon_n \le \left\|d_n\right\|_{L^1(\Omega)}\left(\|w_n\|_{L^\infty(A_n)} + \left\|w_n^i\right\|_{L^\infty(A_n)}\|\nabla u_0\|_{L^\infty(K)}\right) \tag{29}$$

*and, for $p>d$,*

$$\varepsilon_n \le \left\|d_n\right\|_{L^1(\Omega)}^{1+\eta}\|d_n\|_{L^p(A_n)}^{\frac{1}{2}}\left(\left\|\nabla u_0\right\|_{L^\infty(K)} + \|u_0\|_{L^\infty(\partial\Omega')}\right). \tag{30}$$

*where $\eta>0$ depends only on $p$.*

**Remark.** Note that estimates (29) and (30) imply that $\epsilon_n\le 0$ when $A_n=\emptyset$.

**Proof.** We write $Z$ as a shorthand for $\|\nabla u_0\|_{L^\infty(K)} + \|u_0\|_{L^\infty(\partial\Omega')}$. A computation shows that

$$\int_\Omega\left(\left(\gamma_n-\gamma_0\right)\nabla w_n\cdot\nabla x_i\right)\phi\,\mathrm{d}x = \int_\Omega\left(\left(\gamma_0-\gamma_n\right)\nabla w_n^i\cdot\nabla u_0\right)\phi\,\mathrm{d}x + \int_\Omega r_n\cdot\nabla\phi\,\mathrm{d}x$$

where the remainder term $r_n\in L^1(\Omega)$ is

$$r_n = \left(\gamma_n-\gamma_0\right)\left(w_n^i\nabla u_0 - w_n\nabla x_i\right) + w_n^i\gamma_n\nabla w_n - w_n\gamma_n\nabla w_n^i.$$

Now, write $T_1 = \mathbf{1}_{B_n}\left(r_n\cdot\nabla\phi\right)$ and $T_2 = r_n\cdot\nabla\phi - T_1$.

$$\begin{aligned}\left\|T_1\right\|_{L^1(\Omega)} &\le \int_{B_n}\left|w_n\left(\gamma_n\nabla w_n^i\right)\cdot\nabla\phi\right|\mathrm{d}x + \int_{B_n}\left|w_n^i\left(\gamma_n\nabla w_n\right)\cdot\nabla\phi\right|\mathrm{d}x\\ &\quad+\int_{B_n}\left|w_n^i\left(\gamma_n-\gamma_0\right)\nabla u_0\cdot\nabla\phi\right|\mathrm{d}x + \int_{B_n}\left|w_n\left(\gamma_n-\gamma_0\right)\nabla x_i\cdot\nabla\phi\right|\mathrm{d}x\\ &\lesssim \left\|\nabla\phi\right\|_{L^\infty(\Omega)}\left(\|w_n\|_{L^2(\Omega)}E\left(w_n^i\right)^{\frac{1}{2}} + \left\|w_n^i\right\|_{L^2(\Omega)}E\left(w_n\right)^{\frac{1}{2}}\right.\\ &\quad\left.+\left\|w_n^i\right\|_{L^2(\Omega)}\left\|d_n\right\|_{L^1(\Omega)}^{\frac{1}{2}}\left\|\nabla u_0\right\|_{L^\infty(K)} + \|w_n\|_{L^2(\Omega)}\left\|d_n\right\|_{L^1(\Omega)}^{\frac{1}{2}}\right)\end{aligned}$$

Thanks to estimate (6) and (14) (applied to $u_0 = x_i$ for the corrector terms $w_n^i$) we find

$$\left\|T_1\right\|_{L^1(\Omega)} \lesssim \left\|d_n\right\|_{L^1(\Omega)}^{\frac{1}{2}+\frac{1}{2}\tau'}\left\|\nabla\phi\right\|_{L^\infty(\Omega)}Z,$$

with $\tau'\in[1,\frac{d}{d-1})$, so that $\tau=\frac{1+\tau'}{2}\in[1,\frac{2d-1}{2d-2})$. We now turn to the other term. The triangle inequality gives

$$\begin{aligned}\left\|T_2\right\|_{L^1(\Omega)} &\le \int_{A_n}\left|w_n\left(\gamma_n\nabla w_n^i\right)\cdot\nabla\phi\right|\mathrm{d}x + \int_{A_n}\left|w_n^i\left(\gamma_n\nabla w_n\right)\cdot\nabla\phi\right|\mathrm{d}x\\ &\quad+\int_{A_n}\left|w_n^i\left(\gamma_n-\gamma_0\right)\nabla u_0\cdot\nabla\phi\right|\mathrm{d}x + \int_{A_n}\left|w_n\left(\gamma_n-\gamma_0\right)\nabla x_i\cdot\nabla\phi\right|\mathrm{d}x.\end{aligned} \tag{31}$$

Recall that thanks to (12), $|\gamma_n-\gamma_0|_F < |d_n|_F$. Thus using (24) with $p=1$, and (12), we deduce from (31) that

$$\left\|T_2\right\|_{L^1(\Omega)} \lesssim \left\|d_n\right\|_{L^1(\Omega)}\left(\|w_n\|_{L^\infty(A_n)}+\left\|w_n^i\right\|_{L^\infty(A_n)}\left\|\nabla u_0\right\|_{L^\infty(K)}\right)\left\|\nabla\phi\right\|_{L^\infty(K)},$$

which corresponds to estimate (29).

Alternatively, applying Hölder's inequality, then the $L^p$ bound (24) and the $L^q$ bound (21) with the conjugate exponent, we find for any $p\geq 1$, and any $\theta\in[1,\frac{d}{d-1})$,

$$\begin{aligned}\int_{A_n}\left|w_n\gamma_n\nabla w_n^i\cdot\nabla\phi\right|\mathrm{d}x &\leq \left\|\gamma_n\nabla w_n^i\right\|_{L^{\frac{2p}{p+1}}(A_n)}\left\|w_n\right\|_{L^{\frac{2p}{p-1}}(A_n)}\left\|\nabla\phi\right\|_{L^\infty(K)}\\ &\lesssim \left\|d_n\right\|_{L^1(\Omega)}^{\frac12}\left\|d_n\right\|_{L^p(A_n)}^{\frac12}\left\|d_n\right\|_{L^1(\Omega)}^{\left(\frac12-\frac1{2p}\right)\theta}Z\left\|\nabla\phi\right\|_{L^\infty(K)}.\end{aligned}$$

Similarly

$$\int_{A_n}\left|w_n^i\gamma_n\nabla w_n\cdot\nabla\phi\right|\mathrm{d}x \lesssim \left\|d_n\right\|_{L^1(\Omega)}^{\frac12}\left\|d_n\right\|_{L^p(A_n)}^{\frac12}\left\|d_n\right\|_{L^1(\Omega)}^{\left(\frac12-\frac1{2p}\right)\theta}Z\left\|\nabla\phi\right\|_{L^\infty(K)}.$$

Using (12), Hölder's inequality and the $L^q$ bound (21), we write

$$\begin{aligned}\int_{A_n}\left|w_n^i\left(\gamma_n-\gamma_0\right)\nabla u_0\cdot\nabla\phi\right|\mathrm{d}x &\leq \left\|d_n^{\frac12}\right\|_{L^2(A_n)}\left\|d_n^{\frac12}w_n^i\right\|_{L^2(A_n)}\left\|\nabla u_0\right\|_{L^\infty(K)}\left\|\nabla\phi\right\|_{L^\infty(K)}\\ &\lesssim \left\|d_n\right\|_{L^1(A_n)}^{\frac12}\left\|d_n\right\|_{L^p(A_n)}^{\frac12}\left\|w_n^i\right\|_{L^{\frac{2p}{p-1}}(A_n)}\left\|\nabla u_0\right\|_{L^\infty(K)}\left\|\nabla\phi\right\|_{L^\infty(K)}\\ &\lesssim \left\|d_n\right\|_{L^1(\Omega)}^{\frac12}\left\|d_n\right\|_{L^p(A_n)}^{\frac12}\left\|d_n\right\|_{L^1(\Omega)}^{\left(\frac12-\frac1{2p}\right)\theta}Z\left\|\nabla\phi\right\|_{L^\infty(K)},\end{aligned}$$

and by the same argument,

$$\int_{A_n}\left|w_n\left(\gamma_n-\gamma_0\right)\nabla x_i\cdot\nabla\phi\right|\mathrm{d}x \lesssim \left\|d_n\right\|_{L^1(\Omega)}^{\frac12}\left\|d_n\right\|_{L^p(A_n)}^{\frac12}\left\|d_n\right\|_{L^1(\Omega)}^{\left(\frac12-\frac1{2p}\right)\theta}Z\left\|\nabla\phi\right\|_{L^\infty(K)}.$$

Altogether, for any $p\geq 1$, and any $\theta\in[1,\frac{d}{d-1})$,

$$\left\|T_2\right\|_{L^1(\Omega)} \lesssim \left\|d_n\right\|_{L^1(\Omega)}^{\frac12}\left\|d_n\right\|_{L^p(A_n)}^{\frac12}\left\|d_n\right\|_{L^1(\Omega)}^{\left(\frac12-\frac1{2p}\right)\theta}Z\left\|\nabla\phi\right\|_{L^\infty(K)}.$$

For any $p>d$, pick $\theta=\frac12(\frac{p}{p-1}+\frac{d}{d-1})$, then

$$\eta=\frac12\left(\frac{d}{d-1}\frac{p-1}{p}-1\right)>0,$$

and

$$\left\|T_2\right\|_{L^1(\Omega)} \leq \left\|d_n\right\|_{L^1(\Omega)}^{1+\eta}\left\|d_n\right\|_{L^p(A_n)}^{\frac12}Z\left\|\nabla\phi\right\|_{L^\infty(K)},$$

which concludes the proof of estimate (30). □

**Proposition 19.** *Suppose Assumptions* (1), (2), *and* (3a) *hold. Given $\Omega'$ a smooth open subset of $\Omega$ containing $K$ such that $d(\Omega',\partial\Omega)>\frac13 d(K,\partial\Omega)$ and $d(K,\partial\Omega')>\frac13 d(K,\partial\Omega)$, there holds*

$$\int_\Omega\left(\left(\gamma_n-\gamma_0\right)\nabla w_n\cdot\nabla x_i\right)\phi\,\mathrm{d}x=\int_\Omega\left(\left(\gamma_n-\gamma_0\right)\nabla w_n^i\cdot\nabla u_0\right)\phi\,\mathrm{d}x+\int_\Omega r_n\cdot\nabla\phi\,\mathrm{d}x$$

*with*

$$\|r_n\|_{L^1(\Omega)}\leq C\left\|d_n\right\|_{L^1(\Omega)}^{1+\eta}\left(\left\|\nabla u_0\right\|_{L^\infty(K)}+\|u_0\|_{L^\infty(\partial\Omega')}\right),$$

*where the positive constants $C$ and $\eta$ may depend only on $\Omega$, $K$, $\|\gamma_0\|_{W^{2,d}(\Omega)}$, $\Lambda_0$, $\lambda_0$, and $p$ and $B$ as introduced in* (3).

**Proof.** This is an immediate consequence of Proposition 18. □

### 3.1. *The high conductivity inclusion case when $d=2$*

This section addresses the case when (3b) holds. When $d=2$, as it is well known, there is a direct relation between high and low conductivity problem, by means of stream functions (see e.g. [13]). We use this indirect method to obtain the polarisability result under (3b). We remind the reader of the following classical result.

**Lemma 20 ( [3, Lemma I.1]).** *Let $\Omega$ be any smooth open set in $\mathbb{R}^2$, not necessarily simply connected, and $D$ be a vector field such that*

$$\operatorname{div}(D)=0 \quad on\ \Omega, \quad and \quad \int_{\Gamma_i} D\cdot n \mathrm{d}\sigma = 0$$

*on each connected component $\Gamma_i$ of $\partial\Omega$. Then, there exists a function $H$ such that*

$$D = \left(-\partial_{x_2}H, \partial_{x_1}H\right) \quad on\ \Omega.$$

Let $(\Gamma_i)_{1\, leq\, i\leq N}$ the connected components of $\partial\Omega$ and let $Fb_n$ and $Fb_0$ the unique solutions of

$$\begin{cases} \operatorname{div}\left(\gamma_n\nabla Fb_n\right)=0 & \text{on } \Omega', \\ \gamma_n\nabla Fb_n\cdot n = \frac{1}{|\Gamma_i|}\int_{\Gamma_i}\gamma_n\nabla u_n\cdot n\mathrm{d}\sigma & \text{on each } \Gamma_i. \\ \int_\Omega Fb_n \mathrm{d}x = 0. \end{cases} \tag{32}$$

and

$$\begin{cases} \operatorname{div}\left(\gamma_0\nabla Fb_0\right)=0 & \text{on } \Omega', \\ \gamma_0\nabla Fb_0\cdot n = \frac{1}{|\Gamma_i|}\int_{\Gamma_i}\gamma_0\nabla u_0\cdot n\mathrm{d}\sigma & \text{on each } \Gamma_i. \\ \int_\Omega Fb_0 \mathrm{d}x = 0. \end{cases} \tag{33}$$

Then applying Lemma 20 to $\gamma_n\nabla(u_n - Fb_n)$ and $\gamma_0\nabla(u_0 - Fb_0)$ there exist stream functions $\psi_n, \psi_0 \in H^1(\Omega')$ such that

$$\gamma_n\nabla\left(u_n - Fb_n\right) = J\nabla\psi_n \text{ and } \gamma_0\nabla\left(u_0 - Fb_0\right) = J\nabla\psi_0 \quad \text{a.e. in } \Omega'. \tag{34}$$

where $J$ is the antisymmetric matrix $\left(\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right)$. As the stream functions may be chosen uniquely up to an additive constant, we may assume without loss of generality that they satisfy the constraint

$$\int_\Omega \psi_n \mathrm{d}x = 0 = \int_\Omega \psi_0 \mathrm{d}x.$$

Thus, $\psi_n$ and $\psi_0$ are weak solutions of

$$-\operatorname{div}\left(\sigma_n\nabla\psi_n\right) = 0 \text{ in } \Omega'$$
$$-\operatorname{div}\left(\sigma_0\nabla\psi_0\right) = 0 \text{ in } \Omega'$$

where the conductivity matrices $\sigma_n$ and $\sigma_0$ are defined as

$$\sigma_n := J^T\gamma_n^{-1}J \quad \text{and} \quad \sigma_0 := J^T\gamma_0^{-1}J.$$

When then define $\Sigma_n$ as $d_n$ was with respect to $\gamma_0$ and $\gamma_n$.

**Definition 21.** *We set*

$$\Sigma_n = \left(\sigma_n + \sigma_0\sigma_n^{-1}\sigma_0\right)1_{\omega_n}.$$

**Proposition 22.** *Given $\Omega'$ a smooth domain as defined in Proposition 12, given $\psi_n$ and $\psi_0$ be the stream functions defined in* (34). *The function $\varphi_n = \psi_n - \psi_0$ satisfies*

$$-\operatorname{div}\left(\sigma_n\nabla\varphi_n\right) = \operatorname{div}\left((\sigma_n - \sigma_0)\nabla\psi_0\right) \ in\, \mathscr{D}'\left(\Omega'\right) \tag{35}$$

*and for any $\tau \in \left(0, \frac{1}{2}\right)$ there holds*

$$\left\|\sigma_n\nabla\varphi_n\cdot\nu\right\|_{H^{-\frac{1}{2}}(\partial\Omega')} \leq C\left\|d_n\right\|_{L^1(\Omega)}^{\frac{1}{2}+\tau}\left\|g\right\|_{H^{\frac{1}{2}}(\partial\Omega')}, \tag{36}$$

*where the constant $C$ may depend only on $\tau$, $\Omega$, $K$, $\|\gamma_0\|_{W^{2,d}(\Omega)}$, $\Lambda_0$ and $\lambda_0$.*

**Proof.** Thanks to (34), since $d(\partial\Omega', K) > 0$, on $\partial\Omega'$

$$\begin{aligned}\sigma_n \nabla\varphi_n = \sigma_0\nabla\varphi_n &= J^T\nabla\big(u_n - Fb_n - u_0 - Fb_0\big)\\ &= J^T\nabla\big(w_n + Fb_n - Fb_0\big).\end{aligned}$$

Thanks to estimate (22) applied to $w_n$ and to $Fb_n$ and $Fb_0$, there holds

$$\left\|\sigma_n\nabla\varphi_n\right\|_{L^\infty(\partial\Omega')} \le C\left\|d_n\right\|_{L^1(\Omega)}^{\frac{1}{2}+\tau}\left(\left\|\nabla u_0\right\|_{L^\infty(K)} + \|u_0\|_{L^\infty(\partial\Omega')}\right).$$

which implies (36). □

When considering (35) rather than (3), $C_n$ plays the role of $A_n$ and $D_n$ plays the role of $B_n$. The polarisability for $\varphi_n$ is therefore established from Proposition 19 provided $\|d_n\|_{L^p(C_n)} < \infty$ for some $p > 2$.

**Corollary 23.** *Suppose that Assumptions* (1) *and* (2) *are satisfied. Additionally assume that $d = 2$ and that there exists $p > 2$ and $C \in \mathbb{R}$ such that*

$$C = \sup_n \left\|\Sigma_n\right\|_{L^p(B_n)}.$$

*The function $\varphi_n = \psi_n - \psi_0$, weak solution to* (35), *satisfies*

$$\frac{1}{\left\|\Sigma_n\right\|_{L^1(\Omega)}}\Big(\sigma_0 - \sigma_n\Big)\nabla\varphi_n\,\mathrm{d}x \quad \stackrel{*}{\rightharpoonup} \quad \widetilde{N}\nabla\psi_0\,\mathrm{d}\nu \tag{37}$$

*in the space of bounded Radon measures where $\widetilde{N} \in L^2(\Omega, \mathbb{R}^{d\times d}; \mathrm{d}\nu)$, and $\nu$ is the Radon measure generated by the sequence $\frac{1}{\|\Sigma_n\|_{L^1(\Omega)}}\Sigma_n$. The convergence is uniform with respect to $g \in H^{1/2}(\partial\Omega)$ provided*

$$\left\|g\right\|_{H^{\frac{1}{2}}(\partial\Omega)} \le 1.$$

**Proof.** This follows directly from Proposition 18 and Lemma 9. □

**Lemma 24.** *The symmetric positive definite matrix $\Sigma_n$ given by Definition 21 satisfies*

$$\Sigma_n = \sigma_n + \sigma_0\sigma_n^{-1}\sigma_0 = J^T\gamma_0^{-1}d_n\gamma_0^{-1}J.$$

*As a consequence, denoting $\nu$ and $\mu$ to be the Radon measures generated by the sequences*

$$\frac{\Sigma_n}{\left\|\Sigma_n\right\|_{L^1(\Omega)}} \quad \textit{and} \quad \frac{d_n}{\left\|d_n\right\|_{L^1(\Omega)}}$$

*respectively, the Radon–Nikodym derivatives $\frac{\mathrm{d}\nu}{\mathrm{d}}\mu$ and $\frac{\mathrm{d}\mu}{\mathrm{d}\nu}$ belongs to $L^\infty(\Omega; \mathrm{d}\mu)$ and $L^\infty(\Omega; \mathrm{d}\nu)$ respectively, and the spaces $L^p(\Omega; \mathrm{d}\mu)$ are equivalent to $L^p(\Omega; \mathrm{d}\nu)$ for any $p \ge 1$.*

**Proof.** The formula $\Sigma_n = J^T\gamma_0^{-1}d_n\gamma_0^{-1}J$ is straightforward to verify. It follows that

$$\left|d_n\right|_F\left(\min_{\overline{\Omega}}\lambda\left(\gamma_0^{-1}\right)\right)^2 \le \left|\Sigma_n\right|_F \le \left|d_n\right|_F\left(\max_{\overline{\Omega}}\lambda\left(\gamma_0^{-1}\right)\right)^2. \tag{38}$$

Since these two quantities are equivalent, the conclusion follows. □

**Proposition 25.** *Suppose assumptions* (1), (2) *and* (3b) *are satisfied. Given $\Omega'$ a smooth domain as defined in Proposition 12, there holds*

$$\int_\Omega \Big(\left(\gamma_n - \gamma_0\right)\nabla w_n\cdot\nabla x_i\Big)\phi\,\mathrm{d}x = \int_\Omega\Big(\left(\gamma_n - \gamma_0\right)\nabla w_n^i\cdot\nabla u_0\Big)\phi\,\mathrm{d}x + \int_\Omega r_n\cdot\nabla\phi\,\mathrm{d}x$$

*with*

$$\|r_n\|_{L^1(\Omega)} \le C\left\|d_n\right\|_{L^1(\Omega)}^{1+\eta}\left(\left\|\nabla u_0\right\|_{L^\infty(K)} + \|u_0\|_{L^\infty(\partial\Omega')}\right),$$

*where the positive constants $C$ and $\eta$ may depend only on $\Omega$, $K$, $\|\gamma_0\|_{W^{2,d}(\Omega)}$, $\Lambda_0$, $\lambda_0$, $p$ and $B$.*

The proof of this result is given in Appendix C.

## 3.2. *The non finely intertwined case*

The main result of this section is Proposition 12 in the final case, namely when (3c) holds. Example 26 is an illustration of such a configuration.

**Example 26.** Suppose that $\Omega\subset\mathbb{R}^d$ is the ball $B(0,d)$ of radius $d$ centred at the origin. Assume that $\gamma_0=I_d$. Given $\epsilon>0$, for $n\geq 2$, we set

$$A_n=D_n=\bigcup_{k=1}^{n}\left(\frac{k}{n},\frac{k}{n}+\frac{1}{n^{d+1+\epsilon}}\right)\times(0,1)^{d-1},$$
$$B_n=C_n=\bigcup_{k=1}^{n}\left(\frac{k}{n}+\frac{1}{2n},\frac{k}{n}+\frac{3}{4n}\right)\times(0,1)^{d-1},$$

and

$$\gamma_n=\left(\left(n\frac{i-1}{d-1}+\frac{d-i}{d-1}\right)\delta_{ij}\right)_{1\leq i,j\leq d}\text{ on }A_n,\quad \gamma_n=\frac{\ln n}{n}I_d\text{ on }B_n.$$

We have $\omega_n\subset(0,1)^d\subset\Omega$. We have

$$\left\|d_n\right\|_{L^1(A_n)}\propto\frac{1}{n^{d-1+\epsilon}},\quad \left\|d_n\right\|_{L^1(B_n)}\propto\frac{1}{\ln n},$$

therefore $\|d_n\|_{L^1(\Omega)}\to 0$. The insulating and conductive strips are separated by a distance

$$d\left(A_n,C_n\right)\propto\frac{1}{n}>\left\|d_n\right\|^{\tau}_{L^1(A_n)}\quad\text{for}\quad\tau\in\left(0,\frac{1}{d-1}\right).$$

We compute that $\|d_n\|_{L^p(A_n)}\propto n^{p-(d+\epsilon)}$. In particular for $p=d>\frac{d}{2}$ there holds $\|d_n\|_{L^p(A_n)}\to 0$. Notice that the conductive strips are narrowed to accomodate the extra integrability, whereas the insulating one are just chosen to so that $\|d_n\|_{L^1(\Omega)}\to 0$.

**Proposition 27.** *Suppose Assumptions* (1) *and* (2) *are satisfied. Suppose additionally that for some* $p>\frac{d}{2}$ *and* $B\in\mathbb{R}$ *and* $\tau\in(0,\frac{1}{(d-1)}t)$ *there holds*

$$B=\limsup_n\left\|d_n\right\|^{\frac{1}{2}}_{L^p(A_n)},$$

*and that there exists a sequence of function* $(\chi_n)_{n\in\mathbb{N}}\in(W^{1,\infty}(\Omega;[0,1]))^{\mathbb{N}}$ *such that* $\chi_n\equiv 0$ *on* $B_n$, $\chi_n=1$ *on* $A_n$ *and*

$$\left\|d_n\right\|^{\tau}_{L^1(\Omega)}\left\|\nabla\chi_n\right\|_{L^\infty(\Omega)}<\infty,$$

*Given* $\Omega'$ *a smooth domain as defined in Proposition 12, there holds*

$$\int_\Omega\left(\left(\gamma_n-\gamma_0\right)\nabla w_n\cdot\nabla x_i\right)\phi\mathrm{d}x=\int_\Omega\left(\left(\gamma_n-\gamma_0\right)\nabla w_n^i\cdot\nabla u_0\right)\phi\mathrm{d}x+\int_\Omega r_n\cdot\nabla\phi\mathrm{d}x$$

*with*

$$\|r_n\|_{L^1(\Omega)}\leq C\left\|d_n\right\|^{1+\eta}_{L^1(\Omega)}\left(\left\|\nabla u_0\right\|_{L^\infty(K)}+\|u_0\|_{L^\infty(\partial\Omega')}\right),$$

*where the positive constants* $C$ *and* $\eta$ *may depend only on* $\tau$, $\Omega$, $K$, $\left\|\gamma_0\right\|_{W^{2,d}(\Omega)}$, $\Lambda_0$ *and* $\lambda_0$, $p$, $B$ *and* $\tau$ *only.*

**Proof.** This a direct consequence of estimate (29) in Proposition 18 and Lemma 28. □

**Lemma 28.** *Suppose that for some* $p>\frac{d}{2}$ *and* $A\in\mathbb{R}$*, there holds*

$$\sup_n\left\|d_n\right\|^{\frac{1}{2}}_{L^p(A_n)}<A,$$

*and that there exists a sequence of function*

$$\left(\chi_n\right)_{n\in\mathbb{N}}\in\left(W^{1,\infty}(\Omega;[0,1])\right)^{\mathbb{N}}$$

*such that* $\chi_n \equiv 0$ *on* $B_n$, $\chi_n = 1$ *on* $A_n$ *with*

$$\|d_n\|^{\tau}_{L^1(\Omega)} \|\nabla \chi_n\|_{L^\infty(\Omega)} < A,$$

*for some* $\tau < \frac{1}{(d-1)}$. *Then there exists* $\eta > 0$ *depending on* $p$ *and* $\tau$ *only such that*

$$\|w_n\|_{L^\infty(A_n)} \le C\|d_n\|^{\eta}_{L^1(\Omega)} \left(\|\nabla u_0\|_{L^\infty(K)} + \|u_0\|_{L^\infty(\partial\Omega')}\right),$$

*where* $C$ *depends on* $K, \Omega$, $\Lambda_0$, $\lambda_0$, $\|\gamma_0\|_{W^{2,d}(\Omega)}$, $p$, $A$ *and* $\tau$ *only.*

**Proof.** We apply Stampacchia's truncation method [21]. We denote $u \to G_k(u)$ to be the truncation operator, that is,

$$G_k(u) = \begin{cases} u & |u| \le k \\ k & u > k \\ -k & u < -k \end{cases} \quad \text{with } k > 0,$$

and we write $m_k = \{x \in \Omega : |u_n| > k\}$. We test equation (3) against $\chi_n^2 v_n$, with $v_n = w_n - G_k(w_n)$, and obtain, on one hand

$$\int_\Omega \gamma_n \nabla w_n \cdot \nabla\left(\chi_n^2 v_n\right) \mathrm{d}x$$
$$= \int_\Omega \chi_n \left(\gamma_0 - \gamma_n\right) \nabla u_0 \cdot \nabla\left(\chi_n v_n\right) \mathrm{d}x + \int_\Omega \chi_n v_n \left(\gamma_0 - \gamma_n\right) \nabla u_0 \cdot \nabla \chi_n \mathrm{d}x,$$

and on the other

$$\int_\Omega \gamma_n \nabla w_n \cdot \nabla\left(\chi_n^2 v_n\right) \mathrm{d}x = \int_\Omega \gamma_n \nabla\left(\chi_n v_n\right) \cdot \nabla\left(\chi_n v_n\right) \mathrm{d}x - \int_\Omega \gamma_n \nabla \chi_n \cdot \nabla \chi_n v_n^2 \mathrm{d}x.$$

Since $v_n$ is supported on $m_k$, $\chi_n$ is supported on $D_n$, and $\nabla\chi_n$ is supported on $B_n \cap D_n$, we may simplify the above identities to

$$\int_\Omega \gamma_n \nabla\left(\chi v_n\right) \cdot \nabla\left(\chi v_n\right) \mathrm{d}x \lesssim \int_{m_k} \left|\nabla \chi_n\right|^2 v_n^2 \mathrm{d}x + \int_{m_k \cap D_n} \left|\left(\gamma_0 - \gamma_n\right) \nabla u_0 \cdot \nabla\left(\chi_n v_n\right)\right| \mathrm{d}x$$
$$+ \int_{m_k \cap D_n} \left|v_n \left(\gamma_0 - \gamma_n\right) \nabla u_0 \cdot \nabla \chi_n\right| \mathrm{d}x.$$

Using Cauchy–Schwarz, we find

$$\int_{m_k \cap D_n} \left|\left(\gamma_0 - \gamma_n\right) \nabla u_0 \cdot \nabla\left(\chi_n v_n\right)\right| \mathrm{d}x \le \left(\int_{m_k \cap D_n} d_n \nabla u_0 \cdot \nabla u_0 \mathrm{d}x\right)^{\frac{1}{2}} \left(\int_\Omega \gamma_n \nabla\left(\chi v_n\right) \cdot \nabla\left(\chi v_n\right) \mathrm{d}x\right)^{\frac{1}{2}}$$
$$\int_{m_k \cap D_n} \left|v_n \left(\gamma_0 - \gamma_n\right) \nabla u_0 \cdot \nabla \chi_n\right| \mathrm{d}x \lesssim \left(\int_{m_k \cap D_n} d_n \nabla u_0 \cdot \nabla u_0 \mathrm{d}x\right)^{\frac{1}{2}} \left(\int_{m_k} \left|\nabla \chi_n\right|^2 v_n^2 \mathrm{d}x\right)^{\frac{1}{2}}$$

which shows in turn,

$$\lambda_0 \int_\Omega \left|\nabla\left(\chi v_n\right)\right|^2 \mathrm{d}x \le \int_\Omega \gamma_n \nabla\left(\chi v_n\right) \cdot \nabla\left(\chi v_n\right) \mathrm{d}x \lesssim \left(\int_{m_k \cap D_n} d_n \nabla u_0 \cdot \nabla u_0 \mathrm{d}x + \int_{m_k} \left|\nabla \chi_n\right|^2 v_n^2 \mathrm{d}x\right).$$

Using Hölder's inequality and the fact that $|v_n| \le |w_n|$,we write

$$\int_{m_k \cap D_n} d_n \nabla u_0 \cdot \nabla u_0 \mathrm{d}x + \int_{m_k} \left|\nabla \chi_n\right|^2 v_n^2 \mathrm{d}x$$
$$\le \|d_n\|_{L^p(D_n)} \|\nabla u_0\|^2_{L^\infty(K)} |m_k|^{1-\frac{1}{p}} + \|w_n\|^2_{L^{2p}(\Omega)} \|\nabla \chi_n\|^2_{L^\infty(\Omega)} |m_k|^{\frac{p-1}{p}},$$

Whereas for any $h > k$, thanks to the Sobolev embedding $H^1(\Omega) \hookrightarrow L^q(\Omega)$ for $q = (\frac{p}{p-1} + \frac{d}{d-2})$ if $d > 2$ and $q = \frac{2p}{p-1} + 1$ if $d = 2$,

$$|k-h|^2 |m_h|^{\frac{2}{q}} \lesssim \|\chi v_n\|^2_{L^{3+\frac{2}{s}}(m_k)} \lesssim \int_\Omega \lambda_0 \left|\nabla\left(\chi v_n\right)\right|^2 \mathrm{d}x.$$

This shows that $m_k = 0$, for $k$ large enough, that is,

$$\|\chi_n w_n\|_{L^\infty(\Omega)} \lesssim \left(\|d_n\|_{L^p(D_n)}^{\frac{1}{2}} \|\nabla u_0\|_{L^\infty(K)} + \|w_n\|_{L^{2p}(\Omega)} \|\nabla \chi_n\|_{L^\infty(\Omega)}\right),$$

Thanks to estimate (21), for any $\zeta \in [1, \frac{1}{(d-1)})$ there holds

$$\|w_n\|_{L^{2p}(\Omega)} \lesssim \|d_n\|_{L^1(\Omega)}^{\frac{d\zeta}{2p}} \left(\|\nabla u_0\|_{L^\infty(K)} + \|u_0\|_{L^\infty(\partial\Omega')}\right).$$

Altogether,

$$\|w_n\|_{L^\infty(A_n)} \lesssim C\left(\|d_n\|_{L^p(D_n)}^{\frac{1}{2}} + \|d_n\|_{L^1(\Omega)}^{\zeta} \|\nabla \chi_n\|_{L^\infty(\Omega)}\right)\left(\|\nabla u_0\|_{L^\infty(K)} + \|u_0\|_{L^\infty(\partial\Omega')}\right). \tag{39}$$

Note that

$$\sup_n \|d_n\|_{L^p(D_n)} \le \sup_n \|d_n\|_{L^p(A_n)} + \frac{\Lambda_0^2}{\lambda_0} |\Omega|^{\frac{1}{p}} \lesssim 1.$$

write

$$\kappa = \sup_n \|d_n\|_{L^1(\Omega)}^{\tau} \|\nabla \chi_n\|_{L^\infty(\Omega)} + \|d_n\|_{L^p(D_n)}$$

and $p_1 = \frac{1}{2}\min(\frac{d}{2}\frac{1}{\tau(d-1)}, p) + \frac{d}{4}$. By interpolation between $L^1(D_n)$ and $L^p(D_n)$ we have

$$\|d_n\|_{L^{p_1}(D_n)}^{\frac{1}{2}} \le \|d_n\|_{L^1(D_n)}^{\theta_1} \kappa^{\frac{1}{2}-\theta_1},$$

with $\theta_1 = \frac{p-p_1}{2p_1(p-1)} > 0$ and

$$\|d_n\|_{L^1(\Omega)}^{\frac{d\tau}{2p_1}} \|\nabla \chi_n\|_{L^\infty(\Omega)} \le \|d_n\|_{L^1(\Omega)}^{\theta_2} \kappa,$$

with

$$\theta_2 = \left(\frac{d}{2p_1} - 1\right)\tau > 0.$$

Estimate (39) with $p = p_1$ and $\zeta = \tau$ concludes the proof, with $\eta = \min(\theta_1, \theta_2)$. □

## 4. Properties of the polarisation tensor $M$

Thanks to Lemma 9, we may consider alternative definitions for the tensor $M$. The most convenient is the periodic one, namely, embedding $\Omega$ in a large cube $Q$, we set

$$H^1_\#(Q) := \left\{\phi \in H^1_{\mathrm{loc}}\left(\mathbb{R}^d\right) : \int_{Q\setminus K} \phi \,\mathrm{d}x = 0 \quad \text{and} \quad \phi\ Q-\text{periodic}\right\},$$

and $M_{ij} = D_{ij} - W_{ij} \in L^2(\Omega, d\mu)$ is the scalar weak* limit of

$$\frac{1}{\|d_n\|_{L^1(\Omega)}} \left(\left(\nabla w_n^i + \mathbf{e}_i\right) \cdot (\gamma_n - \gamma_0)\mathbf{e}_j\right),$$

where $w_n^i$ is the unique weak solution to

$$\int_Q \gamma_n \nabla w_n^i \cdot \nabla\phi \,\mathrm{d}x = \int_Q \left(\gamma_0 - \gamma_n\right)\mathbf{e}_j \cdot \nabla\phi \,\mathrm{d}x \text{ for all } \phi \in H^1_\#(Q). \tag{40}$$

In [8] another version $\mathbb{M}$ of this tensor is introduced, and $M$ a natural extension to this context.

Assuming $\gamma_n = ((\gamma_1 - \gamma_0)1_{\omega_n} + \gamma_0) I_d$ for some regular functions $\gamma_1$ and $\gamma_0$, then the tensor $\mathbb{M}$ introduced in [8] is defined as the weak* limit in $L^2(\Omega, d\mu)$ of

$$\frac{1}{|\omega_n|}\left(\nabla w_n^i + \mathbf{e}_i\right) \cdot \mathbf{e}_j$$

To compare both formulas, suppose $\gamma_1$ and $\gamma_0$ are constant. Then

$$\frac{1}{\|d_n\|_{L^1(\Omega)}}\left(\left(\nabla w_n^i+\mathbf{e}_i\right)\cdot\left(\gamma_n-\gamma_0\right)\mathbf{e}_j\right)=\frac{1}{|\omega_n|}\frac{1}{\sqrt{d}}\frac{\gamma_1}{\gamma_1^2+\gamma_0^2}\left(\gamma_1-\gamma_0\right)\left(\nabla w_n^i+\mathbf{e}_i\right)\cdot\mathbf{e}_j,$$

thus the two tensors are related by the simple formula

$$M=\frac{1}{\sqrt{d}}\frac{\gamma_1}{\gamma_1^2+\gamma_0^2}\left(\gamma_0-\gamma_1\right)\mathbb{M}, \tag{41}$$

and most properties can be read directly from [10], with the appropriate changes.

**Lemma 29 ( [8, Theorem 1]).** *The entries of the polarisation tensor $M$ satisfies $M_{ij}=M_{ji}$ $\mu$-almost everywhere in $\Omega$.*

**Lemma 30 (See [10, Lemma 4]).** *For every $\phi\in C_c^1(Q)$, $\phi\geq 0$, and every $\zeta\in\mathbb{R}^d$, there holds*

$$\begin{aligned}\int_Q W\zeta\cdot\zeta\phi d\mu=&\frac{1}{\|d_n\|_{L^1(Q)}}\int_Q d_n'\zeta\cdot\zeta\phi\,\mathrm{d}x\\&-\frac{1}{\|d_n\|_{L^1(Q)}}\min_{u\in H_\#^1(Q)^d}\int_Q\gamma_n\left(\nabla u-\gamma_n^{-1}\left(\gamma_n-\gamma_0\right)\zeta\right)\cdot\left(\nabla u-\gamma_n^{-1}\left(\gamma_n-\gamma_0\right)\zeta\right)\phi\,\mathrm{d}x+o(1),\end{aligned}$$

*with*

$$d_n'=\left(\gamma_n-\gamma_0\right)\gamma_n^{-1}\left(\gamma_n-\gamma_0\right)=d_n-2\gamma_0\geq 0.$$

*In particular, the tensor $M$ is positive semi-definite and satisfies*

$$0\leq W\leq I_d\quad \mu\text{ a.e. in } Q.$$

*If $\gamma_n$ and $\gamma_0$ are multiples of the identity matrix, that is, the material is isotropic, then*

$$0\leq W\leq\frac{1}{\sqrt{d}}I_d\quad \mu\text{ a.e. in } Q.$$

**Proof.** The derivation of the identity is, *mutatis mutandis*, done in [10, Lemma 4]. Choosing $u=0$, we find

$$\begin{aligned}&\frac{1}{\|d_n\|_{L^1(\Omega)}}\min_{u\in H_\#^1(Q)^d}\int_Q\gamma_n\left(\nabla u-\gamma_n^{-1}\left(\gamma_n-\gamma_0\right)\zeta\right)\cdot\left(\nabla u-\gamma_n^{-1}\left(\gamma_n-\gamma_0\right)\zeta\right)\phi\,\mathrm{d}x\\\leq&\frac{1}{\|d_n\|_{L^1(\Omega)}}\min_{u\in H_\#^1(Q)^d}\int_Q d_n'\phi\,\mathrm{d}x,\end{aligned}$$

and therefore

$$\int_Q W\zeta\cdot\zeta\phi d\mu\geq 0.$$

Since the second term is negative, we find

$$\int_Q W\zeta\cdot\zeta\phi d\mu\leq\lim_{n\to\infty}\frac{1}{\|d_n\|_{L^1(Q)}}\int_Q\phi d_n'\zeta\cdot\zeta\,\mathrm{d}x.$$

We compute

$$\frac{1}{\|d_n\|_{L^1(\Omega)}}\int_Q\phi d_n'\zeta\cdot\zeta\,\mathrm{d}x=\int_Q\phi\frac{d_n'\zeta\cdot\zeta}{|d_n|_F}\frac{|d_n|_F}{\|d_n\|_{L^1(\Omega)}}\mathrm{d}x\leq\int_Q\phi\frac{d_n'\zeta\cdot\zeta}{|d_n'|_F}\frac{|d_n|_F}{\|d_n\|_{L^1(\Omega)}}\mathrm{d}x,$$

and if $\lambda_1\leq\cdots\leq\lambda_d$ are the eigenvalues of $d_n'$ at $x$,

$$\frac{d_n'\zeta\cdot\zeta}{|d_n'|_F}\leq|\zeta|^2\frac{\lambda_d}{\sqrt{\sum_{i=1}^d\lambda_i^2}}\leq|\zeta|^2\begin{cases}1&\text{in general,}\\\frac{1}{\sqrt{d}}&\text{if }\lambda_1=\cdots=\lambda_d\end{cases}.$$

All eigenvalues are equal when $\gamma_0$ and $\gamma_n$ are isotropic, therefore

$$\int_Q W\zeta\cdot\zeta\phi d\mu \le \lim_{n\to\infty}\frac{1}{\left\|d_n\right\|_{L^1(Q)}}\int_Q \phi d_n'\zeta\cdot\zeta \mathrm{d}x \le C|\zeta|^2\int_Q \phi \mathrm{d}\mu,$$

with $C=1$ in general and $C=d^{-\frac{1}{2}}$ in isotropic media. □

## 5. An example

We revisit an example already considered in [4, 9], namely, elliptic inclusions. In a domain

$$\Omega=\left\{(x,y)\subset\mathbb{R}^2 : \frac{x^2}{\cosh^2(2)}+\frac{y^2}{\sinh^2(2)}\le 1\right\},$$

consider heterogeneities in a homogeneous medium located in the set

$$E_n=\left\{(x,y)\subset\mathbb{R}^2 : \frac{x^2}{\cosh^2\left(n^{-1}\right)}+\frac{y^2}{\sinh^2\left(n^{-1}\right)}\le 1\right\},$$

which collapses to the line segment $(-1,1)\times\{0\}$ as $n\to\infty$. Consider an isotropic inhomogeneity, with conductivity

$$\gamma_n(x)=\begin{cases}1 & x\in\Omega\setminus Q_n\\ \lambda_n & x\in Q_n,\end{cases}$$

where $\lambda_n\in(0,1)\cup(1,\infty)$. In this case,

$$d_n=\left(\lambda_n+\lambda_n^{-1}\right)I_2.$$

and $\|d_n\|_{L^1(\Omega)}\to 0$ means $\max(n^{-1}\lambda_n, n^{-1}\lambda_n^{-1})\to 0$. The solution $u_n^i$ to the equation

$$\begin{aligned}-\nabla\cdot\left(\gamma_n\nabla u_n^i\right)&=0 \quad \text{in} \quad \Omega\\ u_n^i&=x_i \quad \text{on} \quad \partial\Omega\end{aligned} \tag{42}$$

can be computed explicitly in elliptic coordinates. In particular we find that

$$\frac{1}{\left|d_n\right|_F}\left(1-\gamma_n\right)\partial_{x_j}w_n^i=\frac{1}{\sqrt{2}}\frac{\lambda_n}{1+\lambda_n^2}\left(1-\gamma_n\right)1_{E_n}\left(\partial_{x_j}u_n^i-\delta_{ij}\right)=\delta_{ij}\ell^i_n 1_{E_n},$$

with

$$\begin{aligned}&\ell_n^1=O\left(\frac{\lambda_n}{n}\right) \quad\text{and}\quad \ell_n^2=\frac{1}{\sqrt{2}}+O\left(\frac{\lambda_n}{n}\right) &&\text{when } \lambda_n>1,\\ &\ell_n^1=O\left(\frac{1}{n^2}\right) \quad\text{and}\quad \ell_n^2=O\left(\frac{1}{n\lambda_n}\right) &&\text{when } 0<\lambda_n<1,\end{aligned}$$

As a consequence, when $n\lambda_n\to 0$ with $\lambda_n\to\infty$

$$W=\begin{pmatrix}0&0\\0&\frac{1}{\sqrt{2}}\end{pmatrix},\quad D=\begin{pmatrix}\frac{1}{\sqrt{2}}&0\\0&\frac{1}{\sqrt{2}}\end{pmatrix}\quad M=\begin{pmatrix}\frac{1}{\sqrt{2}}&0\\0&0\end{pmatrix},$$

Whereas when $\lambda_n\to 0$, we obtain

$$W=\begin{pmatrix}0&0\\0&0\end{pmatrix},\quad D=-\begin{pmatrix}\frac{1}{\sqrt{2}}&0\\0&\frac{1}{\sqrt{2}}\end{pmatrix}\quad M=-\begin{pmatrix}\frac{1}{\sqrt{2}}&0\\0&\frac{1}{\sqrt{2}}\end{pmatrix},$$

and both results corresponds extreme cases with respect to the isotropic pointwise bounds derived in Lemma 30.

## 6. Conclusion*

The energy comparison approach we follow in the spirit of arguments originally introduced by Jacques-Louis Lions. While our results can be extended to the context of linear elasticity, the extension to oscillatory problems, such as the Helmholtz equation, is much less certain. Recent developments in cloaking by transformation optics have shown that spurious resonances may appear [14, 16, 17]. In the context of a single inclusion in two and three dimensions, quasi-resonance phenomenons and rich variety of asymptotic behaviours can be observed [1, 5–7, 15, 20]. This question certainly calls for further developments.

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## Appendix A. Additional proofs

**Proof of Lemma 14.** Let $\Omega''$ be an open domain such that $K \subset \Omega'' \subset \Omega' \subset \Omega$, with $9d(\Omega'', \partial\Omega') >$

$$d(K, \partial\Omega)$$

and $9d(K, \partial\Omega'') > d(K, \partial\Omega)$. Since

$$-\operatorname{div}\left(\gamma_0 \nabla w_n\right) = 0 \quad \text{on} \quad \Omega'' \setminus \Omega'$$

and $\gamma_0 \in W^{2,d}(\Omega)$, classical regularity theory shows that

$$\| w_n \|_{C^1\left(\overline{\Omega' \setminus \Omega''}\right)} \lesssim \| w_n \|_{L^2(\Omega \setminus K)}. \tag{43}$$

By Poincaré's inequality (or Poincaré-Wirtinger's inequality depending on $X$) since $w_n$ (or $\gamma_0 \nabla w_n \cdot n$) vanishes on $\partial\Omega$, there holds

$$\| w_n \|_{L^2(\Omega \setminus K)} \lesssim \left\| \nabla w_n \right\|_{L^2(\Omega \setminus K)}.$$

On the other hand, using the fact that $\gamma_n = \gamma_0 \geq \lambda_0 I_d$ on $\Omega \setminus K$, there holds

$$\begin{aligned} &\lesssim \left\| d_n \right\|_{L^1(\Omega)}^{\frac{1}{2}} \left\| \nabla u_0 \right\|_{L^\infty(K)} \\ &\lesssim \left\| \nabla u_0 \right\|_{L^\infty(K)}, \end{aligned}$$

where we used (6) for the penultimate inequality and the fact that the sequence $\| d_n \|_{L^1(\Omega)}$ is bounded on the last line. Therefore on $\Omega \setminus \Omega'$, the function $w_n$ satisfies $\operatorname{div}(\gamma_0 \nabla w_n) = 0$ with $|w_n| \lesssim \| \nabla u_0 \|_{L^\infty(K)}$ on $\partial\Omega'$ and satisfies a homogeneous boundary condition on $\partial\Omega$ (or periodicity). By comparison, this implies

$$\| w_n \|_{L^\infty(\Omega \setminus \Omega')} \lesssim \left\| \nabla u_0 \right\|_{L^\infty(K)}.$$

Furthermore, $u_n = w_n + u_0$ satisfies $\| u_n \|_{C^1(\partial\Omega')} \leq \| w_n \|_{C^1(\partial\Omega')} + \| u_0 \|_{C^1(\partial\Omega')}$. Finally, since $\operatorname{div}(\gamma_n \nabla u_n) = 0$ on $\Omega'$, by comparison $\| u_n \|_{L^\infty(\Omega')} = \| u_n \|_{C(\partial\Omega')}$, and $\| w_n \|_{L^\infty(\Omega)} \leq \| w_n \|_{L^\infty(\Omega')} + \| w_n \|_{L^\infty(\Omega \setminus \Omega')} \lesssim \| \nabla u_0 \|_{L^\infty(K)} + \| u_0 \|_{L^\infty(\partial\Omega')}$ and the conclusion follows. $\square$

**Proof of Lemma 5.** The convergence (4) is a direct consequence of the Banach–Alaoglu's theorem and the continuous embedding between $L^1(\Omega) \hookrightarrow C^0(\overline{\Omega})^*$, where we have identified the continuous dual space of $C^0(\overline{\Omega})$ as the space of bounded Radon measures on $\Omega$. We know from (12) that $|(\gamma_0-\gamma_n)_{ij}| \le |d_n|_F$, therefore $\|(\gamma_n-\gamma_0)_{ij}\|_{L^1(\Omega)} \le \|d_n\|_{L^1(\Omega)}$. We may extract a subsequence in which

$$\frac{1}{\left\|d_n\right\|_{L^1(\Omega)}}\left(\gamma_n-\gamma_0\right)_{ij} \overset{*}{\rightharpoonup} \mathrm{d}\mathscr{D}_{ij}$$

in the space of bounded vector Radon measures.

$$\begin{aligned}
\int_\Omega \phi\, d\mathscr{D}_{ij} &= \lim_{n\to\infty}\int_\Omega \frac{1}{\left\|d_n\right\|_{L^1(\Omega)}}\left(\gamma_0-\gamma_n\right)_{ij}\phi\,\mathrm{d}x\\
&\le \lim_{n\to\infty}\int_\Omega \frac{1}{\left\|d_n\right\|_{L^1(\Omega)}}\left|d_n\right|_F\phi\,\mathrm{d}x\\
&\le \lim_{n\to\infty}\left(\int_\Omega \frac{1}{\left\|d_n\right\|_{L^1(\Omega)}}\left|d_n\right|_F\phi^2\,\mathrm{d}x\right)^{\frac12}\\
&= \left(\int_\Omega \phi^2\,\mathrm{d}\mu\right)^{\frac12},
\end{aligned}$$

where we used Cauchy–Schwarz in the penultimate line. It follows that the functional

$$\phi \to \int_\Omega \phi\cdot \mathrm{d}\mathscr{D}_{ij}$$

may be extended to a bounded linear functional on $[L^2(\Omega,\mathrm{d}\mu)]^d$. Hence, by Riesz's Representation Theorem, we may identify

$$\mathrm{d}\mathscr{D}_{ij} = D_{ij}\mathrm{d}\mu$$

for some function $D_{ij}\in L^2(\Omega,\mathrm{d}\mu)$, which is our statement. □

## Appendix B. Proof of Proposition 7

**Proof.** We write

$$d_n' = \left(\gamma_n-\gamma_0\right)\gamma_n^{-1}\left(\gamma_n-\gamma_0\right),$$

and observe that $0\le d_n'\le d_n$. Note that $w_n$ is the unique minimiser over $X$ of the functional

$$J(w) = \int_\Omega \gamma_n\Big(\nabla w+\gamma_n^{-1}\left(\gamma_n-\gamma_0\right)\nabla u_0\Big)\cdot\Big(\nabla w+\gamma_n^{-1}\left(\gamma_n-\gamma_0\right)\nabla u_0\Big)\mathrm{d}x,$$

Necessarily $J(w_n)\ge 0$, thus

$$-\int_\Omega \gamma_n\nabla w_n\cdot\nabla w_n\,\mathrm{d}x + 2\int_\Omega \gamma_n\Big(\nabla w_n+\gamma_n^{-1}\left(\gamma_n-\gamma_0\right)\nabla u_0\Big)\cdot\nabla w_n\mathrm{d}x + \int_\Omega d_n'\nabla u_0\cdot\nabla u_0\mathrm{d}x \ge 0,$$

which shows

$$\int_\Omega \gamma_n\nabla w_n\cdot\nabla w_n\,\mathrm{d}x \le \int_\Omega d_n'\nabla u_0\cdot\nabla u_0\,\mathrm{d}x. \tag{44}$$

Thus, as $u_0\in C^1(K)$

$$\int_\Omega \gamma_n(x)\nabla w_n\cdot\nabla w_n\,\mathrm{d}x \le \left\|\nabla u_0\right\|^2_{L^\infty(K)}\int_\Omega \left|d_n\right|_F\mathrm{d}x.$$

We now turn to the second estimate. Using Cauchy–Schwarz we find

$$\begin{aligned}
\left\|\left(\gamma_n-\gamma_0\right)\nabla w_n\right\|_{L^1(\Omega)} &= \int_\Omega\sqrt{\left|\left(\gamma_n-\gamma_0\right)\gamma_n^{-\frac12}\gamma_n^{\frac12}\nabla w_n\right|^2}\,\mathrm{d}x\\
&\le \sqrt{\int_\Omega\left|\left(\gamma_n-\gamma_0\right)\gamma_n^{-1}\left(\gamma_n-\gamma_0\right)\right|_F\mathrm{d}x}\sqrt{\int_\Omega \gamma_n\nabla w_n\cdot\nabla w_n\mathrm{d}x}\\
&\le \left\|d_n\right\|_{L^1(\Omega)}\left\|\nabla u_0\right\|_{L^\infty(K)}.
\end{aligned} \tag{45}$$

Since $\frac{1}{\|d_n\|_{L^1(\Omega)}}(\gamma_n-\gamma_0)\nabla w_n$ is uniformly bounded in $L^1(\Omega)$, we may extract a subsequence in which

$$\frac{1}{\left\|d_n\right\|_{L^1(\Omega)}}\left(\gamma_n-\gamma_0\right)\nabla w_n \overset{*}{\rightharpoonup} \mathrm{d}\mathscr{M}$$

in the space of bounded vector Radon measures. Moreover, for any $\Psi\in C^0(\overline{\Omega};\mathbb{R}^d)$,

$$\begin{aligned}\int_\Omega \Psi\cdot\mathrm{d}\mathscr{M} &= \lim_{n\to\infty}\int_\Omega \frac{1}{\left\|d_n\right\|_{L^1(\Omega)}}\left(\gamma_n-\gamma_0\right)\nabla w_n\cdot\Psi\mathrm{d}x\\ &\leq \lim_n\left(\frac{1}{\left\|d_n\right\|_{L^1(\Omega)}}\int_\Omega \gamma_n\nabla w_n\cdot\nabla w_n\mathrm{d}x\right)^{\frac12}\left(\frac{1}{\left\|d_n\right\|_{L^1(\Omega)}}\int_\Omega d_n'\Psi\cdot\Psi\mathrm{d}x\right)^{\frac12}\\ &\leq C\left(\int_\Omega |\Psi|^2\,\mathrm{d}\mu\right)^{\frac12}\end{aligned}$$

thanks to the estimate above. As a consequence of this estimate, it follows that the functional

$$\Psi\to\int_\Omega\Psi\cdot\mathrm{d}\mathscr{M}$$

may be extended to a bounded linear functional on $[L^2(\Omega,\mathrm{d}\mu)]^d$. Hence, by Riesz's Representation Theorem, we may identify

$$\mathrm{d}\mathscr{M} = M\mathrm{d}\mu$$

for some function $\mathscr{M}\in[L^2(\Omega,\mathrm{d}\mu)]^d$, which is our statement. □

## Appendix C. Proof of Proposition 25

**Remark.** Note that if $\Omega'$ is simply connected, $Fb_n = Fb_0 = 0$. Remark that

$$\frac{1}{|\Gamma_i|}\int_{\Gamma_i}\gamma_0\nabla u_0\cdot n\mathrm{d}\sigma = \int_{\Gamma_i}\gamma_n\nabla u_n\cdot n\mathrm{d}\sigma.$$

Let $I_i$ be the solution of

$$\operatorname{div}\left(\gamma_0\nabla I_i\right)=0\quad\text{on}\quad\Omega'\quad\text{and}\quad I_i=1\quad\text{on}\quad\Gamma_i.$$

By an integration by parts,

$$\begin{aligned}\int_{\Gamma_i}\gamma_0\nabla u_0\cdot n\mathrm{d}\sigma-\int_{\Gamma_i}\gamma_n\nabla u_n\cdot n\mathrm{d}\sigma &= \int_\Omega\gamma_0\nabla u_0\cdot\nabla I_i\mathrm{d}x-\int_\Omega\gamma_n\nabla u_n\cdot\nabla I_i\mathrm{d}x.\\ &=\int_\Omega gI_i\mathrm{d}\sigma-\int_\Omega gI_i\mathrm{d}\sigma\\ &=0.\end{aligned}$$

Thus $Fb_0=0$ implies $Fb_n=0$. Imposing $Fb_0=0$ corresponds to $N-1$ constraints in an infinite dimensional space and therefore is not a loss of generality. We shall make that assumption in the rest of this section.

**Proof.** By the inequality in (38), we have

$$\frac{\left\|\Sigma_n\right\|_{L^1(\Omega)}}{\left\|d_n\right\|_{L^1(\Omega)}}\leq\left(\max_{\overline{\Omega}}\lambda_d\left(\gamma_0^{-1}\right)\right)^2,$$

thus taking a convergent subsequence of $\frac{\|\Sigma_n\|_{L^1(\Omega)}}{\|d_n\|_{L^1(\Omega)}} \to a_0$ and a possible further extraction of the subsequence $\frac{1}{\|\Sigma_n\|_{L^1(\Omega)}}(\sigma_n - \sigma_0)\nabla\phi_n$, Corollary 23 implies that, if $\Xi \in C^0(\overline{\Omega},\mathbb{R}^2)$ is an arbitrary vector field,

$$\begin{aligned}
&\lim_{n\to\infty}\int_\Omega \left(\frac{1}{\|d_n\|_{L^1(\Omega)}}(\sigma_0-\sigma_n)\nabla\varphi_n\cdot\Xi\right)\mathrm{d}x\\
&= \lim_{n\to\infty}\int_\Omega \left(\frac{\|\Sigma_n\|_{L^1(\Omega)}}{\|d_n\|_{L^1(\Omega)}}\frac{1}{\|d_n\|_{L^1(\Omega)}}(\sigma_n-\sigma_0)\nabla\varphi_n\cdot\Xi\right)\mathrm{d}x\\
&= a_0\lim_{n\to\infty}\int_\Omega \left(\frac{1}{\|d_n\|_{L^1(\Omega)}}(\sigma_0-\sigma_n)\nabla\varphi_n\cdot\Xi\right)\mathrm{d}x\\
&= a_0\int_\Omega \widetilde{N}\nabla\psi_0\cdot\Xi\,\mathrm{d}\nu\\
&= \int_\Omega N\nabla\psi_0\cdot\Xi\,\mathrm{d}\mu.
\end{aligned}$$

Where $N = a_0\frac{\mathrm{d}\nu}{\mathrm{d}\mu}\widetilde{N}$ belongs to $L^2(\Omega;\mathrm{d}\mu)$. Alternatively testing against $(J^T\gamma_0)\Xi$ we find

$$\begin{aligned}
&\int_\Omega \frac{1}{\|d_n\|_{L^1(\Omega)}}(\sigma_0-\sigma_n)\nabla\psi_n\cdot(J^T\gamma_0)\,\Xi\,\mathrm{d}x\\
&=\int_\Omega \frac{1}{\|d_n\|_{L^1(\Omega)}}\left(J^T\gamma_0^{-1}(\gamma_n-\gamma_0)\gamma_n^{-1}J\right)\left(J^T\gamma_n\nabla u_n\right)\cdot\left(J^T\gamma_0\right)\Xi\,\mathrm{d}x\\
&=\int_\Omega \frac{1}{\|d_n\|_{L^1(\Omega)}}J^T\gamma_0^{-1}(\gamma_0-\gamma_n)\nabla u_n\cdot J^T\gamma_0\Xi\,\mathrm{d}x.\\
&=\int_\Omega \frac{1}{\|d_n\|_{L^1(\Omega)}}(\gamma_0-\gamma_n)\nabla u_n\cdot\Xi\,\mathrm{d}x
\end{aligned}$$

whereas

$$\begin{aligned}
&\int_\Omega \frac{1}{\|d_n\|_{L^1(\Omega)}}(\sigma_n-\sigma_0)\nabla\psi_0\cdot\left(J^T\gamma_0\right)\Xi\,\mathrm{d}x\\
&=\int_\Omega \frac{1}{\|d_n\|_{L^1(\Omega)}}\gamma_0\gamma_n^{-1}(\gamma_0-\gamma_n)\nabla u_0\cdot\Xi\,\mathrm{d}x.\\
&=\int_\Omega \frac{1}{\|d_n\|_{L^1(\Omega)}}\left((\gamma_0-\gamma_n)+d_n\right)\nabla u_0\cdot\Xi\,\mathrm{d}x.
\end{aligned}$$

We write $\mathscr{D}$ as the limit limiting tensor corresponding to $\|d_n\|_{L^1(\Omega)}^{-1}d_n$ in $L^2(\Omega,d\mu)^{d\times d}$, that is,

$$\lim_{n\to\infty}\int_\Omega \frac{d_n}{\|d_n\|_{L^1(\Omega)}}\nabla u_0\cdot\Xi\,\mathrm{d}x = \int_\Omega \mathscr{D}\nabla u_0\cdot\Xi\,\mathrm{d}\mu$$

Altogether, we have obtained

$$\begin{aligned}
\lim_{n\to\infty}\int_\Omega \frac{1}{\|d_n\|_{L^1(\Omega)}}(\gamma_0-\gamma_n)\nabla w_n\cdot\Xi\mathrm{d}x &= -\int_\Omega \mathscr{D}\nabla u_0\cdot\Xi\,\mathrm{d}\mu+\int_\Omega N\nabla\psi_0\cdot\left(J^T\gamma_0\right)\Xi\,\mathrm{d}\mu\\
&=\int_\Omega\left(\left(\gamma_0 J\right)N\left(\gamma_0 J\right)^T-\mathscr{D}\right)\nabla u_0\cdot\Xi\,\mathrm{d}\mu
\end{aligned}$$

which concludes our proof. □

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