



**HAL**  
open science

# THE DIMENSIONAL BRUNN-MINKOWSKI INEQUALITY IN GAUSS SPACE

Alexandros Eskenazis, Georgios Moschidis

► **To cite this version:**

Alexandros Eskenazis, Georgios Moschidis. THE DIMENSIONAL BRUNN-MINKOWSKI INEQUALITY IN GAUSS SPACE. *Journal of Functional Analysis*, 2021, 280 (6), pp.108914. 10.1016/j.jfa.2020.108914 . hal-03182077

**HAL Id: hal-03182077**

**<https://hal.sorbonne-universite.fr/hal-03182077v1>**

Submitted on 26 Mar 2021

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# THE DIMENSIONAL BRUNN–MINKOWSKI INEQUALITY IN GAUSS SPACE

ALEXANDROS ESKENAZIS AND GEORGIOS MOSCHIDIS

ABSTRACT. Let  $\gamma_n$  be the standard Gaussian measure on  $\mathbb{R}^n$ . We prove that for every symmetric convex sets  $K, L$  in  $\mathbb{R}^n$  and every  $\lambda \in (0, 1)$ ,

$$\gamma_n(\lambda K + (1 - \lambda)L)^{\frac{1}{n}} \geq \lambda \gamma_n(K)^{\frac{1}{n}} + (1 - \lambda) \gamma_n(L)^{\frac{1}{n}},$$

thus settling a problem raised by Gardner and Zvavitch (2010). This is the Gaussian analogue of the classical Brunn–Minkowski inequality for the Lebesgue measure. We also show that, for a fixed  $\lambda \in (0, 1)$ , equality is attained if and only if  $K = L$ .

*2020 Mathematics Subject Classification.* Primary: 52A40; Secondary: 52A20, 28C20, 47F10.

*Key words.* Brunn–Minkowski inequality, symmetric convex sets, Gaussian measure, Gardner–Zvavitch problem.

## 1. INTRODUCTION

The classical Brunn–Minkowski inequality asserts that for every nonempty compact sets  $A, B$  in  $\mathbb{R}^n$  and every  $\lambda \in (0, 1)$ ,

$$|\lambda A + (1 - \lambda)B|^{\frac{1}{n}} \geq \lambda |A|^{\frac{1}{n}} + (1 - \lambda) |B|^{\frac{1}{n}}, \quad (1)$$

where  $|\cdot|$  denotes Lebesgue measure and the Minkowski convex combination of sets is given by

$$\lambda A + (1 - \lambda)B = \{\lambda a + (1 - \lambda)b : a, b \in A\}. \quad (2)$$

In view of the importance of the Brunn–Minkowski inequality in convex geometric analysis (see the monographs [Gar06, Sch14] and the surveys [Gar02, Mau05, Bar06]), the last decades have seen a surge of activity around refinements and extensions of (1) in different contexts. We refer to [KM17, KL18, HKL20] for an up to date account of some important achievements in this area.

One prominent direction in modern Brunn–Minkowski theory is the study of inequalities relating the “size” of the Minkowski sum of subsets of  $\mathbb{R}^n$  to the “sizes” of the individual summands, where “size” can be interpreted more loosely than in the sense of the usual Euclidean volume. In this paper, we will be interested in the case where “size” refers to the standard Gaussian measure  $\gamma_n$  given by  $d\gamma_n(x) = \frac{\exp(-|x|^2/2)}{(2\pi)^{n/2}} dx$ ; here  $|x|$  denotes the Euclidean length of a vector  $x \in \mathbb{R}^n$ . An example of a profound geometric inequality for  $\gamma_n$  is Ehrhard’s inequality [Ehr83], which states that for every Borel measurable sets  $A, B$  in  $\mathbb{R}^n$  and every  $\lambda \in (0, 1)$ ,

$$\Phi^{-1}(\gamma_n(\lambda A + (1 - \lambda)B)) \geq \lambda \Phi^{-1}(\gamma_n(A)) + (1 - \lambda) \Phi^{-1}(\gamma_n(B)), \quad (3)$$

where  $\Phi^{-1}$  is the inverse of the Gaussian distribution function  $\Phi(x) = \gamma_1((-\infty, x])$ . Inequality (3) is known to capture many delicate probabilistic and geometric properties of the Gaussian measure. For instance, it implies the Gaussian isoperimetric inequality, which asserts that half-spaces have minimal Gaussian surface area among all sets of fixed measure. Ehrhard’s original proof of (3) proceeded via Gaussian symmetrization and required both sets  $A$  and  $B$  to be convex, an assumption which was later removed by Borell in [Bor03] (see also [Lat96] for a partial result).

While Ehrhard’s inequality (3) captures the optimal dimension-free convexity of the Gaussian measure, its validity for general Borel subsets  $A, B$  of  $\mathbb{R}^n$  shows that it is oblivious to additional

---

A. E. was supported by a postdoctoral fellowship of the Fondation Sciences Mathématiques de Paris. G. M. acknowledges support from the Miller Institute for Basic Research in Science, UC Berkeley.

geometric properties of the underlying sets, such as convexity. In [GZ10], Gardner and Zvavitch undertook a detailed investigation of Gaussian inequalities in (dual) Brunn–Minkowski theory, which they concluded by asking (see [GZ10, Question 7.1]) whether the dimensional Brunn–Minkowski inequality (1) holds with the Lebesgue measure  $|\cdot|$  replaced by  $\gamma_n$  with the assumption that the underlying sets are convex and contain the origin. A counterexample to this statement was produced by Nayar and Tkocz in [NT13], yet the possibility of such an inequality being true for all origin symmetric convex sets remained open. This problem is settled in the affirmative here.

**Theorem 1.** *For every  $n \in \mathbb{N}$ , every symmetric convex sets  $K, L$  in  $\mathbb{R}^n$  and every  $\lambda \in (0, 1)$ ,*

$$\gamma_n(\lambda K + (1 - \lambda)L)^{\frac{1}{n}} \geq \lambda \gamma_n(K)^{\frac{1}{n}} + (1 - \lambda) \gamma_n(L)^{\frac{1}{n}}. \quad (4)$$

By taking  $K$  and  $L$  to be small coordinate boxes around the origin, it becomes clear that the exponent  $\frac{1}{n}$  is optimal (i.e. maximal) in inequality (4). As was already observed in [GZ10], the dimensional Gaussian Brunn–Minkowski inequality (4) neither trivially follows nor implies Ehrhard’s inequality (3) for origin symmetric convex sets.

**1.1. Symmetry in Brunn–Minkowski theory.** The dimensional Brunn–Minkowski inequality (4) is a refinement of the log-concavity of the Gaussian measure, that is, the fact that for every Borel sets  $A, B$  in  $\mathbb{R}^n$  and every  $\lambda \in (0, 1)$ ,

$$\gamma_n(\lambda A + (1 - \lambda)B) \geq \gamma_n(A)^\lambda \gamma_n(B)^{1-\lambda}, \quad (5)$$

for the class of symmetric convex sets. Strengthenings of measure-theoretic inequalities such as (5) under convexity and symmetry assumptions repeatedly appear in Brunn–Minkowski theory and the geometry of the Gaussian measure. To illustrate this phenomenon, we recall the deep B-inequality of Cordero-Erausquin, Fradelizi and Maurey [CEFM04], which states that for every symmetric convex set  $K$  in  $\mathbb{R}^n$ , the map  $\alpha(t) = \gamma_n(e^t K)$  is a log-concave function on  $\mathbb{R}$ . In this setting, the log-concavity of the function  $\beta(s) = \gamma_n(sK)$  on  $\mathbb{R}_+$  for every convex set  $K$  in  $\mathbb{R}^n$  is a straightforward consequence of (5), yet the additional symmetry assumption is necessary for the log-concavity of  $\alpha$  as can be seen by taking  $K = [-1, \infty)$  on  $\mathbb{R}$ . Other important examples of inequalities in Gauss space (see also [Lat02]) which crucially rely on the symmetry and convexity of the underlying sets are the S-inequality, proven by Latała and Oleszkiewicz [LO99], and the Gaussian correlation inequality, proven by Royen [Roy14].

Theorem 1 belongs in a large network of (largely conjectural) inequalities involving log-concave measures of various notions of convex combinations of symmetric convex sets in  $\mathbb{R}^n$ . At the top of the hierarchy of these inequalities lies the celebrated log-Brunn–Minkowski conjecture of Böröczky, Lutwak, Yang and Zhang [BLYZ12], which asserts that the Euclidean volume of the geometric mean of two symmetric convex sets can be bounded below by the geometric mean of their volumes. Its validity would, for instance, imply the dimensional Brunn–Minkowski inequality (4) and the B-inequality with  $\gamma_n$  replaced by any symmetric log-concave measure  $\mu$ . Surveying in detail all recent developments in this area lies beyond the scope of this paper, so we refer to [KM17, YZ19, CHLL18, Put19, BK20, HKL20, KL20] for recent results and further bibliographical information.

**1.2. Approaches towards the Gardner–Zvavitch problem.** Since the formulation of the problem in [GZ10], there have been several partial results towards the general statement of Theorem 1. Gardner and Zvavitch themselves proved the Gaussian Brunn–Minkowski inequality (4) in the special cases that the sets  $K, L$  are either coordinate boxes containing the origin or dilates of a fixed symmetric convex set. These results were later generalized to more general measures by Marsiglietti [Mar16]. In [CLM17], Colesanti, Livshyts and Marsiglietti showed that (4) holds when both symmetric bodies  $K, L$  are small perturbations of the Euclidean ball. Moreover, Livshyts,

Marsiglietti, Nayar and Zvavitch [LMNZ17] have used a clever variant of the Prékopa–Leindler inequality [Sch14, Theorem 7.1.2] to show that inequality

$$\forall \lambda \in (0, 1), \quad \mu(\lambda A + (1 - \lambda)B)^{\frac{1}{n}} \geq \lambda \mu(A)^{\frac{1}{n}} + (1 - \lambda) \mu(B)^{\frac{1}{n}} \quad (6)$$

holds true when  $\mu$  is an unconditional product measure on  $\mathbb{R}^n$  and  $A, B$  are ideals in  $\mathbb{R}^n$ . Their result was later extended to weakly unconditional sets by Ritoré and Yepes Nicolás [RYN18]. The planar case of inequality (6) can be derived for any symmetric log-concave measure  $\mu$  on  $\mathbb{R}^2$  and all symmetric convex sets  $K, L$  in  $\mathbb{R}^2$  by combining [LMNZ17, Proposition 1] with [BLYZ12, Theorem 1.7] and a result of Saroglou [Sar16, Theorem 3.1]. Finally, the Gardner–Zvavitch problem was recently settled affirmatively for a class of symmetric convex sets with many hyperplane symmetries by Böröczky and Kalantzopoulos [BK20].

1.2.1. *The local Gardner–Zvavitch problem, after Kolesnikov and Livshyts.* With the exception of the planar inequality of [LMNZ17], the proofs of all the aforementioned results crucially require additional symmetries of the underlying sets which are not available in the general setting of Theorem 1. In [KL18], Kolesnikov and Livshyts took a different route to attack the Gardner–Zvavitch problem, by studying how inequalities of the form

$$\forall \lambda \in (0, 1), \quad \mu(\lambda K + (1 - \lambda)L)^{\frac{\delta}{n}} \geq \lambda \mu(K)^{\frac{\delta}{n}} + (1 - \lambda) \mu(L)^{\frac{\delta}{n}} \quad (7)$$

behave infinitesimally when the convex bodies  $K$  and  $L$  are small perturbations of each other and then proving a local-to-global principle. This is also the approach which we shall be taking. From now on, we will refer to twice continuously differentiable functions simply as smooth functions.

To illustrate this technique, we briefly return to the B-inequality of [CEFM04], asserting that for every symmetric convex set  $K$  in  $\mathbb{R}^n$ , the function  $\alpha(t) = \gamma_n(e^t K)$  is log-concave on  $\mathbb{R}$ . It is straightforward to observe (see [CEFM04, p. 413]) that the log-concavity of  $\alpha$  on  $\mathbb{R}$  for an arbitrary  $K$  is equivalent to its infinitesimal log-concavity at  $t = 0$ , that is,  $\alpha''(0)\alpha(0) \leq \alpha'(0)^2$ . An explicit calculation now shows that the latter inequality can be equivalently rewritten as

$$\text{Var}_{\gamma_K}(|x|^2) \leq \frac{1}{2} \int |\nabla |x|^2|^2 \, d\gamma_K(x), \quad (8)$$

where  $\gamma_K$  is the rescaled restriction of  $\gamma_n$  on a symmetric convex set  $K$  with nonempty interior, that is, the measure given by  $\gamma_K(A) = \frac{\gamma_n(A \cap K)}{\gamma_n(K)}$  for Borel subsets  $A$  of  $\mathbb{R}^n$ . The delicate aspect of inequality (8) lies in the constant  $\frac{1}{2}$  on the right hand side. Indeed, the same Poincaré inequality with constant 1 is valid in much greater generality by a classical result of Brascamp and Lieb [BL76], which implies that if  $\mu$  is a measure of the form  $d\mu(x) = e^{-V(x)} dx$  with a potential whose Hessian satisfies  $\nabla^2 V \geq \beta \text{Id}$ , then every smooth  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  satisfies

$$\text{Var}_{\mu}(f) \leq \frac{1}{\beta} \int |\nabla f|^2 \, d\mu. \quad (9)$$

Equivalently, (9) says that the spectral gap of such a measure is at least  $\sqrt{\beta}$ . In order to prove (8), Cordero-Erausquin, Fradelizi and Maurey succeeded to realize this inequality as special case of a second eigenvalue problem for even functions and then crucially used the symmetry of both the measure  $\gamma_K$  and the function  $x \mapsto |x|^2$ . Such an analytic use of the underlying symmetry of the problem also lies at the heart of the proof of Theorem 1 (see also Theorem 3 below).

Recall that the generator of the Ornstein–Uhlenbeck semigroup is the elliptic differential operator  $\mathcal{L}$  whose action on a smooth function  $u : \mathbb{R}^n \rightarrow \mathbb{R}$  is given by

$$\forall x \in \mathbb{R}^n, \quad \mathcal{L}u(x) = \Delta u(x) - \sum_{i=1}^n x_i \partial_i u(x). \quad (10)$$

We also denote by  $\|A\|_{\text{HS}}$  the Hilbert–Schmidt norm of a matrix  $A$ , i.e.  $\|A\|_{\text{HS}}^2 = \sum_{i,j} a_{ij}^2$ . In [KL18], Kolesnikov and Livshyts proved the following local-to-global principle (see also [Col08, CLM17, KM18] for closely related infinitesimal versions of other Brunn–Minkowski-type inequalities).

**Proposition 2** (Kolesnikov, Livshyts). *Fix  $n \in \mathbb{N}$ . Let  $\delta \in [0, 1]$  be such that for every symmetric convex set  $K$  in  $\mathbb{R}^n$ , every smooth symmetric function  $u : K \rightarrow \mathbb{R}$  with  $\mathcal{L}u = 1$  on  $K$  satisfies*

$$\int \|\nabla^2 u\|_{\text{HS}}^2 + |\nabla u|^2 \, d\gamma_K \geq \frac{\delta}{n}. \quad (11)$$

Then, for every symmetric convex sets  $K, L$  in  $\mathbb{R}^n$  and every  $\lambda \in (0, 1)$ ,

$$\gamma_n(\lambda K + (1 - \lambda)L)^{\frac{\delta}{n}} \geq \lambda \gamma_n(K)^{\frac{\delta}{n}} + (1 - \lambda) \gamma_n(L)^{\frac{\delta}{n}}. \quad (12)$$

The main result of [KL18] was a proof of (11) with  $\delta = \frac{1}{2}$  for all convex sets  $K$  containing the origin. Then, a local-to-global principle for such convex sets (similar to Proposition 2) implies the corresponding Gaussian Brunn–Minkowski inequality (12) with  $\delta = \frac{1}{2}$ . The main technical result of the present paper is the following refinement of the inequality of Kolesnikov and Livshyts for origin symmetric convex sets and symmetric (i.e. even) solutions of the equation  $\mathcal{L}u = 1$  on  $K$ .

**Theorem 3.** *For every  $n \in \mathbb{N}$  and every symmetric convex set  $K$  in  $\mathbb{R}^n$ , every smooth symmetric function  $u : \mathbb{R}^n \rightarrow \mathbb{R}$  with  $\mathcal{L}u = 1$  on  $K$ , satisfies*

$$\int \|\nabla^2 u\|_{\text{HS}}^2 + |\nabla u|^2 \, d\gamma_K \geq \frac{1}{n}. \quad (13)$$

**1.3. Beyond power-type concavity.** While the exponent  $\frac{1}{n}$  in inequality (4) cannot be improved, Ehrhard’s inequality (3) suggests that it is worth investigating potential strengthenings of (4), where the power function  $t \mapsto t^{\frac{1}{n}}$  is replaced by a more general function of the Gaussian measure of the sets. Notice that if  $\zeta_n : [0, 1] \rightarrow \mathbb{R}$  is such that for every Borel sets  $A, B$  in  $\mathbb{R}^n$ , the inequality

$$\zeta_n(\gamma_n(\lambda A + (1 - \lambda)B)) \geq \lambda \zeta_n(\gamma_n(A)) + (1 - \lambda) \zeta_n(\gamma_n(B)), \quad (14)$$

holds true, then choosing  $A$  and  $B$  to be half-spaces of the form  $A = \{x \in \mathbb{R}^n : x_1 \leq a\}$  and  $B = \{x \in \mathbb{R}^n : x_1 \leq b\}$ , we see that  $\zeta_n \circ \Phi$  is concave; here  $\Phi$  is the Gaussian distribution function  $\Phi(x) = \gamma_1((-\infty, x])$ . In this sense, the choice  $\zeta_n = \Phi^{-1}$  in (14), encapsulated by Ehrhard’s inequality (3), captures the optimal convexity of  $\gamma_n$  over all Borel sets in  $\mathbb{R}^n$ . Bearing this in mind as a motivating example, we ask the following (purposefully vague) question.

*Question 4.* Fix  $n \in \mathbb{N}$ . Is there an “optimal” increasing function  $\xi_n : [0, 1] \rightarrow \mathbb{R}$  such that for every origin symmetric convex sets  $K, L$  in  $\mathbb{R}^n$  and every  $\lambda \in (0, 1)$ , the inequality

$$\xi_n(\gamma_n(\lambda K + (1 - \lambda)L)) \geq \lambda \xi_n(\gamma_n(K)) + (1 - \lambda) \xi_n(\gamma_n(L)) \quad (15)$$

is satisfied?

Ideally, such an inequality should be a joint refinement of Ehrhard’s inequality (3) and of the dimensional Brunn–Minkowski inequality (4) which becomes an equality for some nontrivial pairs of symmetric convex sets  $K, L$  in  $\mathbb{R}^n$ . In [GZ10, p. 5350], Gardner and Zvavitch presented an argument of Barthe which implies that (15) is not satisfied for  $\xi_n = \Psi_n^{-1}$ , where  $\Psi_n(r) = \gamma_n(rB_2^n)$  and  $rB_2^n = \{x \in \mathbb{R}^n : |x| \leq r\}$  is the Euclidean ball of radius  $r$ . We refer to Section 4 for some additional observations of this kind. While we have no conjecture as to what the optimal symmetric improvement (15) of Ehrhard’s inequality (3) might be, we obtain the following strengthening of (4).

**Theorem 5.** *Fix  $n \in \mathbb{N}$  and let  $\sigma_n : [0, 1] \rightarrow \mathbb{R}$  be a strictly increasing function satisfying*

$$\forall r \in (0, \infty), \quad 1 + \frac{\sigma_n''(\Psi_n(r))\Psi_n(r)}{\sigma_n'(\Psi_n(r))} = \frac{2}{n} - \frac{c_n}{n^2\Psi_n(r)} r^n e^{-r^2/2}, \quad (16)$$

where  $c_n^{-1} = 2^{\frac{n}{2}-1}\Gamma(n/2)$ . Then, for every symmetric convex sets  $K, L$  in  $\mathbb{R}^n$  and every  $\lambda \in (0, 1)$ ,

$$\sigma_n(\gamma_n(\lambda K + (1 - \lambda)L)) \geq \lambda \sigma_n(\gamma_n(K)) + (1 - \lambda) \sigma_n(\gamma_n(L)). \quad (17)$$

An explicit calculation (see Remark 10) reveals that the function  $y \mapsto \sigma_n(y^n)$  is convex, hence (17) is indeed a genuine improvement of the dimensional Brunn–Minkowski inequality (4). Moreover, the *strict* convexity of  $y \mapsto \sigma_n(y^n)$  readily implies the following corollary, settling the equality cases of the dimensional Brunn–Minkowski inequality (4) for the Gaussian measure.

**Corollary 6.** *Fix  $n \in \mathbb{N}$  and  $\lambda \in (0, 1)$ . If  $K, L$  are two symmetric convex sets in  $\mathbb{R}^n$  satisfying*

$$\gamma_n(\lambda K + (1 - \lambda)L)^{\frac{1}{n}} = \lambda \gamma_n(K)^{\frac{1}{n}} + (1 - \lambda) \gamma_n(L)^{\frac{1}{n}}, \quad (18)$$

then  $K = L$ .

The proof of Theorem 3 will be presented in Section 2 and the proofs of Theorem 5 and Corollary 6 in Section 3. Some additional remarks are postponed to Section 4.

**Acknowledgements.** We are grateful to Ramon van Handel for helpful discussions.

## 2. PROOF OF THEOREM 3

It follows readily from Proposition 2 that Theorem 1 is a formal consequence of Theorem 3, so in this section we shall only establish the latter. For an  $n \times n$  matrix  $A$ , we will denote by  $\widehat{A}$  the traceless part of  $A$ , that is  $\widehat{A} = A - \frac{\text{tr}(A)}{n} \text{Id}$ , where  $\text{Id}$  is the identity matrix. In particular, for a smooth function  $u : K \rightarrow \mathbb{R}$ , where  $K$  is a symmetric convex set, we write

$$\widehat{\nabla}^2 u \stackrel{\text{def}}{=} \nabla^2 u - \frac{\Delta u}{n} \text{Id} \quad (19)$$

for the traceless part of its Hessian. Then, orthogonality implies the pointwise identity

$$\|\nabla^2 u\|_{\text{HS}}^2 = \|\widehat{\nabla}^2 u\|_{\text{HS}}^2 + \frac{(\Delta u)^2}{n}. \quad (20)$$

Consider  $r : \mathbb{R}^n \rightarrow \mathbb{R}$  to be  $r(x) = \frac{|x|^2}{2n}$ , which satisfies  $\widehat{\nabla}^2 r \equiv 0$  and  $\Delta r \equiv 1$ . Then, we have

$$\|\widehat{\nabla}^2 u\|_{\text{HS}}^2 = \|\widehat{\nabla}^2(u - r)\|_{\text{HS}}^2 \stackrel{(20)}{=} \|\nabla^2(u - r)\|_{\text{HS}}^2 - \frac{(\Delta(u - r))^2}{n} = \|\nabla^2(u - r)\|_{\text{HS}}^2 - \frac{(\Delta u - 1)^2}{n}, \quad (21)$$

so that combining (20) and (21), we get

$$\|\nabla^2 u\|_{\text{HS}}^2 = \|\nabla^2(u - r)\|_{\text{HS}}^2 + \frac{2}{n} \Delta u - \frac{1}{n}. \quad (22)$$

Taking into account that  $\mathcal{L}u(x) = \Delta u(x) - \sum_{i=1}^n x_i \partial_i u(x) = 1$ , we can then write

$$\forall x \in K, \quad \|\nabla^2 u(x)\|_{\text{HS}}^2 = \|\nabla^2(u - r)(x)\|_{\text{HS}}^2 + \frac{2}{n} \sum_{i=1}^n x_i \partial_i u(x) + \frac{1}{n}. \quad (23)$$

Since  $u$  is an even function, the partial derivative  $\partial_i(u - r)$  is an odd function on  $K$  and thus has expectation 0 with respect to  $\gamma_K$  for every  $i \in \{1, \dots, n\}$ . Therefore, the Brascamp–Lieb inequality (9) gives

$$\sum_{j=1}^n \int (\partial_j \partial_i(u - r))^2 d\gamma_K \geq \text{Var}_{\gamma_K}(\partial_i(u - r)) = \int \left( \partial_i u(x) - \frac{x_i}{n} \right)^2 d\gamma_K(x). \quad (24)$$

Summing (24) over  $i \in \{1, \dots, n\}$ , we get

$$\begin{aligned} \int \|\nabla^2(u-r)\|_{\text{HS}}^2 d\gamma_K &\geq \sum_{i=1}^n \int \left(\partial_i u(x) - \frac{x_i}{n}\right)^2 d\gamma_K(x) \\ &= \int |\nabla u(x)|^2 - \frac{2}{n} \sum_{i=1}^n x_i \partial_i u(x) + \frac{|x|^2}{n^2} d\gamma_K(x) \end{aligned} \quad (25)$$

Combining (23) and (25), we finally deduce that

$$\int \|\nabla^2 u\|_{\text{HS}}^2 + |\nabla u|^2 d\gamma_K \geq \int 2|\nabla u(x)|^2 + \frac{|x|^2}{n^2} + \frac{1}{n} d\gamma_K(x) \geq \frac{1}{n} \quad (26)$$

and the proof is complete.  $\square$

*Remark 7.* Notice that the only property of  $K$  and  $u$  that was used in the proof of Theorem 3 is the fact that  $\int \nabla(u-r) d\gamma_K = 0$ , which allows us to use the Brascamp–Lieb inequality.

### 3. PROOF OF THEOREM 5

The proof of Theorem 5, in analogy with that of Theorem 1, will proceed in two steps: first we will establish a suitable Poincaré-type inequality for solutions of the equation  $\mathcal{L}u = 1$  and then the geometric inequality (17) will be a consequence of the following local-to-global principle.

**Proposition 8.** *Fix  $n \in \mathbb{N}$ . Let  $\psi : [0, 1] \rightarrow \mathbb{R}$  be an increasing function such that for every symmetric convex set  $K$ , every smooth symmetric function  $u : K \rightarrow \mathbb{R}$  with  $\mathcal{L}u = 1$  on  $K$  satisfies*

$$\int \|\nabla^2 u\|_{\text{HS}}^2 + |\nabla u|^2 d\gamma_K \geq 1 + \frac{\psi''(\gamma_n(K))\gamma_n(K)}{\psi'(\gamma_n(K))}. \quad (27)$$

*Then, for every symmetric convex sets  $K, L$  in  $\mathbb{R}^n$  and every  $\lambda \in (0, 1)$ ,*

$$\psi(\gamma_n(\lambda K + (1-\lambda)L)) \geq \lambda\psi(\gamma_n(K)) + (1-\lambda)\psi(\gamma_n(L)). \quad (28)$$

In order to prove Proposition 8 one has to repeat the arguments of the proof of Proposition 2 in [KL18] (see also [KM18]) mutatis mutandis, by replacing the power function  $t \mapsto t^{\frac{\delta}{n}}$  by a general increasing function  $\psi$ . As the proof of the more general Proposition 8 does not require any additional ideas, we leave the (trivial) necessary modifications to the interested reader.

Recall that a compact set  $K$  in  $\mathbb{R}^n$  is called star-shaped (with respect to the origin) if  $\lambda x \in K$  for every  $x \in K$  and  $\lambda \in [0, 1]$ .

**Lemma 9.** *Fix  $n \in \mathbb{N}$  and let  $K$  be a star-shaped set in  $\mathbb{R}^n$ . If  $\rho \in [0, \infty]$  is such that  $\gamma_n(K) = \gamma_n(\rho B_2^n)$ , where  $\rho B_2^n = \{x \in \mathbb{R}^n : |x| \leq \rho\}$  is the closed Euclidean ball of radius  $\rho$ , then*

$$\int_K |x|^2 d\gamma_n(x) \geq \int_{\rho B_2^n} |x|^2 d\gamma_n(x). \quad (29)$$

*Proof.* We will denote by  $\mathbb{S}^{n-1}$  the unit sphere in  $\mathbb{R}^n$  and by  $\rho_K : \mathbb{S}^{n-1} \rightarrow [0, \infty]$  the radial function of  $K$ , i.e.  $\rho_K(\theta) = \sup\{r \geq 0 : r\theta \in K\}$ . Let  $A = \{\theta \in \mathbb{S}^{n-1} : \rho \leq \rho_K(\theta)\}$  and  $B = \mathbb{S}^{n-1} \setminus A$ . Then,

$$\begin{aligned} 0 &= (2\pi)^{n/2} (\gamma_n(K) - \gamma_n(\rho B_2^n)) = \int_{\mathbb{S}^{n-1}} \int_0^{\rho_K(\theta)} r^{n-1} e^{-r^2/2} dr d\theta - \int_{\mathbb{S}^{n-1}} \int_0^\rho r^{n-1} e^{-r^2/2} dr d\theta \\ &= \int_A \int_\rho^{\rho_K(\theta)} r^{n-1} e^{-r^2/2} dr d\theta - \int_B \int_\rho^\rho r^{n-1} e^{-r^2/2} dr d\theta. \end{aligned}$$

Therefore, using polar coordinates once again, we get

$$\begin{aligned}
(2\pi)^{n/2} \left( \int_K |x|^2 d\gamma_n(x) - \int_{\rho B_2^n} |x|^2 d\gamma_n(x) \right) \\
= \int_A \int_{\rho}^{\rho_{K(\theta)}} r^{n+1} e^{-r^2/2} dr d\theta - \int_B \int_{\rho_{K(\theta)}}^{\rho} r^{n+1} e^{-r^2/2} dr d\theta \\
\geq \rho^2 \left( \int_A \int_{\rho}^{\rho_{K(\theta)}} r^{n-1} e^{-r^2/2} dr d\theta - \int_B \int_{\rho_{K(\theta)}}^{\rho} r^{n-1} e^{-r^2/2} dr d\theta \right) = 0
\end{aligned} \tag{30}$$

and the conclusion readily follows.  $\square$

We are now well equipped to complete the proof of Theorem 5.

*Proof of Theorem 5.* Let  $K$  be a symmetric convex set in  $\mathbb{R}^n$  and  $u : K \rightarrow \mathbb{R}$  a smooth symmetric function with  $\mathcal{L}u = 1$  on  $K$ . Then, by (26) and Lemma 9, we have

$$\int \|\nabla^2 u\|_{\text{HS}}^2 + |\nabla u|^2 d\gamma_K \stackrel{(26)}{\geq} \frac{1}{n^2} \int |x|^2 d\gamma_K(x) + \frac{1}{n} \stackrel{(29)}{\geq} \frac{1}{n^2 \gamma_n(K)} \int_{\rho B_2^n} |x|^2 d\gamma_n(x) + \frac{1}{n}, \tag{31}$$

where  $\gamma_n(K) = \gamma_n(\rho B_2^n)$  or, equivalently,  $\rho = \Psi_n^{-1}(\gamma_n(K))$ . Moreover, integration by parts gives

$$\begin{aligned}
\int_{\rho B_2^n} |x|^2 d\gamma_n(x) &= \frac{|\mathbb{S}^{n-1}|}{(2\pi)^{n/2}} \int_0^{\rho} r^{n+1} e^{-r^2/2} dr = \frac{|\mathbb{S}^{n-1}|}{(2\pi)^{n/2}} \left( n \int_0^{\rho} r^{n-1} e^{-r^2/2} dr - \rho^n e^{-\rho^2/2} \right) \\
&= n\gamma_n(K) - c_n \rho^n e^{-\rho^2/2},
\end{aligned} \tag{32}$$

where  $c_n^{-1} = 2^{\frac{n}{2}-1} \Gamma(n/2)$ . Combining (31) and (32), we get

$$\begin{aligned}
\int \|\nabla^2 u\|_{\text{HS}}^2 + |\nabla u|^2 d\gamma_K &\geq \frac{2}{n} - \frac{c_n}{n^2 \gamma_n(K)} \rho^n e^{-\rho^2/2} \\
&= \frac{2}{n} - \frac{c_n}{n^2 \gamma_n(K)} \Psi_n^{-1}(\gamma_n(K))^n e^{-\Psi_n^{-1}(\gamma_n(K))^2/2}
\end{aligned} \tag{33}$$

and the conclusion follows readily by the definition (16) of  $\sigma_n$  and Proposition 8.  $\square$

*Remark 10.* To see that (17) is a strict strengthening of (4), it suffices to observe that the function  $\tau_n(x) = \sigma_n^{-1}(x)^{\frac{1}{n}}$  is increasing and concave, since then we can write

$$\begin{aligned}
\gamma_n(\lambda K + (1-\lambda)L)^{\frac{1}{n}} &= \tau_n \circ \sigma_n(\gamma_n(\lambda K + (1-\lambda)L)) \\
&\stackrel{(17)}{\geq} \tau_n(\lambda \sigma_n(\gamma_n(K)) + (1-\lambda) \sigma_n(\gamma_n(L))) \\
&\geq \lambda \tau_n \circ \sigma_n(\gamma_n(K)) + (1-\lambda) \tau_n \circ \sigma_n(\gamma_n(L)) = \lambda \gamma_n(K)^{\frac{1}{n}} + (1-\lambda) \gamma_n(L)^{\frac{1}{n}}.
\end{aligned} \tag{34}$$

In order to prove that  $\tau_n$  is concave, we will instead show that  $\tau_n^{-1}(y) = \sigma_n(y^n)$  is convex. Indeed, the second derivative condition  $(\tau_n^{-1})''(y) \geq 0$  can be equivalently rewritten as

$$\forall y \in (0, \infty), \quad \frac{\sigma_n''(y^n) y^n}{\sigma_n'(y^n)} \geq -\frac{n-1}{n}, \tag{35}$$

which readily follows from (16) and (32).

*Proof of Corollary 6.* Using the notation of the previous remark, we see from (31) and (35) that  $\tau_n$  is in fact a strictly concave and strictly increasing function. Suppose now that for some symmetric convex sets  $K, L$  in  $\mathbb{R}^n$ , we have

$$\gamma_n(\lambda K + (1-\lambda)L)^{\frac{1}{n}} = \lambda \gamma_n(K)^{\frac{1}{n}} + (1-\lambda) \gamma_n(L)^{\frac{1}{n}}. \tag{36}$$



Then, since the second inequality in (34) becomes equality, we deduce that  $\sigma_n(\gamma_n(K)) = \sigma_n(\gamma_n(L))$ , which implies that  $\gamma_n(K) = \gamma_n(L)$ . Combining this with the equality condition (36), we have

$$\gamma_n(K) = \gamma_n(\lambda K + (1 - \lambda)L) = \gamma_n(L), \quad (37)$$

which can only hold when  $K = L$ , e.g. by [Dub77] or [Ehr86].  $\square$

#### 4. FURTHER REMARKS

We conclude with a few additional remarks on Brunn–Minkowski-type inequalities for measures.

1. In [KL18], Kolesnikov and Livshyts showed that when  $K$  is a convex set containing the origin and  $u : K \rightarrow \mathbb{R}$  is a smooth function satisfying  $\mathcal{L}u = 1$  on  $K$ , then

$$\int \|\nabla^2 u\|_{\text{HS}}^2 + |\nabla u|^2 \, d\gamma_K \geq \frac{1}{2n} \quad (38)$$

in the following way. By omitting the traceless part of  $\nabla^2 u$ , (20) gives

$$\int \|\nabla^2 u\|_{\text{HS}}^2 + |\nabla u|^2 \, d\gamma_K \stackrel{(20)}{\geq} \int \frac{(\Delta u)^2}{n} + |\nabla u|^2 \, d\gamma_K = \int \frac{1}{n} \left(1 + \sum_{i=1}^n x_i \partial_i u(x)\right)^2 + |\nabla u(x)|^2 \, d\gamma_K(x),$$

where in the equality we used that  $\mathcal{L}u = 1$ . Pointwise minimizing the right hand side over all  $u_1, \dots, u_n \in \mathbb{R}$  (see [KL18, Lemma 2.4]), they get

$$\mathcal{G}(u) \stackrel{\text{def}}{=} \int \frac{1}{n} \left(1 + \sum_{i=1}^n x_i \partial_i u(x)\right)^2 + |\nabla u(x)|^2 \, d\gamma_K(x) \geq \int \frac{1}{|x|^2 + n} \, d\gamma_K(x) \quad (39)$$

and this quantity is always greater than  $\frac{1}{2n}$  by a simple geometric inequality [KL18, Lemma 5.3]. It can easily be seen that when  $K = \mathbb{R}^n$  this quantity is in fact  $\frac{1}{2n} + o(\frac{1}{n})$ . One indication that this argument can be improved is the fact that inequality (39) is saturated for the function  $u_0(x) = -\frac{1}{2} \log(|x|^2 + n)$  that satisfies

$$\forall x \in \mathbb{R}^n, \quad \mathcal{L}u_0(x) = \frac{|x|^2 - n}{|x|^2 + n} + \frac{2|x|^2}{(|x|^2 + n)^2} \quad (40)$$

and this is very far from the imposed constraint  $\mathcal{L}u = 1$  (e.g. when  $|x| \sim \sqrt{n}$ ).

In the proof of Theorem 3 above, the traceless part  $\widehat{\nabla}^2 u$  of the Hessian of  $u$  is controlled via the Brascamp–Lieb inequality (9). The following proposition shows that by crudely omitting it, one cannot substantially improve the result of [KL18], even for very simple symmetric convex sets. In what follows, we will adopt the following shorthand notation: Given any  $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ , we will denote by  $x'$  the vector  $(x_2, x_3, \dots, x_n) \in \mathbb{R}^{n-1}$ , so that  $x = (x_1, x')$ . For any  $\varepsilon \in (0, 1)$ , let  $\mathcal{S}_\varepsilon = \{|x_1| < \varepsilon\} \cap \{|x'|^2 < 2n\} \subseteq \mathbb{R}^n$  be a truncated slab of width  $\varepsilon$ .

**Proposition 11.** *There exists an absolute constant  $C \in (0, \infty)$ , such that, for every  $n \geq 2$  and every small enough  $\varepsilon \in (0, 1)$ , the functional  $\mathcal{G}$  associated to the slab  $\mathcal{S}_\varepsilon$  in  $\mathbb{R}^n$  satisfies*

$$\inf \left\{ \mathcal{G}(u) : u \in C^2(\mathcal{S}_\varepsilon) \text{ with } \mathcal{L}u = 1 \right\} \leq \frac{1}{2n} + C\varepsilon^2 + o\left(\frac{1}{n}\right). \quad (41)$$

*Proof.* Let us define the function  $u : \mathcal{S}_\varepsilon \rightarrow \mathbb{R}$  by the relation  $u \stackrel{\text{def}}{=} u_0 + v$ , where we denote by  $u_0(x) = -\frac{1}{2} \log(|x|^2 + n)$  the pointwise minimizer of the functional  $\mathcal{G}$  and the function  $v$  is the unique solution of the boundary value problem

$$\begin{cases} \mathcal{L}v = R, \\ v|_{\partial\mathcal{S}_\varepsilon} = 0, \end{cases} \quad (42)$$

where

$$R(x) \stackrel{\text{def}}{=} \frac{2n}{|x|^2 + n} - \frac{2|x|^2}{(|x|^2 + n)^2} \quad (43)$$

(the existence and uniqueness of  $v$  follows from [GT01, Theorem 6.13]). Note that the function  $u$  satisfies  $\mathcal{L}u = 1$  and that, by straightforward algebra,  $\mathcal{G}(u)$  can be expressed as

$$\mathcal{G}(u) = \frac{1}{n} \int H\left(\frac{|x|^2}{n}\right) d\gamma_{\mathcal{S}_\varepsilon}(x) + \int \frac{1}{n} \left( \sum_{i=1}^n x_i \partial_i v(x) \right)^2 + |\nabla v(x)|^2 d\gamma_{\mathcal{S}_\varepsilon}(x), \quad (44)$$

where  $H(t) = \frac{1}{t+1}$  for  $t \geq 0$ .

Using the fact that  $|x'|^2 \leq |x|^2 \leq |x'|^2 + \varepsilon^2$  and  $|x|^2 < 2n$  on  $\mathcal{S}_\varepsilon$ , we can bound

$$\int H\left(\frac{|x|^2}{n}\right) d\gamma_{\mathcal{S}_\varepsilon}(x) \leq \int H\left(\frac{|x'|^2}{n}\right) d\gamma_{\mathcal{S}_\varepsilon}(x) + \frac{\varepsilon^2}{n}, \quad (45)$$

since  $H$  is 1-Lipschitz on the interval  $[0, 2]$ . Furthermore, we can estimate

$$\begin{aligned} \int H\left(\frac{|x'|^2}{n}\right) d\gamma_{\mathcal{S}_\varepsilon}(x) &= \frac{1}{\gamma_n(\mathcal{S}_\varepsilon)} \int_{-\varepsilon}^{\varepsilon} \int_{\{|x'|^2 < 2n\}} H\left(\frac{|x'|^2}{n}\right) d\gamma_{n-1}(x') d\gamma_1(x_1) \\ &= \int_{\{|x'|^2 < 2\}} H\left(\frac{|x'|^2}{n}\right) d\gamma_{n-1}(x') = H(1) + \int_{\{|x'|^2 < 2\}} \left( H\left(\frac{|x'|^2}{n}\right) - H(1) \right) d\gamma_{n-1}(x') \\ &\leq H(1) + \int_{\{|x'|^2 < 2\}} \left| \frac{|x'|^2}{n} - 1 \right| d\gamma_{n-1}(x') \leq \frac{1}{2} + o(1), \end{aligned} \quad (46)$$

as  $n \rightarrow \infty$ . Thus, from (45) and (46), we obtain the following bound for the first summand of (44),

$$\frac{1}{n} \int H\left(\frac{|x|^2}{n}\right) d\gamma_{\mathcal{S}_\varepsilon}(x) \leq \frac{1}{2n} + \frac{\varepsilon^2}{n} + o\left(\frac{1}{n}\right), \quad (47)$$

as  $n \rightarrow \infty$ . In order to complete the proof of (41), it suffices to show that the second summand of (44) satisfies

$$\int \frac{1}{n} \left( \sum_{i=1}^n x_i \partial_i v(x) \right)^2 + |\nabla v(x)|^2 d\gamma_{\mathcal{S}_\varepsilon}(x) \leq C\varepsilon^2 \quad (48)$$

for some absolute constant  $C > 0$ , provided that  $\varepsilon \ll 1$ . In view of the bound

$$\int \frac{1}{n} \left( \sum_{i=1}^n x_i \partial_i v(x) \right)^2 + |\nabla v(x)|^2 d\gamma_{\mathcal{S}_\varepsilon}(x) \leq \int \left( \frac{|x|^2}{n} + 1 \right) |\nabla v(x)|^2 d\gamma_{\mathcal{S}_\varepsilon}(x) \leq 4 \int |\nabla v|^2 d\gamma_{\mathcal{S}_\varepsilon}, \quad (49)$$

which follows from the upper bound  $|x|^2 < 2n + \varepsilon^2$  on  $\mathcal{S}_\varepsilon$ , it suffices to establish that

$$\int |\nabla v|^2 d\gamma_{\mathcal{S}_\varepsilon} \leq C\varepsilon^2. \quad (50)$$

Therefore, the following lemma completes the proof of (41).  $\square$

**Lemma 12.** *For any  $\varepsilon > 0$ , the function  $v : \mathcal{S}_\varepsilon \rightarrow \mathbb{R}$  defined by (42) satisfies*

$$\int |\nabla v|^2 d\gamma_{\mathcal{S}_\varepsilon} \leq 36e^{\varepsilon^2/2} \varepsilon^2. \quad (51)$$

*Proof.* Let us multiply equation (42) with  $v$  and integrate over  $\mathcal{S}_\varepsilon$ , obtaining the relation

$$\int v \cdot \mathcal{L}v d\gamma_{\mathcal{S}_\varepsilon} = \int v \cdot R d\gamma_{\mathcal{S}_\varepsilon}. \quad (52)$$

Integrating by parts in the left hand side of (52) then yields

$$\int |\nabla v|^2 d\gamma_{\mathcal{S}_\varepsilon} = - \int v \cdot R d\gamma_{\mathcal{S}_\varepsilon} \leq \left( \int v^2 d\gamma_{\mathcal{S}_\varepsilon} \right)^{\frac{1}{2}} \left( \int R^2 d\gamma_{\mathcal{S}_\varepsilon} \right)^{\frac{1}{2}}. \quad (53)$$

Since  $v$  vanishes on  $\{x_1 = -\varepsilon\} \cap \{|x'|^2 < 2n\} \subseteq \partial\mathcal{S}_\varepsilon$ , for any  $x = (x_1, x') \in \mathcal{S}_\varepsilon$ , we have

$$\begin{aligned} |v(x_1, x')| &= \left| \int_{-\varepsilon}^{x_1} \partial_1 v(s, x') ds \right| \leq \int_{-\varepsilon}^{\varepsilon} |\partial_1 v(s, x')| ds \\ &\leq e^{\varepsilon^2/4} \int_{-\varepsilon}^{\varepsilon} e^{-s^2/4} |\partial_1 v(s, x')| ds. \end{aligned} \quad (54)$$

Therefore, using (54), we can estimate

$$\begin{aligned} \int v^2 d\gamma_{\mathcal{S}_\varepsilon} &= \frac{1}{\gamma_n(\mathcal{S}_\varepsilon)} \int_{\mathcal{S}_\varepsilon} v^2(x_1, x') d\gamma_n(x_1, x') \\ &\stackrel{(54)}{\leq} \frac{1}{\gamma_n(\mathcal{S}_\varepsilon)} \int_{\mathcal{S}_\varepsilon} e^{\varepsilon^2/2} \left( \int_{-\varepsilon}^{\varepsilon} e^{-s^2/4} |\partial_1 v(s, x')| ds \right)^2 d\gamma_n(x_1, x') \\ &\leq \frac{e^{\varepsilon^2/2}}{\gamma_n(\mathcal{S}_\varepsilon)} \int_{\mathcal{S}_\varepsilon} 2\varepsilon \left( \int_{-\varepsilon}^{\varepsilon} e^{-s^2/2} (\partial_1 v(s, x'))^2 ds \right) d\gamma_n(x_1, x') \\ &= \frac{2e^{\varepsilon^2/2}\varepsilon}{\gamma_n(\mathcal{S}_\varepsilon)} \int_{\{|x'|^2 < 2n\}} \int_{-\varepsilon}^{\varepsilon} \left( \int_{-\varepsilon}^{\varepsilon} e^{-s^2/2} (\partial_1 v(s, x'))^2 ds \right) d\gamma_1(x_1) d\gamma_{n-1}(x') \\ &= \frac{2e^{\varepsilon^2/2}\varepsilon}{\gamma_n(\mathcal{S}_\varepsilon)} \int_{\{|x'|^2 < 2n\}} \left( \int_{-\varepsilon}^{\varepsilon} d\gamma_1(x_1) \right) \left( \int_{-\varepsilon}^{\varepsilon} e^{-s^2/2} (\partial_1 v(s, x'))^2 ds \right) d\gamma_{n-1}(x') \\ &\leq \frac{2e^{\varepsilon^2/2}\varepsilon}{\gamma_n(\mathcal{S}_\varepsilon)} \int_{\{|x'|^2 < 2n\}} \frac{2\varepsilon}{\sqrt{2\pi}} \left( \int_{-\varepsilon}^{\varepsilon} e^{-s^2/2} (\partial_1 v(s, x'))^2 ds \right) d\gamma_{n-1}(x') \\ &= \frac{4e^{\varepsilon^2/2}\varepsilon^2}{\gamma_n(\mathcal{S}_\varepsilon)} \int_{\{|x'|^2 < 2n\}} \int_{-\varepsilon}^{\varepsilon} (\partial_1 v(s, x'))^2 d\gamma_1(s) d\gamma_{n-1}(x') \leq 4e^{\varepsilon^2/2}\varepsilon^2 \int |\nabla v|^2 d\gamma_{\mathcal{S}_\varepsilon}. \end{aligned} \quad (55)$$

Finally, using the trivial pointwise bound

$$|R(x)| \leq \frac{2n}{|x|^2 + n} + \frac{|x|^2}{(|x|^2 + n)^2} < 2 + \frac{1}{n} \leq 3 \quad (56)$$

along with (55), we obtain from (53) that

$$\int |\nabla v|^2 d\gamma_{\mathcal{S}_\varepsilon} \leq 6e^{\varepsilon^2/4}\varepsilon \left( \int |\nabla v|^2 d\gamma_{\mathcal{S}_\varepsilon} \right)^{\frac{1}{2}} \quad (57)$$

and the conclusion readily follows.  $\square$

**2.** The bound of Theorem 3 can be sharpened when  $K$  is  $\rho B_2^n$ , the Euclidean ball of radius  $\rho \in (0, \infty)$ . Let  $u$  be a smooth symmetric function with  $\mathcal{L}u = 1$  on  $\rho B_2^n$  and  $u(0) = 0$ . Then, for every  $r \in [0, \rho)$  and  $\theta \in \mathbb{S}^{n-1}$ , we can write  $u(r\theta) = u_0(r) + v(r\theta)$ , where

$$\forall r \in [0, \rho), \quad u_0(r) = \frac{1}{|\mathbb{S}^{n-1}|} \int_{\mathbb{S}^{n-1}} u(r\phi) d\phi \quad \text{and} \quad \int_{\mathbb{S}^{n-1}} v(r\phi) d\phi = 0. \quad (58)$$

For every  $r \in (0, \rho)$ , we have

$$u_0''(r) + \left( \frac{n-1}{r} - r \right) u_0'(r) = \mathcal{L}u_0(r) = \frac{1}{|\mathbb{S}^{n-1}|} \int_{\mathbb{S}^{n-1}} \mathcal{L}u(r\phi) d\phi = 1, \quad (59)$$

hence also  $\mathcal{L}v = \mathcal{L}u - \mathcal{L}u_0 = 0$ . Moreover, notice that  $u_0(0) = 0$  and

$$u'_0(0) = \lim_{r \rightarrow 0} \frac{1}{|\mathbb{S}^{n-1}|} \int_{\mathbb{S}^{n-1}} \sum_{i=1}^n \frac{\phi_i x_i}{r} \partial_i u(r\phi) \, d\phi = \frac{1}{|\mathbb{S}^{n-1}|} \int_{\mathbb{S}^{n-1}} \sum_{i=1}^n \phi_i \lim_{r \rightarrow 0} \frac{x_i}{r} \partial_i u(r\phi) \, d\phi = 0, \quad (60)$$

since  $u$  is even and thus  $\partial_i u(0) = 0$ . Additionally, It can be shown that we can write

$$\int \|\nabla^2 u\|_{\text{HS}}^2 + |\nabla u|^2 \, d\gamma_{\rho B_2^n} = \int \|\nabla^2 u_0\|_{\text{HS}}^2 + |\nabla u_0|^2 \, d\gamma_{\rho B_2^n} + \int \|\nabla^2 v\|_{\text{HS}}^2 + |\nabla v|^2 \, d\gamma_{\rho B_2^n}. \quad (61)$$

Combining the above observations, we deduce that

$$\min \left\{ \int \|\nabla^2 u\|_{\text{HS}}^2 + |\nabla u|^2 \, d\gamma_{\rho B_2^n} : u \in C^2(\rho B_2^n) \text{ with } \mathcal{L}u = 1 \right\} = \int \|\nabla^2 u_0\|_{\text{HS}}^2 + |\nabla u_0|^2 \, d\gamma_{\rho B_2^n},$$

where  $u_0$  is the unique solution of (59) satisfying  $u_0(0) = u'_0(0) = 0$ . An explicit calculation now shows that this function  $u_0$  satisfies

$$\int \|\nabla^2 u_0\|_{\text{HS}}^2 + |\nabla u_0|^2 \, d\gamma_{\rho B_2^n} = 1 - \left( \frac{n-1}{\rho} - \rho \right) \frac{\int_0^\rho r^{n-1} e^{-r^2/2} \, dr}{\rho^{n-1} e^{-\rho^2/2}} > \frac{1}{n}. \quad (62)$$

**3.** If  $\lambda \in (0, 1)$ , the  $\lambda$ -geometric mean of two symmetric convex sets  $K, L$  in  $\mathbb{R}^n$  is the set

$$K^\lambda L^{1-\lambda} \stackrel{\text{def}}{=} \bigcap_{\theta \in \mathbb{S}^{n-1}} \{x \in \mathbb{R}^n : \langle x, \theta \rangle \leq h_K(\theta)^\lambda h_L(\theta)^{1-\lambda}\}, \quad (63)$$

where the support function  $h_M : \mathbb{S}^{n-1} \rightarrow \mathbb{R}_+$  of a symmetric convex set  $M$  is  $h_M(\theta) = \sup_{y \in M} \langle y, \theta \rangle$ . The log-Brunn–Minkowski conjecture of [BLYZ12], asserts that for any such sets and  $\lambda \in (0, 1)$ ,

$$|K^\lambda L^{1-\lambda}| \geq |K|^\lambda |L|^{1-\lambda}. \quad (64)$$

In [Sar16], Saroglou proved that the log-Brunn–Minkowski conjecture formally implies the validity of (64), with the Lebesgue measure  $|\cdot|$  replaced by any symmetric log-concave measure  $\mu$ , that is

$$\mu(K^\lambda L^{1-\lambda}) \geq \mu(K)^\lambda \mu(L)^{1-\lambda}. \quad (65)$$

Then, in [LMNZ17, Proposition 1], the authors showed that (65) implies the dimensional Brunn–Minkowski inequality (7) with  $\delta = 1$  for all symmetric convex sets  $K, L$  in  $\mathbb{R}^n$ . We note here that in the special case where  $\mu = \gamma_n$ , one gets a slightly stronger result in the spirit of Question 4. Recall that  $\Psi_n : [0, \infty) \rightarrow [0, 1]$  is given by  $\Psi_n(r) = \gamma_n(rB_2^n)$ , where  $rB_2^n = \{x \in \mathbb{R}^n : |x| \leq r\}$ . We will need the following well-known lemma, which we could not locate in the literature.

**Lemma 13.** *Fix  $n \in \mathbb{N}$  and let  $A$  be a Borel measurable set in  $\mathbb{R}^n$ . If  $\rho \in [0, \infty]$  is such that  $\gamma_n(A) = \gamma_n(\rho B_2^n)$ , then for every  $t \in [0, 1]$ , we have  $\gamma_n(tA) \geq \gamma_n(t\rho B_2^n)$ .*

*Proof.* Let  $B = \rho B_2^n$ . The assumption implies that  $\gamma_n(A \setminus B) = \gamma_n(B \setminus A)$ . Therefore,

$$\begin{aligned} (2\pi)^{n/2} (\gamma_n(tA) - \gamma_n(tB)) &= \int_{t(A \setminus B)} e^{-|x|^2/2} \, dx - \int_{t(B \setminus A)} e^{-|x|^2/2} \, dx \\ &= t^n \left( \int_{A \setminus B} e^{-t^2|y|^2/2} \, dy - \int_{B \setminus A} e^{-t^2|y|^2/2} \, dy \right) \\ &\geq t^n e^{(1-t^2)\rho^2/2} \left( \int_{A \setminus B} e^{-|y|^2/2} \, dy - \int_{B \setminus A} e^{-|y|^2/2} \, dy \right) = 0 \end{aligned} \quad (66)$$

and the proof is complete.  $\square$

**Proposition 14.** Fix  $n \in \mathbb{N}$ . If (65) holds true for  $\mu = \gamma_n$ , then for every symmetric convex sets  $K, L$  in  $\mathbb{R}^n$  and every  $\lambda \in (0, 1)$ , we have

$$\Psi_n^{-1}(\gamma_n(\lambda K + (1 - \lambda)L)) \geq \sup_{p \in [0, 1]} \left(\frac{\lambda}{p}\right)^p \left(\frac{1 - \lambda}{1 - p}\right)^{1-p} \Psi_n^{-1}(\gamma_n(K)^p \gamma_n(L)^{1-p}). \quad (67)$$

*Proof.* Fix  $p \in (0, 1)$  and let  $t_p = \left(\frac{\lambda}{p}\right)^p \left(\frac{1 - \lambda}{1 - p}\right)^{1-p} \in [0, 1]$ . Then, we have the inclusion

$$\lambda K + (1 - \lambda)L = p \cdot \left(\frac{\lambda}{p}K\right) + (1 - p) \cdot \left(\frac{1 - \lambda}{1 - p}L\right) \supseteq \left(\frac{\lambda}{p}K\right)^p \left(\frac{1 - \lambda}{1 - p}L\right)^{1-p} = t_p K^p L^{1-p}. \quad (68)$$

Since  $t_p \in [0, 1]$ , Lemma 13 implies that

$$\gamma_n(t_p K^p L^{1-p}) \geq \gamma_n\left(t_p \Psi_n^{-1}(\gamma_n(K^p L^{1-p})) B_2^n\right) = \Psi_n\left(t_p \Psi_n^{-1}(\gamma_n(K^p L^{1-p}))\right), \quad (69)$$

Hence combining (65) with (68), (69) and the monotonicity of  $\Psi_n^{-1}$ , we derive (67).  $\square$

**4.** In [GZ10, p. 5350], it was shown that inequality (15) is not satisfied for  $\xi_n = \Psi_n^{-1}$ , where  $\Psi_n(r) = \gamma_n(rB_2^n)$ . As was communicated to us by Ramon van Handel, this observation formally implies that if  $M$  is any Borel set in  $\mathbb{R}^n$  and  $\Xi_n(r) = \gamma_n(rM)$ , then (15) is not satisfied for  $\xi_n = \Xi_n^{-1}$ . Indeed, assuming the contrary and plugging  $K = aB_2^n$  and  $L = bB_2^n$  in (15), we see that  $\Xi_n^{-1} \circ \Psi_n$  is concave. Therefore, for every  $r \in (0, \infty)$  and  $t \in (0, 1)$ , we have

$$t\Xi_n^{-1}(\Psi_n(r)) \leq \Xi_n^{-1}(\Psi_n(tr)). \quad (70)$$

Moreover, Lemma 13 is equivalent to the fact that for every  $p \in (0, 1)$  and  $t \in (0, 1)$ ,

$$\Xi_n(t\Xi_n^{-1}(p)) \geq \Psi_n(t\Psi_n^{-1}(p)) \quad (71)$$

Choosing  $p = \Psi_n(r)$  in (71) and combining it with (70), we deduce that  $\Xi_n^{-1} \circ \Psi_n$  is an affine function, which readily implies that  $\Xi_n(r) = \Psi_n(ar)$  for some  $a \in (0, \infty)$ , thus contradicting [GZ10].

This simple argument reveals that the nontrivial equality cases of the sought-for symmetric Ehrhard inequality (15) must be more complicated than the one-parameter family  $\{sM\}_{s \geq 0}$  of dilates of a given Borel set  $M$ . The possibility of such extremals is reminiscent of the conjectured multiscale solutions of the symmetric Gaussian isoperimetric problem [Bar01, Hei17].

## REFERENCES

- [Bar01] F. Barthe. An isoperimetric result for the Gaussian measure and unconditional sets. *Bull. London Math. Soc.*, 33(4):408–416, 2001.
- [Bar06] F. Barthe. The Brunn-Minkowski theorem and related geometric and functional inequalities. In *International Congress of Mathematicians. Vol. II*, pages 1529–1546. Eur. Math. Soc., Zürich, 2006.
- [BK20] K. J. Böröczky and P. Kalantzopoulos. Log-Brunn-Minkowski inequality under symmetry. Preprint available at <https://arxiv.org/abs/2002.12239>, 2020.
- [BL76] H. J. Brascamp and E. H. Lieb. On extensions of the Brunn-Minkowski and Prékopa-Leindler theorems, including inequalities for log concave functions, and with an application to the diffusion equation. *J. Functional Analysis*, 22(4):366–389, 1976.
- [BLYZ12] K. J. Böröczky, E. Lutwak, D. Yang, and G. Zhang. The log-Brunn-Minkowski inequality. *Adv. Math.*, 231(3-4):1974–1997, 2012.
- [Bor03] C. Borell. The Ehrhard inequality. *C. R. Math. Acad. Sci. Paris*, 337(10):663–666, 2003.
- [CEFM04] D. Cordero-Erausquin, M. Fradelizi, and B. Maurey. The (B) conjecture for the Gaussian measure of dilates of symmetric convex sets and related problems. *J. Funct. Anal.*, 214(2):410–427, 2004.
- [CHLL18] S. Chen, Y. Huang, Q. Li, and J. Liu.  $L_p$ -Brunn-Minkowski inequality for  $p \in (1 - \frac{c}{n^{\frac{3}{2}}}, 1)$ . Preprint available at <https://arxiv.org/abs/1811.10181>, 2018.
- [CLM17] A. Colesanti, G. V. Livshyts, and A. Marsiglietti. On the stability of Brunn-Minkowski type inequalities. *J. Funct. Anal.*, 273(3):1120–1139, 2017.
- [Col08] A. Colesanti. From the Brunn-Minkowski inequality to a class of Poincaré-type inequalities. *Commun. Contemp. Math.*, 10(5):765–772, 2008.

- [Dub77] S. Dubuc. Critères de convexité et inégalités intégrales. *Ann. Inst. Fourier (Grenoble)*, 27(1):x, 135–165, 1977.
- [Ehr83] A. Ehrhard. Symétrisation dans l’espace de Gauss. *Math. Scand.*, 53(2):281–301, 1983.
- [Ehr86] A. Ehrhard. Éléments extrémaux pour les inégalités de Brunn-Minkowski gaussiennes. *Ann. Inst. H. Poincaré Probab. Statist.*, 22(2):149–168, 1986.
- [Gar02] R. J. Gardner. The Brunn-Minkowski inequality. *Bull. Amer. Math. Soc. (N.S.)*, 39(3):355–405, 2002.
- [Gar06] R. J. Gardner. *Geometric tomography*, volume 58 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, New York, second edition, 2006.
- [GT01] D. Gilbarg and N. Trudinger. *Elliptic partial differential equations of second order*. Classics in Mathematics. Springer-Verlag Berlin Heidelberg, 2nd edition, 2001.
- [GZ10] R. J. Gardner and A. Zvavitch. Gaussian Brunn-Minkowski inequalities. *Trans. Amer. Math. Soc.*, 362(10):5333–5353, 2010.
- [Hei17] S. Heilman. Symmetric convex sets with minimal Gaussian surface area. To appear in *Amer. J. Math.*. Preprint available at <https://arxiv.org/abs/1705.06643>, 2017.
- [HKL20] J. Hosle, A. Kolesnikov, and G. Livshyts. On the  $L_p$ -Brunn-Minkowski and dimensional Brunn-Minkowski conjectures for log-concave measures. Preprint available at <https://arxiv.org/abs/2003.05282>, 2020.
- [KL18] A. Kolesnikov and G. Livshyts. On the Gardner-Zvavitch conjecture: symmetry in inequalities of Brunn-Minkowski type. To appear in *Adv. Math.*. Preprint available at <https://arxiv.org/abs/1807.06952>, 2018.
- [KL20] A. Kolesnikov and G. Livshyts. On the local version of the log-Brunn-Minkowski conjecture and some new related geometric inequalities. Preprint available at <https://arxiv.org/abs/2004.06103>, 2020.
- [KM17] A. Kolesnikov and E. Milman. Local  $L^p$ -Brunn-Minkowski inequalities for  $p < 1$ . To appear in *Mem. Amer. Math. Soc.*. Preprint available at <https://arxiv.org/abs/1711.01089>, 2017.
- [KM18] A. V. Kolesnikov and E. Milman. Poincaré and Brunn-Minkowski inequalities on the boundary of weighted Riemannian manifolds. *Amer. J. Math.*, 140(5):1147–1185, 2018.
- [Lat96] R. Latała. A note on the Ehrhard inequality. *Studia Math.*, 118(2):169–174, 1996.
- [Lat02] R. Latała. On some inequalities for Gaussian measures. In *Proceedings of the International Congress of Mathematicians, Vol. II (Beijing, 2002)*, pages 813–822. Higher Ed. Press, Beijing, 2002.
- [LMNZ17] G. Livshyts, A. Marsiglietti, P. Nayar, and A. Zvavitch. On the Brunn-Minkowski inequality for general measures with applications to new isoperimetric-type inequalities. *Trans. Amer. Math. Soc.*, 369(12):8725–8742, 2017.
- [LO99] R. Latała and K. Oleszkiewicz. Gaussian measures of dilatations of convex symmetric sets. *Ann. Probab.*, 27(4):1922–1938, 1999.
- [Mar16] A. Marsiglietti. On the improvement of concavity of convex measures. *Proc. Amer. Math. Soc.*, 144(2):775–786, 2016.
- [Mau05] B. Maurey. Inégalité de Brunn-Minkowski-Lusternik, et autres inégalités géométriques et fonctionnelles. Number 299, pages Exp. No. 928, vii, 95–113. 2005. Séminaire Bourbaki. Vol. 2003/2004.
- [NT13] P. Nayar and T. Tkocz. A note on a Brunn-Minkowski inequality for the Gaussian measure. *Proc. Amer. Math. Soc.*, 141(11):4027–4030, 2013.
- [Put19] E. Putterman. Equivalence of the local and global versions of the  $L^p$ -Brunn-Minkowski inequality. Preprint available at <https://arxiv.org/abs/1909.03729>, 2019.
- [Roy14] T. Royen. A simple proof of the Gaussian correlation conjecture extended to some multivariate gamma distributions. *Far East J. Theor. Stat.*, 48(2):139–145, 2014.
- [RYN18] M. Ritoré and J. Yepes Nicolás. Brunn-Minkowski inequalities in product metric measure spaces. *Adv. Math.*, 325:824–863, 2018.
- [Sar16] C. Saroglou. More on logarithmic sums of convex bodies. *Mathematika*, 62(3):818–841, 2016.
- [Sch14] R. Schneider. *Convex bodies: the Brunn-Minkowski theory*, volume 151 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, expanded edition, 2014.
- [YZ19] Y. Yang and D. Zhang. The log-Brunn-Minkowski inequality in  $\mathbb{R}^3$ . *Proc. Amer. Math. Soc.*, 147(10):4465–4475, 2019.

(A. E.) INSTITUT DE MATHÉMATIQUES DE JUSSIEU, SORBONNE UNIVERSITÉ, 75252 PARIS CEDEX 05, FRANCE  
*Email address:* alexandros.eskenazis@imj-prg.fr

(G. M.) DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, BERKELEY, CA, 94720-3840, USA  
*Email address:* gmoschidis@berkeley.edu