Comptes Rendus Mathématique

Alessandro Chiodo

INSTITUT DE FRANCE
Académie des sciences

Comptes Rendus
Mathématique

Alessandro Chiodo

On the construction of the *Śrī Yantra*

Volume 359, issue 4 (2021), p. 377-397

<https://doi.org/10.5802/crmath.163>

Comptes Rendus
Mathématique
2021, 359, nº 4, p. 377-397
https://doi.org/10.5802/crmath.163

Dissemination of mathematics, History of mathematics / *Diffusion des mathématiques, Histoire des mathématiques*

On the construction of the *Śrī Yantra*

Alessandro Chiodo[a]

[a] Institut de Mathématiques de Jussieu — Paris Rive Gauche, Sorbonne Université, UMR 7586 C.N.R.S., 4 Pl. Jussieu,75252 Paris cedex, France
E-mail: alessandro.chiodo@imj-prg.fr

Abstract. The *Śrī Yantra* (or *Śrī Cakra*) is a sacred diagram of Tantric Hinduism. Its study stimulated a vast effort of specialists from different fields. In mathematics, its construction sets an elementary and nontrivial problem. In this note, we work out a straightedge and compass method for constructing concurrent models of *Śrī Yantras.* The question is equivalent to the circle-line-point problem of Apollonious.

Résumé. Le *śrīyantra* (ou *śrīcakra*) est un diagramme sacré dans les traditions hindoues tantriques. Il a fait l'objet de nombreuses études dans différentes disciplines. En mathématiques, sa construction pose un problème élémentaire et non trivial. Dans cette note, on fournit une méthode de construction à la règle et au compas. La question est équivalente à celle d'un problème d'Apollonius qui consiste à trouver un cercle tangent à un cercle donné, à une droite donnée et passant par un point donné.

Manuscript received 6th October 2020, revised 21st October 2020 and 30th November 2020, accepted 30th November 2020.

Version française abrégée

Les *yantras* sont des objets sacrés dans l'hindouisme. De façon remarquable, plusieurs d'entre eux possèdent des propriétés mathématiques intéressantes. Le *śrīyantra* est l'un des plus fameux parmi les *yantras.* Il peut être décrit ainsi : un point au centre, le *bindu*, un diagramme polygonal qui l'entoure et qui est lui-même inscrit dans un motif circulaire. Celui-ci est constitué de huit pétales de lotus, encerclés à leur tour par un lotus à seize pétales et par un triple carré dit *bhūpura.*

Le polygone fait l'objet de ce papier. Il est issu de la réunion de neuf triangles maximaux $t_1, \dots, t_9$ tous symétriques par rapport à un même axe vertical et numérotés de 1 à 9 selon la position de leurs bases à commencer par celle qui se situe le plus en haut.[1] Les propriétés de concurrence suivantes sont satisfaites (on appelle point de base d'un triangle isocèle le milieu de la base).

[1] Dans les Figures 2 et 8, on utilise pour les contours de $t_1, \dots, t_9$ la séquence de neuf couleurs suivante : gris (t_1), violet (t_2), bleu (t_3), vert (t_4), jaune (t_5), rose (t_6), orange (t_7), rouge (t_8) et marron (t_9). Avec notre choix de notation, t_i est orienté vers le bas si $i \leq 5$; autrement, il est orienté vers le haut.

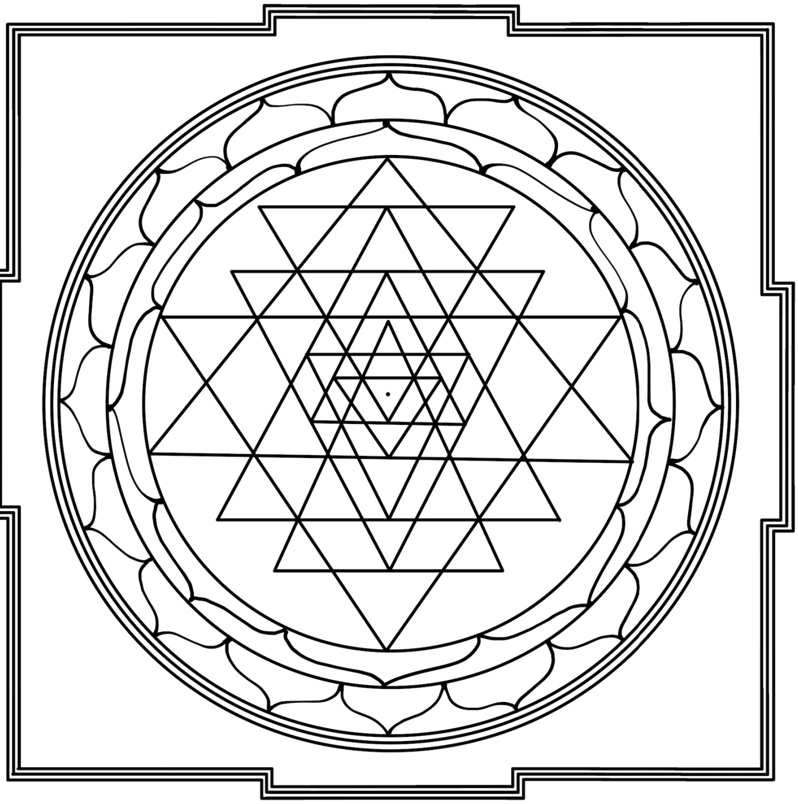


Figure 1. *Śrī Yantra*

(i) Les triangles t_3 et t_7 sont inscrits[2] dans le même cercle.

(ii) Le sommet du triangle t' est le point de base du triangle t'' pour tout couple (t', t'') égal à (t_8, t_1), (t_6, t_2), (t_9, t_3), (t_1, t_6), (t_5, t_7), (t_4, t_8) et (t_2, t_9).

(iii) De chaque côté de l'axe de symétrie, la jambe[3] du triangle orienté vers le bas t', la jambe du triangle orienté vers le haut t''' et la base du triangle t'' ont exactement un point en commun pour tout triple (t', t'', t''') égal à (t_1, t_2, t_7), (t_2, t_3, t_7), (t_1, t_3, t_8), (t_1, t_4, t_6), (t_1, t_5, t_9), (t_4, t_6, t_9), (t_2, t_7, t_9), (t_3, t_7, t_8), (t_3, t_8, t_9), (t_4, t_4, t_8), (t_5, t_5, t_6) et (t_2, t_6, t_6).

Nous nous référons aux conditions ci-dessus avec les notations (i), ((ii) ; t_i, t_j) et ((iii) ; t_i, t_j, t_k). À titre d'exemple, nous avons détaillé la condition ((iii) ; t_5, t_5, t_6) dans la Figure 3. Le dessin de la Figure 1, basé sur des réalisations faites à la main, illustre ces concurrences, mais met aussi en évidence de légères imprécisions visibles à l'œil nu (vérifier par exemple la condition ((iii) ; t_3, t_7, t_8)). Le programme GeoGebra [10] que nous avons écrit ici [https://www.geogebra.org/m/zdvxtdvv] permet au lecteur de comprendre ces contraintes en faisant varier la position des triangles enchevêtrés tout en respectant (i), (ii) et (iii). Le lecteur peut déjà le consulter sans lire davantage, en ignorant dans un premier temps la construction géométrique qui y apparait en traits légers (voir aussi l'Appendice A pour quelques outils supplémentaires sur la construction du *śrīyantra*).

[2] Un triangle est inscrit dans un cercle si le cercle passe par ses sommets. Dans ce cas, le cercle est dit circonscrit au triangle et est uniquement déterminé par ce dernier.

[3] L'axe de symétrie étant fixé, on appelle jambes d'un triangle isocèle les deux côtés symétriques.

Les origines

Les origines du *śrīyantra* sont inconnues. On doit les chercher dans les rituels qui lui ont été associés (voir Padoux [17]).

Michaël [14] décrit l'un de ces rituels à partir du texte *Saundarya Laharī*, « Les vagues de la beauté ». Il s'agit d'un poème attribué à l'un des disciples du philosophe du VIII$^{\text{ème}}$ siècle, Ādi Śaṅkara (voir aussi [13]).

Dans [6], R. C. Gupta considère un large nombre de *yantras* de grand intérêt mathématique ; (parmi eux, on en trouve certains, comme le *chautisa yantra*, qui ont déjà fait l'objet d'études purement mathématiques, voir G. Bhowmik [1] et A. Navas [16]). Quant au *śrīyantra*, Gupta le décrit comme l'un des plus importants et des plus fameux parmi les *yantras*. La présentation qu'il en donne permet d'établir quelques points cruciaux de sa définition au cours de l'histoire. On suit son texte, mais on se réfère aussi à Huet [11], Mookerjee et Khanna [15], Rao [19] et Zimmer [26].

Gupta [6, p. 180] cite le *Rudrāyamala Tantra* et notamment le vers faisant référence à un *yantra* constitué par un *bindu*, un triangle central et des « enceintes » de 8, puis 10, puis encore 10, puis 14 triangles, entourées de trois cercles et de trois *bhūpura*. Ces quatre rangées de 8, 10, 10 et 14 triangles apparaissent effectivement dans le *śrīyantra* et sont mises en évidence dans la Figure 2.b par deux tons alternés de gris. Elles sont souvent regardées en tant que *cakras* (roues).[4]

Gupta [6, p. 181] mentionne aussi trois variantes tridimensionnelles, *kūrma* (la tortue), *padma* (le lotus) et *meru* (la montagne fabuleuse), alternatives à la variante plane que nous considérons ici (dite *bhū*, c.-à-d. la Terre). Dans la version dite *kūrma*, les segments sont remplacés par des arcs de cercle sur une calotte sphérique (sur ce point Gupta se réfère au *Gaurīyāmala Tantra*).

Kulaichev fait référence à de Casparis : on trouverait une mention du *śrīyantra* dans une inscription bouddhiste de Sumatra du VII$^{\text{ème}}$ siècle (voir Kulaichev [13, p. 279] et de Casparis [3, p. 34 et p. 41]). Dans [13, p. 279], il fait aussi allusion à un hymne dans *Atharva Veda* (XII$^{\text{ème}}$ siècle av. J.-C.) qui porterait sur une figure analogue constituée de neuf triangles. Toujours selon Kulaichev [13], une représentation datant du XVII$^{\text{ème}}$ siècle est située dans le monastère de Śṛṅgārī Maṭha fondé par Ādi Śaṅkara.

On conclut en rappelant, comme le fait Huet, que, même si plusieurs sources suggèrent que ce symbole est très ancien, nous ne connaissons pas de représentations publiées antérieures au XVII$^{\text{ème}}$ siècle (Huet, [11, p. 622]). La question de la datation des premières créations du *śrīyantra* demeure ouverte et l'intérêt que l'on porte à cette question est temoigné par ce passage dans Mookerjee et Khanna [15, p. 61] : « Dans son contenu formel, le *śrīyantra* est un chef d'œuvre d'abstraction, et doit avoir été créé par révélation plutôt que par l'ingéniosité et l'habilité de l'homme ».

L'histoire de la construction du śrīyantra

Dessiner un *śrīyantra* pose un problème mathématique élémentaire et non trivial. En 2007, Gupta dresse la liste des différentes méthodes connues à ce jour.

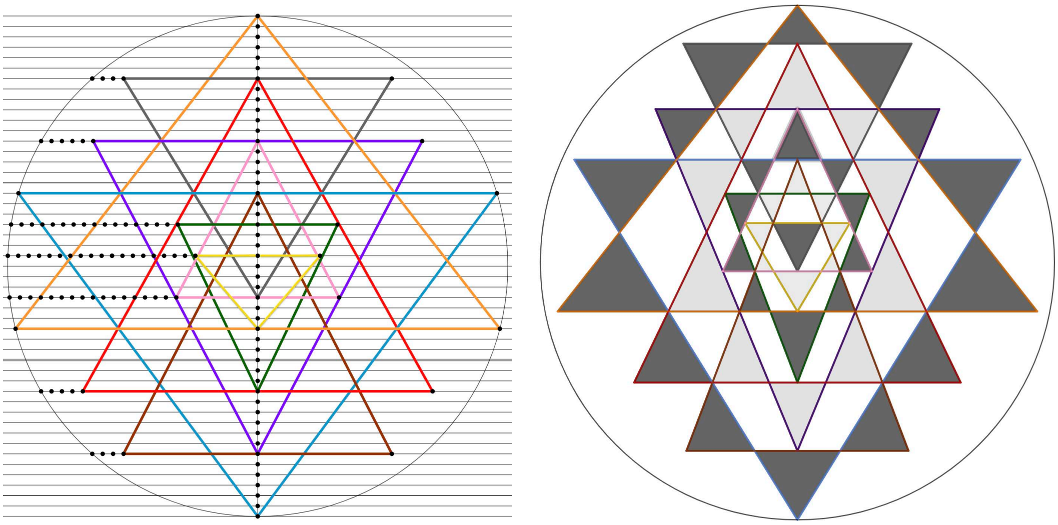
Méthodes traditionnelles

Des méthodes traditionnelles sont dues à deux commentateurs du texte *Saundarya Laharī* : Kaivalyāśrama Figure 2.a et Lakṣmīdhara Figure 2.b.

[4]Le *śrīyantra* est décrit en termes de *cakras* par exemple dans Mookerjee–Khanna [15] et Zimmer [26] (voir aussi Jung [12]). Les neuf *cakras* (roues) inscrits les uns dans les autres et entourant le *bindu* sont le *bhūpura*, le triplet de cercles, les 16 pétales, les 8 pétales, l'enceinte de 14 triangles, celle de 10, puis 10, puis 8 triangles et le triangle central. Gupta signale que le terme *cakra* est aussi parfois attribué aux neuf triangles t_i.

La méthode A, décrite par Kaivalyāśrama est exécutée dans la Figure 2.a. Elle consiste à préciser la position des sommets des triangles dans un repère constitué par le diamètre vertical découpé en 48 parties égales et par le cercle. On peut voir à l'œil nu que cette méthode ne respecte pas exactement la condition de concurrence ci-dessus (observons par exemple l'intersection entre la base de t_8 et les jambes de t_3 et t_9 — propriété ((iii) ; t_3, t_8, t_9)).

La méthode B, transmise par Lakṣmīdhara, part du triangle central et l'entoure quatre fois par des rangées de 8, 10, 10 et 14 triangles comme dans le passage du *Rudrāyamala Tantra* mentionné ci-dessus. Ainsi, les conditions de concurrence ((ii)–(iii)) sont satisfaites. Cependant, la Figure 2.b montre bien que, en suivant cette méthode, la condition (i) peut échouer (le cercle ne passe pas par les sommets latéraux de t_3 et t_7).



(a) On dessine la figure en positionnant les sommets à l'intérieur d'une grille fixée. Les intersections telles que celles des triangles bleu, rouge et marron ((iii) ; t_3, t_8, t_9) sont imprécises.

(b) On peut partir des conditions de concurrence entre les triangles ((ii)–(iii)). En général, la condition (i) qui impose le passage du cercle extérieur par les sommets de t_3 et t_7 échoue.

FIGURE 2. Les méthodes de Kaivalyāśrama et Lakṣmīdhara

Ces méthodes sont imparfaites, mais elles ont le mérite de porter jusqu'à nos jours un archétype : l'idée bien identifiée d'une collection de figures géométriques qu'on appelle *śrīyantra*. Pour faire un exemple simple, on peut comparer la collection des *śrīyantras* à l'ensemble des triangles isocèles à homothétie et à congruence près. Cet ensemble est défini par une propriété élémentaire de symétrie mais contient plusieurs formes (triangles obtus, triangles aigus ...) non équivalentes même après homothétie et congruence. Les propriétés caractérisant un *śrīyantra* ne sont pas si élémentaires, mais ces deux méthodes permettent de comprendre les conditions de contact et de concurrence qu'il faut imposer pour qu'un *śrīyantra* soit admissible, ou - comme on écrira dorénavant - *concurrent*. Nous les avons résumées en ((i)–(iii)) ci-dessus. À compter de la fin des années 1970, d'autres auteurs les ont formalisées mathématiquement de façon différente mais compatible. On illustre ci-dessous ces approches.

Constructions à la règle et au compas

Si l'on peut aujourd'hui suivre facilement les constructions A et B, c'est grâce à Bolton *et al* [2] et à Fonseca [5]. On trouve, traduites en anglais et listées, les coordonnées de A dans [2, p. 68–69] et [5, p. 35–36] (voir aussi Gupta [6, Section 5]). On trouve aussi un tableau et une figure, [2, Tableau 2, Figure 2], qui précisent la méthode B suivant une communication non publiée de A. West. Notons au passage que ce tableau, montre que les figures satisfaisant (ii) et (iii) constituent une collection de formes géométriques à six degrés de liberté (à homothétie et congruence près) ; l'imposition de la condition (i) reviendrait à éliminer deux paramètres. On obtiendrait ainsi une famille à *quatre* paramètres de *śrīyantras* concurrents.

Bolton *et al* [2] et puis Fonseca [5] proposent également des constructions à la règle et au compas basées sur celle du nombre d'or. Elles échouent de façon invisible à l'œil nu : cela a motivé l'hypothèse [2, p. 75], encore considérée aujourd'hui qu'une construction analogue du *śrīyantra* remonterait à l'Égypte ancienne.

Un germe d'une variété de śrīyantras

En 1990, Huet a formalisé le problème et dressé une liste de liens logico-géométriques [11, p. 611] équivalente à ((i)–(iii)) et basée sur le choix de cinq paramètres initiaux. Le dessin se complète en appliquant cette liste qui traduit toutes les concurrences ((i)–(iii)) ci-dessus *sauf une :* la condition ((ii) ; t_2, t_6). Enfin, Huet impose cette dernière condition, ce qui revient à poser une équation qui lie les cinq paramètres initiaux. Les programmes informatiques qu'il en dérive calculent par approximation et dessinent chaque élément de l'ensemble des *śrīyantras* concurrents.[5] On peut alors conclure que l'ensemble des solutions, issu de cinq paramètres satisfaisant une équation, se réduit effectivement à *quatre* choix[6] de paramètres réels. Huet remarque que l'on doit choisir ces quatre paramètres dans des très petits intervalles. On commence à apercevoir l'espace des paramètres des *śrīyantras* sur lequel on reviendra dans quelques lignes.

La recherche d'un unique śrīyantra

En 1984, Kulaichev aborde un problème plus difficile encore. Celui d'imposer à la fois les conditions ((i)–(iii)) mais aussi *quatre* conditions supplémentaires.[7] Il fournit un raisonnement qui indique l'existence d'une solution et il se pose la question (toujours ouverte) de l'unicité.

Kulaichev considère aussi le problème du point de vue de l'approximation des solutions algébriques et des constructions tridimensionnelles *kūrma*. Des travaux systématiques dans ces directions ont été menés par C. S. Rao [20]. Plus récemment, dans [18, 21, 23, 24], on explore davantage l'approche algèbrique déjà abordée par Kulaichev et Rao.

Le śrīyantra *et le problème d'Apollonius*

À ma connaissance, la question naturelle de savoir si le *śrīyantra* est constructible à la règle et au compas demeurait ouverte. Cette question est traitée ici sous des hypothèses équivalentes à celles de Huet, c.-à-d. sans imposer de conditions supplémentaires. On construit à la règle et au compas la famille de tous les *śrīyantras* qui satisfont les conditions minimales de concurrence ((i)–(iii)).

---

[5] L'un de ces dessins constitue la couverture d'un volume en l'honneur de l'informaticien Maurice Nivat. C'est dans ce volume que paraît, en 2002, le papier de Huet des années 1980 que l'on cite aujourd'hui [11].

[6] On note au passage que quatre des cinq paramètres choisis par Huet, à savoir Y_Q, Y_P, X_A, et Y_J dans ses notations, sont interchangeables avec les quatre données de départ de notre construction, à savoir Y_Q, Y_P, Y_L, et Y_J, toujours dans ses notations.

[7] Les conditions sont les suivantes. On impose que (a) les triangles t_3 et t_7 soient congruents (comme le font déjà les auteurs de [2]), (b) que le cercle circonscrit à t_1 coïncide avec le cercle extérieur $\mathscr{E}$ (circonscrit à t_3 et t_7), (c) que le cercle circonscrit à t_9 coïncide aussi avec $\mathscr{E}$, (d) que le cercle inscrit au triangle central du diagramme (c.-à-d. $t_1 \cap t_5$) ait le même centre que $\mathscr{E}$. Notons que certains auteurs considèrent aussi l'équilatéralité de t_1. Et mentionnons aussi qu'il semblerait naturel d'étudier également une condition de concurrence supplémentaire : ((iii) ; t_4, t_5, t_6).

On démontre que la construction du *śrīyantra* est équivalente à la solution de l'un des problèmes des contacts posés par Apollonius de Perga au III$^{\text{ème}}$ siècle av. J.-C. Ce problème consiste à trouver les cercles Ξ passant par un point ϕ fixé initialement, tangent à une droite Δ et à un cercle Π qui sont aussi fixés au préalable. On voit ci-dessous que ce problème se pose lors de la construction d'un *śrīyantra* à partir des points de base de t_3, t_6, t_7 et t_9 le long du diamètre vertical (voir la Figure 6). Ces paramètres identifient une droite Δ et, dans le même demi-plan délimité par celle-ci, un cercle Π et un point ϕ externe au cercle. Le dessin du *śrīyantra* découle alors d'une solution particulière parmi les quatre cercles qui résolvent ce problème d'Apollonius.[8]

Dans les années 1980, grâce aux développements majeurs de la géométrie énumérative des courbes algébriques, Eisenbud et Harris [4] montrent que les solutions au problème d'Apollonius forment les feuillets d'un revêtement ramifié d'un espace qui paramétrise tout choix de Π, Δ et ϕ.[9] Cet espace complète la variété des *śrīyantras* décrite par Huet : on ne voyait qu'une famille de *śrīyantras* paramétrée par un voisinage suffisamment petit pour qu'il puisse se relever au revêtement à quatre feuillets mentionné ci-dessus. En fait, tous les points de ce revêtement admettent une interpretation géométrique qui prolonge la famille des *śrīyantras*.

La construction

On présente la construction dans les Figures 5, 6, 7 et 8. Dans toutes ces figures, on considère la moitié droite du diagramme qu'on place à l'horizontal en opérant une rotation d'un angle droit dans le sens inverse des aiguilles d'une montre. On redresse la construction dans la Figure finale 11. On commente ici ces figures brièvement et nous nous référons à la Section 2.2 pour des instructions détaillées.

Le début

À homothétie et à translation près, le diagramme dépend de quatre paramètres. Sur le diamètre orienté qui rejoint le sommet O de t_7 et le sommet T de t_3, on fixe P, Q, R, et S dans l'ordre croissant : ce sont les points de base de t_3 t_6, t_7 et t_9. Si l'on veut obtenir un diagramme de la famille des *śrīyantras* similaire à ceux qui apparaissent traditionnellement (comme ceux des figures de cet article) on peut placer P, Q, R et S dans les positions $\frac{17}{48}$, $\frac{27}{48}$, $\frac{30}{48}$ et $\frac{42}{48}$ du diamètre $[0,1] = \overline{OT}$, comme dans la méthode A, Figure 2.a. Dans les figures qui suivent, à partir de la Figure 4, nous plaçons P, Q, R, S en position $0,324$; $0,517$; $0,592$; $0,864$ afin de mieux dégager les droites qui apparaissent au cours de la construction. Enfin, dans la figure finale (Figure 11) on utilise les paramètres suggérés par Huet. On procède ensuite suivant les légendes des six diagrammes qui apparaissent à l'intérieur de la Figure 4. Ceux-ci permettent de positionner le sommet du triangle t_4 et la base de t_6 en entier.

Du śrīyantra *à Apollonius*

Les Figures 5, 6 et 7 servent à poser et à résoudre le problème d'Apollonius. Dans cette digression, au lieu des lettres, on emploie des nombres de 1 à 28 pour dénoter les points : (n, m) est la droite qui passe par n et m, $\overline{n,m}$ est le segment de n à m, $\overline{n, m, \ldots, k}$ est le polygone de sommets $n, m, \ldots, k$. Cette digression a le seul but (crucial) de définir la position du point 28 qui permet de reprendre la construction en plaçant le point A.

[8] Les théorèmes écrits par Apollonius ont été perdus comme le témoigne Pappus d'Alexandrie au IV$^{\text{ème}}$ siècle. Ils ont été redémontrés par Viète en 1600.

[9] Harris [7] interprète aussi cela en termes d'espaces des modules (c.-à-d. espaces de paramètres) des courbes algébriques spin, c.-à-d. des courbes C munies d'une théta caractéristique L qui satisfait $L^{\otimes 2} \cong \omega_C$.

D'Apollonius au śrīyantra

On conclut la construction sur la Figure 8 en revenant aux points dénotés par des lettres. On dispose des points O, P, Q, R, S, T, U, V, W, X, Y, Z, E, F, G. On résume brièvement les étapes finales en renvoyant le lecteur à la Section 2.2 pour des instructions détaillées. On projette horizontalement le centre (point numéro 28) du cercle Ξ, solution au problème d'Apollonius. On obtient ainsi le point A sur PU. On déduit B en prolongeant le segment AQ jusqu'à OV. Puis on projette B verticalement sur OT et on obtient C. On pose $D = CG \cap AQ$. La droite ZA rencontre l'axe OT au point H. On pose $I = ZA \cap EF$. On pose $L = BC \cap SW$ et $K = BQ \cap h$, où h est la droite verticale qui passe par H. On définit M comme le point où la droite verticale passant par J rencontre CD. On pose $N = HA \cap EY$ et enfin $\square = PY \cap s$, où s est la droite verticale qui passe par S.

Les triangles $t_1, \ldots, t_9$ sont déterminés sur la Figure 8 avec leurs bases ordonnées de gauche à droite dans les couleurs gris, violet, bleu, jaune, vert, orange, rouge et marron. On renvoie à la Section 2.3 pour la preuve de la validité de cette construction.

1. Introduction

Yantras are sacred objects in Hinduism. Remarkably, many of them have very interesting mathematical properties. The *Śrī Yantra* is one of the most popular *yantras*; it is used in yoga meditation. It can be described from the exterior to the interior as follows. The outermost motive is the square of defence or *bhūpura*. A triple circle circumscribes the core of the diagram. Then, sixteen petals of lotus surround an eight-petalled lotus and a polygonal diagram containing a central point, the *bindu* (see Figure 1).

This article deals with the polygon, which is indeed the union of nine maximal triangles $t_1, \ldots, t_9$ possessing a common vertical axis of symmetry and ordered according to the position of their base from the top downwards. In Figures 2 and 8, for the edges of $t_1, \ldots, t_9$, we use the following sequence of colours: gray (t_1), violet (t_2), blue (t_3), green (t_4), yellow (t_5), rose (t_6), orange (t_7), red (t_8), and brown (t_9). With these notations t_i points downward if $i \leq 5$ and points upward otherwise.

The following concurrency properties are required to be satisfied (see Figures 1 and 3).

Throughout, *base point* denotes the midpoint of the base of an isosceles triangle.

(i) The triangles t_3 and t_7 share the same circumscribed circle.

(ii) The apex of the triangle t' is the base point of the triangle t'' for $(t', t'') = (t_8, t_1)$, (t_6, t_2), (t_9, t_3), (t_1, t_6), (t_5, t_7), (t_4, t_8), and (t_2, t_9).

(iii) On each side of the axis, the intersection between the leg[10] of the downward triangle t', the leg of the upward triangle t''', and the base of t'' consists of a single point for $(t', t'', t''') = (t_1, t_2, t_7)$, (t_2, t_3, t_7), (t_1, t_3, t_8), (t_1, t_4, t_6), (t_1, t_5, t_9), (t_4, t_6, t_9), (t_2, t_7, t_9), (t_3, t_7, t_8), (t_3, t_8, t_9), (t_4, t_4, t_8), (t_5, t_5, t_6), and (t_2, t_6, t_6).

We will refer to the above conditions as (i), ((ii); t_i, t_j), and ((iii); t_i, t_j, t_k). Figures 1 and 3 are based on a handmade realisation of the diagram that enables us to both check the concurrency conditions and appreciate the slight imperfections, as for instance ((iii); t_3, t_7, t_8).

In the GeoGebra [10] programme (available here [https://www.geogebra.org/m/zdvxtdvv] and based on the present paper) we can move the triangles without breaking the concurrency conditions. This allows us to appreciate conditions (i), (ii) et (iii) and carry out experimentation (see also Appendix A for more information and GeoGebra programmes illustrating the *Śrī Yantra* and its construction).

[10] Since an axis of symmetry is fixed, we can refer to the symmetrical edges as the legs.

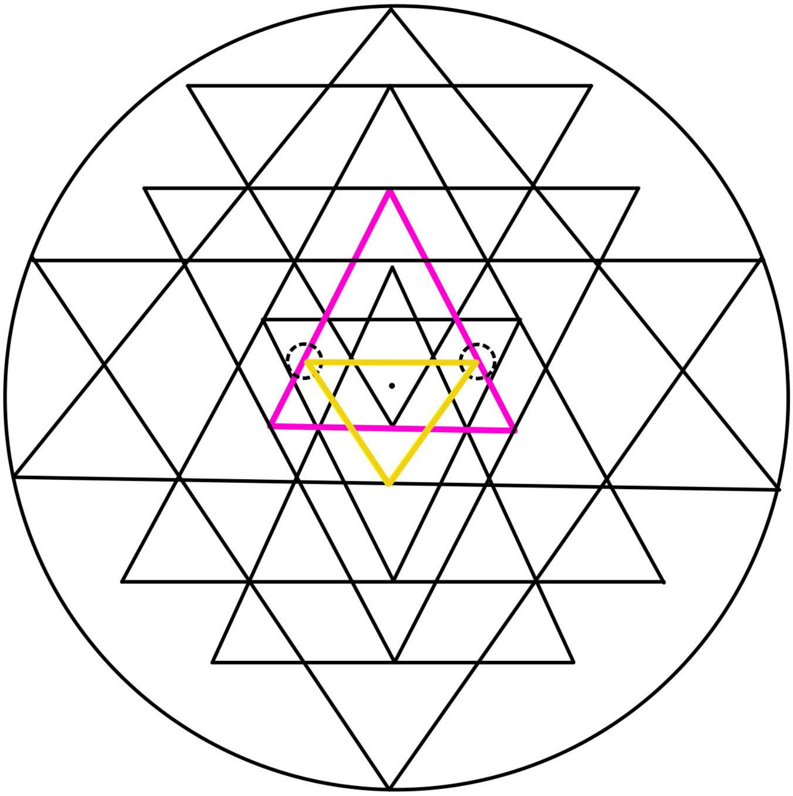


Figure 3. We illustrate the concurrency condition ((iii); t_5, t_5, t_6). The intersection between the legs of the (yellow) triangle t_5, the base of the (yellow) triangle t_5 and the legs of the (rose) triangle t_6 consists of exactly two points highlighted by the dashed circles.

1.1. *The origins*

The origins of the *Śrī Yantra* are unknown. We have to trace them back among the rituals that have been associated to it (see Padoux [17]).

Michaël [14] describes one of these rituals based on the text *Saundarya Laharī* ("The waves of beauty") attributed to a disciple of the 8$^{\text{th}}$ century philosopher Ādi Śaṅkara.

In [6], R. C. Gupta considers a large variety of *yantras* of great mathematical interest (some of them, such as the *chautisa yantra*, have been already studied from a purely mathematical point of view, see G. Bhowmik [1] and A. Navas [16]). Gupta's presentation of the properties of the *Śrī Yantra* enables us to reconstruct some of the crucial moments of its definition throughout history. We mostly follow Gupta's presentation, but also refer to Huet [11], Mookerjee and Khanna [15], Rao [19], and Zimmer [26].

Gupta [6, p. 180] quotes the *Rudrāyamala Tantra* and in particular a verse referring to a *yantra* consisting of a *bindu*, a central triangle, and then "*enclosures formed by 8, 10, 10, and 14 triangles*," and finally "*three circles, and three* bhūpura." This does indeed describe the *Śrī Yantra*, see Figure 2.b in which the central triangle is $t_1 \cap t_5$ and the "enclosures" are highlighted by two different shades. Traditionally these circular rows of triangles are described as *cakras* (wheels).[11]

[11] The *Śrī Yantra* is described in terms of *cakras* for instance in Mookerjee and Khanna [15] and Zimmer [26], (see

Gupta also mentions three three-dimensional variants, *kūrma* (turtle), *padma* (lotus) and *meru* (the fabulous mountain), of the planar *Śrī Yantra* considered here, usually referred to as *bhū* (the Earth). In the *kūrma* version the segments are replaced by cross sections of a sphere (see [6, p. 181] and references to *Gaurīyāmala Tantra*).

Kulaichev refers to the work of de Casparis [3, p. 34 and p. 41] reporting mentions of the *Śrī Yantra* in 7$^{\text{th}}$ century inscriptions dating back to the time of the kingdom of Śrīvijaya in south Sumatra.

Kulaichev [13, p. 279] also alludes to a hymn from *Atharva Veda* (c. 12$^{\text{th}}$ century BC) dedicated to a *Śrī Yantra*-like figure consisting of nine triangles. He also mentions a representation dated to the 17$^{\text{th}}$ century to be found in the religious institution of Śṛṅgārī Maṭha established by Ādi Śaṅkara, [13].

Finally, we recall, as Huet does in [11, p. 622], that although it is hinted in several sources that this symbol is very old we do not know of any published representation before the 17$^{\text{th}}$ century. Therefore, the problem of determining the date of creation remains open. The interest that is brought to this question is well illustrated by the following quote of Mookerjee and Khanna [15]: "the *Śrī Yantra*, in its formal content, is a visual masterpiece of abstraction, and must have been created through revelation rather than by human ingenuity and craft."

1.2. *The history of the construction of the* Śrī Yantra

Drawing a *Śrī Yantra* is an elementary, but nontrivial, mathematical problem. In 2007, Gupta lists the various known methods [6].

1.2.1. *The traditional methods*

Two traditional methods are reported by two commentators of *Saundarya Laharī:* Lakṣmīdhara et Kaivalyāśrama, see Figure 2 and [22].

Method A, is due to Kaivalyāśrama: it is applied in Figure 2.a.[12] The vertices of the triangles are placed within a frame consisting of the vertical diameter subdivided in 48 equal units and the circle (Gupta mentions also a 42-unit version). As Figure 2.a shows, the unit is used to specify the distance from the circle along the horizontal parallel lines. Figure 2.a shows that condition ((iii); t_3, t_8, t_9) can fail (and indeed there is no reason for any concurrency of (iii) to hold).

Method B is due to Lakṣmīdhara: it is applied in Figure 2.b.[13] It starts from the central triangle and surrounds it four times by sequences of 8, 10, 10, and 14 triangles as in the above quote from *Rudrāyamala Tantra*. The concurrency conditions ((ii)–(iii)) are satisfied by construction. However Figure 2.b shows that condition (i) can fail (the circle does not pass through the side vertices of t_3 and t_7).

These methods are not precise, as illustrated by Figures 2.a and 2.b, but they have the merit of yielding an archetype: a precisely identified set of geometric figures. As a simple example, we can compare the collection of *Śrī Yantras* to the set of isosceles triangles up to rescaling and congruence. This set is, of course, defined by a much more elementary symmetry property, but also contains several nonequivalent forms (acute triangles, obtuse triangles…) even after

---

also Jung [12]). The nine *cakras* (wheels) surrounding the *bindu* are the *bhūpura*, the triplet of circles, the 16 petals, the 8 petals, the enclosure of 14 triangles, than that of 10, again 10, and 8 triangles, and finally the central triangle. Gupta points out that the term *cakra* is also attributed sometimes to the nine triangles t_i.

[12] The caption reads: we draw the picture by placing the vertices within a fixed grid. The concurrency conditions, such as those between the blue, red and maroon triangles ((iii); t_3, t_8, t_9), fail.

[13] The caption reads: we can impose all concurrency conditions ((ii)–(iii)) to the triangles. In general, condition (i) requiring that the exterior circle passes through the vertices of t_3 and t_7 fails.

reducing by rescaling and congruence. The properties of the *Śrī Yantra* are not as elementary, but these two methods clarify which minimal set of conditions of contact and intersection should be imposed today in order to consider a *Śrī Yantra* admissible, or—as we will write here—*concurrent*. These properties were stated above as ((i)–(iii)). Since 1970 other authors have formalised the problem in different but equivalent terms. We now review the modern approaches.

1.2.2. *Straightedge and compass construction*

Although imprecise, the traditional methods can be performed with straightedge and compass. During the seventies, Nicolas J. Bolton and D. Nicol G. Macleod [2, p. 68–69] and Fonseca [5, p. 35–36] rewrote independently the list of vertex coordinates of Method A. The first paper also contains an optimisation of Method B: the table and the figure [2, tab. 2, Fig. 2] attributed to A. West produces a collection of diagrams satisfying all the concurrency conditions ((ii)–(iii)). The method "has no errors in the intersections" of the triangles [2, p. 7], but does not take into account the external circle condition (i). A simple examination allows us to notice that the forms obtained in this way (up to rescaling and congruence) depend on six parameters; imposing condition (i) would reduce to a *four*-parameter space of solutions.[14]

Nicolas J. Bolton and D. Nicol G. Macleod [2] and Fonseca [5] propose a straightedge and compass construction based on the golden ratio.[15] These approaches fail to meet concurrency conditions in a way that is not visible to the eye. This motivated the hypothesis still considered today of a possible analogue, perhaps perfect, construction believed to have originated in ancient Egypt (numerous attempts of this kind can be found on the internet).

1.2.3. *The germ of a variety of* Śrī Yantras

In 1990, G. Huet provided a formal definition of the mathematical problem posed by the *Śrī Yantra*. He listed a sequence of logical and geometrical links between the vertices [11, p. 611]. This is equivalent to our concurrency conditions ((i)–(iii)). Indeed, in order to start his sequence one has to fix five parameters.[16] All conditions ((i)–(iii)) *except* ((ii); 2,6) can be satisfied as long as each of the five parameters are chosen in a sufficiently small interval.[17] He finally translates ((ii); 2,6) into an equation relating the initial parameters. The computer programme issued from this argument expresses the coordinates of the vertices by Newton approximation and draws all possible *Śrī Yantras*.[18] Note that the set of drawings of *Śrī Yantras* produced by five parameters subject to an equation has *four* degrees of freedom.

[14]Instead of fixing all the vertices we just set the position (along the vertical diameter) of the bases of t_1, t_2, t_4, t_6, t_8 and t_9 as well as their lengths for triangles t_4 and t_6. In order to complete the solution, to these *eight* variables, we would have to impose *t*wo independent conditions equivalent to condition (i). This happens because, in order to coincide, the circumscribed circles should be concentric and congruent. We observe that, up to rescaling (one less parameter) and translation (again one less parameter), we obtain a set of solutions depending on $8-2-1-1=4$ parameters in total.

[15]The first construction is approached by replacing at a first stage the circumscribed circle by a square. The second construction is inscribed in a circle and appears as a refinement of the first.

[16]Huet's parameters are the distances of the three base points of t_3, t_6 and t_7 from the apex of t_3 on the diameter and the distance from the diameter of the concurrency points identified above as ((iii); t_1, t_3, t_8) and ((iii); t_3, t_8, t_9). In his text, these are referred to as Y_Q, Y_P, Y_J, X_F, X_A.

[17]Huet fixes the midpoint of this neighbourhood at $Y_Q = 0.668$; $Y_P = 0.463$; $Y_J = 0.398$; $X_F = 0.126$; $X_A = 0.187$ with respect to the length-1 diameter of the circle.

[18]One of these automated drawings is the cover of a volume in the honour of the computer scientist Maurice Nivat. This is where Huet's work from the eighties [11] finally appeared in 2002.

1.2.4. *The search for a unique* Śrī Yantra

In 1984, Kulaichev considers the more difficult problem of imposing ((i)–(iii)) plus *four* extra conditions.[19] He shows that it should be possible to obtain all these conditions simultaneously (p. 284) by iterated approximation. He discusses whether the solution is unique (p. 285), estimates the complexity, and argues that the answer is out of reach. This problem is not considered here, but the complexity estimated by Kulaichev is reduced. This has been the subject of recent work: there is a short addendum by Kulaichev listing further methods of approximation suggested by readers (the English translation of this text is easily available on the internet under the title "*Addition to Sri Yantra and its mathematical properties*").

Hoping to gain more flexibility and overcome the computational difficulties of the two-dimensional problem, Kulaichev also studies the problem of constructing three-dimensional models (namely, he considers the *kūrma* version mentioned above). This line of attack has been systematically investigated by C. S. Rao [20]. We point out that the hypothesis of a three-dimensional origin would have been reinforced by the statement that the *Śrī Yantra* could not be constructed on a plane by compass and straightedge but only through projection to the plane of a three-dimensional construction. Instead, the constructibility shown here implies that the coordinates of the diagram are contained in the the field of constructible numbers, [25]. This may illuminate the attempt by [13, Section Analysis], [18, 21, 23, 24] to write explicit equations for it.

2. Our construction and the problem of Apollonius

To the best of my knowledge, the natural question of whether the *Śrī Yantra* is constructible by straightedge and compass has remained open until now. Here, we treat the problem under the same hypotheses as Huet [11], i.e. without imposing any extra conditions. We construct with straightedge and compass the family of all *Śrī Yantras* satisfying the minimal concurrency requirements ((i)–(iii)).

2.1. *Apollonius*

The construction of the *Śrī Yantra* turns out to reduce to the solution of the so-called circle-line-point problem posed by Apollonius from Perga in the 3rd century BC. In its general form, the problem consists in finding the circles tangent to three given plane circles. The circle-line-point problem is analogous, but imposes the passage through a given point, and tangency to a given straight line and to a given circle. Apollonius solved these problems, but his work was lost as documented by Pappus from Alexandria in the 4th century. The solutions were worked out again by Viète in 1600.

Below, we see that the problem arises in the construction of the *Śrī Yantra* starting from the base points of t_3, t_6, t_7 and t_9 along the vertical diameter (see Figure 4). Through an elementary—but not straightforward—argument detailed in Figures 5 and 6, these parameters identify a line Δ and—within one of the Δ-bounded half-planes—a circle Π and a point ϕ exterior to the circle.

[19]The extra conditions are the following: (a) triangles t_3 and t_7 are congruent, (b) the circumcircle of t_1 coincides with the external cercle $\mathscr{E}$ (circumscribed to t_3 and t_7), (c) the circumcircle of t_9 also coincides with $\mathscr{E}$, (d) the incircle to the innermost triangle (i.e. $t_1 \cap t_5$) is concentric to $\mathscr{E}$. In some works using this approach, condition (a) is replaced with a condition imposing that t_1 should be equilateral. Let us point out that the congruence $t_3 \equiv t_7$ is also considered in [2] but not in [5]. It is immediately satisfied by our construction if we modify the initial steps in such a way that $d(O,P) = d(R,T)$. This is a good spot to mention that it might be interesting to study the interplay between conditions (a–d) and an extra condition ((iii), t_4, t_5, t_6).

In this configuration, there are exactly four circles Ξ satisfying the required contact with ϕ, Δ, and Π. The exact drawing of the *Śrī Yantra* is based on *one* of them (we explain this in Section 2.2.3). We conclude by reviewing the original *Śrī Yantra* problem in the light of the Apollonius problem.

In the eighties, the enormous developments in the enumerative geometry of algebraic curves brought us at least two beautiful geometric pictures representing the Apollonius problem. Eisenbud and Harris [4, Section 2.3] describe the space parametrising all the solutions of the Apollonius problem as ϕ, Δ, and Π vary. This family of solutions gives rise to a multiple-sheeted cover of a space parametrising conics, see [4, Section 2.3]. Harris [7] provides another interpretation of the same space in termes of moduli spaces of algebraic spin curves, i.e. curves C equipped with a theta characteristic L satisfying $L^{\otimes 2} \cong \omega_C$.

This space, with its two descriptions, completes the variety of *Śrī Yantras* described by Huet. Previously, we could only see a family of *Śrī Yantras* parametrised by a small enough neighbourhood lifting to such a multi-sheeted cover. Now all points of this cover admit a geometric interpretation extending the family of *Śrī Yantras*.

2.2. *The construction*

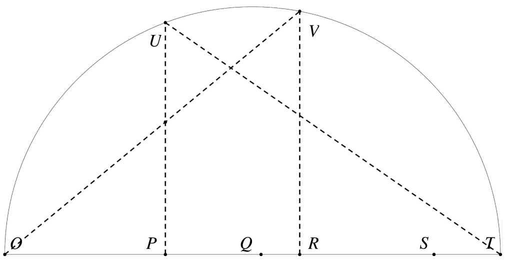
In this section, we present the construction following Figures 4, 5, 6, 7, and 11.

2.2.1. *An anticlockwise turn*

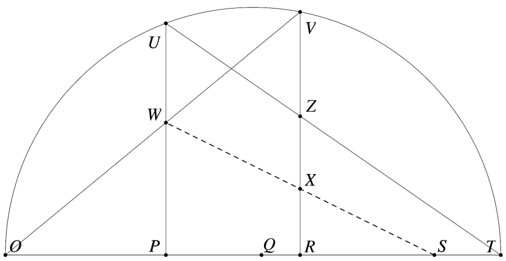
From now on, and in all the figures except the final Figure 11, we consider only the right-hand half of the diagram. We place it horizontally after a right angle rotation in anticlockwise direction.

2.2.2. *The setup*

Up to rescaling and translation, the diagram depends on four parameters. In the construction presented here we fix the four points P, Q, R, S on the diameter $\overline{OT} = [0,1]$. These are the base points of t_3, t_6, t_7 and t_9. Each choice of such parameters,[20] in increasing order and within the circle, leads to a diagram provided we allow degenerate cases.



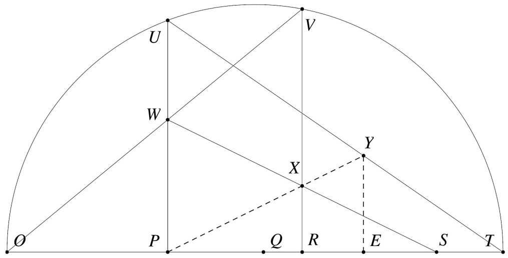
(a) The vertical lines through P and R meet the half-circle at U and V. Join V to O and U to T.



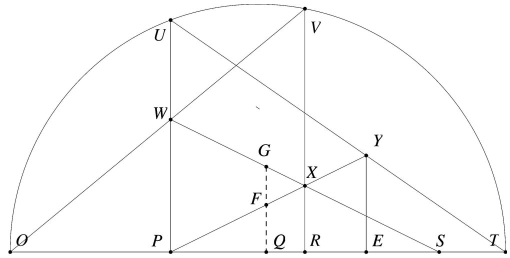
(b) Join S to $W = PU \cap VO$. Set $X = SW \cap RV$.

Figure 4. The first steps - Part 1

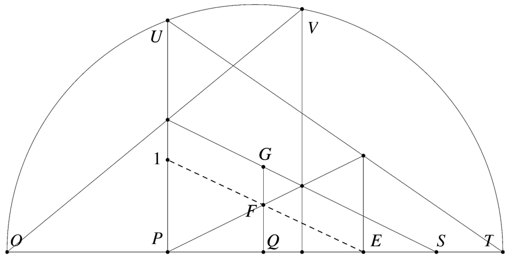
[20]Choosing the base point of t_3, t_6, t_7 and t_9 is equivalent to choosing the base points of t_3, t_6 and t_7 and the concurrency point ((iii); t_3, t_8, t_9) as in Huet. Furthermore Huet's choice is equivalent to drawing the bases t_3, t_6, t_7 and t_9 at position $0.332; 0.537; 0.602; 0.835$ on the segment $\overline{OT} = [0,1]$. In the figures below we prefer to use $0.324; 0.517; 0.692; 0.866$ for sake of clarity in the final drawings of the construction.



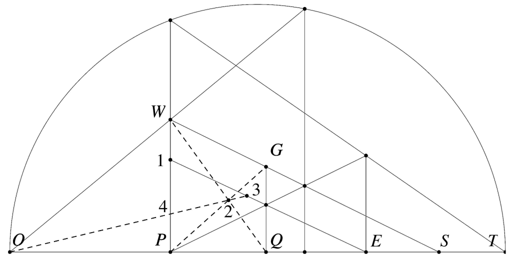
(c) The ray from P through X intersects UT at Y. The point Y projects vertically to E in OT.



(d) The vertical line passing through Q intersects PY and SW at F and G.



(e) The ray originating at E and containing F intersects UP at point 1.



(f) Draw a ray from O to the intersection 2 of the diagonals of $PWGQ$. It meets the line $(E, 1)$ at 3.

Figure 4. The first steps - Part 2

We proceed following the captions of the six diagrams appearing in Figure 4. Points P and R are the base points of the triangles t_3 and t_7, which can be drawn entirely by intersecting the vertical lines through P and R with the semicircle at U and V, Figure 4.a. Point S is the apex of the triangle t_2, whose leg passes through point $W = OV \cap PU$, Figure 4.b. Point P is the apex of t_9, whose leg passes through $X = SW \cap RV$, Figure 4.b–c. Then, we get $Y = PX \cap UT$, its vertical projection E on OT, and F (resp. G), the intersection of the vertical line through Q with PX (resp. SW), Figure 4.d.

We get point $1 = FE \cap PU$, point $2 = QW \cap PG$, and point 3 (resp. point 4) by intersecting the line through the points O and 2 with the line through E and 1 (resp. the line PW); see Figure 4.e–f. These four points $1, 2, 3$, and 4 allow us to introduce the Apollonius problem.

Here we label points by numbers; (n, m) is the line through n and m, $\overline{n, m}$ is the segment joining n and m, $\mathrm{mid}(n, m)$ is its middle point, and $\overline{n, m, \dots, k}$ is the polygon whose vertices are $n, m, \dots, k$.

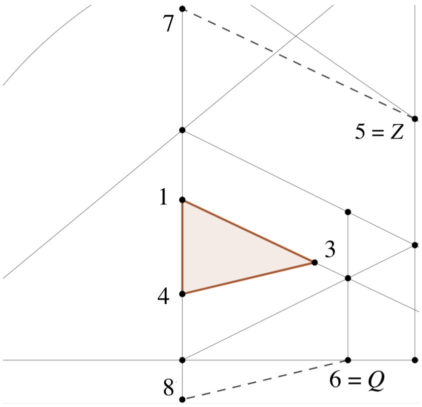
2.2.3. *From the* Śrī Yantra *to Apollonius*

We state a circle-line-point (CLP) Apollonius problem with respect to a point ϕ, a line Δ, and a circle Π determined in Figures 5 and 6.

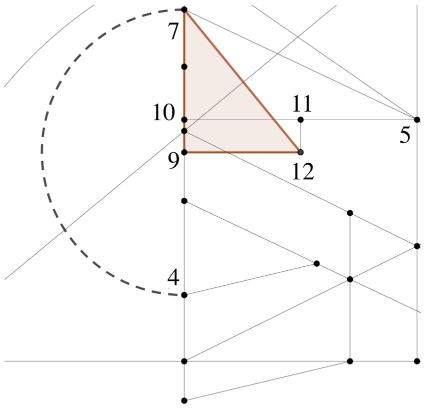
Consider the triangle of vertices $1, 3, 4$ in Figure 5 and the points $5 = Z$ and $6 = Q$. We define points $7, \dots, 19$ allowing us to determine the relevant circle Π, line Δ and point ϕ. The construction is two-fold: on the one hand it starts from point 5 and its crucial steps are points $7, 10, 12, 14$ and 15. On the other hand it starts from point 6 and its crucial steps are points $8, P, 16, 18$ and 19.

The first part is as follows: the line through 1 and 4 intersects at 7 the line through $5 = Z$ directed by $\overline{1, 3}$; see Figure 5.a. Point 10 is aligned with 4 and horizontally aligned with 5. Point 12 is the intersection between the perpendicular bisector of $\overline{5, 10}$ and the perpendicular bisector of $\overline{4, 7}$. Point 14 is the intersection between the perpendicular bisector of $\overline{7, 12}$ and the horizontal line through point 12. Point 15 is the leftmost point of the circle centred at point 7 and passing through 14. The second part is identical, but $1,3,5,7,10,12,14,15$ are replaced by $4,3,6,8,P,16,18,19$.

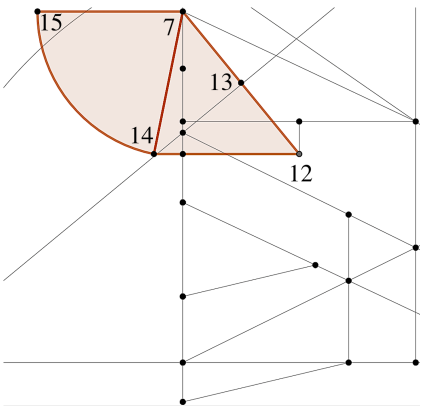
On the plane compare the abscissæ x_{15} and x_{19} of points 15 and 19. If $x_{15} < x_{19}$ (resp. $x_{15} < x_{19}$) let Δ and Δ' be the vertical lines through 19 and 15 (resp. 15 and 19), let ϕ and ϕ' be the points 18 and 14 (resp. 14 and 18), and let Π be the circle through ϕ of ray $d(\Delta,\Delta') = |x_{15} - x_{19}|$. In Figures 5 and 6, we illustrate the above procedure. Point 19 lies on the right of point 15; so, $\phi = 18$, Δ passes through 19 and Π is centred at 18 and has ray $x_{19} - x_{15} = d(15,20)$.



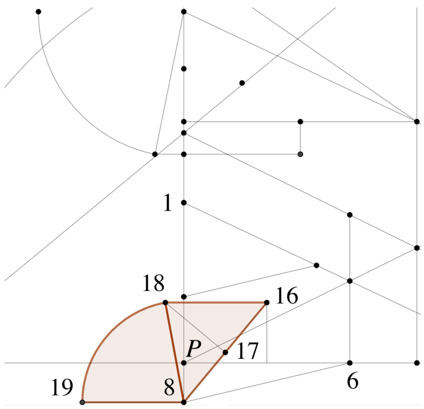
(a) Parallel lines through 5 and 6 to $\overline{1,3}$ and $\overline{4,3}$ meet the line $(1,4)$ at 7 and 8, respectively.



(b) Draw the right triangle $\overline{7,9,12}$, where $9 = \mathrm{mid}(4,7)$. Point 12 is the vertical projection of the midpoint between 5 and the line $(7,4)$.



(c) We extend the triangle $\overline{7,9,12}$ to an isosceles triangle $\overline{7,12,14}$. We draw the circular sector bounded by $15,7,14$ with $(15,7) \parallel (14,12)$.



(d) We follow the same procedure for 6. We get the isosceles triangle $8,16,18$ and the sector $18,8,19$, with $(18,16) \parallel (19,8)$.

Figure 5. Setting up the Apollonius problem

In Figure 6, the dashed circle Ξ is a special solution characterised by two properties:

(a) the circle Ξ is *external* to the circle Π, and
(b) the center of Ξ projects horizontally *within* the vertical base of the dashed triangle.

It is well known that condition (a) holds for two out of four of Apollonius circles. As shown in Section 2.3, condition (b) singles out exactly one of the two remaining solutions.

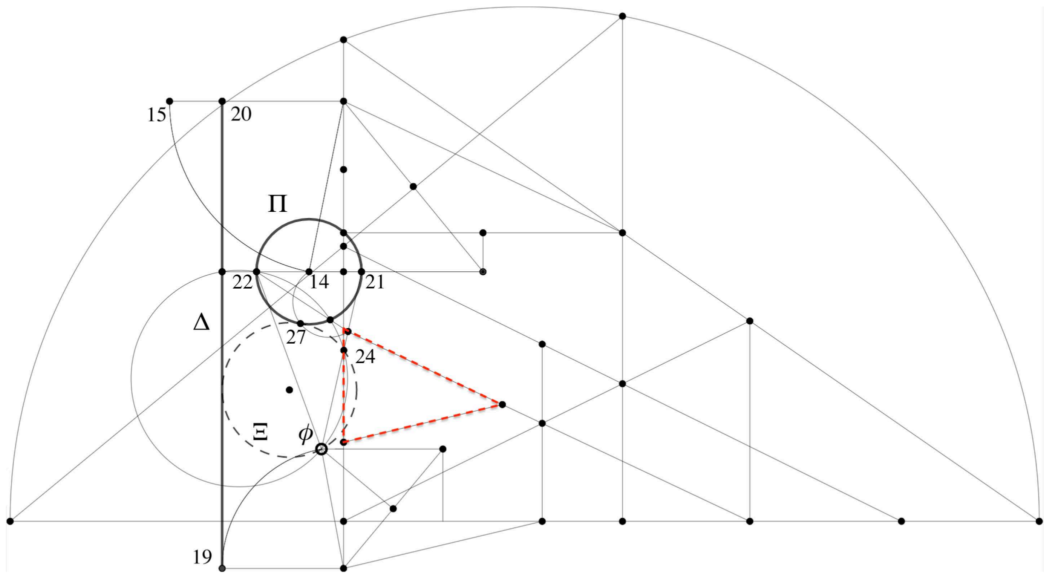


Figure 6. The dashed circle in the figure is the Apollonius circle externally tangent to the circle Π, tangent to the line Δ, and passing through the point ϕ. The circle is characterised by the fact that its center projects horizontally onto the vertical edge of the dashed triangle. The figure contains the entire construction, but Figure 7 details each step.

Indeed, Figure 7 is an application of the classical sequence of nested solutions of Apollonius problems (see [8, 9]): in order to solve the Apollonius problem we begin by reducing the Circle-Line-Point problem to a Circle-Point-Point problem, and then to the elementary Point-Point-Point problem, whose solution is the (unique) circle passing through three given points.

2.2.4. *From Apollonius to the* Śrī Yantra

We conclude the construction in Figure 8 by applying the concurrency constraints ((i)–(iii)).

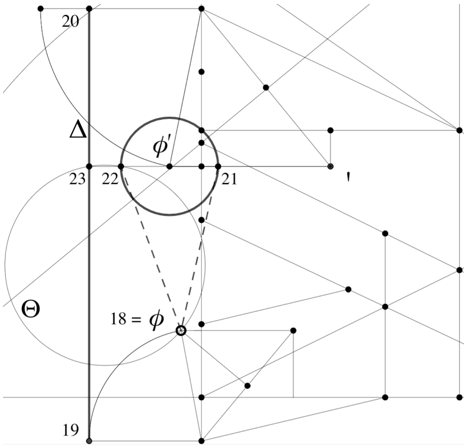
As mentioned above the setup (1–14) determines the points O, P, Q, R, S, T, U, V, W, X, Y, Z, E, F, and G. We project horizontally the center of Apollonius circle Ξ and we obtain A. We deduce B, C, D, H, I, J, K, L, M, N, and, finally, point □ by a sequence of simple straightedge operations. Point B is where AQ meets OV. It projects vertically to C on OT. We set $D = CG \cap AQ$. The line ZA meets the horizontal axis at H. We set $I = ZA \cap EF$, $L = BC \cap SW$, and $K = BQ \cap h$, where h is the vertical line through H. Point M is where the vertical line through J meets CG. We set $N = HA \cap EY$ and, finally, $\square = PY \cap s$, where s is the vertical line through S.

2.3. *Justification of the construction*

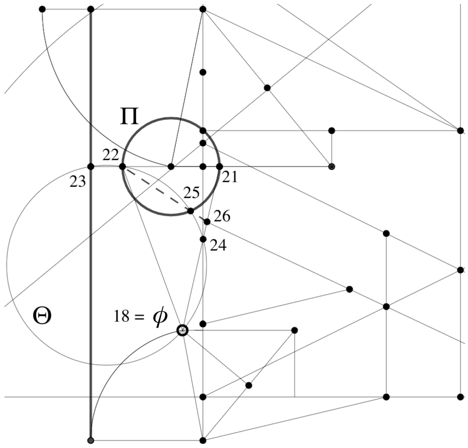
On the plane, the abscissa and the ordinate are taken with respect to P. By π, we denote the vertical projection $(x, y) \mapsto (x, 0)$.

2.3.1. *Justification of the setup*

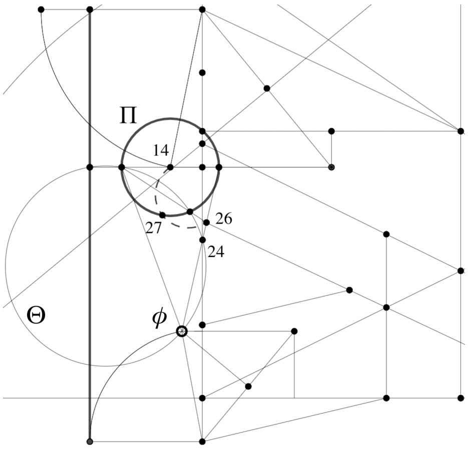
In Figure 4.a–e, we apply concurrency conditions ((i)–(iii)) directly. We provide an example: at Figure 4.c we get E by $\pi(Y) = E$. This happens because E is the base point of t_8 (red) and $Y = TU \cap PX$ lies on the legs of t_3 (blue) and t_9 (maroon). By ((iii); t_3, t_8, t_9), Y belongs to the base of t_8 (red), which implies $\pi(Y) = E$.



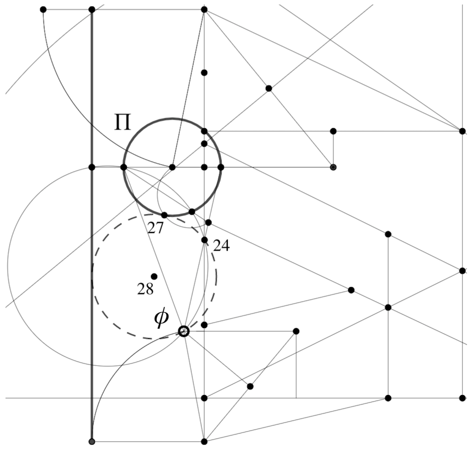
(a) The circle Θ passes through 22, 23 and $\phi = 18$. Join ϕ to 21. Set $24 = \Theta \cap (\phi, 21)$.



(b) Set point 26, the intersection between the lines $(\phi, 24)$ and $(22, 25)$.



(c) Point 27 is the intersection between Π and the above semicircle joining 26 and 14.



(d) The desired Apollonius circle is the dashed circle Ξ passing through the points ϕ, 24 and 27. Point 28 is its centre.

Figure 7. We solve the Apollonius CLP problem. Point 23 is the horizontal projection of ϕ' on Δ. The circle Θ passes through point ϕ, point 23, and the *leftmost* point of Π. We draw a line through ϕ and the *rightmost* point of Π, which we label by 21. Such a line meets Θ at point 24 (and, of course, at ϕ). Point 26 is the intersection between the line through ϕ and 24 and the line through the two points of $\Pi \cap \Theta$. The circle of diameter $\overline{26, \phi'}$ intersects Π at two points. The first point is 27 in Figure (c) and the desired Apollonius circle is the dashed circle through ϕ, 24, and 27 appearing in Figure (d). The second point is not drawn in the picture; it leads to a second circle externally tangent to Π, whose center does not project within the edge $\overline{1,4}$ as required. There are two more solutions to this CLP Apollonius problem; they are given by exchanging "*rightmost*" and "*leftmost*" in the above instructions. These solutions should be ignored, because the corresponding circles contain Π.

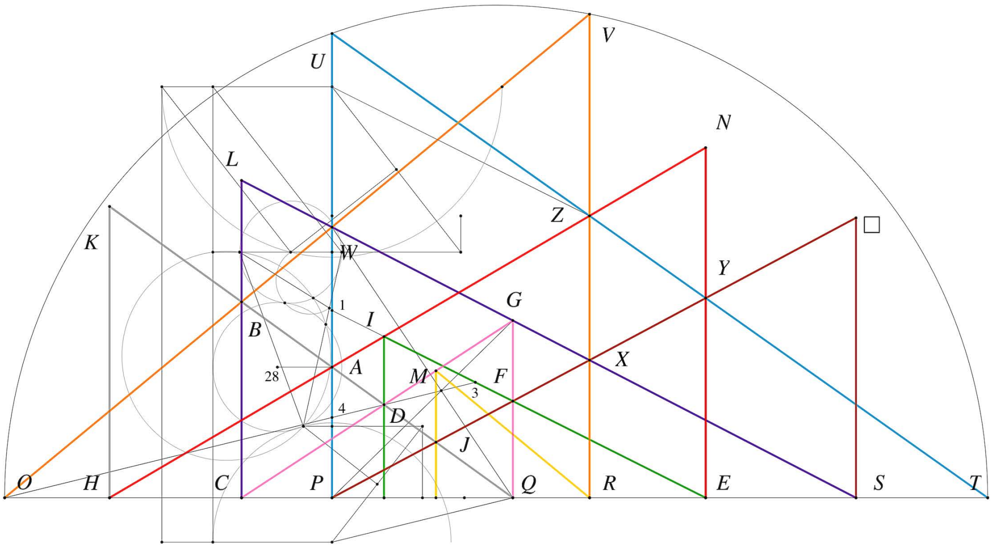


Figure 8. We deduce points $A, B, C, D, H, I, J, K, L, M, N$, and $\square$.

2.3.2. *The key part of the construction*

This entire part of the construction (Figures 5, 6, and 7) is about choosing the right slope of the leg of t_1 (grey triangle).

This amounts to identifying a suitable ray r stemming from Q. Notice that the choice of r determines D and I. Point D is defined as the center of mass of the polygon of vertices G, $B = r \cap OV$, $C = \pi(B)$, and Q; in other words D is the intersection $r \cap G\pi(r \cap OV)$ between the legs of t_1 (grey) and t_6 (pink). Point I is the intersection of the legs of t_4 (green) and t_8 (red); hence $I = Z(PU \cap r) \cap EF$, see Figure 8. By conditions ((iii); t_1, t_4, t_6) and ((iii); t_4, t_4, t_8), D and I lie on the base of t_4 (green), which is perpendicular to the diameter; therefore, points I and D should have the same abscissæ.

We simplify the definition of D as follows. Indeed, when the slope of r varies, D moves along a ray stemming from O as Figure 9 shows. In Figure 4.f, we set point $2 = WQ \cap PG$; then, D can be described as $D = O2 \cap r$.

2.3.3. *The triangle* $\overline{1,3,4}$

We refer to the triangle $\overline{1,3,4}$, which appears in Figure 5.a. The choice of the ray r is equivalent to the choice of the point A on the line PU so that $r = QA$. Therefore, the problem can be restated as follows. We should choose A on the vertical axis PU in such a way that ZA and QA intersect the sides $\overline{1,3}$ and $\overline{4,3}$ of the triangle $\overline{1,3,4}$ of Figure 5.a at two points (I and D) with the same abscissæ x_1 and x_2. The reader can refer to Figures 10 or 8, which contain the solution.

Let us denote the ordinates of the points $8, 4, 1, 7$ by v, l, l', v'. Let m' and m be the abscissæ of $Z = 5$ and $Q = 6$ (note that $m' > m$ for our initial choices). Let t be the ordinate of A along PU. In this way the abscissæ of $I = ZA \cap (13)$ and $D = QA \cap (43)$ can be written[21] in terms of t

[21] In Figure 5, let τ be a point on the segment joining 1 and 4; consider the triangle $\overline{\tau,7,5}$ with base $\overline{\tau,7}$ and the similar triangle $\overline{\tau,7,5} \cap \overline{1,3,4}$ with base $\overline{\tau,1}$. The identity between the ratios base/height for the two triangles yields the desired identity for $i = 1$. For $i = 2$, use point 4, 6 and 8 instead of 1, 5, and 7.

as $x = m(t-l)/(t-v)$ and $x = m'(t-l')/(t-v')$. The position of A is the solution of

$$m\frac{t-l}{t-v} = m'\frac{t-l'}{t-v'}.$$

This amounts to intersecting the parabola p with roots at v and l' and leading term t^2/m and the parabola p' with roots at v' and l and leading term t^2/m'. Since $m < m'$, there are two solutions, one before l and one within $]l, l'[$.

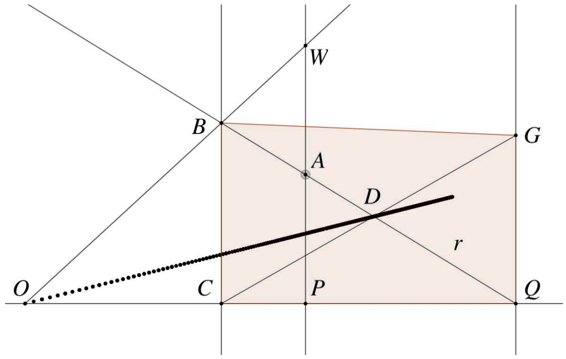


Figure 9. As the slope of the ray r varies, the center of mass D of the right trapezoid $BGQC$ follows a line through O of slope $mg/(g+mq)$, where m is the slope of OG, and we have $g = \overline{QG}$ and $q = \overline{OQ}$.

2.3.4. *Two parabolæ*

The directrix and the focus of p (resp. p') are Δ and ϕ (resp. Δ' and ϕ'). Indeed, since t^2/m and t^2/m' are the leading terms of the parabolæ, the distance $d(\phi,\Delta)$ and $d(\phi',\Delta')$ equal $m/2$ and $m'/2$. By the conditions $p \ni (0,v)$ and $p' \ni (0,v')$, each focus determines the corresponding directrix. Indeed, by $p \ni (0,v)$, the line Δ is the vertical axis through the leftmost point of the circle centred in $(0,v)$ and passing through ϕ. The same holds for Δ'. The axis of each parabola is determined by the roots. It remains to place ϕ on the axis so that $d(\phi,\Delta) = m/2$. The geometric construction of Figure 10 identifies a unique choice. This happens because the polygon of vertices $8, 16, 18, 19$ is a parallelogram and the distance of $18 = \phi$ from the vertical line Δ through 19 is the same as the distance of 8 from the vertical line through 16. By construction, the latter is $m/2$. So we have $d(\phi,\Delta) = m/2$. The argument for p' is identical.

We now show that point 28 in the construction is indeed $p \cap p'$. A point κ of the parabola p is, by definition, the center of a circle tangent to the line Δ, passing through ϕ, and having ray $\rho = d(\kappa,\phi) = d(\kappa,\Delta)$. If κ also lies on p', its distance from ϕ' equals the distance from the directrix Δ', which is a vertical line lying on the left of Δ. Hence $d(\kappa,\Delta') = (x_{19} - x_{15}) + d(\kappa,\Delta) = (x_{19} - x_{15}) + \rho$. In this way the circle Ξ of centre κ and radius ρ intersects the circle Π of centre ϕ' and radius $x_{19} - x_{15}$ at a single point. This amounts to saying that Ξ is the Apollonius circle tangent to Δ, externally tangent to Π, and passing through ϕ. By the external tangency condition, there are exactly two such circles (see [9]). By the argument above we know that only one Apollonius circle is centred at a point whose ordinate fit within $]l_2, l_1[$. Apollonius CLP problem is solved in Figure 7 by following [9, Section 7.38]. □

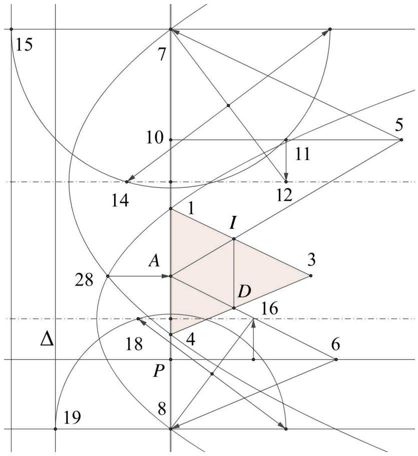


Figure 10. In this figure we treat independently the problem of placing point A along the vertical axis passing through the triangle edge $\overline{1,4}$. We require that $(A,5)$ and $(A,6)$ meet the two remaining edges of the triangle at two vertically aligned points D and I. We find the foci and the directrices of the parabolæ p and p'. Point 28 is the intersection $p \cap p'$ projecting on $A \in \overline{1,4}$.

Acknowledgements

I would like to thank Daniele Faenzi for useful mathematical conversations on this work. I am extremely grateful to Gérard Huet for his very encouraging feedback on an early version and for allowing me to understand the history of the problem. I thank Paul Delisle, Catherine Goldstein, Agathe Keller, Kim Plofker, Claire Voisin for positive feedback, references and advices. I thank Émilie Jacquemot and Catriona Maclean for their careful corrections, and Zoé Blumenfeld for her constant help with the writing of this paper. GeoGebra [10] has helped me a great deal in writing and sharing this work. Finally, I would like to thank Étienne Ghys for his work as an editor, the anonymous referee for his precious remarks, and Alexandre Moeschler (layout editor, Centre Mersenne) for taking an enormous amount of care in improving the readability of this paper.

Appendix A. Annexes

The material below has been useful in explaining this work.

A.1. *The* Śrī Yantra *with a single stroke*

The triangles and the circle of the *Śrī Yantra* can be drawn with a single continuous line respecting the following rule. *The line switches from a figure to another (circle or triangle) if and only if it reaches an apex of a triangle.* See [https://www.geogebra.org/m/cmny6ypg].

A.2. *Traditional methods*

Method A (from the commentator Kaivalyāśrama and Figure 2) is illustrated here: [https://www.geogebra.org/m/dbaykgx4]. Method B (from the commentator Lakṣmīdhara and Figure 2) is illustrated here: [https://www.geogebra.org/m/txq5uuzb].

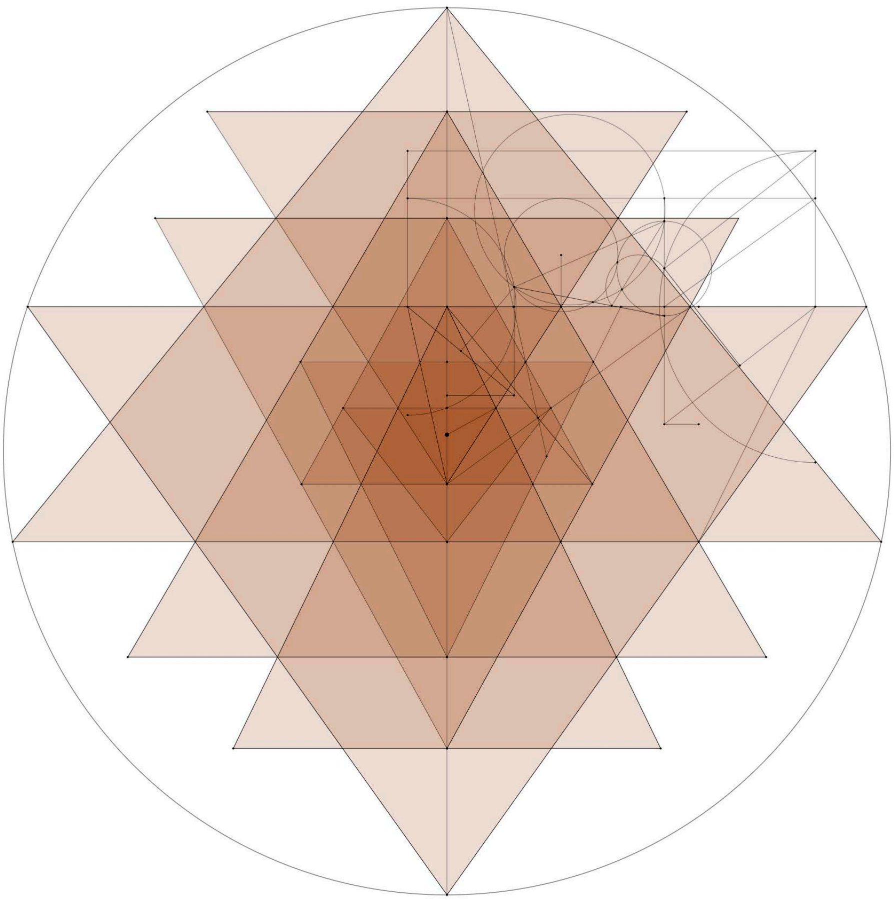


Figure 11. The complete construction

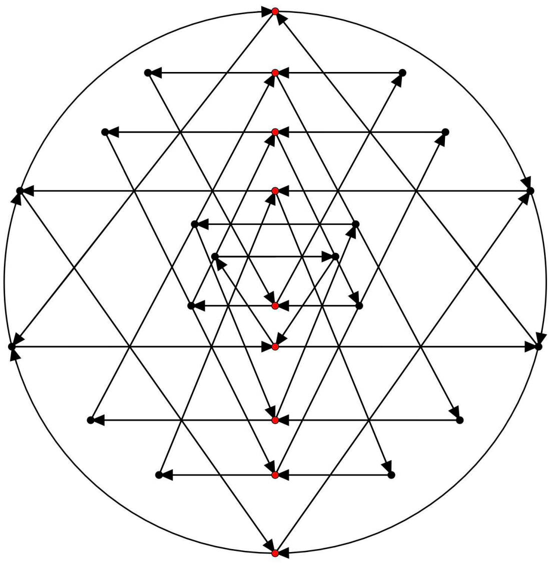


Figure 12. The *Śrī Yantra* in a single stroke

A.3. *The complete construction*

The complete construction is available on this public GeoGebra activity: [https://www.geogebra.org/m/zdvxtdvv]. It allows us to move the bases of t_3, t_6, t_7, and t_2. An animated picture of the construction is available here: [https://webusers.imj-prg.fr/~alessandro.chiodo/sriyantra.gif].

References

[1] G. Bhowmik, "Les carrés magiques de Nārāyaṇa", 2020, published by *CNRS Images des Mathématiques - La recherche mathématique en mots et en images*, http://images.math.cnrs.fr/Les-carres-magiques-de-Nārāyaṇa.html.

[2] N. J. Bolton, D. N. G. Macleod, "The geometry of the Śrīyantra", *Religion* **7** (1977), no. 1, p. 66-85.

[3] J. G. de Casparis, *Selected inscriptions from the 7th to the 9th century A. D.*, Bandung, Masa baru, 1956.

[4] D. Eisenbud, J. Harris, *3264 and All That: A Second Course in Algebraic Geometry*, Cambridge University Press, 2016.

[5] R. Fonseca, "Constructive geometry of the Śrī-cakra diagram", *Religion* **16** (1986), p. 33-49.

[6] R. C. Gupta, "Yantras and mystic diagrams: a wild area of study for ancient and medieval Indian mathematics", *Indian Journal of History of Science* **42** (2007), no. 2, p. 163-204.

[7] J. Harris, "Theta-characteristics on algebraic curves", *Trans. Am. Math. Soc.* **271** (1982), no. 2, p. 611-638.

[8] R. Hartshorne, *Companion to Euclid*, Berkeley Mathematics Lecture Notes, vol. 9, American Mathematical Society, 1997.

[9] ——— , *Geometry: Euclid and Beyond*, Undergraduate Texts in Mathematics, Springer, 2000.

[10] M. Hohenwarter, "GeoGebra: Ein Softwaresystem für dynamische Geometrie und Algebra der Ebene", 2002, Paris Lodron University, Salzburg, Austria, http://www.geogebra.org/.

[11] G. Huet, "Śrī Yantra Geometry", *Theor. Comput. Sci.* **281** (2002), no. 1-2, p. 609-628.

[12] C. G. Jung, *Psychology of Yoga and Meditation: Lectures Delivered at ETH Zurich, Volume 6: 1938–1940*, Philemon Foundation Series, Princeton University Press, 2020, translated by John Peck and Heather McCartney, edited by Martin Liebscher.

[13] A. P. Kulaichev, "Śrīyantra and its Mathematical Properties", *Indian J. Hist. Sci.* **19** (1984), no. 3, p. 279-292.

[14] T. Michaël, "Le Śrīcakra dans la Saudrarya-Laharī", in *Mantras et diagrammes rituels dans l'Hindouisme* (A. Padoux, ed.), CNRS Editions, 1986, p. 127-139.

[15] A. Mookerjee, M. Khanna, *The Tantric Way*, Thames and Hudson, London, 1977.

[16] A. Navas, "Le carré magique de Kharujaho est un hypercube", 2020, with the collaboration of María José Moreno. Published by *CNRS Images des Mathématiques - La recherche mathématique en mots et en images*, http://images.math.cnrs.fr/Le-carre-magique-de-Kharujaho-est-un-hypercube.html.

[17] A. Padoux, *Le Cœur de la Yoginī — Yoginīhṛdaya, avec le commentaire Dīpikā d'Amaṛtānanda*, Publications de l'Institut de Civilisation Indienne, Institut de civilisations Indienne (Collège de France) - Publications, 1994.

[18] V.-V. Pustynski, "On mathematical complexity of Śrīyantra", *Indian Journal of History of Science* **49** (2014), no. 3, p. 268-277.

[19] S. K. Ramachandra Rao, *The Yantras*, Sri Garib Dass Oriental Series, vol. 48, Sri Satguru Publications, 1988.

[20] C. S. Rao, "Śrīyantra — A study of spherical and plane forms", *Indian Journal of History of Science* **33** (1998), no. 3, p. 203-227.

[21] B. Santhi, N. Rajesh Kumar, C. Bharathy, R. Bala Krishnan, "Generation of Divine Image–Sri Yantra", *Research Journal of Applied Sciences, Engineering and Technology* **4** (2012), no. 14, p. 2241-2246.

[22] P. S. S. Sastri, T. R. S. Ayyangar, *Saundarya Laharī*, Theosophical Publishing House, 1948.

[23] D. Tahta, "On the geometry of the sriyantra", *The Mathematical Gazette* **76** (1992), no. 475, p. 49-60.

[24] G. A. Tularam, "Investigating the Development of Arithmetic and Algebra in Vedic India, Tribute to Swami Dayananda Saraswati", *International Journal of Mathematics, Game Theory and Algebra* **20** (2012), no. 2, p. 163-187.

[25] P. L. Wantzel, "Recherches sur les moyens de reconnaître si un Problème de Géométrie peut se résoudre avec la règle et le compas", *J. Math. Pures Appl.* **2** (1837), p. 366-372.

[26] H. Zimmer, *Artistic Form and Yoga in the Sacred Images of India*, Princeton University Press, NJ, 1984, trans. by G. Chapple, J. B. Lawson.