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HAUSDORFF DIMENSION OF FREQUENCY SETS OF UNIVOQUE SEQUENCES

YAO-QIANG LI

ABSTRACT. We study the set Γ consisting of univoque sequences, the set Λ consisting of sequences in which the lengths of consecutive zeros and consecutive ones are bounded, and their frequency subsets Γ_a, $\underline{\Gamma}_a$, $\overline{\Gamma}_a$ and Λ_a, $\underline{\Lambda}_a$, $\overline{\Lambda}_a$ consisting of sequences respectively in Γ and Λ with frequency, lower frequency and upper frequency of zeros equal to some $a \in [0,1]$. The Hausdorff dimension of all these sets are obtained by studying the dynamical system $(\Lambda^{(m)}, \sigma)$ where σ is the shift map and $\Lambda^{(m)} = \{w \in \{0,1\}^{\mathbb{N}} : w \text{ does not contain } 0^m \text{ or } 1^m\}$ for integer $m \geq 3$, studying the Bernoulli-type measures on $\Lambda^{(m)}$ and finding out the unique equivalent σ-invariant ergodic probability measure.

1. INTRODUCTION

Let $\mathbb{N}$ be the set of positive integers $\{1, 2, 3, \cdots\}$ and define

$$\Gamma := \Big\{ w \in \{0,1\}^{\mathbb{N}} : \overline{w} < \sigma^k w < w \text{ for all } k \geq 1 \Big\}$$

where σ is the shift map on $\{0,1\}^{\mathbb{N}}$, $\overline{0} := 1$, $\overline{1} := 0$ and $\overline{w} := \overline{w_1}\,\overline{w_2} \cdots$ for all $w = w_1 w_2 \cdots \in \{0,1\}^{\mathbb{N}}$.

The set Γ is strongly related to two well known research topics, iterations of unimodal functions and unique expansions of 1 (see [4] for more details).

On the one hand, in 1985, Cosnard [8] proved that a sequence $\alpha = (\alpha_n)_{n\geq 1} \in \{0,1\}^{\mathbb{N}}$ is the kneading sequence of 1 for some unimodal function f if and only if $\tau(\alpha) \in \Gamma'$, where $\tau : \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$ is a bijection defined by $\tau(w) := (\sum_{i=1}^{n} w_i \pmod 2)_{n \geq 1}$ and

$$\Gamma' := \Big\{ w \in \{0,1\}^{\mathbb{N}} : \overline{w} \leq \sigma^k w \leq w \text{ for all } k \geq 0 \Big\}$$

is similar to Γ in the sense that $\Gamma' \setminus \{\text{periodic sequences}\} = \Gamma$. The structure of $\Gamma' \setminus \{(10)^\infty\}$ was studied in detail by Allouche [1] (see also [3]). The generalizations of Γ and Γ' (to more than two digits) were studied in [1, 5].

On the other hand, expansions of real numbers in non-integer bases were introduced by Rényi [29] in 1957 and then widely studied until now (see for examples [2, 6, 7, 22, 26, 27, 28, 30]). In 1990, Erdös, Joó and Komornik [16] proved that a sequence $\alpha = (\alpha_n)_{n \geq 1} \in \{0,1\}^{\mathbb{N}}$ is the unique expansion of 1 in some base $q \in (1,2)$ if and only if $\alpha \in \Gamma$. Thus we call Γ the set of *univoque sequences* in this paper. Note that the term "univoque sequence" is different in some other papers [9, 11, 12].

For any $a \in [0,1]$, the *frequency subsets* of Γ are defined by

$$\Gamma_a := \Big\{ w \in \Gamma : \lim_{n \to \infty} \frac{\#\{k : 1 \leq k \leq n, w_k = 0\}}{n} = a \Big\},$$

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$$\underline{\Gamma}_a := \Big\{ w \in \Gamma : \varliminf_{n\to\infty} \frac{\#\{k : 1 \le k \le n, w_k = 0\}}{n} = a \Big\},$$
$$\overline{\Gamma}_a := \Big\{ w \in \Gamma : \varlimsup_{n\to\infty} \frac{\#\{k : 1 \le k \le n, w_k = 0\}}{n} = a \Big\},$$

and the *frequency subsets* of

$\Lambda := \Big\{ w \in \{0,1\}^{\mathbb{N}}$: the lengths of consecutive 0's and consecutive 1's in w are bounded$\Big\}$

are defined by

$$\Lambda_a := \Big\{ w \in \Lambda : \lim_{n\to\infty} \frac{\#\{k : 1 \le k \le n, w_k = 0\}}{n} = a \Big\},$$
$$\underline{\Lambda}_a := \Big\{ w \in \Lambda : \varliminf_{n\to\infty} \frac{\#\{k : 1 \le k \le n, w_k = 0\}}{n} = a \Big\},$$
$$\overline{\Lambda}_a := \Big\{ w \in \Lambda : \varlimsup_{n\to\infty} \frac{\#\{k : 1 \le k \le n, w_k = 0\}}{n} = a \Big\},$$

where $\#$ denotes the cardinality. It is straightforward to check $\Gamma \subset \Lambda$. Let

$$\mathcal{U} := \Big\{ q \in (1,2) : 1 \text{ has a unique } q\text{-expansion} \Big\}$$

be the set of *univoque bases*. It is proved in [10, 21] that $\mathcal{U}$ is of full Hausdorff dimension. That is,

$$\dim_H \mathcal{U} = 1.$$

For more research on $\mathcal{U}$, we refer the reader to [13, 23, 24].

On frequency sets, there is a well known result given by Eggleston [15] saying that for any $a \in [0,1]$, the *classical Eggleston-Besicovitch set* has Hausdorff dimension

$$\dim_H \Big(\Big\{ x \in [0,1) : \lim_{n\to\infty} \frac{\#\{k : 1 \le k \le n, \varepsilon_k(x) = 0\}}{n} = a \Big\} \Big) = \frac{-a\log a - (1-a)\log(1-a)}{\log 2}, \tag{1.1}$$

where $\varepsilon_1(x)\varepsilon_2(x)\cdots\varepsilon_n(x)\cdots$ is the *greedy binary expansion* of x, and $0\log 0 := 0$.

Motivated by the above mentioned results, correspondingly, we study the set of univoque sequences Γ and the larger set Λ, study their frequency subsets Γ_a, $\underline{\Gamma}_a$, $\overline{\Gamma}_a$, Λ_a, $\underline{\Lambda}_a$, $\overline{\Lambda}_a$ and give the following theorem as the main result in this paper. Let $\dim_H(\cdot, d_2)$ denote the Hausdorff dimension in $\{0,1\}^{\mathbb{N}}$ equipped with the usual metric d_2.

Theorem 1.1. (1) *We have* $\dim_H(\Gamma, d_2) = \dim_H(\Lambda, d_2) = 1$.
(2) *For all* $a \in [0,1]$ *we have*

$$\begin{aligned} &\dim_H(\Gamma_a, d_2) = \dim_H(\underline{\Gamma}_a, d_2) = \dim_H(\overline{\Gamma}_a, d_2) \\ &= \dim_H(\Lambda_a, d_2) = \dim_H(\underline{\Lambda}_a, d_2) = \dim_H(\overline{\Lambda}_a, d_2) = \frac{-a\log a - (1-a)\log(1-a)}{\log 2}, \end{aligned}$$

where $0\log 0 := 0$.

It is known that by defining Bernoulli measures, and then calculating the lower local dimension of the measures and using Billingsley Lemma [18, Proposition 2.3], the Hausdorff dimension of classical Eggleston-Besicovitch sets mentioned above can be obtained. But this is based on the fact that only expansions in integer bases are considered in classical Eggleston-Besicovitch sets, there are no forbidden words in the symbolic space and the Bernoulli measures are invariant and ergodic with respect to the shift map. Ergodicity garuantees that classical Eggleston-Besicovitch sets have positive Bernoulli measures, which is a condition needed for applying Billingsley Lemma to get the lower bound of the Hausdorff dimension. If there are forbidden words, such as expansions in non-integer bases [25], the corresponding Bernoulli-type measures are not ergodic (actually not invariant). This makes some difficulties to be overcome. In [25], after defining Bernoulli-type measures, the authors found out the equivalent invariant ergodic measures, studied the relation between the equivalent measures and the original measures and obtained the Hausdorff dimension of Eggleston-Besicovitch (frequency) sets for a class of non-integer bases (see [25, Theorem 1.2]) by applying an avatar of the Billingsley Lemma. This paper follows a similar framework and construction, but most of the details we need to confirm are different.

For any $a \in [0,1]$ we define the *global frequency sets* in $\{0,1\}^{\mathbb{N}}$ by

$$G_a := \Big\{ w \in \{0,1\}^{\mathbb{N}} : \lim_{n\to\infty} \frac{\#\{k : 1 \le k \le n, w_k = 0\}}{n} = a \Big\},$$

$$\underline{G}_a := \Big\{ w \in \{0,1\}^{\mathbb{N}} : \varliminf_{n\to\infty} \frac{\#\{k : 1 \le k \le n, w_k = 0\}}{n} = a \Big\},$$

$$\overline{G}_a := \Big\{ w \in \{0,1\}^{\mathbb{N}} : \varlimsup_{n\to\infty} \frac{\#\{k : 1 \le k \le n, w_k = 0\}}{n} = a \Big\},$$

for any integer $m \ge 3$ we define

$$\Lambda^{(m)} := \Big\{ w \in \{0,1\}^{\mathbb{N}} : w \text{ does not contain } 0^m \text{ or } 1^m \Big\},$$

and we let

$$\Lambda_a^{(m)} := \Lambda^{(m)} \cap G_a.$$

Here we give an outline for the proof of Theorem 1.1 (2) to explain how the concepts in this paper interact. Following the simple argument at the beginning of the *Proof of Theorem 1.1* in Section 5, we know that it suffices to consider the lower bound of $\dim_H(\Gamma_a, d_2)$. Since (5.2) says that $\dim_H(\Gamma_a, d_2) \ge \dim_H(\Lambda_a^{(m)}, d_2)$ for any integer $m \ge 3$, we only need to find a good lower bound for $\dim_H(\Lambda_a^{(m)}, d_2)$. Hence the main we need to prove is Lemma 5.3. By the Billingsley Lemma in metric space (Proposition 2.6), this can be done by constructing a suitable measure on $(\Lambda^{(m)}, d_2)$ such that $\Lambda_a^{(m)}$ has positive measure. Thus we define the Bernoulli-type measure μ_p in Section 4. To guarantee that $\Lambda_a^{(m)}$ has positive measure, we want ergodicity (see (2) ② in the *Proof of Lemma 5.3*). Hence we find out the unique σ-invariant ergodic measure λ_p equivalent to μ_p in Section 4, and calculate its value on the cylinder $[0]$ in Lemma 5.2, which is in fact a relation between μ_p and λ_p. Finally we apply Billingsley Lemma to obtain Lemma 5.3.

This paper is organized as follows. In Section 2, we give some basic notation and preliminaries on dynamical systems and measure theory. In Section 3, we study related digit occurrence parameters and their properties which will be used later. In Section 4 we study Bernoulli-type measures, and finally we prove our main result in Section 5.

2. NOTATION AND PRELIMINARIES

Let $\{0,1\}^* := \bigcup_{n=1}^{\infty} \{0,1\}^n$ and $\{0,1\}^{\mathbb{N}}$ be the sets of finite words and infinite sequences respectively on two digits $\{0,1\}$. For any integer $m \ge 3$, recall

$$\Lambda^{(m)} = \Big\{ w \in \{0,1\}^{\mathbb{N}} : w \text{ does not contain } 0^m \text{ or } 1^m \Big\},$$

and define
$$\Lambda^{(m),*} := \Big\{ w \in \{0,1\}^* : w \text{ does not contain } 0^m \text{ or } 1^m \Big\}$$
and
$$\Lambda^{(m),n} := \Big\{ w \in \{0,1\}^n : w \text{ does not contain } 0^m \text{ or } 1^m \Big\}$$
where $n \in \mathbb{N}$. For a finite word $w \in \{0,1\}^*$, we use $|w|$, $|w|_0$ and $|w|_1$ to denote its length, the number of 0's in w and the number of 1's in w respectively. Besides, $w|_k := w_1 w_2 \cdots w_k$ denotes the prefix of w with length k for $w \in \{0,1\}^{\mathbb{N}}$ or $w \in \{0,1\}^n$ where $n \geq k$.

Let $\sigma : \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$ be the *shift map* defined by
$$\sigma(w_1 w_2 \cdots) = w_2 w_3 \cdots \quad \text{for } w \in \{0,1\}^{\mathbb{N}}$$
and d_2 be the *usual metric* on $\{0,1\}^{\mathbb{N}}$ defined by
$$d_2(w,v) := 2^{-\inf\{k \geq 0 : w_{k+1} \neq v_{k+1}\}} \quad \text{for } w, v \in \{0,1\}^{\mathbb{N}},$$
where $2^{-\infty} = 0$. Then σ is continuous on $(\{0,1\}^{\mathbb{N}}, d_2)$. By $\sigma(\Lambda^{(m)}) = \Lambda^{(m)}$, we know that $(\Lambda^{(m)}, \sigma)$ is a dynamical system. It is straightforward to check that the *natural projection map* $\pi_2 : \{0,1\}^{\mathbb{N}} \to [0,1]$, defined by
$$\pi_2(w) := \sum_{n=1}^{\infty} \frac{w_n}{2^n} \quad \text{for } w \in \{0,1\}^{\mathbb{N}},$$
is surjective and continuous. Besides, we need the following concepts and notation.

Definition 2.1 (Cylinder)**.** Let $m \geq 3$ be an integer and $w \in \Lambda^{(m),*}$. We call
$$[w] := \Big\{ v \in \Lambda^{(m)} : v \text{ begins with } w \Big\}$$
the *cylinder* in $\Lambda^{(m)}$ generated by w.

Definition 2.2 (Absolute continuity and equivalence)**.** Let μ and ν be measures on a measurable space $(X, \mathcal{F})$. We say that μ is *absolutely continuous* with respect to ν and denote it by $\mu \ll \nu$ if, for any $A \in \mathcal{F}$, $\nu(A) = 0$ implies $\mu(A) = 0$. Moreover, if $\mu \ll \nu$ and $\nu \ll \mu$ we say that μ and ν are *equivalent* and denote this property by $\mu \sim \nu$.

Definition 2.3. Let $\mathcal{C}$ be a family of certain subsets of a set X.

(1) $\mathcal{C}$ is called a *monotone class* on X if
  ① $\{A_n\}_{n\geq 1} \subset \mathcal{C}$ and $A_1 \subset A_2 \subset \cdots \Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{C}$;
  ② $\{A_n\}_{n\geq 1} \subset \mathcal{C}$ and $A_1 \supset A_2 \supset \cdots \Rightarrow \bigcap_{n=1}^{\infty} A_n \in \mathcal{C}$.

(2) $\mathcal{C}$ is called a *semi-algebra* on X if
  ① $\varnothing \in \mathcal{C}$;
  ② $A, B \in \mathcal{C} \Rightarrow A \cap B \in \mathcal{C}$;
  ③ $A \in \mathcal{C} \Rightarrow A^c \in \mathcal{C}_{\Sigma f}$

  where $A^c := X \setminus A$ and $\mathcal{C}_{\Sigma f} := \Big\{ \bigcup_{i=1}^{n} C_i : C_1, \cdots, C_n \in \mathcal{C} \text{ are disjoint}, n \in \mathbb{N} \Big\}$.
  (The subscript $_{\Sigma f}$ means finite disjoint union.)

(3) $\mathcal{C}$ is called an *algebra* on X if
  ① $\varnothing, X \in \mathcal{C}$;
  ② $A \in \mathcal{C} \Rightarrow A^c \in \mathcal{C}$;
  ③ $A, B \in \mathcal{C} \Rightarrow A \cap B \in \mathcal{C}$.

(4) $\mathcal{C}$ is called a *sigma-algebra* on X if
  ① $\varnothing, X \in \mathcal{C}$;
  ② $A \in \mathcal{C} \Rightarrow A^c \in \mathcal{C}$;

③ $A_1, A_2, A_3 \dots \in \mathcal{C} \Rightarrow \bigcap_{n=1}^{\infty} A_n \in \mathcal{C}$.

The following useful approximation lemma follows from [31, Theorem 0.1 and 0.7].

Lemma 2.4. *Let $(X, \mathcal{B}, \mu)$ be a probability space, $\mathcal{C}$ be a semi-algebra which generates the sigma-algebra $\mathcal{B}$ and $\mathcal{A}$ be the algebra generated by $\mathcal{C}$. Then*

(1) $\mathcal{A} = \mathcal{C}_{\Sigma f} := \Big\{ \bigcup_{i=1}^{n} C_i : C_1, \cdots, C_n \in \mathcal{C}$ are disjoint, $n \in \mathbb{N} \Big\}$;
(2) for each $B \in \mathcal{B}$ and each $\varepsilon > 0$, there is some $A \in \mathcal{A}$ with $\mu(A \triangle B) < \varepsilon$.

In order to extend some properties from a small family to a larger one in some proofs in Section 4, we recall the following well known Monotone Class Theorem (see for example [20, Page 66]).

Theorem 2.5 (Monotone Class Theorem). *Let $\mathcal{A}$ be an algebra. Then the smallest monotone class containing $\mathcal{A}$ is precisely the smallest sigma-algebra containing $\mathcal{A}$.*

Let $B(x, r)$ denote the closed ball centered on x with radius r. The following version of the Billingsley Lemma in metric space follows in the same way as the classical one in Euclidean space.

Proposition 2.6 ([17, 18]). *Let (X, d) be a metric space, $E \subset X$ be a Borel set, μ be a finite Borel measure on X and $s \geq 0$. If*

$$\mu(E) > 0 \quad \text{and} \quad \varliminf_{r \to 0} \frac{\log \mu(B(x, r))}{\log r} \geq s \text{ for all } x \in E,$$

then the Hausdorff dimension of E in (X, d) is no less than s.

3. Digit occurrence parameters

The digit occurrence parameters and their properties studied in this section will be used in Sections 4 and 5.

Definition 3.1 (Digit occurrence parameters). Let $m \geq 3$ be an integer. For any $w \in \Lambda^{(m),*}$, define

$$\mathcal{N}_0^{(m)}(w) := \Big\{ k : 1 \leq k \leq |w|, w_k = 0 \text{ and } w_1 \dots w_{k-1}1 \in \Lambda^{(m),*} \Big\},$$
$$\mathcal{N}_1^{(m)}(w) := \Big\{ k : 1 \leq k \leq |w|, w_k = 1 \text{ and } w_1 \dots w_{k-1}0 \in \Lambda^{(m),*} \Big\},$$

and let

$$N_0^{(m)}(w) := \#\mathcal{N}_0^{(m)}(w) \quad \text{and} \quad N_1^{(m)}(w) := \#\mathcal{N}_1^{(m)}(w)$$

where $\#\mathcal{N}$ denotes the cardinality of the set $\mathcal{N}$.

Proposition 3.2. *Let $m \geq 3$ be an integer and $w, v \in \Lambda^{(m),*}$ such that $wv \in \Lambda^{(m),*}$. Then*

(1) $N_0^{(m)}(w) + N_0^{(m)}(v) - 1 \leq N_0^{(m)}(wv) \leq N_0^{(m)}(w) + N_0^{(m)}(v)$;
(2) $N_1^{(m)}(w) + N_1^{(m)}(v) - 1 \leq N_1^{(m)}(wv) \leq N_1^{(m)}(w) + N_1^{(m)}(v)$.

Proof. Let $a = |w|$ and $b = |v|$.
(1) ① Prove $N_0^{(m)}(wv) \leq N_0^{(m)}(w) + N_0^{(m)}(v)$.
It suffices to prove $\mathcal{N}_0^{(m)}(wv) \subset \mathcal{N}_0^{(m)}(w) \cup (\mathcal{N}_0^{(m)}(v) + a)$, where $\mathcal{N}_0^{(m)}(v) + a := \{j + a : j \in \mathcal{N}_0^{(m)}(v)\}$. Let $k \in \mathcal{N}_0^{(m)}(wv)$.

i) If $1 \leq k \leq a$, then $w_k = 0$, $w_1 \cdots w_{k-1}1 \in \Lambda^{(m),*}$ and we get $k \in \mathcal{N}_0^{(m)}(w)$.

ii) If $a+1 \le k \le a+b$, then $v_{k-a} = 0$ and $w_1 \cdots w_a v_1 \cdots v_{k-a-1}1 \in \Lambda^{(m),*}$. It follows from $v_1 \cdots v_{k-a-1}1 \in \Lambda^{(m),*}$ that $k-a \in \mathcal{N}_0^{(m)}(v)$ and $k \in \mathcal{N}_0^{(m)}(v) + a$.

② Prove $N_0^{(m)}(w) + N_0^{(m)}(v) \le N_0^{(m)}(wv) + 1$.
When $v = 1^b$, we get $N_0^{(m)}(v) = 0$ and then the conclusion follows immediately from $N_0^{(m)}(w) \le N_0^{(m)}(wv)$. Thus it suffices to consider $v \ne 1^b$ in the following. Let $s \in \{1, \cdots, b\}$ be the smallest such that $v_1 = \cdots = v_{s-1} = 1$ and $v_s = 0$. In order to get the conclusion, it suffices to show $\mathcal{N}_0^{(m)}(w) \cup (a + \mathcal{N}_0^{(m)}(v)) \subset \mathcal{N}_0^{(m)}(wv) \cup \{a+s\}$. Since $\mathcal{N}_0^{(m)}(w) \subset \mathcal{N}_0^{(m)}(wv)$, we only need to prove $(a + \mathcal{N}_0^{(m)}(v)) \subset \mathcal{N}_0^{(m)}(wv) \cup \{a+s\}$. Let $k \in \mathcal{N}_0^{(m)}(v) \setminus \{s\}$. It suffices to check $a+k \in \mathcal{N}_0^{(m)}(wv)$. By $v_k = 0$, we only need to prove $w_1 \cdots w_a v_1 \cdots v_{k-1}1 \in \Lambda^{(m),*}$. (By contradiction) Assume $w_1 \cdots w_a v_1 \cdots v_{k-1}1 \notin \Lambda^{(m),*}$. Then $w_1 \cdots w_a v_1 \cdots v_{k-1}1$ contains 0^m or 1^m.

i) If $w_1 \cdots w_a v_1 \cdots v_{k-1}1$ contains 0^m, then $w_1 \cdots w_a v_1 \cdots v_{k-1}$ contains 0^m. This contradicts $wv \in \Lambda^{(m),*}$.

ii) If $w_1 \cdots w_a v_1 \cdots v_{k-1}1$ contains 1^m, by $k \ge s+1$, we know that
$$w_1 \cdots w_a v_1 \cdots v_{s-1} 0 v_{s+1} \cdots v_{k-1} 1$$
contains 1^m. Thus $w_1 \cdots w_a v_1 \cdots v_{s-1}$ contains 1^m or $v_{s+1} \cdots v_{k-1}1$ contains 1^m. But $w_1 \cdots w_a v_1 \cdots v_{s-1}$ contains 1^m will contradict $wv \in \Lambda^{(m),*}$, and $v_{s+1} \cdots v_{k-1}1$ contains 1^m will imply $v_1 \cdots v_{k-1}1$ contains 1^m which contradicts $k \in \mathcal{N}_0^{(m)}(v)$.

(2) follows in the same way as (1). □

Proposition 3.3. *Let $m \ge 3$ be an integer and $w \in \Lambda^{(m),*}$. Then*

(1) $m \cdot |w|_0 \le (m-1)N_0^{(m)}(w) + |w|$;
(2) $m \cdot |w|_1 \le (m-1)N_1^{(m)}(w) + |w|$.

Proof. (1) Let $n = |w|$. If $n \le m-1$, the conclusion follows immediately from $N_0^{(m)}(w) = |w|_0$. In the following, we assume $n \ge m$. Recall
$$\mathcal{N}_0^{(m)}(w) = \Big\{k : 1 \le k \le n, w_k = 0, w_1 \cdots w_{k-1}1 \in \Lambda^{(m),*}\Big\} \quad \text{and} \quad N_0^{(m)}(w) = \#\mathcal{N}_0^{(m)}(w).$$
We define
$$\mathcal{N}_{1^{m-1}0}^{(m)}(w) := \Big\{k : m \le k \le n, w_{k-m+1} \cdots w_{k-1} w_k = 1^{m-1}0\Big\} \quad \text{and} \quad N_{1^{m-1}0}^{(m)} := \#\mathcal{N}_{1^{m-1}0}^{(m)}(w).$$

① Prove $\{k : 1 \le k \le n, w_k = 0\} = \mathcal{N}_0^{(m)}(w) \cup \mathcal{N}_{1^{m-1}0}^{(m)}(w)$.
$\boxed{\supset}$ Obvious.
$\boxed{\subset}$ Let $k \in \{1, \cdots, n\}$ such that $w_k = 0$. If $k \notin \mathcal{N}_0^{(m)}(w)$, then $k \ge m$ and $w_1 \cdots w_{k-1}1 \notin \Lambda^{(m),*}$. By $w_1 \cdots w_{k-1} \in \Lambda^{(m),*}$, we get $w_{k-m+1} \cdots w_{k-1} = 1^{m-1}$. This implies $k \in \mathcal{N}_{1^{m-1}0}^{(m)}(w)$.

② Prove $\mathcal{N}_0^{(m)}(w) \cap \mathcal{N}_{1^{m-1}0}^{(m)}(w) = \varnothing$.
(By contradiction) Assume that there exists $k \in \mathcal{N}_0^{(m)}(w) \cap \mathcal{N}_{1^{m-1}0}^{(m)}(w)$. Then $k \ge m$, $w_{k-m+1} \cdots w_{k-1} = 1^{m-1}$ and $w_1 \cdots w_{k-1}1 \in \Lambda^{(m),*}$. These imply $w_1 \cdots w_{k-m}1^m \in \Lambda^{(m),*}$, which contradicts the definition of $\Lambda^{(m),*}$.

Combining ① and ②, we get $|w|_0 = N_0^{(m)}(w) + N_{1^{m-1}0}^{(m)}(w)$. It follows from $(m-1)N_{1^{m-1}0}^{(m)}(w) \le |w|_1 = |w| - |w|_0$ that $(m-1)(|w|_0 - N_0^{(m)}(w)) \le |w| - |w|_0$, i.e., $m \cdot |w|_0 \le (m-1)N_0^{(m)}(w) + |w|$.
(2) follows in the same way as (1). □

4. Bernoulli-type measures on $\Lambda^{(m)}$

Let $m \geq 3$ be an integer, $\mathcal{B}(\Lambda^{(m)})$ be the Borel sigma-algebra on $\Lambda^{(m)}$ (equipped with the usual metric d_2) and $p \in (0,1)$. We define the $(p, 1-p)$ *Bernoulli-type measure* μ_p on $(\Lambda^{(m)}, \mathcal{B}(\Lambda^{(m)}))$ as follows:

I. Let

$$\mu_p(\varnothing) = 0, \quad \mu_p(\Lambda^{(m)}) = 1, \quad \mu_p[0] = p, \quad \text{and} \quad \mu_p[1] = 1 - p.$$

II. Suppose μ_p has been defined for all cylinders of order $n \in \mathbb{N}$. For any $w \in \Lambda^{(m),n}$, if $w0, w1 \in \Lambda^{(m),n+1}$, we define

$$\mu_p[w0] := p\mu_p[w] \quad \text{and} \quad \mu_p[w1] := (1-p)\mu_p[w];$$

if $w0 \in \Lambda^{(m),n+1}$ but $w1 \notin \Lambda^{(m),n+1}$, then $[w1] = \varnothing$, $[w0] = [w]$ and naturally we have

$$\mu_p[w0] = \mu_p[w];$$

if $w1 \in \Lambda^{(m),n+1}$ but $w0 \notin \Lambda^{(m),n+1}$, then $[w0] = \varnothing$, $[w1] = [w]$ and naturally we have

$$\mu_p[w1] = \mu_p[w].$$

III. By Carathéodory's measure extension theorem, we uniquely extend μ_p from its definition on the family of cylinders to become a measure on $\mathcal{B}(\Lambda^{(m)})$.

Remark 4.1. By the definition of μ_p, we have

$$\mu_p[w] = p^{N_0^{(m)}(w)}(1-p)^{N_1^{(m)}(w)} \quad \text{for all } w \in \Lambda^{(m),*}.$$

Note that μ_p is not σ-invariant. In fact, $\mu_p[0^{m-2}1] = p^{m-2}(1-p)$ but

$\mu_p(\sigma^{-1}[0^{m-2}1]) = \mu_p[0^{m-1}1] + \mu_p[10^{m-2}1] = p^{m-1} + p^{m-2}(1-p)^2 \neq p^{m-2}(1-p)$ for all $p \in (0,1)$.

The main result in this section is the following.

Theorem 4.2. *Let $m \geq 3$ be an integer and $p \in (0,1)$. Then there exists a unique σ-invariant ergodic probability measure λ_p on $(\Lambda^{(m)}, \mathcal{B}(\Lambda^{(m)}))$ equivalent to μ_p, where λ_p is defined by*

$$\lambda_p(B) := \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \sigma^k \mu_p(B) \quad \text{for } B \in \mathcal{B}(\Lambda^{(m)}).$$

The proof is based on the following lemmas.

Lemma 4.3. *Let $m \geq 3$ be an integer, $p \in (0,1)$ and $w, v \in \Lambda^{(m),*}$ such that $wv \in \Lambda^{(m),*}$. Then*

$$\mu_p[w]\mu_p[v] \leq \mu_p[wv] \leq p^{-1}(1-p)^{-1}\mu_p[w]\mu_p[v].$$

Proof. It follows from Remark 4.1 and Proposition 3.2. □

Lemma 4.4. *Let $m \geq 3$ be an integer and $p \in (0,1)$. Then there exists a constant $c > 1$ such that*

$$c^{-1}\mu_p(B) \leq \sigma^k \mu_p(B) \leq c\mu_p(B)$$

for all $k \in \mathbb{N}$ and $B \in \mathcal{B}(\Lambda^{(m)})$.

Proof. Let $c = p^{-2}(1-p)^{-2} > 1$.
(1) Prove $c^{-1}\mu_p[w] \le \sigma^k \mu_p[w] \le c\mu_p[w]$ for any $k \in \mathbb{N}$ and $w \in \Lambda^{(m),*}$.
Fix $w \in \Lambda^{(m),*}$ and $k \in \mathbb{N}$. Note that

$$\sigma^{-k}[w] = \bigcup_{u_1\cdots u_k w \in \Lambda^{(m),*}} [u_1 \cdots u_k w]$$

is a disjoint union.
① Estimate the upper bound of $\sigma^k \mu_p[w]$:

$$\begin{aligned}
\mu_p \sigma^{-k}[w] &= \sum_{u_1\cdots u_k w \in \Lambda^{(m),*}} \mu_p[u_1 \cdots u_k w] \\
&\overset{(\star)}{\le} \sum_{u_1\cdots u_k w \in \Lambda^{(m),*}} p^{-1}(1-p)^{-1}\mu_p[u_1\cdots u_k]\mu_p[w] \\
&\le p^{-1}(1-p)^{-1} \sum_{u_1\cdots u_k \in \Lambda^{(m),*}} \mu_p[u_1\cdots u_k]\mu_p[w] \\
&= p^{-1}(1-p)^{-1}\mu_p[w] \\
&\le c\mu_p[w]
\end{aligned}$$

where $(\star)$ follows from Lemma 4.3.
② Estimate the lower bound of $\sigma^k \mu_p[w]$:

i) Prove $\mu_p\sigma^{-k}[0] \ge p^2(1-p)$ and $\mu_p\sigma^{-k}[1] \ge p(1-p)^2$. In fact, when $k = 1$, the conclusion is obvious. When $k \ge 2$, we have

$$\begin{aligned}
\mu_p \sigma^{-k}[0] &= \sum_{u_1\cdots u_k 0 \in \Lambda^{(m),*}} \mu_p[u_1 \cdots u_k 0] \\
&\ge \sum_{u_1\cdots u_{k-1}\overline{u}_{k-1} 0 \in \Lambda^{(m),*}} \mu_p[u_1 \cdots u_{k-1}\overline{u}_{k-1}0] \\
&\overset{(\star)}{=} \sum_{u_1\cdots u_{k-1} \in \Lambda^{(m),*}} \mu_p[u_1 \cdots u_{k-1}\overline{u}_{k-1}0] \\
&\overset{(\star\star)}{\ge} \mu_p[0] \sum_{u_1\cdots u_{k-1} \in \Lambda^{(m),*}} \mu_p[u_1 \cdots u_{k-1}]\mu_p[\overline{u}_{k-1}] \\
&\ge p \sum_{u_1\cdots u_{k-1} \in \Lambda^{(m),*}} \mu_p[u_1 \cdots u_{k-1}] \cdot p(1-p) \\
&= p^2(1-p),
\end{aligned}$$

where $(\star)$ follows from

$$u_1 \cdots u_{k-1}\overline{u}_{k-1}0 \in \Lambda^{(m),*} \Leftrightarrow u_1 \cdots u_{k-1} \in \Lambda^{(m),*}$$

and $(\star\star)$ follows from Lemma 4.3. In the same way, we can get $\mu_p\sigma^{-k}[1] \ge p(1-p)^2$.

ii) Prove $\mu_p\sigma^{-k}[w] \geq c^{-1}\mu_p[w]$. In fact, when $w_1 = 0$, we have

$$\begin{aligned}
\mu_p\sigma^{-k}[w] &= \sum_{u_1\cdots u_k w\in\Lambda^{(m),*}} \mu_p[u_1\cdots u_k w] \\
&\geq \sum_{u_1\cdots u_{k-1}1w\in\Lambda^{(m),*}} \mu_p[u_1\cdots u_{k-1}1w] \\
&\overset{(\star)}{=} \sum_{u_1\cdots u_{k-1}1\in\Lambda^{(m),*}} \mu_p[u_1\cdots u_{k-1}1w] \\
&\overset{(\star\star)}{\geq} \sum_{u_1\cdots u_{k-1}1\in\Lambda^{(m),*}} \mu_p[u_1\cdots u_{k-1}1]\mu_p[w] \\
&= \mu_p\sigma^{-(k-1)}[1]\mu_p[w] \\
&\overset{(\star\star\star)}{\geq} p(1-p)^2\mu_p[w].
\end{aligned}$$

where $(\star)$ follows from $w_1 = 0$ and $w \in \Lambda^{(m),*}$, $(\star\star)$ follows from Lemma 4.3 and $(\star\star\star)$ follows from i). When $w_1 = 1$, in the same way, we can get $\mu_p\sigma^{-k}[w] \geq p^2(1-p)\mu_p[w]$.

(2) Prove $c^{-1}\mu_p(B) \leq \sigma^k\mu_p(B) \leq c\mu_p(B)$ for all $k \in \mathbb{N}$ and $B \in \mathcal{B}(\Lambda^{(m)})$. Let

$$\mathcal{C} := \Big\{[w] : w \in \Lambda^{(m),*}\Big\} \cup \Big\{\varnothing\Big\},$$

$$\mathcal{C}_{\Sigma f} := \Big\{\bigcup_{i=1}^n C_i : C_1, \cdots, C_n \in \mathcal{C} \text{ are disjoint, } n \in \mathbb{N}\Big\}$$

and

$$\mathcal{G} := \Big\{B \in \mathcal{B}(\Lambda^{(m)}) : c^{-1}\mu_p(B) \leq \sigma^k\mu_p(B) \leq c\mu_p(B) \text{ for all } k \in \mathbb{N}\Big\}.$$

Then $\mathcal{C}$ is a semi-algebra on $\Lambda^{(m)}$, $\mathcal{C}_{\Sigma f}$ is the algebra generated by $\mathcal{C}$ (by Lemma 2.4 (1)) and $\mathcal{G}$ is a monotone class. Since in (1) we have already proved $\mathcal{C} \subset \mathcal{G}$, it follows that $\mathcal{C}_{\Sigma f} \subset \mathcal{G} \subset \mathcal{B}(\Lambda^{(m)})$. Noting that $\mathcal{B}(\Lambda^{(m)})$ is the smallest sigma-algebra containing $\mathcal{C}_{\Sigma f}$, it follows from the Monotone Class Theorem (Theorem 2.5) that $\mathcal{G} = \mathcal{B}(\Lambda^{(m)})$. ☐

Lemma 4.5 ([14]). *Let $(X, \mathcal{B}, \mu)$ be a probability space and T be a measurable transformation on X satisfying $\mu(T^{-1}B) = 0$ whenever $B \in \mathcal{B}$ with $\mu(B) = 0$. If there exists a constant $M > 0$ such that for any $B \in \mathcal{B}$ and any $n \in \mathbb{N}$,*

$$\frac{1}{n}\sum_{k=0}^{n-1} \mu(T^{-k}B) \leq M\mu(B),$$

then for any real integrable function f on X, the limit

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} f(T^k x)$$

exists for μ-almost every $x \in X$.

Lemma 4.6. *Let $m \geq 3$ be an integer and $p \in (0,1)$. For any $B \in \mathcal{B}(\Lambda^{(m)})$ satisfying $\sigma^{-1}B = B$, we have $\mu_p(B) = 0$ or 1.*

Proof. Let $\alpha = p^2(1-p)^2 > 0$.

(1) Let $w \in \Lambda^{(m),*}$ and $n = |w|$. For any $A \in \mathcal{B}(\Lambda^{(m)})$, we prove $\alpha\mu_p[w]\mu_p(A) \leq \mu_p([w] \cap \sigma^{-(n+2)}A)$.

① For any $v \in \Lambda^{(m),*}$, prove $\alpha\mu_p[w]\mu_p[v] \leq \mu_p([w] \cap \sigma^{-(n+2)}[v])$.
In fact, it follows from $w\overline{w}_n\overline{v}_1 v \in \Lambda^{(m),*}$ and $[w] \cap \sigma^{-(n+2)}[v] \supset [w\overline{w}_n\overline{v}_1 v]$ that

$$\mu_p([w] \cap \sigma^{-(n+2)}[v]) \geq \mu_p[w\overline{w}_n\overline{v}_1 v] \overset{(\star)}{\geq} \mu_p[w]\mu_p[\overline{w}_n]\mu_p[\overline{v}_1]\mu_p[v] \geq (p(1-p))^2\mu_p[w]\mu_p[v]$$

where $(\star)$ follows from Lemma 4.3.

② Let

$$\mathcal{C} := \Big\{[v] : v \in \Lambda^{(m),*}\Big\} \cup \Big\{\varnothing\Big\}$$

and

$$\mathcal{G}_w := \Big\{A \in \mathcal{B}(\Lambda^{(m)}) : \alpha\mu_p[w]\mu_p(A) \leq \mu_p([w] \cap \sigma^{-(n+2)}A)\Big\}.$$

Then $\mathcal{G}_w$ is a monotone class. Since in ① we have already proved $\mathcal{C} \subset \mathcal{G}_w$, in the same way as the end of the proof of Lemma 4.4, we get $\mathcal{G}_w = \mathcal{B}(\Lambda^{(m)})$.

(2) We use B^c to denote the complement of B in $\Lambda^{(m)}$. For any $\varepsilon > 0$, by Lemma 2.4, there exist finitely many disjoint cylinders $\big\{[w^{(i)}]\big\} \subset \mathcal{C}$ such that $\mu_p(B^c \Delta E_\varepsilon) < \varepsilon$ where $E_\varepsilon = \bigcup_i [w^{(i)}]$.

(3) Let $B \in \mathcal{B}(\Lambda^{(m)})$ with $\sigma^{-1}B = B$. For any $w \in \Lambda^{(m),*}$, by $B = \sigma^{-(|w|+2)}B$ and (1) we get

$$\alpha\mu_p(B)\mu_p[w] \leq \mu(\sigma^{-(|w|+2)}B \cap [w]) = \mu_p(B \cap [w]).$$

Thus

$$\alpha\mu_p(B)\mu_p(E_\varepsilon) = \sum_i \alpha\mu_p(B)\mu_p[w^{(i)}] \leq \sum_i \mu_p(B \cap [w^{(i)}]) = \mu_p(B \cap \bigcup_i [w^{(i)}]) = \mu_p(B \cap E_\varepsilon).$$

Let $a = \mu_p((B \cup E_\varepsilon)^c)$, $b = \mu_p(B \cap E_\varepsilon)$, $c = \mu_p(B \setminus E_\varepsilon)$ and $d = \mu_p(E_\varepsilon \setminus B)$. Then we already have

$$\alpha(b+c)(b+d) \leq b, \quad a+b < \varepsilon \text{ (by } \mu_p(B^c \Delta E_\varepsilon) < \varepsilon) \quad \text{and} \quad a+b+c+d = 1.$$

It follows from

$$\alpha(b+c)(a+d-\varepsilon) \leq \alpha(b+c)(b+d) \leq b < \varepsilon$$

that

$$(b+c)(a+d) < (\frac{1}{\alpha} + b + c)\varepsilon \leq (\frac{1}{\alpha} + 1)\varepsilon.$$

This implies $\mu_p(B)\mu_p(B^c) \leq (\frac{1}{\alpha} + 1)\varepsilon$ for any $\varepsilon > 0$. Therefore $\mu_p(B)(1 - \mu_p(B)) = 0$ and then $\mu_p(B) = 0$ or 1. □

Proof of Theorem 4.2. (1) For any $n \in \mathbb{N}$ and $B \in \mathcal{B}(\Lambda^{(m)})$, define

$$\lambda_p^n(B) := \frac{1}{n}\sum_{k=0}^{n-1} \mu_p(\sigma^{-k}B).$$

Then λ_p^n is a probability measure on $(\Lambda^{(m)}, \mathcal{B}(\Lambda^{(m)}))$. By Lemma 4.4, there exists $c > 0$ such that

$$c^{-1}\mu_p(B) \leq \lambda_p^n(B) \leq c\mu_p(B) \quad \text{for any } B \in \mathcal{B}(\Lambda^{(m)}) \text{ and } n \in \mathbb{N}. \tag{4.1}$$

(2) For any $B \in \mathcal{B}(\Lambda^{(m)})$, prove that $\lim_{n\to\infty} \lambda_p^n(B)$ exists.
Let $\mathbb{1}_B : \Lambda^{(m)} \to \{0, 1\}$ be defined by

$$\mathbb{1}_B(w) := \begin{cases} 1 & \text{if } w \in B \\ 0 & \text{if } w \notin B \end{cases}$$

for any $w \in \Lambda^{(m)}$. Then

$$\begin{aligned}
\lim_{n\to\infty} \lambda_p^n(B) &= \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \int \mathbb{1}_{\sigma^{-k}B} \, d\mu_p \\
&= \lim_{n\to\infty} \int \frac{1}{n} \sum_{k=0}^{n-1} \mathbb{1}_B(\sigma^k w) \, d\mu_p(w) \\
&= \int \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \mathbb{1}_B(\sigma^k w) \, d\mu_p(w)
\end{aligned}$$

where the last equality is an application of the dominated convergence theorem, in which the μ_p-a.e. (almost every) existence of $\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \mathbb{1}_B(\sigma^k w)$ follows from Lemma 4.5, Lemma 4.4 and (4.1).
(3) For any $B \in \mathcal{B}(\Lambda^{(m)})$, define

$$\lambda_p(B) := \lim_{n\to\infty} \lambda_p^n(B).$$

By the well known Vitali-Hahn-Saks Theorem, λ_p is a probability measure on $(\Lambda^{(m)}, \mathcal{B}(\Lambda^{(m)}))$.
(4) The fact $\lambda_p \sim \mu_p$ on $\mathcal{B}(\Lambda^{(m)})$ follows from (4.1) and the definition of λ_p.
(5) Prove that λ_p is σ-invariant.
In fact, for any $B \in \mathcal{B}(\Lambda^{(m)})$ and $n \in \mathbb{N}$, we have

$$\lambda_p^n(\sigma^{-1}B) = \frac{1}{n} \sum_{k=1}^{n} \mu_p(\sigma^{-k}B) = \frac{1}{n} \sum_{k=0}^{n} \mu_p(\sigma^{-k}B) - \frac{\mu_p(B)}{n} = \frac{n+1}{n} \lambda_p^{n+1}(B) - \frac{\mu_p(B)}{n}.$$

Let $n \to \infty$, we get $\lambda_p(\sigma^{-1}B) = \lambda_p(B)$.
(6) Prove that $(\Lambda^{(m)}, \mathcal{B}(\Lambda^{(m)}), \lambda_p, \sigma)$ is ergodic.
In fact, for any $B \in \mathcal{B}(\Lambda^{(m)})$ satisfying $\sigma^{-1}B = B$, by Lemma 4.6 we get $\mu_p(B) = 0$ or 1, which implies $\lambda_p(B) = 0$ or 1 since $\lambda_p \sim \mu_p$.
(7) Prove that such λ_p is unique on $\mathcal{B}(\Lambda^{(m)})$.
Let λ_p' be a σ-invariant ergodic probability measure on $(\Lambda^{(m)}, \mathcal{B}(\Lambda^{(m)}))$ equivalent to μ_p. Then for any $B \in \mathcal{B}(\Lambda^{(m)})$, by the Birkhoff Ergodic Theorem, we get

$$\lambda_p'(B) = \int \mathbb{1}_B \, d\lambda_p' = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \mathbb{1}_B(\sigma^k w) \quad \text{for } \lambda_p'\text{-a.e. } w \in \Lambda^{(m)}$$

and

$$\lambda_p(B) = \int \mathbb{1}_B \, d\lambda_p = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \mathbb{1}_B(\sigma^k w) \quad \text{for } \lambda_p\text{-a.e. } w \in \Lambda^{(m)}.$$

Since $\lambda_p' \sim \mu_p \sim \lambda_p$, there exists $w \in \Lambda^{(m)}$ such that $\lambda_p'(B) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \mathbb{1}_B(\sigma^k w) = \lambda_p(B)$. It means that λ_p' and λ_p are the same on $\mathcal{B}(\Lambda^{(m)})$. $\square$

5. PROOF OF THE MAIN RESULT

For any $a \in [0,1]$, recall the definition of the global frequency sets $G_a, \underline{G}_a$ and $\overline{G}_a$ from the introduction. The following lemma follows immediately from (1.1), [25, Theorem 6.1] and the invariance of Hausdorff dimension under the projection π_2. (See also [19, Theorem 3.3].)

Lemma 5.1. *For any $a \in [0,1]$, we have*

$$\dim_H(G_a, d_2) = \dim_H(\underline{G}_a, d_2) = \dim_H(\overline{G}_a, d_2) = \frac{-a\log a - (1-a)\log(1-a)}{\log 2}.$$

To prove Theorem 1.1, we also need the next two lemmas, which will be proved later.

Lemma 5.2. *Let $m \geq 3$ be an integer, $p \in (0,1)$ and λ_p be the measure on $(\Lambda^{(m)}, \mathcal{B}(\Lambda^{(m)}))$ defined in Theorem 4.2. Then*

$$\lambda_p[0] = \frac{p - p^m}{1 - p^m - (1-p)^m}.$$

For any integer $m \geq 3$, we recall

$$\Lambda^{(m)} = \Big\{ w \in \{0,1\}^{\mathbb{N}} : w \text{ does not contain } 0^m \text{ or } 1^m \Big\}$$

and

$$\Lambda_a^{(m)} = \Lambda^{(m)} \cap G_a \quad \text{for } a \in [0,1].$$

Lemma 5.3. *Let $a \in (0,1)$ and integer $m \geq 3$ be large enough such that $\frac{1}{m} < a < 1 - \frac{1}{m}$. Define $f_m : (0,1) \to \mathbb{R}$ by*

$$f_m(x) := \frac{x - x^m}{1 - x^m - (1-x)^m} \quad \text{for } x \in (0,1).$$

Then there exists $q_m \in (0,1)$ such that $f_m(q_m) = a$ and

$$\dim_H(\Lambda_a^{(m)}, d_2) \geq \frac{-(ma-1)\log q_m - (m - ma - 1)\log(1-q_m)}{(m-1)\log 2}.$$

Moreover, $q_m \to a$ as $m \to \infty$.

Proof of Theorem 1.1. First we prove (2). Let $a \in [0,1]$. Since it is straightforward to check $\Gamma \subset \Lambda$, we have

$$\Gamma_a \subset \Lambda_a \subset G_a, \quad \Gamma_a \subset \underline{\Gamma}_a \subset \underline{\Lambda}_a \subset \underline{G}_a \quad \text{and} \quad \Gamma_a \subset \overline{\Gamma}_a \subset \overline{\Lambda}_a \subset \overline{G}_a.$$

By Lemma 5.1, we only need to prove

$$\dim_H(\Gamma_a, d_2) \geq \frac{-a\log a - (1-a)\log(1-a)}{\log 2}. \tag{5.1}$$

If $a = 0$ or 1, this follows immediately from $0 \log 0 := 0$ and $1 \log 1 = 0$. So we only need to consider $0 < a < 1$ in the following. For any integer $m \geq 3$, we define

$$\Theta_a^{(m)} := \Big\{ w \in G_a : w_1 \cdots w_{2m} = 1^{2m}, w_{km+1} \cdots w_{km+m} \notin \{0^m, 1^m\} \text{ for all } k \geq 2 \Big\}$$

and

$$\Xi_a^{(m)} := \Big\{ w \in G_a : w_{km+1} \cdots w_{km+m} \notin \{0^m, 1^m\} \text{ for all } k \geq 0 \Big\}.$$

Then

$$\dim_H(\Gamma_a, d_2) \overset{(\star)}{\geq} \dim_H(\Theta_a^{(m)}, d_2) \overset{(\star\star)}{\geq} \dim_H(\Xi_a^{(m)}, d_2) \overset{(\star\star\star)}{\geq} \dim_H(\Lambda_a^{(m)}, d_2) \tag{5.2}$$

where $(\star)$ follows from $\Gamma_a \supset \Theta_a^{(m)}$, $(\star\star\star)$ follows from $\Xi_a^{(m)} \supset \Lambda_a^{(m)}$, and $(\star\star)$ follows from $\sigma^{2m}(\Theta_a^{(m)}) = \Xi_a^{(m)}$ and the fact that σ^{2m} is Lipschitz continuous (since $d_2(\sigma^{2m}(w), \sigma^{2m}(v)) \le 2^{2m} d_2(w,v)$ for all $w, v \in \{0,1\}^{\mathbb{N}}$). By (5.2) and Lemma 5.3, for m large enough, there exists $q_m \in (0,1)$ such that $q_m \to a$ (as $m \to \infty$) and

$$\dim_H(\Gamma_a, d_2) \ge \frac{-(ma-1)\log q_m - (m-ma-1)\log(1-q_m)}{(m-1)\log 2}.$$

Let $m \to \infty$, we get (5.1).

Finally we deduce (1) from (2). In fact, since (2) implies $\dim_H(\Gamma_{\frac{1}{2}}, d_2) = 1$, it follows from $\Gamma_{\frac{1}{2}} \subset \Gamma \subset \Lambda \subset \{0,1\}^{\mathbb{N}}$ that $\dim_H(\Gamma, d_2) = \dim_H(\Lambda, d_2) = 1$. □

Finally we prove Lemmas 5.2 and 5.3 to end this paper.

Proof of Lemma 5.3. Since f_m is continuous on $(0,1)$, $\lim_{x\to 0^+} f_m(x) = \frac{1}{m}$, $\lim_{x\to 1^-} f_m(x) = 1 - \frac{1}{m}$ and $\frac{1}{m} < a < 1 - \frac{1}{m}$, there exists $q_m \in (0,1)$ such that $f_m(q_m) = a$.

(1) Prove $q_m \to a$ as $m \to \infty$. Notice that

$$|q_m - a| = |q_m - f_m(q_m)| = \left| \frac{q_m^m(1-q_m) - q_m(1-q_m)^m}{1 - q_m^m - (1-q_m)^m} \right|.$$

Let

$$g_m(x) := \frac{x^m(1-x) - x(1-x)^m}{1 - x^m - (1-x)^m} \quad \text{for } x \in (0,1).$$

Then

$$|q_m - a| = |g_m(q_m)| \le \sup_{x\in(0,1)} |g_m(x)|.$$

In order to prove $q_m \to a$, it suffices to check $|g_m(x)| \le \frac{1}{m}$ for all $x \in (0,1)$. That is,

$$m \cdot |x^m(1-x) - x(1-x)^m| \le 1 - x^m - (1-x)^m \quad \text{for all } x \in (0,1).$$

① When $x \in (0, \frac{1}{2}]$, we get $x^m(1-x) - x(1-x)^m \le 0$. It suffices to prove $(m - mx - 1)x^m + 1 - (mx+1)(1-x)^m \ge 0$. Since $m - mx - 1 > 0$, we only need to prove $h_m(x) := (mx+1)(1-x)^m \le 1$ for all $x \in [0, \frac{1}{2}]$. This follows from $h_m(0) = 1$ and $h_m'(x) = -m(m+1)x(1-x)^{m-1} \le 0$ for all $x \in [0, \frac{1}{2}]$.

② When $x \in (\frac{1}{2}, 1)$, we get $x^m(1-x) - x(1-x)^m \ge 0$. It suffices to prove $(mx - 1)(1-x)^m + 1 - (1 + m - mx)x^m \ge 0$. Since $mx - 1 > 0$, we only need to prove $h_m(x) := (1 + m - mx)x^m \le 1$ for all $x \in [\frac{1}{2}, 1]$. This follows from $h_m(1) = 1$ and $h_m'(x) = m(m+1)(1-x)x^{m-1} \ge 0$ for all $x \in [\frac{1}{2}, 1]$.

(2) We apply Proposition 2.6 to get the lower bound of $\dim_H(\Lambda_a^{(m)}, d_2)$. Let μ_{q_m} be the $(q_m, 1-q_m)$ Bernoulli-type measure on $(\Lambda^{(m)}, \mathcal{B}(\Lambda^{(m)}))$ defined in Section 4.

① The fact that $\Lambda_a^{(m)} = \Lambda^{(m)} \cap G_a$ is a Borel set in $(\Lambda^{(m)}, d_2)$ follows from the fact that G_a is a Borel set in $(\{0,1\}^{\mathbb{N}}, d_2)$.

② Prove $\mu_{q_m}(\Lambda_a^{(m)}) = 1$.
Let λ_{q_m} be the measure defined in Theorem 4.2 such that $(\Lambda^{(m)}, \mathcal{B}(\Lambda^{(m)}), \lambda_{q_m}, \sigma)$ is ergodic. It follows from the Birkhoff Ergodic Theorem that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \mathbb{1}_{[0]}(\sigma^k w) = \int \mathbb{1}_{[0]} d\lambda_{q_m} = \lambda_{q_m}[0] \xlongequal[\text{Lemma 5.2}]{\text{by}} \frac{q_m - q_m^m}{1 - q_m^m - (1-q_m)^m} = f_m(q_m) = a$$

for λ_{q_m}-almost every $w \in \Lambda^{(m)}$. By $\frac{|w_1\cdots w_n|_0}{n} = \frac{1}{n}\sum_{k=0}^{n-1} \mathbb{1}_{[0]}(\sigma^k w)$, we get

$$\lim_{n\to\infty} \frac{|w_1\cdots w_n|_0}{n} = a \quad \text{for } \lambda_{q_m}\text{-almost every } w \in \Lambda^{(m)},$$

which implies $\lambda_{q_m}(\Lambda_a^{(m)}) = 1$. It follows from $\lambda_{q_m} \sim \mu_{q_m}$ that $\mu_{q_m}(\Lambda_a^{(m)}) = 1$.

③ For all $w \in \Lambda_a^{(m)}$, we have

$$\begin{aligned}
& \varliminf_{r\to\infty} \frac{\log \mu_{q_m}(B(w,r))}{\log r} \\
\overset{(\star)}{\geq}\ & \varliminf_{n\to\infty} \frac{\log \mu_{q_m}[w_1\cdots w_n]}{\log 2^{-n}} \\
=\ & \varliminf_{n\to\infty} \frac{-\log q_m^{N_0^{(m)}(w_1\cdots w_n)}(1-q_m)^{N_1^{(m)}(w_1\cdots w_n)}}{n\log 2} \\
\geq\ & \frac{\varliminf_{n\to\infty} \frac{N_0^{(m)}(w_1\cdots w_n)}{n}(-\log q_m) + \varliminf_{n\to\infty} \frac{N_1^{(m)}(w_1\cdots w_n)}{n}(-\log(1-q_m))}{\log 2} \\
\overset{(\star\star)}{\geq}\ & \frac{\varliminf_{n\to\infty} \left(\frac{m\cdot|w_1\cdots w_n|_0}{(m-1)n} - \frac{1}{m-1}\right)(-\log q_m) + \varliminf_{n\to\infty} \left(\frac{m\cdot|w_1\cdots w_n|_1}{(m-1)n} - \frac{1}{m-1}\right)(-\log(1-q_m))}{\log 2} \\
\overset{(\star\star\star)}{=}\ & \frac{-(ma-1)\log q_m - (m-ma-1)\log(1-q_m)}{(m-1)\log 2}
\end{aligned}$$

where $(\star\star\star)$ follows from $w \in \Lambda_a^{(m)}$, $(\star\star)$ follows from Proposition 3.3 and $(\star)$ can be proved as follows. For any $r \in (0,1)$, there exists $n = n(r) \in \mathbb{N}$ such that $\frac{1}{2^n} \leq r < \frac{1}{2^{n-1}}$. Then by $B(w,r) = [w_1\cdots w_n]$ and $\log \mu_{q_m}[w_1\cdots w_n] < 0$, we get $\frac{\log \mu_{q_m}(B(w,r))}{\log r} \geq \frac{\log \mu_{q_m}[w_1\cdots w_n]}{\log 2^{-n}}$. (In fact, $(\star)$ can take "=".)

Thus the lower bound of $\dim_H(\Lambda_a^{(m)}, d_2)$ follows from ①, ②, ③ and Proposition 2.6. □

Proof of Lemma 5.2. By the definition of λ_p, we know

$$\lambda_p[0] = \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \mu_p\sigma^{-k}[0].$$

For any integer $k \geq 0$, let

$$a_k := \mu_p\sigma^{-k}[0] = \sum_{u_1\cdots u_k 0 \in \Lambda^{(m),*}} \mu_p[u_1\cdots u_k 0], \quad b_k := \mu_p\sigma^{-k}[1] = \sum_{u_1\cdots u_k 1 \in \Lambda^{(m),*}} \mu_p[u_1\cdots u_k 1],$$

$$c_k := \mu_p\sigma^{-k}[01] = \sum_{u_1\cdots u_k 01 \in \Lambda^{(m),*}} \mu_p[u_1\cdots u_k 01], \quad d_k := \mu_p\sigma^{-k}[10] = \sum_{u_1\cdots u_k 10 \in \Lambda^{(m),*}} \mu_p[u_1\cdots u_k 10].$$

By Theorem 4.2, the following limits exist:

$$a := \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} a_k = \lambda_p[0], \quad b := \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} b_k = \lambda_p[1],$$

$$c := \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} c_k = \lambda_p[01], \quad d := \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} d_k = \lambda_p[10].$$

(1) We have $a + b = 1$ since $\lambda_p[0] + \lambda_p[1] = \lambda_p(\Lambda^{(m)})$.
(2) We have $c = d$ since $\lambda_p[00] + \lambda_p[01] = \lambda_p[0] = \lambda_p\sigma^{-1}[0] = \lambda_p[00] + \lambda_p[10]$.

(3) Prove $(1-p)a+p^{m-1}d=c$ and $pb+(1-p)^{m-1}c=d$.
① For $k\geq m$, we have

$$a_k=d_{k-1}+pd_{k-2}+\cdots+p^{m-3}d_{k-m+2}+p^{m-2}d_{k-m+1},$$

since

$$\begin{aligned}
&\sum_{u_1\cdots u_k0\in\Lambda^{(m),*}}\mu_p[u_1\cdots u_k0]\\
&=\sum_{u_1\cdots u_{k-1}10\in\Lambda^{(m),*}}\mu_p[u_1\cdots u_{k-1}10]+\sum_{u_1\cdots u_{k-1}00\in\Lambda^{(m),*}}\mu_p[u_1\cdots u_{k-1}00]\\
&=d_{k-1}+\sum_{u_1\cdots u_{k-2}100\in\Lambda^{(m),*}}\mu_p[u_1\cdots u_{k-2}100]+\sum_{u_1\cdots u_{k-2}000\in\Lambda^{(m),*}}\mu_p[u_1\cdots u_{k-2}000]\\
&\overset{(\star)}{=}d_{k-1}+\sum_{u_1\cdots u_{k-2}10\in\Lambda^{(m),*}}p\mu_p[u_1\cdots u_{k-2}10]+\sum_{u_1\cdots u_{k-2}000\in\Lambda^{(m),*}}\mu_p[u_1\cdots u_{k-2}000]\\
&=d_{k-1}+pd_{k-2}+\sum_{u_1\cdots u_{k-3}1000\in\Lambda^{(m),*}}\mu_p[u_1\cdots u_{k-3}1000]+\sum_{u_1\cdots u_{k-3}0000\in\Lambda^{(m),*}}\mu_p[u_1\cdots u_{k-3}0000]\\
&\overset{(\star\star)}{=}d_{k-1}+pd_{k-2}+\sum_{u_1\cdots u_{k-3}10\in\Lambda^{(m),*}}p^2\mu_p[u_1\cdots u_{k-3}10]+\sum_{u_1\cdots u_{k-3}0^4\in\Lambda^{(m),*}}\mu_p[u_1\cdots u_{k-3}0^4]\\
&=\cdots\\
&=d_{k-1}+pd_{k-2}+\cdots+p^{m-3}d_{k-m+2}+\sum_{u_1\cdots u_{k-m+2}0^{m-1}\in\Lambda^{(m),*}}\mu_p[u_1\cdots u_{k-m+2}0^{m-1}]\\
&=d_{k-1}+pd_{k-2}+\cdots+p^{m-3}d_{k-m+2}+\sum_{u_1\cdots u_{k-m+1}10^{m-1}\in\Lambda^{(m),*}}\mu_p[u_1\cdots u_{k-m+1}10^{m-1}]\\
&\overset{(\star\star\star)}{=}d_{k-1}+pd_{k-2}+\cdots+p^{m-3}d_{k-m+2}+\sum_{u_1\cdots u_{k-m+1}10\in\Lambda^{(m),*}}p^{m-2}\mu_p[u_1\cdots u_{k-m+1}10]\\
&=d_{k-1}+pd_{k-2}+\cdots+p^{m-3}d_{k-m+2}+p^{m-2}d_{k-m+1},
\end{aligned}$$

where $(\star)$, $(\star\star)$ and $(\star\star\star)$ follow from

$$\begin{aligned}
&u_1\cdots u_{k-2}100\in\Lambda^{(m),*}\Leftrightarrow u_1\cdots u_{k-2}10\in\Lambda^{(m),*}\\
&\Rightarrow u_1\cdots u_{k-2}101\in\Lambda^{(m),*},
\end{aligned}$$

$$\begin{aligned}
&u_1\cdots u_{k-3}1000\in\Lambda^{(m),*}\Leftrightarrow u_1\cdots u_{k-3}10\in\Lambda^{(m),*}\\
&\Rightarrow u_1\cdots u_{k-3}101,u_1\cdots u_{k-3}1001\in\Lambda^{(m),*}
\end{aligned}$$

and

$$\begin{aligned}
&u_1\cdots u_{k-m+1}10^{m-1}\in\Lambda^{(m),*}\Leftrightarrow u_1\cdots u_{k-m+1}10\in\Lambda^{(m),*}\\
&\Rightarrow u_1\cdots u_{k-m+1}101,u_1\cdots u_{k-m+1}1001,\cdots,u_1\cdots u_{k-m+1}10^{m-2}1\in\Lambda^{(m),*}
\end{aligned}$$

respectively, recalling the definition of μ_p. ② For $k\geq m$, we have

$$c_k=(1-p)d_{k-1}+(1-p)pd_{k-2}+\cdots+(1-p)p^{m-3}d_{k-m+2}+p^{m-2}d_{k-m+1},$$

since

$$\begin{aligned}
&\sum_{u_1\cdots u_k01\in\Lambda^{(m),*}} \mu_p[u_1\cdots u_k01]\\
=&\sum_{u_1\cdots u_{k-1}101\in\Lambda^{(m),*}} \mu_p[u_1\cdots u_{k-1}101]+\sum_{u_1\cdots u_{k-1}001\in\Lambda^{(m),*}} \mu_p[u_1\cdots u_{k-1}001]\\
\overset{(\star)}{=}&\sum_{u_1\cdots u_{k-1}10\in\Lambda^{(m),*}} (1-p)\mu_p[u_1\cdots u_{k-1}10]+\sum_{u_1\cdots u_{k-1}001\in\Lambda^{(m),*}} \mu_p[u_1\cdots u_{k-1}001]\\
=&(1-p)d_{k-1}+\sum_{u_1\cdots u_{k-2}1001\in\Lambda^{(m),*}} \mu_p[u_1\cdots u_{k-2}1001]+\sum_{u_1\cdots u_{k-2}0001\in\Lambda^{(m),*}} \mu_p[u_1\cdots u_{k-2}0001]\\
\overset{(\star\star)}{=}&(1-p)d_{k-1}+\sum_{u_1\cdots u_{k-2}10\in\Lambda^{(m),*}} p(1-p)\mu_p[u_1\cdots u_{k-2}10]+\sum_{u_1\cdots u_{k-2}0^31\in\Lambda^{(m),*}} \mu_p[u_1\cdots u_{k-2}0^31]\\
=&(1-p)d_{k-1}+p(1-p)d_{k-2}+\sum_{u_1\cdots u_{k-3}10^31\in\Lambda^{(m),*}} \mu_p[u_1\cdots u_{k-3}10^31]+\sum_{u_1\cdots u_{k-3}0^41\in\Lambda^{(m),*}} \mu_p[u_1\cdots u_{k-3}0^41]\\
=&\cdots\\
=&(1-p)d_{k-1}+(1-p)pd_{k-2}+\cdots+(1-p)p^{m-3}d_{k-m+2}+\sum_{u_1\cdots u_{k-m+2}0^{m-1}1\in\Lambda^{(m),*}} \mu_p[u_1\cdots u_{k-m+2}0^{m-1}1]\\
=&(1-p)d_{k-1}+(1-p)pd_{k-2}+\cdots+(1-p)p^{m-3}d_{k-m+2}+\sum_{u_1\cdots u_{k-m+1}10^{m-1}1\in\Lambda^{(m),*}} \mu_p[u_1\cdots u_{k-m+1}10^{m-1}1]\\
\overset{(\star\star\star)}{=}&(1-p)d_{k-1}+(1-p)pd_{k-2}+\cdots+(1-p)p^{m-3}d_{k-m+2}+\sum_{u_1\cdots u_{k-m+1}10\in\Lambda^{(m),*}} p^{m-2}\mu_p[u_1\cdots u_{k-m+1}10]\\
=&(1-p)d_{k-1}+(1-p)pd_{k-2}+\cdots+(1-p)p^{m-3}d_{k-m+2}+p^{m-2}d_{k-m+1},
\end{aligned}$$

where $(\star)$, $(\star\star)$ and $(\star\star\star)$ follow from

$$\begin{aligned}
&u_1\cdots u_{k-1}101\in\Lambda^{(m),*}\Leftrightarrow u_1\cdots u_{k-1}10\in\Lambda^{(m),*}\\
&\Rightarrow u_1\cdots u_{k-1}100\in\Lambda^{(m),*},\\
&u_1\cdots u_{k-2}1001\in\Lambda^{(m),*}\Leftrightarrow u_1\cdots u_{k-2}10\in\Lambda^{(m),*}\\
&\Rightarrow u_1\cdots u_{k-2}101, u_1\cdots u_{k-2}1000\in\Lambda^{(m),*}
\end{aligned}$$

and

$$\begin{aligned}
&u_1\cdots u_{k-m+1}10^{m-1}1\in\Lambda^{(m),*}\Leftrightarrow u_1\cdots u_{k-m+1}10\in\Lambda^{(m),*}\\
&\Rightarrow u_1\cdots u_{k-m+1}101,\cdots,u_1\cdots u_{k-m+1}10^{m-2}1\in\Lambda^{(m),*}\\
&\text{but } u_1\cdots u_{k-m+1}10^{m-1}0\notin\Lambda^{(m),*}
\end{aligned}$$

respectively, recalling the definition of μ_p.

Combining ① and ② we get $(1-p)(a_k-p^{m-2}d_{k-m+1})=c_k-p^{m-2}d_{k-m+1}$,

$$\text{i.e.,}\quad (1-p)a_k+p^{m-1}d_{k-m+1}=c_k\quad\text{for any } k\geq m.$$

That is,

$$(1-p)a_{k+m}+p^{m-1}d_{k+1}=c_{k+m}\quad\text{for any } k\geq 0,$$

which implies

$$(1-p)\frac{1}{n}\sum_{k=0}^{n-1}a_{k+m}+p^{m-1}\frac{1}{n}\sum_{k=0}^{n-1}d_{k+1}=\frac{1}{n}\sum_{k=0}^{n-1}c_{k+m}.$$

Let $n\to\infty$, we get $(1-p)a+p^{m-1}d=c$. It follows in the same way that $pb+(1-p)^{m-1}c=d$.

Combining (1), (2) and (3) we get $a=\frac{p-p^m}{1-p-(1-p)^m}$. □

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