On maximal det-independent (res-independent) sets in graphs

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Abstract

In this writing, we point out some errors made in [D. L. Boutin, Determining sets, resolving sets and the exchange property, *Graphs and Combin.*, 25(2009), 789-806], where the author claims that a maximal independent set in a hereditary system is a minimal determining (resolving) set. Further more, the author claims that if the exchange property holds at the level of minimal resolving sets, then, the corresponding hereditary system is a matroid. We give counter examples to disprove both of her claims. Besides, we prove that there exist graphs having such maximal independent sets which are not necessarily determining (resolving) sets. Also, we give necessary and sufficient conditions for a class of graphs to have a maximal independent set which is not minimal determining (resolving).

Keywords: determining set, resolving set, independent set, hereditary system

2010 AMS Subject Classification Numbers: 05C12, 05B35

1 Introduction

A set U of vertices of a connected graph G is called a determining set for G if every automorphism of G is uniquely determined by its action on the vertices of U. If the vertices in U are unable to determine all the automorphisms of G, then we call U a non-determining set for G. The minimum

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cardinality of a minimal determining set for G is called the determining number of G, denoted by $D(G)$ [2]. A set W of vertices of G is called a resolving set for G if every vertex in G is uniquely determined by its distances to the vertices of W. If the vertices in W are unable to determine all the vertices of G, then we call W a non-resolving set for G. The minimum cardinality of a minimal resolving set for G is called the metric dimension of G, denoted by $\beta(G)$ [3, 4].

Determining (resolving) sets are said to have the exchange property in G if whenever S and R are minimal determining (resolving) sets for G and $r \in R$, then there exists $s \in S$ so that $S - \{s\} \cup \{r\}$ is a minimal determining (resolving) set [1]. It is noteworthy that if determining (resolving) sets have the exchange property in a given graph, then every minimal determining (resolving) set for that graph has the same cardinality.

A set system is a finite set H together with a family $\mathcal{F}$ of subsets of H and is denoted by the pair $(H, \mathcal{F})$. A set system $(H, \mathcal{P})$ is said to be a *hereditary system* (an independence system) if for every subset X of H possessing the property $\mathcal{P}$, each subset of X also possesses the property $\mathcal{P}$. That is, for each $X \subseteq H$ such that $X \in \mathcal{P}$, $Y \in \mathcal{P}$ for all $Y \subseteq X$. In fact, in a hereditary system $(H, \mathcal{P})$, $\mathcal{P}$ is identified by the family of subsets of H possessing the property $\mathcal{P}$. A subset X of H which possesses the property $\mathcal{P}$ is said to be an *independent set*, and a *dependent set* otherwise.

A subset H of the vertex set V of a graph G is a *det-independent* (*res-independent*) set if no proper subset of H is a determining (resolving) set for G. This concept was first introduced by Boutin in [1]. If we denote the property of being det-independent (res-independent) by *det* (*res*), then the hereditary system in G corresponds to this property is as follows:

$$(V, det\ (res)) = \{H \subset V \mid H \text{ possesses the property } det\ (res)\},$$

where V is the vertex set of G. It is also worth mentioning here that, by definition, a non-determining (non-resolving) set is always a det-independent (res-independent) set.

2 Counter examples

Formally, determining and resolving sets are defined as follows: Let $Aut(G)$ be the automorphism group of G and U be a set of vertices of G. For every two automorphisms $\phi, \psi \in Aut(G)$, if $\phi(u) = \psi(u)$ for all $u \in U$ implies $\phi = \psi$, then U is called a determining set for G [2]. Let $d(x, y)$ be the number of edges in a geodesic (shortest path) between the vertices x and y, and is called the distance between x and y in G. A set W of vertices of G is a resolving set for G if for every two vertices v and z of G, there is a vertex w in W such that $d(v, w) \neq d(z, w)$ [3, 4].

Boutin formally defines *det-independence* (*res-independence*), so that a set S of vertices in a graph G is a det-independent (res-independent), if for every $s \in S$, $S - \{s\}$ is not a determining (resolving) set for G [1]. With this definition, she claims that a maximal det-independent (res-independent) set is a minimal determining (resolving) set. (line -6 at page 791 of [1]). But, her claim is unfortunately wrong, as we observe in the following two examples.

Example 1. Let us consider the graph G of Figure 1. Minimal determining sets for G are: $\{v_4\}$ and $\{v_5\}$. Thus, $D(G) = 1$. According to the property of det-independence, we have the following hereditary system for G:

$$(V, des) \quad = \quad \{\{v_i\}\ ;\ 1 \leq i \leq 5\} \cup \{\{v_1, v_2\}, \{v_1, v_3\}, \{v_2, v_3\}\} \cup \{\{v_1, v_2, v_3\}\}. \tag{1}$$

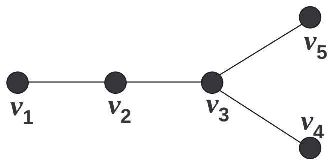

Figure 1: A graph G

It can be observed that the sets $\{v_1, v_2\}$, $\{v_1, v_3\}$, $\{v_2, v_3\}$ and $\{v_1, v_2, v_3\}$ are maximal det-independent sets, but not minimal determining sets.

Example 2. Consider the graph G of Figure 1. Minimal resolving sets for G are: $\{v_1, v_4\}$, $\{v_2, v_4\}$, $\{v_1, v_5\}$, $\{v_2, v_5\}$ and $\{v_4, v_5\}$. Thus, $\beta(G) = 2$. According to the property of res-independence, we have the following hereditary system for G:

$$\begin{aligned} (V, res) &= \{\{v_i\}\ ;\ 1 \leq i \leq 5\} \\ &\cup\ \{\{v_1, v_2\}, \{v_1, v_3\}, \{v_1, v_4\}, \{v_1, v_5\}, \{v_2, v_3\}, \{v_2, v_4\}, \{v_2, v_5\}\{v_3, v_4\}, \{v_3, v_5\}, \{v_4, v_5\}\} \\ &\cup\ \{\{v_1, v_2, v_3\}\}. \end{aligned} \tag{2}$$

Here the set $\{v_1, v_2, v_3\}$ is a maximal res-independent, but not a minimal resolving set.

In [1], the author also claims that if the exchange property holds at the level of minimal determining (resolving) sets, then the property holds for maximal independent sets in the corresponding hereditary system (lines 1-5 at page 792 [1]). Consequently, the hereditary system is a matroid [6]. But, it is not true regrettably, because Examples 1 and 2 provide that a maximal det-independent (res-independent) set need not to be a minimal determining (resolving) set, and so the corresponding hereditary systems (1) and (2) are not matroids. Since the graph considered in Examples 1 and 2 is a tree graph, so the hereditary system (produced due to Theorem 3 at page 793 of [1]) for tree graphs is not a matroid.

3 Necessary and sufficient conditions

Let v be a vertex of a graph G having the vertex set V. Then, the open neighborhood of v is $N(v) = \{u \in V\ :\ u \text{ is adjacent with } v \text{ in } G\}$ and the closed neighborhood of v is $N[v] = N(v) \cup \{v\}$. Two vertices of G are said to be twin vertices (simply called twins) if they have the same (open or closed) neighborhoods. The relation of twins between the vertices of G is an equivalence relation, which produces classes of twin vertices in G, called the twin classes. That is, a twin class in G is a set of vertices of G such that its every two elements are twins [5]. A non-singleton twin contains two or more elements. The next lemma is due to the definition of twins.

Lemma 1. [5] *If u and v are twins in a graph G, then $d(u, x) = d(v, x)$ for each $x \in V - \{u, v\}$.*

Due to Lemma 1, the following remark is directly followed.

Remark 2. *If u and v are twins in a graph G and U (or W) is a determining (resolving) set for G, then either $u \in U$ (W) or $v \in U$ (W). In other words, if T is a twin class of order $t \geq 2$ in G, then every determining (resolving) set for G must contain at least $t-1$ elements of T.*

The removal of only two twins from the vertex set of a graph G makes it a maximal det-independent (res-independent) set for G.

Lemma 3. *Let T be a twin class of order $t \geq 2$ in a graph G with vertex set V. Then, for any two elements $u, v \in T$, the set $V - \{u, v\}$ is a maximal det-independent (res-independent) set for G.*

Proof. Since $d(u, x) = d(v, x)$ for all $x \in V - \{u, v\}$, by Lemma 1, so no subset of $V - \{u, v\}$ is a determining (resolving) set for G. It follows the result. □

In next result, we observe the condition when a hereditary system for a graph corresponds to the det-independence (res-independence) property is not a matroid.

Theorem 4. *If there is a non-singleton twin class in non-complete graph (not a complete graph), then there is a maximal det-independent (res-independent) set which is not a minimal determining (resolving) set. In fact, the corresponding hereditary system for the graph is not a matroid.*

Proof. Let G be a non-complete graph with vertex set V and let T be a twin class in G of cardinality at least two. Then, for any $u, v \in T$, the set $X = V - \{u, v\}$ is a maximal det-independent (res-independent) set for G, by Lemma, 3. But, X is not a determining (resolving) set for G, because $d(u, x) = d(v, x)$ for all $x \in X$ due to Lemma 1 and Remark 2. Thus, the hereditary system for G corresponds to det-independence (res-independence) property contains a det-independent (res-independent) set which is not a minimal determining (resolving) set, and hence it is not a matroid. □

The next result provides necessary and sufficient conditions for a class of graphs to have a maximal independent set which is not minimal determining (resolving).

Theorem 5. *The following assertions are equivalent for a non-complete graph G in which determining (resolving) sets have the exchange property.*

1. *G has a non-determining (non-resolving) set of cardinality greater than $D(G)$ (or $\beta(G)$).*
2. *G has a maximal det-independent (res-independent) set which is not minimal determining (resolving).*

Proof. Suppose that G has a non-determining (non-resolving) set Y such that $|Y| > D(G)$ (or $\beta(G)$). Since the exchange property holds in G for determining (resolving) sets, so the cardinality of all the minimal determining (resolving) sets is the same, and is $D(G)$ (or $\beta(G)$). Thus, any non-determining (non-resolving) set Y with $|Y| > D(G)$ (or $\beta(G)$) is a a maximal det-independent (res-independent) set which is not minimal determining (resolving).

Conversely, suppose that G has a maximal det-independent (res-independent) set X which is not minimal determining (resolving). Then, of course, X is a non-determining (non-resolving) set with $|X| > D(G)$ (or $\beta(G)$) due to holding of the exchange property. □

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