Colouring finite products

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COLOURING FINITE PRODUCTS

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Abstract. We consider finite colourings of finite products $X_1 \times X_2 \times \cdots \times X_n$ of infinite sets and determine what is the minimal number of colours a subproduct $Y_1 \times Y_2 \times \cdots \times Y_n$ of infinite subsets could achieve.

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1. Introduction

It is well known that if $X_1, X_2, ..., X_n$ is a finite sequence of countable infinite sets then there is a colouring of their product $\prod_{i=1}^n X_i$ with $n!$ colours each of which shows up in any subproduct $\prod_{i=1}^n Y_i$ with $Y_i \subseteq X_i$ are infinite. For example, letting $X_i = \mathbb{N}$ for all i and colouring a given one-to-one sequence $(k_1, k_2, ..., k_n)$ of integers by the permutation σ of $\{1, 2, ..., n\}$ such that $\sigma(i) < \sigma(j)$ is equivalent to $k_i < k_j$ for all $i < j$, it is clear that all permutations show up in any n-product of infinite subsets of $\mathbb{N}$. On the other hand, a simple application of Ramsey's theorem shows that for every finite colouring of $\prod_{i=1}^n X_i$ there exist infinite $Y_i \subseteq X_i$ such that the subproduct $\prod_{i=1}^n Y_i$ uses no more than $n!$ colours. In this note we investigate this phenomenon in the case when some of the sets X_i are uncountable. The following is an interpretation of a result of R. Laver (see page 31 of [1]) showing that in this case the critical number $n!$ drops to $(n-1)!$.

1.1 Theorem (Laver). *Suppose $X_1, X_2, ..., X_n$ is a finite sequence of infinite sets with at least one of them uncountable. Then for every finite colouring of $\prod_{i=1}^n X_i$ there exist infinite $Y_i \subseteq X_i$ $(i = 1, 2, ..., n)$ such that the subproduct uses no more than $(n-1)!$ colours.*

Here, we supplement this result by proving the following.

1.2 Theorem. *Let Ω denote the set of countable ordinals and let n be a positive integer. Then there is a finite colouring of Ω^n which has at least $(n-1)!$ colours on any subproduct $\prod_{i=1}^n Y_i$ with $Y_i \subseteq \Omega$ infinite.*

The case $n = 3$ of Theorem 1.2 (see page 108 of [4]) is an unpublished result of K. Prikry and C. Mills but, as we shall see from the analysis below, the higher dimensional case is a bit more challenging.

2. An operation on the set Ω of countable ordinals

We start this section by listing some properties of the mapping[1]

$$\bar{\rho} : \Omega^2 \to \omega$$

which can be found in Section 3.2 of [2]. In order to simplify the notation, we write $\alpha\beta$ instead of $\bar{\rho}(\alpha\beta)$ and $\xi \vee \eta$ instead of $\max\{\xi, \eta\}$.

2.1 Lemma ([2]; Section 3.2). *For all $\alpha < \beta < \gamma$,*

(1) $\alpha\gamma \neq \beta\gamma$ *and* $\alpha\beta \neq \beta\gamma$,
(2) $\alpha\gamma \leq \alpha\beta \vee \beta\gamma$,
(3) $\alpha\beta \leq \alpha\gamma \vee \beta\gamma$.

From now on we analyze the behaviour of this operation on infinite subsets $A, B, C, D, ...$ of Ω of order-type ω which we enumerate increasingly as (α_i), (β_i), (γ_i), (δ_i), ..., respectively.

2.2 Lemma. *By refining A and B, we may assume that either for every i the sequence $(\alpha_i\beta_k)_{k>i}$ is constant, or for every i the sequence $(\alpha_i\beta_k)_{k>i}$ is strictly increasing. Moreover, if for every i the sequence $(\alpha_i\beta_k)_{k>i}$ is constant then for $i < j$ the constant value of $(\alpha_i\beta_k)_{k>i}$ is strictly smaller than the constant value of $(\alpha_j\beta_k)_{k>j}$.*

Proof. Note that if $\sup(A) > \sup(B)$ the conclusion follows easily from Lemma 2.1 (1) (giving us refinement with the property that $(\alpha_i\beta_k)_{k>i}$ is strictly increasing for all i), so we may assume that $\sup(A) \leq \sup(B)$. Colouring a triple $i < k < l$ by three colours according whether $\alpha_i\beta_k$ is smaller, equal, or larger than $\alpha_i\beta_l$ and applying the Ramsey theorem to refine A and B, we may assume that for all $i < k < l$ exactly one of the following three possibilities happen,

$$\alpha_i\beta_k < \alpha_i\beta_l, \alpha_i\beta_k = \alpha_i\beta_l, \text{ or } \alpha_i\beta_k > \alpha_i\beta_l.$$

Since $\alpha_i\beta_k > \alpha_i\beta_l$ cannot hold for all $i < k < l$, we are left with the first two possibilities which is exactly the conclusion of the first sentence of the lemma. Towards the conclusion of the second sentence, we assume that $\alpha_i\beta_k = \alpha_i\beta_l$ holds for all $i < k < l$ and apply the Ramsey theorem again to refine further and get that for all $i < j < k$, exactly one of the following three possibilities happen,

$$\alpha_i\beta_k < \alpha_j\beta_k, \alpha_i\beta_k = \alpha_j\beta_k, \text{ or } \alpha_i\beta_k > \alpha_j\beta_k.$$

Since the conclusion of the second sentence is equivalent to the first possibility, we have to eliminate the other two. Note that $\alpha_i\beta_k = \alpha_j\beta_k$ contradicts the property (1) of Lemma 2.1, so towards a contradiction let us assume that we have $\alpha_i\beta_k > \alpha_j\beta_k$ for all $i < j < k$. Let n_0 be the constant value of the sequence $(\alpha_0\beta_k)_{k>0}$. Fix a $k > n_0 + 2$. Then $\alpha_j\beta_k$ $(0 < j < k)$ is a sequence of distinct integers below n_0, a contradiction. □

Note that the proof of the second sentence of Lemma 2.2 also shows the following

[1] symmetric and constantly equal to 0 on the diagonal

2.3 Lemma. *Assume that for some A, B, C with $\sup(A) \leq \sup(B) \leq \sup(C)$, we have that for every i the sequences $(\alpha_i\gamma_k)_{k>i}$ and $(\beta_i\gamma_k)_{k>i}$ are constant. Then by refining, we may assume that for $i < j$ the constant value of $(\alpha_i\gamma_k)_{k>i}$ is strictly smaller than the constant value of $(\beta_j\gamma_k)_{k>j}$.*

2.4 Lemma. *Suppose that $\sup(A) \leq \sup(B) \leq \sup(C)$. Then by refining the sets, we may assume that $\alpha_i\gamma_k < \beta_j\gamma_k$ for all $i < j < k$.*

Proof. By refining the three sets we assume that they satisfy the conclusion of Lemma 2.3, and moreover, that each of the pairs (A, B), (A, C) and (B, C) satisfies the conclusion of Lemma 2.2. Suppose towards contradiction that the refinement of the Lemma does not exist. Then applying Ramsey's theorem, we can find a refinement such that $\alpha_i\gamma_k > \beta_j\gamma_k$ for all $i < j < k$. Our assumption $\sup(A) \leq \sup(B) \leq \sup(C)$ and a further refinement of these three sets allow us to assume that the enumerations (α_i), (β_i), (γ_i) of these sets have the property that $\alpha_i \leq \beta_i \leq \gamma_i$ for all i. So, in particular $i < j$ implies $\alpha_i < \beta_j$, so the properties (2) and (3) of Lemma 2.1, give us

$$\alpha_i\beta_j \leq \alpha_i\gamma_k \vee \beta_j\gamma_k = \alpha_i\gamma_k,$$

and

$$\alpha_i\gamma_k \leq \alpha_i\beta_j \vee \beta_j\gamma_k$$

Note that since $\alpha_i\gamma_k > \beta_j\gamma_k$, from the last inequality, we conclude that

$$\alpha_i\beta_j \vee \beta_j\gamma_k = \alpha_i\beta_j.$$

Combining the two inequalities, we conclude that

$$\alpha_i\beta_j = \alpha_i\gamma_k \text{ for all } i < j < k.$$

It follows, in particular, that for every i, the sequences $(\alpha_i\gamma_k)_{k>i}$ and $(\alpha_i\beta_k)_{k>i}$ are constant with the same constant values. Note that by our assumption $\beta_j\gamma_k < \alpha_i\gamma_k$ for all $i < j < k$, we conclude that for every j the sequence $(\beta_j\gamma_k)_{k>j}$ is also constant and its constant value is bounded by the constant value of the sequence $(\alpha_i\gamma_k)_k$ for any $i < j$. This contradicts Lemma 2.3

□

2.5 Lemma. *Suppose that for some A, B, C with $\sup(A) \leq \sup(B) \leq \sup(C)$, we have that $\alpha_j\gamma_k < \beta_i\gamma_k$ for all $i < j < k$. Suppose that A' is an infinite set with $\sup(A') \leq \sup(A)$. Then these sets can be refined in such a way such that $\alpha'_j\gamma_k < \beta_i\gamma_k$ for all $i < j < k$.*

Proof. Suppose that such refinements cannot be found. Applying Ramsey's theorem, we may assume that $\alpha'_j\gamma_k > \beta_i\gamma_k$ for all $i < j < k$. Applying Lemma 2.4 to A', A and C and going to refinements, we may assume that $\alpha'_i\gamma_k < \alpha_j\gamma_k$ for all $i < j < k$. So, for every $i < j - 1 < j < k$, we have

$$\alpha'_{j-1}\gamma_k < \alpha_j\gamma_k < \beta_i\gamma_k.$$

It follows that $\alpha'_{j-1}\gamma_k < \beta_i\gamma_k$, contradicting our assumption that $\alpha'_{j'}\gamma_{k'} > \beta_{i'}\gamma_{k'}$ for all $i' < j' < k'$. □

2.6 Lemma. *Suppose that for some* A, B, C *with* $\sup(A) \leq \sup(B) \leq \sup(C)$, *we have that* $\alpha_j\gamma_k < \beta_i\gamma_k$ *for all* $i < j < k$. *Let* B' *be an infinite set such that* $\sup(B) \leq \sup(B') \leq \sup(C)$. *Then these sets can be refined in such a way that* $\alpha_j\gamma_k < \beta'_i\gamma_k$ *for all* $i < j < k$.

Proof. Applying Lemma 2.4 to B, B' and C, by refining the sets, we may assume that $\beta_i\gamma_k < \beta'_j\gamma_k$ for all $i < j < k$. Combining this with the assumption of the Lemma we get that

$$\alpha_j\gamma_k < \beta_i\gamma_k < \beta'_{i+1}\gamma_k$$

for all $i < i+1 < j < k$. It follows that the triple $i+1 < j < k$ satisfies the conclusion of the Lemma. Since such triple can be found in any refinement of these sets, an application of Ramsey's theorem will give us refinements satisfying the conclusion of the Lemma.

□

2.7 Lemma. *Suppose that for some* A, B, C *with* $\sup(A) \leq \sup(B) \leq \sup(C)$, *we have that* $\alpha_j\gamma_k < \beta_i\gamma_k$ *for all* $i < j < k$. *Then the sets can be refined in such way that for every* i *the sequence* $(\beta_i\gamma_k)_{k>i}$ *is strictly increasing.*

Proof. We assume that the sets have already been refined in such way that for every i, the sequence $(\beta_i\gamma_k)_{k>i}$ is either constant or strictly increasing. So suppose that for all i the sequence $(\beta_i\gamma_k)_{k>i}$ is constant. Fix i. Let l be the constant value of the sequence $(\beta_i\gamma_k)_{k>i}$. Choose $k > l+i+1$. Then by the assumption of the Lemma $\alpha_j\gamma_k < l$ for all $i < j < k$. So there must be $i < j' < j < k$ such that $\alpha_j\gamma_k = \alpha_{j'}\gamma_k$, contradicting the property (1) of the operation $\xi\eta$ of Ω.

□

Conclusion. A given finite sequence $X_1, X_2, ..., X_n$ of infinite subsets of Ω can be refined in such a way that for every three sets A, B and C in the refinement with property that $\sup(A) \leq \sup(B) \leq \sup(C)$ we have that $\alpha_i\gamma_k < \beta_j\gamma_k$ for all $i < j < k$. If for some A, B and C in the refinement with $\sup(A) \leq \sup(B) \leq \sup(C)$ we have that $\beta_i\gamma_k > \alpha_j\gamma_k$ for some $i < j < k$ then this holds for all $i < j < k$. Moreover, if for some A, B and C in the refinement with $\sup(A) \leq \sup(B) \leq \sup(C)$ we have that $\beta_i\gamma_k > \alpha_j\gamma_k$ holds for all $i < j < k$, then this is true for A', B' and C in place of A, B and C provided that $\sup(A') \leq \sup(A)$ and $\sup(B) \leq \sup(B')$. Moreover, when this happens, the sequence $(\beta_i\gamma_k)_{k>i}$ is strictly increasing for every i.

3. Proof of Theorem 1.1

We shall show that for every positive integer n and for every finite colouring c of the product $\Omega \times \mathbb{N}^n$ there exist infinite $X \subseteq \Omega$ and $M \subseteq \mathbb{N}$ such that the range of c on the product $X \times M^n$ has cardinality at most $((n+1)-1)! = n!$. To prove this, fix a selective ultrafilter $\mathcal{U}$ on $\mathbb{N}$. Note that for every $\xi \in \Omega$ the section $c_\xi(\xi_1, \xi_2, ..., \xi_n) = c(\xi, \xi_1, \xi_2, ..., \xi_n)$ is a finite colouring of $\mathbb{N}^n$, so an application of the selective version of Ramsey's theorem (see Chapter 7 of [3]), we can fix $M_\xi \in \mathcal{U}$ such that the colouring c_ξ on the power M_ξ^n has range F_ξ of cardinality at most $n!$. Find an uncountable set Γ of Ω and a subset F of the range of c such that $F_\xi = F$ for all $\xi \in \Gamma$. Since the directed set $[\Omega]^{<\omega}$ is not Tukey reducible to $\mathcal{U}$, there is an infinite subset X of Γ such that the set $M = \bigcap_{\xi\in X} M_\xi$ belongs to $\mathcal{U}$. Then the colouring c on the product $X \times M^n$ has range F whose cardinality is at most $n!$.

4. Proof of Theorem 1.2

Fix a positive integer n. We may assume $n \geq 3$. Let S_{n-1} be the permutation group of a set with $n-1$ elements. Given a sequence $(\xi_1, \xi_2, ..., \xi_n)$ of elements of Ω, we let $c(\xi_1, \xi_2, ..., \xi_n)$ be the sequence (i, σ_i) $(1 \leq i \leq n)$ of elements of $\bigcup_{i=1}^n \{i\} \times S_{n-1}$, where for each i, the permutation σ_i codes in the natural way the mapping $j \mapsto \xi_i\xi_j$ from the ordered set $\{1, ..., i-1, i+1, ..., n\}$ into the ordered set ω. Let $X_1, X_2, ..., X_n$ be a sequence of infinite subsets of Ω. We have to show that the range of c on $\prod_{i=1}^n X_i$ has cardinality at least $(n-1)!$ Note that we may assume that $\sup(X_i) \leq \sup(X_j)$ whenever $i < j$. We also assume that the sets X_i are refined in order to satisfy the Conclusion of the previous section. Let $m \leq n$ be minimal such that $\sup(X_m) = \sup(X_n)$.

Case 1. For all $A, B, C \in \{X_1, ..., X_n\}$ with $\sup(A) \leq \sup(B) \leq \sup(C)$ and $C = X_n$, we have that $\alpha_j\gamma_k > \beta_i\gamma_k$ for all $i < j < k$. Let ξ_n be the nth element of X_n according to the increasing enumeration. Let σ be a given permutation of the set $\{1, 2, ..., n-1\}$ viewed as bijection from this set into itself. For $1 \leq i < n$, let ξ_i be the $\sigma(i)$th element of the set X_i according to the increasing enumeration of this set. Then the nth term of the sequence $c(\xi_1, \xi_2, ..., \xi_n)$ is equal to (n, σ). Since there are $(n-1)!$ such σ's, we are done.

Case 2. There exist $A, B, C \in \{X_1, ..., X_n\}$ with $\sup(A) \leq \sup(B) \leq \sup(C)$ and $C = X_n$ such that $\alpha_j\gamma_k < \beta_i\gamma_k$ for all $i < j < k$.[2] Note that by Lemma 2.4 the set A cannot be in $\{X_m, ..., X_n\}$, so if we let ξ_n be the first element of X_n, then from the Conclusion of previous section, we know that for every $m \leq i \leq n$ the map $\xi \mapsto \xi\xi_n$ is strictly increasing on $X_i \setminus \{\min(X_i)\}$. Moreover, note that by the property (1) of the operation and the Conclusion, this map is strictly increasing on $X_i \setminus \{\min(X_i)\}$ also for $1 \leq i < m$. So it is clear that for every permutation σ of $\{1, 2, ..., n-1\}$ we can find $\xi_i \in X_i$ $(1 \leq i < n)$ such that $\sigma(i) < \sigma(j)$ is equivalent to $\xi_i\xi_n < \xi_j\xi_n$ for all $1 \leq i < j < n$. It follows that the nth term of the sequence $c(\xi_1, \xi_2, ..., \xi_n)$ is equal to (n, σ). So, as before, this shows that the range of c on $\prod_{i=1}^n X_i$ has cardinality at least $(n-1)!$ and we are done also in this case.

4.1 Remark. As we have indicated in the Introduction there is indeed difference between the case $n = 3$ and the case when n is large. In this case a given triple (ξ_1, ξ_2, ξ_3) is coloured by the permutation σ of the two-element set $I = \{1, 2, 3\} \setminus \{k\}$ where $k \in \{1, 2, 3\}$ is such that ξ_k is the maximal ordinal in the sequence (ξ_1, ξ_2, ξ_3) , or more precisely, $\sigma(i) < \sigma(j)$ is equivalent to $\xi_i\xi_k < \xi_j\xi_k$ for $i < j$ in I. Then it is clear from the argument above that every permutation of a two-element set shows up in any product of three infinite subsets of Ω.

[2] It is worth mentioning that such a triple A, B, C can be guaranteed not just with $C = X_n$ but also for any $C \in \{X_m, ..., X_{n-1}\}$ since otherwise we end up with Case 1 after reindexing the sets. Then the second to last sentence of the Conclusion ensures that the choice $B = X_n$ is compatible with every $C \in \{X_m, ..., X_{n-1}\}$.

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