Mesures centrales pour les graphes multiplicatifs, représentations d'algèbres de Lie et polytopes des poids

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MESURES CENTRALES POUR LES GRAPHES MULTIPLICATIFS, REPRÉSENTATIONS D'ALGÈBRES DE LIE ET POLYTOPES DES POIDS

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ABSTRACT. — To each finite-dimensional representation of a simple Lie algebra is associated a multiplicative graph in the sense of Kerov and Vershik defined from the decomposition of its tensor powers into irreducible components. It was shown in [11] and [12] that the conditioning of natural random Littelmann paths to stay in their corresponding Weyl chamber is controlled by central measures on this type of graphs. Using the K-theory of associated C^*-algebras, Handelman [8] established a homeomorphism between the set of central measures on these multiplicative graphs and the weight polytope of the underlying representation. In the present paper, we make explicit this homeomorphism independently of Handelman's results by using Littelmann's path model. As a by-product we also get an explicit parametrization of the weight polytope in terms of drifts of random Littelmann paths. This explicit parametrization yields a complete description of harmonic and c-harmonic functions for the Littelmann path model describing the iterated tensor product of an irreducible representation.

RÉSUMÉ. — Nous associons un graphe multiplicatif au sens de Vershik et Kerov à chaque représentation de dimension finie d'une algèbre de Lie simple en considérant la décomposition de ses produits tensoriels successifs en représentations irréductibles. Pour chacune de ces représentations de dimension finie, il a été montré en [11] et [12] que le conditionnement d'un chemin de Littelmann aléatoire à rester dans la chambre de Weyl est décrit par les mesures centrales sur le graphe multiplicatif associé. En utilisant la K-théorie des C^*-algèbres correspondantes, Handelman a établi un homéomorphisme entre l'ensemble des mesures centrales sur un de ces graphes multiplicatifs et le polytope des poids de la représentation sous-jacente. Dans cet article, nous rendons explicite l'homéomorphisme d'Handelman en utilisant les modèles de chemins de Littelmann. On obtient en conséquence une paramétrisation du polytope des poids en termes de dérives de chemins de Littelmann aléatoires. La paramétrisation explicite donne une description complète des fonctions harmoniques et c-harmoniques pour les modèles de chemins de Littelmann décrivant les itérations de produits tensoriels d'une représentation irréductible.

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1. Introduction

Consider a simple finite-dimensional Lie algebra $\mathfrak{g}$ of rank d over $\mathbb{C}$ and its root system in $\mathbb{R}^d$. Let P be the corresponding weight lattice and fix Δ a dominant Weyl chamber. Then $P_+ = P \cap \Delta$ is the cone of dominant weights of $\mathfrak{g}$. Denote by $S = \{\alpha_1, \ldots, \alpha_d\}$ the underlying set of simple roots. To each dominant weight $\delta \in P_+$ corresponds a finite-dimensional representation $V(\delta)$ of $\mathfrak{g}$ of highest weight δ. In [14] Littelmann associated to every representation $V(\delta)$ a set $B(\delta)$ of paths in $\mathbb{R}^d$ with length 1 starting at 0 and ending in the set Π_δ of weights of $V(\delta)$. Random Littelmann paths can then be defined first by endowing $B(\delta)$ with a suitable probability distribution, next by considering random concatenations of paths in $B(\delta)$. In [11] and [12] distributions on the set $B(\delta)$ are defined as morphisms from P to $\mathbb{R}_{>0}$. This is equivalent to associating to each simple root α_i a real t_i in $]0, +\infty[$. It is then shown that these random paths and their conditioning to stay in the Weyl chamber Δ are controlled by the representation theory of $\mathfrak{g}$. In fact, one obtains particular central distributions on the set $\Gamma_n^{\mathbb{R}^d}$ of paths of any length $n \geqslant 1$ (obtained by concatenating n paths in $B(\delta)$). By central distributions we mean that the probability of a finite path only depends on its length and its end. Equivalently, we get a central measure on the set of infinite concatenations $\Gamma^{\mathbb{R}}$ of paths in $B(\delta)$ (see Section 2).

Write $\mathcal{H}(\mathbb{R}^d)$ for the set of central measures on $\Gamma^{\mathbb{R}}$ and $\mathcal{H}(\Delta)$ for the subset of $\mathcal{H}(\mathbb{R}^d)$ of central measures on Γ^{Δ}, the set of infinite trajectories remaining in Δ. By Choquet's Theorem both sets $\mathcal{H}(\mathbb{R}^d)$ and $\mathcal{H}(\Delta)$ are simplices so they are essentially determined by their minimal boundaries $\partial \mathcal{H}(\mathbb{R}^d)$ and $\partial \mathcal{H}(\Delta)$. Write $K(\delta)$ for the convex hull of Π_δ and set $K(\delta)^+ = \Delta \cap K(\delta)$. For walks in the Weyl chambers, the characterization of the sets $\partial \mathcal{H}(\mathbb{R}^d)$ and $\partial \mathcal{H}(\Delta)$ has been obtained by Handelman in [8] and [9] using an important work of Price [17, 18], by proving that they are respectively homeomorphic to $K(\delta)$ and $K(\delta)^+$. Nevertheless, Handelman did not explicit the homeomorphisms. Their existence is established by considering the central measures as traces on certain C^*-algebras and then using analytic tools. In particular, a central element of the proof is the extension of traces on C^*-algebras using K-theory (a short explanation of these arguments is given in Section 3.4).

The goal of this paper is essentially threefold: first we make explicit both homeomorphisms by using the Weyl characters of $\mathfrak{g}$ (see Theorem 3.1), next we give an algebraic proof of Handelman's results and finally we connect them with more recent works on conditioned random walks or Brownian motions, generalizations of the Pitman transform and asymptotic Young

tableaux (see [2, 4, 5, 11, 12, 13, 16, 20]). As a corollary of these results, we describe the set of harmonic and c-harmonic functions corresponding to the aforementioned random walks. Finally, we get a law of large numbers for random walks distributed according the central measures we obtain. Our two last results seem quite disconnected from the initial algebraic setting in representation theory, and we conjecture that they still hold for a very broad class of random paths. Our approach extends that of Kerov and Vershik to which it essentially reduces when $V(\delta)$ is the defining representation of $\mathfrak{g} = \mathfrak{sl}_n$. Nevertheless, numerous difficulties arise when considering the general case of dominant weights of any simple algebra $\mathfrak{g}$, which explains the involved machinery used in the proof of Handelman. Our methods to determine $\partial\mathcal{H}(\mathbb{R}^d)$ and $\partial\mathcal{H}(\Delta)$ are quite similar. So we will now give its main steps only in the case of $\partial\mathcal{H}(\Delta)$.

We first need to show that the characterization of $\partial\mathcal{H}(\Delta)$ is equivalent to that of the extremal harmonic functions on the growth graph $\mathcal{G}(\Delta)$ associated with Γ^Δ. This growth graph is rooted, graded and multiplicative: its vertices label the basis $\mathcal{B} = \{(s_\lambda, n) \mid V(\lambda) \text{ irreducible component of } V(\delta)^{\otimes n}$ and $n \geqslant 1\}$ of a commutative algebra $\widehat{T}_\delta^+$ (here s_λ is the Weyl character of $V(\lambda)$). We then establish that the extremal nonnegative harmonic functions on $\mathcal{G}(\Delta)$ are in bijection with the algebra morphisms from $\widehat{T}_\delta^+$ to $\mathbb{R}$ that are nonnegative on $\mathcal{B}$. Next, we prove that all these morphisms are obtained by associating to each simple root $\alpha_i, i = 1, \ldots, n$ a real in $[0, 1]$. The difficulty here comes from the fact that two such associations can yield the same morphism. So to obtain a genuine parametrization, we need to restrict ourselves to a subset $[0, 1]_\delta^d$ (see (4.5) for a precise definition) of $[0, 1]^d$ whose combinatorial description is in terms of the δ-admissible subsets of S introduced in [21]. Finally, in Proposition 6.4, we show that our set $[0, 1]_\delta^d$ also parametrizes the simplex $K(\delta)^+$ by considering, for each d-tuple in $[0, 1]_\delta^d$, the drift of the corresponding random Littelmann path appearing in the construction of [11] and [12].

The paper is organized as follows. In Section 2, we recall some background on random paths and central measures on multiplicative graphs. We also apply the Ring Theorem of Kerov and Vershik to relate extremal harmonic functions on a multiplicative graph to nonnegative morphisms of the underlying algebra. The main result is written down in Section 3 where we also introduce the algebras $\widehat{T}_\delta$ and $\widehat{T}_\delta^+$; a sketch of Handelman's arguments is proposed at the end of Section 3. Section 4 gives the description of $\partial\mathcal{H}(\mathbb{R}^d)$. Here, we define our set $[0, 1]_\delta^d$ and relate it to the geometry of the polytope $K(\delta)$. The description of $\partial\mathcal{H}(\Delta)$ is deduced from that of

$\partial\mathcal{H}(\mathbb{R}^d)$ in Section 5. It is worth noticing that we need here (as in the result of Kerov and Vershik) a classical theorem relating polynomials with non positive roots to totally positive sequences. Another important ingredient in the proof is the use of certain plethyms of Schur and Weyl characters of $\mathfrak{g}$. Finally, Section 6 relates both descriptions of $\partial\mathcal{H}(\mathbb{R}^d)$ and $\partial\mathcal{H}(\Delta)$ to the drift of random Littelmann paths. Notably it explains how the polytope $K(\delta)$ can be simply parametrized by using the set $[0,1]_\delta^d$. A nomenclature with all recurring notations is provided at the end of the manuscript.

2. General probabilistic framework

We present here a general probabilistic model of random paths in a domain, which is well suited to study probabilistic aspects of Littelmann paths and their asymptotics. We introduce first a discrete version of paths in a vector space.

2.1. Random paths on a lattice

Let $d \geqslant 0$ and let Λ be a lattice of $\mathbb{R}^d$ with rank d. We shall denote by $\#S$ the cardinality of any set S.

DEFINITION 2.1.

(1) *Let $n \geqslant 0$. A path π on Λ of length n is a piecewise linear function $\pi : [0,n] \to \mathbb{R}^d$ with $\pi(0) = 0$, $\pi(i) \in \Lambda$ for all $i \in \{0,\dots,n\}$, and $\pi(x) \in \mathbb{Q}^d$ for all x on which π is not differentiable. The path π is called infinitesimal if $n = 1$.*
(2) *An infinite path on Λ is a piecewise linear function $\pi : [0,+\infty[\to \mathbb{R}^d$ with $\pi(0) = 0$, $\pi(i) \in \Lambda$ for all $i \in \mathbb{N}$, and $\pi(x) \in \mathbb{Q}^d$ for all x on which π is not differentiable.*

The length of the path π on Λ is denoted by $l(\pi) \in \mathbb{N} \cup \{+\infty\}$. We write $\pi.\tau$ for the concatenation of two finite paths π and τ. Let X be a countable set of infinitesimal paths and let Ω be a domain of $\mathbb{R}^d$ such that $0 \in \Omega$; from now on, the set X is fixed and is not mentioned in the various notations. A path π is called X-valued if π is the concatenation of infinitesimal paths coming from X: equivalently, $\left(\pi_{|[i,i+1]} - \pi(i)\right) \in X$ for all $i \geqslant 1$. In the sequel, any path is always considered as X-valued. The set of infinite X-valued paths (resp. finite X-valued paths and X-valued paths of length

$n \geqslant 1$) whose image is included in Ω is denoted by Γ^Ω (resp. by $\Gamma^\Omega_{\text{fin}}$ and Γ^Ω_n). For $x, y \in \Lambda$, we denote by $\Gamma^\Omega(x, y)$ the set of infinitesimal paths $\pi \in X$ such that $\pi(1) = y - x$ and $x + \pi \subset \Omega$, and we write $x \nearrow y$ when $\#\Gamma^\Omega(x, y) \neq 0$. Finally, we denote by $\Gamma^\Omega_n(y)$ the set of finite paths of length n ending at y.

In order to consider random infinite paths in Ω, we need to define a σ-algebra on Γ^Ω. Let τ be a finite path of length n, and let $\Gamma^\Omega(\tau)$ be the set $\{\pi \in \Gamma^\Omega \,|\, l(\pi) \geqslant n, \pi_{|[0,n]} = \tau\}$. We consider the coarsest σ-algebra containing all the sets $\Gamma^\Omega(\tau)$ for $\tau \in \Gamma^\Omega_{\text{fin}}$. The set $\mathcal{M}_1(\Gamma^\Omega)$ of probability measures on Γ^Ω is considered with the initial topology with respect to the evaluation maps on the sets $\Gamma^\Omega(\tau), \tau \in \Gamma^\Omega_{\text{fin}}$. By Tychonov's Theorem, $\mathcal{M}_1(\Gamma^\Omega)$ is a compact set with respect to this topology.

2.2. Central random paths

DEFINITION 2.2. — *A probability measure $\mathbf{P}$ on Γ^Ω is called central if there is a function $p : \Lambda \times \mathbb{N} \to \mathbb{R}^+$ such that*

$$\mathbf{P}\left(\Gamma^\Omega(\pi)\right) = p(y, n),$$

for all $\pi \in \Gamma^\Omega_n(y)$ with $y \in \Lambda, n \geqslant 0$. A random path in Γ^Ω is called central if the induced measure is central.

Similarly, we could have defined central measure on $\Gamma^\Omega_{\text{fin}}$ by similar means. It is then easily seen that any central measure supported on $\Gamma_{\text{fin}}(\Omega)$ is a convex combination of uniform measures on the sets $\Gamma^\Omega_n(y)$ with $y \in \Lambda$ and $n \geqslant 1$. Therefore, the main interesting phenomena arise for central measures supported on Γ^Ω. The set of central measures supported on Γ^Ω is denoted by $\mathcal{H}(\Omega)$.

Let $\mathbf{P} \in \mathcal{H}(\Omega)$. Then, by Definition 2.2 there exists a function $p : \Lambda \times \mathbb{N} \to \mathbb{R}^+$ such that

$$\mathbf{P}(\Gamma^\Omega(\pi)) = p(y, n),$$

for all $\pi \in \Gamma^\Omega_n(y)$ with $y \in \Lambda$ and $n \geqslant 1$. Let $x \in \Lambda$, and suppose that π is a finite path in $\Gamma^\Omega_n(x)$. A path τ of length $n + 1$ ending at $y \in \Lambda$ satisfies $\tau_{|[0,n]} = \pi$ if and only if $\tau_{|[0,n]} = \pi$ and $\tau_{[n,n+1]}$ is an infinitesimal path joining x to y. Therefore, $\Gamma^\Omega(\pi)$ can be decomposed as

$$\Gamma^\Omega(\pi) = \coprod_{y \in \Lambda} \coprod_{\tau \in \Gamma^\Omega(x,y)} \Gamma^\Omega(\pi.\tau).$$

Thus,

$$\mathbf{P}\left(\Gamma^{\Omega}(\pi)\right) = \sum_{y \in \Lambda} \sum_{\tau \in \Gamma^{\Omega}(x,y)} \mathbf{P}\left(\Gamma^{\Omega}(\pi.\tau)\right),$$

which translates into the relation

$$(2.1) \qquad\qquad p(x,n) = \sum_{y \in \Lambda} \#\Gamma^{\Omega}(x,y) p(y, n+1)$$

for $x \in \Lambda \cap \Omega$ such that $\Gamma_n^{\Omega}(x) \neq \emptyset$. Describing the set of solutions to (2.1) is very complicated in general. It is known however that $\mathcal{H}(\Omega)$ is a convex subset of $\mathcal{M}_1(\Gamma^{\Omega})$ and even a Choquet simplex.

DEFINITION 2.3. — *The minimal boundary of Γ^{Ω} is the unique subset $\partial\mathcal{H}(\Omega) \subset \mathcal{H}(\Omega)$, such that any central measure $\mathbf{P}_0$ in $\mathcal{H}(\Omega)$ admits a unique integral representation*

$$\mathbf{P}_0 = \int_{\partial\mathcal{H}(\Omega)} \mathbf{P} d\mu(\mathbf{P}),$$

where μ is a probability measure on the set $\partial\mathcal{H}(\Omega)$.

2.3. Central measures and Doob's conditioning

We establish here some connections between central measures on random paths and random walks on lattices. Indeed, any random path $\pi \in \Gamma^{\Omega}$ following a central measure $\mathbf{P} \in \mathcal{H}(\Omega)$ yields a random walk $Z = (\pi(0) = 0, \pi(1), \dots)$ on the lattice $\Lambda \cap \Omega$. The family of Markov kernels $(Q_n)_{n \geqslant 0}$ of Z can be explicitly given from the function $p : \Lambda \times \mathbb{N} \to \mathbb{R}^+$ associated to the central measure $\mathbf{P}$. Indeed one can show that

$$Q_n(x,y) = \mathbf{1}_{p(x,n) \neq 0} \frac{\#\Gamma^{\Omega}(x,y) p(y, n+1)}{p(x,n)}.$$

By the equality $p(x,n) = \sum_{y \in \Lambda} \#\Gamma^{\Omega}(x,y) p(y, n+1)$, Q_n is a well-defined Markov kernel. Note that this random walk is generally not homogeneous in time, since the kernel Q_n depends on n through p.

Doob's conditioning is a standard way to produce random walks on $\Lambda \cap \Omega$ coming from central measures. Let Z be the random walk on $\mathbb{R}^d$ starting at 0 and with Markov kernel

$$\mathbb{P}(Z_{n+1} = y \mid Z_n = x) = \frac{\Gamma^{\mathbb{R}}(x,y)}{\#X}$$

for any $x, y \in \Lambda$. Note that this random walk actually comes from the central measure $\mathbf{P}$ whose value is

$$\mathbf{P}(\Gamma^{\mathbb{R}}(\pi)) = \frac{1}{(\#X)^n}$$

for all $\pi \in \Gamma_n^{\mathbb{R}^d}$ with $n \geqslant 0$.

DEFINITION 2.4. — Let $c > 0$. A function $h : \Omega \cap \Lambda \to \mathbb{R}^+$ is a c-harmonic function for the random path Z killed when exiting Ω if and only if

$$h(x) = \frac{1}{c(\#X)} \sum_{\substack{\pi \in X \\ x + \pi \subset \Omega}} h(x + \pi(1)) = \frac{1}{c(\#X)} \sum_{y \in \Lambda \cap \Omega} \#\Gamma^\Omega(x, y) h(y).$$

Let $c > 0$ and assume that h is a c-harmonic function for Z. Then, the Doob conditioning Z^h of Z in Ω is given by the Markov kernel

$$\mathbb{P}(Z_{n+1}^h = y \mid Z_n^h = x) = \mathbf{1}_{h(x)>0} \frac{1}{c(\#X)} \frac{\#\Gamma^\Omega(x, y) h(y)}{h(x)},$$

for $y \in \Lambda \cap \Omega$. The random walk Z^h is well-defined because h is c-harmonic and is time homogeneous. This random walk comes from the central measure $\mathbf{P}^h$ whose value is

$$\mathbf{P}^h(\Gamma^\Omega(\pi)) = \left(\frac{1}{c(\#X)} \right)^n h(y)$$

for $\pi \in \Gamma_n^\Omega(y)$ with $n \geqslant 0$ and $y \in \Lambda \cap \Omega$.

Conversely, suppose that $\mathbf{P}$ is a central measure with an associated function p such that $p(x, n) = a^n h(x)$ for some function $h : \Lambda \cap \Omega \to \mathbb{R}^+$ and $a > 0$. Then, (2.1) yields

$$a^n h(x) = \sum_{y \in \Lambda} \#\Gamma^\Omega(x, y) a^{n+1} h(y),$$

which is equivalent to the relation $h(x) = a \sum_{y \in \Lambda} \#\Gamma^\Omega(x, y) h(y)$. Thus, the function h is $(a \#X)^{-1}$-harmonic.

Hence, the set of c-harmonic functions for Z is homeomorphic to the set of central measures $\mathbf{P} \in \mathcal{H}(\Omega)$ whose associated functions p have the form $p(x, n) = \left(\frac{1}{\#X \cdot c} \right)^n h(x)$ with $h : \Lambda \cap \Omega \to \mathbb{R}^+$.

For any real $c > 0$, denote by $\mathcal{H}_c(\Omega)$ the set of central measures coming from c-harmonic functions. A quick computation shows that the random walk Z induced by a central measure $\mathbf{P}$ is time homogeneous if and only if $\mathbf{P} \in \mathcal{H}_c(\Omega)$ for some $c > 0$.

It is easily seen that $\mathcal{H}_c(\Omega)$ is a convex subset of $\mathcal{H}(\Omega)$ and we denote by $\partial \mathcal{H}_c(\Omega)$ the set of extreme points of $\mathcal{H}_c(\Omega)$. To the best of our knowledge,

there is no general proof that $\partial\mathcal{H}_c(\Omega) = \partial\mathcal{H}(\Omega) \cap \mathcal{H}_c(\Omega)$; in particular, an answer to the following problem yields a nice description of $\partial\mathcal{H}(\Omega)$.

PROBLEM. — *Do we have the decomposition $\partial\mathcal{H}(\Omega) = \overline{\sqcup_{c>0}\partial\mathcal{H}_c(\Omega)}$?*

In our case of study, this equality is proven by explicitly describing both sets (see Section 3.3). As we shall explain in the following we do need here to consider the closure of $\sqcup_{c>0}\partial\mathcal{H}_c(\Omega)$.

2.4. Central measures on multiplicative graphs

We investigate here the general solution of (2.1) when the path concatenation on $\Gamma_{\text{fin}}^{\Omega}$ encodes the multiplicative structure of an algebra. Let us first give a brief overview of the general theory in the setting of graded graphs before applying it to our situation.

So consider a rooted graded graph $\mathcal{G} = \{*\} \sqcup \coprod_{n\geqslant 1}\mathcal{G}_n$ where $\mathcal{G}_n$ is the set of vertices in level n ($\mathcal{G}_0 = \{*\}$ by convention). For any $n \geqslant 0$, we can only have directed weighted arrows between vertices $\lambda \in \mathcal{G}_n$ and $\mu \in \mathcal{G}_{n+1}$ with weight $e(\lambda, \mu)$. Such a graph is called multiplicative if there exists a commutative algebra A and an injective map $\iota : \mathcal{G} \to A$ such that $\iota(\lambda)\iota(*) = \sum_{\lambda \nearrow \mu} e(\lambda, \mu)\iota(\mu)$. Here $\lambda \nearrow \mu$ means we consider all the neighbors μ of the vertex λ. We suppose that the graph is connected, which means that for all $\mu \in \mathcal{G}$, the number of paths between the root and μ is positive. The weight $e(\pi)$ of a path π between the root and a vertex μ is the product of all the weights of the edges of π.

Let K be the positive cone spanned by $\iota(\mathcal{G})$, and let $A_{\mathcal{G}}$ be the unital subalgebra of A generated by K. Denote by $\text{Mult}(A_{\mathcal{G}})^+ \subset A_{\mathcal{G}}^*$ the set of multiplicative functions on $A_{\mathcal{G}}$ which are nonnegative on K and equal to 1 on $\iota(*)$. Note that $\iota : \mathcal{G} \to A_{\mathcal{G}}$ induces a map $\iota^* : A_{\mathcal{G}}^* \to F(\mathcal{G}, \mathbb{R})$ such that $\iota^*(\phi) = \phi \circ \iota$ for any linear map $\phi : A_{\mathcal{G}} \to \mathbb{R}$. Now denote by $\mathcal{H}(\mathcal{G})$ the set of functions $p : \mathcal{G} \to \mathbb{R}^+$ such that

$$(2.2) \qquad \begin{cases} p(*) = 1, \\ p(\lambda) = \displaystyle\sum_{\lambda \nearrow \mu} e(\lambda, \mu)p(\mu) \text{ for any } \lambda \in \mathcal{G}. \end{cases}$$

We can characterize the set $\partial\mathcal{H}(\mathcal{G})$ of extremal points in $\mathcal{H}(\mathcal{G})$:

PROPOSITION 2.5. — *Suppose that $K.K \subset K$. Then, the map ι^* yields an homeomorphism between $\text{Mult}^+(A_{\mathcal{G}})$ and the set $\partial\mathcal{H}(\mathcal{G})$.*

The proof of this proposition is an application of the Ring Theorem of Kerov and Vershik.

THEOREM 2.6 ([7, Section 8.4]). — *Let B be a unital commutative algebra over $\mathbb{R}$ and $K \subset B$ a convex cone satisfying the following conditions:*

- $K - K = B$ *(K generates B).*
- $K.K \subset K$ *(K is stable by multiplication).*
- K *is spanned by a countable set of elements.*
- *For all $a \in B$, there exists $\epsilon > 0$ such that $1 - \epsilon a \in K$.*

If L denotes the convex set of linear forms on B which are nonnegative on K and map 1_B to 1, then ϕ is an extreme point of L if and only if ϕ is multiplicative (meaning that $\phi(ab) = \phi(a)\phi(b)$ for all $a, b \in B$).

We give now the proof of Proposition 2.5.

Proof. — Let $B = A_{\mathcal{G}}/\langle \iota(*) = 1 \rangle$ and let $\mathrm{pr} : A_{\mathcal{G}} \to B$ be the canonical projection; denote by $\widetilde{K}$ the projection of the cone $\mathbb{R}^+ \, \mathrm{Id} + K$ in B. Since $K.K \subset K$ and $\{1, K\}$ spans $A_{\mathcal{G}}$, $\widetilde{K}.\widetilde{K} \subset \widetilde{K}$ and $\widetilde{K}$ spans B. Since $\mathcal{G}$ has a countable set of vertices, $\widetilde{K}$ is spanned by a countable set of elements. Note that there is a bijection between the elements of $\mathcal{H}(\mathcal{G})$ and the linear forms on B which are nonnegative on $\widetilde{K}$ and equal to 1 on 1: indeed $h \in \mathcal{H}(\mathcal{G})$ if and only if $h(\mu) = \sum_{\mu \nearrow \nu} e(\mu, \nu) h(\nu)$. Thus, for $f \in A_{\mathcal{G}}^*$, $\iota^*(f) \in \mathcal{H}(\mathcal{G})$ if and only if $f(\iota(*)\iota(\mu)) = f(\iota(\mu))$; equivalently, this means that f factors through B. The fact $\iota^*(f)$ is nonnegative on $\mathcal{G}$ is then equivalent to the fact it is nonnegative on $\widetilde{K}$. We also have $[\iota^*(f)](*) = 1$ if and only if $f(\mathrm{pr} \circ \iota(*)) = f(1) = 1$.

Let $a \in B$, and let us show that there exists ϵ such that $1 - \epsilon a \in \widetilde{K}$. Since $\widetilde{K} - \widetilde{K} = B$, and $1 - b \in \widetilde{K}$ for all $b \in -\widetilde{K}$, we can suppose without loss of generality that $a \in \widetilde{K}$. It is thus enough to prove that for $\mu \in \mathcal{G}$, there exists ϵ such that $1 - \epsilon \, \mathrm{pr} \circ \iota(\mu) \in K$. Suppose that μ has rank n. Since the graph is connected, there exists a path π_0 of weight $e(\pi_0)$ between $*$ and μ. By iteration of the relation coming from the multiplicative structure of $\mathcal{G}$, $\iota(*)^n = \sum_{\nu \in \mathcal{G}, \, rk(\mu)=n} (\sum_{\pi : * \to \mu} e(\pi)) \iota(\nu)$. Thus $\iota(*)^n - e(\pi_0) \iota(\mu)$ belongs to K. Since $\mathrm{pr}(\iota(*)^n) = 1$, $1 - e(\pi_0) \, \mathrm{pr} \circ \iota(\mu)$ belongs to $\widetilde{K}$. Therefore, we can apply Theorem 2.6 to $(B, \widetilde{K})$, which yields that the extreme linear maps among the set of linear maps on B which are nonnegative on $\widetilde{K}$ and equal to 1 on 1 are the multiplicative ones. Since there is a bijection between multiplicative maps on B which are nonnegative on $\widetilde{K}$ and multiplicative maps on $A_{\mathcal{G}}$ which are nonnegative on K and equal to 1 on $\iota(*)$, the proof is complete. $\qquad\square$

In order to apply the previous result to central random paths on $\Lambda \cap \Omega$, we need to relate Γ^Ω to a graded graph.

DEFINITION 2.7. — *The growth graph of Γ^Ω is the rooted graded graph* $\mathcal{G}(\Omega)$ *with*

- *set of vertices of rank n: the set $\mathcal{G}_n(\Omega) = \{(x,n) \in \Lambda \cap \Omega \times \mathbb{N}$, $\Gamma_n^\Omega(x) \neq \emptyset\}$,*
- *a directed weighted edge between $(x,n) \in \mathcal{G}_n(\Omega)$ and $(y,n+1) \in \mathcal{G}_{n+1}(\Omega)$ with weight $\#\Gamma^\Omega(x,y)$.*

It is readily seen that the weighted sum on paths in $\mathcal{G}(\Omega)$ between the root $(0,0)$ and (y,n) is equal to $\#\Gamma_n^\Omega(y)$. Moreover, the sets $\mathcal{H}(\Omega)$ and $\mathcal{H}(\mathcal{G}(\Omega))$ are canonically homeomorphic through the equivalence between (2.1) and (2.2).

3. Littelmann paths in Weyl chambers

We describe a class of random paths coming from the representation theory of semi-simple Lie algebras.

3.1. Background

We consider a simple Lie group G over $\mathbb{C}$ and its Lie algebra $\mathfrak{g}$. Let $R \subset V$ be the set of roots of $\mathfrak{g}$ regarded as a finite subset of the Euclidean vector space V with scalar product $\langle \cdot, \cdot \rangle$. We fix R_+ a subset of positive roots and $S = \{\alpha_1, \alpha_2, \ldots, \alpha_d\} \subset R_+$ a basis of simple roots in R. The Weyl group of $\mathfrak{g}$ is denoted by W. This is the Coxeter group generated by the reflections s_{α_i} associated to the simple roots. Thus for any $x \in V$ and any $\alpha \in S$, we have

$$(3.1) \qquad s_\alpha(x) = x - \frac{2\langle \alpha, x \rangle}{\langle \alpha, \alpha \rangle} \alpha$$

Denote by ℓ the length function on W defined from S.

Write P for the weight lattice of $\mathfrak{g}$ and $\omega_1, \ldots, \omega_d$ for its fundamental weights so that we have

$$P = \bigoplus_{i=1}^{d} \mathbb{Z}\omega_i.$$

Let us denote by $\leqslant$ the dominant order on P such that $\gamma \leqslant \gamma'$ if and only if $\gamma' - \gamma$ is a sum of simple roots. Let Δ be the fundamental Weyl chamber

of $\mathfrak{g}$ with respect to S, which corresponds to the positive orthant on the weight space $\bigoplus_{i=1}^{d} \mathbb{R}\omega_i$. The cone of dominant weights is then

$$P^+ = P \cap \Delta = \bigoplus_{i=1}^{d} \mathbb{Z}_{\geqslant 0}\omega_i.$$

Write Q^+ the subset of P spanned by linear combinations of the simple roots with nonnegative coefficients. We denote by $\mathbb{R}[P]$ the ring group of P over $\mathbb{R}$ with basis $\{e^\beta \mid \beta \in P\}$, and by $\mathbb{R}[Q^+]$ the subalgebra of $\mathbb{R}[P]$ generated by Q^+. Then

$$\mathbb{R}^W[P] = \{u \in \mathbb{R}[P] \mid w(u) = u, w \in W\}$$

is the character ring of $\mathfrak{g}$. To each $\lambda \in P^+$ corresponds a simple finite-dimensional representation of $\mathfrak{g}$ we denote by $V(\lambda)$. The Weyl character of $V(\lambda)$ is

$$s_\lambda = \sum_{\gamma \in P} K_{\lambda,\gamma} e^\gamma$$

where $K_{\lambda,\gamma}$ is the dimension of the weight space γ in $V(\lambda)$. For $\mathbf{t} \in \mathbb{R}_{\geqslant 0}^d$ and $\gamma \in P$ with $\gamma = \sum_{i=1}^{d} \gamma_i \omega_i$, set $\mathbf{t}^\gamma = \prod_{1 \leqslant i \leqslant d} \exp(\gamma_i \log(t_i))$, with the convention $\mathbf{t}^\gamma = +\infty$ when there exists $1 \leqslant i \leqslant d$ such that $t_i = 0$ and $\gamma_i < 0$. It is then possible to evaluate s_λ on $\mathbf{t} \in (\mathbb{R}^+)^d$ as $s_\lambda(\mathbf{t}) = \sum_{\gamma \in P} K_{\lambda,\gamma} \mathbf{t}^\gamma$. Hence, with our convention, $s_\lambda(\mathbf{t}) = +\infty$ as soon as some t_i vanishes, since for any $\lambda \in P^+$ and any $1 \leqslant i \leqslant d$, there exists $\gamma \in P$ such that $K_{\lambda,\gamma} \neq 0$ and $\gamma_i < 0$. For $\mu \geqslant \lambda$, denote by $S_{\lambda,\mu}$ the function

$$(3.2) \qquad S_{\lambda,\mu} = e^{-\mu} s_\lambda = \sum_{\gamma \in P} K_{\lambda,\gamma} e^{\gamma - \mu}$$

where for any γ such that $K_{\lambda,\gamma} > 0$, $\gamma - \mu$ is a linear combination of the simple roots with nonpositive coefficients; for $\mu = \lambda$, we simply write S_λ, instead of $S_{\lambda,\lambda}$. By setting $T_i = e^{-\alpha_i}$ we thus obtain that $S_{\lambda,\mu} = S_{\lambda,\mu}(T_1, \ldots, T_d)$ is polynomial in the variables $T_1, \ldots, T_d$ with nonnegative integer coefficients. Recall also the Weyl dimension formula

$$\dim(V(\lambda)) = \prod_{\alpha \in R_+} \frac{(\lambda + \rho, \alpha)}{(\rho, \alpha)},$$

where $\rho = \frac{1}{2} \sum_{\alpha \in R_+} \alpha$. In particular, $\dim(V(\lambda))$ is polynomial in the coordinates of λ on the basis of fundamental weights.

3.2. Random Littelmann paths

Now, fix a dominant weight $\delta \in P^+$ and denote by Π_δ the set of weights of the irreducible representation $V(\delta)$. Let P_δ be the sublattice of P generated by Π_δ. This defines subalgebras

$$\mathbb{R}[P_\delta] = \{e^\beta \,|\, \beta \in P_\delta\} \subset \mathbb{R}[P]$$

$$\text{and} \ \ \mathbb{R}^W[P_\delta] = \{u \in \mathbb{R}[P_\delta] \,|\, w(u) = u\} \subset \mathbb{R}^W[P].$$

Finally write T_δ^+ the subset of P^+ of weights λ such that $V(\lambda)$ appears as an irreducible component in a tensor power $V(\delta)^{\otimes n}, n \geqslant 0$. Given λ and μ in T_δ^+, we clearly have $\lambda + \mu$ in T_δ^+.

Now let $\mathcal{A}_\delta$ be the subalgebra of $\mathbb{R}^W[P]$ generated by the Weyl character s_λ with $\lambda \in T_\delta^+$. We have the inclusions

$$\mathcal{A}_\delta \subset \mathbb{R}^W[P_\delta] \subset \mathbb{R}[P_\delta] \subset \mathbb{R}[P].$$

We denote by $K(\delta)$ the convex hull of the set Π_δ: $K(\delta)$ is a polytope whose extreme points are the elements $w(\delta)$ for $w \in W$. The intersection of $K(\delta)$ with the Weyl chamber Δ is denoted by $K(\delta)^+$. By Littelmann's paths theory, there is a set $B(\delta) = \{\pi_i\}_{1 \leqslant i \leqslant \dim V(\delta)}$ of infinitesimal paths on P_δ, with the following properties:

- $\pi_i(1) \in \Pi_\delta$ for all $1 \leqslant i \leqslant \dim V(\delta)$,
- the multiplicity of the weight μ in $V(\delta)^{\otimes n}$ is equal to $\#\Gamma_n^{\mathbb{R}^d}(\mu)$,
- the multiplicity of the irreducible representation $V(\nu)$ in $V(\mu) \otimes V(\delta)$ is equal to $\#\Gamma^\Delta(\mu, \nu)$ and the multiplicity of the irreducible representation $V(\nu)$ in $V(\delta)^{\otimes n}$ is equal to $\#\Gamma_n^\Delta(\nu)$ for all $\mu, \nu \in P^+$ and $n \geqslant 0$.

The set of infinite paths we are interested in is the set of infinite paths starting at 0 with set of infinitesimal paths $B(\delta)$.

3.3. Statements of the results

We recall that we consider the space of probability measures on each Γ^Δ with the initial topology with respect to the evaluation maps on the cylinders $\Gamma^\Delta(\tau), \tau \in \Gamma_{\text{fin}}^\Delta$. We give an algebraic proof of the identification of the minimal boundaries for random paths in $\Gamma^\mathbb{R}$ and Γ^Δ with the topological spaces $K(\delta)$ and $K(\delta)^+$, respectively. In both cases, the homeomorphism can be made explicit by the introduction of a natural parametrization

$$t : K(\delta) \longrightarrow [0, 1]^d \times W$$

of $K(\delta)$ such that $t(K(\delta)^+) \subset [0,1]^d \times \mathrm{Id}_W$ (this parametrization is explained in Section 5). For $m \in K(\delta)$, we set $t(m) = (\mathbf{t}_m, w_m)$. The main result of the paper is summarized in the following theorem:

THEOREM 3.1.

(1) *The map*
$$\mathbb{P} : \begin{cases} K(\delta) \longrightarrow \partial\mathcal{H}(\mathbb{R}^d) \\ m \longmapsto \mathbb{P}_m \end{cases}$$
with $\mathbb{P}_m(\Gamma^{\mathbb{R}}(\pi)) = \frac{\mathbf{t}_m^{n\delta - w_m(\lambda)}}{S_\delta(\mathbf{t}_m)}$ *for* $n \geqslant 0$ *and* $\pi \in \Gamma_n^{\mathbb{R}^d}(\lambda)$ *is a homeomorphism between the set of extremal measures* $\partial\mathcal{H}(\mathbb{R}^d)$ *and* $K(\delta)$.

(2) *The map*
$$\mathbb{P}^+ : \begin{cases} K(\delta)^+ \longrightarrow \partial\mathcal{H}(\Delta) \\ m \longmapsto \mathbb{P}_m^+ \end{cases}$$
with $\mathbb{P}_m^+(\Gamma^{\Delta}(\pi)) = \frac{S_{\lambda,n\delta}(\mathbf{t}_m)}{S_\delta(\mathbf{t}_m)^n}$ *for* $n \geqslant 0$ *and* $\pi \in \Gamma_n^{\Delta}(\lambda)$ *is a homeomorphism between the set of extremal measures* $\partial\mathcal{H}(\Delta)$ *and* $K(\delta)^+$.

It is easy to see that the measures $\mathbb{P}_m$ and $\mathbb{P}_m^+$ are indeed central. Note moreover that for $m \in K(\delta)^+$, Littelmann's theory yields that for $\pi \in \Gamma_n^{\Delta}(y)$,

$$\sum_{\tilde{\pi} \in \Gamma_{n+1}^{\Delta}, \tilde{\pi}_{|[0,n]} = \pi} \mathbb{P}_m^+(\Gamma^{\Delta}(\tilde{\pi}))$$

$$= \frac{1}{S_\delta(\mathbf{t}_m)^{n+1}} \sum_{\mu \in B(\delta), \pi.\mu \in \Gamma_{n+1}^{\Delta}(y)} S_{\pi(n)+\mu(1), n\delta + x}(\mathbf{t}_m)$$

$$= \frac{1}{S_\delta(\mathbf{t}_m)^{n+1}} S_{\pi(n), n\delta}(\mathbf{t}_m) S_\delta(\mathbf{t}_m)$$

$$= \frac{S_{\pi(n), n\delta}(\mathbf{t}_m)}{S_\delta(\mathbf{t}_m)^n} = \mathbb{P}_m^+(\Gamma^{\Delta}(\pi)),$$

so that $\mathbb{P}_m^+$ is a well defined probability measure on Γ^{Δ}. The main point of the result is to prove that $\mathbb{P}$ and $\mathbb{P}^+$ are bijective.

Remark 3.2. — In type A_d, when $\delta = \omega_1$ is the first fundamental weight, $V(\delta)$ can be regarded as the defining representation of $\mathfrak{sl}_{d+1}$ or more conveniently, of $\mathfrak{gl}_{d+1}$. The set $\partial\mathcal{H}(\Delta)$ is then homeomorphic to

$$K(\delta)^+ = \left\{ (p_1, \ldots, p_{n+1}) \in \mathbb{R}^{d+1} \,\middle|\, \begin{matrix} p_1 \geqslant \cdots \geqslant p_{n+1} \geqslant 0 \\ \text{and } p_1 + \cdots + p_{n+1} = 1 \end{matrix} \right\}$$

and we recover the finite-dimensional version of the Thoma simplex.

As a corollary of Theorem 3.1, we get the complete characterization of c-harmonic measures killed when exiting Δ. Define the function $\widehat{s}_\delta$: $\partial\mathcal{H}(\Delta) \to \mathbb{R}^+ \cup \{\infty\}$ by $\widehat{s}_\delta(\mathbb{P}_m^+) = s_\delta(\mathbf{t}_m)$.

COROLLARY 3.3. — *For $c > 0$, the set $\partial\mathcal{H}_c(\Delta)$ is equal to*

$$\widehat{s}_\delta^{-1}(\{c \dim V(\delta)\}).$$

In particular,

- $\mathcal{H}_1(\Delta) = \partial\mathcal{H}_1(\Delta)$ *is a singleton corresponding to* $\mathbb{P}_\mathbf{0}^+$,
- *and for $c < 1$, $\mathcal{H}_c(\Delta) = \emptyset$.*

This corollary gives a positive answer to the our Problem 2.3. We prove Corollary 3.3 in Section 6.3. We discuss here a possible generalization of the latter result. Let X be an arbitrary multiset of infinitesimal paths (or alternatively a weight set of such paths). Set $\widehat{Z}(\mathbf{t}) := \sum_{\pi \in X} \mathbf{t}^{\pi(1)}$ and $z = \widehat{Z}(1)$. Finally, fix a cone $\mathcal{C}$ centered at 0 and denote by $K_\mathcal{C}$ the set of elements $\mathbf{t} \in \mathbb{R}^d$ such that $\sum_{\pi \in X} \mathbf{t}^{\pi(1)}\pi(1) \in \mathcal{C}$.

A function f is c-harmonic for these paths if and only if

$$f(x) = \sum_{\substack{\pi \in X \\ x+\pi \subset \mathcal{C}}} \frac{1}{cz} \operatorname{wt}(\pi) f(x + \pi(1)).$$

We can use the same notation as in the case where $X = B(\delta)$ is the a set of Littelmann paths associated to δ. Then, we conjecture that the following general result holds:

CONJECTURE 3.4. — *For $c > 0$, the set $\partial\mathcal{H}_c(\Delta)$ is homeomorphic to $\widehat{Z}^{-1}(\{cz\}) \cap K_\mathcal{C}$. In particular,*

- *for $u = \min_{K_\mathcal{C}} \widehat{Z}$, $\mathcal{H}_{u/z}(\Delta)$ is a singleton.*
- *and for $c < u/z$, $\mathcal{H}_c(\Delta) = \emptyset$.*

This conjecture is a generalization of the conjecture of Raschel [19, Conjecture 1] for two dimensional random walks with bounded increments, which asserts that such a random walk admits a unique harmonic function killed on the boundary of a quarter plane. This special situation can be seen in the above conjecture, in which case the minimum of $\widehat{Z}$ is exactly z.

3.4. The approach of Handelman and Price

The existence of the homeomorphisms of Theorem 3.1 can also be deduced from the main results of [8, 9], themselves based on fundamental results of [17, 18]. We review here their approach, and the reader could read the aforementioned articles and references therein for a detailed proof.

Let n denote the dimension of $V(\delta)$, and consider the adjoint representation $\rho : G \to GL(M_n(\mathbb{C}))$ which is defined by $\rho(g)(M) = u_\delta(g)Mu_\delta(g)^{-1}$, where u_δ is the irreducible representation associated with δ. Form the infinite tensor product $A := \bigotimes M_n(\mathbb{C})$ as an inductive limit of the sequence of finite-dimensional C^*-algebras $(M_n(\mathbb{C})^{\otimes k})_{k \geqslant 1}$, where $M_n(\mathbb{C})^{\otimes k}$ embeds in $M_n(\mathbb{C})^{\otimes k+1}$ with the map $X \mapsto X \otimes \mathrm{Id}_n$. We can canonically associate a structure of C^*-algebra to this inductive limit of C^*-algebras. Then, G acts continuously on each $M_n(\mathbb{C})^{\otimes k}$ and on A with the map $\widetilde{\rho}(g) := \bigotimes \rho(g)$ (which means that g acts as $\rho(g)$ on each component of the tensor product), and we can therefore consider the C^*-algebra A^δ (resp. A_k^δ) of elements of A (resp. $M_n(\mathbb{C})^{\otimes k}$) fixed by ρ. The algebra A^δ is the inductive limit of the finite-dimensional C^*-algebras $(A_k^\delta)_{k \geqslant 1}$, and the Bratteli diagram of this inductive limit is exactly the growth graph of Γ^Δ. Therefore, the set of central measures on Γ^Δ is in bijection with the set of traces on A^δ.

Doing the same construction for the restriction of the representation δ to the maximal torus $T \subset G$, we get another sequence of finite dimensional C^*-algebras $(A_k^T)_{k \geqslant 1}$, whose inductive limit is denoted by A^T. Similarly, the Bratteli diagram of A^T is exactly the growth graph of $\Gamma^\mathbb{R}$, and the set of central measures on $\Gamma^\mathbb{R}$ is in bijection with the set of traces on A^T.

Note that we have the natural inclusion of C^*-algebras $A^\delta \subset A^T$. The main result of [8] is that any extremal trace on A^δ extends to an extremal trace on A^T. To prove this, the author uses the bijection between the set of traces on an approximately finite C^*-algebra A and the set of states on its associated dimension group $K_0(A)$. Let us quickly explain the nature of $K_0(A)$: a dimension group is a group with a notion of positive cone. By considering equivalence classes of projections on the $*$-algebra $\bigoplus_{k \geqslant 1} M_k(A)$, one can canonically associate a dimension group $K_0(A)$ to each C^*-algebra A; this dimension group is always a ring in our case. An important fact is that an inclusion of C^*-algebras induces an inclusion of the associated dimension groups, and therefore the problem reduces to extend any state on $K_0(A^\delta)$ to a state on $K_0(A^T)$. Handelman managed to prove this in [8], and the main ingredient of the proof is the non-trivial property that $K_0(A^T)$ is a finitely generated $K_0(A^\delta)$-module.

Once proven that any trace on A^δ extends to a trace on A^T, the problem amounts to describe the set of traces on A^T. In [9], the author achieves this by proving that the set of faithful traces on A^T is in bijection with the interior of $K(\delta)$. Then, the identification of the set of faithful traces on A^δ with the interior of $K(\delta)^+$ is done thanks to a result of [18], which asserts that the Weyl group W acts transitively on the set of traces extending a

particular faithful trace on A^T. Finally, the case of non-faithful traces is done by considering parabolic subgroups of G.

3.5. The extended algebra of characters

Our proof of Theorem 3.1 will mainly use algebraic properties of the representations of the Lie algebra $\mathfrak{g}$. We define the extended algebra of characters $\widehat{\mathcal{A}}_\delta$ as follows:

- $\widehat{\mathcal{A}}_\delta$ is isomorphic to $\mathcal{A}_\delta \times \mathbb{R}[T]$ as a vector space; for $x \in \mathcal{A}_\delta$, we simply denote by (x, n) the element (x, T^n). A basis of $\widehat{\mathcal{A}}_\delta$ is given by the set $\mathcal{B} = \{(s_\lambda, n)\}_{n \geqslant 0, \lambda \in T_\delta^+}$ and the multiplicative structure of $\widehat{\mathcal{A}}_\delta$ is defined on $\mathcal{B}$ by the product
$$(s_\lambda, n) \times (s_\mu, m) = (s_\lambda s_\mu, n + m).$$

- We also denote by $\widehat{T}_\delta^+$ the subalgebra of $\widehat{\mathcal{A}}_\delta$ spanned by $\{(s_\lambda, n) \mid \lambda \in T_{\delta,n}^+\}$ where $T_{\delta,n}^+$ is the set of dominant weights λ such that $V(\lambda)$ is an irreducible component of $V(\delta)^{\otimes n}$.

Likewise, we define the extended algebra of weights $\widehat{P}_\delta$ as follows

- $\widehat{P}_\delta$ is isomorphic to $\mathbb{R}[P_\delta] \times \mathbb{R}[T]$ as a vector space. A basis of $\widehat{P}_\delta$ is given by the set $\{(e^\gamma, n) \mid n \geqslant 0, \gamma \in P_\delta\}$. The multiplicative structure of $\widehat{P}_\delta$ is defined by the product
$$(e^\gamma, n) \times (e^{\gamma'}, m) = (e^{\gamma+\gamma'}, n + m).$$

Write $T_{\delta,n}$ for the set of weights γ appearing with nonzero multiplicity in the representation $V(\delta)^{\otimes n}$. We shall also need the algebra $\widehat{T}_\delta$ defined as follows

- $\widehat{T}_\delta$ is the subalgebra of $\widehat{P}_\delta$ spanned by the elements $\{(e^\gamma, n) \mid n \geqslant 1, \gamma \in T_{\delta,n}\}$.

Note that the inclusion $\mathcal{A}_\delta \subset \mathbb{R}[P_\delta]$ translates naturally into the inclusion $\widehat{\mathcal{A}}_\delta \subset \widehat{P}_\delta$ and $\widehat{T}_\delta^+ \subset \widehat{T}_\delta$.

We can write the multiset of weights of δ in $\widehat{T}_\delta$ as $\Pi_\delta = \{(e^{\gamma_1}, 1), \ldots, (e^{\gamma_N}, 1)\}$ where each weight appears a number of times equal to its multiplicity. For any $k = 0, \ldots, N$, let $e_k(X_1, \ldots, X_N)$ be the k-th elementary symmetric function in the variables $X_1, \ldots, X_N$. Define the polynomial $\Phi(X) \in \widehat{T}_\delta[X]$ by
$$\Phi(X) = \prod_{\gamma \in \Pi_\delta} (X + (e^\gamma, 1)).$$

PROPOSITION 3.5. — *We have*

(3.3)
$$\Phi(X) = \sum_{k=0}^{N}(e_k(e^{\gamma_1}, \ldots, e^{\gamma_N}), k)X^{N-k}$$

and for any $k = 0, \ldots, N$, *the expression* $(e_k(e^{\gamma_1}, \ldots, e^{\gamma_N}), k)$ *decomposes as a sum of elements* $(s_\lambda, n) \in \widehat{T}_\delta^+$ *with positive integer coefficients. In particular, we have* $\Phi(X) \in \widehat{T}_\delta^+[X]$.

Proof. — Recall that $e_k((e^{\gamma_1}, 1), \ldots, (e^{\gamma_N}, 1))$ is the k-th elementary symmetric function in minus the roots of the polynomial Φ. Hence, $e_k(e^{\gamma_1}, \ldots, e^{\gamma_N})$ is the plethysm of the elementary symmetric function e_k by s_δ. This means that

$$e_k(e^{\gamma_1}, \ldots, e^{\gamma_N}) = \mathrm{char}\left(\bigwedge^{k} V(\delta)\right)$$

is the character of the k-th exterior power of the representation $V(\delta)$. Since $\bigwedge^k V(\delta)$ is a submodule of $V(\delta)^{\otimes k}$, its character indeed decomposes as a sum of characters in $\{s_\lambda \mid V(\lambda) \in V(\delta)^{\otimes k}\}$ with positive integer coefficients. $\square$

COROLLARY 3.6. — $\widehat{T}_\delta^+$ *is integrally closed in* $\widehat{T}_\delta$.

Proof. — Let $\overline{\widehat{T}_\delta^+}$ denote the integral closure of $\widehat{T}_\delta^+$ in $\widehat{T}_\delta$. We have $\overline{\widehat{T}_\delta^+} \subset \widehat{T}_\delta$ by definition. Conversely, since $\overline{\widehat{T}_\delta^+}$ is a ring and $\widehat{T}_\delta$ is generated by the monomials $(e^\gamma, 1)$ with $\gamma \in \Pi_\delta$, it suffices to prove that each such $(e^\gamma, 1)$ belongs to $\overline{\widehat{T}_\delta^+}$. But $-(e^\gamma, 1)$ is a root of $\Phi(X)$ which is, by the previous proposition, a monic polynomial with coefficients in $\overline{\widehat{T}_\delta^+}$. Therefore $-(e^\gamma, 1)$ and $(e^\gamma, 1)$ are integers over $\widehat{T}_\delta^+$ and thus belong to $\overline{\widehat{T}_\delta^+}$. $\square$

4. Minimal boundary of $\Gamma^\mathbb{R}$

4.1. Algebraic description of the growth graph

Let $\mathcal{G}(\mathbb{R}^d)$ be the growth graph of $\Gamma^\mathbb{R}$ and $\mathcal{G}(\Delta)$ be the one of Γ^Δ. Namely, the set $\mathcal{G}_n(\mathbb{R}^d)$ of vertices of rank n of the graph $\mathcal{G}(\mathbb{R}^d)$ are pairs (γ, n) where γ is a weight of P_δ such that $\Gamma_n^{\mathbb{R}^d}(\gamma) \neq \emptyset$, and the weight of the edge between (γ, n) and $(\gamma', n+1)$ is $\#\Gamma^\mathbb{R}(\gamma, \gamma')$. From the graph embedding of Section 2.4, the set of extreme central measures on $\Gamma^\mathbb{R}$ is in bijection with the set of extreme points of the convex set $\partial\mathcal{H}(\mathcal{G}(\mathbb{R}^d))$ of nonnegative harmonic functions $p : \mathcal{G}(\mathbb{R}^d) \to \mathbb{R}^+$ with $p(0,0) = 1$ and the same

holds for $\mathcal{G}(\Delta)$. An important feature of $\mathcal{G}(\mathbb{R}^d)$ is that this graded graph is multiplicative: it is related to the algebra $\widehat{T}_\delta$ as follows.

PROPOSITION 4.1. — $\mathcal{G}(\mathbb{R}^d)$ *is a multiplicative graph associated with the algebra* $\widehat{P}_\delta$ *with the injective map*

$$\iota : \begin{cases} \mathcal{G}(\mathbb{R}^d) \longrightarrow \widehat{P}_\delta \\ (\gamma, n) \longmapsto (e^\gamma, n), n \geqslant 1 \\ \longmapsto (s_\delta, 1), \end{cases}$$

and $\iota(\mathcal{G}(\mathbb{R}^d)) = \widehat{T}_\delta$. *In particular,* $\partial\mathcal{H}(\mathcal{G}(\mathbb{R}^d))$ *is homeomorphic to* $\mathrm{Mult}(\widehat{T}_\delta)^+$ *through the map*

$$\iota^* : \begin{cases} \mathrm{Mult}(\widehat{T}_\delta)^+ \longrightarrow \partial\mathcal{H}(\mathcal{G}(\mathbb{R}^d)) \\ f \longmapsto f \circ \iota. \end{cases}$$

Proof. — Since $\#\Gamma^\mathbb{R}(\gamma, \gamma') = K_{\delta, \gamma' - \gamma}$, the following equality holds for $(\gamma, n) \in \mathcal{G}_n(\mathbb{R}^d)$:

$$\iota(\gamma, n)\iota(*) = (e^\gamma, n)\left(\sum_{\kappa \in \Pi_\delta} K_{\delta, \kappa} e^\kappa, 1 \right) = \sum_{\kappa \in \Pi_\delta} K_{\delta, \kappa}(e^{\gamma + \kappa}, n + 1)$$

$$= \sum_{\substack{\gamma' \in P_\delta \\ \gamma' - \gamma \in \Pi_\delta}} K_{\delta, \gamma' - \gamma}(e^{\gamma'}, n + 1)$$

$$= \sum_{\gamma' \in P_\delta} \#\Gamma^\mathbb{R}(\gamma, \gamma')\iota(\gamma', n + 1).$$

Thus, $\mathcal{G}(\mathbb{R}^d)$ is a multiplicative graph associated with $\widehat{P}_\delta$ through the map ι. Note that by construction, the sub-algebra of $\widehat{P}_\delta$ generated by the elements $\{\iota(\gamma, n)\}_{(\gamma, n) \in \mathcal{G}(\mathbb{R}^d)}$ is precisely $\widehat{T}_\delta$: the last part of the proposition is deduced from Proposition 2.5. □

4.2. Characterization of the multiplicative maps on $\widehat{T}_\delta$

The set of extreme central measures on $\mathcal{G}(\mathbb{R}^d)$ is thus given by the set of positive morphisms from $\widehat{T}_\delta$ to $\mathbb{R}$ which take the value 1 on $(s_\delta, 1)$. We will prove in this subsection the following result:

PROPOSITION 4.2. — *Let $f \in \mathrm{Mult}(\widehat{T}_\delta)^+$. There exists a multiplicative map $\phi : \mathbb{R}[Q^+] \to \mathbb{R}^+$ and an element $w \in W$ such that*

$$f(e^\gamma, n) = \frac{1}{\phi(S_\delta)^n}\phi(e^{n\delta - w(\gamma)}),$$

for all $(e^\gamma, n) \in \widehat{T}_\delta$.

Note that the element $\phi(e^{n\delta - w(\gamma)})$ is well-defined: indeed, if $(e^\gamma, n) \in \widehat{T}_\delta$, then the weight γ appears in the representation $V(\delta)^{\otimes n}$ and $w(\gamma)$ is thus smaller than $n\delta$ with respect to the roots order relative to the set of simple roots S. Therefore, $n\delta - w(\gamma) \in Q^+$.

Let f be a multiplicative map on $\widehat{T}_\delta$. Since f is multiplicative and $\widehat{T}_\delta$ is generated by the set $\widetilde{\Pi}_\delta := \{(e^\gamma, 1), \gamma \in \Pi_\delta\}$, f is completely determined by its values on $\widetilde{\Pi}_\delta$. We can also extend naturally the action of W by setting $w.(e^\gamma, n) = (e^{w(\gamma)}, n)$. We suppose from now on that $f \in \mathrm{Mult}(\widehat{T}_\delta)^+$. Let

$$(4.1) \qquad M_f = \sum_{\gamma \in \Pi_\delta} K_{\delta,\gamma} f(\gamma, 1)\gamma.$$

The vector M_f belongs to $\mathbb{R}^d$, thus there exists $w \in W$ such that $w(M_f) \in \Delta$. Replacing f by $f \circ w^{-1}$ gives another multiplicative map on $\widehat{T}_\delta$ such that

$$M_{f \circ w^{-1}} = \sum_{\gamma \in \Pi_\delta} K_{\delta,\gamma}(f \circ w^{-1})(e^\gamma, 1)\gamma \in \Delta$$

and we have $f = (f \circ w^{-1}) \circ w$.

LEMMA 4.3. — *Assume that $M_f \in \Delta$ and let $\alpha \in S$. For all $\gamma \in \Pi_\delta$ such that*

$$f(e^\gamma, 1) = 0 \longrightarrow f(e^{\gamma - \alpha}, 1) = 0.$$

In particular, $f(e^\delta, 1) \neq 0$.

Proof. — It is a classical result in the representation theory of $\mathfrak{g}$ that for any weight $\gamma \in \Pi_\delta$ and any simple root $\alpha \in S$ such that $\gamma - \alpha \notin \Pi_\delta$, we must have $\langle \gamma, \alpha \rangle \leqslant 0$. Also if both γ and $\gamma - \alpha$ belong to Π_δ but $\gamma - 2\alpha$ does not, one has $\langle \gamma - \alpha, \alpha \rangle < 0$.

Now let $\alpha \in S$, and suppose that there exists $\gamma \in \Pi_\delta$ such that $\gamma - \alpha \in \Pi_\delta, f(e^\gamma, 1) = 0$ and $f(e^{\gamma - \alpha}, 1) \neq 0$. If γ' is another weight of Π_δ such that $f(e^{\gamma'}, 1) \neq 0$, then necessarily $\gamma' - \alpha \notin \Pi_\delta$: indeed, if $\gamma' - \alpha \in \Pi_\delta$, then

$$f(e^{\gamma' - \alpha}, 1)f(e^\gamma, 1) = f(e^{\gamma + \gamma' - \alpha}, 2) = f(e^{\gamma - \alpha}, 1)f(e^{\gamma'}, 1) \neq 0,$$

which contradicts the fact that $f(e^\gamma, 1) = 0$. For all $\gamma' \in \Pi_\delta$, $\gamma' - \alpha \notin \Pi_\delta$ implies that $\langle \gamma', \alpha \rangle \leqslant 0$: thus, $f(e^{\gamma'}, 1) \neq 0$ implies that $\langle \gamma', \alpha \rangle \leqslant 0$. We get

$$\left\langle \sum_{\gamma' \neq \gamma - \alpha} K_{\delta, \gamma'} f(e^{\gamma'}, 1) \gamma', \alpha \right\rangle \leqslant 0.$$

Since $f(e^{\gamma - \alpha}, 1) \neq 0$ and from the previous argument, we get $\gamma - 2\alpha \notin \Pi_\delta$. This impose $\langle \gamma - \alpha, \alpha \rangle < 0$ as claimed below. Finally,

$$\langle M_f, \alpha \rangle = K_{\delta, \gamma - \alpha} f(e^{\gamma - \alpha}, 1) \langle \gamma - \alpha, \alpha \rangle + \sum_{\gamma' \neq \gamma - \alpha} K_{\delta, \gamma'} f(e^{\gamma'}, 1) \langle \gamma', \alpha \rangle < 0,$$

which contradicts the fact that $M_f \in \Delta$. Let $\gamma \in \Pi_\delta$ be such that $f(e^\gamma, 1) \neq 0$. Since $\gamma \in \Pi_\delta$, there exists a finite sequence $(x_i)_{1 \leqslant i \leqslant r}$ in S such that $\delta - \sum_{i=1}^j x_i \in \Pi_\delta$ for all $1 \leqslant j \leqslant r$ and $\delta - \sum_{i=1}^r x_i = \gamma$. Thus, from the first part of the lemma, $f(e^{\delta - \sum_{i=1}^j x_i}, 1) \neq 0$ for all $1 \leqslant j \leqslant r$; in particular, $f(e^{\delta - x_1}, 1) \neq 0$, and applying again the first part of the lemma yields that $f(e^\delta, 1) \neq 0$. $\square$

We can now prove Proposition 4.2:

Proof of Proposition 4.2. — Let $f \in \mathrm{Mult}(\widehat{T}_\delta)^+$ be such that $M_f \in \Delta$. Let $\alpha \in S$. If for all $\gamma \in \Pi_\delta$ such that $f(e^\gamma, 1) \neq 0$ we have $\gamma - \alpha \notin \Pi_\delta$, then set $\phi(e^\alpha) = 0$. Otherwise, let $\gamma \in \Pi_\delta$ be such that $f(e^\gamma, 1) \neq 0$ and such that $\gamma - \alpha \in \Pi_\delta$, and set $\phi(e^\alpha) = \frac{f(e^{\gamma - \alpha}, 1)}{f(e^\gamma, 1)}$. Then, $\phi(e^\alpha)$ is independent of the choice of γ. Indeed, if γ' is another weight satisfying the same hypothesis, then

$$f(e^\gamma, 1) f(e^{\gamma' - \alpha}, 1) = f(e^{\gamma + \gamma' - \alpha}, 2) = f(e^{\gamma - \alpha}, 1) f(e^{\gamma'}, 1),$$

so that finally,

$$\frac{f(e^{\gamma - \alpha}, 1)}{f(e^\gamma, 1)} = \frac{f(e^{\gamma' - \alpha}, 1)}{f(e^{\gamma'}, 1)}.$$

Note that we have in particular proven that for all $\gamma \in \Pi_\delta$ such that $\gamma + \alpha \in \Pi_\delta$ and $f(e^{\gamma + \alpha}, 1) \neq 0$, we have

$$(4.2) \qquad\qquad \frac{f(e^\gamma, 1)}{f(e^{\gamma + \alpha}, 1)} = \phi(e^\alpha).$$

Let $\phi : \mathbb{R}[Q^+] \to \mathbb{R}^+$ be the multiplicative map obtained by extending multiplicatively the map ϕ defined on $\{e^\alpha, \alpha \in S\}$ and by specifying the value $\phi(1) = 1$. Recall the dominant order $\leqslant$ on the weight lattice P: $\gamma \leqslant \gamma'$ if and only if $\gamma' - \gamma$ is a sum of simple roots. Let us prove by induction on the dominant order that $f(e^\gamma, 1) = f(e^\delta, 1) \phi(e^{\delta - \gamma})$ for $\gamma \in \Pi_\delta$. For $\gamma = \delta$ the result is straightforward. Let $\gamma \in \Pi_\delta$ and suppose that the result is true for all $\gamma' > \gamma$. There exists $\alpha \in S$ such that $\gamma + \alpha \in \Pi_\delta$. If $f(e^{\gamma + \alpha}, 1) = 0$,

then $f(e^\gamma, 1) = 0$ by Lemma 4.3; in particular, $f(e^\gamma, 1) = \phi(e^\alpha)f(e^{\gamma+\alpha}, 1)$. By the induction hypothesis, $f(e^{\gamma+\alpha}, 1) = f(e^\delta, 1)\phi(e^{\delta-(\gamma+\alpha)})$, and finally,

$$f(e^\gamma, 1) = \phi(e^\alpha)f(e^\delta, 1)\phi(e^{\delta-(\gamma+\alpha)}) = f(e^\delta, 1)\phi(e^{\delta-\gamma}).$$

If $f(e^{\gamma+\alpha}, 1) \neq 0$, then by (4.2) and by the induction hypothesis,

$$f(e^\gamma, 1) = \phi(e^\alpha)f(e^{\gamma+\alpha}, 1) = \phi(e^\alpha)f(e^\delta, 1)\phi(e^{\delta-(\gamma+\alpha)}) = f(e^\delta, 1)\phi(e^{\delta-\gamma}).$$

Let $(\gamma, n) \in \widehat{T}_\delta$, and let $\gamma_1, \ldots, \gamma_n \in \Pi_\delta$ such that $\gamma = \sum_{i=1}^n \gamma_i$. Then, by multiplicativity of f and the result above, we have

$$f(e^\gamma, n) = f\left(e^{\sum_{i=1}^n \gamma_i}, n\right) = \prod_{i=1}^n f(e^{\gamma_i}, 1) = \prod_{i=1}^n f(e^\delta, 1)\phi(e^{\delta-\gamma_i})$$

$$= f(e^\delta, 1)^n \phi\left(e^{n\delta - \sum_{i=1}^n \gamma_i}\right) = f(e^\delta, 1)^n \phi(e^{n\delta-\gamma})$$

Since $f(s_\delta, 1) = 1$, we have on the one hand

$$\sum_{\gamma \in \Pi_\delta} K_{\delta,\gamma} f(e^\gamma, 1) = 1.$$

On the other hand, from the previous result,

$$\sum_{\gamma \in \Pi_\delta} K_{\delta,\gamma} f(e^\gamma, 1) = \sum_{\gamma \in \Pi_\delta} K_{\delta,\gamma} f(e^\delta, 1)\phi(e^{\delta-\gamma}) = f(e^\delta, 1)\phi(S_\delta).$$

Thus, $f(e^\delta, 1) = \frac{1}{\phi(S_\delta)}$, which ends the proof of the proposition in the case $M_f \in \Delta$.

Now assume that f is a general nonnegative multiplicative function on $\widehat{T}_\delta$. Let $w \in W$ be such that $M_{f \circ w^{-1}} \in \Delta$. By the first part of the proof, there exists $\phi \in \text{Mult}(\mathbb{R}[Q+])^+$ such that $f \circ w^{-1}(e^\gamma, n) = \frac{1}{\phi(S_\delta)^n}\phi(e^{n\delta-\gamma})$. Thus, composing $f \circ w^{-1}$ with w yields that $f(e^\gamma, n) = \frac{1}{\phi(S_\delta)^n}\phi(e^{n\delta-w(\gamma)})$ for $(\gamma, n) \in \widehat{T}_\delta$. $\qquad\square$

Remark 4.4. — Suppose that $\phi(e^\alpha) \neq 0$ for all $\alpha \in S$. Then, the map ϕ extends to a homomorphism $\phi : \mathbb{R}[P] \to \mathbb{R}^+$ with the formula

$$\phi(e^\gamma) = \prod_{\alpha \in S} \phi(e^\alpha)^{r_\alpha}$$

for $\gamma = \sum_{\alpha \in S} r_\alpha \alpha$ with $r_\alpha \in \mathbb{Q}$ for any $\alpha \in S$. In this case,

$$f(e^\gamma, n) = f(e^\delta, 1)^n \phi(e^{\lambda-n\delta}) = \left(\frac{f(e^\delta, 1)}{\phi(e^\delta)}\right)^n \phi(e^\gamma).$$

Since, $f(s_\delta, 1) = 1$, $\frac{f(e^\delta, 1)}{\phi(e^\delta)} = \phi(s_\delta)^{-1}$. Hence, when $\phi(e^\alpha) > 0$ for all $\alpha \in S$, f can be written on $\widehat{T}_\delta$ as

$$f(e^\gamma, n) = \frac{\phi(e^\gamma)}{\phi(s_\delta)^n},$$

with $\phi : P \to \mathbb{R}^+$ a multiplicative map.

In this case, for $(s_\lambda, n) \in \widehat{T}_\delta^+$, we also have

$$f(s_\lambda, n) = \frac{\phi(s_\lambda)}{\phi(s_\delta)^n},$$

where the restriction $\phi : \mathcal{A}_\delta \to \mathbb{R}^+$ is again a multiplicative map.

Let us reformulate the results of this section by introducing the map $\Phi : \mathrm{Mult}(\mathbb{R}[Q^+])^+ \times W \to \mathrm{Mult}(\widehat{T}_\delta)^+$ defined by

$$(4.3) \qquad\qquad \Phi(\phi, w)(e^\gamma, n) = \frac{1}{\phi(S_\delta)} \phi(e^{n\delta - w(\gamma)}).$$

Proposition 4.2 yields that the map Φ is surjective. Since $\mathbb{R}[Q^+]$ is the free commutative algebra generated by $\{e^\alpha, \alpha \in S\}$, $\mathrm{Mult}(\mathbb{R}[Q^+])^+$ is isomorphic to $(\mathbb{R}^+)^d$ through the map $\theta : \mathrm{Mult}(\mathbb{R}[Q^+])^+ \to (\mathbb{R}^+)^d$ given by

$$(4.4) \qquad\qquad \theta(\phi) = (\phi(e^{\alpha_i}))_{1 \leqslant i \leqslant d}$$

for $\phi \in \mathrm{Mult}(\mathbb{R}[Q^+])^+$. The composition of Φ with θ^{-1} yields thus a surjective map $(\mathbb{R}^+)^d \times W \to \mathrm{Mult}(\widehat{T}_\delta)^+$. Since Φ is not necessarily injective, the latter map is not bijective. The lack of injectivity comes from two facts: first, if M_f lies at the intersection of two Weyl chambers, then $M_{f \circ w^{-1}} \in \Delta$ for several $w \in W$. Next, some degeneracy may occur when δ is orthogonal to some simple roots. The goal of the next subsection is to overcome the second problem.

4.3. Dominant faces of the weight polytope

Let $f \in \mathrm{Mult}(\widehat{T}_\delta)^+$ such that $M_f \in \Delta$; it is possible to give a geometric description of the set $\Pi_\delta(f) := \{\gamma \in \Pi_\delta \mid f(e^\gamma, 1) \neq 0)\}$. A face of the polytope $K(\delta)$ is the intersection of $K(\delta)$ with a supporting hyperplane (that is $K(\delta)$ is contained in one of the two half-spaces defined by this hyperplane). A dominant face F is a face of the polytope $K(\delta)$ such that $F \cap \Delta \neq 0$. We denote by Π_F the intersection of Π_δ with F. By an indecomposable component of a subset $S' \subset S$ we mean a maximal subset of S' consisting of mutually non-orthogonal simple roots.

DEFINITION 4.5. — *A subset $S' \subset S$ of simple roots is δ-admissible if each indecomposable component of S' contains a simple root which is not orthogonal to δ.*

Observe that, according to this definition, the empty set is a δ-admissible subset, since it has no indecomposable component. Write $\langle S' \rangle$ for the linear span of S'. For each subset $S' \subset S$, denote by $W_{S'}$ the Weyl group generated by the elements $s_{\alpha'}, \alpha' \in S'$ (where $W_\emptyset$ is simply $\{\mathrm{Id}\}$). We will use the following results which comes from [21].

THEOREM 4.6. — *Assigning to each δ-admissible subset $S' \subset S$ the polytope $F_{S'} = \mathrm{Conv}(w'\delta \,|\, w' \in W_{S'})$ yields a one-to-one correspondence between δ-admissible subsets of S and dominant faces of the polytope $K(\delta)$. Moreover, the set $\Pi_{F_{S'}} = \Pi_\delta \cap F_{S'}$ coincides with the set $(\delta + \langle S' \rangle) \cap \Pi_\delta$ and $\dim F_{S'} = \#S'$*

Let us denote by S_F the δ-admissible subset of simple roots associated to the dominant face F by the previous theorem. We get the following characterization $\Pi_\delta(f)$.

PROPOSITION 4.7. — *Assume M_f belongs to Δ. Then, there exists a dominant face F of the weight polytope $K(\delta)$ such that $\Pi_\delta(f) = \Pi_F$.*

Before proving Proposition 4.7, let us prove the following lemma:

LEMMA 4.8. — *Let $S' \subset S$ and $\gamma \in \Pi_\delta$ such that $\delta - \gamma = \sum_{\alpha \in S'} k_\alpha \alpha$ with $k_\alpha > 0$ for all $\alpha \in S'$. Then, S' is δ-admissible and $\gamma \in F_{S'}$.*

The proof of this lemma uses ingredients similar to those of Vinberg in [21].

Proof. — Suppose γ can be written as

$$\gamma = \delta - \sum_{\alpha \in S'} k_\alpha \alpha,$$

with S' a subset of S and $k_\alpha \in \mathbb{N}^*$ for $\alpha \in S'$. Since $\gamma \in \Pi_\delta$, there exists a sequence $(\gamma_i)_{0 \leqslant i \leqslant t}$ with $t = \sum_{\alpha \in S'} k_\alpha$ such that $\gamma_i \in \Pi_\delta$, $\gamma_0 = \gamma$, $\gamma_t = \delta$ and $\gamma_{i+1} - \gamma_i \in S$. Since for all $\gamma \in \Pi_\delta$, $\delta - \gamma$ is a sum of simple roots with nonnegative coefficients, for all $0 \leqslant i \leqslant t-1$ we have $\gamma_{i+1} - \gamma_i \in S'$ and $\#\{0 \leqslant i \leqslant t-1 \,|\, \gamma_{i+1} - \gamma_i = \alpha\} = k_\alpha$ for $\alpha \in S'$. This implies in particular that $\gamma_i \in \delta + \langle S' \rangle$ for all $0 \leqslant i \leqslant t$. Let $\alpha \in S'$: since $k_\alpha > 0$, there exists $0 \leqslant i_\alpha \leqslant t-1$ such that $\gamma_{i_\alpha+1} - \gamma_{i_\alpha} = \alpha$. This yields that $\dim(K(\delta) \cap (\delta + \langle S' \rangle)) = \#S'$. Let L be the linear form such that $L(\alpha) = 1$ for $\alpha \in S \setminus S'$ and $L(\alpha) = 0$ for $\alpha \in S$. For $\gamma \in \Pi_\delta$, $\delta - \gamma$ is a sum of simple roots with positive coefficients, thus $L(\gamma) \leqslant L(\delta)$, with equality if and only

if $\gamma \in \delta + \langle S' \rangle$. Thus, $(K(\delta) \cap (\delta + \langle S' \rangle))$ is a face of the polytope $K(\delta)$. Since $\dim(K(\delta) \cap (\delta + \langle S' \rangle)) = \#S'$, the set S' is δ-admissible by [21, p. 10]. Finally, $K(\delta) \cap (\delta + \langle S' \rangle) = F_{S'}$ and $\gamma \in F_{S'}$ because $F_{S'}$ has dimension $\#S'$.   □

LEMMA 4.9. — $\Pi_F \subset \Pi_\delta(f)$ if and only if ϕ is nonzero on S_F.

Proof. — Suppose that $\Pi_F \subset \Pi_\delta(f)$. Let $\alpha_{i_0} \in S_F$. Since $f(e^\gamma, 1)$ is nonzero for $\gamma \in \Pi_F$, by Lemma 4.3 and the definition of ϕ it suffices to prove that there exists $\gamma \in \Pi_F$ such that $\gamma + \alpha_{i_0} \in \Pi_F$ or $\gamma - \alpha_{i_0} \in \Pi_F$. Since S_F is δ-admissible, $\dim F = \#S_F$; $F = \mathrm{Conv}(w.\delta \,|\, w \in W_{S_F})$ and $\dim F = \#S_F$, thus there exists $w \in W_F$ such that $\delta - w.\delta = \sum_{\alpha \in S_F} k_\alpha \alpha$ with $k_{\alpha_{i_0}} > 0$. This implies the existence of $\gamma \in \Pi_F$ such that $\gamma + \alpha_0 \in \Pi_F$. Since $\Pi_F \subset \Pi_\delta(f)$, $f(e^{\gamma + \alpha_{i_0}}, 1) \neq 0$ and $f(e^\gamma, 1) \neq 0$, and thus

$$\phi(\alpha_{i_0}) = \frac{f(e^\gamma, 1)}{f(e^{\gamma + \alpha_{i_0}}, 1)} \neq 0.$$

Conversely, suppose that ϕ is nonzero on S_F. By Theorem 4.6, $\Pi_F = (\delta + \langle S_F \rangle) \cap \Pi_\delta$. Since $f(e^\delta, 1) \neq 0$ and ϕ is nonzero on S_F, f is nonzero on Π_F by Proposition 4.2.   □

We turn now to the proof of Proposition 4.7.

Proof. — We order the set of dominant faces by the inclusion order; note that the set of dominant faces is a lattice with respect to this order, and we denote by $F \vee F'$ the supremum of two dominant faces F and F': $F \vee F'$ is the smallest dominant face containing both F and F'. Let $\gamma \in \Pi_\delta$ such that $f(e^\gamma, 1) \neq 0$, and let F be the smallest dominant face containing γ. The weight γ can be written as

$$\gamma = \delta - \sum_{\alpha \in S_F} k_\alpha \alpha$$

with $k_\alpha \in \mathbb{N}$. Necessarily, we have $k_\alpha > 0$ for all $\alpha \in S_F$. Otherwise, Lemma 4.8 would imply that γ belongs to a smaller dominant face of $K(\delta)$. Let $(\gamma_i)_{0 \leqslant i \leqslant t}$ with $t = \sum_{\alpha \in S'} k_\alpha$ be a sequence of Π_δ such that $\gamma_i \in \Pi_\delta$, $\gamma_0 = \gamma$, $\gamma_t = \delta$ and $\gamma_{i+1} - \gamma_i \in S_F$. Since $f(e^{\gamma_0}, 1) \neq 0$, Lemma 4.3 yields that $f(e^{\gamma_i}, 1) \neq 0$ for $0 \leqslant i \leqslant t$. Let $\alpha \in S_F$: since $k_\alpha > 0$, a similar deduction as in the proof of the previous lemma yields that there exists $0 \leqslant i \leqslant t - 1$ such that $\gamma_{i+1} - \gamma_i = \alpha$. Therefore,

$$\phi(e^\alpha) = \frac{f(e^{\gamma_i}, 1)}{f(e^{\gamma_{i+1}}, 1)} \neq 0.$$

Since $\phi(e^\alpha) \neq 0$ for $\alpha \in S_F$, $f(e^\gamma, 1)$ is nonzero on $\Pi_\delta \cap (\delta - \langle S_F \rangle)$, and $\Pi_F \subset \Pi_\delta(f)$. We have thus proven that if a weight γ is in $\Pi_\delta(f)$, then the

intersection of Π_δ with the smallest dominant face containing γ is also included in $\Pi_\delta(f)$; hence, $\Pi_\delta(f)$ is an union of sets Π_F, where F are dominant faces.

Let F and F' be two dominant faces such that $\Pi_F, \Pi_{F'} \subset \Pi_\delta(f)$, and let us show that $\Pi_{F \vee F'} \subset \Pi_\delta(f)$. Note first that $F \vee F' = F_{S_F \cup S_{F'}}$: on the first hand, the smallest vector space containing both $\langle S_F \rangle$ and $\langle S_{F'} \rangle$ is $\langle S_F \cup S_{F'} \rangle$. On the other hand, since S_F and $S_{F'}$ are δ-admissible, $S_F \cup S_{F'}$ is again δ-admissible. It suffices thus to show that $\Pi_{F_{S_F \cup S_{F'}}} \subset \Pi_\delta(f)$. But Lemma 4.9 yields that $\phi(e^\alpha)$ is nonzero for $\alpha \in S_F$ and $\alpha \in S_{F'}$. Thus, $\phi(e^\alpha)$ is nonzero for $\alpha \in S_F \cup S_{F'}$ and $\Pi_{F_{S_F \cup S_{F'}}} \subset \Pi_\delta(f)$. Let F_0 be the supremum of $\{F$ dominant face of $K(\delta), \Pi_F \subset \Pi_\delta(f)\}$. By the previous argument, $\Pi_{F_0} \subset \Pi_\delta(f)$. Let $\gamma \in \Pi_\delta(f)$ and let F be the smallest dominant face of $K(\delta)$ containing γ. By the first part of the proof, $\Pi_F \subset \Pi_\delta(f)$. Thus $F \subset F_0$ and $\gamma \in F_0$: this proves that $\Pi_\delta(f) \subset \Pi_{F_0}$, and finally $\Pi_\delta(f) = \Pi_{F_0}$. □

COROLLARY 4.10. — *Let $f \in \mathrm{Mult}(\widehat{T}_\delta)^+$ be such that $M_f \in \Delta$. There exists a unique $\phi \in \mathrm{Mult}(\mathbb{R}[Q^+])^+$ such that $\Phi(\phi, \mathrm{Id}) = f$ and $\{\alpha, \phi(e^\alpha) \neq 0\}$ is a δ-admissible subset of S.*

Proof. — Let ϕ be such that $\Phi(\phi, \mathrm{Id}) = f$. By Proposition 4.7, there exists a dominant face F of $K(\delta)$ such that $\Pi_F = \Pi_\delta(f)$. Lemma 4.9 yields that ϕ is nonzero on S_F. Let $\alpha \in S_F$: then, there exists $\gamma \in \Pi_\delta$ such that $\gamma \in \Pi_F, \gamma - \alpha \in \Pi_F$; thus, $f(e^\gamma, 1) \neq 0$ and $f(e^{\gamma - \alpha}, 1) \neq 0$. Therefore, the value of ϕ on α has to be equal to $\frac{f(e^{\gamma - \alpha}, 1)}{f(e^\gamma, 1)}$. Hence, there exists at most one ϕ such that $\{\alpha, \phi(e^\alpha) \neq 0\}$ is the δ-admissible subset S_F. Such a map ϕ exists, since f is zero on $\Pi_\delta \setminus \Pi_\delta(f)$.

Suppose that there exists a bigger δ-admissible subset $S_F \subsetneq S'$ such that ϕ is nonzero on S'. Then by Lemma 4.9, $\Pi_{F_{S'}} \subset \Pi_\delta(f)$. But by Theorem 4.6, there is a bijection between dominant faces and δ-admissible subsets: therefore, $\Pi_\delta(f) = \Pi_F \subsetneq \Pi_{F_{S'}} \subset \Pi_\delta(f)$, which is a contradiction. Thus, there exists exactly one map ϕ such that $\Phi(\phi, \mathrm{Id}) = f$ and $\{\alpha \in S, \phi(e^\alpha) \neq 0\}$ is a δ-admissible subset (and this δ-admissible subset has to be S_F). □

4.4. Identification of the minimal boundary

We give in this subsection a complete description of the minimal boundary by describing $\mathrm{Mult}(\widehat{T}_\delta)^+$.

LEMMA 4.11. — *Let $f \in \mathrm{Mult}(\widehat{T}_\delta)^+$ be such that $M_f \in \Delta$, and let $\phi \in \mathrm{Mult}(\mathbb{R}[Q^+])^+$ be such that $\Phi(\phi, \mathrm{Id}) = f$. Then $\phi(e^\alpha) \in [0,1]$.*

Proof. — Let $f \in \mathrm{Mult}(\widehat{T}_\delta)^+$ be such that $M_f \in \Delta$ (see (4.1)). Let $\phi \in \mathrm{Mult}(\mathbb{R}[Q^+])^+$ be a morphism associated with f by Proposition 4.2, and let $\alpha \in S$. Since $M_f \in \Delta$, $\langle M_f, \alpha \rangle \geqslant 0$. Moreover,

$$\langle M_f, \alpha \rangle = \sum_{\gamma \in \Pi_\delta} K_{\delta,\gamma} f(e^\delta, 1) \phi(e^{\delta-\gamma}) \langle \gamma, \alpha \rangle.$$

By invariance of Π_δ under the symmetry s_α, $\gamma \in \Pi_\delta$ implies that $s_\alpha(\gamma) \in \Pi_\delta$. Since $s_\alpha^2 = \mathrm{Id}$ and since $s_\alpha(\gamma) = \gamma$ if and only if $\langle \alpha, \gamma \rangle = 0$, we have by (3.1)

$$\langle M_f, \alpha \rangle = f(e^\delta, 1) \sum_{\substack{\gamma \in \Pi_\delta \\ \langle \gamma, \alpha \rangle > 0}} K_{\delta,\gamma} (\phi(e^{\delta-\gamma}) - \phi(e^{\delta-s_\alpha(\gamma)})) \langle \gamma, \alpha \rangle$$

$$= f(e^\delta, 1) \sum_{\substack{\gamma \in \Pi_\delta \\ \langle \gamma, \alpha \rangle > 0}} K_{\delta,\gamma} \phi(e^{\delta-\gamma}) (1 - \phi(e^\alpha)^{\frac{2\langle \gamma, \alpha \rangle}{\langle \alpha, \alpha \rangle}}) \langle \gamma, \alpha \rangle.$$

If $\phi(e^\alpha) > 1$, then $(1 - \phi(e^\alpha)^{\frac{2\langle \gamma, \alpha \rangle}{\langle \alpha, \alpha \rangle}}) < 0$ for all $\gamma \in \Pi_\delta$ such that $\langle \gamma, \alpha \rangle > 0$, and thus $\langle M_f, \alpha \rangle < 0$: this would contradict the choice of f. Therefore, $\phi(e^\alpha) \leqslant 1$. $\qquad\square$

The set $\{1, \ldots, d\}$ is identified with S by ordering the set of simple roots, and for $S' \subset S$, we denote by $W^{S'}$ the set of minimal right-coset representatives with respect to S' : namely,

$$W^{S'} = \{w \in W \mid \ell(sw) > \ell(w) \text{ for } s \in S'\}$$

where ℓ is the length function on the Coxeter group W. For $x \in \Delta$, we denote by S_x the set $\{\alpha \in S, \langle \alpha, x \rangle = 0\}$.

LEMMA 4.12. — *Let $x \in \mathbb{R}^d$ and let y be the unique element of Wx belonging to Δ. There exists a unique element $w \in W^{S_y}$ such that $wx = y$.*

Proof. — Let W_y be the parabolic subgroup generated by S_y. Then, W_y is the stabilizer of y. In particular, the set $\{w \in W, w(y) = x\}$ is a left coset of W_y in W, and thus the set $\{w \in W, w(x) = y\}$ is a right coset of W_y in W. By [10, 1.10], there exists a unique $\widetilde{w} \in W^{S_y}$ such that $\{w \in W, w(x) = y\} = \widetilde{w} W_y$. Thus, there exists a unique $\widetilde{w} \in W^{S_y}$ such that $\widetilde{w}(x) = y$. $\qquad\square$

For each d-tuple $\mathbf{t} = (t_1, \ldots, t_d)$, denote by $\mathbf{0}^c(\mathbf{t})$ the set of indices i such that $t_i \neq 0$ and by $\mathbf{1}(\mathbf{t})$ the set of indices i such that $t_i = 1$. Now consider the set $[0, 1]_\delta^d$ such that

$$(4.5) \qquad [0, 1]_\delta^d := \{\mathbf{t} \in [0, 1]^d \mid \mathbf{0}^c(\mathbf{t}) \text{ is } \delta\text{-admissible}\}.$$

Example 4.13. — Assume, the root system considered is of type C_2 realized in $\mathbb{R}^2 = \mathbb{R}\varepsilon_1 + \mathbb{R}\varepsilon_2$. Then, the simple roots are $\alpha_1 = \varepsilon_1 - \varepsilon_2$ and $\alpha_2 = 2\varepsilon_2$. If we choose $\delta = \omega_1 = \varepsilon_1$, the δ-admissible subsets of $\{\alpha_1, \alpha_2\}$ are $\emptyset$, $\{\alpha_1\}$ and $\{\alpha_1, \alpha_2\}$. Indeed, $\{\alpha_2\}$ is not δ-admissible since ω_1 is orthogonal to α_2. The condition $\mathbf{0}^c(\mathbf{t})$ is δ-admissible is equivalent to $(t_1, t_2) \in]0, 1[^2$, $(t_1, t_2) = (t_1, 0)$ with $t_1 > 0$ and $(t_1, t_2) = (0, 0)$. Hence $[0, 1]_\delta^d = [0, 1]_{\omega_1}^2 =]0, 1] \times [0, 1] \cup \{(0, 0)\}$.

The set $[0, 1]_\delta^d$ will turn out to be a natural parametrization of $K(\delta)^+$. Then, we will prove in Section 6 that there exists a natural map $t : K(\delta) \to [0, 1]_\delta^d \times W$, written as $t(m) = (\mathbf{t}_m, w_m)$, such that $t(K(\delta)^+) = [0, 1]_\delta^d \times \text{Id}$.

Remind the definition of the maps Φ and θ in (4.3) and (4.4).

PROPOSITION 4.14. — *The map $\Phi \circ \theta^{-1}$ yields a bijection Ψ between* $\text{Mult}(\widehat{T}_\delta)^+$ *and*

$$(4.6) \qquad \mathcal{S} = \{(\mathbf{t}, w) \in [0, 1]_\delta^d \times W \mid w \in W^{\mathbf{1}(\mathbf{t})}\}.$$

Proof. — Let $f \in \text{Mult}(\widehat{T}_\delta)^+$. Let y be the unique point in $W(M_f) \cap \Delta$ and denote by S_y the set $\{\alpha \in S \mid \langle \alpha, y \rangle = 0\}$. By Lemma 4.12, there exists a unique $w \in W^{S_y}$ such that $w(M_f) = y$. Thus, by Proposition 4.2, Corollary 4.10 and Lemma 4.11, there exists a unique $\phi \in \text{Mult}(\mathbb{R}[Q^+])^+$ such that $\Phi(\phi, w) = f$ and $\{\alpha \in S \mid \phi(e^\alpha) \neq 0\}$ is a δ-admissible subset. In order to conclude, we just have to show that $\phi(e^\alpha) = 1$ if and only if $\langle \alpha, w(M_f) \rangle = 0$: but, as in the proof of Lemma 4.11, we have

$$\langle \alpha, w(M_f) \rangle = \langle w^{-1}(\alpha), M_f \rangle = \sum_{\gamma \in \Pi_\delta} K_{\delta, \gamma} f(e^\gamma, 1) \langle w^{-1}(\alpha), \gamma \rangle$$

$$= \sum_{\gamma \in \Pi_\delta} K_{\delta, \gamma} \frac{1}{\phi(S_\delta)} \phi(e^{\delta - w(\gamma)}) \langle w^{-1}(\alpha), \gamma \rangle$$

$$= \sum_{\gamma \in \Pi_\delta} K_{\delta, \gamma} \frac{1}{\phi(S_\delta)} \phi(e^{\delta - w(\gamma)}) \langle \alpha, w(\gamma) \rangle$$

$$= \sum_{\gamma \in \Pi_\delta} K_{\delta, \gamma} \frac{1}{\phi(S_\delta)} \phi(e^{\delta - \gamma}) \langle \alpha, \gamma \rangle$$

$$= \frac{1}{\phi(S_\delta)} \sum_{\substack{\gamma \in \Pi_\delta \\ \langle \alpha, \gamma \rangle > 0}} K_{\delta, \gamma}(\phi(e^{\delta - \gamma}) - \phi(e^{\delta - s_\alpha(\gamma)})) \langle \alpha, \gamma \rangle$$

$$= \frac{1}{\phi(S_\delta)} \sum_{\substack{\gamma \in \Pi_\delta \\ \langle \gamma, \alpha \rangle > 0}} K_{\delta, \gamma} \phi(e^{\delta - \gamma})(1 - \phi(e^\alpha)^{\frac{2\langle \gamma, \alpha \rangle}{\langle \alpha, \alpha \rangle}}) \langle \alpha, \gamma \rangle,$$

where the fourth inequality is due to the fact that w yields a bijection on the set of weights which satisfies the relation $K_{\delta, w(\gamma)} = K_{\delta, \gamma}$ for each $\gamma \in \Pi_\delta$. Thus, $\langle M_f, \alpha \rangle = 0$ if and only $\phi(e^\alpha) = 1$. □

Note that the bijection Ψ in the above proposition is explicitly given by Proposition 4.2: for $\mathbf{t} \in [0,1]_\delta^d$, denote by $\phi_\mathbf{t}$ the unique element of $\mathrm{Mult}(\mathbb{R}[Q^+])^+$ such that $\{\phi_t(\alpha) \neq 0\}$ is δ- admissible and $\Phi(\phi_t, w) = \Psi(\mathbf{t}, w)$. Then,

$$(4.7) \qquad \Psi(\mathbf{t}, w)(e^\gamma, n) = \frac{1}{\phi_\mathbf{t}(S_\delta)^n} \phi_\mathbf{t}(e^{n\delta - w(\gamma)}) = \frac{\mathbf{t}^{n\delta - w(\gamma)}}{S_\delta(\mathbf{t})^n},$$

for $(e^\gamma, n) \in \widehat{T}_\delta$.

Remark 4.15. — The restriction of the set of parameters $(\mathbf{t}, w) \in [0,1]^d \times W$ to the set $\mathcal{S}$ defined in (4.6) is only useful to ensure the injectivity of the map Ψ. It is however still possible to define an element of $\mathrm{Mult}(\widehat{T}_\delta)^+$ by applying the map $\Phi \circ \theta^{-1}$ to any element $(\mathbf{t}, w)$. The lack of injectivity without the restriction of the parameters can be seen in the following example: consider the Lie algebra of type A_2 with set of simple roots $\{\alpha_1, \alpha_2\}$ and choose $\delta = \omega_1$, the first fundamental weight. Then, $(\omega_1, \alpha_2) = 0$, and thus any weight $\gamma \neq \omega_1$ of Π_{ω_1} is written $\gamma = \omega_1 - k_1 \alpha_1 - k_2 \alpha_2$ with $k_1 > 0$: hence, if $t_1 = 0$, we have $\phi(e^{\omega_1 - \gamma}) = \delta_{\gamma, \omega_1}$ for all value of t_2. On the other hand, the ω_1 -admissible subsets of $\{\alpha_1, \alpha_2\}$ are $\emptyset$, $\{\alpha_1\}$ and $\{\alpha_1, \alpha_2\}$. Thus the empty ω_1-admissible subset $\emptyset$ yields the unique choice of t_2 such that $t_1 = 0$ and $\mathbf{0}^c(\mathbf{t})$ is ω_1 -admissible, namely $t_2 = 0$. The latter procedure has singled out a particular choice of parameters $t_1 = 0, t_2 = 0$ among all the choices of $\mathbf{t}$ yielding the map $\phi(e^{\omega_1 - \gamma}) = \delta_{\gamma, \omega_1}$.

A straightforward application of Proposition 4.1 yields the following corollary:

COROLLARY 4.16. — *The map $\iota \circ \Psi$ gives a bijection between the minimal boundary $\partial \mathcal{H}(\mathbb{R}^d)$ and the set*

$$\mathcal{S} = \{(\mathbf{t}, w) \in [0,1]_\delta^d \times W \mid w \in W^{\mathbf{1}(\mathbf{t})}\}.$$

5. Minimal boundary of Γ^Δ

In this section, we use the description of $\partial\mathcal{H}(\mathbb{R}^d)$ to get the one of $\partial\mathcal{H}(\Delta)$.

5.1. Algebraic description of the growth graph of Γ^Δ

The growth graph $\mathcal{G}(\Delta)$ of Γ^Δ admits a description similar to the one of $\Gamma^\mathbb{R}$. The set $\mathcal{G}_n(\Delta)$ of vertices of rank n of the graph $\mathcal{G}(\Delta)$ are pairs (λ, n) where λ is a weight of P_δ^+ such that $\Gamma_n^\Delta(\lambda) \neq \emptyset$ (i.e., there is at least a path of length n from 0 to λ). The weight of the edge between (λ, n) and $(\mu, n+1)$ is just $\#\Gamma^\Delta(\lambda, \mu)$. Moreover, we have the following algebraic description of $\mathcal{G}(\Delta)$:

PROPOSITION 5.1. — $\mathcal{G}(\Delta)$ is a multiplicative graph associated with the algebra $\widehat{\mathcal{A}}_\delta$ with the injective map

$$\iota : \begin{cases} \mathcal{G}(\Delta) \longrightarrow \widehat{\mathcal{A}}_\delta \\ (\lambda, n) \longmapsto (s_\lambda, n), n \geqslant 1 \\ \qquad\quad \longmapsto (s_\delta, 1), \end{cases}$$

and $\iota(\mathcal{G}(\Delta)) = \widehat{T}_\delta^+$. In particular, $\partial\mathcal{H}(\mathcal{G}(\Delta))$ is isomorphic to $\mathrm{Mult}(\widehat{T}_\delta^+)^+$ through the map

$$\iota^* : \begin{cases} \mathrm{Mult}(\widehat{T}_\delta^+)^+ \longrightarrow \partial\mathcal{H}(\mathcal{G}(\Delta)) \\ \qquad\qquad f \longmapsto f \circ \iota. \end{cases}$$

Proof. — By Littelmann's path theory, the following equality holds for $(\lambda, n) \in \mathcal{G}_n(\Delta)$:

$$\iota(\lambda, n)\iota(*) = (s_\lambda, n)(s_\delta, 1) = \sum_{\mu \in \mathcal{A}_\delta} \#\Gamma^\Delta(\lambda, \mu)(s_\mu, n+1).$$

Thus, $\mathcal{G}(\Delta)$ is a multiplicative graph associated with $\widehat{\mathcal{A}}_\delta$ with the map ι. We have $\iota(\mathcal{G}(\Delta)) = \widehat{T}_\delta^+$ by construction and the last part of the proposition follows from Proposition 2.5. $\qquad\square$

Now we are going to connect the sets $\mathrm{Mult}(\widehat{T}_\delta^+)^+$ and $\mathrm{Mult}(\widehat{T}_\delta)^+$ of nonnegative multiplicative maps defined on $\widehat{T}_\delta^+$ and $\widehat{T}_\delta$, respectively.

5.2. Relation between $\mathrm{Mult}(\widehat{T}_\delta)^+$ and $\mathrm{Mult}(\widehat{T}_\delta^+)^+$

Recall that $\widehat{T}_\delta^+$ is a subalgebra of $\widehat{T}_\delta$. Thus we have a restriction map $\mathrm{Res} : \mathrm{Mult}(\widehat{T}_\delta)^+ \to \mathrm{Mult}(\widehat{T}_\delta^+)^+$. The following proposition is an important step in the description of $\mathrm{Mult}(\widehat{T}_\delta^+)^+$.

PROPOSITION 5.2. — *The restriction map* Res *yields a surjection from* $\{f \in \mathrm{Mult}(\widehat{T}_\delta)^+ \mid M_f \in \Delta\}$ *to* $\mathrm{Mult}(\widehat{T}_\delta^+)^+$.

The proof of this proposition requires some preparation. Let f be a multiplicative map from $\widehat{T}_\delta^+$ to $\mathbb{R}_+$. By Corollary 3.6 and by Corollary 4 in [3, p. 35], f can be extended to a morphism $\widetilde{f}$ from $\widehat{T}_\delta$ to $\mathbb{C}$. The first task is to prove that $\widetilde{f} \in \mathrm{Mult}(\widehat{T}_\delta)^+$.

We need to recall a classical result by Aissen, Edrei, Schoenberg and White on polynomials with real coefficients having negative zeros.

THEOREM 5.3 ([1]). — *Consider a polynomial* $P(T) = a_m T^m + a_{m-1}T^{m-1} + \cdots + a_1 T + a_0 \in \mathbb{R}[T]$. *Then* P *has only real and nonpositive zeros if and only if the sequence* $a_0, a_1, \ldots, a_m, 0, 0, 0, \ldots$ *is totally positive, that is if and only if all the minors of the infinite matrix*

$$\begin{pmatrix} a_0 & 0 & 0 & 0 & \cdots \\ a_1 & a_0 & 0 & 0 & \cdots \\ a_2 & a_1 & a_0 & 0 & \cdots \\ a_3 & a_2 & a_1 & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots & \cdots \end{pmatrix}$$

are nonnegative.

PROPOSITION 5.4. — *Any morphism* $\widetilde{f}$ *defined on* $\widehat{T}_\delta$ *which extends the positive morphism* f *belongs to* $\mathrm{Mult}(\widehat{T}_\delta)^+$.

Proof. — Let $\widetilde{f}$ be a morphism extending f. Set $\varphi(T) = \widetilde{f}(\Phi)(T)$ that is

$$\varphi(T) = \prod_{\gamma \in \Pi_\delta} (T + \widetilde{f}(e^\gamma, 1)).$$

By using the same arguments as in the proof of Proposition 3.5, we obtain that the coefficients of $\varphi(T)$ are the

$$\widetilde{f}(e_k(e^{\gamma_1}, \ldots, e^{\gamma_N}), k) \in \mathbb{C}, k = 0, \ldots, N.$$

The minors of the matrix defined from the coefficients of $\varphi(T)$ as in Theorem 5.3 admit a classical description in terms of Schur functions (see [15, p. 42]). These Schur functions are symmetric polynomials in infinitely many

variables indexed by partitions (nonincreasing sequences of nonnegative integers). Let us write $\mathcal{P}_N$ the set of partitions with length at most N and for such a partition L let $|L|$ be the sum of its components. We then have

$$\widetilde{f}(\boldsymbol{s}_L(e^{\gamma_1}, \ldots, e^{\gamma_N}), |L|), L \in \mathcal{P}_N$$

where $\boldsymbol{s}_L(e^{\gamma_1}, \ldots, e^{\gamma_N})$ is the plethysm of the Schur function $\boldsymbol{s}_\lambda$ in N variables $X_1, \ldots, X_N$ by the Weyl character s_δ. If we consider any young symmetrizer c_L of shape L in $\mathbb{R}[S_l]$, the group algebra of the symmetric group S_l (see [6]), the space

$$c_L \cdot V(\delta)^{\otimes l} \text{ such that } l = |L|$$

has indeed the structure of a $\mathfrak{g}$-module and

$$\boldsymbol{s}_L(e^{\gamma_1}, \ldots, e^{\gamma_N}) = \operatorname{char}\left(c_L \cdot V(\delta)^{\otimes l}\right).$$

This shows that $\boldsymbol{s}_L(e^{\gamma_1}, \ldots, e^{\gamma_N})$ decomposes as a sum of characters in $\{s_\lambda \mid \lambda \in \delta^{\otimes|L|}\}$ with nonnegative integer coefficients. In particular, $(\boldsymbol{s}_L(e^{\gamma_1}, \ldots, e^{\gamma_N}), |L|)$ belongs to $\widehat{T}_\delta^+$ and therefore we get that $\widetilde{f}(\boldsymbol{s}_L(e^{\gamma_1}, \ldots, e^{\gamma_N}), |L|)$, equal to $f(\boldsymbol{s}_L(e^{\gamma_1}, \ldots, e^{\gamma_N}), |L|)$, is real nonnegative since f is assumed nonnegative. By Theorem 5.3 this shows that $-\widetilde{f}(e^\gamma, 1)$ is real nonpositive for any $\gamma \in \Pi_\delta$. Finally we obtain that $\widetilde{f}(e^\gamma, 1)$ is real nonnegative for any $\gamma \in \Pi_\delta$ and thus $\widetilde{f}$ takes real nonnegative values on $\widehat{T}_\delta$. $\qquad\square$

Proof of Proposition 5.2. — Let $f \in \operatorname{Mult}(\widehat{T}_\delta^+)^+$. By Proposition 5.3, there exists $\widetilde{f} \in \operatorname{Mult}(\widehat{T}_\delta)^+$ such that $\widetilde{f}(s_\lambda, n) = f(s_\lambda, n)$ for $(s_\lambda, n) \in \widehat{T}_\delta^+$. Let $w \in W$ and $(s_\lambda, n) \in \widehat{T}_\delta^+$: since w^{-1} yields a multiplicity preserving bijection on Π_λ,

$$(\widetilde{f} \circ w)(s_\lambda, n) = \sum_{\gamma \in \Pi_\lambda} K_{\lambda,\gamma} \widetilde{f}(e^{w(\gamma)}, n) = \sum_{\gamma \in \Pi_\lambda} K_{\lambda, w^{-1}(\gamma)} \widetilde{f}(e^\gamma, n)$$

$$= \sum_{\gamma \in \Pi_\lambda} K_{\lambda,\gamma} \widetilde{f}(e^\gamma, n) = \widetilde{f}(s_\lambda, n).$$

Thus, for all $w \in W$, $(\widetilde{f} \circ w)_{|\widehat{T}_\delta^+} = \widetilde{f}_{|\widehat{T}_\delta^+}$. Let $w \in W$ be such that $M_{\widetilde{f} \circ w} \in \Delta$ and set $g = \widetilde{f} \circ w$. Then, g is an element of $\operatorname{Mult}(\widehat{T}_\delta)^+$ such that $\operatorname{Res}(g) = f$ and $M_g \in \Delta$. $\qquad\square$

Consider the map

(5.1) $$\Psi^+ : \begin{cases} [0,1]_\delta^d \longrightarrow \operatorname{Mult}(\widehat{T}_\delta^+)^+ \\ \mathbf{t} \longmapsto f_\mathbf{t} \end{cases}$$

where $f_\mathbf{t}$ is such that $f_\mathbf{t}(s_\lambda, n) = \frac{S_{\lambda,n\delta}(\mathbf{t})}{S_\delta(\mathbf{t})^n}$.

COROLLARY 5.5. — *The map Ψ^+ is surjective.*

Proof. — Consider f in $\text{Mult}(\widehat{T}_\delta^+)^+$. By Proposition 5.2, f can be regarded as the restriction of a multiplicative function in $\text{Mult}(\widehat{T}_\delta)^+$ we shall also denote by f. By the previous proposition, we know there then exists $\mathbf{t} \in [0,1]_\delta^d$ such that $f(s_\lambda, n) = \frac{S_{\lambda,n\delta}(\mathbf{t})}{(S_\delta(\mathbf{t}))^n}$ for any $(s_\lambda, n) \in \mathcal{G}(\Delta)$. $\square$

5.3. Injectivity of the map Ψ^+

It remains to show that the map Ψ^+ is injective.

LEMMA 5.6. — *Let $\mathbf{t} \in [0,1]_\delta^d$. For any $(\lambda, n) \in \widehat{T}_\delta^+$ we have*

$$1 \leqslant \frac{S_{\lambda,n\delta}(\mathbf{t})}{\mathbf{t}^{n\delta-\lambda}} \leqslant \dim(V(\lambda)).$$

Proof. — On the first hand,

$$S_{\lambda,n\delta}(\mathbf{t}) = \sum_{\gamma \in \Pi_\lambda} K_{\lambda,\gamma} \mathbf{t}^{n\delta-\gamma} \geqslant \mathbf{t}^{n\delta-\lambda},$$

which yields $1 \leqslant \frac{S_{\lambda,n\delta}(\mathbf{t})}{\mathbf{t}^{n\delta-\lambda}}$. On the other hand, since $t_i \leqslant 1$ for all $1 \leqslant i \leqslant d$ and $\gamma \leqslant \lambda$ for the dominant order,

$$S_{\lambda,n\delta}(\mathbf{t}) = \sum_{\gamma \in \Pi_\lambda} K_{\lambda,\gamma} \mathbf{t}^{n\delta-\gamma} \leqslant \sum_{\gamma \in \Pi_\lambda} K_{\lambda,\gamma} \mathbf{t}^{n\delta-\lambda} \leqslant \dim V(\lambda) \mathbf{t}^{n\delta-\lambda},$$

yielding the other inequality

$$\frac{S_{\lambda,n\delta}(\mathbf{t})}{\mathbf{t}^{n\delta-\lambda}} \leqslant \dim(V(\lambda)).$$ $\square$

COROLLARY 5.7. — *Let $t = (t_1, \ldots, t_d), \tau = (\tau_1, \ldots, \tau_d)$ be such that $\Psi^+(\mathbf{t}) = \Psi^+(\tau)$. Then $S_\delta(\mathbf{t}) = S_\delta(\tau)$.*

Proof. — For all $n \geqslant 1$, $(n\delta, n) \in \widehat{T}_\delta^+$. Thus, by Lemma 5.6, we have

$$1 \leqslant S_{n\delta,n\delta}(\mathbf{t}) \leqslant \dim(V(n\delta)) \text{ and } 1 \leqslant S_{n\delta,n\delta}(\tau) \leqslant \dim(V(n\delta)).$$

This yields

$$\frac{1}{\dim(V(n\delta))} \leqslant \frac{S_{n\delta,n\delta}(\mathbf{t})}{S_{n\delta,n\delta}(\tau)} \leqslant \dim(V(n\delta)).$$

But

$$\frac{S_{n\delta,n\delta}(\mathbf{t})}{S_{n\delta,n\delta}(\tau)} = \frac{S_\delta(\tau)^n \Psi(t_1, \ldots, t_d)(s_{n\delta}, n)}{S_\delta(\mathbf{t})^n \Psi(\tau_1, \ldots, \tau_d)(s_{n\delta}, n)} = \frac{S_\delta(\tau)^n}{S_\delta(\mathbf{t})^n},$$

for $\Psi(t_1, \ldots, t_d) = \Psi(\tau_1, \ldots, \tau_d)$. Therefore, we have the inequality

$$\frac{1}{\dim(V(n\delta))} \leqslant \left(\frac{S_\delta(\tau)}{S_\delta(\mathbf{t})}\right)^n \leqslant \dim(V(n\delta)).$$

Since $\dim(V(n\delta))$ is polynomial in n, necessarily $S_\delta(\mathbf{t}) = S_\delta(\tau)$. $\square$

The proof of the injectivity uses the combinatorics of Littelmann paths. We refer the reader to [14] for an introduction, the definition and the basic properties of the operators $f_\alpha, \alpha \in S$ we shall need in our proofs. We recall that $B(\delta)$ denotes the set of Littelmann paths obtained by applying any sequence of operators f_α to a fixed infinitesimal path π_0 in Δ such that $\pi_0(1) = \delta$. The weight $\mathrm{wt}(\mu)$ of any path $\pi \in B(\delta)$ is then defined by $\mathrm{wt}(\pi) = \pi(1)$.

We introduce moreover the following decomposition of a δ-admissible subset $S' \subset S$.

DEFINITION 5.8. — *Let $S' \subset S$ be δ-admissible and $\alpha \in S$. A Dynkin subchain of type α and length r is a sequence $(\alpha_{i_1}, \alpha_{i_2} \ldots, \alpha_{i_r})$ of simple roots in S' such that $\alpha_{i_1} = \alpha$, $\langle \alpha_{i_r}, \delta \rangle \neq 0$ and $\langle \alpha_{i_j}, \alpha_{i_{j+1}} \rangle \neq 0$ for $1 \leqslant i \leqslant r-1$. The depth $d(\alpha)$ of α (relatively to S') is the minimal length of a Dynkin subchain of type α.*

Alternatively, the depth of α relatively to a δ-admissible subset $S' \subset S$ is the distance of α to a simple root non-orthogonal to δ in the Dynkin subdiagram induced by S'. Note that any simple root of a δ-admissible subset belongs to at least one Dynkin subchain, since it belongs to an indecomposable root system which is not orthogonal to δ.

LEMMA 5.9. — *Let $\delta \in P^+$ and $\alpha \in S$ be such that $\langle \delta, \alpha \rangle \neq 0$. Then there exists $(\mu, n) \in \widehat{T}_\delta^+$ such that $n\lambda - \mu = \alpha$.*

Proof. — Suppose that $\langle \delta, \alpha \rangle > 0$, and thus $\delta - \alpha \in \Pi_\delta$. Recall that We denote by π_0 the Littelmann path of $B(\delta)$ with weight δ. Then, $\mathrm{wt}(f_\alpha(\pi_0)) = \delta - \alpha$. Also one can write $f_\alpha(\pi_0) = \pi_0 - v\alpha$ where v is a continuous function from $[0, 1]$ to itself such that $v(1) = 1$. We have

$$\langle f_\alpha(\pi_0)(t), \alpha \rangle = \langle \pi_0(t), \alpha \rangle - v(t)\langle \alpha, \alpha \rangle \geqslant -\langle \alpha, \alpha \rangle,$$

and for all simple root $\alpha' \neq \alpha$ and $t \in [0, 1]$, we have

$$\langle f_\alpha(\pi_0)(t), \alpha' \rangle = \langle \pi_0(t), \alpha' \rangle - v(t)\langle \alpha', \alpha \rangle \geqslant 0,$$

because π_0 lies in the Weyl chamber Δ and $\langle \alpha, \alpha' \rangle \leqslant 0$. Consider an integer $n \geqslant 2$ such that $\langle (n-1)\delta, \alpha \rangle \geqslant \langle \alpha, \alpha \rangle$. Then, from the two previous inequalities, $\pi_0^{*(n-1)} * f_\alpha(\pi_0)$ lies in Δ. Thus, $\mathrm{wt}\left(\pi_0^{*(n-1)} * f_\alpha(\pi_0)\right) = (n-1)\delta + (\delta - \alpha)$ is the highest weight of an irreducible component of $B(\delta)^{\otimes n}$, and $((n-1)\delta + (\delta - \alpha), n) = (n\delta - \alpha, n) \in \widehat{T}_\delta^+$. One concludes by setting $\mu = n\delta - \alpha$. $\qquad\square$

The latter result can be generalized along a Dynkin subchain and yields the following Lemma:

LEMMA 5.10. — *Let $S' \subset S$ be δ-admissible and let $\alpha \in S'$. There exists $(\lambda, n) \in \widehat{P}_\delta^+$ such that $n\delta - \lambda = \alpha + \sum_{\alpha' \in S', d(\alpha') < d(\alpha)} k_{\alpha'} \alpha'$.*

Proof. — Let $S' \subset S$ be a δ-admissible subset. We will prove the result by induction on the depth of the simple root. For $d(\alpha) = 1$, the result is given by Lemma 5.9. Let $r \geqslant 2$. Suppose that the result is proven for all roots of depth at most $r - 1$, and let α be a root in S' of depth r. Let $(\alpha = \alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_r})$ be a Dynkin chain of minimal length of type α. By minimality, α_{i_j} has depth $r - j + 1$ for $2 \leqslant j \leqslant r$. Since $d(\alpha_{i_2}) = r - 1$, there exists $(\lambda', l) \in \widehat{T}_\delta^+$ such that $l\delta - \lambda' = \alpha_{i_2} + \sum_{d(\alpha') < d(\alpha_{i_2})} k_{\alpha'} \alpha'$. If α' is such that $d(\alpha') < d(\alpha_{i_2})$, then necessarily $\langle \alpha, \alpha' \rangle = 0$ (otherwise, there would exist a Dynkin subchain of type α and length smaller than r); likewise, since $d(\alpha) \geqslant 2$, $\langle \delta, \alpha \rangle = 0$. Thus,

$$\langle \lambda', \alpha \rangle = \left\langle l\delta - \alpha_{i_2} - \sum_{d(\alpha') < d(\alpha_{i_2})} k_{\alpha'} \alpha', \alpha \right\rangle = -\langle \alpha_{i_2}, \alpha \rangle > 0.$$

Let π be a Littelmann path in $B(\delta)^{\otimes l}$ lying in Δ with weight λ'. One can consider π as a Littelmann path of highest weight for the irreducible representation $V(\lambda')$. Since $\langle \alpha, \lambda' \rangle > 0$, $\lambda' - \alpha$ is a weight of $V(\lambda')$. Applying Lemma 5.9 yields the existence of $m \geqslant 1$ such that $\pi^{*m} * f_\alpha(\pi)$ lies in the Weyl chamber. Thus, $\pi^{*m} * f_\alpha(\pi)$ correspond to a highest weight path in $(B(\delta)^{\otimes l})^{\otimes m}$. On the other hand,

$$wt(\pi^{*m} * f_\alpha(\pi)) = m\lambda' - \alpha = lm\delta - \alpha - m\alpha_{i_2} - m \sum_{d(\alpha') < d(\alpha_{i_2})} k_{\alpha'} \alpha'$$

$$= lm\delta - \alpha - \sum_{d(\alpha') < d(\alpha)} k'_{\alpha'} \alpha',$$

with $k'_\alpha \geqslant 0$. Setting $\lambda = lm\delta - \alpha - \sum_{d(\alpha') < d(\alpha)} k'_{\alpha'} \alpha'$ and $n = lm$, we get an element $(\lambda, n) \in \widehat{T_\delta}^+$ satisfying the hypothesis of the Lemma. □

COROLLARY 5.11. — *Let $(t_1, \ldots, t_d), (\tau_1, \ldots, \tau_d) \in [0,1]_\delta^d$ and $i \in \{1, \ldots, d\}$ be such that $t_i = 0$ and $\tau_i \neq 0$. Then, $\Psi(t_1, \ldots, t_d) \neq \Psi(\tau_1, \ldots, \tau_d)$.*

Proof. — Note that $\mathbf{0}_{\mathbf{t}}^c$ and $\mathbf{0}_\tau^c$ are δ-admissible subsets by definition of $[0,1]_\delta^d$. Since $\tau_i \neq 0$, $i \in \mathbf{0}_\tau^c$. Thus, by Lemma 5.10, there exists $(\lambda, n) \in \widehat{T}_\delta^+$ such that $\lambda = n\delta - \alpha_i - \sum_{\substack{j \in \mathbf{0}_\tau^c \\ d(\alpha_{i_j}) < d(\alpha_i)}} k_{\alpha_{i_j}} \alpha_{i_j}$. Since $\tau_j > 0$ for all $j \in \mathbf{0}_\tau^c$,

$$\Psi(\tau_1, \ldots, \tau_d)(s_\lambda, n) = \frac{S_{\lambda, n\delta}(\tau)}{S_\delta(\tau)^n} \geqslant \frac{\tau^{n\delta - \lambda}}{S_\delta(\tau)^n} = \frac{1}{S_\delta(\tau)^n} \tau_i \prod_{\substack{j \in \mathbf{0}_\tau^c \\ d(\alpha_j) < d(\alpha_i)}} \tau_j^{k_{\alpha_j}} > 0.$$

On the other hand, any weight of $V(\lambda)$ has the form $\lambda - \sum_{\alpha \in S} r_\alpha \alpha$ for some integer coefficients $r_\alpha \geqslant 0$; thus, since $t_i = 0$, for any weight $\mu = \lambda - \sum_{\alpha \in S} r_\alpha \alpha$ of $V(\lambda)$ we have

$$\mathbf{t}^{n\delta - \mu} = t_i \prod_{\substack{j \in \mathbf{0}_\tau^c \\ d(\alpha_j) < d(\alpha_i)}} t_j^{k_{\alpha_j}} \prod_{\alpha_j \in S} t_j^{r_{\alpha_j}} = 0.$$

Thus, $\Psi(t_1, \ldots, t_d)(s_\lambda, n) = 0 \neq \Psi(\tau_1, \ldots, \tau_d)(s_\lambda, n)$. This yields that $\Psi(t_1, \ldots, t_d) \neq \Psi(\tau_1, \ldots, \tau_d)$. $\qquad\square$

PROPOSITION 5.12. — *The map Ψ^+ is injective.*

Proof. — Let $(t_1, \ldots, t_d), (\tau_1, \ldots, \tau_d) \in [0, 1]_\delta^d$ be such that $\Psi(t_1, \ldots, t_d) = \Psi(\tau_1, \ldots, \tau_d)$. In this case, Corollary 5.7 yields that $S_\delta(\tau) = S_\delta(\mathbf{t})$. By Corollary 5.11, we can assume that $\mathbf{0}_\mathbf{t}^c = \mathbf{0}_\tau^c$, and we will denote this set S': we recall that the set of simple roots is identified with $\{1, \ldots, d\}$, so that S' corresponds to a δ-admissible subset of S. We shall prove that $t_j = \tau_j$ for any $j \in S'$ by induction on the depth of the simple root α_j. Suppose that $\alpha_j \in S'$ is such that $d(\alpha_j) = 1$. By Lemma 5.10, there exists $n \geqslant 1$ such that $(n\delta - \alpha_j, n) \in \widehat{T}_\delta^+$. Thus, $(nk\delta - k\alpha_j, kn) \in \widehat{T}_\delta^+$ for all $k \geqslant 1$. Since $\Psi(t_1, \ldots, t_d) = \Psi(\tau_1, \ldots, \tau_d)$, we have

$$\frac{1}{S_\delta(\mathbf{t})^{kn}} S_{kn\delta - k\alpha_j, kn\delta}(\mathbf{t}) = \frac{1}{S_\delta(\tau)^{kn}} S_{kn\delta - k\alpha_j, kn\delta}(\tau),$$

which simplifies into $S_{kn\delta - k\alpha_j, kn\delta}(\mathbf{t}) = S_{kn\delta - k\alpha_j, kn\delta}(\tau)$ because $S_\delta(\mathbf{t}) = S_\delta(\tau)$. By Lemma 5.6, we have

$$1 \leqslant \frac{S_{kn\delta - k\alpha_j, kn\delta}(\mathbf{t})}{t_j^k} \leqslant \dim V(kn\delta - k\alpha_j),$$

and

$$1 \leqslant \frac{S_{kn\delta - k\alpha_j, kn\delta}(\tau)}{\tau_j^k} \leqslant \dim V(kn\delta - k\alpha_j).$$

Thus,

$$\frac{1}{\dim V(kn\delta - k\alpha_j)} \leqslant \left(\frac{t_j}{\tau_j}\right)^k \leqslant \dim V(kn\delta - k\alpha_j).$$

Since $\dim V(kn\delta - k\alpha_j)$ is polynomial in k, necessarily $t_j = \tau_j$. Let $i \geqslant 2$, and suppose that we have proven that $t_j = \tau_j$ for all j such that $d(\alpha_j) < i$. Let α_l be such that $d(\alpha_l) = i$. By Lemma 5.10, there exists $(\lambda, n) \in \widehat{T}_\delta^+$ such that $n\delta - \lambda = \alpha_l + \sum_{\alpha' \in S', d(\alpha') < d(\alpha)} k_{\alpha'} \alpha'$, with $k_{\alpha'} \geqslant 0$. Thus, for all $k \geqslant 1$, $(k\lambda, kn) \in \widehat{T}_\delta^+$. As in the previous case, this implies that

$$S_{k\lambda, kn\delta}(\mathbf{t}) = S_{k\lambda, kn\delta}(\tau),$$

yielding together with Lemma 5.6 the inequality

$$(5.2) \qquad \frac{1}{\dim V(k\lambda)} \leqslant \left(\frac{\mathbf{t}}{\tau}\right)^{nk\delta-\lambda} \leqslant \dim V(k\lambda).$$

But $nk\delta - k\lambda = k\alpha_l + k\sum_{j\in\mathbf{0}_\mathbf{t}^c,\, d(\alpha_j)<d(\alpha_l)} k_{\alpha_j}\alpha_j$, and by the induction hypothesis, $t_j = \tau_j$ for all $j \in \mathbf{0}_\mathbf{t}^c, d(\alpha_j) < d(\alpha_l)$. Thus

$$\frac{\mathbf{t}^{nk\delta-\lambda}}{\tau^{nk\delta-\lambda}} = \frac{t_l^k}{\tau_l^k}.$$

Since $\dim V(k\lambda)$ is polynomial in k, (5.2) yields that $\frac{t_l}{\tau_l} = 1$ as expected. $\qquad\square$

COROLLARY 5.13. — *The map Ψ^+ is a bijection from $[0,1]_\delta^d$ to* $\mathrm{Mult}(\widehat{T}_\delta^+)^+$. *In particular, $\partial\mathcal{H}(\Delta)$ is isomorphic to $[0,1]_\delta^d$.*

6. Mean vector of a central measure

In this section, we identify the set $\mathcal{S}$ with $K(\delta)$ in order to complete the proof of Theorem 3.1. At the end of this section we prove Corollary 3.3.

6.1. The mean vector $\vec{M}$

Let us introduce the map

$$\vec{M} : \begin{cases} \mathcal{S} \longrightarrow K(\delta) \\ (\mathbf{t}, w) \longmapsto M_{\Psi(\mathbf{t},w)}. \end{cases}$$

Recall that the mean vector M_f has been defined in (4.1) for any multiplicative map $f \in \mathrm{Mult}(\widehat{T}_\delta)^+$. Here we have

$$(6.1) \qquad M_{\Psi(\mathbf{t},w)} = \frac{1}{S_\delta(\mathbf{t})} \sum_{\gamma\in\Pi_\delta} K_{\delta,\gamma} \mathbf{t}^{\delta-w(\gamma)} \gamma.$$

For $I \subset \{1, \ldots, d\}$, denote by W_I the parabolic subgroup generated by the simple roots α_i for $i \in I$.

Example 6.1. — Resume Example 4.13 with $\delta = \omega_1$ in type C_2. Then $V(\delta)$ is the defining representation of $\mathfrak{g} = \mathfrak{sp}_4(\mathbb{C})$ whose weights γ are $\pm\varepsilon_1 \pm \varepsilon_2$. Thus $\delta - \gamma$ runs over the set $\{0, \alpha_1, \alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2\}$. For $(\mathbf{t}, w) = ((t_1, t_2), \mathrm{Id})$ on gets

$$(6.2) \qquad \vec{M}(\mathbf{t}, \mathrm{Id}) = \frac{1}{1 + t_1 + t_1 t_2 + t_1^2 t_2} \left((1 - t_1^2 t_2)\varepsilon_1 + (t_1 - t_1 t_2)\varepsilon_2\right).$$

LEMMA 6.2. — *Let $(\mathbf{t}, w) \in \mathcal{S}$. Then $M_{\Psi(\mathbf{t},w)} \in (w')^{-1}(\Delta)$ if and only if $w' \in W_{\mathbf{1}(\mathbf{t})}w$.*

Proof. — Let $\alpha_i \in S$. By (6.1), we get as in the proof of Proposition 4.14

$$\langle M_{\Psi(\mathbf{t},w)}, w^{-1}(\alpha_i) \rangle = \frac{1}{S_\delta(\mathbf{t})} \sum_{\substack{\gamma \in \Pi_\delta \\ \langle \gamma, \alpha_i \rangle > 0}} K_{\delta,\gamma} \mathbf{t}^{\delta-\gamma} (1 - t_i^{2 \frac{\langle \gamma, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle}}) \langle \gamma, \alpha_i \rangle.$$

Since each t_i is in $[0,1]$, $\langle M_{\Psi(\mathbf{t},w)}, w^{-1}(\alpha_i) \rangle \geqslant 0$ for $1 \leqslant i \leqslant d$ and hence $w(M_{\Psi(\mathbf{t},w)}) \in \Delta$. Moreover, $\langle M_{\Psi(\mathbf{t},w)}, w^{-1}(\alpha_i) \rangle = 0$ if and only if $t_i = 1$. Therefore, $w(M_{\Psi(\mathbf{t},w)}) \in w'(\Delta)$ if and only if w' is a product of reflections s_{α_i} such that $t_i = 1$. Applying w^{-1} to the latter result yields the proof of the Lemma. $\qquad\square$

PROPOSITION 6.3. — *The map $\vec{M}$ is injective.*

Proof. — Let $(\mathbf{t}, w)$ and $(\mathbf{t}', w')$ be two elements of $\mathcal{S}$ such that $\vec{M}(\mathbf{t}, w) = \vec{M}(\mathbf{t}', w')$. We simply denote by M this common value. Lemma 6.2 implies that $W_{\mathbf{1}(\mathbf{t})}w = W_{\mathbf{1}(\mathbf{t}')}w'$. Since w and w' are both minimal right coset representatives of $W_{\mathbf{1}(\mathbf{t}')}w$, we must have $w = w'$. Thus, $W_{\mathbf{1}(\mathbf{t})} = W_{\mathbf{1}(\mathbf{t}')}$ which implies that $\mathbf{1}(\mathbf{t}) = \mathbf{1}(\mathbf{t}')$. Let F and F' be the dominant faces corresponding to the δ-admissible set $\mathbf{0}_{\mathbf{t}}^c$ and $\mathbf{0}_{\mathbf{t}'}^c$, respectively. By the results of Section 4.3, $M \in w^{-1}(\overset{\circ}{F})$ (where $\overset{\circ}{F}$ the interior of the face F) and $M \in w'^{-1}(\overset{\circ}{F'})$. Since $w = w'$, we must have $F = F'$ and thus $\mathbf{0}_{\mathbf{t}}^c = \mathbf{0}_{\mathbf{t}'}^c$. Let $(X_l)_{l \geqslant 0}$, $(X_l')_{l \geqslant 0}$ be two random walks with initial position $X_0 = X_0' = 0$ and respective transition matrices

$$\mathbb{P}(X_{l+1} = \gamma \mid X_l = \gamma') = K_{\delta, \gamma-\gamma'} \frac{\mathbf{t}^{\delta - w(\gamma - \gamma')}}{S_\delta(\mathbf{t})}, \mathbb{P}(X_{l+1}' = \gamma \mid X_l' = \gamma')$$

$$= K_{\delta, \gamma-\gamma'} \frac{\mathbf{t}'^{\delta - w(\gamma - \gamma')}}{S_\delta(\mathbf{t}')}.$$

Both random walks have mean M, thus it follows by the local limit theorem for large deviations (see for instance Theorem 4.2.1 in [11]) that for any sequences of weights $(\gamma_l)_{l \geqslant 1}, (\gamma_l')_{l \geqslant 1}$ such that $\gamma_l - lM = o(l^{2/3}), \gamma_l' - lM = o(l^{2/3})$, and $\mathbb{P}(X_l = \gamma_l) \neq 0, \mathbb{P}(X_l = \gamma_l') \neq 0$, we have

(6.3) $$\mathbb{P}(X_l = \gamma_l) \sim \mathbb{P}(X_l = \gamma_l'),$$

and the same relation holds for $(X_l')_{l \geqslant 1}$. Let $i \in \mathbf{0}_{\mathbf{t}}^c$. For $l \geqslant 1$, let $(\gamma_l, l) \in \widehat{T}_\delta^+$ be such that γ_l is an element of $P_\delta \cap lF$ at minimal distance from lM and set $\gamma_l' = \gamma_l - \alpha_i$. Then, $\mathbb{P}(X_l = \gamma_l) \neq 0$. Since M belongs to the interior of F, $\gamma_l' \in P_\delta \cap lF$ for l large enough: thus, $\mathbb{P}(X_l = \gamma_l') \neq 0$ for l large

enough. The sequences $(\gamma_l - lM)_{l \geqslant 1}$ and $(\gamma_l' - lM)_{l \geqslant 1}$ are bounded, thus the local limit Theorem applies and

$$(6.4) \qquad\qquad \mathbb{P}(X_l = \gamma_l) \sim \mathbb{P}(X_l = \gamma_l')$$

as l goes to infinity. Since X comes from a central measure,

$$(6.5) \qquad\qquad \mathbb{P}(X_l = \gamma_l) = \#\Gamma_l^{\mathbb{R}^d}(\gamma_l) \frac{\mathbf{t}^{n\delta - \gamma_l}}{\left(S_\delta(\mathbf{t})\right)^l}.$$

Using (6.3) with (6.5) yields that

$$\frac{\#\Gamma_l^{\mathbb{R}^d}(\gamma_l)}{\#\Gamma_l^{\mathbb{R}^d}(\gamma_l')} \sim \mathbf{t}^{\gamma_l' - \gamma_l} = t_i^{-1} \quad \text{and} \quad \frac{\#\Gamma_l^{\mathbb{R}^d}(\gamma_l)}{\#\Gamma_l^{\mathbb{R}^d}(\gamma_l')} \sim \mathbf{t'}^{\gamma_l' - \gamma_l} = t_i'^{-1}.$$

Thus $t_i = t_i'$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

We can now prove the main result of this subsection:

PROPOSITION 6.4. — *The map $\vec{M}$ is a bijective map from $\mathcal{S}$ to $K(\delta)$ such that $\vec{M}([0,1]_\delta^d \times \mathrm{Id}) = K(\delta)^+$.*

Proof. — The injectivity of $\vec{M}$ has already been proven in Proposition 6.3. Let us prove that $\vec{M}$ is surjective. Recall that Ψ is a restriction of the map $\Phi \circ \theta^{-1} : [0,1]^d \times W \to \mathrm{Mult}(\widehat{T}_\delta)^+$ defined at the end of Section 4.2, and that both maps have the same image; thus, it is enough to prove that the map $\vec{M}$ extended to the domain $[0,1]^d \times W$ by the formula $\vec{M}(\mathbf{t}, w) = M_{\Phi \circ (\theta^{-1}(\mathbf{t}), w)}$ is surjective. Let us first prove that $\vec{M}_{|[0,1]^d \times \mathrm{Id}}$ is surjective onto $K(\delta)^+$. Let $1 \leqslant i \leqslant d$ be such that $\langle \delta, \alpha_i \rangle \neq 0$: then, α_i is a δ-admissible set, and the dominant face associated with α_i is one-dimensional. Let $\Sigma : \mathbb{R}^d \to \mathbb{R}$ be the function defined by

$$\Sigma(\mathbf{u}) = \log(S_\delta(e^{u_1}, \ldots, e^{u_d})) = \log\left(\sum_{\gamma \in \Pi_\delta} K_{\delta, \gamma} e^{\mathbf{u} \cdot (\delta - \gamma)} \right).$$

Then,

$$\nabla \Sigma(\mathbf{u}) = \left(\frac{1}{S_\delta(e^{u_1}, \ldots, e^{u_d})} \sum_{\gamma \in \Pi_\delta} K_{\delta, \gamma} (\delta_i - \gamma_i) e^{\mathbf{u} \cdot (\delta - \gamma)} \right)_{1 \leqslant i \leqslant d}$$

$$= \delta - \vec{M}((e^{u_1}, \ldots, e^{u_d}), \mathrm{Id}).$$

Moreover, we can show that Σ is a convex function. To do this, introduce the random variable X such that $\mathbb{P}(X = \delta - \gamma) = \frac{K_{\delta, \gamma}}{S_\delta(e^{u_1}, \ldots, e^{u_d})} e^{\mathbf{u} \cdot (\delta - \gamma)}$. The Hessian matrix of Σ at $\mathbf{u}$ is exactly the covariance matrix of the random variable X, which is nonnegative. Since this is true for all vector $\mathbf{u} \in \mathbb{R}^d$, Σ is indeed convex. Since Σ is a convex function and $(\mathbb{R}^-)^d$ is convex,

the set $\nabla\Sigma(\mathbb{R}^-)^d)$ is a convex set. We have thus proven that the set $\{\delta - \vec{M}(e^{u_1}, \ldots, e^{u_d}, \mathrm{Id}) | \mathbf{u} \in (\mathbb{R}^-)^d\} = \delta - \vec{M}(]0, 1]^d, \mathrm{Id})$ is convex, which implies by continuity that $\vec{M}([0, 1]^d, \mathrm{Id})$ is also convex.

Let $x_i = F_i \cap \partial\Delta$ (that is, x_i is the projection of δ on $\alpha_i^\perp$). Then, a classical reasoning yields that $K(\delta)^+$ is a convex polytope whose extreme points are the elements $\delta, 0$ and $\{x_i\}_{1 \leqslant i \leqslant d, \langle\alpha_i,\delta\rangle>0}$. Let us prove that the extreme points of $K(\delta)^+$ are in the image of $\vec{M}$. Note first that $M(\mathbf{1}, \mathrm{Id}) = \mathbf{0}$, yielding that $\mathbf{0} \in \vec{M}([0, 1]^d \times \mathrm{Id})$. Moreover, since $\vec{M}(\mathbf{0}, \mathrm{Id}) = \delta$, $\delta \in \vec{M}([0, 1]^d \times \mathrm{Id})$. Let $1 \leqslant i \leqslant d$ be such that $\langle\alpha_i, \delta\rangle \neq 0$. Let us set $(\alpha_i, \delta) := \frac{2\langle\alpha_i,\delta\rangle}{\langle\alpha_i,\alpha_i\rangle}$. For $1 \leqslant l \leqslant (\delta, \alpha_i)$, $K_{\delta,\delta-l\alpha_i} = 1$, since the only element of $B(\delta)$ ending at $\delta - l\alpha_i$ is $f_{\alpha_i}^l(\pi_0)$; thus, we have

$$
\begin{aligned}
\vec{M}((\delta_{ij})_{1\leqslant j \leqslant d}, \mathrm{Id}) &= \frac{1}{S_\delta((\delta_{ij})_{1\leqslant j \leqslant d})} \sum_{l=0}^{(\alpha_i,\delta)} K_{\delta,\delta-l\alpha_i}(\delta - l\alpha_i) \\
&= \frac{1}{\sum_{l=0}^{(\alpha_i,\delta)} K_{\delta,\delta-l\alpha_i}} \left(\sum_{l=0}^{(\alpha_i,\delta)} \delta - \sum_{l=0}^{(\alpha_i,\delta)} l\alpha_i \right) \\
&= \frac{1}{(\alpha_i, \delta) + 1} \left(((\alpha_i, \delta) + 1)\delta - \frac{(\alpha_i, \delta).((\alpha_i, \delta) + 1)}{2}\alpha_i \right) \\
&= \delta - \frac{(\alpha_i, \delta)}{2}\alpha_i = x_i,
\end{aligned}
$$

and x_i belongs to $\vec{M}([0, 1]^d \times \mathrm{Id})$. Similarly, if $(\mathbf{t}_l)$ is a sequence of $[0, 1]^d$ converging to $\mathbf{0}$, then $\vec{M}(\mathbf{t}_l, \mathrm{Id})$ converges to $\vec{M}(\mathbf{0}, \mathrm{Id})$. Hence, $\mathbf{0}, \delta$ and $\{x_i\}_{1 \leqslant i \leqslant d, \langle\alpha_i,\delta\rangle>0}$ are in the closure of $\vec{M}([0, 1]^d \times \mathrm{Id})$. Since $\vec{M}([0, 1]^d \times \mathrm{Id})$ is convex, this yields that $K(\delta)^+ \subset \vec{M}([0, 1]^d \times \mathrm{Id})$. By Lemma 6.2, $\vec{M}([0, 1]^d, \mathrm{Id}) \subset K(\delta)^+$, so that finally $\vec{M}([0, 1]^d, \mathrm{Id}) = K(\delta)^+$. Since $\vec{M}(\mathbf{t}, w) = w^{-1}\vec{M}(\mathbf{t}, \mathrm{Id})$, $\vec{M}([0, 1]^d \times W) = \bigcup_{w \in W} w(K(\delta)^+) = K(\delta)$. $\square$

Example 6.5. — With the notation and results of Examples 4.13 and 6.1, one verifies that all the pairs $(t_1, t_1) \in [0, 1]^2$ with $t_1 = 0$ give the same mean vector, namely $(0, 0)$ which is already obtained by considering $(0, 0) \in [0, 1]^2_{\omega_1}$. The set $K(\delta)^+$ coincide with the triangle with vertices $(0, 0), (0, 1),$ $(1/2, 1/2)$ intersection of the convex hull of the vectors $\pm\varepsilon_1, \pm\varepsilon_2$ with the Weyl chamber $\Delta = \{(x_1, x_2) \mid x_1 \geqslant x_2 \geqslant 0\}$. It is parametrized by $[0, 1]^2_{\omega_1}$ through the map $\vec{M}$ detailed in (6.2).

6.2. Proof of Theorem 3.1

We give the proof of Theorem 3.1 by gathering the different results we have established in the previous sections. We only detail the proof for $\partial\mathcal{H}(\mathbb{R}^d)$, the arguments for $\partial\mathcal{H}(\Delta)$ being similar.

- By Corollary 4.1, $\partial\mathcal{H}(\mathbb{R}^d)$ is homeomorphic to $\mathrm{Mult}(\widehat{T}_\delta)^+$ through the map $\eta : \mathrm{Mult}(\widehat{T}_\delta)^+ \to \partial\mathcal{H}(\mathbb{R}^d)$ defined by

$$\eta(f)(\Gamma^{\mathbb{R}}(\tau)) = f(\gamma, n)$$

for any path $\tau \in \Gamma^{\mathbb{R}}$ of length n ending at γ. Since $\mathrm{Mult}(\widehat{T}_\delta)^+$ is compact, $\partial\mathcal{H}(\mathbb{R}^d)$ is a compact space.

- By Proposition 4.14 the map $\Psi : \mathcal{S} \to \mathrm{Mult}(\widehat{T}_\delta)^+$ given by

$$\Psi(\mathbf{t}, w)(\gamma, n) = \frac{1}{S_\delta(\mathbf{t})^n} \mathbf{t}^{n\delta - w(\gamma)}$$

is a bijection.

- Finally, by Proposition 6.4, the map $\vec{M} : \mathcal{S} \to K(\delta)$ given by

$$\vec{M}(\mathbf{t}, w) = \frac{1}{S_\delta(\mathbf{t})} \sum_{\gamma \in \Pi_\delta} K_{\delta,\gamma} \mathbf{t}^{\delta - w(\gamma)} \gamma$$

is also a bijection.

Therefore, the map $\mathbb{P} : K(\delta) \to \partial\mathcal{H}(\mathbb{R}^d)$ given by $\mathbb{P} = \eta \circ \Psi \circ (\vec{M}^{-1})$ is also a bijection. Note that from the previous results, for $m \in K(\delta)$,

$$\mathbb{P}_m(\Gamma^{\mathbb{R}}(\tau)) = \frac{1}{S_\delta(\mathbf{t}_m)^n}(\mathbf{t}_m)^{n\delta - w(\gamma)},$$

for all paths τ of length n ending at γ. It remains to show that $\mathbb{P}$ is indeed an homeomorphism. Since $K(\delta)$ and $\partial\mathcal{H}(\mathbb{R}^d)$ are compact, it suffices to prove that $\mathbb{P}$ or $\mathbb{P}^{-1}$ is continuous. But for $\mathbf{P} \in \partial\mathcal{H}(\mathbb{R}^d)$,

$$\mathbb{P}^{-1}(\mathbf{P}) = \sum_{\tau \in B(\delta)} \mathbf{P}\big(\Gamma^{\mathbb{R}}(\tau)\big)\tau(1).$$

Thus $\mathbb{P}^{-1}$ is continuous, which concludes the proof of Theorem 3.1. The same proof holds for $\partial\mathcal{H}(\Delta)$ with $K(\delta)^+$ and the map $\mathbb{P}^+$ introduced in the statement of the Theorem.

For a metric space X, denote by $\mathcal{M}_1(X)$ the set of probability measures on X with respect to its Borel σ-algebra; we consider $\mathcal{M}_1(X)$ as a topological space with the weak convergence topology. As a straightforward corollary of Theorem 3.1, we get the following integral representation of $\mathcal{H}(\mathbb{R}^d)$ and $\mathcal{H}(\Delta)$.

COROLLARY 6.6. — *The topological spaces $\mathcal{H}(\mathbb{R}^d)$ and $\mathcal{H}(\Delta)$ are homeomorphic to $\mathcal{M}_1(K(\delta))$ and $\mathcal{M}_1(K(\delta)^+)$, respectively through the maps*

$$\mathcal{P} : \begin{cases} \mathcal{M}_1(K(\delta)) \longrightarrow \mathcal{H}(\mathbb{R}^d) \\ \\ \mu \longmapsto \displaystyle\int_{K(\delta)} \mathbb{P}_m \mathrm{d}\mu(m) \end{cases}$$

and

$$\mathcal{P} : \begin{cases} \mathcal{M}_1(K(\delta)^+) \longrightarrow \mathcal{H}(\Delta) \\ \\ \mu \longmapsto \displaystyle\int_{K(\delta)^+} \mathbb{P}_m^+ \mathrm{d}\mu(m). \end{cases}$$

We prove now that a random path in Γ^Δ following the harmonic measure $\mathbb{P}_m^+$ admits a law of large numbers with drift m. In the case of a random path in $\Gamma^\mathbb{R}$ following the central measure $\mathbb{P}_m$, the result is clear from the definition of $\mathbb{P}_m$ and the classical law of large numbers for random walks. The case of $\mathbb{P}_m^+$ is more complicated, since the random path is constrained to remain in a domain. However, the result still holds true.

PROPOSITION 6.7. — *Let π_m be a random path in Γ^Δ following the harmonic measure $\mathbb{P}_m^+$. Denote by $\tau_m(n)$ the position of the path after n steps. Then, almost surely,*

$$\frac{1}{n}\tau_m(n) \longrightarrow m,$$

as n goes to infinity.

Proof. — Denote by $\widetilde{\tau}_m$ the random path in $\Gamma^\mathbb{R}$ following the harmonic measure $\mathbb{P}_m$. By [13, Theorem 4.12], we have the equality in law

$$\tau_m = \mathcal{P}_{\alpha_{i_1}} \dots \mathcal{P}_{\alpha_{i_r}}(\widetilde{\tau}_m),$$

where $w_0 = s_{\alpha_{i_1}} \dots s_{\alpha_{i_r}}$ is a minimal length decomposition of the longest element of W, and each operator $\mathcal{P}_\alpha$ is the Pitman transformation associated with the root α. We recall that the definition of the operator $\mathcal{P}_\alpha$ on a path $\tau \in \Gamma^\mathbb{R}$ is given by

$$\mathcal{P}_\alpha(\tau)(t) = \tau(t) - \left(\inf_{s\in[0,t]} \frac{2\langle \tau(s), \alpha \rangle}{\langle \alpha, \alpha \rangle} \right)\alpha$$

and we set $\mathcal{P} = \mathcal{P}_{\alpha_{i_1}} \dots \mathcal{P}_{\alpha_{i_r}}$. By a large deviation principle,

$$\left\| \frac{1}{t}(\widetilde{\tau}_m)_{|[0,t]} - m\,\mathrm{Id}_{|[0,t]} \right\|_\infty \underset{t\to+\infty}{\longrightarrow} 0$$

with probability one. Thus, for $s \in [0,t]$ and $\alpha \in S$,

$$\left| \frac{1}{t}\frac{2\langle \widetilde{\tau}_m(s), \alpha \rangle}{\langle \alpha, \alpha \rangle} - \frac{2\langle ms, \alpha \rangle}{\langle \alpha, \alpha \rangle} \right| \leqslant \epsilon(t),$$

with $\epsilon(t)$ converging to 0 when t goes to infinity with probability one. Since $m \in K(\delta)^+$, $\langle m, \alpha \rangle \geqslant 0$, and thus $\inf_{s \in [0,t]} s \frac{2\langle m, \alpha \rangle}{\langle \alpha, \alpha \rangle} = 0$. Hence,

$$\left| \frac{1}{t} \inf_{s \in [0,t]} \frac{2\langle \widetilde{\tau}_m(s), \alpha \rangle}{\langle \alpha, \alpha \rangle} \alpha \right| \leqslant \epsilon(t)|\alpha| \xrightarrow[t \to +\infty]{} 0,$$

and finally $\frac{1}{t} \mathcal{P}_\alpha(\widetilde{\tau}_m)(t) \sim \frac{1}{t} \widetilde{\tau}_m(t) \to m$ as t goes to $+\infty$, with probability one. Iterating this result for $\mathcal{P}_{\alpha_{i_1}}, \dots, \mathcal{P}_{\alpha_{i_r}}$ yields that

$$\frac{1}{t} \mathcal{P}(\widetilde{\tau}_m)(t) \xrightarrow[t \to +\infty]{} m$$

with probability one. We conclude by observing that $\mathcal{P}(\widetilde{\tau}_m)$ is equal in law to τ_m. $\qquad\square$

6.3. c-harmonic functions killed on the boundary of Δ

We end this section by proving Corollary 3.3. We recall that $\widehat{s}$ is the map from $\partial \mathcal{H}(\Delta)$ to $\mathbb{R}^+ \cup \{\infty\}$ defined by

$$\widehat{s}(\mathbb{P}_m^+) = \sum_{\gamma \in P} K_{\delta,\gamma} \mathbf{t}_m^\gamma.$$

Note that the range of $\mathbf{t}_m$ is exactly $[0,1]_\delta^d$ by Proposition 6.4.

Proof of Corollary 3.3. — Suppose that $\mathbf{P} \in \mathcal{H}_c(\Delta)$. By Corollary 6.6, there exists $\mu \in \mathcal{M}_1(K(\delta)^+)$ such that $\mathbf{P} = \int_{K(\delta)^+} \mathbb{P}_m^+ d\mu(m)$. Since $\mathbf{P} \in \mathcal{H}_c(\Delta)$, there exists $p : \Lambda \cap \Delta \to \mathbb{R}^+$ such that

$$\mathbf{P}(\Gamma^\Delta(\pi)) = \frac{p(x)}{(c \dim V(\delta))^n},$$

for $\pi \in \Gamma_n^\Delta(x)$.

We first prove that the support of μ is included in $\{\mathbb{P}_m^+ | \mathbf{t}_m \in]0,1]^d\}$. Let $x \in \Lambda \cap \Delta$ and let $n_0 \geqslant 1$ be such that $\Gamma_{n_0}^\Delta(x) \neq \emptyset$. Let $\overline{\pi} = (\pi_n)_{n \geqslant 1}$ be a sequence of paths such that $\pi_n \in \Gamma_{u_n}^\Delta(x)$ with $u_n \to +\infty$ when n goes to infinity (u_n being such that $\Gamma_{u_n}^\Delta(x)$ is nonempty, which is always possible since $\Gamma_{n_0}^\Delta(x) \neq \emptyset$). Then $(c \dim V(\delta))^{u_n} \mathbf{P}(\Gamma^\Delta(\pi_n))$ is constant and equal to $p(x)$. Hence, by the expression of $\mathbf{P}$ and Theorem 3.1,

$$(6.6) \quad p(x) = (c \dim V(\delta))^{u_n} \mathbf{P}(\Gamma^\Delta(\pi_n))$$

$$= (c \dim V(\delta))^{u_n} \int_{K(\delta)^+} \mathbb{P}_m^+(\Gamma^\Delta(\pi)) d\mu(m).$$

Suppose that m is such that $\widehat{s}_\delta(\mathbb{P}_m^+) = +\infty$. This means that there exists $1 \leqslant i \leqslant d$ such that $(\mathbf{t}_m)_i = 0$. Thus, by Proposition 4.7, $\mathbb{P}_m^+(\Gamma^\Delta(\pi))$ is

nonzero only if $\pi \in \Gamma_n^\Delta(y)$ for $y \in n\delta + n\Pi_F$, where F is a fixed dominant face whose dimension is strictly smaller than $K(\delta)$. Hence, the distance of x to $n\delta + n\Pi_F$ goes to infinity with n, which implies that $\mathbb{P}_m^+(\Gamma^\Delta(\pi_n)) = 0$ for n large enough. We can thus assume that the support of μ is included in $\{\mathbb{P}_m^+ \mid \mathbf{t}_m \in]0,1]^d\}$.

For $\mathbf{t}_m \in]0,1]^d$, $s_x(\mathbf{t}_m) \in \mathbb{R}$ for all $x \in P^+$ and we have

$$\frac{S_{x,u_n\delta}(\mathbf{t}_m)}{S_\delta(\mathbf{t}_m)^{u_n}} = \frac{\mathbf{t}_m^{u_n\delta} s_x(\mathbf{t}_m)}{\mathbf{t}_m^{u_n\delta}(s_\delta(\mathbf{t}_m))^{u_n}} = \frac{s_x(\mathbf{t}_m)}{\widehat{s}_\delta(\mathbb{P}_m^+)^{u_n}},$$

where $S_{x,u_n\delta}$ is the normalized character as defined in (3.2). Thus, with Theorem 3.1, (6.6) becomes

$$(c \dim V(\delta))^{u_n} \mathbf{P}(\Gamma^\Delta(\pi_n)) = \int_{K(\delta)^+} (c \dim V(\delta))^{u_n} \frac{S_{x,u_n\delta}(\mathbf{t}_m)}{S_\delta(\mathbf{t}_m)^{u_n}} d\mu(m)$$

$$= \int_{K(\delta)^+} (c \dim V(\delta))^{u_n} \frac{s_x(\mathbf{t}_m)}{\widehat{s}_\delta(\mathbb{P}_m^+)^{u_n}} d\mu(m),$$

where we have used that the support of μ is in $\{\mathbb{P}_m^+ \mid \mathbf{t}_m \in]0,1]^d\}$. In order that the right hand-side of the latter expression does not go to infinity, we must have $\mu(\widehat{s}_\delta^{-1}[0, c \dim V(\delta)[) = 0$. Then, we have

$$p(x) = \lim_{n \to +\infty} \int_{K(\delta)^+} \left(\frac{c \dim V(\delta)}{\widehat{s}_\delta(\mathbb{P}_m^+)} \right)^{u_n} s_x(\mathbf{t}_m) d\mu(m)$$

$$= \int_{\widehat{s}_\delta^{-1}(\{c \dim V(\delta)\})} s_x(\mathbf{t}_m) d\mu(m),$$

and the support of μ is $\widehat{s}_\delta^{-1}(\{c \dim V(\delta)\})$. Finally,

$$\partial \mathcal{H}_c(\Delta) \subset \widehat{s}_\delta^{-1}(\{c \dim V(\delta)\}).$$

Its is readily seen that $\widehat{s}_\delta^{-1}(\{c \dim V(\delta)\}) \subset \mathcal{H}_c(\Delta) \cap \partial \mathcal{H}(\Delta)$ by the expression of $\mathbb{P}_m^+$ from Theorem 3.1, and by the characterization of c-harmonic functions from Section 2.3; since $\mathcal{H}_c(\Delta)$ is a convex subset of $\mathcal{H}(\Delta)$, $\mathcal{H}_c(\Delta) \cap \partial \mathcal{H}(\Delta) \subset \partial \mathcal{H}_c(\Delta)$. Finally, $\partial \mathcal{H}_c(\Delta) = \widehat{s}_\delta^{-1}(\{c \dim V(\delta)\})$, which proves the first part of the corollary.

For the second part of the corollary, a quick computation yields that $\widetilde{s}_\delta : (u_1, \ldots, u_d) \mapsto \log(s_\delta(e^{u_1}, \ldots, e^{u_d}))$ is strictly convex on $\mathbb{R}^d$: as in the proof of Proposition 6.4, the Hessian matrix of $\widetilde{s}_\delta$ at $\mathbf{u}$ is actually the covariance matrix of a non-degenerate random variable. Thus, $\widetilde{s}_\delta$ admits a unique minimum on $\mathbb{R}^d$, which is located at the unique vector $\mathbf{u}_0$ such that $\nabla \widetilde{s}_\delta(\mathbf{u}_0) = \mathbf{0}$. Since

$$\nabla \widetilde{s}_\delta(\mathbf{t}) = \frac{1}{s_\delta(t)} \sum_{\gamma \in P} K_{\delta,\gamma} e^{\mathbf{u} \cdot \gamma} \gamma = \vec{M}(e^{-u_1}, \ldots, e^{-u_d}, \mathrm{Id}) e^{\mathbf{u} \cdot \delta}$$

and $\vec{M}(\mathbf{1}) = \mathbf{0}$, the minimum of $\tilde{s}_\delta$ is at $\mathbf{0}$. Hence, the minimum for s_δ is at $\mathbf{1}$, and thus $\min_{\partial \mathcal{H}_\infty(\Delta)} \hat{s}_\delta = \hat{s}_\delta \left(\mathbb{P}_{\mathbf{1}}^+ \right) = \dim V(\delta)$. The second part of the corollary is a straightforward deduction of this fact. $\qquad\square$

Note that the proof of the latter corollary gives an explicit expression of the harmonic function associated with the central measure $\mathbb{P}_m^+$.

α_i	simple roots, p. 2362
Δ	dominant Weyl chamber, p. 2362
ℓ	length function on W, p. 2370
$\Gamma_n^\Omega(y)$	set of paths of length n in Ω ending at y, p. 2365
γ, β	weight of $\mathfrak{g}$, p. 2370
Γ^Ω	set of infinite paths included in Ω, p. 2365
$\Gamma^\Omega(\tau)$	paths of $\Gamma(\Omega)$ starting by τ, p. 2365
$\Gamma^\Omega(x, y)$	set of infinitesimal paths in Ω between x and y, p. 2365
$\Gamma_{\mathrm{fin}}^\Omega$	set of finite paths included in Ω, p. 2365
Γ_n^Ω	set of paths of length n in Ω, p. 2365
$\widehat{\mathcal{A}}_\delta$	extended algebra of characters, p. 2376
$\widehat{P}_\delta$	extended algebra of weights, p. 2376
Λ	lattice in $\mathbb{R}^d$, p. 2364
λ, δ	dominant weights of $\mathfrak{g}$, p. 2371
$[0, 1]_\delta^d$	subset of $[0, 1]^d$ defined in (4.5), p. 2387
$\mathbb{P}_m$	harmonic measure of $\partial \mathcal{H}(\mathbb{R}^d)$ with drift $m \in \mathbb{R}^d$, p. 2373
$\mathbb{P}_m^+$	harmonic measure of $\partial \mathcal{H}(\Delta)$ with drift $m \in \Delta$, p. 2373
$\mathbb{R}[P]^W$	character ring of $\mathfrak{g}$, p. 2371
$\mathcal{A}_\delta$	subalgebra of $\mathbb{R}^W[P]$ generated by s_λ with $\lambda \in T_\delta^+$, p. 2372
$\mathcal{G}(\Omega)$	rooted graph associated to Γ^Ω, p. 2370
$\mathcal{H}(\mathcal{G})$	harmonic functions on the multiplicative graph $\mathcal{G}$, p. 2368
$\mathcal{H}(\Omega)$	minimal boundary of Γ^Ω, p. 2366
$\mathcal{H}(\Omega)$	set of central measures on Γ^Ω, p. 2365
$\mathcal{H}_c(\Omega)$	set of central measures coming from c-harmonic functions, p. 2367
$\mathcal{M}_1(\Gamma^\Omega)$	space of probability measures on Γ^Ω, p. 2365
$\mathrm{Mult}(A)^+$	multiplicative functions on A nonnegative on the cone $K \subset A$, p. 2368
ω_i	fundamental weight of $\mathfrak{g}$, p. 2370
$\partial \mathcal{H}(\mathcal{G})$	extremal points of $\partial \mathcal{H}(\mathcal{G})$, p. 2368
$\mathbf{P}, \mathbf{P}_0$	central/harmonic probability measure on Γ^Ω, p. 2365
π, τ	paths on Λ, p. 2364
Π_δ	set of weights of $V(\delta)$, p. 2362
$\Pi_\delta(f)$	subset of Π_δ on which f is nonzero, p. 2382
Π_F	intersection of Π_δ with the dominant face F, p. 2382

Ψ^+ map defined in Corollary 5.5, p. 2392

$\vec{M}$ mean vector map defined in Section 6.1, p. 2396

$A_{\mathcal{G}}$ algebra generated by the vertices of a multiplicative graph, p. 2368

$B(\delta)$ set of Littelmann paths associated to $V(\delta)$, p. 2362

d dimension of $\mathfrak{g}$, p. 2362

e^β abstract variable corresponding to the weight β, p. 2371

F dominant face of $K(\delta)$, p. 2382

h c-harmonic function with respect to Z, p. 2367

$K(\delta)$ convex hull of Π_δ, p. 2362

$K(\delta)^+$ intersection of $K(\delta)$ with Δ, p. 2362

$K_{\lambda,\gamma}$ dimension of the weight space γ in $V(\lambda)$, p. 2371

$l(\pi)$ length of π, p. 2364

M_f drift associated to the multiplicative function f, p. 2379

P weight lattive, p. 2362

p multiplicative function associated to $\mathbf{P}$, p. 2365

P_+ cone of dominant weights, p. 2362

P_δ sublattice of P generated by Π_δ, p. 2372

Q^+ subset of P^+ spanned by S over $\mathbb{Z}$, p. 2371

R set of roots of $\mathfrak{g}$, p. 2370

R_+ set of positive roots, p. 2370

S set of simple roots, p. 2362

s_{α_i} reflection associated to α_i, p. 2370

$S_{\lambda,\mu}, S_\lambda$ normalized character defined in (3.2), p. 2371

s_λ Weyl character of $V(\lambda)$, p. 2371

T_δ^+ dominant weights λ such that $V(\lambda)$ appears in some tensor power $V(\delta)^{\otimes n}, n \geqslant 0$, p. 2372

$T_{\delta,n}$ weights appearing with nonzero multiplicity in $V(\delta)^{\otimes n}$, p. 2376

T_i shortcut for $e^{-\alpha_i}$, p. 2371

$V(\delta)$ irreducible representation associated to δ, p. 2362

W Weyl group of $\mathfrak{g}$, p. 2370

$W^{S'}$ set of minimal right-coset representative with respect to $S' \subset S$, p. 2386

X set of infinitesimal paths, p. 2364

Z canonical random walk generated by X, p. 2366

$\mathfrak{g}$ finite dimensional Lie algebra, p. 2362

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