



HAL
open science

Stratified Radiative Transfer for Multidimensional Fluids and numerical applications to earth science

François Golse, Olivier Pironneau

► **To cite this version:**

François Golse, Olivier Pironneau. Stratified Radiative Transfer for Multidimensional Fluids and numerical applications to earth science. 2023. hal-03306461v2

HAL Id: hal-03306461

<https://hal.sorbonne-universite.fr/hal-03306461v2>

Preprint submitted on 7 Nov 2023

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

1 **STRATIFIED RADIATIVE TRANSFER IN A FLUID AND**
2 **NUMERICAL APPLICATIONS TO EARTH SCIENCE***

3 FRANÇOIS GOLSE[†] AND OLIVIER PIRONNEAU[‡]

4 **Abstract.** New mathematical results are given for the Radiative Transfer equations alone and
5 coupled with the temperature equation of a fluid: existence, uniqueness, a maximum principle and
6 a convergent monotone iterative scheme. Thanks to these new results, a numerical method using
7 an integro-differential formulation is proved to be stable, convergent and accurate. For climate,
8 a robust numerical method is important because the difference between an atmosphere with and
9 without greenhouse gases easily falls below the precision of the numerical schemes. Numerical tests
10 for Earth’s atmosphere and the heating of a pool by the Sun are included and discussed.

11 **Key words.** Radiative transfer, Temperature equation, Integral equation, Numerical analysis,
12 Climate modelling

13 **AMS subject classifications.** 3510, 35Q35, 35Q85, 80A21, 80M10

14 **1. Introduction.** Radiative transfer is an important field of physics. It appears
15 in astronomy, nuclear physics and heat transfer in fluid mechanics. It is also a key
16 ingredient of climate models.

17 Books on radiative transfer for the atmosphere are numerous, such as [22],[15],
18 [4], the numerically oriented [28] and the two mathematically oriented [6] and [9].

19 When Planck’s theory of black bodies is used, radiation involves a continuum of
20 frequencies governed by the temperature of the emitting bodies. Studies based on the
21 interactions of the photons with the atoms of the medium, such as [3], are currently
22 unusable numerically in large physical domains. A much simpler formulation has
23 been proposed a hundred years ago, known as the radiative transfer equations, which
24 is based on the energy conservation principles of continuum mechanics.

25 Even when the interactions with the background fluid are neglected, the radiative
26 transfer equations involves 5 “spatial” variables (3 coordinates for the position of each
27 photon, and the 2 components of its direction). Existence of solutions of the radiative
28 transfer equations can be proved by a Schauder-type compactness argument (see [1]),
29 with uniqueness under appropriate additional boundedness (see Proposition 2 in [23]
30 and [27]), or monotonicity assumptions (see Corollary 2 in [23], together with [12]).

31 Given the intricacy of the radiative transfer equations, several simplifying assump-
32 tions have been studied in the literature. If the scattering and absorption coefficients
33 do not depend on the frequencies of the radiation source, the radiative transfer equa-
34 tions can be averaged in the frequency variable, leading to a closed system of equations
35 for the temperature and frequency-averaged radiative intensity, known as the “grey”
36 model. However the frequency dependence of the scattering and absorption coeffi-
37 cients is fundamental to understand several important effects in Earth’s atmosphere.
38 For instance, Rayleigh explained the blue color of the sky by the fact that the scatter-
39 ing coefficient is proportional to the fourth power of the radiation frequency. Likewise,
40 the fact that some components of Earth’s atmosphere are opaque to infrared radia-
41 tions seems important to understand the greenhouse effect. Another simplification, of

*SUBMITTED TO THE EDITORS DATE.

[†]CMLS, École Polytechnique & CNRS, Institut polytechnique de Paris, 91128 Palaiseau Cedex, France (francois.golse@polytechnique.edu).

[‡]Applied Mathematics, Jacques-Louis Lions Lab, Sorbonne Université, 75252 Paris cedex 5, France (olivier.pironneau@sorbonne-universite.fr, <https://www.ljll.math.upmc.fr/pironneau/>).

42 a purely geometric nature, consists in assuming that the temperature and radiative
 43 intensity are uniform on a foliation of the space by parallel planes, and therefore dep-
 44 end on a single position variable. As a result, the radiative intensity depends only on
 45 the projection of the photon’s direction on the orthogonal axis to these planes. This
 46 is known as the “slab symmetry” assumption, which appears in the “Milne problem”
 47 for planetary or stellar atmospheres (see [6] for a detailed physical discussion of the
 48 Milne problem, and [11] for the corresponding mathematical theory).

49 The term “radiative transfer” usually refers to the interaction of radiation with
 50 a fixed background material. But of course, radiation obviously deposits energy in
 51 the background fluid, gas or plasma, as well as momentum, through the radiation
 52 pressure, and conversely, high speed fluid motion obviously modifies such processes
 53 as Compton scattering (scattering of a photon by a free electron at rest) by Doppler
 54 effect. Therefore, in full generality, the equation for the radiation intensity are coupled
 55 with the fluid equations. This coupling is studied under the name of “radiation
 56 hydrodynamics” (see [26] for the coupling with ideal fluids, and [24]).

57 The most general studies of radiation hydrodynamics mentioned above involve
 58 high speed (possibly relativistic) fluid motion. In the present paper, we consider
 59 radiation passing through an incompressible fluid, or a compressible fluid at low Mach
 60 number. Thus our setting will be intermediate between radiation hydrodynamics as
 61 [26],[24], and as in [10]. This last reference considers the coupling of the grey model of
 62 radiative transfer with a background material at rest. See also [27]¹ for an existence
 63 results for the general system in 3D, yet without the monotone properties used by
 64 the numerical algorithm, which is at the core of this study. The radiation energy is
 65 deposited in the background medium in the form of heat, and appears as a source
 66 term in the heat equation for the temperature, while the black body radiation of
 67 the background medium appears as a source term in the radiative transfer equation
 68 for the radiative intensity. Our model retains the fluid motion equation, as well as
 69 the frequency dependence of the radiation field, which is essential for applications to
 70 Earth’s climate.

71 We shall however make another simplification, referred to as the “stratification or
 72 parallel plane assumption”: while the radiation intensity and temperature depend on
 73 all 3 position coordinates, only one of these coordinates is retained in the computation
 74 of the streaming operator acting on the radiative intensity, while the two other coord-
 75 inates appear only as parameters in the radiative transfer equation. The stratified
 76 approximation is used when the radiation source is far — as in the case of the Sun —
 77 and the radiative intensity deposited at the boundary of the computational domain
 78 is uniform or at least slowly varying in the tangential directions to this surface.

79 In 2005 K. Evans and A. Marshak wrote in chapter 4 of [22] a review of the
 80 numerical methods available for Radiative Transfer alone. Today, judging from [5], the
 81 situation has not changed: SHDOM (Spherical Harmonic Discrete Ordinate Method)
 82 and Monte-Carlo are the two most popular methods. While reviewing the current
 83 situation for the radiative transfer equations in [2] we implemented a finite element
 84 version of SHDOM and found that the method was incapable, unless a huge number
 85 of degree of freedom is used, of giving results with the accuracy needed to differentiate
 86 between small variations on the absorption coefficient.

87 On the other hand an integral formulation present in [6] turned out to be much
 88 more precise and also computationally much cheaper. A fixed-point iteration of this
 89 nonlinear integral formulation, known in the RT community as “iterations on the

¹While this paper was being reviewed, [27] was brought to our attention.

90 sources” was shown to be monotone in [25], a property which seems to have escaped
 91 earlier studies. Finally in [14] the method was extended to include the temperature
 92 equation of the fluid and also to handle Rayleigh scattering while retaining monotonic-
 93 ity. While [14] is more numerically oriented, the present article gives the convergence
 94 proofs as well.

95 The radiative transfer equations are presented in section 2. After this, a cascade
 96 of simplifications are discussed: the stratified approximation, the decoupling from the
 97 fluid, and Milne problem techniques originating from [11] (see also [23]).

98 In section 3, the stratified radiative transfer decoupled from the fluid is analyzed
 99 in the case of isotropic scattering. Existence of a solution is proved by using the
 100 convergent monotone iterative scheme proposed in [2]. A maximum principle in the
 101 line of [23, 11] is also presented.

102 Uniqueness issues are discussed in section 4. The proofs are far from straightfor-
 103 ward, and heavily rely on ideas in [23]. It may be interesting to compare Mercier’s
 104 monotonicity structure for the radiative transfer equation, which is quite involved,
 105 with the general observation [7] on order preserving maps in L^1 leaving the integral
 106 invariant.

107 In section 5 the above results are extended to the non isotropic case of scattering
 108 with the Rayleigh phase function.

109 Finally in section 6 existence, uniqueness and monotone convergence of the fixed-
 110 point iterations are proved for the radiative transfer equation coupled with the tem-
 111 perature equation of a fluid whose velocity field is known.

112 Three numerical applications are presented in section 7. The first one is a nu-
 113 merical simulation of the radiative transfer in the atmosphere with real data for the
 114 frequency dependent absorption coefficient κ_ν . The numerical method is sufficiently
 115 accurate to study the effect of variations of κ_ν in part of the spectrum, much like
 116 changing the composition of the atmosphere by adding more CO₂ or other greenhouse
 117 gases. The problem is one dimensional in space. The second example is the study
 118 of the temperature in a pond heated by the Sun. For this problem radiative transfer
 119 is coupled with the Navier-Stokes equations. The geometry is academic, in 2D; its
 120 object is to show the feasibility of the numerical method for such coupled problems.
 121 The third problem is also a feasibility study which shows that it is possible to make
 122 a 3D computation of the wind in the atmosphere of a planet heated by the Sun and
 123 subject to thermal diffusion. The computing times show that the method could be
 124 used in real life situations.

125 **2. Fundamental equations and approximations.** Finding the temperature
 126 T in a fluid heated by electromagnetic radiations is a complex problem because in-
 127 teractions of photons with atoms of the medium involve rather intricate quantum
 128 phenomena. A first simplifying assumption is that of local thermodynamic equilib-
 129 rium (LTE): at each point in the fluid, there is a well-defined electronic temperature.
 130 In that case, one can write a kinetic equation for the radiative intensity $I_\nu(\mathbf{x}, \boldsymbol{\omega}, t)$
 131 at time t , at position \mathbf{x} and in the direction $\boldsymbol{\omega}$ for photons of frequency ν , in terms of
 132 the temperature field $T(\mathbf{x}, t)$:

$$(2.1) \quad \frac{1}{c} \partial_t I_\nu + \boldsymbol{\omega} \cdot \nabla I_\nu + \rho \bar{\kappa}_\nu a_\nu \left[I_\nu - \frac{1}{4\pi} \int_{\mathbb{S}^2} p_\nu(\boldsymbol{\omega}, \boldsymbol{\omega}') I_\nu(\boldsymbol{\omega}') d\boldsymbol{\omega}' \right] \\ = \rho \bar{\kappa}_\nu (1 - a_\nu) [B_\nu(T) - I_\nu].$$

133 In this equation, ∇ designates the gradient with respect to the position \mathbf{x} , while

$$(2.2) \quad B_\nu(T) = \frac{2h\nu^3}{c^2[e^{\frac{h\nu}{kT}} - 1]}$$

134 is the Planck function at temperature T , with h the Planck constant, c the speed of
135 light in the medium (assumed to be constant) and k the Boltzmann constant. Notice
136 that

$$(2.3) \quad \int_0^\infty B_\nu(T) d\nu = \bar{\sigma} T^4, \quad \bar{\sigma} = \frac{2\pi^4 k^4}{15c^2 h^3},$$

137 where $\pi\bar{\sigma}$ is the Stefan-Boltzmann constant.

138 The intricacy of the interaction of photons with atoms of the medium is contained
139 in 3 quantities: 1/ the mass-absorption $\bar{\kappa}_\nu$ which is the fraction of radiative intensity
140 at frequency ν that is absorbed per unit length, 2/ the scattering albedo a_ν and a
141 probability of scattering from directions $\boldsymbol{\omega}'$ to $\boldsymbol{\omega}$. Indeed, a photon of frequency ν
142 travelling in a direction $\boldsymbol{\omega}'$ may be deflected by the atoms of the medium in a new
143 direction $\boldsymbol{\omega}$. The proportion of deflected photons $a_\nu \in (0, 1)$ is called the scattering
144 albedo. Furthermore if $p_\nu(\boldsymbol{\omega}, \boldsymbol{\omega}') \geq 0$ is the probability density of scattering from $\boldsymbol{\omega}'$
145 to $\boldsymbol{\omega}$ the scattered intensity is (see [9], p 74): $\frac{a_\nu \bar{\kappa}_\nu}{4\pi} \int_{\mathbb{S}^2} p_\nu(\boldsymbol{\omega}, \boldsymbol{\omega}') I_\nu(\boldsymbol{\omega}') d\boldsymbol{\omega}'$. Probabilities
146 sum up to 1, so $\frac{1}{4\pi} \int_{\mathbb{S}^2} p_\nu(\boldsymbol{\omega}, \boldsymbol{\omega}') d\boldsymbol{\omega}' = \frac{1}{4\pi} \int_{\mathbb{S}^2} p_\nu(\boldsymbol{\omega}, \boldsymbol{\omega}') d\boldsymbol{\omega} = 1$.

147 The kinetic equation (2.1) is coupled to the fluid equations solely by the local
148 conservation of energy. When the fluid is incompressible, density ρ , pressure p and
149 velocity fields \mathbf{u} satisfy the Navier-Stokes equations

$$(2.4) \quad \begin{cases} \partial_t \rho + \mathbf{u} \cdot \nabla \rho = 0, & \nabla \cdot \mathbf{u} = 0, \\ \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} - \frac{\mu_F}{\rho} \Delta \mathbf{u} + \frac{1}{\rho} \nabla p = \mathbf{g}, \end{cases}$$

150 where Δ is the Laplacian in the \mathbf{x} variable. Here, \mathbf{g} is the gravity, while μ_F is the fluid
151 viscosity. For the applications discussed in Section 7, namely the Earth atmosphere
152 below 12km and lakes, air and water are incompressible to a very good precision (see
153 the low Mach number limit theorem in [18]).

154 The total energy density is the sum of the kinetic energy density of the fluid, of
155 the internal energy of the fluid, and of the radiative energy. Subtracting the kinetic
156 energy balance equation from the local conservation of energy, neglecting the viscous
157 heating term $\frac{1}{2}\mu_F |\nabla \mathbf{u} + (\nabla \mathbf{u})^T|^2$ on the right hand side of the equality above, which
158 is legitimate assuming that the variations of $|\mathbf{u}|^2$ times μ_F are small, we arrive at

$$(2.5) \quad \begin{aligned} \rho c_V (\partial_t T + \mathbf{u} \cdot \nabla T) &= \nabla \cdot (\rho c_P \kappa_T \nabla T) \\ &+ \int_0^\infty \rho \bar{\kappa}_\nu (1 - a_\nu) \left(\int_{\mathbb{S}^2} I_\nu(\boldsymbol{\omega}) d\boldsymbol{\omega} - 4\pi B_\nu(T) \right) d\nu, \end{aligned}$$

159 where T is the temperature, while c_V, c_P are the specific heat capacity at constant
160 volume and constant pressure respectively, and κ_T is the thermal diffusivity.

161 Summarizing, the kinetic equation (2.1) for the radiative intensity is coupled to
162 the incompressible Navier-Stokes equations (2.4) and to the drift diffusion equation

163 (2.5) for the temperature. The resulting system is

$$(2.6) \quad \begin{cases} \frac{1}{c} \partial_t I_\nu + \boldsymbol{\omega} \cdot \nabla I_\nu + \rho \bar{\kappa}_\nu a_\nu \left[I_\nu - \frac{1}{4\pi} \int_{\mathbb{S}^2} p_\nu(\boldsymbol{\omega}, \boldsymbol{\omega}') I_\nu(\boldsymbol{\omega}') d\boldsymbol{\omega}' \right] \\ \qquad \qquad \qquad = \rho \bar{\kappa}_\nu (1 - a_\nu) [B_\nu(T) - I_\nu], \\ \rho c_V (\partial_t T + \mathbf{u} \cdot \nabla T) - \nabla \cdot (\rho c_P \kappa_T \nabla T) \\ \qquad \qquad \qquad = \int_0^\infty \rho \bar{\kappa}_\nu (1 - a_\nu) \left(\int_{\mathbb{S}^2} I_\nu(\boldsymbol{\omega}) d\boldsymbol{\omega} - 4\pi B_\nu(T) \right) d\nu, \\ \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} - \frac{\mu_F}{\rho} \Delta \mathbf{u} + \frac{1}{\rho} \nabla p = \mathbf{g}, \\ \partial_t \rho + \mathbf{u} \cdot \nabla \rho = 0, \qquad \nabla \cdot \mathbf{u} = 0. \end{cases}$$

164 This system is supplemented with appropriate initial and boundary conditions.
 165 Assuming for instance that the spatial domain is an open subset Ω of \mathbb{R}^3 with C^1 , or
 166 piecewise C^1 boundary $\partial\Omega$, and denoting by \mathbf{n} the outward unit normal field on $\partial\Omega$,
 167 the following boundary conditions are natural:

$$(2.7) \quad \begin{aligned} I_\nu(\mathbf{x}, \boldsymbol{\omega}, t) &= Q_\nu(x, \omega, t), & x \in \partial\Omega, \boldsymbol{\omega} \cdot \mathbf{n}_x < 0, \nu > 0, \\ \mathbf{u}|_{\partial\Omega} &= 0, & \frac{\partial T}{\partial n} \Big|_{\partial\Omega} &= 0. \end{aligned}$$

168 The first boundary condition tells us that the radiative intensity of incoming photons
 169 ($\boldsymbol{\omega} \cdot \mathbf{n}_x < 0$) at the boundary of the spatial domain is known, which is a typical
 170 admissible boundary condition for kinetic models; the second boundary condition is
 171 the classical Dirichlet boundary condition for the velocity field, solution of the Navier-
 172 Stokes equations, while the last boundary condition, the Neuman condition for the
 173 temperature, corresponds to the absence of heat flux at the boundary of the spatial
 174 domain. (Of course, this is just one example of boundary condition for the heat
 175 equation, other boundary conditions could also be considered — for instance, one
 176 could have mixed Dirichlet-Neuman, or even Robin conditions on the temperature.)
 177 Notice that there is no boundary condition for the density ρ , since the velocity field
 178 \mathbf{u} is tangent (and even vanishes) at the boundary $\partial\Omega$.

179 Finally, one should specify initial conditions of the form

$$(2.8) \quad \begin{aligned} I_\nu(\mathbf{x}, \boldsymbol{\omega}, 0) &= I_\nu^{in}(x, \omega), & x \in \Omega, \boldsymbol{\omega} \in \mathbb{S}^2, \nu > 0, \\ \rho|_{t=0} &= \rho^{in}, & \mathbf{u}|_{t=0} &= \mathbf{u}^{in}, & T|_{t=0} &= T^{in}. \end{aligned}$$

180 Neglecting the viscous heating term as explained above has an important conse-
 181 quence on the structure of this system, which can be thought of as “block triangular”.
 182 In other words, one can first solve for ρ, \mathbf{u}, p the Navier-Stokes equations (2.4), then
 183 the last three equations in the system (2.6) above. The mathematical theory of (2.4)
 184 has been discussed in great detail by P.-L. Lions in [21]. Then, the density ρ and
 185 velocity field \mathbf{u} are known, and appear as coefficients in the coupled system of the
 186 radiative transfer equation (2.1) and of the heat drift-diffusion equation (2.5). This
 187 coupling must be studied in detail. In the next two sections, we discuss simplified
 188 model equations deduced from (2.6).

189 **2.1. Stratified radiative transfer.** Let (x, y, z) be the Cartesian coordinates
 190 of the point $\mathbf{x} \in \mathbb{R}^3$, with z denoting the altitude/depth.

191 Assume that the radiation source (henceforth referred to as “the Sun”) is far away
 192 in the direction $z > 0$, and is independent of x and y . The radiation spectrum of this

193 source is that of a black body at temperature T_S , that is, the Planck function $B_\nu(T_S)$.
 194 With such a radiation source, it is natural to assume that the temperature field T is
 195 slowly varying with x and y , so that $|\partial_x T| + |\partial_y T| \ll |\partial_z T|$ and that I_ν is also slowly
 196 varying in x and y so that $|\partial_x I_\nu| + |\partial_y I_\nu| \ll |\partial_z I_\nu|$.

197 Similarly, we further assume that $|\frac{1}{c}\partial_t I_\nu| \ll |\partial_z I_\nu|$, and forget the initial condition
 198 on I_ν , so that the time dependence of the radiative intensity is governed solely by the
 199 evolution of the temperature field through the radiative transfer equation (2.1).

200 With this assumption, the streaming term $\frac{1}{c}\partial_t I_\nu + \boldsymbol{\omega} \cdot \nabla I_\nu$ reduces to $\mu\partial_z I_\nu$, where
 201 μ is the cosine of the angle of $\boldsymbol{\omega}$ with the z axis. Henceforth, the spatial domain is
 202 $\Omega = \mathbb{O} \times (z_m, z_M)$, where \mathbb{O} is an open subset of \mathbb{R}^2 with C^1 boundary.

203 Then (2.6) becomes (see [28]):

$$(2.9) \quad \begin{cases} \mu\partial_z I_\nu + \rho\bar{\kappa}_\nu I_\nu = \rho\bar{\kappa}_\nu(1 - a_\nu)B_\nu(T) + \frac{1}{2}\rho\bar{\kappa}_\nu a_\nu \int_{-1}^1 p_\nu(\mu, \mu') I_\nu(z, \mu', t) d\mu', \\ \partial_t T + \mathbf{u} \cdot \nabla T - \frac{c_E}{c_V} \kappa_T \Delta T = \frac{4\pi}{c_V} \int_0^\infty \bar{\kappa}_\nu(1 - a_\nu) \left(\frac{1}{2} \int_{-1}^1 I_\nu d\mu - B_\nu(T) \right) d\nu, \\ I_\nu(x, y, z_M, \mu, t)|_{\mu < 0} = Q^-(\mu)B_\nu(T_S), \quad I_\nu(x, y, z_m, \mu, t)|_{\mu > 0} = Q_\nu^+(\mu), \\ \frac{\partial T}{\partial n} \Big|_{\partial\Omega} = 0, \quad T|_{t=0} = T^{in}. \end{cases}$$

204 That $I_\nu(z_m, \mu, t)|_{\mu > 0} = 0$, i.e. $Q_\nu^+(\mu) = 0$, is natural since no radiation comes from
 205 the bottom of the spatial domain. Yet, by the law of black bodies, radiation could
 206 also come from the bottom but more general boundary conditions could be handled
 207 by the same analysis. In fact in [9] and other references, it is assumed that most of
 208 the energy from the Sun is in the form of visible light and is essentially unaffected
 209 by crossing the atmosphere, so that it is equivalent to a source of energy located at
 210 $z = 0$. Recall that it make physical sense to take $Q^-(\mu) = \mu Q' \cos \theta$, where θ is the
 211 latitude on Earth, while μ is the cosine of the observation angle. The fluid velocity
 212 field \mathbf{u} is given, assumed to be divergence-free and regular enough for (2.9) to make
 213 sense. Note that by rescaling the time variable, \mathbf{u} and κ_T appropriately, the factor
 214 $4\pi/\rho c_V$ can be replaced with 1.

2.2. Radiative transfer decoupled from hydrodynamics. When $\kappa_T = 0$,
 and the fluid is at rest, the left-hand side of temperature equation is zero, so that
 the fluid equations are decoupled from the radiative transfer equation (2.1). Let us
 consider first the case of isotropic scattering, namely $p_\nu(\mu, \mu') = 1$ at all frequencies
 ν . Then the system becomes (see [2])

$$(2.10) \quad (\mu\partial_\tau + \kappa_\nu)I_\nu(\tau, \mu) = \kappa_\nu a_\nu J_\nu(\tau) + \kappa_\nu(1 - a_\nu)B_\nu(T(\tau)),$$

$$(2.11) \quad I_\nu(0, \mu) = Q_\nu^+(\mu), \quad I_\nu(Z, -\mu) = Q_\nu^-(\mu), \quad 0 < \mu < 1,$$

$$(2.12) \quad \int_0^\infty \kappa_\nu(1 - a_\nu)B_\nu(T(\tau))d\nu = \int_0^\infty \kappa_\nu(1 - a_\nu)J_\nu(\tau)d\nu,$$

215 with the notation $Q_\nu^-(\mu) = Q^-(-\mu)B_\nu(T_S)$ and

$$(2.13) \quad J_\nu(\tau) := \frac{1}{2} \int_{-1}^1 I_\nu(\tau, \mu) d\mu.$$

216 In these equations, we have replaced $\bar{\kappa}_\nu$ by κ_ν and the height $z \in (z_m, z_M)$ by τ ,
 217 analogous to the ‘‘optical depth’’ (see for instance [9], or formula (51) in chapter I of
 218 [6]), defined as follows.

219 Pick $\rho_0 > 0$, some “reference” density of the fluid. For instance, ρ_0 could be
 220 the average density in the fluid, or the density at some reference altitude z . Indeed,
 221 the following expressions for the atmospheric density ρ in terms of the altitude z are
 222 found in the literature: $\rho(z) = \rho_0 e^{-z}$ or $\rho(z) = \rho_0 - \rho_1 z$. The new variable τ , and the
 223 absorption coefficient κ_ν are defined as follows:

$$(2.14) \quad \tau := \int_{z_m}^z \frac{\rho(\zeta)}{\rho_0} d\zeta, \quad \text{and } \kappa_\nu := \rho_0 \bar{\kappa}_\nu.$$

224 Equations (2.10) and (2.12) imply that

$$(2.15) \quad \partial_\tau \int_0^\infty \int_{-1}^1 \mu I_\nu(\tau, \mu) d\mu d\nu = 0.$$

225 We have ignored the dependence in x, y of T and I_ν , since x, y are mere parameters
 226 in these equations, which are anyway completely decoupled from the fluid equations.

227 Assuming that $0 < \kappa_\nu \leq \kappa_M$ and $0 \leq a_\nu < 1$ for all $\nu > 0$, we see that (2.12)
 228 and (2.13) define T as a functional of I , henceforth denoted $T[I]$. Equivalently, one
 229 can consider J_ν as a radiative intensity independent of μ , and observe that (2.12) and
 230 (2.13) imply that $T[I]$ is also a $T[J]$. Thus (2.10), (2.11), (2.12) can be recast as

$$(2.16) \quad \begin{cases} (\mu \partial_\tau + \kappa_\nu) I_\nu(\tau, \mu) = \kappa_\nu \mathcal{S}_\nu[J] := \kappa_\nu (a_\nu J_\nu(\tau) + \kappa_\nu (1 - a_\nu) B_\nu(T[J](\tau))) , \\ I_\nu(0, \mu) = Q_\nu^+(\mu), \quad I_\nu(Z, -\mu) = Q_\nu^-(\mu), \quad 0 < \mu < 1. \end{cases}$$

231 Throughout this article we use the exponential integrals

$$(2.17) \quad E_p(X) := X^{1-p} \int_X^\infty \frac{e^{-z}}{z^p} dz = \int_0^1 e^{-X/\mu} \mu^{p-2} d\mu, \quad X > 0.$$

LEMMA 2.1. *The following inequality holds:*

$$\frac{1}{2} \sup_{0 \leq t \leq Z} \int_0^Z E_1(\kappa|\tau - t|) \kappa d\tau \leq C_1(\kappa),$$

232 where $\kappa \mapsto C_1(\kappa)$ is monotone increasing from \mathbb{R}^+ to \mathbb{R}^+ , and less than 1.

233 *Proof* With $s = \kappa t$, observe that

$$(2.18) \quad \begin{aligned} \int_0^Z E_1(\kappa|\tau - t|) \kappa d\tau &= \int_0^{\kappa Z} E_1(|\sigma - s|) d\sigma = \int_{\mathbf{R}} E_1(|\sigma - s|) 1_{[0, \kappa Z]}(\sigma) d\sigma \\ &= \int_{\mathbf{R}} E_1(|\theta|) 1_{[-s, \kappa Z - s]}(\theta) d\theta \leq \int_{\mathbf{R}} E_1(|\theta|) 1_{[-\kappa Z/2, \kappa Z/2]}(\theta) d\theta \\ &= 2 \int_0^{\kappa Z/2} E_1(\theta) d\theta \leq 2 \int_0^{Z\kappa_M/2} E_1(\theta) d\theta =: 2C_1(\kappa). \end{aligned}$$

The first inequality above is the elementary rearrangement inequality (Theorem 3.4 in [20]). Now C_1 is obviously increasing since $E_1 > 0$, and

$$C_1(\kappa) = \int_0^{Z\kappa/2} E_1(\theta) d\theta < \int_0^\infty E_1(\theta) d\theta = \int_0^\infty \left(\int_1^\infty \frac{e^{-\theta y}}{y} dy \right) d\theta = \int_1^\infty \frac{dy}{y^2} = 1.$$

234

□

235

236 LEMMA 2.2. *Let*

$$(2.19) \quad S_\nu(\tau) = \frac{1}{2} \int_0^1 \left(e^{-\frac{\kappa_\nu \tau}{\mu}} Q_\nu^+(\mu) + e^{-\frac{\kappa_\nu(Z-\tau)}{\mu}} Q_\nu^-(\mu) \right) d\mu.$$

237 *Problem (2.10),(2.11),(2.12),(2.13) is equivalent to (2.12), plus the integral equation*

$$(2.20) \quad J_\nu(\tau) = S_\nu(\tau) + \frac{1}{2} \int_0^Z E_1(\kappa_\nu|\tau-t|) \kappa_\nu (a_\nu J_\nu(t) + (1-a_\nu) B_\nu(T(t))) dt.$$

238 *Proof* Applying the method of characteristics shows that

$$(2.21) \quad \begin{aligned} I_\nu(\tau, \mu) &= e^{-\frac{\kappa_\nu \tau}{\mu}} Q_\nu^+(\mu) \mathbf{1}_{\mu>0} + e^{-\frac{\kappa_\nu(Z-\tau)}{|\mu|}} Q_\nu^- (|\mu|) \mathbf{1}_{\mu<0} \\ &\quad + \mathbf{1}_{\mu>0} \int_0^\tau e^{-\frac{\kappa_\nu(\tau-t)}{\mu}} \frac{\kappa_\nu}{\mu} \mathcal{S}_\nu[J](t) dt + \mathbf{1}_{\mu<0} \int_\tau^Z e^{-\frac{\kappa_\nu(t-\tau)}{|\mu|}} \frac{\kappa_\nu}{\mu} \mathcal{S}_\nu[J](t) dt. \end{aligned}$$

239 One integrates both sides of this identity in μ , exchange the order of integration by
240 Tonelli's theorem, and change variables in the inner integral, observing that

$$\int_0^1 e^{-\frac{x}{\mu}} \frac{d\mu}{\mu} = \int_1^\infty \frac{e^{-Xy}}{y} dy = \int_X^\infty \frac{e^{-z}}{z} dz = E_1(X).$$

241 Thus (2.20) holds □

242

243 **3. Analysis of problem (2.10)-(2.12).** In order to solve numerically (2.10)-
244 (2.12), one uses the method of iteration on the sources. Starting from some appropri-
245 ate (I_ν^0, T^0) , one constructs a sequence (I_ν^n, T^n) by the following prescription:

$$(3.1) \quad \begin{cases} (\mu \partial_\tau + \kappa_\nu) I_\nu^{n+1}(\tau, \mu) = \kappa_\nu \mathcal{S}_\nu[J^n] \\ I_\nu^{n+1}(0, \mu) = Q_\nu^+(\mu), \quad I_\nu^{n+1}(Z, -\mu) = Q_\nu^-(\mu), \quad 0 < \mu < 1, \end{cases}$$

246 Note that $\mathcal{S}_\nu[J^n] := a_\nu J_\nu^n(t) + (1-a_\nu) B_\nu(T^n(t))$ does not depend on μ . Hence, it is

$$(3.2) \quad \begin{aligned} J_\nu^{n+1}(\tau) &= S_\nu(\tau) + \frac{1}{2} \int_0^Z E_1(\kappa_\nu|\tau-t|) \kappa_\nu (a_\nu J_\nu^n(t) + (1-a_\nu) B_\nu(T^n(t))) dt, \\ &\int_0^\infty \kappa_\nu (1-a_\nu) B_\nu(T^{n+1}(\tau)) d\nu = \int_0^\infty \kappa_\nu (1-a_\nu) J_\nu^{n+1}(\tau) d\nu. \end{aligned}$$

247 As in (2.21), the method of characteristics shows that

$$(3.3) \quad \begin{aligned} I_\nu^{n+1}(\tau, \mu) &= e^{-\frac{\kappa_\nu \tau}{\mu}} Q_\nu^+(\mu) \mathbf{1}_{\mu>0} + e^{-\frac{\kappa_\nu(Z-\tau)}{|\mu|}} Q_\nu^- (|\mu|) \mathbf{1}_{\mu<0} \\ &\quad + \mathbf{1}_{\mu>0} \int_0^\tau e^{-\frac{\kappa_\nu(\tau-t)}{\mu}} \frac{\kappa_\nu}{\mu} \mathcal{S}_\nu[J^n] dt + \mathbf{1}_{\mu<0} \int_\tau^Z e^{-\frac{\kappa_\nu(t-\tau)}{|\mu|}} \frac{\kappa_\nu}{|\mu|} \mathcal{S}_\nu[J^n] dt. \end{aligned}$$

Since $B_\nu \geq 0$, this formula shows, by a straightforward induction argument, that

$$I_\nu^0 \geq 0, \quad T^0 \geq 0, \quad Q_\nu^\pm \geq 0 \implies I_\nu^n \geq 0.$$

Moreover

$$\begin{aligned}
 I_\nu^{n+1}(\tau, \mu) - I_\nu^n(\tau, \mu) &= \mathbf{1}_{\mu>0} \int_0^\tau e^{-\frac{\kappa_\nu(\tau-t)}{\mu}} \frac{\kappa_\nu}{\mu} a_\nu (J_\nu^n(t) - J_\nu^{n-1}(t)) dt \\
 &+ \mathbf{1}_{\mu>0} \int_0^\tau e^{-\frac{\kappa_\nu(\tau-t)}{\mu}} \frac{\kappa_\nu}{\mu} (1 - a_\nu) (B_\nu(T^n(t)) - B_\nu(T^{n-1}(t))) dt \\
 &+ \mathbf{1}_{\mu<0} \int_\tau^Z e^{-\frac{\kappa_\nu(t-\tau)}{|\mu|}} \frac{\kappa_\nu}{|\mu|} a_\nu (J_\nu^n(t) - J_\nu^{n-1}(t)) dt \\
 &+ \mathbf{1}_{\mu<0} \int_\tau^Z e^{-\frac{\kappa_\nu(t-\tau)}{|\mu|}} \frac{\kappa_\nu}{|\mu|} (1 - a_\nu) (B_\nu(T^n(t)) - B_\nu(T^{n-1}(t))) dt.
 \end{aligned}$$

Since B_ν is nondecreasing for each $\nu > 0$, formula (2.12) shows that

$$J_\nu^n \geq J_\nu^{n-1} \implies T^n \geq T^{n-1},$$

and we conclude from the equality above that

$$I_\nu^0 = 0, T^0 = 0, Q_\nu^\pm \geq 0 \implies \begin{cases} 0 \leq I_\nu^1 \leq I_\nu^2 \leq \dots \leq I_\nu^n \leq \dots \\ 0 \leq T^1 \leq T^2 \leq \dots \leq T^n \leq \dots \end{cases}$$

Integrating both sides of (3.2) over $[0, Z]$ in τ implies that

$$\begin{aligned}
 \int_0^Z J_\nu^{n+1}(\tau) d\tau &= \int_0^Z S_\nu(\tau) d\tau + \frac{1}{2} \int_0^Z \left(\int_0^Z E_1(\kappa_\nu|\tau-t|) \kappa_\nu d\tau \right) \mathcal{S}_\nu[J^n] dt \\
 &\leq \int_0^Z S_\nu(\tau) d\tau + \frac{1}{2} \sup_{0 \leq t \leq Z} \int_0^Z E_1(\kappa_\nu|\tau-t|) \kappa_\nu d\tau \int_0^Z \mathcal{S}_\nu[J^n] dt.
 \end{aligned}$$

Thus by Lemma 2.1

$$\int_0^Z J_\nu^{n+1}(\tau) d\tau \leq \int_0^Z S_\nu(\tau) d\tau + C_1(\kappa_\nu) \int_0^Z \mathcal{S}_\nu[J^n] dt.$$

Multiply both sides of this inequality by κ_ν and integrate in ν : one finds that

$$\int_0^\infty \int_0^Z \kappa_\nu J_\nu^{n+1}(\tau) d\tau d\nu \leq \int_0^\infty \int_0^Z (\kappa_\nu S_\nu(\tau) + C_1(\kappa_M) \kappa_\nu \mathcal{S}_\nu[J^n]) dt d\nu.$$

248 At this point, we recall that $T^n = T[J_\nu^n]$, so that

$$(3.4) \quad \int_0^\infty \kappa_\nu (1 - a_\nu) B_\nu(T^n(t)) d\nu = \int_0^\infty \kappa_\nu (1 - a_\nu) J_\nu^n(t) d\nu,$$

and hence

$$\int_0^\infty \int_0^Z \kappa_\nu J_\nu^{n+1}(\tau) d\tau d\nu \leq C_1(\kappa_M) \int_0^\infty \int_0^Z \kappa_\nu J_\nu^n(t) dt d\nu + \int_0^\infty \int_0^Z \kappa_\nu S_\nu(\tau) d\tau d\nu.$$

The expression of the source term can be slightly reduced, by integrating out the τ variable:

$$\int_0^Z \kappa_\nu e^{-\frac{\kappa_\nu \tau}{\mu}} d\tau = \int_0^Z \kappa_\nu e^{-\frac{\kappa_\nu(Z-\tau)}{\mu}} d\tau = \mu \left(1 - e^{-\frac{\kappa_\nu Z}{\mu}} \right),$$

so that

$$\begin{aligned} 0 &\leq \int_0^\infty \kappa_\nu \int_0^Z S_\nu(\tau) d\tau d\nu \leq \frac{1}{2} \int_0^\infty \kappa_\nu \int_0^1 (Q_\nu^+(\mu) + Q_\nu^-(\mu)) \mu d\mu d\nu =: \mathcal{Q}. \\ \implies &\int_0^\infty \int_0^Z \kappa_\nu J_\nu^{n+1}(\tau) d\tau d\nu \leq C_1(\kappa_M) \int_0^\infty \int_0^Z \kappa_\nu J_\nu^n(t) dt d\nu + \mathcal{Q}. \end{aligned}$$

Initializing the sequence I_ν^n with $I_\nu^0 = 0$ and $T^0 = T[J_\nu^0] = 0$, one finds that

$$\int_0^\infty \int_0^Z \kappa_\nu J_\nu^1(\tau) d\tau d\nu \leq \mathcal{Q}, \quad \int_0^\infty \int_0^Z \kappa_\nu J_\nu^2(\tau) d\tau d\nu \leq C_1(\kappa_M) \mathcal{Q} + \mathcal{Q}$$

and by induction

$$\int_0^\infty \int_0^Z \kappa_\nu J_\nu^{n+1}(\tau) d\tau d\nu \leq \mathcal{Q} \sum_{j=0}^n C_1(\kappa_M)^j.$$

Since $C_1(\kappa_M) < 1$, the series above converges and one has the uniform bound

$$\int_0^\infty \int_0^Z \kappa_\nu J_\nu^{n+1}(\tau) d\tau d\nu \leq \frac{\mathcal{Q}}{1 - C_1(\kappa_M)}.$$

Furthermore, as

$$0 \leq I_\nu^1 \leq I_\nu^2 \leq \dots \leq I_\nu^n \leq I_\nu^{n+1} \leq \dots$$

the bound above and the Monotone Convergence Theorem implies that the sequence $I_\nu^{n+1}(\tau, \mu)$ converges for a.e. $(\tau, \mu, \nu) \in (0, Z) \times (-1, 1) \times (0, +\infty)$ to a limit denoted $I_\nu(\tau, \mu)$ as $n \rightarrow \infty$. Since

$$0 \leq T^1 \leq T^2 \leq \dots \leq T^n \leq T^{n+1} \leq \dots$$

we conclude from (2.15) and the Monotone Convergence Theorem that $T^{n+1}(\tau)$ converges for a.e. $\tau \in (0, Z)$ to a limit denoted $T(\tau)$ as $n \rightarrow \infty$.

Then we can pass to the limit in (3.3) as $n \rightarrow \infty$ by monotone convergence, so that (2.21) holds for a.e. $(\tau, \mu, \nu) \in (0, Z) \times (-1, 1) \times (0, +\infty)$. One recognizes in this equality the integral formulation of (2.10)-(2.12). Besides, we have seen that

$$\begin{aligned} 0 &= I_\nu^0 \leq I_\nu^1 \leq I_\nu^2 \leq \dots \leq I_\nu^n \leq I_\nu^{n+1} \leq \dots \leq I_\nu, \\ 0 &= T^0 \leq T^1 \leq T^2 \leq \dots \leq T^n \leq T^{n+1} \leq \dots \leq T, \end{aligned}$$

so that

$$\begin{aligned} 0 &\leq \int_0^Z (J_\nu^{n+1} - J_\nu^n)(\tau) d\tau = \frac{1}{2} \int_0^Z \left(\int_0^Z E_1(\kappa_\nu|\tau - t|) \kappa_\nu d\tau \right) a_\nu (J_\nu^n - J_\nu^{n-1})(t) dt \\ &\quad + \frac{1}{2} \int_0^Z \left(\int_0^Z E_1(\kappa_\nu|\tau - t|) \kappa_\nu d\tau \right) (1 - a_\nu) (B_\nu(T^n(t)) - B_\nu(T^{n-1}(t))) dt \\ &\leq C_1(\kappa_M) \int_0^Z (a_\nu (J_\nu^n - J_\nu^{n-1})(t) + (1 - a_\nu) (B_\nu(T^n(t)) - B_\nu(T^{n-1}(t)))) dt. \end{aligned}$$

Using again (3.4), we conclude that

$$0 \leq \int_0^Z \int_0^\infty \kappa_\nu (J_\nu^{n+1} - J_\nu^n)(\tau) d\nu d\tau \leq C_1(\kappa_M) \int_0^Z \int_0^\infty \kappa_\nu (J_\nu^n - J_\nu^{n-1})(t) dt.$$

Hence

$$0 \leq \int_0^Z \int_0^\infty \kappa_\nu (J_\nu^{n+1} - J_\nu^n)(\tau) d\nu d\tau \leq C_1(\kappa_M)^n \int_0^\infty \kappa_\nu J_\nu^1(\tau) d\nu d\tau \leq C_1(\kappa_M)^n \mathcal{Q},$$

so that

$$0 \leq \int_0^Z \int_0^\infty \kappa_\nu (J_\nu - J_\nu^n)(\tau) d\nu d\tau \leq C_1(\kappa_M)^n \int_0^\infty \kappa_\nu J_\nu^1(\tau) d\nu d\tau \leq \frac{C_1(\kappa_M)^n \mathcal{Q}}{1 - C_1(\kappa_M)}.$$

251 Summarizing, we have proved the following result.

THEOREM 3.1. *Assume that $0 < \kappa_\nu \leq \kappa_M$, while $0 \leq a_\nu < 1$ for all $\nu > 0$. Let $Q_\nu^\pm(\mu)$ satisfy*

$$\mathcal{Q} := \frac{1}{2} \int_0^\infty \kappa_\nu \int_0^1 (Q_\nu^+(\mu) + Q_\nu^-(\mu)) \mu d\mu < \infty.$$

Choose $I_\nu^0 = 0$ and $T^0 = 0$, and let I_ν^n and $T^n = T[J_\nu^n]$ be the solution of (3.1). Then

$$I_\nu^n(\tau, \mu) \rightarrow I_\nu(\tau, \mu) \quad \text{and} \quad T^n(\tau) \rightarrow T(\tau)$$

for $(\tau, \mu, \nu) \in (0, Z) \times (-1, 1) \times (0, +\infty)$ as $n \rightarrow \infty$, where (I_ν, T) is a solution of (2.10)-(2.12). This method converges exponentially fast, in the sense that

$$0 \leq \int_0^Z \int_0^\infty \kappa_\nu (J_\nu - J_\nu^n)(\tau) d\nu d\tau \leq \frac{C_1(\kappa_M)^n \mathcal{Q}}{1 - C_1(\kappa_M)},$$

and, if $0 \leq a_\nu \leq a_M < 1$ while $0 < \kappa_m \leq \kappa_\nu$, one has

$$0 \leq \int_0^Z \bar{\sigma} (T(t)^4 - T^n(t)^4) dt \leq \frac{C_1(\kappa_M)^n \mathcal{Q}}{\kappa_m(1 - a_M)(1 - C_1(\kappa_M))}.$$

The last bound comes from the defining equality for the temperature in terms of the radiative intensity

$$\begin{aligned} \kappa_m(1 - a_M) \bar{\sigma} (T^4 - (T^n)^4) &= \kappa_m(1 - a_M) \int_0^\infty (B_\nu(T) - B_\nu(T^n)) d\nu \\ &\leq \int_0^\infty \kappa_\nu(1 - a_\nu) (B_\nu(T) - B_\nu(T^n)) d\nu = \int_0^\infty \kappa_\nu(1 - a_\nu) (J_\nu - J_\nu^n) d\nu \\ &\leq \int_0^\infty \kappa_\nu (J_\nu - J_\nu^n) d\nu. \end{aligned}$$

252 **4. Uniqueness, Maximum Principle for (2.10)-(2.12).** This section follows
 253 computations in [11] (in the case $Z = +\infty$ and with $a_\nu = 0$) and the rather subtle
 254 monotonicity structure of the radiative transfer equations, a striking result² found
 255 by Mercier in [23]. The following theorem shows that two solutions of the problem
 256 (2.10)-(2.12) are ordered exactly as their boundary data. (This situation is analogous
 257 to the case of harmonic functions, except that the radiative transfer equations (2.10)-
 258 (2.12) are nonlinear, at variance with the Laplace equation.)

²In fact, Mercier's original argument is even more complex, because he assumes that the opacity $K_\nu := \kappa_\nu(1 - a_\nu)$ depends on the temperature T , and is a nonincreasing function of T for each $\nu > 0$ while $T \mapsto K_\nu(T)B_\nu(T)$ is nondecreasing.

THEOREM 4.1. *Assume that $0 < \kappa_\nu \leq \kappa_M$, while $0 \leq a_\nu < 1$ for all $\nu > 0$. Let $Q^\pm, Q'^\pm \in L^1((0, 1) \times (0, \infty))$ satisfy*

$$0 \leq Q_\nu^\pm(\mu) \leq Q'_\nu^\pm(\mu) \quad \text{for a.e. } (\mu, \nu) \in (0, 1) \times (0, \infty).$$

Then, the solutions $(I_\nu, T[I])$ of (2.10)-(2.12), and $(I'_\nu, T[I'])$ of (2.10)-(2.12), with boundary data $Q_\nu^\pm(\mu)$ replaced with $Q'_\nu^\pm(\mu)$ satisfy

$$I_\nu(\tau, \mu) \leq I'_\nu(\tau, \mu) \text{ and } T[I](\tau) \leq T[I'](\tau) \quad \text{for a.e. } (\tau, \mu) \in (-1, 1) \times (0, \infty).$$

In particular,

$$Q_\nu^\pm(\mu) = Q'_\nu^\pm(\mu) \text{ a.e. } \mu, \nu \implies I_\nu(\tau, \mu) = I'_\nu(\tau, \mu) \text{ and } T[I](\tau) = T[I'](\tau) \\ \text{for a.e. } \tau, \mu \in (-1, 1) \times (0, \infty).$$

259 The proof of this result is deferred to the appendix at the very end of this paper.

260 One has also the following form of Maximum Principle for the radiative transfer
261 equation. (If one keeps in mind the analogy with harmonic functions recalled before
262 **Theorem 4.1**, the Maximum Principle below is a *consequence* of the monotonicity of
263 the dependence of the solution of (2.10)-(2.12) in terms of its boundary data, whereas
264 the analogous monotonicity in the case of harmonic functions is *deduced* from the
265 Maximum Principle for the Laplace equation.)

Corollary 4.2. *Assume that $0 < \kappa_\nu \leq \kappa_M$, while $0 \leq a_\nu < 1$ for all $\nu > 0$. Let $Q_\nu^\pm(\mu) \leq B_\nu(T_M)$ (resp. $Q_\nu^\pm(\mu) \geq B_\nu(T_m)$) for a.e. $(\mu, \nu) \in (0, 1) \times (0, \infty)$. Then*

$$I_\nu(\tau, \mu) \leq B_\nu(T_M) \text{ and } T[I](\tau) \leq T_M \\ \text{(resp. } I_\nu(\tau, \mu) \geq B_\nu(T_m) \text{ and } T[I](\tau) \geq T_m) \\ \text{for a.e. } (\tau, \mu) \in (-1, 1) \times (0, \infty).$$

266 *Proof* Indeed, $I'_\nu = B_\nu(T_M)$ and $T[I'] = T_M$ (resp. $I'_\nu = B_\nu(T_m)$ and $T[I'] = T_m$)
267 is the solution of (2.11) with boundary data $Q'_\nu^\pm(\mu) = B_\nu(T_M)$ (resp. $Q'_\nu^\pm(\mu) =$
268 $B_\nu(T_m)$). The announced inequalities follow from the comparison of solutions ob-
269 tained in **Theorem 4.1**. \square

270

Remark 4.3. In **Theorem 3.1**, if one has the stronger condition

$$0 \leq Q_\nu^\pm(\mu) \leq B_\nu(T_M) \quad \text{for a.e. } (\mu, \nu) \in (0, 1) \times (0, \infty),$$

one obtains the following bound for the numerical and theoretical solutions

$$0 \leq I_\nu^1 \leq \dots \leq I_\nu^n \leq \dots \leq I_\nu \leq B_\nu(T_M), \text{ and } 0 \leq T^1 \leq \dots \leq T^n \leq \dots \leq T \leq T_M.$$

5. Radiative Transfer with Rayleigh Phase Function. In this section, we discuss the same problem as in the previous section, with the isotropic scattering kernel replaced by the Rayleigh phase function. In the case of slab symmetry, the Rayleigh phase function is

$$p(\mu, \mu') = \frac{3}{16}(3 - \mu^2) + \frac{3}{16}(3\mu^2 - 1)\mu'^2$$

271 (see section 11.2 in chapter I of [6]). Observe that

$$(5.1) \quad p(\mu, \mu') = \frac{3}{16}(3 + 3\mu^2\mu'^2 - \mu^2 - \mu'^2) \geq \frac{3}{16} > 0,$$

272 while

$$(5.2) \quad \frac{1}{2} \int_{-1}^1 p(\mu, \mu') d\mu = \frac{3}{16} (6 + 3 \cdot \frac{2}{3} \mu'^2 - \frac{2}{3} - 2\mu'^2) = 1.$$

273 Keeping (2.12) as the defining equation for $T[I]$, the problem becomes

$$(5.3) \quad \begin{cases} (\mu \partial_\tau + \kappa_\nu) I_\nu(\tau, \mu) = \frac{3}{8} \kappa_\nu a_\nu ((3 - \mu^2) J_\nu(\tau) + (3\mu^2 - 1) K_\nu(\tau)) \\ \quad + \kappa_\nu (1 - a_\nu) B_\nu(T[J](\tau)), \\ I_\nu(0, \mu) = Q_\nu^+(\mu), \quad I_\nu(Z, -\mu) = Q_\nu^-(\mu), \quad 0 < \mu < 1, \end{cases}$$

274 with

$$(5.4) \quad J_\nu := \frac{1}{2} \int_{-1}^1 \mu I_\nu d\mu, \quad K_\nu = \frac{1}{2} \int_{-1}^1 \mu^2 I_\nu d\mu$$

275 and (2.12). Starting from $I_\nu^0(\tau, \mu) = 0$ and $T^0(\tau) = 0$, one solves for I_ν^{n+1}

$$(5.5) \quad \begin{cases} (\mu \partial_\tau + \kappa_\nu) I_\nu^{n+1}(\tau, \mu) = \frac{3}{8} \kappa_\nu a_\nu ((3 - \mu^2) J_\nu^n(\tau) + (3\mu^2 - 1) K_\nu^n(\tau)) \\ \quad + \kappa_\nu (1 - a_\nu) B_\nu(T^n(\tau)), \quad T^n := T[I^n] \\ I_\nu^{n+1}(0, \mu) = Q_\nu^+(\mu), \quad I_\nu^{n+1}(Z, -\mu) = Q_\nu^-(\mu), \quad 0 < \mu < 1. \end{cases}$$

Since B_ν is nondecreasing for each $\nu > 0$, one easily checks with (5.1) that

$$\begin{aligned} 0 &= I_\nu^0 \leq I_\nu^1 \leq I_\nu^2 \leq \dots \leq I_\nu^n \leq I_\nu^{n+1} \leq \dots \\ 0 &= T^0 \leq T^1 \leq T^2 \leq \dots \leq T^n \leq T^{n+1} \leq \dots \end{aligned}$$

276 The construction of these sequences is straightforward:

$$(5.6) \quad \begin{aligned} J_\nu^{n+1}(\tau) &= S_\nu(\tau) + \frac{3}{16} \int_0^Z E_1(\kappa_\nu |\tau - t|) \kappa_\nu a_\nu (3J_\nu^n(t) - K_\nu^n(t)) dt \\ &\quad + \frac{3}{16} \int_0^Z E_3(\kappa_\nu |\tau - t|) \kappa_\nu a_\nu (3K_\nu^n(t) - J_\nu^n(t)) dt \\ &\quad + \frac{1}{2} \int_0^Z E_1(\kappa_\nu |\tau - t|) \kappa_\nu (1 - a_\nu) B_\nu(T^n(t)) dt, \\ K_\nu^{n+1}(\tau) &= \frac{1}{2} \int_0^1 \left(e^{-\frac{\kappa_\nu \tau}{\mu}} Q_\nu^+(\mu) \mathbf{1}_{\mu > 0} + e^{-\frac{\kappa_\nu (Z - \tau)}{|\mu|}} Q_\nu^-(|\mu|) \mathbf{1}_{\mu < 0} \right) \mu^2 d\mu \\ &\quad + \frac{3}{16} \int_0^Z E_3(\kappa_\nu |\tau - t|) \kappa_\nu a_\nu (3J_\nu^n(t) - K_\nu^n(t)) dt \\ &\quad + \frac{3}{16} \int_0^Z E_5(\kappa_\nu |\tau - t|) \kappa_\nu a_\nu (3K_\nu^n(t) - J_\nu^n(t)) dt \\ &\quad + \frac{1}{2} \int_0^Z E_3(\kappa_\nu |\tau - t|) \kappa_\nu (1 - a_\nu) B_\nu(T^n(t)) dt, \\ \int_0^\infty \kappa_\nu (1 - a_\nu) B_\nu(T^{n+1}) d\nu &= \int_0^\infty \kappa_\nu (1 - a_\nu) J_\nu^{n+1} d\nu. \end{aligned}$$

277 Notice that the radiative intensity is eliminated, but can be recovered by

$$\begin{aligned}
(5.7) \quad I_\nu^{n+1}(\tau, \mu) &= e^{-\frac{\kappa_\nu \tau}{\mu}} Q_\nu^+(\mu) \mathbf{1}_{\mu > 0} + e^{-\frac{\kappa_\nu (Z-\tau)}{|\mu|}} Q_\nu^- (|\mu|) \mathbf{1}_{\mu < 0} \\
&+ \mathbf{1}_{\mu > 0} \int_0^\tau e^{-\frac{\kappa_\nu (\tau-t)}{\mu}} \frac{\kappa_\nu}{\mu} \frac{3}{8} a_\nu ((3-\mu^2) J_\nu^n(t) + (3\mu^2-1) K_\nu^n(t)) dt \\
&+ \mathbf{1}_{\mu > 0} \int_0^\tau e^{-\frac{\kappa_\nu (\tau-t)}{\mu}} \frac{\kappa_\nu}{\mu} (1-a_\nu) B_\nu(T^n(t)) dt \\
&+ \mathbf{1}_{\mu < 0} \int_t^Z e^{-\frac{\kappa_\nu |t-\tau|}{|\mu|}} \frac{\kappa_\nu}{|\mu|} \frac{3}{8} a_\nu ((3-\mu^2) J_\nu^n(t) + (3\mu^2-1) K_\nu^n(t)) dt \\
&+ \mathbf{1}_{\mu < 0} \int_0^Z e^{-\frac{\kappa_\nu |t-\tau|}{|\mu|}} \frac{\kappa_\nu}{|\mu|} (1-a_\nu) B_\nu(T^n(t)) dt.
\end{aligned}$$

278 Assume that $0 \leq Q_\nu^\pm \leq B_\nu(T_M)$, $0 \leq I_\nu^n \leq B_\nu(T_M)$ and $0 \leq T^n \leq T_M$. Thus
279 $0 \leq J_\nu^n \leq B_\nu(T_M)$ and $0 \leq K_\nu^n \leq \frac{1}{3} B_\nu(T_M)$, so that

$$\begin{aligned}
(5.8) \quad I_\nu^{n+1}(\tau, \mu) &\leq \left(e^{-\frac{\kappa_\nu \tau}{\mu}} \mathbf{1}_{\mu > 0} + e^{-\frac{\kappa_\nu (Z-\tau)}{|\mu|}} \mathbf{1}_{\mu < 0} \right) B_\nu(T_M) \\
&+ \mathbf{1}_{\mu > 0} \int_0^\tau e^{-\frac{\kappa_\nu (\tau-t)}{\mu}} \frac{\kappa_\nu}{\mu} \frac{3}{8} a_\nu ((3-\mu^2) B_\nu(T_M) + (\mu^2 - \frac{1}{3}) B_\nu(T_M)) dt \\
&+ \mathbf{1}_{\mu > 0} \int_0^\tau e^{-\frac{\kappa_\nu (\tau-t)}{\mu}} \frac{\kappa_\nu}{\mu} (1-a_\nu) B_\nu(T_M) dt \\
&+ \mathbf{1}_{\mu < 0} \int_\tau^Z e^{-\frac{\kappa_\nu (t-\tau)}{|\mu|}} \frac{\kappa_\nu}{|\mu|} \frac{3}{8} a_\nu ((3-\mu^2) B_\nu(T_M) + (\mu^2 - \frac{1}{3}) B_\nu(T_M)) dt \\
&+ \mathbf{1}_{\mu < 0} \int_\tau^Z e^{-\frac{\kappa_\nu (t-\tau)}{|\mu|}} \frac{\kappa_\nu}{|\mu|} (1-a_\nu) B_\nu(T_M) dt \\
&= B_\nu(T_M) \mathbf{1}_{\mu > 0} \left(e^{-\frac{\kappa_\nu \tau}{\mu}} + \int_0^\tau e^{-\frac{\kappa_\nu (\tau-t)}{\mu}} \frac{\kappa_\nu}{\mu} \left(\frac{3}{8} a_\nu (3 - \frac{1}{3}) + (1-a_\nu) \right) dt \right) \\
&+ B_\nu(T_M) \mathbf{1}_{\mu < 0} \left(e^{-\frac{\kappa_\nu (Z-\tau)}{|\mu|}} + \int_\tau^Z e^{-\frac{\kappa_\nu (t-\tau)}{|\mu|}} \frac{\kappa_\nu}{|\mu|} \left(\frac{3}{8} a_\nu (3 - \frac{1}{3}) + (1-a_\nu) \right) dt \right) = B_\nu(T_M).
\end{aligned}$$

Besides, using again that $T \mapsto B_\nu(T)$ is increasing for each $\nu > 0$ while $\kappa_\nu(1-a_\nu) > 0$ for all $\nu > 0$,

$$T^{n+1} = T[I^{n+1}] \leq T[B_\nu(T_M)] = T_M.$$

280 Summarizing, we have proved the following result.

THEOREM 5.1. *Assume that $\kappa_\nu > 0$ while $0 \leq a_\nu < 1$ for all $\nu > 0$. Let the boundary data Q_ν^\pm satisfy*

$$0 \leq Q_\nu^\pm(\mu) \leq B_\nu(T_M) \quad \text{for all } \mu \in (-1, 1) \text{ and } \nu > 0.$$

281 (5.6) defines an increasing sequence of radiative intensities I_ν^n and temperatures T^n
282 converging pointwise to I_ν and $T = T[I]$ respectively, which is a solution of (5.3).

283 The argument above is based on the monotonicity of the sequences I_ν^n and T^n ,
284 and does not give any information on the convergence rate.

285 *Remark 5.2.* One easily checks that the uniqueness [Theorem 4.1](#) holds verbatim
286 for the problem (5.3) with Rayleigh phase function. See the appendix at the end of
287 this paper for the proof.

288 **6. Radiative transfer in a fluid with thermal diffusion.** For clarity we
 289 consider the case of a lake; we neglect the wind above the lake and we assume that
 290 the sunlight hits the surface of the lake with a given energy. The depth of the lake
 291 should vary slowly with x, y , but for the sake of simplicity, it is assumed to be uniform:
 292 $\Omega = \mathbb{O} \times (0, Z)$, for some open set $\mathbb{O} \subset \mathbb{R}^2$ with C^1 boundary, or piecewise C^1
 293 boundary.

294 With $\mathbf{u} \in H^1(\Omega)$ satisfying $\nabla \cdot \mathbf{u} = 0$ and $\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0$, consider again the system
 295 (2.9). Throughout this section, we assume isotropic scattering, with

$$(6.1) \quad 0 \leq a_\nu \leq a_M < 1, \quad 0 < \kappa_m \leq \kappa_\nu \leq \kappa_M, \quad \nu > 0.$$

296 Here, ρ is assumed to be a constant, and we choose $\rho_0 = \rho$ in (2.14), so that $\kappa_\nu = \rho \bar{\kappa}_\nu$,
 297 and $\tau = z$.

We further assume that the fluid flow is steady, and consider the system

$$(6.2) \quad \mu \partial_z I_\nu + \kappa_\nu I_\nu = \kappa_\nu (1 - a_\nu) B_\nu(T) + \kappa_\nu a_\nu J_\nu, \quad J_\nu := \frac{1}{2} \int_{-1}^1 I_\nu d\mu,$$

$$(6.3) \quad \mathbf{u} \cdot \nabla T - \frac{c_P}{c_V} \kappa_T \Delta T = \frac{4\pi}{\rho c_V} \int_0^\infty \kappa_\nu (1 - a_\nu) (J_\nu - B_\nu(T)) d\nu,$$

$$(6.4) \quad I_\nu|_{z=Z, \mu < 0} = Q_\nu^-(x, y, -\mu), \quad I_\nu|_{z=0, \mu > 0} = Q_\nu^+(x, y, \mu), \quad \frac{\partial T}{\partial n}|_{\partial\Omega} = 0.$$

298 The boundary sources $Q_\nu^\pm(x, y, \mu)$ are bounded, measurable, nonnegative functions
 299 defined a.e. on $\mathbb{O} \times (-1, 1) \times (0, \infty)$.

300 As a first reduction, we solve (6.2) for the radiative intensity I_ν in terms of
 301 the angle-averaged intensity J_ν and of the temperature T , and average the resulting
 302 expression in μ : proceeding as in Lemma 2.2, we arrive at the system

$$(6.5) \quad \begin{cases} J_\nu(x, y, z) = S_\nu(x, y, z) \\ + \frac{1}{2} \int_0^Z \kappa_\nu E_1(\kappa_\nu |z - \zeta|) (a_\nu J_\nu(x, y, \zeta) + (1 - a_\nu) B_\nu(T(x, y, \zeta))) d\zeta, \\ \mathbf{u}(\mathbf{x}) \cdot \nabla T(\mathbf{x}) - \frac{c_P}{c_V} \kappa_T \Delta T(\mathbf{x}) = \frac{4\pi}{\rho c_V} \int_0^\infty \kappa_\nu (1 - a_\nu) (J_\nu(\mathbf{x}) - B_\nu(T(\mathbf{x}))) d\nu, \\ \frac{\partial T}{\partial n}|_{\partial\Omega} = 0, \end{cases}$$

303 where

$$(6.6) \quad S_\nu(x, y, z) := \frac{1}{2} \int_0^1 \left(e^{-\frac{\kappa_\nu z}{\mu}} Q_\nu^+(x, y, \mu) + e^{-\frac{\kappa_\nu (Z-z)}{\mu}} Q_\nu^-(x, y, \mu) \right) d\mu.$$

304 Once the angle-averaged radiative intensity is known J_ν , the radiative intensity
 305 I_ν itself is easily obtained by solving the transfer equation (6.2) by the method of
 306 characteristics: see (2.21).

THEOREM 6.1. *Assume that the absorption coefficient κ_ν and the scattering albedo a_ν satisfy (6.1). Let the boundary source terms Q_ν^\pm satisfy: for some T_M ,*

$$0 \leq Q_\nu^\pm(\mu) \leq B_\nu(T_M), \quad 0 < \mu < 1, \quad \nu > 0.$$

307 Consider $\{J_\nu^n, T^n\}_{n \geq 0}$ initiated by T^0 given and generated by

$$(6.7) \quad \begin{aligned} J_\nu^{n+1}(x, y, z) &= S_\nu(x, y, z) + \\ &\frac{1}{2} \int_0^Z \kappa_\nu E_1(\kappa_\nu |z - \zeta|) (a_\nu J_\nu^n(x, y, \zeta) + (1 - a_\nu) B_\nu(T^n(x, y, \zeta))) d\zeta. \end{aligned}$$

$$(6.8) \quad \begin{cases} \mathbf{u} \cdot \nabla T^{n+1} - \frac{c_P}{c_V} \kappa_T \Delta T^{n+1} + \frac{4\pi}{\rho c_V} \int_0^\infty \kappa_\nu (1 - a_\nu) B_\nu(T_+^{n+1}) d\nu \\ = \frac{4\pi}{\rho c_V} \int_0^\infty \kappa_\nu (1 - a_\nu) J_\nu^{n+1} d\nu, \quad \frac{\partial T}{\partial n} \Big|_{\partial\Omega} = 0. \end{cases}$$

Then

$$\begin{aligned} S_\nu(\mathbf{x}) &= J_\nu^0(\mathbf{x}) \leq J_\nu^1(\mathbf{x}) \leq \dots \leq J_\nu^n(\mathbf{x}) \leq J_\nu^{n+1}(\mathbf{x}) \leq \dots \leq B_\nu(T_M), \quad \nu > 0, \\ 0 &= T^0 \leq T^1(\mathbf{x}) \leq \dots \leq T^n(\mathbf{x}) \leq T^{n+1}(\mathbf{x}) \leq \dots \leq T_M, \quad \mathbf{x} \in \Omega, \end{aligned}$$

308 and convergence to a solution (J, T) of the system (6.5) holds.

Define

$$\mathcal{B}(T) := \int_0^\infty \kappa_\nu (1 - a_\nu) B_\nu(T_+) d\nu.$$

Observe that

$$\kappa_m (1 - a_M) \bar{\sigma} T_+^4 \leq \mathcal{B}(T) \leq \kappa_M \bar{\sigma} T_+^4,$$

309 where $\pi \bar{\sigma}$ is the Stefan-Boltzmann constant (see (2.3)). Observe also that the function
310 $\mathcal{B} : \mathbf{R} \rightarrow \mathbf{R}$ is nondecreasing, and increasing on $(0, +\infty)$ by construction, since B_ν is
311 increasing on $[0, +\infty)$ for each $\nu > 0$.

312 For the sake of notational simplicity, in order to keep the number of physical
313 constants to a strict minimum, we assume henceforth that $\rho c_P \kappa_T / 4\pi = 1$, and replace
314 \mathbf{u} with $\rho c_V \mathbf{u} / 4\pi$.

315 The key argument in the proof of this theorem is the following lemma.

LEMMA 6.2. *Let $R \in L^{6/5}(\Omega)$. There exists at least one weak solution of*

$$-\Delta T + \mathbf{u} \cdot \nabla T + \mathcal{B}(T) = R, \quad \frac{\partial T}{\partial n} \Big|_{\partial\Omega} = 0.$$

316 *If $R \geq 0$ a.e. and $|\{x \in \Omega \text{ s.t. } R(x) > 0\}| > 0$, the weak solution of the problem
317 above is unique and satisfies $T \geq 0$ a.e. on Ω .*

318 *Moreover, if $R' \in L^{6/5}(\Omega)$ and $R' \geq R$ a.e. on Ω , the weak solution T' of the
319 problem above with right hand side R' satisfies $T \leq T'$ a.e. on Ω .*

Proof For each $0 < \varepsilon < 1$, the problem

$$\varepsilon T_\varepsilon - \Delta T_\varepsilon + \mathbf{u} \cdot \nabla T_\varepsilon + \mathcal{B}(T_\varepsilon) = R, \quad \frac{\partial T}{\partial n} \Big|_{\partial\Omega} = 0$$

320 has a weak solution in $H^1(\Omega)$.

To see this, apply Theorem 1 of [19] with $V = H^1(\Omega)$ to the nonlinear operator $\mathcal{A}_\varepsilon : V \mapsto V'$ defined by

$$\langle \mathcal{A}_\varepsilon T, \phi \rangle_{V', V} = \int_\Omega (\varepsilon T \phi + \nabla T \cdot \nabla \phi + \phi \mathbf{u} \cdot \nabla T + \mathcal{B}(T) \phi) d\mathbf{x}.$$

That \mathcal{A}_ε is continuous from V to V' easily follows from the Sobolev embedding $H^1(\Omega) \subset L^6(\Omega)$, which implies by duality the continuous inclusion $L^{6/5}(\Omega) \subset V'$. Since $\mathbf{u} \in H^1(\Omega) \subset L^6(\Omega)$, one has

$$\mathbf{u} \cdot \nabla T \in L^{3/2}(\Omega) \subset L^{6/5}(\Omega) \subset V' \quad \text{with } \|\mathbf{u} \cdot \nabla T\|_{L^{3/2}(\Omega)} \leq \|\mathbf{u}\|_{L^6(\Omega)} \|T\|_{H^1(\Omega)},$$

and

$$\mathcal{B}(T) \in L^{3/2}(\Omega) \subset L^{6/5}(\Omega) \subset V' \quad \text{with } \|\mathcal{B}(T)\|_{L^{3/2}(\Omega)} \leq \kappa_M \bar{\sigma} \|T_+\|_{L^6(\Omega)}^4.$$

Since \mathbf{u} is a divergence free vector in $H^1(\Omega)$ satisfying $\mathbf{u} \cdot \mathbf{n} = 0$ on $\partial\Omega$, the bilinear functional

$$H^1(\Omega) \times H^1(\Omega) \ni (T, \phi) \mapsto \int_{\Omega} \phi \mathbf{u} \cdot \nabla T \, d\mathbf{x} \in \mathbf{R}$$

is skew-symmetric, and $\mathcal{B}(T(x)) = 0$ if $T(x) \leq 0$ by definition, so that

$$\langle \mathcal{A}_\varepsilon T, T \rangle_{V', V} = \varepsilon \|T\|_{L^2(\Omega)}^2 + \|\nabla T\|_{L^2(\Omega)}^2 + \int_{\Omega} \mathcal{B}(T) T \, d\mathbf{x} \geq \varepsilon \|T\|_{H^1(\Omega)}^2.$$

Hence \mathcal{A}_ε is coercive on V . Besides, for all $T_1, T_2 \in H^1(\Omega)$

$$\begin{aligned} \langle \mathcal{A}_\varepsilon T_1 - \mathcal{A} T_2, T_1 - T_2 \rangle_{V', V} &= \varepsilon \|T_1 - T_2\|_{L^2(\Omega)}^2 + \|\nabla(T_1 - T_2)\|_{L^2(\Omega)}^2 \\ &\quad + \int_{\Omega} (T_1 - T_2)(\mathcal{B}(T_1) - \mathcal{B}(T_2)) \, d\mathbf{x} \geq 0. \end{aligned}$$

³²¹ Theorem 1 in [19], implies the desired existence result for each $\varepsilon \in (0, 1)$.

Then, since $R \geq 0$ a.e. on Ω , one has $RT_\varepsilon \leq RT_{\varepsilon+}$ a.e. on Ω , and therefore

$$\begin{aligned} \varepsilon \|T_\varepsilon\|_{L^2(\Omega)}^2 + \|\nabla T_\varepsilon\|_{L^2(\Omega)}^2 + \bar{\sigma} \kappa_m (1 - a_M) \int_{\Omega} T_\varepsilon(\mathbf{x})_+^5 \, d\mathbf{x} &\leq \langle \mathcal{A}_\varepsilon T, T \rangle_{V', V} \\ &\leq \int_{\Omega} R(\mathbf{x}) T_\varepsilon(\mathbf{x})_+ \, d\mathbf{x} \leq \|R\|_{L^{6/5}(\Omega)} \|T_{\varepsilon+}\|_{L^6(\Omega)} \leq C_S \|R\|_{L^{6/5}(\Omega)} \|T_{\varepsilon+}\|_{H^1(\Omega)}. \end{aligned}$$

By Hölder's inequality

$$\int_{\Omega} T_\varepsilon(\mathbf{x})_+^5 \, d\mathbf{x} \geq \frac{1}{|\Omega|^{3/2}} \|T_{\varepsilon+}\|_{L^2(\Omega)}^5,$$

and since $\|\nabla T_{\varepsilon+}\|_{L^2(\Omega)} \leq \|\nabla T_\varepsilon\|_{L^2(\Omega)}$, we see that

$$\|\nabla T_\varepsilon\|_{L^2(\Omega)}^2 + \frac{\bar{\sigma} \kappa_m (1 - a_M)}{|\Omega|^{3/2}} \|T_{\varepsilon+}\|_{L^2(\Omega)}^5 \leq C_S \|R\|_{L^{6/5}(\Omega)} \left(\|T_{\varepsilon+}\|_{L^2(\Omega)}^2 + \|\nabla T_\varepsilon\|_{L^2(\Omega)}^2 \right)^{\frac{1}{2}}$$

so that

$$\sup_{0 < \varepsilon < 1} (\|\nabla T_\varepsilon\|_{L^2(\Omega)} + \|T_{\varepsilon+}\|_{L^2(\Omega)}) < \infty.$$

By the Banach-Alaoglu and the Rellich theorems, there exists a subsequence of T_ε (still denoted T_ε for simplicity) such that

$$T_{\varepsilon+} \rightarrow T_+ \quad \text{in } L^p(\Omega) \quad \text{and} \quad \nabla T_\varepsilon \rightarrow \nabla T \quad \text{weakly in } L^2(\Omega)$$

for all $p \in [1, 6)$ while $\varepsilon^{1/2} T_\varepsilon$ is bounded in $L^2(\Omega)$. Hence, for each $\phi \in H^1(\Omega)$, one has

$$\begin{aligned} 0 &= \int_{\Omega} (\varepsilon T_\varepsilon \phi + \nabla T_\varepsilon \cdot \nabla \phi + \phi \mathbf{u} \cdot \nabla T_\varepsilon + \mathcal{B}(T_\varepsilon) \phi) \, d\mathbf{x} \\ &\rightarrow \int_{\Omega} (\nabla T \cdot \nabla \phi + \phi \mathbf{u} \cdot \nabla T + \mathcal{B}(T) \phi) \, d\mathbf{x} =: \langle \mathcal{A} T, \phi \rangle_{V', V} \end{aligned}$$

in the limit as $\varepsilon \rightarrow 0$, so that T is a weak solution of

$$-\Delta T + \mathbf{u} \cdot \nabla T + \mathcal{B}(T) = R, \quad \frac{\partial T}{\partial n} \Big|_{\partial\Omega} = 0.$$

Observe that

$$\langle \mathcal{A}T - \mathcal{A}T', (T - T')_+ \rangle_{V',V} = \|\nabla(T - T')_+\|_{L^2(\Omega)}^2 + \int_{\Omega} (\mathcal{B}(T) - \mathcal{B}(T'))(T - T')_+ d\mathbf{x} \geq 0,$$

since

$$\int_{\Omega} (T - T')_+ \mathbf{u} \cdot \nabla(T - T') d\mathbf{x} = \int_{\Omega} \mathbf{u} \cdot \nabla \frac{1}{2} (T - T')_+^2 d\mathbf{x} = \int_{\partial\Omega} \frac{1}{2} (T - T')_+^2 \mathbf{u} \cdot n d\sigma(\mathbf{x}) = 0,$$

denoting by $d\sigma(\mathbf{x})$ the surface element on $\partial\Omega$. Hence

$$R \leq R' \text{ a.e. on } \Omega \implies \langle (R - R'), (T - T')_+ \rangle_{V',V} = \|\nabla(T - T')_+\|_{L^2(\Omega)} = 0.$$

322 Since Ω is connected, $(T - T')_+ = c$ a.e. on Ω for some constant $c \geq 0$.

A first consequence of this remark is that, if $R' \geq 0$ a.e. on Ω , weak solutions of

$$-\Delta T' + \mathbf{u} \cdot \nabla T' + \mathcal{B}(T') = R', \quad \frac{\partial T'}{\partial n} \Big|_{\partial\Omega} = 0$$

satisfy

$$T' \geq 0 \text{ a.e. on } \Omega, \quad \text{unless } R' = 0 \text{ a.e. on } \Omega, \quad \text{in which case } T' = \text{Const.} \leq 0.$$

A second consequence is that, if $R' \geq R \geq 0$, with $|\{x \in \Omega \text{ s.t. } R \geq 0\}| > 0$, the solutions T and T' of

$$-\Delta T + \mathbf{u} \cdot \nabla T + \mathcal{B}(T) = R, \quad \frac{\partial T}{\partial n} \Big|_{\partial\Omega} = 0,$$

satisfy $T \geq 0$ and $T' \geq 0$, and $(T - T')_+ = c$ a.e. on Ω for some constant $c \geq 0$. Besides

$$\begin{aligned} 0 &= \langle R - R', (T - T')_+ \rangle_{V',V} = \langle \mathcal{A}T - \mathcal{A}T', (T - T')_+ \rangle_{V',V} = \|\nabla(T - T')_+\|_{L^2(\Omega)}^2 \\ &+ \int_{\Omega} (\mathcal{B}(T) - \mathcal{B}(T'))(T - T')_+ d\mathbf{x} = c \int_{\Omega} (\mathcal{B}(T' + c) - \mathcal{B}(T')) d\mathbf{x}. \end{aligned}$$

Since $T' \geq 0$ a.e. on Ω , and since \mathcal{B} is increasing, this implies that $c = 0$. Therefore

$$R' \geq R \geq 0 \text{ with } |\{x \in \Omega \text{ s.t. } R \geq 0\}| > 0 \implies (T - T')_+ = 0.$$

323 Hence $T \leq T'$ a.e. on Ω . □

324

Proof [Proof of [Theorem 6.1](#)] For the sake of clarity, we systematically omit the tangential variables x, y in the integral equations for the averaged radiative intensity J_{ν}^n (as well as for the radiative intensity I_{ν} itself), since these variables are only parameters in all these formulas. Start from

$$T^0 \equiv 0, \quad J_{\nu}^0(z) = S_{\nu}(z) > 0.$$

Construct iteratively $(T^n, J_\nu^n)_{n \geq 0}$ by the following recursion formula: first, compute

$$J_\nu^{n+1}(z) = S_\nu(z) + \frac{1}{2} \int_0^Z \kappa_\nu E(\kappa_\nu |z - t|) (a_\nu J_\nu^n(t) + (1 - a_\nu) B_\nu(T^n(t))) dt;$$

325 and then let T^{n+1} be the solution of

$$(6.9) \quad -\Delta T^{n+1} + \mathbf{u} \cdot \nabla T^{n+1} + \mathcal{B}(T^{n+1}) = \int_0^\infty \kappa_\nu (1 - a_\nu) J_\nu^{n+1} d\nu, \quad \left. \frac{\partial T^{n+1}}{\partial n} \right|_{\partial\Omega} = 0.$$

Obviously $J_\nu^1 \geq J_\nu^0 > 0$, and applying Lemma 6.2 implies that $T^1 \geq T^0$ a.e. on Ω . Moreover

$$T^n \geq T^{n-1} \quad \text{and} \quad J_\nu^n \geq J_\nu^{n-1} > 0 \implies J_\nu^{n+1} \geq J_\nu^n > 0,$$

326 and applying the Lemma 6.2 shows that $T^{n+1} \geq T^n$ a.e. on Ω .

Assume that $Q_\nu^\pm(\mu) \leq B_\nu(T_M)$. It will be more convenient to deal with radiative intensities I_ν instead of their angle-averaged variants J_ν . Therefore, we define I_ν^n to be the solution of

$$\begin{aligned} (\mu \partial_z + \kappa_\nu) I_\nu^{n+1} &= \kappa_\nu (1 - a_\nu) B_\nu(T^n) + \kappa_\nu a_\nu J_\nu^n, & J_\nu^n &= \tilde{I}_\nu^n, \\ I_\nu^{n+1}(Z, -\mu) &= Q_\nu^-(-\mu), & I_\nu^{n+1}(0, +\mu) &= Q_\nu^+(+\mu), \quad 0 < \mu < 1. \end{aligned}$$

Let us prove by induction that

$$\begin{aligned} I_\nu^n &\leq B_\nu(T_M) \text{ a.e. on } \Omega \times (-1, 1) \times (0, +\infty), \\ J_\nu^n &\leq B_\nu(T_M) \text{ a.e. on } \Omega \times (0, +\infty), \quad T^n \leq T_M \text{ a.e. on } \Omega. \end{aligned}$$

This is true for $n = 0$ since $T^0 \equiv 0$, while

$$\begin{aligned} I_\nu^0(z, \mu) &= \mathbf{1}_{0 < \mu < 1} e^{-\kappa_\nu z / \mu} Q_\nu^+(\mu) + \mathbf{1}_{0 < -\mu < 1} e^{-\kappa_\nu (Z - z) / |\mu|} Q_\nu^-(-\mu) \\ &\leq (\mathbf{1}_{0 < \mu < 1} + \mathbf{1}_{0 < -\mu < 1}) B_\nu(T_M), \quad \text{so that } 0 \leq J_\nu^0 \leq B_\nu(T_M). \end{aligned}$$

If this is true for some $n \geq 0$, then

$$\begin{aligned} (\mu \partial_z + \kappa_\nu) I_\nu^{n+1} &= \kappa_\nu \Sigma_\nu^n, & 0 \leq \Sigma_\nu^n &\leq B_\nu(T_M), \\ I_\nu^{n+1}(Z, -\mu) \Big|_{0 < \mu < 1} &= Q_\nu^-(-\mu), & I_\nu^{n+1}(0, +\mu) \Big|_{0 < \mu < 1} &= Q_\nu^+(+\mu). \end{aligned}$$

Thus, proceeding as (5.8) shows that $I_\nu^{n+1} \leq B_\nu(T_M)$. Hence $J_\nu^{n+1} \leq B_\nu(T_M)$, and one solves (6.9) for T^{n+1} . Since $J_\nu^n \geq S_\nu > 0$ and

$$\int_0^\infty \kappa_\nu (1 - a_\nu) J_\nu^{n+1} d\nu \leq \int_0^\infty \kappa_\nu (1 - a_\nu) B_\nu(T_M) d\nu = \mathcal{B}(T_M),$$

we conclude from Lemma 6.2 that T^{n+1} is a.e. less than or equal to the solution of the problem

$$-\Delta T + \mathbf{u} \cdot \nabla T + \mathcal{B}(T) = \mathcal{B}(T_M), \quad \left. \frac{\partial T}{\partial n} \right|_{\partial\Omega} = 0,$$

327 which is obviously the constant T_M . Hence $T^{n+1} \leq T_M$ a.e. on Ω , so that we have
328 proved by induction the desired chain of inequalities.

329 From these inequalities, we conclude that the sequences J_ν^n and T^n converge a.e.
 330 pointwise on $\Omega \times (0, \infty)$ and on Ω respectively to limits denoted J_ν and T , and that this
 331 convergence also holds in $L^p(\Omega \times (0, \infty))$ and $L^p(\Omega)$ for all $p \in [1, \infty)$ by dominated
 332 convergence.

333 Passing to the limit in (6.7) immediately shows that J_ν, T satisfy the first equation
 334 in (6.5). As for the second equation, one can pass to the limit in the right hand side
 335 and in the nonlinear term on the left hand side of (6.8). Since T^{n+1} is a weak solution
 336 of (6.8), one has $T^{n+1} \in H^1(\Omega)$ and

$$(6.10) \quad \int_{\Omega} \nabla T^{n+1}(\mathbf{x}) \cdot \nabla \phi(\mathbf{x}) d\mathbf{x} - \int_{\Omega} T^{n+1}(\mathbf{x}) \mathbf{u}(\mathbf{x}) \cdot \nabla \phi(\mathbf{x}) d\mathbf{x} = \int_{\Omega} h_{n+1}(\mathbf{x}) \phi(\mathbf{x}) d\mathbf{x}$$

for all $\phi \in H^1(\Omega)$, with

$$h_{n+1} := \int_0^\infty \kappa_\nu (1 - a_\nu) (J_\nu^{n+1} - B_\nu(T^{n+1})) d\nu$$

so that h_{n+1} is bounded in $L^p(\Omega)$ for all $p \in [1, \infty)$. Taking $\phi = T^{n+1}$, and observing that

$$\int_{\Omega} T^{n+1}(\mathbf{x}) \mathbf{u}(\mathbf{x}) \cdot \nabla T^{n+1}(\mathbf{x}) d\mathbf{x} = \int_{\partial\Omega} \frac{1}{2} T^{n+1}(\mathbf{x})^2 \mathbf{u}(\mathbf{x}) \cdot \mathbf{n}_x d\sigma(\mathbf{x}) = 0$$

337 since $\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0$ shows that T^{n+1} is bounded, and therefore weakly relatively
 338 compact in $H^1(\Omega)$. Since we already know that $T^{n+1} \rightarrow T$ in $L^p(\Omega)$ for all $p \in [1, \infty)$
 339 as $n \rightarrow \infty$, we conclude that $T^{n+1} \rightarrow T$ weakly in $H^1(\Omega)$. At this point, we can pass
 340 to the limit in the weak formulation of (6.10), and this shows that T satisfies the
 341 second equation in (6.5). \square

342

Next we discuss the convergence rate of (6.7). We shall use the monotonic structure of the radiative transfer equations. Consider the upper approximating sequence

$$\begin{aligned} \mu \partial_z H_\nu^n &= \kappa_\nu (a_\nu K_\nu^{n-1} + (1 - a_\nu) B_\nu(\Theta^{n-1}) - H_\nu^n), \quad K_\nu = \frac{1}{2} \int_{-1}^1 H_\nu d\mu, \\ \mathbf{u} \cdot \nabla \Theta^n - \Delta \Theta^n &= \int_0^\infty \kappa_\nu (1 - a_\nu) (K_\nu^n - B_\nu(\Theta^n)) d\nu, \\ H_\nu^n(0, \mu) &= Q_\nu^+(\mu), \quad H_\nu^n(Z, -\mu) = Q_\nu^-(\mu), \quad 0 < \mu < 1, \quad \frac{\partial \Theta^n}{\partial n} \Big|_{\partial\Omega} = 0, \end{aligned}$$

343 for all $n \geq 1$, initialized with $\Theta^0 = T_M$ and $H_\nu^0 = K_\nu^0 = B_\nu(\Theta^0)$.

344 **THEOREM 6.3.** *Assume that the absorption coefficient κ_ν and the scattering albedo*
 345 *a_ν satisfy (6.1). Assume moreover that the constant C_1 defined in (2.18) satisfies*

$$(6.11) \quad 0 \leq \gamma := \left(\sup_{\nu > 0} (1 - a_\nu) C_1(\kappa_\nu) + \sup_{\nu > 0} a_\nu C_1(\kappa_\nu) \right) < 1.$$

Let the boundary source terms Q_ν^\pm satisfy the bound

$$0 \leq Q_\nu^\pm(\mu) \leq B_\nu(T_M), \quad 0 < \mu < 1, \quad \nu > 0.$$

346 Then one has

$$\begin{aligned}
(6.12) \quad & 0 \leq T^0 \leq \dots \leq T^{n-1} \leq \Theta^n \leq \dots \Theta^1 \leq T_M, \\
& 0 \leq J_\nu^0 \dots \leq J_\nu^{n-1} \leq K_\nu^n \leq \dots \leq K_\nu^1 \leq B_\nu(T_M); \\
& \|\mathcal{B}(T^{n+1}) - \mathcal{B}(T^n)\|_{L^1(\Omega)} \leq \|\mathcal{B}(\Theta^{n+1}) - \mathcal{B}(T^n)\|_{L^1(\Omega)} \leq \gamma^n |\Omega| \mathcal{B}(T_M), \\
& \|J_\nu^{n+1} - J_\nu^n\|_{L^1(\Omega \times (0, +\infty))} \leq \|K_\nu^{n+1} - J_\nu^n\|_{L^1(\Omega \times (0, +\infty))} \leq \frac{\gamma^n |\Omega| \mathcal{B}(T_M)}{\kappa_m(1 - a_M)}; \\
& \|\mathcal{B}(T) - \mathcal{B}(T^n)\|_{L^1(\Omega)} \leq \frac{\gamma^n}{1 - \gamma} |\Omega| \mathcal{B}(T_M), \\
& \|J_\nu - J_\nu^n\|_{L^1(\Omega \times (0, +\infty))} \leq \frac{\gamma^n |\Omega| \mathcal{B}(T_M)}{\kappa_m(1 - a_M)(1 - \gamma)}.
\end{aligned}$$

Proof First, one has

$$\begin{aligned}
& \mu \partial_z H_\nu^1 + \kappa_\nu H_\nu^1 = \kappa_\nu B_\nu(T_M) \geq 0, \quad 0 < z < Z, \\
& 0 \leq H_\nu^1(0, +\mu) = Q_\nu^+(\mu) \leq B_\nu(T_M), \quad 0 < \mu < 1, \\
& 0 \leq H_\nu^1(Z, -\mu) = Q_\nu^-(\mu) \leq B_\nu(T_M), \quad 0 < \mu < 1, \\
& \implies H_\nu^1(z, \mu) = 1_{0 < \mu < 1} \left(e^{-\kappa_\nu z / \mu} Q_\nu^+(\mu) + (1 - e^{-\kappa_\nu z / \mu}) B_\nu(T_M) \right) \\
& \quad + 1_{0 < -\mu < -1} \left(e^{-\kappa_\nu(Z-z)/|\mu|} Q_\nu^-(\mu) + (1 - e^{-\kappa_\nu(Z-z)/\mu}) B_\nu(T_M) \right) \\
& 0 \leq I_\nu^0 \leq H_\nu^1 \leq B_\nu(T_M), \quad 0 \leq J_\nu^0 \leq K_\nu^1 \leq B_\nu(T_M).
\end{aligned}$$

Hence

$$\mathcal{B}(\Theta^1) + \mathbf{u} \cdot \nabla \Theta^1 - \Delta \Theta^1 = \int_0^\infty \kappa_\nu (1 - a_\nu) K_\nu^1 d\nu \leq \mathcal{B}(T_M),$$

so that $0 \leq T^0 \leq \Theta^1 \leq T_M$ by [Lemma 6.2](#). The same induction argument as in the proof of [Theorem 6.1](#) shows that

$$\begin{aligned}
& 0 \leq \dots \leq \Theta^n \leq \Theta^{n-1} \leq T_M, \\
& 0 \leq \dots \leq H_\nu^n \leq H_\nu^{n-1} \leq B_\nu(T_M), \quad 0 \leq \dots \leq K_\nu^n \leq K_\nu^{n-1} \leq B_\nu(T_M).
\end{aligned}$$

Moreover, assume that we have proved that

$$\begin{aligned}
& 0 \leq T^0 \leq \dots \leq T^{n-1} \leq \Theta^n \leq \dots \Theta^1 \leq T_M, \\
& 0 \leq I_\nu^0 \leq \dots \leq I_\nu^{n-1} \leq H_\nu^n \leq \dots H_\nu^1 \leq B_\nu(T_M), \\
& 0 \leq J_\nu^0 \dots \leq J_\nu^{n-1} \leq K_\nu^n \leq \dots \leq K_\nu^0 \leq B_\nu(T_M).
\end{aligned}$$

Then

$$\begin{aligned}
& \mu \partial_z (H_\nu^{n+1} - I_\nu^n) + \kappa_\nu (H_\nu^{n+1} - I_\nu^n) = \kappa_\nu a_\nu (K_\nu^n - J_\nu^{n-1}) \\
& \quad + \kappa_\nu (1 - a_\nu) (B_\nu(\Theta^n) - B_\nu(T^{n-1})) \geq 0, \\
& (H_\nu^{n+1} - I_\nu^n)(0, +\mu) = (H_\nu^{n+1} - I_\nu^n)(Z, -\mu) = 0, \quad 0 < \mu < 1,
\end{aligned}$$

so that $I_\nu^n \leq H_\nu^{n+1}$, and $J_\nu^n \leq K_\nu^{n+1}$. Then $\frac{\partial \Theta^{n+1}}{\partial n} \Big|_{\partial \Omega} = \frac{\partial T^n}{\partial n} \Big|_{\partial \Omega} = 0$ and

$$\begin{aligned}
& \mathcal{B}(\Theta^{n+1}) + \mathbf{u} \cdot \nabla \Theta^{n+1} - \Delta \Theta^{n+1} = \int_0^\infty \kappa_\nu (1 - a_\nu) K_\nu^{n+1} d\nu, \\
& \mathcal{B}(T^n) + \mathbf{u} \cdot \nabla T^n - \Delta T^n = \int_0^\infty \kappa_\nu (1 - a_\nu) J_\nu^n d\nu,
\end{aligned}$$

and [Lemma 6.2](#) implies that $T^n \leq \Theta^{n+1}$. Hence we have proved by induction that,

$$\begin{aligned} 0 &\leq T^0 \leq \dots \leq T^{n-1} \leq \Theta^n \leq \dots \leq \Theta^1 \leq T_M, \\ 0 &\leq I_\nu^0 \leq \dots \leq I_\nu^{n-1} \leq H_\nu^n \leq \dots \leq H_\nu^1 \leq B_\nu(T_M), \\ 0 &\leq J_\nu^0 \leq \dots \leq J_\nu^{n-1} \leq K_\nu^n \leq \dots \leq K_\nu^1 \leq B_\nu(T_M), \text{ for all } n \geq 1, \end{aligned}$$

³⁴⁷ which implies the two first chains of inequalities in [\(6.12\)](#).

Then

$$\begin{aligned} &\mathcal{B}(\Theta^{n+1}) - \mathcal{B}(T^n) + \mathbf{u} \cdot \nabla(\Theta^{n+1} - T^n) - \Delta(\Theta^{n+1} - T^n) \\ &= \int_0^\infty \kappa_\nu(1 - a_\nu)(K_\nu^{n+1} - J_\nu^n) d\nu, \quad \frac{\partial(\Theta^{n+1} - T^n)}{\partial n} \Big|_{\partial\Omega} = 0, \\ \implies &\int_\Omega (\mathcal{B}(\Theta^{n+1}) - \mathcal{B}(T^n)) d\mathbf{x} = \int_\Omega \int_0^\infty \kappa_\nu(1 - a_\nu)(K_\nu^{n+1} - J_\nu^n) d\nu d\mathbf{x}, \end{aligned}$$

because
$$\int_{\partial\Omega} \left((\Theta^{n+1} - T^n) \mathbf{u} \cdot \mathbf{n}_\mathbf{x} - \frac{\partial(\Theta^{n+1} - T^n)}{\partial n} \right) d\sigma(\mathbf{x}) = 0.$$

Then

$$\begin{aligned} &K_\nu^{n+1}(\mathbf{x}) - J_\nu^n(\mathbf{x}) \\ &= \frac{1}{2} \int_0^Z \kappa_\nu E_1(\kappa_\nu |z - \zeta|) (1 - a_\nu) (B_\nu(\Theta^n) - B_\nu(T^{n-1}))(x, y, \zeta) d\zeta \\ &\quad + \frac{1}{2} \int_0^Z \kappa_\nu E_1(\kappa_\nu |z - \zeta|) a_\nu (K_\nu^n - J_\nu^{n-1})(x, y, \zeta) d\zeta. \\ \implies \quad \epsilon_n &:= \int_\Omega \int_0^\infty \kappa_\nu(1 - a_\nu)(K_\nu^{n+1} - J_\nu^n) d\nu d\mathbf{x} = \frac{1}{2} \int_\mathbb{O} dx dy \int_0^\infty d\nu \int_0^Z dz \int_0^Z \\ &\quad \kappa_\nu^2 E_1(\kappa_\nu |z - \zeta|) \cdot (1 - a_\nu)^2 (B_\nu(\Theta^n) - B_\nu(T^{n-1}))(x, y, \zeta) d\zeta \\ &+ \frac{1}{2} \int_\mathbb{O} dx dy \int_0^\infty d\nu \int_0^Z dz \int_0^Z \kappa_\nu^2 E_1(\kappa_\nu |z - \zeta|) \cdot (1 - a_\nu) a_\nu (K_\nu^n - J_\nu^{n-1})(x, y, \zeta) d\zeta. \end{aligned}$$

At this point, we integrate first in z and use [\(2.18\)](#), to obtain

$$\begin{aligned} \epsilon_n &= \int_\Omega \int_0^\infty \kappa_\nu(1 - a_\nu)(K_\nu^{n+1} - J_\nu^n) d\nu d\mathbf{x} \\ &\leq \int_\mathbb{O} dx dy \int_0^\infty d\nu \int_0^Z C_1(\kappa_\nu) \kappa_\nu(1 - a_\nu)^2 (B_\nu(\Theta^n) - B_\nu(T^{n-1}))(x, y, \zeta) d\zeta \\ &\quad + \int_\mathbb{O} dx dy \int_0^\infty d\nu \int_0^Z C_1(\kappa_\nu) \kappa_\nu(1 - a_\nu) a_\nu (K_\nu^n - J_\nu^{n-1})(x, y, \zeta) d\zeta \\ &\leq \sup_{\nu>0} (1 - a_\nu) C_1(\kappa_\nu) \int_\Omega \int_0^\infty \kappa_\nu(1 - a_\nu) (B_\nu(\Theta^n) - B_\nu(T^{n-1}))(\mathbf{x}) d\nu d\mathbf{x} \\ &\quad + \sup_{\nu>0} a_\nu C_1(\kappa_\nu) \int_\Omega \int_0^\infty \kappa_\nu(1 - a_\nu) (K_\nu^n - J_\nu^{n-1})(\mathbf{x}) d\nu d\mathbf{x} \\ &\leq \sup_{\nu>0} (1 - a_\nu) C_1(\kappa_\nu) \int_\Omega (\mathcal{B}(\Theta^n) - \mathcal{B}(T^{n-1}))(\mathbf{x}) d\mathbf{x} \\ &\quad + \sup_{\nu>0} a_\nu C_1(\kappa_\nu) \int_\Omega \int_0^\infty \kappa_\nu(1 - a_\nu) (K_\nu^n - J_\nu^{n-1})(\mathbf{x}) d\nu d\mathbf{x} \\ &= \epsilon_{n-1} \left(\sup_{\nu>0} (1 - a_\nu) C_1(\kappa_\nu) + \sup_{\nu>0} a_\nu C_1(\kappa_\nu) \right). \end{aligned}$$

Hence $\epsilon_n \leq \epsilon_0 \gamma^n$ with $\gamma := (\sup_{\nu>0} (1 - a_\nu) C_1(\kappa_\nu) + \sup_{\nu>0} a_\nu C_1(\kappa_\nu)) \in [0, 1)$, while $\epsilon_0 \leq |\Omega| \mathcal{B}(T_M) < \infty$. Hence the sequence $(K_\nu^n, \Theta^n)_{n \geq 1}$ of upper approximations and the sequence (J_ν^n, T^n) of lower approximations provided by (6.7) are adjacent. In particular

$$\begin{aligned} \|\mathcal{B}(T^{n+1}) - \mathcal{B}(T^n)\|_{L^1(\Omega)} &= \int_{\Omega} (\mathcal{B}(T^{n+1}) - \mathcal{B}(T^n)) d\mathbf{x} \\ &\leq \int_{\Omega} (\mathcal{B}(\Theta^{n+1}) - \mathcal{B}(T^n)) d\mathbf{x} \leq \epsilon_0 \gamma^n \end{aligned}$$

for all $n \geq 1$, so that $\|\mathcal{B}(T) - \mathcal{B}(T^n)\|_{L^1(\Omega)} \leq \frac{\epsilon_0 \gamma^n}{1 - \gamma}$. Similarly

$$\begin{aligned} &\int_{\Omega} \int_0^\infty \kappa_\nu (1 - a_\nu) (J_\nu^{n+1} - J_\nu^n) d\nu d\mathbf{x} \\ &\leq \int_{\Omega} \int_0^\infty \kappa_\nu (1 - a_\nu) (K_\nu^{n+1} - J_\nu^n) d\nu d\mathbf{x} \leq \epsilon_0 \gamma^n, \\ \kappa_m (1 - a_M) \|J_\nu - J_\nu^n\|_{L^1(\Omega \times (0, \infty))} &\leq \sum_{m \geq n} \int_{\Omega} \int_0^\infty \kappa_\nu (1 - a_\nu) (J_\nu^{m+1} - J_\nu^m) d\nu d\mathbf{x} \\ &\leq \sum_{m \geq n} \int_{\Omega} \int_0^\infty \kappa_\nu (1 - a_\nu) (K_\nu^{m+1} - J_\nu^m) d\nu d\mathbf{x} \leq \frac{\epsilon_0 \gamma^n}{1 - \gamma}. \end{aligned}$$

348 This concludes the proof of the convergence statements in (6.12). \square

349

350 *Remark 6.4.* The condition $\sup_{\nu>0} (1 - a_\nu) C_1(\kappa_\nu) < 1$ implies that the absorption-
 351 emission nonlinearity is a contraction, while $\sup_{\nu>0} a_\nu C_1(\kappa_\nu) < 1$ implies that the
 352 scattering term is also a contraction. The condition $\gamma < 1$ implies that these two
 353 terms are contractions separately, leading to the exponential rate in Theorem 6.3 (3).
 354 As $a_\nu \in [0, 1]$ and $\kappa_\nu \mapsto C_1(\kappa_\nu)$ is monotone increasing from 0 to 1, for a given a_ν
 355 there is always a κ^* such that (6.11) holds for all $\kappa_\nu < \kappa^*$. Conversely, if it is known
 356 that $\kappa_\nu < \kappa^*$, for some κ^* , for all ν , there is a maximum a^* for which (6.11) for all
 357 $a_\nu < a^*$. By Lemma 2.1, $C_1 < 1$. Hence $\gamma < 1$ if a_ν is independent of ν , whatever the
 358 upper bound κ_M in (6.1). The more a_ν varies between 0 and 1, the lower κ_M must
 359 be to satisfy $\gamma < 1$.

360 With the monotonic structure of the radiative transfer equations, our argument
 361 will also provide the uniqueness of the solution of the system (6.2)-(6.3)-(6.4).

THEOREM 6.5. *Under the same assumptions as in Theorem 6.3, there exists at most one solution (I_ν, T) of the problem (6.2)-(6.3)-(6.4) such that $T \in L^\infty(\Omega)$,*

$$I_\nu \geq 0 \text{ a.e. on } \Omega \times (-1, 1) \times (0, \infty) \quad \text{and} \quad T \geq 0 \text{ a.e. on } \Omega.$$

Proof Let (I_ν, T) be a solution of (6.2)-(6.3)-(6.4), and assume that the upper approximating sequence $(H_\nu^n, \Theta^n)_{n \geq 1}$ satisfies $I_\nu \leq H_\nu^n$ and $J_\nu \leq K_\nu^n$, with $T \leq \Theta^n$. Then, one has

$$\begin{aligned} \mu \partial_z (H_\nu^{n+1} - I_\nu) + \kappa_\nu (H_\nu^{n+1} - I_\nu) &= \kappa_\nu a_\nu (K_\nu^n - J_\nu) \\ &\quad + \kappa_\nu (1 - a_\nu) (B_\nu(\Theta^n) - B_\nu(T)) \geq 0, \\ (H_\nu^{n+1} - I_\nu)(0, +\mu) &= (H_\nu^{n+1} - I_\nu)(Z, -\mu) = 0, \quad 0 < \mu < 1. \end{aligned}$$

Solving this equation for $(H_\nu^{n+1} - I_\nu)$ by the method of characteristics shows that $I_\nu \leq H_\nu^{n+1}$ and therefore $J_\nu \leq K_\nu^{n+1}$. Next, one has

$$\begin{aligned} & \mathcal{B}(\Theta^{n+1}) - \mathcal{B}(T) + \mathbf{u} \cdot \nabla(\Theta^{n+1} - T) - \Delta(\Theta^{n+1} - T) \\ &= \int_0^\infty \kappa_\nu(1 - a_\nu)(K_\nu^{n+1} - J_\nu) d\nu \geq 0, \quad \frac{\partial(\Theta^{n+1} - T)}{\partial n} \Big|_{\partial\Omega} = 0, \end{aligned}$$

so that $T \leq \Theta^{n+1}$ according to [Lemma 6.2](#).

It remains to check the initial step of this induction argument. Since $T \in L^\infty(\Omega)$, we pick $\Theta^0 = \max(T_M, \|T\|_{L^\infty(\Omega)})$ and $H_\nu^0 = K_\nu^0 = B_\nu(\Theta^0)$. Hence $T \leq \Theta^0$ by construction. Next we prove that $I_\nu \leq B_\nu(\Theta^0)$. Multiplying both sides of [\(6.2\)](#) by $s_+(I_\nu - B_\nu(\Theta^0))$, we repeat the argument of the proof of [Theorem 4.1](#):

$$\begin{aligned} & \partial_z \langle \mu(I_\nu - B_\nu(\Theta^0))_+ \rangle \\ &= -\langle \kappa_\nu(1 - a_\nu)(I_\nu - B_\nu(\Theta^0)) - (B_\nu(T) - B_\nu(\Theta^0)) \rangle s_+(I_\nu - B_\nu(\Theta^0)) \\ & - \langle \kappa_\nu a_\nu(I_\nu - B_\nu(\Theta^0)) - (J_\nu - B_\nu(\Theta^0)) \rangle s_+(I_\nu - B_\nu(\Theta^0)) = -D_1 - D_2. \end{aligned}$$

We have seen in the proof of [Theorem 4.1](#) that

$$\begin{aligned} D_2 &= \langle \kappa_\nu a_\nu(I_\nu - B_\nu(\Theta^0)) - (J_\nu - B_\nu(\Theta^0)) \rangle s_+(I_\nu - B_\nu(\Theta^0)) \\ &= \langle \kappa_\nu a_\nu(I_\nu - B_\nu(\Theta^0)) - (J_\nu - B_\nu(\Theta^0)) \rangle (s_+(I_\nu - B_\nu(\Theta^0)) - s_+(J_\nu - B_\nu(\Theta^0))) \geq 0. \end{aligned}$$

As for D_1 , observe that

$$D_1 = \langle \kappa_\nu(1 - a_\nu)((I_\nu - B_\nu(\Theta^0)) - (B_\nu(T) - B_\nu(\Theta^0))) \rangle (s_+(I_\nu - B_\nu(\Theta^0)) - s_+(T - \Theta^0))$$

which is positive by our assumption on T which implies that $s_+(T - \Theta^0) = 0$. Integrating on Ω , we conclude that

$$\int_{\mathbb{O}} \langle \mu_+(I_\nu - B_\nu(\Theta^0))_+ \rangle(x, y, Z) dx dy = \int_{\mathbb{O}} \langle \mu_-(I_\nu - B_\nu(\Theta^0))_+ \rangle(x, y, 0) dx dy = 0$$

and that $D_1 = D_2 = 0$ a.e. on Ω . Now, since $\kappa_\nu(1 - a_\nu) \geq \kappa_m(1 - a_M) > 0$, the condition $D_1 = 0$ implies that

$$\begin{aligned} & ((I_\nu - B_\nu(\Theta^0)) - (B_\nu(T) - B_\nu(\Theta^0))) (s_+(I_\nu - B_\nu(\Theta^0)) - s_+(T - \Theta^0)) = 0 \\ & \text{which implies in turn that } s_+(I_\nu - B_\nu(\Theta^0)) = s_+(T - \Theta^0) = 0 \end{aligned}$$

Hence $I_\nu \leq B_\nu(\Theta^0)$, which completes the proof of the initialization of our induction argument. Summarizing, we have proved that, if one chooses $\Theta^0 = \max(T_M, \|T\|_{L^\infty(\Omega)})$, the solution (I_ν, T) of [\(6.2\)](#)-[\(6.3\)](#)-[\(6.4\)](#) considered satisfies

$$I_\nu \leq H_\nu^n \leq H_\nu^{n-1} \leq \dots \leq H_\nu^0 = B_\nu(\Theta^0), \quad \text{while } T \leq \Theta^n \leq \Theta^{n-1} \leq \dots \leq \Theta^0,$$

where (H_ν^n, Θ^n) is the upper approximating sequence. A similar argument (with a slightly simpler initialization) shows that

$$I_\nu \geq I_\nu^n \geq I_\nu^{n-1} \geq \dots \geq I_\nu^0 = 0, \quad \text{while } T \geq T^n \geq T^{n-1} \geq \dots \geq T^0 = 0.$$

With this, we easily prove the uniqueness of the solution of [\(6.2\)](#)-[\(6.3\)](#)-[\(6.4\)](#). If (I_ν, T) and (I'_ν, T') are two solutions satisfying the assumptions of [Theorem 6.5](#), we initialize the upper approximating sequence with $\Theta^0 = \max(T_M, \|T\|_{L^\infty(\Omega)}, \|T'\|_{L^\infty(\Omega)})$.

The argument above shows that $I_\nu^n \leq I_\nu$, $I'_\nu \leq H_\nu^{n+1}$ while $T^n \leq T$, $T' \leq \Theta^{n+1}$. Hence

$$\begin{aligned} \|J_\nu - J'_\nu\|_{L^1(\Omega \times (0, \infty))} &\leq \|K_\nu^{n+1} - J_\nu^n\|_{L^1(\Omega \times (0, \infty))} \leq \frac{|\Omega| \gamma^n}{\kappa_m (1 - a_M)} \mathcal{B}(\Theta^0), \\ \|\mathcal{B}(T) - \mathcal{B}(T')\|_{L^1(\Omega)} &\leq \|\Theta^{n+1} - T^n\|_{L^1(\Omega)} \leq \gamma^n |\Omega| \mathcal{B}(\Theta^0). \end{aligned}$$

363 When $n \rightarrow \infty$ it shows that $T = T'$ a.e. on Ω and $J_\nu = J'_\nu$ a.e. on $\Omega \times (0, \infty)$. Once
 364 it is known that $J_\nu = J'_\nu$ a.e. on $\Omega \times (0, \infty)$, solving (6.2) for I_ν and I'_ν by the method
 365 of characteristics shows that $I_\nu = I'_\nu$ a.e. on $\Omega \times (-1, 1) \times (0, \infty)$. \square

366

367 Several remarks regarding Theorems Theorem 6.1, Theorem 6.3 and Theorem 6.5
 368 are in order.

369 **Remarks.**

- 370 (1) One can treat slightly more general situations with the same techniques. For
 371 instance, one could assume that the scattering rate a_ν depends on z , and is a slowly
 372 varying function of x, y . This may be useful to include a layer of clouds in our problem.
 373 Similarly, one can treat the case where ρ is not a constant, but for instance a function
 374 of z , by introducing an optical length defined as in (2.14). Typically, one could assume
 375 that $0 < \rho_m \leq \rho(z) \leq \rho_M < \infty$, and recast the radiative transfer equation in terms of
 376 the variable τ instead of z . Of course, this will modify the drift-diffusion operator in
 377 the left hand side of (6.3), but in a way that should be tractable by the same methods.
 378 (2) One could enrich the class of boundary conditions considered here by taking into
 379 account the albedo coefficients of the boundary at $z = 0$ and $z = Z$. This should
 380 lead to more serious modifications of the strategy discussed above, but we expect that
 381 some of our results can be modified to handle these more general boundary conditions.
 382 (3) Until now, we have treated the case of an incompressible fluid with constant
 383 density. This is the reason for the factor c_P/c_V multiplying the heat diffusivity. One
 384 can treat in the same manner the case of low Mach number flows of a compressible
 385 fluid which could be useful for the stratosphere (In the case of water at 20°C, one finds
 386 that $c_P/c_V = 1.007$, so that this ratio is very close to 1 for all practical purposes.)
 387 (4) Including Boussinesq's approximation in our model in order to take into account
 388 the buoyancy created by the temperature dependence of the density is a more difficult
 389 problem — in the first place because the motion equation of the fluid becomes coupled
 390 to the simple system considered here. We keep this problem for future work.

391 **7. Numerical Simulations.** This section is meant to show that iterations (3.2),
 392 (5.6) and (6.7), proposed in the previous sections, are monotone, implementable, ro-
 393 bust and computationally fairly fast. Here, robustness means that there are no singu-
 394 lar integrals and convergence is not subject to the adjustment of sensitive parameters;
 395 in other words, the mathematical properties derived above are observed numerically.

396 Two computer programs have been written: one in C++ with (3.2) or (5.6) for the
 397 case $\kappa_T = 0$ and the other in the FreeFEM language [17] with (6.7) for the general
 398 case, either in Cartesian coordinates (2D) or in spherical ones (3D).

399 The programming is straightforward except at three places:

- 400 1. Writing a function to compute the exponential integrals is simple due to two

401 formulas

$$(7.1) \quad \begin{aligned} E_1(x) &= -\gamma - \ln x + \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^k}{k k!}, & \gamma &= 0.577215664901533, \\ E_{n+1}(x) &= \frac{e^{-x}}{n} - \frac{x}{n} E_n(x), \end{aligned}$$

402 but the tail of the series falls below machine precision if $x > 18$. From
403 practical purpose keeping $9 + (\text{int}(x) - 1) \cdot 5$ terms in the series is more than
404 enough.

405 2. When thermal diffusion is neglected, one must solve for T , with J_ν given,

$$\int_0^\infty \kappa_\nu(1 - a_\nu) B_\nu(T) d\nu = \int_0^\infty \kappa_\nu(1 - a_\nu) J_\nu d\nu.$$

406 Newton iterations are used combined with dichotomy. The integrals are ap-
407 proximated with the trapezoidal rule on a mesh which is uniform in wave-
408 length with up to 900 points, though 300 are usually more than enough.

409 3. When thermal diffusion is not neglected, the temperature equation has a
410 similar nonlinearity which requires iterations. We use the time dependent
411 problem, discretized by a method of characteristics, as follows, which is un-
412 conditionally stable:

$$(7.2) \quad \begin{aligned} & \frac{1}{\delta t} (T^{m+1}(x) - T^m(x - \delta t u(x)) - \kappa_T \Delta T^{m+1} + \int_0^\infty \kappa_\nu(1 - a_\nu) B_\nu(T^{m+1}) d\nu \\ &= \int_0^\infty \kappa_\nu(1 - a_\nu) J_\nu d\nu, \end{aligned}$$

413 with Dirichlet or Neumann conditions on the boundaries. Then a standard
414 P^1 Finite Element approximation of the temperature equation is applied for
415 the discretization in a finite dimensional space V_h on a triangular (2D) or
416 tetrahedral (3D) mesh. Then the numerical approximation of T^{m+1} is also the
417 solution of the minimization problem below, T^m and J_ν given, which can be
418 solved by a BFGS method:

$$(7.3) \quad \begin{aligned} & \min_{T \in V_h} \int_\Omega \left[\frac{T^2}{2\delta t} + \frac{\kappa_T}{2} |\nabla T|^2 + \int_0^\infty \left(\kappa_\nu(1 - a_\nu) \int_0^T B_\nu(T') dT' \right) d\nu \right] dx \\ & - \int_\Omega T \left(\frac{1}{\delta T} T^m(x - \delta t u(x)) + \int_0^\infty \kappa_\nu(1 - a_\nu) J_\nu d\nu \right) dx. \end{aligned}$$

419 Speed-up can be achieved by using for initial value in BFGS, the temperature
420 computed by the Newton algorithm mentioned above with $\kappa_T = 0$.

421 The first set of tests are for the radiative transfer system decoupled from the temper-
422 ature equation. The second set of test involves the complete system in 2D and the
423 third is also with radiative transfer coupled with the temperature equation but in 3D.

424 7.1. Radiative Transfer in the Troposphere without Thermal Diffusion.

425 The troposphere is roughly 12km thick. When air density is $\rho(z) = \rho_0 e^{-z}$, with
426 $\rho_0 = 1.225 \cdot 10^{-3}$, a change of vertical coordinate is made, $\tau = 1 - e^{-z}$ to remove the
427 exponential from the equations; thus $\tau \in (0, Z)$ with $Z = 1 - e^{-12}$.

428 We wish to study the influence of κ_ν on T . As said earlier, $\bar{\kappa}_\nu$ is the mass-
 429 extinction coefficient and $\kappa_\nu = \rho_0 \bar{\kappa}_\nu$, is the absorption coefficient, defined as a di-
 430 mensionless parameter between 0 and 1 which measures the output to input ratio of
 431 ν -light crossing an horizontal unit length (here 1 km) of air layer. Note however that
 432 we are not restricted to $\kappa_\nu \in (0, 1)$ because of the following observation.

433 *Remark 7.1.* When Z is large, $T(\tau)$ computed by (3.2) with κ_ν is equal to $T(\frac{\tau}{L})$
 434 computed with by (3.2) with $\kappa_\nu L$.

435 Incidentally, it implies that if $\tau \mapsto T(\tau)$ is decreasing, increasing κ uniformly in ν will
 436 cause a uniform decrease of temperature.

437 The problem is: find $I_\nu(\tau, \mu)$ and $T(\tau)$ verifying (2.10), (2.12) and the boundary
 438 conditions used in [9]:

$$(7.4) \quad I(0, \mu)|_{\mu>0} = Q_\nu \mu, \quad I(Z, \mu)|_{\mu<0} = 0.$$

439 The first one implies that the Earth receives sunlight on its surface and that the
 440 computation does not include the effect of the atmosphere on the sun rays during
 441 their downward travel ($\mu < 0$). It is generally assumed that visible light is unaffected
 442 by air.

443 Due to Planck's law for black bodies, Earth radiates ($\mu > 0$) infrared radiations
 444 upward ; the second boundary condition says that these escapes at $\tau = Z$ without
 445 back-scattering.

446 The frequency spectrum of interest is $\nu \in (0, 20 \cdot 10^{14})$. It is convenient to rescale
 447 some variables:

$$\nu' = 10^{-14} \nu, \quad T' = 10^{-14} \frac{k}{h} T = 10^{-14} \frac{1.381 \cdot 10^{-23}}{6.626 \cdot 10^{-34}} T = \frac{T}{4798},$$

448 so as to write

$$B_\nu(T) = B_0 \frac{\nu'^3}{e^{\frac{\nu'}{T'}} - 1}, \quad \text{with } B_0 = \frac{2h}{c^2} 10^{42} = \frac{2 \times 6.626 \cdot 10^{-34}}{2.998^2 \cdot 10^{16}} 10^{42} = 1.4744 \cdot 10^{-8}.$$

449 We may work with B_ν/B_0 and I_ν/B_0 so that, forgetting the primes, we have (2.10)
 450 with (2.12) and (7.4) with

$$(7.5) \quad B_\nu(T) = \frac{\nu^3}{e^{\frac{\nu}{T}} - 1}, \quad Q_\nu = Q_0 B_\nu(1.209), \quad Q_0 = 2 \cdot 10^{-5},$$

451 because T_{Sun} being $5800^0 K$, it is now $5800/4798 = 1.209$; Q_0 is found from the
 452 sunlight energy sent to Earth, $Q_{sun} = 1370 \text{ Watt}/m^2$:

$$(7.6) \quad Q_{sun} = \int_0^\infty Q_0 B_0 B_\nu(1.209) 10^{14} d\nu = Q_0 1.4744 \cdot 10^6 \frac{(1.209\pi)^4}{15} = 1.023 \cdot 10^7 Q_0.$$

453 This leads to $Q_0 = 13.4 \cdot 10^{-5}$, but the Sun sees Earth as a disk of surface πR^2
 454 while the Earth surface reemitting radiations is $2\pi R^2$, so $6.7 \cdot 10^{-5}$ should be used
 455 instead. Yet this value is too high as it gives an Earth temperature around 400K. It
 456 comes down to 3.1 when it is corrected by the latitude, $\frac{1}{\sqrt{2}}$ at 45° , and by the Earth
 457 albedo: 35% of the Sun energy is reflected, i.e. not absorbed, by the Earth surface.
 458 Furthermore due to the alternation of days and nights only a portion of the final value
 459 should be retained [9]. Thus Q_0 is in the range $(1.5, 3) \cdot 10^{-5}$. A reasonable value is
 460 $Q_0 = 2 \cdot 10^{-5}$, because, with a constant $\kappa = 0.5$, the temperature near the ground is

461 found to be around 24°C ; but it should not be taken for its face value because rains,
 462 clouds etc, are not accounted for.

463 Scattering is the sum of an isotropic part and a Rayleigh part; both have their
 464 own a_{ν} , function of altitude (i.e. τ) and ν .

465 To simulate clouds, isotropic scattering is activated between altitude Z_1 and $Z_2 >$
 466 Z_1 and

$$a_{\nu}(z) = \alpha[4 \max(z - Z_1, 0) \max(Z_2 - z, 0)/(Z_2 - Z_1)^2].$$

467 It is known that Rayleigh scattering is a function of ν^4 in the ultraviolet range at
 468 high altitude, so it is switched on above altitude Z_2 and is $O(\nu^4)$ for $\nu \in (0.8, 1.2)$:

$$a'_{\nu}(z) = \alpha[40 \max(\nu - 0.8, 0)^2 \max(1.2 - \nu, 0)^2 \max(z - Z_2, 0)/(Z - Z_2)].$$

469 The values of the physical and numerical parameters are

- 470 • $\alpha = \frac{1}{2}$ or zero; , $Z_1 = 6\text{km}$, $Z_2 = 9\text{km}$
- 471 • Absorption coefficient κ_{ν} digitalized from Gemini measurements.
- 472 • Discretization: 60 altitude stations, 485 frequencies corresponding to a uni-
 473 form grid in wavelength in $(1,20)\mu\text{m}$.
- 474 • Number of iterations 20.

475 The Gemini measurements of the absorption are posted on wikipedia in

<https://www.gemini.edu/observing/telescopes-and-sites/sites#Transmission>

476 Figure 1 shows κ_{ν}^0 versus wavelength c/ν . Recall that visible light is in the range
 477 $0.4 - 0.7\mu\text{m}$ (i.e. 450-750 THz) and relevant infrared radiations are in the range
 478 $0.8 - 20\mu\text{m}$ (i.e. 0.03 - 0.4 THz).

479 To assess the sensitivity of the temperature to gas like carbon dioxide opaque,
 480 for wavelengths in $7-9\mu\text{m}$, and $1-3\mu\text{m}$ we constructed κ_{ν}^1 by increasing κ_{ν}^0 by a factor
 481 3, and capped at 1, in the infrared range $7 - 8\mu\text{m}$. Similarly we construct κ_{ν}^2 by
 482 increasing κ_{ν}^0 by a factor of 3 , and capped at 1, in the range $1 - 3\mu\text{m}$. These are
 483 displayed in Figure 1.

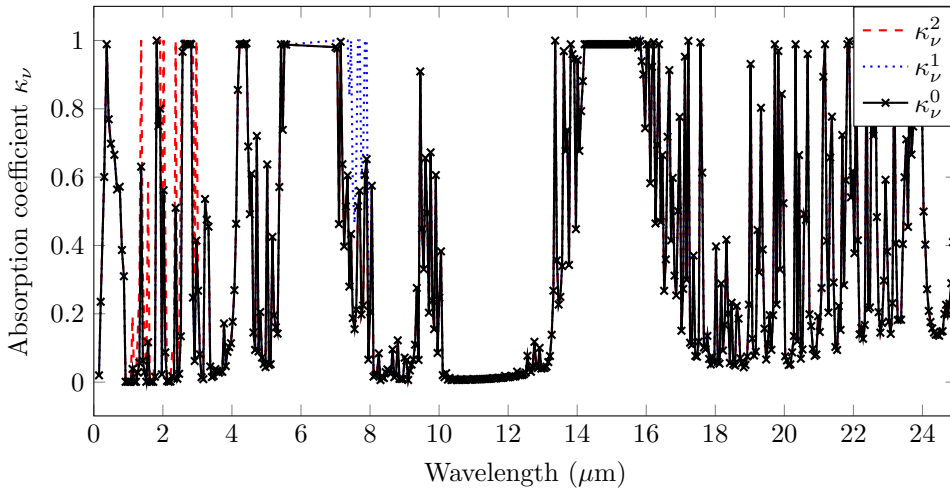


FIG. 1. Absorption κ_{ν}^0 versus wavelength ($3/\nu$) read from Gemini measurements; κ_{ν}^1 , is κ_{ν}^0 increased in the infrared range $2 - 3\mu\text{m}$ and κ_{ν}^2 is κ_{ν}^0 increased in the range $8 - 14\mu\text{m}$. The \times marks show the 487 grid points for the integrals in ν .

484 Convergence of the lower increasing and upper decreasing sequences is studied
 485 with and without Rayleigh scattering.

486 The convergence of the lower sequences is faster and it is slightly slower in the
 487 presence of scattering. Yet, for both 20, iterations seem appropriate for a 3 digits
 488 precision.

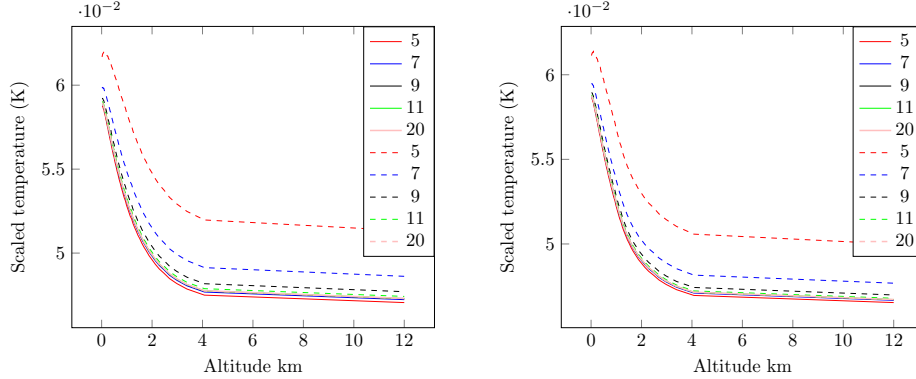


FIG. 2. *Temperatures scaled by 4798 without (left) and with (right) scattering: convergence history. The dashed curves are computed with an initial $T^0 = T_{Sun}/10$ and the solid curves with $T^0 = 0$. Notice the monotonic convergence towards a solution after 20 iterations. The iterations shown for the upper and lower solutions are (5,7,9,11,20). This computation has used $Q_0 = 3 \cdot 10^{-5}$.*

489 Next, results are shown with κ_ν^0 , κ_ν^1 and κ_ν^2 , with and without scattering. Figures
 490 3 and 4 shows the mean radiation intensity J_ν versus wavelength at altitude 0 and
 491 12km. Notice the dramatic changes when going from κ_ν^0 to κ_ν^1 and the smaller changes
 492 in the opposite direction when going from κ_ν^0 to κ_ν^2 . Note too that scattering decreases
 493 J_ν . It is also interesting to note that in the frequency range where κ_ν^0 is very small
 494 such as wavelength 3-4 μm and 10-14 μm , J_ν is also small; it is because the Planck
 495 function with the Earth temperature (3.2) cannot create ν -waves in regions where
 496 κ_ν is small.

497 Figure 5 shows the scaled temperatures versus altitude computed with κ_ν^0 , κ_ν^1 and
 498 κ_ν^2 with and without scattering. Note that going from κ_ν^0 to κ_ν^1 decreases the temper-
 499 atures by 5%. On the other hand going from κ_ν^0 to κ_ν^2 increases the temperatures by
 500 2%.

501 Comments.

- 502 • CPU time is 20" on an Macbook air M1, but with a smoother κ_ν , 50 ν -
 503 integration points are sufficient, cutting the CPU time by 10 to 2".
- 504 • We observed that a highly oscillating κ_ν did not cause any programming or
 505 convergence problems. The total light intensities J plotted on Figures 3 and
 506 4 show clearly that the method traces the small or large changes on κ_ν .
- 507 • Figure 2: Monotone convergence from below and from above is observed. The
 508 convergence from below, i.e. starting with $T^0 = 0$, is faster than the one from
 509 above, starting from $T = T_{sun}/10$, and it is slightly slower in the presence of
 510 scattering.
- 511 • Figure 5: Increasing κ_ν in the Earth infrared range can cause either an in-
 512 crease or a decrease of temperature, depending on the position of the change
 513 in the infrared spectrum.
- 514 • Isotropic and Rayleigh scattering did not change the above conclusion (see
 515 Figure 5).

516 Finally, note that the Earth albedo and the clouds seem to play an important role on

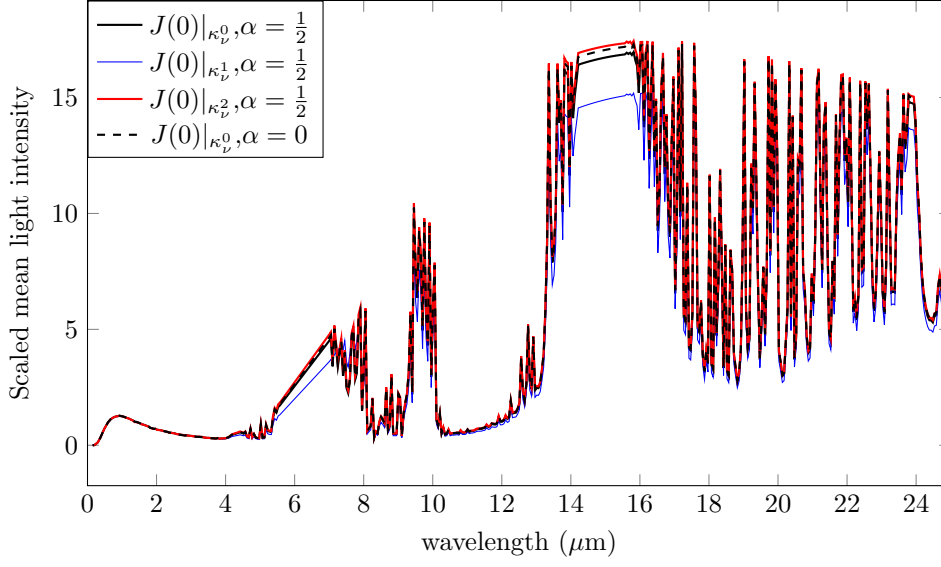


FIG. 3. Computed mean radiation intensities $10^5 \cdot J_\nu(0)$ at the ground level for κ_ν^0 , κ_ν^1 , κ_ν^2 with scattering ($\alpha = \frac{1}{2}$) and for κ_ν^0 without scattering.

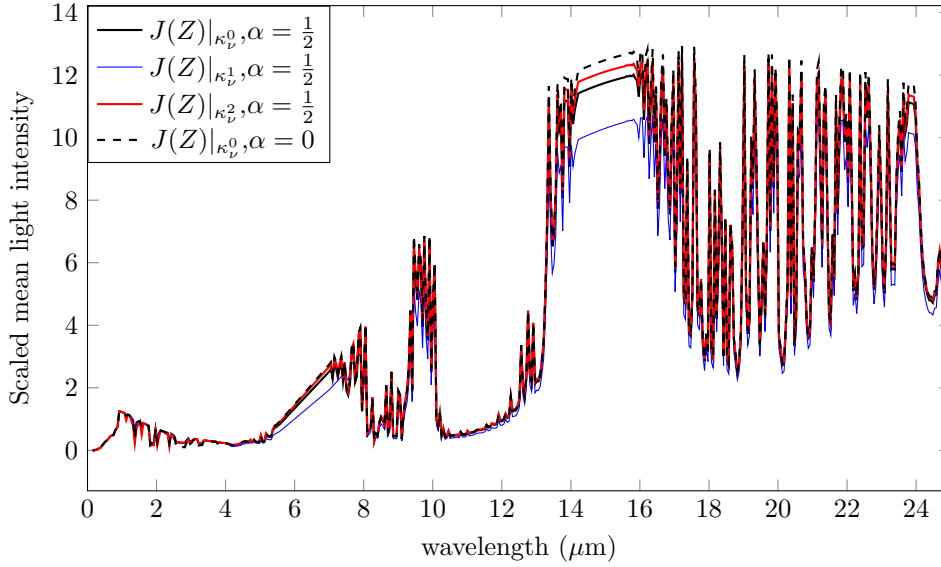


FIG. 4. Computed mean radiation intensities $10^5 \cdot J_\nu(Z)$ at the top of the troposphere for κ_ν^0 , κ_ν^1 , κ_ν^2 with scattering ($\alpha = \frac{1}{2}$) and for κ_ν^0 without scattering.

517 the effect of the greenhouse gases on the temperature of the atmosphere [8]. If it is
518 modeled by a Lambert condition of the type

$$I_\nu(0, \mu) - \beta I_\nu(0, -\mu) = \mu Q_0 B_\nu(T_{Sun}), \quad \forall \mu > 0,$$

519 then the present numerical method can handle it and our preliminary test show an
520 increase of temperature when β increases; while this is another story, it is yet another

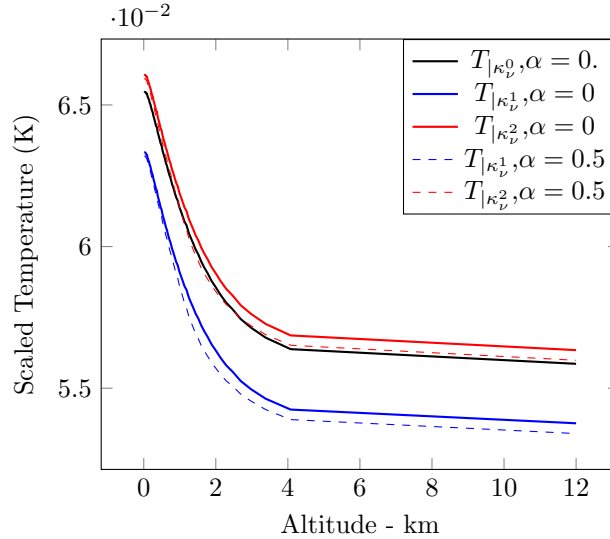


FIG. 5. Temperatures in Kelvin divided by 4798 versus altitude, computed with κ_ν^0 , κ_ν^1 and κ_ν^2 without scattering ($\alpha = 0$) and with a scattering $\alpha = \frac{1}{2}$.

521 proof of the versatility of the present numerical formulation for climate modeling.

522 **7.1.1. Relevance to Global Warming.** The simulations made above indicate
 523 that an increase of opacity in the atmosphere may cause cooling or warming depending
 524 on the range of frequencies where the change of opacity occurs. It is known that CO_2
 525 is opaque to wavelengths around $\lambda_1 = 2\mu\text{m}$ and around $\lambda_2 = 6\mu\text{m}$. According to
 526 Figure 1 the λ_1 peak heats the atmosphere and the λ_2 peak cools it. Cooling does
 527 not go against the physical observations because it is known that CO_2 cools the high
 528 atmosphere: see figure 13 in [8] and this Belgium website, for instance:

529 www.aeronomie.be/en/news/2021/rising-co2-levels-also-cause-cooling-upper-layers-atmosphere

530 What differentiates high and low altitudes? Clouds, for one thing, probably play
 531 a big role; also the absorption coefficient depends on the pressure, i.e. on altitude.
 532 The present formulation does not allow it, but it is not hard to see that by taking the
 533 greatest value for each frequency on the left hand side of (3.2) and compensate for
 534 the difference on the right hand side, the iterations on the source are still convergent.
 535 Thus there are many opportunities for future developments; we will show also, in [13],
 536 that the method is not confined to stratified atmospheres and that the full 3D problem
 537 can be solved by iterations on the source in an integral formulation; it is much more
 538 expensive computationally but still a lot cheaper than SHDOM and Monte-Carlo.

539 One should be cautious not to draw early conclusions before the full problem is
 540 solved; the purpose of the present study is to show that here is a method which is
 541 mathematically well understood and numerically faster than others.

542 **7.2. Radiative Transfer with Thermal Diffusion in a Pool.** Consider the
 543 vertical cross-section of a pool, Ω , heated from above, possibly by the Sun, and
 544 subject to wind on its surface, but without evaporation. The bottom is elliptical
 545 with maximum length 3 and height 1.

546 The time dependent Navier-Stokes equations is solve with a kinematic viscosity
 547 $\nu_F = 0.05$. A no-slip condition $\mathbf{u} = (0, 0)^T$ is applied on the bottom boundary and
 548 a Dirichlet condition on the horizontal boundary $\mathbf{u} = (10, 0)^T$ to simulate the wind

549 velocity.

550 The Taylor-Hood finite element method is used with the space V_h of continuous
551 piecewise quadratic velocities on a triangulation and the space Q_h of piecewise linear
552 pressures on the same triangulation. Galerkin-characteristics discretization in time
553 are used: at each time step $n+1$, find $\mathbf{u}_h^{n+1} \in V_h$, satisfying the boundary conditions,
554 and $p_h^{n+1} \in Q_h$, such that

$$(7.7) \quad \int_{\Omega_h} \left(\frac{1}{\delta t} \mathbf{u}_h^{n+1} \cdot \hat{\mathbf{u}}_h + \nu_F \nabla \mathbf{u}_h^{n+1} \cdot \nabla \hat{\mathbf{u}}_h - p_h^{n+1} \nabla \cdot \hat{\mathbf{u}}_h + \hat{p}_h \nabla \cdot \mathbf{u}_h^{n+1} \right) dx \\ = \int_{\Omega_h} \frac{1}{\delta t} \mathbf{u}_h^n(\mathbf{x} - \mathbf{u}_h^n(\mathbf{x})\delta t) \cdot \hat{\mathbf{u}}_h dx, \quad \forall \hat{p}_h \in Q_h, \forall \hat{\mathbf{u}}_h \in V_h, \text{ with } \hat{\mathbf{u}}_h|_{\partial\Omega} = 0.$$

555 There are 764 vertices in the triangulation; Figure 6 displays the velocity vectors after
556 50 time steps of size 0.02; stationarity is reached. The computation takes 12”.

557 For the temperature (6.5) is rescaled and discretized by (7.3). We chose $\kappa_T =$
558 0.5, $a_\nu = 0$, with vertical radiative transfer in the fluid, from its surface down into
559 the liquid and Dirichlet conditions on the bottom boundary $T = 0.057$ which is
560 approximately the reduced temperature found in the previous section.

561 The liquid water absorption parameter κ_ν can be found in

562 https://en.wikipedia.org/wiki/Electromagnetic_absorption_by_water

563 It turned out to be CPU prohibitive to solve the problem with such a detailed
564 κ_ν ; the bottleneck is in the computation of the integral in T of $B_\nu(T)$ required by the
565 variational principle (7.3). Hence we approximated κ_ν by its regression line in the
566 range $\nu \in (0.02, 7)10^{-14}$:

$$\kappa_\nu = \kappa_0 - \kappa_1\nu \text{ with } \nu \in (0.02, \nu_{max}) \quad \nu_{max} = 7, \quad \kappa_0 = 0.7, \quad \kappa_1 = 0.5/\nu_{max}.$$

567 Then the integral of $\kappa_\nu B_\nu(T)$ can be computed analytically:

$$\int_0^\infty \kappa_\nu B_\nu(T) d\nu = T^4 \kappa_0 \frac{\pi^4}{15} - 24T^5 \kappa_1 \zeta(5).$$

568 where ζ is the Riemann function, $\zeta(5) = 1.03693$.

569 The time dependent temperature equation is solved until convergence to a sta-
570 tionary state with 50 time steps of size 0.1. The convection terms are treated explicitly
571 so as to use (7.3) which is solved by the BFGS module in FreeFEM++. The computation
572 takes 326”. The solution is shown on Figure 6. One sees the effect of the current in the
573 fluid on the temperature distribution which has shifted to the right. Note that with a
574 Neumann condition on the bottom the temperature would keep rising with time and
575 even with a Dirichlet condition on the bottom boundary there is a critical value for
576 κ_T and/or Q_0 below which the temperature rises with time. Here $Q_0 = 0.02$, which
577 is much bigger than the value for the sunlight, but with the later the temperature is
578 almost constant everywhere, equal to its bottom value 0.06.

579 **7.3. Radiative Transfer with Thermal Diffusion in the Atmosphere of**
580 **a planet.** Consider the atmosphere of a spherical planet, heated by the Sun, with
581 a known ground temperature T_e . The computational domain is the space between a
582 sphere of radius R_2 and a sphere of radius $R_1 < R_2$.

583 As before the sunrays travel unaffected and hit the ground; so the radiative part
584 is governed by the first equation in (2.10) with (2.12) and (7.4), i.e. the second
585 equation in (2.10) is replaced by (7.2). The density of the atmosphere is constant

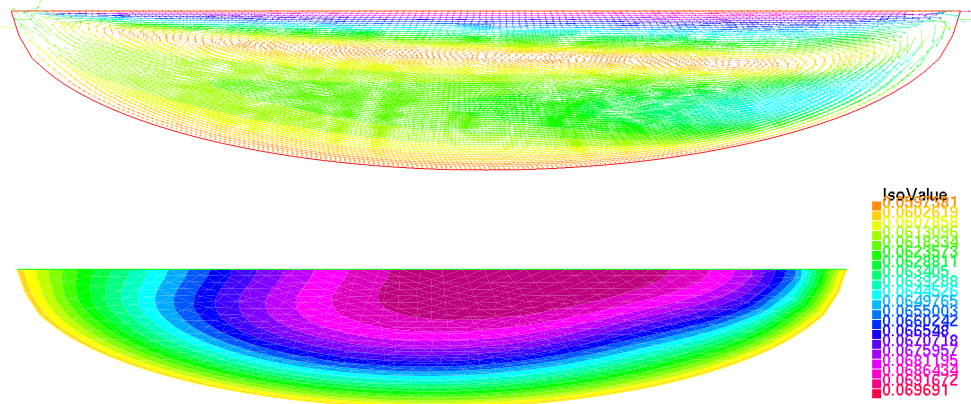


FIG. 6. Velocity vectors and Temperature in a pool subject to wind on its top boundary and given temperature on the bottom. The wind creates a large eddy rotating clockwise which, in turn, moves the hotter fluid region to the right.

rather than decaying exponentially with altitude. The absorption parameter chosen for the computation is also constant $\kappa = 0.5$. The wind velocity is a rotating Poiseuille flow around an axis $(\sin \bar{\psi}, 0, \cos \bar{\psi})^T$ which is not aligned with the direction of the Sun. In spherical coordinates it is

$$u = r(H - r)[\cos \psi, \sin \psi, 0]^T, \quad r \text{ is the distance to the ground.}$$

where $H = R_2 - R_1$. The time dependent equation (7.2), is solved in spherical coordinates (details can be found in [16] -appendix A). The computational domain becomes a solid rectangle with periodic conditions; it is discretized with a uniform distribution of vertices $16 \times 8 \times 8$ in the domain $(0, 2\pi) \times (0, \pi) \times (0, Z)$ with $Z = 1$.

The equations are discretized in time and space by a Galerkin-Characteristic method and piecewise linear conforming finite elements on tetraedras. The time step is $\delta t = 0.1$, the thermal diffusion is $\kappa_T = 0.01$. The stratified approximation requires R_1 to be large and H small. For the visualizations, however, we map the solid rectangle onto the spherical domain with $R_1 = 1$ and $R_2 = 2$. As before $T_{Sun} = 1.209$ and $Q_0 = 2 \cdot 10^{-5}$. Initially $T_{t=0}$ is set to $T_e = T_{sun} \frac{\kappa}{2} (Q^0 E_3(\kappa z))^{\frac{1}{4}}$. On the surface of the planet T is set to $0.95T_e(0)$.

Figure 7 shows the temperatures after 15 iterations without wind. The computing time is 357". The Sun is at infinity in the direction opposite to the blue region. Blue means cold; it corresponds to the night on this part of the planet. Yet with more time iterations we would see this zone heated by thermal diffusion due to the fixed temperature of the planet.

Figure 8 compares the temperatures with and without wind. The planar views correspond to cross sections of the domain by the plane $z = 0$. Here, the Sun in the horizontal direction on the right but the wind transports its heat counterclockwise.

7.4. Conclusion. In this article a special case of radiative and heat transport has been studied, the so called stratified approximation. The one dimensional radiative transfer equations are coupled with the temperature equation. Existence and uniqueness have been established with almost no restriction on the absorption and scattering parameters. Furthermore the proofs are based on a formulation of the problem which gives rise to an efficient numerical algorithm for radiative transfer

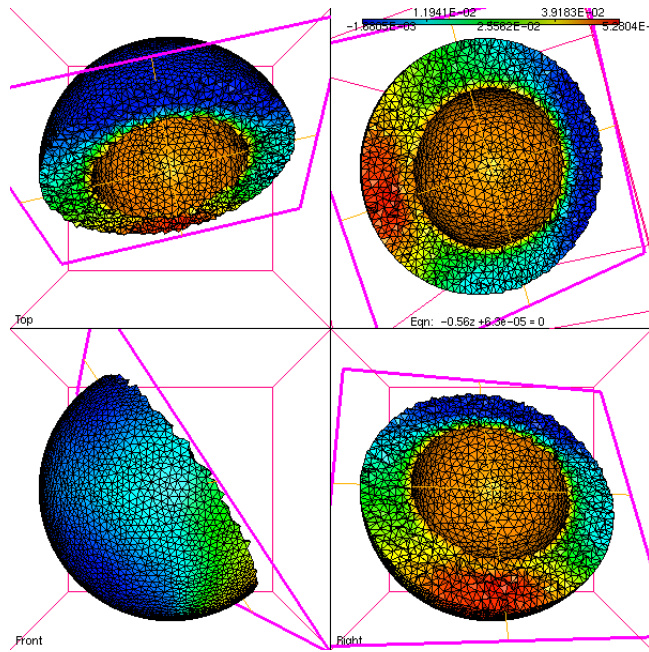


FIG. 7. *Temperature in the atmosphere of a planet heated by a Sun, when thermal diffusion propagates heat in unlit regions and also in the presence of a counter clockwise rotating wind. Note that the thickness of the atmosphere has been expanded for readability.*

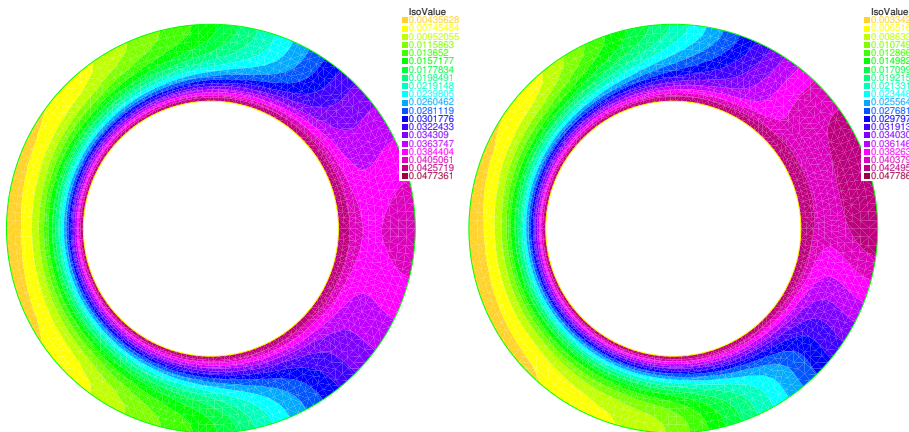


FIG. 8. *Temperature in the atmosphere of a planet heated by a Sun on the right with wind (right) and without wind (left); it is a counterclockwise rotating wind around an axis almost (but not quite) perpendicular to the figure. Thermal diffusion propagates heat in unlit regions and the wind transports the heat counterclockwise. Note that the thickness of the atmosphere has been expanded for readability.*

615 coupled with the heat equation for a fluid. Upper and lower positive solutions can be
 616 computed and the convergence to the unique solution is polynomial.

617 The method has been implemented numerically and indeed arbitrary precision can
 618 be obtained, even with highly oscillating absorption or scattering coefficients. Fur-
 619 thermore it is computationally very fast when the thermal diffusion is neglected and

620 reasonably fast otherwise, at least with absorption coefficients which are polynomial
 621 functions of the frequencies.

622 It has been applied to the computation of the temperature in the Earth atmos-
 623 phere, to that of a pool heated from above and to the atmosphere of a planet with
 624 a large thermal diffusion. However these are test cases rather than a full solution of
 625 physical problems and so, one should be cautious not to draw early conclusions from
 626 these computations; the purpose of the present study is to show that here is a method
 627 which is mathematically well understood and numerically faster than others.

628 There are many other applications, especially for climate modelling and in nuclear
 629 engineering for which these new mathematical and numerical results should be useful.

Acknowledgments. The authors would like to thank Prof. Claude Bardos and Prof. Guy Lucazeau for the numerous discussions and references.

8. Appendix: Proof of Theorem 4.1. Set $s_+(z) = 1_{z \geq 0}$. We recall that $z_+ = \max(z, 0) = z s_+(z)$ while $z_- = \max(-z, 0)$. Multiply both sides of the radiative transfer equation for two solutions I_ν and I'_ν by $s_+(I_\nu - I'_\nu)$ and integrate in all variables, with the notation

$$\langle \Phi \rangle := \int_0^\infty \int_{-1}^1 \Phi(\mu, \nu) d\mu d\nu.$$

With $T = T[I]$ and $T' = T[I']$ defined by (2.16), let us compute

$$\begin{aligned} D &:= \langle \kappa_\nu ((I_\nu - I'_\nu) - a_\nu (J_\nu - J'_\nu) - (1 - a_\nu)(B_\nu(T) - B_\nu(T'))) s_+(I_\nu - I'_\nu) \rangle \\ &= \langle \kappa_\nu (1 - a_\nu) ((I_\nu - I'_\nu) - (B_\nu(T) - B_\nu(T'))) s_+(I_\nu - I'_\nu) \rangle \\ &\quad + \langle \kappa_\nu a_\nu ((I_\nu - I'_\nu) - (J_\nu - J'_\nu)) s_+(I_\nu - I'_\nu) \rangle =: D_1 + D_2. \end{aligned}$$

Since

$$\int_{-1}^1 ((I_\nu - I'_\nu)(\tau, \mu) - (J_\nu - J'_\nu)(\tau)) d\mu = 0$$

and since $s_+(J_\nu - J'_\nu)$ is independent of μ , one has

$$D_2 = \langle \kappa_\nu a_\nu ((I_\nu - I'_\nu) - (J_\nu - J'_\nu)) (s_+(I_\nu - I'_\nu) - s_+(J_\nu - J'_\nu)) \rangle \geq 0$$

since the function $z \mapsto s_+(z)$ is nondecreasing and $\kappa_\nu a_\nu \geq 0$. Similarly

$$T = T[I] \text{ and } T' = T[I'] \implies \langle \kappa_\nu (1 - a_\nu) ((I_\nu - I'_\nu) - (B_\nu(T) - B_\nu(T'))) \rangle = 0$$

and since $s_+(T - T')$ is independent of μ and ν , one has

$$D_1 = \langle \kappa_\nu (1 - a_\nu) ((I_\nu - I'_\nu) - (B_\nu(T) - B_\nu(T'))) (s_+(I_\nu - I'_\nu) - s_+(T - T')) \rangle.$$

Since B_ν is increasing for each $\nu > 0$, one has $s_+(T - T') = s_+(B_\nu(T) - B_\nu(T'))$. Hence

$$D_1 = \langle \kappa_\nu (1 - a_\nu) ((I_\nu - I'_\nu) - (B_\nu(T) - B_\nu(T'))) (s_+(I_\nu - I'_\nu) - s_+(B_\nu(T) - B_\nu(T'))) \rangle \geq 0$$

since $\kappa_\nu (1 - a_\nu) \geq 0$ and $z \mapsto s_+(z)$ is nondecreasing.

Let I_ν and I'_ν be two solutions of (2.11) with boundary data

$$\begin{aligned} I_\nu(0, \mu) &= Q_\nu^+(\mu), & I_\nu(Z, -\mu) &= Q_\nu^-(\mu), & 0 < \mu < 1, \\ I'_\nu(0, \mu) &= Q'^+_\nu(\mu), & I'_\nu(Z, -\mu) &= Q'^-_\nu(\mu), & 0 < \mu < 1. \end{aligned}$$

Assume that

$$Q_\nu^\pm(\mu) \leq Q'_\nu{}^\pm(\mu) \quad \text{for a.e. } (\mu, \nu) \in (0, 1) \times (0, \infty).$$

Then

$$\partial_\tau \langle \mu(I_\nu - I'_\nu)_+ \rangle = -D_1 - D_2 \leq 0$$

so that $\tau \mapsto \langle \mu(I_\nu - I'_\nu)_+ \rangle(\tau)$ is nonincreasing. Since

$$\begin{aligned} Q_\nu^- \leq Q'_\nu{}^- &\implies \langle \mu(I_\nu - I'_\nu)_+ \rangle(Z) = \langle \mu_+(I_\nu - I'_\nu)_+ \rangle(Z) \geq 0, \\ Q_\nu^+ \leq Q'_\nu{}^+ &\implies \langle \mu(I_\nu - I'_\nu)_+ \rangle(0) = -\langle \mu_-(I_\nu - I'_\nu)_+ \rangle(0) \leq 0, \end{aligned}$$

one has

$$\begin{aligned} 0 &= \langle \mu(I_\nu - I'_\nu)_+ \rangle = D_1 = D_2 \quad \text{for a.e. } \tau \in (0, Z) \\ (I_\nu - I'_\nu)_+(0, -\mu) &= (I_\nu - I'_\nu)_+(Z, \mu) = 0 \quad \text{for a.e. } \mu \in (0, 1). \end{aligned}$$

Besides, since $\kappa_\nu(1 - a_\nu) > 0$ for all $\nu > 0$

$$D_1 = 0 \implies s_+(I_\nu(\tau, \mu) - I'_\nu(\tau, \mu)) = s_+(T[I] - T[I']) \text{ for a.e. } (\tau, \mu, \nu).$$

Next we use the K -invariant (in the terminology of section 10 in chapter I of Chandrasekhar [6]) for solutions of the radiative transfer equation with slab symmetry. We compute

$$\begin{aligned} \partial_\tau \left\langle \frac{\mu^2}{\kappa_\nu} (I_\nu - I'_\nu)_+ \right\rangle &= -\langle a_\nu \mu ((I_\nu - I'_\nu) - (I'_\nu - \tilde{I}'_\nu)) s_+(T[I] - T[I']) \rangle \\ &\quad - \langle (1 - a_\nu) \mu ((I_\nu - I'_\nu) - (B_\nu(T[I]) - B_\nu(T[I']))) s_+(T[I] - T[I']) \rangle \\ &= -\langle \mu (I_\nu - I'_\nu) s_+(T[I] - T[I']) \rangle = -\langle \mu (I_\nu - I'_\nu)_+ \rangle = 0, \end{aligned}$$

since

$$\int_{-1}^1 \mu (I'_\nu(\tau) - \tilde{I}'_\nu(\tau)) d\mu = \int_{-1}^1 \mu (B_\nu(T[I]) - B_\nu(T[I'])) d\mu = 0.$$

Next we integrate in $\tau \in (0, Z)$, and observe that

$$\begin{aligned} (I_\nu - I'_\nu)_+(0, -\mu) = 0 \text{ and } Q_\nu^+(\mu) &\leq Q'_\nu{}^+(\mu) \quad \text{for a.e. } \mu \in (0, 1) \\ \implies \left\langle \frac{\mu^2}{\kappa_\nu} (I_\nu - I'_\nu)_+ \right\rangle(\tau) &= \left\langle \frac{\mu^2}{\kappa_\nu} (I_\nu - I'_\nu)_+ \right\rangle(0) = 0. \end{aligned}$$

Thus, we have proved that

$$\begin{aligned} Q_\nu^\pm(\mu) &\leq Q'_\nu{}^\pm(\mu) \quad \text{for a.e. } (\mu, \nu) \in (0, 1) \times (0, \infty) \\ \implies I_\nu(\tau, \mu) &\leq I'_\nu(\tau, \mu) \quad \text{for a.e. } (\tau, \mu, \nu) \in (0, Z) \times (-1, 1) \times (0, \infty) \\ \implies T[I](\tau) &\leq T[I'](\tau) \quad \text{for a.e. } \tau \in (0, Z). \end{aligned}$$

Exchanging $Q_\nu^\pm(\mu)$ and $Q'_\nu{}^\pm(\mu)$ above shows that $I_\nu = I'_\nu$ and $T[I] = T[I']$, which is the announced uniqueness.

Proof of Remark 5.2 Let $(I_\nu, T[I])$ and $(I'_\nu, T[I'])$ the solutions of (5.3) corresponding to the boundary data Q_ν^\pm and $Q'_\nu{}^\pm$ respectively, such that $Q_\nu^\pm(\mu) \leq Q'_\nu{}^\pm(\mu)$ for

a.e. $(\mu, \nu) \in (0, 1) \times (0, \infty)$. First, we slightly modify the treatment of D_2 as follows:

$$D_2 = \frac{1}{2} \int_0^\infty \kappa_\nu a_\nu \int_{-1}^1 (I_\nu - I'_\nu)_+(\mu) d\mu d\nu \\ - \frac{1}{2} \int_0^\infty \kappa_\nu a_\nu \int_{-1}^1 \int_{-1}^1 p(\mu, \mu') (I_\nu - I'_\nu)(\mu') s_+(I_\nu - I'_\nu)(\mu) d\mu' d\mu d\nu.$$

Since $p \geq 0$ and $\frac{1}{2} \int_{-1}^1 p(\mu, \mu') d\mu = 1$, one has

$$p(\mu, \mu') (I_\nu - I'_\nu)(\mu') s_+(I_\nu - I'_\nu)(\mu) \leq p(\mu, \mu') (I_\nu - I'_\nu)_+(\mu'),$$

so that

$$D_2 \geq \frac{1}{2} \int_0^\infty \kappa_\nu a_\nu \int_{-1}^1 (I_\nu - I'_\nu)_+(\mu) d\mu d\nu \\ - \frac{1}{2} \int_0^\infty \kappa_\nu a_\nu \int_{-1}^1 \int_{-1}^1 p(\mu, \mu') (I_\nu - I'_\nu)_+(\mu') d\mu' d\mu d\nu = 0,$$

As in the proof of [Theorem 4.1](#), we see that

$$\langle \mu (I_\nu - I'_\nu)_+ \rangle(\tau) = 0 \text{ for a.e. } \tau \in (0, Z),$$

and

$$s_+(I_\nu(\tau, \mu) - I'_\nu(\tau, \mu)) = s_+(T[I](\tau) - T[I'](\tau))$$

for a.e. $(\tau, \mu, \nu) \in (0, Z) \times (-1, 1) \times (0, \infty)$, while

$$(I_\nu - I'_\nu)_+(0, -\mu) = (I_\nu - I'_\nu)_+(Z, \mu) = 0 \quad \text{for a.e. } \mu \in (0, 1).$$

Next we compute

$$\partial_\tau \left\langle \frac{\mu^2}{\kappa_\nu} (I_\nu - I'_\nu)_+ \right\rangle = -\frac{1}{2} \int_0^\infty a_\nu \int_{-1}^1 \mu (I_\nu - I'_\nu)_+(\tau, \mu) d\mu d\nu \\ + \frac{1}{2} \int_0^\infty a_\nu \int_{-1}^1 \mu \int_{-1}^1 p(\mu, \mu') (I_\nu - I'_\nu)_+(\tau, \mu') d\mu' d\mu d\nu s_+(T[I](\tau) - T[I'](\tau)) \\ - \langle (1 - a_\nu) \mu (I_\nu - I'_\nu) - (B_\nu(T[I]) - B_\nu(T[I'])) s_+(T[I] - T[I']) \rangle \\ = -\langle a_\nu \mu (I_\nu - I'_\nu) s_+(T[I] - T[I']) \rangle - \langle (1 - a_\nu) \mu (I_\nu - I'_\nu) s_+(T[I] - T[I']) \rangle \\ = -\langle \mu (I_\nu - I'_\nu) s_+(T[I] - T[I']) \rangle = -\langle \mu (I_\nu - I'_\nu)_+ \rangle = 0,$$

since

$$\int_{-1}^1 \mu p(\mu, \mu') d\mu = \int_{-1}^1 \mu (B_\nu(T[I]) - B_\nu(T[I'])) d\mu = 0.$$

Finally we integrate in $\tau \in (0, Z)$, and conclude as in the previous section that

$$(I_\nu - I'_\nu)_+(0, -\mu) = 0 \text{ and } Q_\nu^+(\mu) \leq Q_\nu'^+(\mu) \quad \text{for a.e. } \mu \in (0, 1) \\ \implies \left\langle \frac{\mu^2}{\kappa_\nu} (I_\nu - I'_\nu)_+ \right\rangle(\tau) = \left\langle \frac{\mu^2}{\kappa_\nu} (I_\nu - I'_\nu)_+ \right\rangle(0) = 0.$$

Hence $Q_\nu^\pm(\mu) \leq Q_\nu'^\pm(\mu)$ for a.e. $(\mu, \nu) \in (0, 1) \times (0, \infty)$ implies that $I_\nu(\tau, \mu) \leq I'_\nu(\tau, \mu)$ for a.e. $(\tau, \mu, \nu) \in (0, Z) \times (-1, 1) \times (0, \infty)$, and $T[I](\tau) \leq T[I'](\tau)$ for a.e. $\tau \in (0, Z)$. This implies the uniqueness of the solution as explained in the proof of [Theorem 4.1](#).

REFERENCES

- [1] C. BARDOS, F. GOLSE, B. PERTHAME, AND R. SENTIS, *The nonaccretive radiative transfer equations: Existence of solutions and Rosseland approximation*, J. Funct. Anal., 77 (1988), pp. 434–460.
- [2] C. BARDOS AND O. PIRONNEAU, *Radiative Transfer for the Greenhouse Effect*. submitted to SeMA J. Springer, 2021, <https://hal.sorbonne-universite.fr/hal-03094855>.
- [3] V. BARLAKASA, A. MACKE, AND M. WENDISCHA, *Sparta – solver for polarized atmospheric radiative transfer applications*, Journal of Quantitative Spectroscopy and Radiative Transfer, 178 (2016), pp. 77–92.
- [4] C. F. BOHREN AND E. E. CLOTHIAUX, *Fundamentals of Atmospheric Radiation*, Wiley-VCH, 2006.
- [5] R. CAHALAN, L. OREOPOULOS, A. MARSHAK, K. EVANS, A. DAVIS, AND ET AL., *Bringing together the most advanced radiative transfer tools for cloudy atmospheres*, American Meteorological Society, 86 (2005), pp. 1275–1294.
- [6] S. CHANDRASEKHAR, *Radiative Transfer*, Clarendon Press, Oxford, 1950.
- [7] M. G. CRANDALL AND L. TARTAR, *Some relations between nonexpansive and order preserving mappings*, Proc. Am. Math. Soc., 78 (1980), pp. 385–390.
- [8] J. DUFRESNE, V. EYMET, C. CREVOISIER, AND J. GRANDPEIX, *Greenhouse effect: The relative contributions of emission height and total absorption*, Journal of Climate, American Meteorological Society, 33 (2020), pp. 3827–3844.
- [9] A. FOWLER, *Mathematical geoscience*, Berlin: Springer, 2011.
- [10] M. GHATTASSI, X. HUO, AND N. MASMOUDI, *On the diffusive limits of radiative heat transfer system I: well prepared initial and boundary conditions*. Preprint 2020, <https://arxiv.org/abs/2007.13209>.
- [11] F. GOLSE, *The Milne Problem for the Radiative Transfer Equations (with Frequency Dependence)*, Trans. Am. Math. Soc., 303 (1987), pp. 125–143.
- [12] F. GOLSE AND B. PERTHAME, *Generalized solutions of the radiative transfer equations in a singular case*, Commun. Math. Phys., 106 (1986), pp. 211–239.
- [13] F. GOLSE AND O. PIRONNEAU, *Radiative transfer in a fluid*, RACSAM, Springer, dedicated to I. Diaz (2021).
- [14] F. GOLSE AND O. PIRONNEAU, *Stratified radiative transfer for multidimensional fluids*. Comptes Rendus de l’Académie des Sciences (Mécanique), to appear 2022.
- [15] R. M. GOODY AND Y. L. YUNG, *Atmospheric Radiation. 2nd ed.*, Oxford: Clarendon Press, 1996.
- [16] J. HAPPEL AND H. BRENNER, *Low Reynolds number hydrodynamics with special applications to particulate media. 2nd rev. ed., 4th printing*, Kluwer Academic Publishers Group, Dordrecht, 1986.
- [17] F. HECHT, *New developments in freefem++*, J. Numer. Math., 20 (2012), pp. 251–265.
- [18] S. KLAINERMAN AND A. MAJDA, *Compressible and incompressible fluids*, Comm Pure and Applied Math, 35 (1982), pp. 629–651.
- [19] J. LERAY AND J.-L. LIONS, *Quelques résultats de Visik sur les problèmes elliptiques non linéaires par les méthodes de Minty-Browder*, Bull. Soc. Math. Fr., 93 (1965), pp. 97–107.
- [20] E. H. LIEB AND M. LOSS, *Analysis. 2nd ed.*, Providence, RI: American Mathematical Society (AMS), 2001.
- [21] P.-L. LIONS, *Mathematical topics in fluid mechanics. Vol. 1: Incompressible models*, Oxford: Clarendon Press, 1996.
- [22] A. MARSHAK AND A. DAVIS, eds., *3D Radiative Transfer in Cloudy Atmospheres*, vol. 5117 of Physics of Earth and Space Environments, Springer, 2005.
- [23] B. MERCIER, *Application of accretive operators theory to the radiative transfer equations*, SIAM J. Math. Anal., 18 (1987), pp. 393–408.
- [24] D. MIHALAS AND B. WEIBEL MIHALAS, *Foundations of radiation hydrodynamics*, New York, Oxford: Oxford University Press, 1984.
- [25] O. PIRONNEAU, *A fast and accurate numerical method for radiative transfer in the atmosphere*. Comptes Rendus de l’académie des sciences (Math), December 2021.
- [26] G. POMRANING, *The equations of Radiation Hydrodynamics*, Pergamon Press, NY, 1973.
- [27] M. PORZIO AND O. LOPEZ-POUSO, *Application of accretive operators theory to evolutive combined conduction, convection and radiation*, Rev. Mat. Iberoamericana, 20 (2004), pp. 257–275.
- [28] W. ZDUNKOWSKI AND T. TRAUTMANN, *Radiation in the Atmosphere. A Course in Theoretical Meteorology*, Cambridge Univ. Press, 2007.