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DEGENERATING KÄHLER-EINSTEIN CONES, LOCALLY SYMMETRIC CUSPS, AND THE TIAN-YAU METRIC

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ABSTRACT. Let X be a complex projective manifold and let $D \subset X$ be a smooth divisor. In this article, we are interested in studying limits when $k \to 0$ of Kähler-Einstein metrics ω_k with a cone singularity of angle $2\pi k$ along D. In our first result, we assume that $X \setminus D$ is a locally symmetric space and we show that ω_k converges to the locally symmetric metric and further give asymptotics of ω_k when $X \setminus D$ is a ball quotient. Our second result deals with the case when X is Fano and D is anticanonical. We prove a folklore conjecture asserting that a rescaled limit of ω_k is the complete, Ricci flat Tian-Yau metric on $X \setminus D$. Furthermore, we prove that (X, ω_k) converges to an interval in the Gromov-Hausdorff sense.

CONTENTS

INTRODUCTION

Let X be a complex projective manifold and let $D \subset X$ be a smooth divisor. In many geometrically meaningful situations, one is able to construct Kähler-Einstein metrics ω on the complement $X^\circ := X \setminus D$ of the divisor D. Unless one imposes some growth condition near D, such a metric ω may not be unique – one can typically find complete and incomplete KE metrics on the same X°, possibly with the same Einstein constant too.

The focus of the present paper is to investigate the relationship between these different metrics in the specific setting of Kähler-Einstein metrics with *cone singularities* along D. Recall that if $k \in (0,1)$, a Kähler metric ω on X° is said to have cone singularities along D with cone angle $2\pi k$ if it is locally quasi-isometric to the model cone metric

$$\omega_{k,\mathrm{mod}} := \frac{idz_1 \wedge d\bar{z}_1}{|z_1|^{2(1-k)}} + \sum_{j\geqslant 2} idz_j \wedge d\bar{z}_j$$

on each coordinate chart $(U, (z_i))$ where $U \cap D = (z_1 = 0)$. Such a metric is incomplete, has finite volume and automatically extends to a closed, positive $(1,1)$-current on X. There is an analogue of the Aubin-Yau (resp. Yau) theorem guaranteeing the existence and uniqueness of a negatively curved (resp. Ricci-flat) Kähler-Einstein metric ω_k with cone angle $2\pi k$ along D under the condition that the adjoint $\mathbb{R}$-line bundle $K_X + (1-k)D$ is ample (resp. numerically trivial), cf e.g. [Bre13, CGP13, GP16, JMR16]. The positive curvature case is more complicated, in analogy with the absolute case $D = \emptyset$ and it involves the properness of some suitable analogue of the Mabuchi or Ding functional.

Let us now shift our focus to the small angle regime, that is when $0 < k \ll 1$. We raise the following broad and somewhat vague question, which is closely related to [CR15, Conjecture 1.11] and [Oda20, Conjecture 1.4] in the positive curvature case.

Question. *Let X be a complex projective manifold and let $D \subset X$ be a smooth divisor. Assume that for any $0 < k \ll 1$, there exists a unique Kähler-Einstein metric ω_k with cone angle $2\pi k$ along D, i.e.*

$$\mathrm{Ric}\,\omega_k = \sigma\omega_k + (1-k)[D] \tag{0.1}$$

for some $\sigma = \pm 1$. Do the metrics ω_k converge when $k \to 0$ to some canonical metric on X°, possibly after rescaling?

The aim of this paper is to provide an answer to the above question in two different geometric situations, one for each sign of the curvature.

The negative case.
As recalled above, the existence of a KE metric solving (0.1) with $\sigma = -1$ is equivalent to $K_X + (1-k)D$ being ample. For instance, if one assumes that $K_X + D$ is ample, then the same will hold true for $K_X + (1-k)D$ as long as k is small enough. In that situation, it was proved in [Gue20] that when $k \to 0$, the KE metric ω_k converges to the complete KE metric with Poincaré growth constructed by R. Kobayashi [Kob84] and Tian-Yau [TY87].

Another interesting example is provided by toroidal compactifications of ball quotients $X^\circ = \Gamma \backslash \mathbb{B}^n$, where $\Gamma \subset \mathrm{Aut}(\mathbb{B}^n)$ is a torsion-free, discrete subgroup. It is well-known that one can embed $X^\circ \hookrightarrow X$ as a Zariski-open subset of a projective orbifold X such that $D := X \setminus X^\circ$ is a disjoint union of abelian varieties, cf § 1.1 for references and more details. Note that the Bergman metric on $\mathbb{B}^n$ descends to the complex (complete) hyperbolic metric ω_{hyp} on X°, which we normalize to have $\mathrm{Ric}\,\omega_{\mathrm{hyp}} = -\omega_{\mathrm{hyp}}$. Moreover, $K_X + (1-k)D$ is ample for $0 < k \ll 1$ (but certainly not for $k = 0$ unless $D = \emptyset$) and therefore X° also comes equipped with KE metrics ω_k with cone angle $2\pi k$ along D whenever $k > 0$ is small enough. The relationship between these metrics is provided by the following

Theorem A. *Let (X,D) be a toroidal compactification of a ball quotient $X^\circ = \Gamma\Big\backslash\mathbb{B}^n$, and let ω_k be the KE metric solving (0.1) for small k, with $\sigma=-1$. Then, we have convergence*

$$\omega_k \xrightarrow[k\to 0]{} \omega_{\mathrm{hyp}}$$

both in $C^\infty_{\mathrm{loc}}(X^\circ)$ and weakly as currents on X. Moreover, we have precise asymptotics of ω_k near D when $k\to 0$.

A few remarks are in order here.

- (i) The asymptotics of ω_k in C^0 are given in Theorem 4.5. They are obtained by constructing a model metric on the normal bundle of D using the Calabi Ansatz, cf § 2.
- (ii) The first half of the statement (i.e. the convergence part) remains true in the more general setting of quotients of bounded symmetric domains, cf Theorem 1.1. In that case, D needs not be smooth anymore but has simple normal crossings up to a finite cover.
- (iii) Assume that the lattice Γ is arithmetic. By choosing the angles carefully along each torus at the boundary, one can find a sequence ω_{k_m} of orbifold KE metrics that can be globally desingularized so that $(X^\circ,\omega_{\mathrm{hyp}})$ is the limit of *smooth, compact* KE spaces up to the action of a larger and larger group of isometries, cf § 4.3. In a nutshell, one can "close the complex hyperbolic cusp". This gives the closest analog to the Dehn filling of real hyperbolic cusps by Einstein manifolds, due to Thurston in dimension 3 and Anderson [And06] in higher dimension: in the complex case, one cannot fill the cusp, but this is possible up to some larger and larger covering. This answers a question of Misha Kapovich to the first author several years ago.

The positive case.
In general, it is not so easy to characterize the existence of a metric ω_k solving (0.1) with $\sigma=1$, for small values of k. However, a result of Berman [Ber13] (later generalized by Song-Wang [SW16]) asserts that if X is a Fano manifold (that is, $-K_X$ is ample) and $D\in|-K_X|$ is smooth, then there exists $k_0>0$ such that for any $0<k<k_0$, there exists a unique Kähler metric ω_k on X° such that $\mathrm{Ric}\,\omega_k=\omega_k$ and ω_k has cone singularities with cone angle $2\pi k$ along D, i.e. ω_k solves (0.1).

The existence of such a metric had been conjectured by Donaldson [Don12, § 6] in relation with his program to prove that a K-stable Fano manifold admits a Kähler-Einstein metric by using the continuity path $\mathrm{Ric}\,\omega_t=t\omega_t+(1-t)[D]$ involving metrics with cone singularities. He also predicted that the (conjectural then) ω_k would actually converge to the Ricci flat complete Kähler metric ω_{TY} constructed by Tian and Yau in [TY90].

If $n=1$, then the metrics ω_k on $\mathbb{P}^1\setminus\{0,\infty\}$ are completely explicit, given by the expression $\omega_k=\frac{k^2 idz\wedge d\bar z}{|z|^{2(1-k)}(1+|z|^{2k})^2}$ and one sees immediately that $k^{-2}\omega_k$ converges locally smoothly to the cylinder $\omega_{\mathrm{cyl}}=\frac{idz\wedge d\bar z}{4|z|^2}$ while $(\mathbb{P}^1,\omega_k)$ converges in the Gromov-Hausdorff sense to the interval $([0,\frac{\pi}{2}],dt^2)$ (set $r=|z|^k$ to that

$g_k = \frac{dr^2+k^2r^2d\theta^2}{(1+r^2)^2}$ and reparametrize by $t = \tan^{-1}(r)$), cf also [RZ20]. Our second main result establishes the conjecture in full generality.

Theorem B. *Let X be a Fano manifold of dimension n and let $D \in |-K_X|$ be a smooth anticanonical divisor. Then up to a rescaling factor, the conic KE metrics ω_k solving (0.1) with $\sigma = 1$ for small k converge to the Tian-Yau metric:*

$$k^{-1-\frac{1}{n}}\omega_k \xrightarrow[k\to 0]{} \omega_{\mathrm{TY}}$$

in $C^\infty_{\mathrm{loc}}(X \setminus D)$. Moreover, we have precise asymptotics of ω_k near D when $k \to 0$.

Finally the metrics ω_k themselves converge in the Gromov-Hausdorff sense to an interval.

As before, a few remarks

(i) The fibers of the collapsing to an interval are the normal circle bundle of the divisor D. The two endpoints of the interval correspond respectively to the conical divisor D itself and to the Tian-Yau metric.

(ii) Several recent papers study cases of collapsing of Ricci flat Kähler metrics to an interval, for K3 surfaces [HSVZ] or in higher dimension [SZ19]. Our theorem probably gives the first general example of collapsing of Kähler-Einstein metrics with positive Ricci: of course this is made possible by the presence of a cone angle going to zero.

Strategy of the proof.

Although Theorem A and Theorem B have a quite different flavor, their proofs share a common approach. Indeed, in both proofs, we rely on the existence of a model metric living in the neighborhood of the zero section in the normal bundle of D. That metric is provided by the Calabi Ansatz, cf § 2, and its curvature is computed in the following section, § 3.

The proof of Theorem A goes as follows. We use pluripotential methods, especially the comparison principle (quite suited in negative curvature), in order to estimate the potential of ω_k with enough precision to establish the weak convergence. The local smooth convergence away from D follows from a suitable use of Chern-Lu formula. In order to further compute the asymptotics of ω_k (at order zero), we show that ω_k is asymptotically close to the Calabi metric constructed and analyzed in § 2. This relies on the previous step as well as the application of the maximum principle and Chern-Lu formula, which in turn uses crucially that the curvature of the Calabi metric is bounded, cf § 3.

The proof of Theorem B, technicallly more involved than the previous one, relies on gluing methods. The general idea is to construct a model cone metric $\tilde{\omega}_k$ by gluing the Calabi metric $\omega_{k,L}$ near D and the Tian-Yau metric ω_{TY} away from D. For $k \ll 1$, the implicit function theorem allows us to find the Kähler-Einstein metric $\omega_k = \tilde{\omega}_k + dd^c\varphi_k$ with a control on φ_k and its covariant derivatives that is sufficiently precise that one can derive the desired smooth convergence $k^{-1-\frac{1}{n}}\omega_k \to \omega_{TY}$ away from D as well as the global Gromov-Hausdorff convergence of (X, ω_k) to an interval.

Some of the main technical steps include: estimating the curvature of the Calabi metric; finely gluing the Calabi metric which lives on the normal bundle L of D onto a neighborhood of D in X using a fibration in extremal disks; establishing a Schauder estimate for the model cone metric on $\mathbb{C}^* \times \mathbb{C}^{n-1}$ with cone angle $2\pi k$ which is uniform in k; establishing a uniform Schauder estimate in suitable weighted Hölder spaces for the family of collapsing cone metrics $\tilde{\omega}_k$ mentioned above. This is similar in spirit to other gluing problems, especially the papers [HSVZ, SZ19] mentioned above, but our techniques are different.

Applying the techniques used for Theorem B to Theorem A would probably enhance our C^0 estimates for the metrics to estimates on all derivatives, at the expense of a much more technical proof. On the other hand, the pluripotential techniques seem to fall short in the context of Theorem B.

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1. Closing the cusps of locally symmetric spaces

1.1. **Setup.** Let $X = \Gamma \backslash \Omega$ be an n-dimensional quotient of a bounded symmetric domain Ω by a torsion-free lattice $\Gamma \subset \mathrm{Aut}(\Omega)^\circ$. It is well-known that X is a quasi-projective variety that can be compactified in several meaningful ways.

The Satake-Baily-Borel compactification $X \hookrightarrow \overline{X}_{\min}$ is a *singular*, minimal compactification in the sense that given any normal compactification $X \hookrightarrow \overline{X}'$, the identity morphism on X extends to a holomorphic map $\overline{X}' \to \overline{X}_{\min}$. The variety $\overline{X}_{\min}$ is normal, has log canonical singularities and $K_{\overline{X}_{\min}}$ is ample. In a modern terminology, $\overline{X}_{\min}$ is a (normal) stable variety.

The Ash-Mumford-Rapoport-Tai [AMRT10] toroidal compactification $X \hookrightarrow \overline{X}$ is a compactification with finite quotient singularities such that $\overline{X} \setminus X$ is a (reduced) divisor with simple normal crossings that we denote by $D = \sum_{\lambda=1}^N D_\lambda$. Moreover, the birational morphism $\pi : \overline{X} \to \overline{X}_{\min}$ satisfies

$$K_{\overline{X}} + D = \pi^* K_{\overline{X}_{\min}}.$$

If Γ is neat, the toroidal compactification $\overline{X}$ is actually smooth. Moreover, any torsion-free lattice $\mathrm{Aut}(\Omega)$ admits a finite index subgroup which is neat. As a result, one can find $\Gamma' < \Gamma$ with finite index such that $Y = \Gamma' \backslash \Omega$ admits a smooth toroidal compactification $(\overline{Y}, D')$. Moreover, the finite étale morphism $f : Y \to X$ extends uniquely to a finite cover $f : \overline{Y} \to \overline{X}$ and one has $K_{\overline{Y}} + D' = f^*(K_{\overline{X}} + D)$.

In the case where $\Omega = \mathbb{B}^n$ is the euclidean unit ball in $\mathbb{C}^n$, $\overline{X}_{\min} \setminus X$ consist of finitely many singular points $\{x_1, \ldots, x_N\}$ and $D = \sqcup_{\lambda=1}^N D_\lambda$ is a disjoint union of abelian varieties D_λ with negative normal bundle, which are contracted onto those singular points by π.

1.2. **The Kähler-Einstein metric.** The Bergman metric ω_{Berg} on Ω is invariant under the action of $\mathrm{Aut}(\Omega)$ hence it descends to a Kähler-Einstein metric ω_{KE} on X. Moreover, one can prove that ω_{KE} extends to a closed, positive current $\omega_{\mathrm{KE}} \in c_1(K_{\overline{X}} + D)$ and

$$\int_X \omega_{\mathrm{KE}}^n = c_1(K_{\overline{X}} + D)^n.$$

In particular, $\omega_{\mathrm{KE}} = \pi^*\omega_{\min}$ for some closed, positive current $\omega_{\min} \in c_1(K_{\overline{X}_{\min}})$ which coincides with the singular Kähler-Einstein metric constructed in [BG14].

If $\Gamma' < \Gamma$ has finite index, then the Kähler-Einstein metric of $Y = {\Gamma'}\backslash\Omega$ is simply $f^*\omega_{\mathrm{KE}}$ where $f : Y \to X$ is the finite étale cover induced by the lattice inclusion.

In the case where $\Omega = \mathbb{B}^n$ and Γ is neat, one has a very precise description of ω_{KE} near D, cf. e.g. [Mok12, Eq. (8)]. In particular, if $(z_1, \ldots, z_n)$ is a system of holomorphic coordinates on some open set $U \subset \overline{X}$ such that $D \cap U = (z_1 = 0)$, then $\omega|_U$ is quasi-isometric to

$$\frac{idz_1 \wedge d\bar{z}_1}{|z_1|^2(-\log|z_1|)^2} + \frac{1}{(-\log|z_1|)} \sum_{k=2}^n idz_k \wedge d\bar{z}_k. \tag{1.1}$$

One can actually say much more and exhibit an exact formula for ω_{KE} on a small enough neighborhood U of D after identifying U with a neighborhood of the zero section in the normal bundle $N_{D/X} \to D$ of D, cf (4.5).

1.3. **Monge-Ampère equation.** One can write down the Monge-Ampère equation satisfied by ω_{KE}. In order to do so, we pick:

· A Kähler metric $\omega_{\overline{X}_{\min}} \in c_1(K_{\overline{X}_{\min}})$ and set $\chi := \pi^*\omega_{\overline{X}_{\min}}$. It is a smooth, semipositive form on $\overline{X}$. Recall that a Kähler metric on a singular complex space Y is defined to be a Kähler metric on the regular locus Y_{reg} which is locally the restriction of an ambient Kähler form under local embeddings $Y \underset{\mathrm{loc}}{\hookrightarrow} \mathbb{C}^N$. In particular, its bisectional curvature is bounded above locally near any point.

· Holomorphic sections $s_\lambda \in H^0(\overline{X}, \mathcal{O}_{\overline{X}}(D_\lambda))$ such that $D_\lambda = (s_\lambda = 0)$ and smooth hermitian metrics h_λ on $\mathcal{O}_{\overline{X}}(D_\lambda)$. with Chern curvature form $\theta_\lambda := i\Theta_{h_\lambda}(D_\lambda)$. We set $s = \bigotimes s_\lambda$, $|s| := \prod_k |s_\lambda|_{h_\lambda}$, $\theta = \sum_k \theta_\lambda$. Up to scaling h_λ, one can assume that $|s_\lambda|_{h_\lambda} < e^{-1}$.

· A smooth volume form dV on $\overline{X}$ satisfying $-\mathrm{Ric}(dV) + \theta = \chi$. Then one can write the Kähler-Einstein metric ω_{KE} on $\overline{X}$ as $\omega_{\mathrm{KE}} = \chi + dd^c\widehat{\varphi}$ for the unique χ-psh function $\widehat{\varphi}$ solution of the (non-pluripolar) Monge-Ampère equation

$$\omega_{\mathrm{KE}}^n = (\chi + dd^c\widehat{\varphi})^n = \frac{e^{\widehat{\varphi}}dV}{|s|^2}. \tag{1.2}$$

One knows that for any $\varepsilon > 0$, there exists a constant C_ε such that the following set of inequalities

$$C_1 \geqslant \widehat{\varphi} \geqslant -(n+1+\varepsilon)\log(-\log|s|) - C_\varepsilon \tag{1.3}$$

hold on $\overline{X}$, cf. [DGG20, Prop. D].

1.4. **Conic approximation.** As π can be obtained as a sequence of blow ups of smooth centers, there exist coefficients $a_\lambda \in \mathbb{Q}_+$ such that $-\sum a_\lambda D_\lambda$ is π-ample. In particular, for $k > 0$ small enough, the $\mathbb{Q}$-line bundle $K_X + \sum_\lambda(1 - ka_\lambda)D_\lambda$ is ample. Moreover, up to scaling down the a_λ (by the same factor), one can assume that $\chi - \widetilde{\theta}$ is a Kähler form, where $\widetilde{\theta} := \sum a_\lambda \theta_\lambda$. The Monge-Ampère equation

$$(\chi - k\widetilde{\theta} + dd^c\widehat{\varphi}_k)^n = \frac{e^{\widehat{\varphi}_k}dV}{\prod_\lambda |s_\lambda|_{h_\lambda}^{2(1-ka_\lambda)}} \tag{1.4}$$

has a unique solution $\widehat{\varphi}_k \in L^\infty(\overline{X}) \cap \mathrm{PSH}(\overline{X}, \chi - k\widetilde{\theta})$ by [Koł98]. Moreover, it is well-known that

$$\widehat{\omega}_k := \chi - k\widetilde{\theta} + dd^c\widehat{\varphi}_k$$

is smooth outside D, has conic singularities along each D_λ with cone angle $2\pi k a_\lambda$ (say if $\overline{X}$ is smooth, otherwise this will be true only after a finite cover) and one has $\operatorname{Ric}\widehat{\omega}_k = -\widehat{\omega}_k$ on $X = \overline{X} \setminus D$, cf e.g. [GP16]. In particular, we have as currents

$$\operatorname{Ric}\widehat{\omega}_k = -\widehat{\omega}_k + \sum_{\lambda=1}^{N}(1 - ka_\lambda)[D_\lambda]. \tag{1.5}$$

1.5. **Main result.** The aim of this section is to prove the following result.

Theorem 1.1. *The conic Kähler-Einstein metrics $\widehat{\omega}_k$ solution of* (1.5) *converge to ω_{KE} when $k \to 0$, both weakly as currents on $\overline{X}$ and locally smoothly on X.*

Proof. We divide the proof in three steps. In the first two steps, we assume that Γ is neat so that $\overline{X}$ is a smooth manifold. In the last step, we will explain how to work with the finite quotient singularities that $\overline{X}$ has in general.

Step 1. Weak convergence.
Let $k \in [0, \frac{1}{2}]$ and let $\tau_k \in (0, 1]$ be a number to be determined later. We set

$$\widehat{\psi}_k := \frac{1}{1-k} \cdot (\widehat{\varphi}_k + \tau_k);$$

this is a $\frac{1}{1-k}\cdot(\chi-k\widetilde{\theta})$-psh function with finite energy (even bounded if $k>0$) satisfying the Monge-Ampère equation

$$\left(\frac{1}{1-k}\cdot(\chi-k\widetilde{\theta})+dd^c\widehat{\psi}_k\right)^n=e^{(1-k)\widehat{\psi}_k+F_k}\cdot\frac{dV}{|s|_h^2} \tag{1.6}$$

where $F_k=k\sum_\lambda a_\lambda\log|s_\lambda|^2_{h_\lambda}-n\log(1-k)-\tau_k$. We claim that for any $k'>k$ small enough, one has

$$\frac{1}{1-k'}\cdot(\chi-k'\widetilde{\theta})+dd^c\widehat{\psi}_k\geqslant 0. \tag{1.7}$$

This follows from the identity

$$\frac{1-k}{1-k'}\cdot(\chi-k'\widetilde{\theta})=(\chi-k\widetilde{\theta})+\frac{k'-k}{1-k'}\cdot\underbrace{(\chi-\widetilde{\theta})}_{>0}. \tag{1.8}$$

More precisely, we get

$$\begin{aligned}\left(\frac{1}{1-k'}\cdot(\chi-k'\widetilde{\theta})+dd^c\widehat{\psi}_k\right)^n&\geqslant(1-k)^{-n}\left(\frac{1-k}{1-k'}\cdot(\chi-k'\widetilde{\theta})+dd^c\widehat{\varphi}_k\right)^n\\&\geqslant(1-k)^{-n}(\chi-k\widetilde{\theta}+dd^c\widehat{\varphi}_k)^n\\&=e^{(1-k')\widehat{\psi}_k+F_{k'}+H_{k',k}}\cdot\frac{dV}{|s|_h^2}\end{aligned}$$

where $H_{k',k}=(k'-k)(\widehat{\psi}_k-\log|s|^2)-n\log\left(\frac{1-k}{1-k'}\right)-\tau_k+\tau_{k'}$.

If we first choose $k=0$, $\tau_{k'}=Ck'-n\log(1-k')$ where $C>0$ is a constant such that $\varphi\geqslant\log|s|_h^2-C$, whose existence is guaranteed by (1.3), then we see that $H_{k',0}\geqslant 0$ so that $\widehat{\varphi}=\widehat{\psi}_0$ is a subsolution of (1.6), hence the comparison principle yields

$$\frac{1}{1-k}\cdot\widehat{\varphi}_k+\frac{Ck-n\log(1-k)}{1-k}\geqslant\widehat{\varphi} \tag{1.9}$$

Using the inequality above, we conclude that there is a constant $C'>0$ such that $\widehat{\psi}_k\geqslant\log|s|^2-C'$ for any k. Then we set $\tau_k:=C'k-n\log(1-k)$ and it follows that $H_{k',k}\geqslant 0$. In other words, $\widehat{\psi}_k$ is a subsolution of the Monge-Ampère equation satisfied by $\hat{\psi}_{k'}$, hence

$$\widehat{\psi}_{k'}\geqslant\widehat{\psi}_k. \tag{1.10}$$

The family $(\widehat{\psi}_k)_{k>0}$ is a decreasing family of quasi-psh functions with complex Hessian uniformly bounded from below. It follows that they converge when k approaches zero to a χ-psh function $\widetilde{\varphi}$. It follows from (1.9) that $\widetilde{\varphi}$ has finite energy, is locally bounded on X has satisfies

$$(\chi+dd^c\widetilde{\varphi})^n=\frac{e^{\widetilde{\varphi}}dV}{|s|^2}$$

on X by Bedford-Taylor theory, hence also globally on $\overline{X}$. By uniqueness of such a solution (cf e.g. [BG14, Prop. 4.1]), we get $\widetilde{\varphi}=\widehat{\varphi}$, which proves the first part of the proposition.

Step 2. Smooth convergence locally on X.
We apply Chern-Lu inequality to the identity map from $(X, \widehat{\omega}_k)$ to (X, χ), cf e.g. [Rub14, Proposition 7.1]. As $\mathrm{Ric}\, \widehat{\omega}_k = -\widehat{\omega}_k$ and the bisectional curvature of (X, χ) is bounded from above, there is a constant $A > 0$ such that

$$\Delta_{\widehat{\omega}_k} \log \mathrm{tr}_{\widehat{\omega}_k} \chi \geqslant -A(1 + \mathrm{tr}_{\widehat{\omega}_k} \chi). \tag{1.11}$$

Next, we have

$$\begin{aligned}\Delta_{\widehat{\omega}_k}(-\widehat{\varphi}_k) &= \mathrm{tr}_{\widehat{\omega}_k} \chi - n - \varepsilon\, \mathrm{tr}_{\widehat{\omega}_k} \widetilde{\theta} \\ &= \mathrm{tr}_{\widehat{\omega}_k} \chi - n + \varepsilon[\mathrm{tr}_{\widehat{\omega}_k}(\chi - \widetilde{\theta}) - \mathrm{tr}_{\widehat{\omega}_k} \chi] \\ &\geqslant (1-k)\, \mathrm{tr}_{\widehat{\omega}_k} \chi - n\end{aligned}$$

and, if $|\widetilde{s}|^2 := \prod |s_\lambda|^{2a_\lambda}_{h_\lambda}$,

$$\begin{aligned}\Delta_{\widehat{\omega}_k} \log |\widetilde{s}|^2 &= \mathrm{tr}_{\widehat{\omega}_k}(-\widetilde{\theta}) \\ &= \mathrm{tr}_{\widehat{\omega}_k}(\chi - \widetilde{\theta}) - \mathrm{tr}_{\widehat{\omega}_k} \chi \\ &\geqslant -\mathrm{tr}_{\widehat{\omega}_k} \chi.\end{aligned}$$

Fix some number $\delta \in (0, \frac{1}{4})$; it follows from the previous inequalities that the following holds on X

$$\Delta_{\widehat{\omega}_k}\left[\log \mathrm{tr}_{\widehat{\omega}_k} \chi - (A+1)\widehat{\varphi}_k + \delta \log |\widetilde{s}|^2\right] \geqslant (1 - (A+1)k - \delta)\, \mathrm{tr}_{\widehat{\omega}_k} \chi - B$$

where $B = (n+1)A + n$. Up to decreasing k, one can assume without loss of generality that $(A+1)k \leqslant 1/2$ so that $(1 - (A+1)k - \delta) \geqslant \frac{1}{4}$. Set $H_{k,\delta} := \log \mathrm{tr}_{\widehat{\omega}_k} \chi - (A+1)\widehat{\varphi}_k + \delta \log |\widetilde{s}|^2$; it is a smooth function on X which tends to $-\infty$ near D thanks to (1.3)-(1.9). At its maximum $x_{\delta,k} \in X \setminus D$, one has $\mathrm{tr}_{\widehat{\omega}_k} \chi(x_{k,\delta}) \leqslant 4B$. Therefore, one has, for any $x \in X$

$$\begin{aligned}\log \mathrm{tr}_{\widehat{\omega}_k} \chi(x) &= H_{k,\delta}(x) + (A+1)\widehat{\varphi}_k(x) - \delta \log |\widetilde{s}|^2(x) \\ &\leqslant H_{k,\delta}(x_{k,\delta}) + (A+1)\widehat{\varphi}_k(x) - \delta \log |\widetilde{s}|^2(x) \\ &\leqslant \log 4B + [\delta \log |\widetilde{s}|^2 - (A+1)\widehat{\varphi}_k](x_{k,\delta}) + (A+1)\widehat{\varphi}_k(x) - \delta \log |\widetilde{s}|^2(x) \\ &\leqslant C_\delta - \delta \log |\widetilde{s}|^2(x)\end{aligned}$$

where we used the fact that $\widehat{\varphi}_k$ is uniformly bounded above (e.g. (1.10)) and $\widehat{\varphi}_k \geqslant -(n+2)\log(-\log|s|) + O(1)$ by (1.9). As a result, we get

$$\chi \leqslant \frac{C_\delta}{|\widetilde{s}|^{2\delta}} \cdot \widehat{\omega}_k$$

uniformly on X, for any $\delta \in (0, 1/4)$. From the Monge-Ampère equation satisfied by $\widehat{\omega}_k$ and the fact that χ is a smooth Kähler form on X, we deduce that given any compact subset $V \Subset X$, there is a constant C_V independent of k such that

$$\sup_V |\Delta_\chi \widehat{\varphi}_k| \leqslant C_V.$$

Using standard bootstrapping arguments, we get uniform bounds on the higher derivatives of φ_k on compact subsets of X, which ends the proof of the theorem in the case where Γ is neat.

Step 3. General case when Γ is not neat.
Let $\Gamma' < \Gamma$ be a neat, finite index sub-lattice. The quotient $Y = {}^{\Gamma'}\backslash^{\Omega}$ admits a smooth toroidal compactification $(\overline{Y}, D')$, and let $f : \overline{Y} \to \overline{X}$ be the associated finite cover. Set m_λ to be the ramification order of f along D_λ, and set $D'_\lambda = f^{-1}(D_\lambda)$. In summary, one has

$$K_{\overline{Y}} + \sum_\lambda (1 - ka_\lambda m_\lambda) D'_\lambda = f^*(K_{\overline{X}} + \sum_\lambda (1 - ka_\lambda) D_\lambda).$$

The Kähler-Einstein metrics $\widehat{\omega}'_k := f^*\widehat{\omega}_k$ have cone singularities along D' with cone angle $2\pi(1 - ka_\lambda m_\lambda)$ along D'_λ. That is, they satisfy $\mathrm{Ric}\,\widehat{\omega}'_k = -\widehat{\omega}'_k + \sum_\lambda (1 - ka_\lambda m_\lambda)[D'_\lambda]$. Thanks to Steps 1-2 above, $\widehat{\omega}'_k$ converge globally weakly on $\overline{Y}$ and locally smoothly on Y towards the hyperbolic metric Y. Of course, one needs to perform a harmless adjustment by replacing χ with $f^*\chi$. Since the hyperbolic metric on Y is nothing but $f^*\omega_{\mathrm{KE}}$, cf. § 1.2, the theorem follows immediately. □

2. The Calabi ansatz

We now construct some explicit model Kähler metrics in the total space of a holomorphic line bundle L over D. This technique goes back to Calabi [Cal79].

Model Setup. Let D be a compact Kähler manifold equipped with a Kähler form θ_D and let (L, h) be a Hermitian holomorphic line bundle over D. We make the following assumptions:

(i) $\mathrm{Ric}(\theta_D) = 0$.
(ii) $i\Theta(L, h) = \sigma\theta_D, \qquad \sigma = \pm 1$.

We think of L as the normal bundle of D which will be a divisor in a compact complex manifold X. The case $\sigma = -1$ corresponds to a quotient of a ball: in that case D will be a torus. The case $\sigma = 1$ corresponds to that of an anticanonical divisor in a Fano manifold X.

We consider the function $t = \log \|v\|_h^2$ defined on $L \setminus D$, the complement of the zero section in the total space of L. We also have on $L \setminus D$ a connection 1-form η which coincides on each fibre of L with the angular form $d\theta$, and satisfies

$$d\eta = -ip^*\Theta(L, h) = -\sigma p^*\theta_D,$$

where p is the projection $p : L \to D$. Then $\xi = \frac{1}{2}dt + i\eta$ is a (1,0)-form on $L \setminus D$, coinciding with $\frac{dz}{z}$ in each fibre. In particular, $dt \wedge \eta = i\xi \wedge \bar{\xi}$ coincides with $\frac{idz \wedge d\bar{z}}{|z|^2}$ in each fiber. In the following, one will identify $p^*\theta_D$ with θ_D and view the latter as a $(1,1)$-form on the total space L.

We are looking for a Kähler metric $\omega = i\partial\bar\partial\varphi$ on $L \setminus D$ whose Kähler potential $\varphi = \varphi(t)$ only depends on t and such that

$$\mathrm{Ric}(\omega) = \sigma\omega. \tag{2.1}$$

One can compute the coefficients of the metric in the frame introduced above as follows. First, $d^c\varphi = \frac{1}{i}(\partial - \bar\partial)\varphi = 2\varphi'(t)\eta$ and, then the associated Kähler form is

$$\omega = i\partial\bar\partial\varphi = \frac{1}{2}dd^c\varphi = \varphi'' dt \wedge \eta - \sigma\varphi'\theta_D. \tag{2.2}$$

In particular, we have as necessary conditions

$$\varphi'' > 0 \quad \text{and} \quad -\sigma\varphi' > 0.$$

Let ϖ be a (maybe local) parallel $(n-1,0)$ form on D, then $\Omega = \xi \wedge \varpi$ satisfies $d\Omega = i d\eta \wedge \varpi = 0$, so Ω is holomorphic. Moreover, up to some positive constant, $|\Omega|^{-2} = (-\sigma\varphi')^{n-1}\varphi''$. As $\mathrm{Ric}\,\omega = i\partial\bar\partial \log|\Omega|^2$, in order for ω to be a solution of (2.1), it is enough to see that φ is a solution of the following equation

$$(-\sigma\varphi')^{n-1}\varphi'' = ce^{-\sigma\varphi} \tag{2.3}$$

for some constant $c > 0$. This can be integrated into

$$(-\sigma\varphi')^{n+1} = a - b\sigma e^{-\sigma\varphi} \tag{2.4}$$

for constants $a \in \mathbb{R}$ and $b = (n+1)c > 0$.

As we shall see, the solutions $\varphi(t)$ will be defined on intervals of the form $(-\infty, t_0)$ for some arbitrary constant $t_0 \in \mathbb{R}$, and they will satisfy $\sigma\varphi(t) \to \infty$ when $t \to -\infty$, that is when we go to the divisor D. It follows that $a \geq 0$ and actually $\varphi(t) \sim -\sigma a^{\frac{1}{n+1}} t$ when $t \to -\infty$, which says that ω extends over D with $\omega|_D = a^{\frac{1}{n+1}}\theta_D$. Coming back to the equation (2.4) we obtain the first terms of the expansion of φ when $t \to -\infty$, for some constant φ_0:

$$\varphi(t) \sim -\sigma a^{\frac{1}{n+1}} t + \varphi_0 + \frac{b}{(n+1)a} e^{-\sigma\varphi_0 + a^{\frac{1}{n+1}} t} + \cdots \tag{2.5}$$

which shows that ω has actually a conical singularity around D with angle $2\pi a^{\frac{1}{n+1}}$, so the angle goes to zero when $a \to 0$, this is the limit we want to study.

Observe that if we have a solution $\varphi_1(t)$ of equation (2.4) with $a = b = 1$, then $\varphi_1(kt) + \varphi_0$ is still a solution with $a = k^{n+1}$ and $b = k^{n+1}e^{\sigma\varphi_0}$. We use this remark to produce our model families $(\varphi_k(t))$ with angle $2\pi k$ degenerating to zero:

Negative case.
(i) The potential φ_1.
This is when $\sigma = -1$. The function φ_1 satisfies $(\varphi_1')^{n+1} = 1 + e^{\varphi_1}$ so we can take as solution $\varphi_1(t) = F_-^{-1}(t) : (-\infty, 0) \to \mathbb{R}$ with $F_- : \mathbb{R} \to (-\infty, 0)$ defined

by

$$F_-(x) = -\int_x^{+\infty} \frac{dx}{(1+e^x)^{\frac{1}{n+1}}}.$$

One can check that the precise behavior of φ_1 at $t=-\infty$ is given by

$$\varphi_1(t) = t + I_n + \frac{e^{I_n}}{n+1}\cdot e^t + O(e^{2t}) \quad \text{when } t\to -\infty, \tag{2.6}$$

where the constant I_n is defined by

$$I_n := \int_0^{+\infty} \frac{du}{(e^u+1)^{\frac{1}{n+1}}} - \int_{-\infty}^0 \frac{(e^u+1)^{\frac{1}{n+1}}-1}{(e^u+1)^{\frac{1}{n+1}}}du \tag{2.7}$$

while at $t=0^-$, one has

(2.8)

$$\varphi_1(t) = -(n+1)\log\left(\frac{-t}{n+1}\right) + \frac{1}{n+2}\cdot\left(\frac{-t}{n+1}\right)^{n+1} + O(t^{2(n+1)}) \quad \text{when } t\to 0^-.$$

(ii) Degeneration.

We choose to fix $b=1$ by taking

$$\varphi_k(t) = \varphi_1(kt) + (n+1)\log k. \tag{2.9}$$

When $k\to 0$, (2.8) implies that $\varphi_k(t)\to -(n+1)\log(\frac{-t}{n+1})$ which is the Kähler potential of the hyperbolic cusp.

Positive case.

(i) The potential φ_1.

In the positive case, $\sigma=1$. The function φ_1 satisfies $(-\varphi_1')^{n+1} = 1-e^{-\varphi_1}$ and we take $\varphi_1(t) = F_+^{-1}(-t) : (-\infty,0)\to\mathbb{R}_+$ with $F_+:\mathbb{R}_+\to\mathbb{R}_+$ by

$$F_+(x) = \int_0^x \frac{dx}{(1-e^{-x})^{\frac{1}{n+1}}}.$$

Again, one can obtain the precise behavior of φ_1 at $t=-\infty$ as

$$\varphi_1(t) = -t - J_n + \frac{e^{J_n}}{n+1}\cdot e^t + O(e^{2t}) \quad \text{when } t\to -\infty, \tag{2.10}$$

where the constant J_n is defined by

$$I_n := \int_0^{+\infty} \frac{1-(1-e^{-u})^{\frac{1}{n+1}}}{(1-e^{-u})^{\frac{1}{n+1}}}du$$

while at $t=0^-$, one has

$$\varphi_1(t) = c_n(-t)^{1+\frac{1}{n}}(1+O((-t)^{1+\frac{1}{n}}) \quad \text{when } t\to 0^-, \tag{2.11}$$

where $c_n = (\frac{n}{n+1})^{\frac{n+1}{n}}$.

(ii) Degeneration.

We now choose the degeneration

$$\varphi_k(t) = \varphi_1(kt). \tag{2.12}$$

Here the limit when $k \to 0$ is just 0. More precisely, the asymptotics (2.11) imply that when $k \to 0$ one has

$$\varphi_k(t) \sim \left(\frac{-nkt}{n+1}\right)^{1+\frac{1}{n}}. \tag{2.13}$$

Therefore the rescaling $k^{-1-\frac{1}{n}}\varphi_k$ converges to $(-\frac{nt}{n+1})^{1+\frac{1}{n}}$ which is the Kähler potential of a Ricci flat metric on $L \setminus D$, the Tian-Yau metric: it gives the asymptotic behaviour of the Tian-Yau metric of $X \setminus D$ near D.

We have seen a limit of $\varphi_k(t)$ when $k \to 0$ on each compact set in t. To understand the global geometry of our models, we write from (2.2)

$$\omega_k = i\partial\overline{\partial}\varphi_k = k^2\varphi_1''(kt)dt \wedge \eta - k\sigma\varphi_1'(kt)\theta_D. \tag{2.14}$$

The geometry is clear when one writes the associated Riemannian metric g_k, after the change of variable $u = kt \in (-\infty, 0)$:

$$g_k = 2\varphi_1''(u)\big(\tfrac{1}{4}du^2 + k^2\eta^2\big) - k\sigma\varphi_1'(u)g_D. \tag{2.15}$$

The geometry collapses at speed $\sqrt{k}$ in the directions of D, and k in the circle directions. Observe that, up to a multiplicative constant, we have when $u \to 0$ the asymptotics $\varphi_1''(u) \sim u^{-2}$ in the negative case, and $u^{-1+\frac{1}{n}}$ in the positive case, from which it follows that the diameter is infinite in the negative case and bounded in the positive case (as it should by Myers's theorem). More precisely, we have at $u = 0$ the following asymptotics

$$\varphi_1'(u) = -c_n'(-u)^{1/n} + O((-u)^{1+\frac{2}{n}}), \quad \varphi_1''(u) = \frac{c_n'}{n}(-u)^{-1+1/n} + O((-u)^{\frac{2}{n}}). \tag{2.16}$$

with $c_n' = \big(\frac{n}{n+1}\big)^{1/n}$.

To see the behaviour of the metric near the divisor D, we consider the expansion at $u = -\infty$ given as in (2.5) by

$$\varphi_1'(u) = -\sigma + \frac{1}{n+1}e^u + O(e^{2u}) \tag{2.17}$$

from which follows, taking $r = e^{\frac{u}{2}} \in (0,1)$,

$$\begin{aligned} g_k &= \tfrac{2}{n+1}e^u\big(\tfrac{1}{4}du^2 + k^2\eta^2\big) + kg_D + O(e^u) && (2.18)\\ &= \tfrac{2}{n+1}\big(dr^2 + k^2r^2\eta^2\big) + kg_D + O(r^2), && (2.19)\end{aligned}$$

where the $O(r^2)$ is with respect to g_k and is uniform with respect to k.

3. Curvature calculations

We now calculate the curvature of our model metrics ω_k on $L \setminus D$. This gives a control of the local geometry and will be also used in the Laplacian estimate.

The Kähler metric $\omega = i\partial\bar{\partial}\varphi$ induces a hermitian metric h on the bundle $\Omega^{1,0}$ of $(1,0)$-forms on $L \setminus D$, and it will be convenient to compute its Chern curvature tensor $F^{\Omega^{1,0}}$. At this point, we take an arbitrary Kähler potential $\varphi = \varphi(t)$. We will use the $\mathcal{C}^\infty$ orthogonal splitting

$$\Omega^{1,0} = \mathbb{C}\xi \oplus p^*\Omega^{1,0}_D.$$

From (2.2) we see that the Hermitian metric preserves this decomposition, and is equal to

$$h = \begin{pmatrix} (\varphi'')^{-1} & \\ & (-\sigma\varphi')^{-1}h^{\Omega^{1,0}_D} \end{pmatrix}.$$

where $h^{\Omega^{1,0}_D}$ is the hermitian metric induced on $\Omega^{1,0}_D$ by the Kähler metric θ_D. From $d\xi = id\eta = -i\sigma\theta_D$ we deduce the $\bar\partial$ and ∂ operators of $\Omega^{1,0}$ in this splitting:

$$\bar\partial = \begin{pmatrix} \bar\partial & 0 \\ a & \bar\partial^{\Omega^{1,0}_D} \end{pmatrix}, \qquad \partial = \begin{pmatrix} \partial - \partial\log\varphi'' & -\frac{\varphi''}{-\sigma\varphi'}a^* \\ 0 & \partial^{\Omega^{1,0}_D} - \partial\log(-\sigma\varphi') \end{pmatrix}$$

Here a is the $(0,1)$-form with values in $\mathrm{Hom}(\mathbb{C},\Omega^{1,0}_D) = \Omega^{1,0}_D$ defined by $a_X = X\lrcorner d\xi = -i\sigma X\lrcorner\theta_D$, and a^* its adjoint $a^*_X\alpha = -i\sigma\Lambda(\alpha\wedge(X\lrcorner\theta))$. The familiar form for the curvature is then

$$F^{\Omega^{1,0}} = \begin{pmatrix} \partial\bar\partial\log\varphi'' - \frac{\varphi''}{-\sigma\varphi'}a^*\wedge a & -\bar\partial(\frac{\varphi''}{-\sigma\varphi'}a^*) \\ \partial^h a & \partial\bar\partial\log\varphi' + F^{\Omega^{1,0}_D} - \frac{\varphi''}{-\sigma\varphi'}a\wedge a^* \end{pmatrix}$$

so that $iF^{\Omega^{1,0}}$ is given by

(3.1)
$$\begin{pmatrix} (\log\varphi'')''i\xi\wedge\bar\xi - \sigma(\log\frac{\varphi''}{-\sigma\varphi'})'\theta_D & -i(\frac{\varphi''}{-\sigma\varphi'})'\bar\xi\wedge a^* \\ i\log(\frac{\varphi''}{-\sigma\varphi'})'\xi\wedge a & \log(-\sigma\varphi')''i\xi\wedge\bar\xi - \sigma\log(-\sigma\varphi')'(\theta_D+\Theta_D) + iF^{\Omega^{1,0}_D} \end{pmatrix}$$

where the last Θ_D is the 2-form with values in the endomorphisms of $\Omega^{1,0}_D$ defined by $(\Theta_D)_{X,Y}(\alpha) = -(X\lrcorner\alpha)(Y\lrcorner\theta_D)$ for $X\in T^{1,0}$ and $Y\in T^{0,1}$.

Lemma 3.1. *One has the following bounds for the curvature of the model Kähler metric ω_k defined in (2.14):*

- *in the negative case ($\sigma=-1$), if D is flat, then the curvature is bounded;*
- *in the positive case ($\sigma=1$), the curvature is bounded by*

$$\mathrm{cst.}\Big(\frac{1}{1-e^{-\varphi_k}} + \frac{1}{k(1-e^{-\varphi_k})^{\frac{1}{n+1}}}\Big).$$

Proof. For now, $\omega = i\partial\bar\partial\varphi$ is still arbitrary, and only at the end we will choose $\varphi := \varphi_k$ to be the potential constructed by the Calabi Ansatz above. Let $X = \lambda\xi^* + v$ with $\lambda\in\mathbb{C}, v\in T^{1,0}_D$ and $Y = \mu\xi+\alpha$ with $\mu\in\mathbb{C},\alpha\in\Omega^{1,0}_D$. We assume that $\|\xi\|_\omega = \|Y\|_h = 1$, so that

$$|\lambda|^2 \leqslant \frac{1}{\varphi''},\ \|v\|^2_{\theta_D} \leqslant \frac{1}{-\sigma\varphi'}, \quad \text{and} \quad |\mu|^2\leqslant\varphi'',\ \|\alpha\|^2_{h_D}\leqslant -\sigma\varphi' \tag{3.2}$$

where h_D is the metric on $\Omega_D^{1,0}$ induced by θ_D. The hermitian matrix $h \cdot iF_{X,\bar{X}}^{\Omega^{1,0}}$ is equal to

$$\begin{pmatrix} |\lambda|^2 \frac{(\log \varphi'')''}{\varphi''} - \frac{\sigma \|v\|^2_{\theta_D}}{\varphi''} \log(\frac{\varphi''}{-\sigma\varphi'})' & -i\bar{\lambda} a_v^* \frac{1}{-\sigma\varphi'} \log(\frac{\varphi''}{-\sigma\varphi'})' \\ i\lambda a_{\bar{v}} \frac{1}{-\sigma\varphi'} \log(\frac{\varphi''}{-\sigma\varphi'})' & \frac{\log(-\sigma\varphi')''}{-\sigma\varphi'}|\lambda|^2 + \frac{\log(-\sigma\varphi')'}{\varphi'}(\theta_D + \Theta_D)_{v,\bar{v}} + \frac{i}{-\sigma\varphi'} F_{v,\bar{v}}^{\Omega_D^{1,0}} \end{pmatrix}$$

As a result, the expansion of $\langle iF_{X,\bar{X}}^{\Omega^{1,0}} Y, Y\rangle_h$ involves the following terms

$$|\lambda\mu|^2 \frac{(\log \varphi'')''}{\varphi''}, |\mu|^2 \frac{\|v\|^2_{\theta_D}}{\varphi''} (\log(\frac{\varphi''}{-\sigma\varphi'})', \lambda\bar{\mu}\langle a_{\bar{v}}, \alpha\rangle_{h_D} \log(\frac{\varphi''}{-\sigma\varphi'})',$$

$$|\lambda|^2 \|\alpha\|^2_{h_D} \frac{\log(-\sigma\varphi')''}{-\sigma\varphi'}, \|\alpha\|^2_{h_D} \frac{\log(-\sigma\varphi')'}{\varphi'}(\theta_D + \Theta_D)_{v,\bar{v}}, \frac{1}{\varphi'}\langle iF_{v,\bar{v}}^{\Omega_D^{1,0}} \alpha, \alpha\rangle_{h_D}.$$

Given the bounds (3.2), we see that we need to bound the quantities

$$\frac{(\log \varphi'')''}{\varphi''}, \quad \frac{\log(-\sigma\varphi')''}{\varphi''}, \quad \frac{(\log \varphi'')'}{\varphi'}, \quad \frac{\log(-\sigma\varphi')'}{\varphi'}, \tag{3.3}$$

while the term $F^{\Omega_D^{1,0}}$ (present only in the positive case) is bounded by $\frac{\text{cst.}}{-\sigma\varphi'}$.

Up to an additive constant, we have $\varphi_k(t) = \varphi_1(kt)$, so we see that the factors k in (3.3) cancel and it is enough to bound these quantities for φ_1, while the term involving the curvature of D is bounded by $\frac{\text{cst.}}{-k\varphi_1'(kt)}$. We now use the equation satisfied by φ_1, that is

$$-\sigma\varphi_1' = (1 - \sigma e^{-\sigma\varphi_1})^{\frac{1}{n+1}}.$$

Taking $x = e^{-\sigma\varphi_1}$ we obtain $x' = x(1-\sigma x)^{\frac{1}{n+1}}$. It is then convenient to write all the quantities in terms of x. We have $-\sigma\varphi_1' = (1-\sigma x)^{\frac{1}{n+1}}$, therefore $-\sigma\varphi_1'' = \frac{1}{n+1} x(1-\sigma x)^{-\frac{n-1}{n+1}}$. Then one calculates all quantities in (3.3):

$$\frac{\log(-\sigma\varphi_1')'}{\varphi_1'} = \frac{x}{1-\sigma x}, \qquad \frac{\log(-\sigma\varphi_1')''}{\varphi_1''} = \frac{n+1-\sigma x}{(n+1)(1-\sigma x)},$$
$$\frac{\log(-\sigma\varphi_1'')'}{\varphi_1'} = \frac{n+1-2\sigma x}{(n+1)(1-\sigma x)}, \qquad \frac{\log(-\sigma\varphi_1'')''}{\varphi_1''} = \frac{-n^2+n+2-2\sigma x}{(n+1)(1-\sigma x)}.$$

If $\sigma = -1$ then $1 - \sigma x = 1 + e^{\varphi} \geq 1$ therefore all these quantities are bounded. If $\sigma = 1$ then $1 - \sigma x = 1 - e^{-\varphi}$ and we obtain the bounds in the statement of the lemma. □

In the positive case, the divisor D corresponds to $\varphi \to +\infty$ so the curvature is $O(k^{-1})$ in all sets of the form $u < -A < 0$. At $u = 0$ the metric g_k from (2.15) degenerates, but this part will be cut out since we will glue with the rest of X.

4. Asymptotics of the conical KE metrics on ball quotients

4.1. Set-up. In this section, we borrow the setup and notation of § 1.1 and we assume additionally that $\Omega = \mathbb{B}^n$ is the complex hyperbolic space of dimension n. In this section, we assume that Γ is neat, so that $X = {}^{\mathbb{B}}\!/_{\Gamma}$ can be compactified smoothly by adding finitely many disjoint tori $D_1, \ldots, D_N$ of dimension $n-1$. In general, this is only true up to the action of a finite group (locally in the neighborhood of each torus). The Kähler-Einstein metric ω_{KE} is, up to a normalizing constant, the *hyperbolic* metric on X, described locally near D by (1.1).

It will be important in the following to allow cone angles along D_λ that are not necessarily of the form $2\pi k a_\lambda$ for some given $a_\lambda > 0$ and a *single* parameter $k > 0$ going to zero. For that reason and from now on, we denote by $k := (k_1, \ldots, k_N)$ a N-tuple of positive numbers. Since the components $D_1, \ldots, D_N$ of the boundary divisor D are disjoint, the divisor $-\sum_\lambda a_\lambda D_\lambda$ is relatively ample for *any* $a_\lambda > 0$. In particular, up to changing h_λ one can find $\delta_0 > 0$ such that $\chi - \delta\theta_\lambda$ is semi-positive globally on $\overline{X}$ and Kähler on $X \setminus \sqcup_{\mu\neq\lambda} D_\mu$, for any $\delta \leqslant 2\delta_0 N$. As a result, $\chi - \sum_\lambda a_\lambda\theta_\lambda$ is globally Kähler on $\overline{X}$ for any $a_\lambda \in (0, 2\delta_0]$. In the following, one will assume that $k_\lambda \leqslant \delta_0/2N$ for any λ.

The Kähler-Einstein metric $\widehat{\omega}_k = \chi - \sum_{\lambda=1}^N k_\lambda\theta_\lambda + dd^c\widehat{\varphi}_k$ solution of

$$\operatorname{Ric}\widehat{\omega}_k = -\widehat{\omega}_k + \sum_{\lambda=1}^N (1-k_\lambda)[D_\lambda] \tag{4.1}$$

for $\|k\|$ small enough solves the following Monge-Ampère equation

$$(\chi - \sum_{\lambda=1}^N k_\lambda\theta_\lambda + dd^c\widehat{\varphi}_k)^n = \frac{e^{\widehat{\varphi}_k} dV}{\prod_\lambda |s_\lambda|_{h_\lambda}^{2(1-k_\lambda)}} \tag{4.2}$$

where dV is a smooth volume form such that $-\operatorname{Ric}(dV) + \sum_{\lambda=1}^N \theta_\lambda = \chi$.

One can reproduce the arguments in the proof of Theorem 1.1 verbatim to show that $\widehat{\varphi}_k$ almost decreases to $\widehat{\varphi}$ when k goes to zero. More precisely, one can find a sequence of real numbers $\tau_k \to 0$ such that $\frac{1}{1-\delta_0^{-1}\sum_\lambda k_\lambda}(\widehat{\varphi}_k + \tau_k)$ decreases to $\widehat{\varphi}$ when $k \searrow 0$ component-wise. The main point is that if $k' > k$ component-wise and if we set $K = \sum k_\lambda$ (resp. $K' = \sum k'_\lambda$), we have

$$\begin{aligned}\frac{1-\delta_0^{-1}K}{1-\delta_0^{-1}K'} \cdot (\chi - \sum k'_\lambda\theta_\lambda) =& (\chi - \sum k_\lambda\theta_\lambda) \\ &+ \frac{K'-K}{\delta_0 - K'} \cdot \Big[\chi - \sum_\lambda \underbrace{\Big(\delta_0 k_\lambda + \frac{\delta_0 - K}{K'-K}\cdot(k'_\lambda - k_\lambda)\Big)}_{\in(0,2\delta_0)} \cdot\theta_\lambda\Big].\end{aligned}$$

which replaces the identity (1.8).

Moreover, the Laplacian estimate from the proof of Theorem 1.1 carries over with no significant change, and therefore

$$\widehat{\omega}_k \underset{k\to 0}{\longrightarrow} \omega_{\mathrm{KE}} \quad \text{in} \quad \mathcal{C}^\infty_{\mathrm{loc}}(U^*) \tag{4.3}$$

4.2. **Comparison to the model metric.** We now aim to compare the global Kähler-Einstein metric $\widehat{\omega}_k$ to the model ω_k constructed via the Calabi Ansatz in § 2. One

Given any torus $D \in \{D_1, \ldots, D_N\}$, one can identify an open neighborhood U of D in $\overline{X}$ to a neighborhood of the zero section in the total space of the normal bundle $L := N_{D/\overline{X}} \to D$. Moreover, L comes naturally equipped with a smooth hermitian metric h such that $\theta_D := \pi \cdot i\Theta(L^{-1}, h^{-1})$ is a flat Kähler metric on D. We let $p : U \to D$ be the projection induced by the identification of U to an open subset of the total space of L. Under this identification and given a point $(x, v) \in U$ (i.e. $x \in D, v \in L_x$), we can consider the quantity $\|v\|_h^2$ and assume that $\|v\|_h^2 < e^{-1}$ on U. On $U^* := U \setminus D$, the smooth function $t = \log \|v\|_h^2 : U^* \to (-\infty, -1)$ satisfies

$$i\partial\bar{\partial} t = p^*\theta_D. \tag{4.4}$$

Moreover, the Kähler-Einstein metric ω_{KE} on X has an *exact* expression in restriction to U^*; namely

$$\begin{aligned} \omega_{\mathrm{KE}}|_{U^*} &= i\partial\bar{\partial}\big[-(n+1)\log(-t)\big]. \\ &= (n+1)\Big[\frac{\xi \wedge \bar{\xi}}{(-t)^2} + \frac{p^*\theta_D}{-t}\Big] \end{aligned} \tag{4.5}$$

where $\xi = \frac{1}{2}dt + i\eta$ has been defined in § 2. We have observed in *ibid.* that the potential $\varphi(t) = -(n+1)\log(-t)$ of ω_{KE} is the limit of the potentials

$$\psi_k := \varphi_k(t) + (n+1)\log(n+1)$$

of ω_k (cf. (2.9)) when $k \to 0$ and that the convergence is smooth on the compact subsets of $\overline{U} \setminus D$. In particular, we get

$$\omega_k \underset{k\to 0}{\longrightarrow} \omega_{\mathrm{KE}} \quad \text{in} \quad \mathcal{C}^\infty_{\mathrm{loc}}(U^*). \tag{4.6}$$

Let $\Omega = \xi \wedge \varpi$ be the holomorphic n-form with logarithmic poles along D constructed on U in the previous section.The Monge-Ampère equation solved by ψ_k reads

$$(i\partial\bar{\partial}\psi_k)^n = e^{\psi_k} i^{n^2} \Omega \wedge \overline{\Omega}.$$

The Monge-Ampère equation solved by $\widehat{\omega}_k$ has a similar form. Indeed, let ψ_χ be a smooth potential for χ on U and let us set $\widehat{\psi}_k := \psi_\chi - kt + \widehat{\varphi}_k$ which is well-defined on U^*. Recall that $\widehat{\varphi}_k \in L^\infty(U^*)$ so that $\widehat{\psi}_k - \varphi_k$ is globally bounded on U^* (only qualitatively at this point). Moreover, we have

$$(i\partial\bar{\partial}\widehat{\psi}_k)^n = e^{\widehat{\psi}_k + F_k} i^{n^2} \Omega \wedge \overline{\Omega}$$

where F_k is a smooth function on U^*, globally bounded independently of k, i.e. $\|F_k\|_{L^\infty(U^*)} \leqslant C_1$.

Lemma 4.1. *The following bound holds*

$$\|\widehat{\psi}_k - \psi_k\|_{L^\infty(U)} \leqslant C_1. \tag{4.7}$$

Proof. This is a simple application of the maximum principle. Indeed, let $\delta > 0$ arbitrarily small and let $H = H_{k,\delta} := \widehat{\psi}_k - \varphi_k + \delta t$. Since $H_{k,\delta}$ goes to $-\infty$ along D, its maximum is attained at a point $x = x_{k,\delta} \in \overline{U} \setminus D$ at which the complex Hessian of H is non-positive. In particular, we get $i\partial\bar{\partial}\widehat{\psi}_k \leqslant i\partial\bar{\partial}\psi_k - \delta\theta_D \leqslant i\partial\bar{\partial}\psi_k$

at x. Taking the top wedge product and using the Monge-Ampère equations above, we find $H(x) \leqslant -F_k(x) + \delta t(x) \leqslant C_1$. In particular, $H \leqslant C_1$ everywhere on U^* and passing to the limit when $\delta \to 0$, we get the first half of (4.7). The other half is obtained in a similar way. $\square$

Remark 4.2. In the lemma above, we could have use Bedford-Taylor's comparison principle instead of the maximum principle (with the tweak by δt), see e.g. [CKZ11, Lemma 3.4].

Next, we claim that $\widehat{\omega}_k$ and ω_k are uniformly quasi-isometric on U^*.

Lemma 4.3. *There exists $C_2 > 0$ independent of k such that*

$$C_2^{-1}\omega_k \leqslant \widehat{\omega}_k \leqslant C_2\omega_k. \tag{4.8}$$

Proof. Consider the smooth function

$$H_k := \log \mathrm{tr}_{\widehat{\omega}_k} \omega_k \quad \text{on } U^*.$$

Since $\mathrm{Ric}\, \widehat{\omega}_k = -\widehat{\omega}_k$ and the holomorphic bisectional curvature of ω_k is bounded above independently of k by Lemma 3.1, an application of Chern-Lu formula (see e.g. [Rub14, Proposition 7.1]) yields a constant $B > 0$ independent of k such that

$$\Delta_{\widehat{\omega}_k} H_k \geqslant -1 - Be^{H_k} \quad \text{on } U^*. \tag{4.9}$$

Thanks to (4.3)- (4.6), we have

$$H_k \leqslant (n+1) \quad \text{on } \partial U \tag{4.10}$$

for k small enough.

Since $\Delta_{\widehat{\omega}_k}(\psi_k - \widehat{\psi}_k) = e^{H_k} - n$ and $i\partial\overline{\partial}t \geqslant 0$, we get for any $\delta > 0$

$$\Delta_{\omega_\varepsilon}\big(H_k - (B+1)\cdot(\psi_k - \widehat{\psi}_k) + \delta t\big) = e^{H_\varepsilon} - n(B+1) - 1.$$

The maximum of the function inside the Laplacian is attained at $x \in \overline{U} \setminus D$. If $x \in \partial U$, then (4.7)-(4.10) and the inequality $t \leqslant 0$ imply that $H_k \leqslant (n+1) + 2(B+1)C_1 - \delta t$ on U^*. If $x \in U$, then the maximum principle implies that $H_k \leqslant (B+1)(n+2C_1) + 1 - \delta t$ on U^*. Passing to the limit when $\delta \to 0$, one finds $H_k \leqslant C_2$ on U^*. The result follows (up to enlarging C_2) since the Monge-Ampère of ω_k and $\widehat{\omega}_k$ are commensurable - which itself relies on the estimate (4.7). $\square$

Since we know that ω_k and $\widehat{\omega}_k$ are asymptotically close at any order away from D, one can improve Lemma 4.3 as follows.

Lemma 4.4. *There exists a sequence of numbers $\varepsilon_k \searrow 0$ such that*

$$(1-\varepsilon_k)\omega_k \leqslant \widehat{\omega}_k \leqslant (1+\varepsilon_k)\omega_k \quad \textit{on } U^*. \tag{4.11}$$

Proof. We introduce for any $\delta > 0$ the quantities

$$F_k := \log\left(\frac{\widehat{\omega}_k^n}{\omega_k^n}\right) \quad \text{and} \quad F_{k,\delta} := F_k + \delta t.$$

The function F_k is bounded on U and smooth away from D. If we can show that F_k converges uniformly to 0 on U, then we will be done since we know that $\widehat{\omega}_k$ and ω_k are uniformly quasi-isometric thanks to Lemma 4.3. First, we observe that

$$\lim_{k\to 0} \|F_k\|_{L^\infty(\partial U)} = 0 \tag{4.12}$$

thanks to (4.3)- (4.6). Let $x \in U$ be a point where $F_{k,\delta}$ attains its maximum. If $x \in \partial U$, we have $F_{k,\delta} \leqslant \|F_k\|_{L^\infty(\partial U)}$ which goes to zero by (4.12). Otherwise, $x \in U^*$ and we have $dd^c F_{k,\delta}(x) \leqslant 0$. Since both metrics are Kähler-Einstein with the same constant, we have $dd^c F_k = \widehat{\omega}_k - \omega_k$. In particular, we get at the point x the following inequality

$$\widehat{\omega}_k(x) \leqslant \omega_k(x) - \delta dd^c t \leqslant \omega_k(x)$$

It follows that $F_k(x) \leqslant 0$, hence $F_{\varepsilon,\delta} \leqslant 0$. Passing to the limit when $\delta \to 0$, we obtain that in any case, $\sup_U F_k \leqslant o(1)$ when $k \to 0$.

One can proceed similarly with $G_{k,\delta} = \log\left(\frac{\omega_k^n}{\widehat{\omega}_k^n}\right) + \delta t$ to see that $\inf_U F_k \geqslant o(1)$ when $k \to 0$. The lemma is proved. □

To finish this section, we put together the Laplacian estimate (4.11) with the asymptotics (2.6)-(2.8), which yields

Theorem 4.5. *The conical Kähler-Einstein metric $\widehat{\omega}_k$ has the following behavior on U as k approaches zero:*

- *On $\{kt \to 0\}$, it is quasi-isometric to*
$$\omega_{\mathrm{KE}} = (n+1)\left[\frac{i\xi \wedge \bar{\xi}}{(-t)^2} + \frac{\theta_D}{-t}\right]$$
with quasi-isometry constant converging to 1 as $kt \to 0$.
- *On $\{kt \to -\infty\}$, it is quasi-isometric to*
$$a_n k^2 \cdot e^{kt} i\xi \wedge \bar{\xi} + k\theta_D$$
with quasi-isometry constant converging to 1 as $kt \to -\infty$ and $k \to 0$ and where $a_n = \frac{e^{I_n}}{n+1}$, I_n being defined in (2.7).
- *Elsewhere, i.e. on $\{-C \leqslant kt \leqslant C^{-1}\}$; it is quasi-isometric to*
$$k^2 \cdot e^{kt} i\xi \wedge \bar{\xi} + k\theta_D$$
with quasi-isometry constant uniformly bounded as $k \to 0$.

4.3. **Ramified covers.** In Set-up 4.1, assume additionally that Γ is arithmetic, so that Γ can be realized as the integral points $G(\mathbb{Z})$ of an algebraic group G defined over $\mathbb{Z}$. Given an integer $m \geqslant 1$, the congruence subgroup $\Gamma(m) = \mathrm{Ker}\Big[G(\mathbb{Z}) \to G\left(\mathbb{Z}/m\mathbb{Z}\right)\Big]$ induces an étale cover

$$\pi_m : \Gamma(m)\Big\backslash \mathbb{B} \to \Gamma\backslash\mathbb{B}.$$

Let $X_m := \Gamma(m)\Big\backslash\mathbb{B}$ and let $\overline{X}_m$ be a log smooth compactification of X_m. The étale cover $\pi_m : X_m \to X$ can be uniquely extended to a cover $\overline{\pi}_m : \overline{X}_m \to \overline{X}$. Up to taking a further cover, one can assume that $\overline{\pi}_m$ is Galois, with group Λ_m.

Moreover, Mumford shows in [Mum77, p270-271] that $\overline{\pi}_m$ is highly ramified along D in the following sense. Let $\nu_{m,k}$ be the ramification order of $\overline{\pi}_m$ along D_λ. Then, given any integer $\ell \geqslant 1$, there exists $m = m(\ell)$ such that $\ell | \nu_{m,\lambda}$ for any $\lambda = 1, \dots, N$.

Pick ℓ arbitrary large and consider the ramified cover $\overline{\pi}_m : \overline{X}_m \to \overline{X}$ for $m = m(\ell)$ as above. Set $k_m := (\frac{1}{\nu_{m,1}}, \dots, \frac{1}{\nu_{m,N}})$ and consider the conical Kähler-Einstein metric ω_{k_m} with cone angles $2\pi(k_m)_\lambda$ along D_λ. By the choice of k_m, ω_{k_m} is an orbifold Kähler metric for the pair $(X, \sum_\lambda (1 - \frac{1}{\nu_{m,\lambda}}) D_\lambda)$, hence $\omega_m := \overline{\pi}_m^* \omega_{k_m}$ is a genuine Kähler-Einstein metric on the compact Kähler manifold $\overline{X}_m$, i.e.

$$\operatorname{Ric} \omega_m = -\omega_m \quad \text{on } \overline{X}_m.$$

As $\ell \to +\infty$, so does m and k_m converges to 0, so that ω_{k_m} converges to the hyperbolic (Bergman) metric on X by the previous results. Schematically, one can summarize the situation as below

$$(\overline{X}_m, \omega_m) \Big/ \Lambda_m \quad \underset{m \to +\infty}{\longrightarrow} (\Gamma \backslash \mathbb{B}, \omega_{\mathrm{Berg}}).$$

5. Gluing with the Tian-Yau metric

We now pass to the setting of a compact Fano manifold X of dimension $n \geqslant 2$ endowed with a *smooth* anticanonical divisor $D \subset X$. Note that D is connected by the Lefschetz hyperplane theorem. We denote by L the normal bundle of D. The objects on L constructed in section 2 will now carry an index L (h_L, Ω_L, $\varphi_{k,L}$, $\omega_{k,L}$, etc.) to distinguish them from the objects constructed on X.

5.1. **The Tian-Yau metric.** The Tian-Yau metric was obtained in [TY90] and precise asymptotics are derived in [Hei12]. A nice summary is written in [HSVZ, § 3], and the asymptotics written below are taken from this reference.

We choose the holomorphic $(n-1)$-form ϖ on D of section 2 so that $\frac{i^{(n-1)^2}}{n} \varpi \wedge \overline{\varpi} = \omega_D^{n-1}$. We have a global holomorphic n-form Ω on X with a simple pole along D, normalized by $\varpi = \operatorname{Res}_D \Omega$, so that the form induced by Ω on L is $\Omega_L = \xi \wedge p^* \varpi$.

The normal bundle L gives the infinitesimal neighbourhood of D in X. One can identify a neighbourhood of D in X with a disc bundle in L: one method uses the Riemannian exponential of a Hermitian metric on X, but we prefer a more intrinsic identification using the theory of extremal discs which produces a (non-holomorphic) fibration in holomorphic discs and simplifies some later calculations. The following proposition can be extracted for example from [Biq02, Theorem 4.1]:

Proposition 5.1. *There exists a diffeomorphism $\Phi : \Delta_L \to U_L \subset X$ from the disc bundle $\Delta_L \subset L$ to a neighbourhood $U_L \subset X$ of D, such that $\phi := \Phi^* J_X - J_L \in \Omega^{0,1}(T^{1,0})$ satisfies $\phi|_D = 0$, ϕ is a section of $(p^* \Omega_D^{0,1}) \otimes \ker \eta^{1,0}$ (that is, ϕ is purely horizontal), and ϕ is holomorphic along the discs of Δ_L.*

Moreover $\Phi^* \Omega = v\,(1 - \phi)^* \Omega_L$, where v is a function on Δ_L such that $v|_D = 1$ and v is holomorphic along the discs.

We still denote $t = \log \|v\|^2_{h_L}$, which via the diffeomorphism Φ we can also see as a function on U_L. We modify the function t on $\{-2 \le t \le -1\}$ to get a smooth function $\tilde{t}$ on $X \setminus D$ such that

$$\tilde{t} = \begin{cases} t & \text{on } t \le -2, \\ -1 & \text{on } t \ge -1 \text{ and } X \setminus U_L. \end{cases} \tag{5.1}$$

We denote $\omega_{TY,L} = (\frac{n}{n+1})^{1+\frac{1}{n}} i\partial\overline{\partial}(-t)^{1+\frac{1}{n}}$ the Tian-Yau metric defined on $\{t < 0\} \subset L$, and $g_{TY,L}$ the corresponding Riemannian metric. From (2.2) it is given by the formula

$$\omega_{TY,L} = (\tfrac{n}{n+1})^{\frac{1}{n}} \big(\tfrac{1}{n}(-t)^{-1+\frac{1}{n}} dt \wedge \eta + (-t)^{\frac{1}{n}} \theta_D\big). \tag{5.2}$$

Take some Hermitian metric h on K_X^{-1} such that $h|_D = h_L$, and with positive Ricci curvature on $X \setminus D$. Note that $h|_D$ is only well-defined up to a constant, which will be fixed later in order to have (5.7). Then, the asymptotics of $\omega_{TY,L}$ coincide with those of the metric

$$\omega_0 = i\partial\overline{\partial}\big(-\tfrac{n}{n+1}\log|\Omega^{-1}|^2_h\big)^{1+\frac{1}{n}} \tag{5.3}$$

on $X \setminus D$, with the corresponding Riemannian metric g_0. More precisely, for any $\varepsilon > 0$:

$$|\nabla^j_{g_{TY,L}}(\Phi^*\Omega - \Omega_L)|_{g_{TY,L}} = O(e^{(\frac{1}{2}-\varepsilon)t}), \tag{5.4}$$

$$|\nabla^j_{g_{TY,L}}(\Phi^* g_0 - g_{TY,L})|_{g_{TY,L}} = O(e^{(\frac{1}{2}-\varepsilon)t}). \tag{5.5}$$

This comes from the fact that the objects on L and on $X \setminus D$ coincide near D up to order $O(z) = O(e^{\frac{t}{2}})$, but then the form of the metric (5.2) introduces powers of t in the estimates for the differences and their derivatives, so we simply write $O(e^{(\frac{1}{2}-\varepsilon)t})$ which will be enough for us.

The Tian-Yau metric on $X \setminus D$ is a Kähler metric $\omega_{TY} = i\partial\overline{\partial}\varphi_{TY}$ satisfying

$$\omega_{TY}^n = \frac{i^{n^2}}{n+1}\Omega \wedge \overline{\Omega} \tag{5.6}$$

and asymptotic to our Tian-Yau metric $\omega_{TY,L}$ near D. Of course (5.6) implies that it is Ricci flat. For a suitable (unique) normalization of h, we have the asymptotics

$$\varphi_{TY} = \big(-\tfrac{n}{n+1}\log|\Omega^{-1}|^2_h\big)^{1+\frac{1}{n}} + \psi, \quad |\nabla^j_{g_{TY,L}}\psi|_{g_{TY,L}} = O(e^{-\varepsilon\sqrt{-t}}) \tag{5.7}$$

for all $j \ge 0$ and for some $\varepsilon > 0$. Compared to [HSVZ], we have a different normalization of the constants in order to match our models of section 2. The rate $e^{-\varepsilon\sqrt{-t}}$ comes from the fact that harmonic functions which go to zero in the metric g_0 have exponential decay in $\sqrt{-t}$.

5.2. **The gluing.** We now define Kähler metrics ω_k on X, with a cone singularity of angle $2\pi k$ around D, which are close to be Kähler-Einstein with constant 1, by gluing the metrics $\omega_{k,L}$ of section 2 with the Tian-Yau metric ω_{TY}. This is done by gluing the corresponding Kähler potentials around $t_k = -k^{-1+\mu}$ for some fixed $\mu \in (0,1)$, for example the reader can take $\mu = 1/2$.

We define on $X \setminus D$

$$\varphi_k = \begin{cases} \chi(\frac{t}{t_k})\Phi_* \varphi_{k,L} + (1-\chi(\frac{t}{t_k}))k^{1+\frac{1}{n}}\varphi_{TY} & \text{on } U_L, \\ k^{1+\frac{1}{n}}\varphi_{TY} & \text{on } X \setminus U_L, \end{cases} \tag{5.8}$$

where $\chi : \mathbb{R}_+ \to \mathbb{R}_+$ is a nondecreasing function such that $\chi(u) = 1$ for $u \geq 2$ and $\chi(u) = 0$ for $u \leq \frac{1}{2}$. We denote $\omega_k = i\partial\bar{\partial}\varphi_k$ and g_k the corresponding Kähler form and Riemannian metric.

This metric is very close to our model $\omega_{k,L}$ for $t < t_k/2$:

Lemma 5.2. *For any $\varepsilon > 0$, one has for k small enough, uniformly with respect to k:*

$$|\nabla^j_{g_{k,L}}(g_k - g_{k,L})|_{g_{k,L}} = \begin{cases} O(e^{(\frac{1}{2}-\varepsilon)t}) & \text{for } t \leq 2t_k, \\ O((-kt)^{(1-\frac{j}{2})(1+\frac{1}{n})}) & \text{for } 2t_k \leq t \leq \frac{1}{2}t_k. \end{cases} \tag{5.9}$$

Proof. The main point here is the uniformity with respect to k. For $t \leq 2t_k$, we have the same potential $\varphi_{k,L}$ but with respect to two different complex structures, that of L and of X, that we shall denote J_L and J_X. It follows from Proposition 5.1 that $J_X - J_L$ vanishes on the vertical directions of L, and reduces to an endomorphism of $\ker \eta$. Since on $t \leq 2t_k$ both Kähler forms have potential $\varphi_{k,L}(t)$, and $J_L dt = J_X dt = 2\eta$, it follows that actually $\omega_{k,L} = \frac{1}{2} dJ_L d\varphi_{k,L}$ coincides with $\omega_k = \frac{1}{2} dJ_X d\varphi_{k,L}$ on $t \leq 2t_k$. Therefore $g_k - g_{k,L} = \omega_{k,L}(\cdot, (J_X - J_L)\cdot)$, so estimating $g_k - g_{k,L}$ on this region is the same as estimating $J_X - J_L$.

Since $J_X - J_L$ vanishes on D, it follows from formula (2.14) that

$$|J_X - J_L|_{g_{k,L}} = O(e^{\frac{t}{2}}) \tag{5.10}$$

uniformly in k, since the factor $-k\varphi_1'(kt)$ in front of θ_D does not change the norm of the endomorphisms. The covariant derivatives include terms $\frac{1}{k\sqrt{\varphi_1''(kt)}}\frac{\partial}{\partial t}$ and $\frac{1}{\sqrt{-k\varphi_1'(kt)}}\frac{\partial}{\partial x}$ (for x coordinate on D). From the behaviour of φ_1 given in (2.10) it follows that the worst coefficient introduced by a covariant derivative is $k^{-1}e^{-k\frac{t}{2}}$. As a result, for any $\varepsilon > 0$, we have for k small enough and $t \leq 2t_k = 2k^{-1+\mu}$, uniformly in k,

$$|\nabla^j_{g_{k,L}}(J_X - J_L)|_{g_{k,L}} = O(e^{(\frac{1}{2}-\varepsilon)t}). \tag{5.11}$$

Now pass to the region $2t_k \leq t \leq \frac{1}{2}t_k$. Here we have

$$\varphi_k = k^{1+\frac{1}{n}}\varphi_{TY}(t) + \chi(\tfrac{t}{t_k})\big(\varphi_{k,L}(t) - k^{1+\frac{1}{n}}\varphi_{TY}(t)\big),$$

with

$$\varphi_{TY}(t) = \big(\tfrac{-nt}{n+1}\big)^{1+\frac{1}{n}} + \psi, \quad \psi = O(e^{-\varepsilon\sqrt{-t}}),$$
$$\varphi_{k,L}(t) = \big(\tfrac{-knt}{n+1}\big)^{1+\frac{1}{n}} + O((-kt)^{2(1+\frac{1}{n})}).$$

(The second line is actually a complete expansion in powers of $(-kt)^{1+\frac{1}{n}}$). Since $t_k = -k^{-1+\mu}$ goes to $-\infty$, the term coming from ψ is negligible and we obtain

$$k^{1+\frac{1}{n}}\varphi_{TY}(t) - \varphi_{k,L}(t) = O\big((-kt)^{2(1+\frac{1}{n})}\big).$$

The Kähler form $\omega_{k,L}$ is asymptotic to the Tian-Yau form

$$k^{1+\frac{1}{n}}\omega_{TY,L} = k^{1+\frac{1}{n}}\big(\tfrac{n}{n+1}\big)^{\frac{1}{n}}\big(\tfrac{1}{n}(-t)^{-1+\frac{1}{n}}dt\wedge\eta + (-t)^{\frac{1}{n}}\theta_D\big).$$

Therefore we have

$$\big|\nabla^j(k^{1+\frac{1}{n}}\varphi_{TY}(t) - \varphi_{k,L}(t))\big|_{k^{1+\frac{1}{n}}g_{TY,L}} = O\big((-kt)^{(2-\frac{j}{2})(1+\frac{1}{n})}\big).$$

On the other hand, $|\partial_t^j(\chi(\frac{t}{t_k}))| = O(t_k^{-j}) = O(t^{-j})$ so we have the same estimate on the derivatives of $\chi(\frac{t}{t_k})$:

$$\big|\nabla^j(\chi(\tfrac{t}{t_k}))\big|_{k^{1+\frac{1}{n}}g_{TY,L}} = O\big((-kt)^{-\frac{j}{2}(1+\frac{1}{n})}\big).$$

Altogether we obtain

$$\big|\nabla^j(k^{1+\frac{1}{n}}\varphi_{TY}(t) - \varphi_{k,L}(t))\big|_{k^{1+\frac{1}{n}}g_{TY,L}} = O\big((-kt)^{(2-\frac{j}{2})(1+\frac{1}{n})}\big). \tag{5.12}$$

Since the difference $J_X - J_L$ is exponentially small, the estimate (5.12) gives the lemma. □

We will solve the Kähler-Einstein equation $\mathrm{Ric}(\omega_k + i\partial\overline{\partial}\varphi) = \omega_k + i\partial\overline{\partial}\varphi$ under the form

$$P_k(\varphi) := \log\frac{(\omega_k + i\partial\overline{\partial}\varphi)^n}{i^{n^2}\Omega\wedge\overline{\Omega}} + (\varphi_k+\varphi) - C_k = 0 \tag{5.13}$$

where the constant C_k is the constant obtained for the model Calabi metric $g_{k,L}$, that is $C_k = C_1 + (n+1)\log k$. One can calculate $C_1 = -\log(n+1)$, in accordance with (5.6) when one checks that the Tian-Yau metric must be the limit of $\frac{\omega_k}{k^{1+1/n}}$ when $k\to 0$. We can now estimate the initial error term:

Lemma 5.3. *For any $\varepsilon > 0$, one has for k small enough, uniformly with respect to k:*

$$|\nabla^j_{g_k}P_k(0)|_{g_k} = \begin{cases} O(e^{(\frac{1}{2}-\varepsilon)t}) & t\le 2t_k, \\ O((-k\tilde{t})^{(1-\frac{j}{2})(1+\frac{1}{n})}) & \tilde{t}\ge 2t_k. \end{cases} \tag{5.14}$$

Proof. This follows from the estimates in Lemma 5.2:

- The form $\omega_{k,L}$ solves $\omega_{k,L}^n = C_k e^{\varphi_{k,L}} i^{n^2}\Omega_L\wedge\overline{\Omega}_L$, therefore on $t\le 2t_k$, since ω_k and Ω differ from $\omega_{k,L}$ and Ω_L respectively by an exponentially decreasing term, we obtain the estimate of the lemma.
- On $\tilde{t}\ge\frac{1}{2}t_k$ the Tian-Yau form ω_{TY} is Ricci flat and solves (5.6), so the error term in (5.14) is $\varphi_k = O((-k\tilde{t})^{1+\frac{1}{n}})$ (and the corresponding estimates for the derivatives).
- On the gluing region $2t_k\le t\le\frac{1}{2}t_k$, the estimates from lemma 5.2 are still sufficient to prove (5.14).

□

6. Uniform Schauder estimate for cones

6.1. **Preliminaries.** In this section, we consider the flat Kähler metric on $\mathbb{C}^* \times \mathbb{C}^{n-1}$ with cone angle $2\pi k$ along $D := (z_1 = 0)$, that is

$$dd^c(|z_1|^{2k} + \|z'\|^2) = k^2|z_1|^{2(k-1)} i dz_1 \wedge d\bar{z}_1 + \sum_{j=2}^n i dz_j \wedge d\bar{z}_j$$

where $z' = (z_2, \dots, z_n)$. Using the real coordinates $r := |z_1|^k, \theta := \arg(z_1)$, the Riemannian metric associated that Kähler metric is

$$\bar{g}_k := (dr^2 + k^2 r^2 d\theta^2) + g_{\mathbb{C}^{n-1}}.$$

It will be convenient to introduce the notation

$$\widehat{g}_k := dr^2 + k^2 r^2 d\theta^2$$

for the one-dimensional complex cone with cone angle $2\pi k$ at $0 \in \mathbb{C}$.

On balls.
We are interested in the behavior of $\bar{g}_k$ near the divisor when k approaches 0. When $n = 1$, the zone $\{C \geqslant |z_1| \geqslant 1/C\}$ is collapsed onto a point which is at distance exactly one of the origin. This means that the asymptotic geometry is concentrated extremely close to the divisor. In the following, we will only consider with points p at distance at most $\frac{1}{2}$ from the origin with respect to $\bar{g}_k$; in particular, $|z_1(p)| \leqslant 2^{-1/k}$ converges exponentially fast to zero.

If $p \in \mathbb{C}^n$ and $\rho > 0$, we denote by $B_p(\rho)$ the geodesic ball of radius ρ centered at p, with respect to $\bar{g}_k$. If $p \in D$, then $B_p(\rho) = \{q = (r_q, \theta_q, z'); r_q^2 + \|z' - z'(p)\|^2 < \rho^2\}$.

In the following, we set $B_k := B_0(\frac{1}{2})$ for the ball centered at the origin with radius $1/2$ with respect to $\bar{g}_k$. It will be convenient to also set $B'_k := B_0(\frac{1}{4})$. As explained above, we will exclusively focus on the behavior of $\bar{g}_k$ on B_k. Punctured balls B^* are defined as $B \setminus D$.

We decompose the gradient of $\bar{g}_k$ as $\nabla^{\bar{g}_k} = (D', D'')$ where $D' = (D_1, D_2)$ with $D_1 = \partial_r, D_2 = \frac{1}{kr}\partial_\theta$ and $D'' = (D_3, \dots, D_{2n})$ where $D_{2j-1} = \partial_{x_j}, D_{2j} = \partial_{y_j}$ if $z_j = x_j + iy_j$. The laplacian is

$$\Delta_{\bar{g}_k} = \underbrace{\partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{k^2r^2}\partial_\theta^2}_{=\Delta_{\widehat{g}_k}} + \Delta_{\mathbb{C}^{n-1}}.$$

In complex coordinates, the first order derivatives are given by

$$\partial_r = \frac{1}{k|z_1|^k}(z_1\partial_{z_1} + \bar{z}_1\partial_{\bar{z}_1}), \quad \text{and} \quad \frac{1}{kr}\partial_\theta = \frac{i}{k|z_1|^k}(z_1\partial_{z_1} - \bar{z}_1\partial_{\bar{z}_1})$$

while the second order derivatives are

$$\partial_r^2 = \frac{1}{k^2|z_1|^{2k}}\left(2|z_1|^2\partial^2_{z_1\bar{z}_1} + (1-k)(z_1\partial_{z_1} + \bar{z}_1\partial_{\bar{z}_1}) + (z_1^2\partial_{z_1} + \bar{z}_1^2\partial_{\bar{z}_1})\right) \tag{6.1}$$

and

$$\frac{1}{k^2r^2}\partial_\theta^2 = \frac{-1}{k^2|z_1|^{2k}}\left(z_1^2\partial_{z_1}^2 + z_1\partial_{z_1} - 2|z_1|^2\partial^2_{z_1\bar{z}_1} + \bar{z}_1\partial_{\bar{z}_1} + \bar{z}_1^2\partial_{\bar{z}_1}\right). \tag{6.2}$$

For any real number $\alpha \in (0,1)$, we define the $\mathcal{C}^\alpha$ norm with respect to $\bar{g}_k$ in the classical way. That is, if $p \in B_k$ i.e. if $f \in \mathcal{C}^0(B_k)$, then

$$\|f\|_\alpha = \sup_{x,y\in B_k} \frac{|f(x)-f(y)|}{d_{\bar{g}_k}(x,y)^\alpha}.$$

We defined the $\mathcal{C}^{2,\alpha}$ norm of a function f on B_k as

$$\|f\|_\alpha = \|f\|_0 + \|\nabla^{\bar{g}_k} f\|_0 + \sum_{i=1}^{2n}\sum_{j=3}^{2n} \|D_iD_jf\|_\alpha + \|\Delta_{\widehat{g}_k} f\|_\alpha.$$

This means that we do not require a control on the derivatives $D_1^2 f$, $D_2^2 f$, $D_1D_2 f$, $D_2D_1 f$, following Donaldson [Don12].

The main result of this section is the following

Theorem 6.1 (Weak Schauder estimate)**.** *Let $u \in L^\infty(B_k)$ solve $\Delta_{\bar{g}_k} u = f$ for some $f \in \mathcal{C}^\alpha(B_k)$. Then $u \in \mathcal{C}^{2,\alpha}(B_k)$ and there exists a constant $C = C(n,\alpha,\delta)$ such that for all $k \in (0, 1-\delta]$ one has*

$$\|u\|_{\mathcal{C}^{2,\alpha}(B_k')} \leqslant C\left(\|f\|_{\mathcal{C}^\alpha(B_k)} + \|u\|_{\mathcal{C}^0(B_k)}\right).$$

The novelty of the result above relies on the uniformity of the "Schauder constant" C above with respect to k (say when $k \to 0$), since the result for fixed k has been known for a while. It is initially due to Donaldson [Don12] and was later reproved and generalized via many different methods, cf [GS16, GY18, dBE20].

We will follow the approach of Bin Guo and Jian Song [GS16], itself based on an original and quite direct proof of the usual Schauder estimate by Xu-Jia Wang [Wan06].

In what follows, we will systematically assume that $k < \frac{1}{2}$.

6.2. **Gradient estimates.** In this section, we provide two types of gradient estimates for the metric $\bar{g}_k$ that will be useful later.

Lemma 6.2. *Assume that $n = 1$. Let $u \in L^\infty(B_0(\rho))$, smooth outside 0 solving $\Delta_{\widehat{g}_k} u = f$. Then there exists a constant $C > 0$ such that for any $r \leqslant \frac{\rho}{2^k}$, one has*

$$|\nabla^{\widehat{g}_k} u|(r,\theta) \leqslant C\Big[\frac{1}{k}\left(\frac{r}{\rho}\right)^{\frac{1}{k}} \cdot \frac{1}{r}\sup_{B_0(\rho)}|u| + r\sup_{B_0(\rho)}|f|.\Big]$$

Remark 6.3. A trivial but crucial observation is that when $r \leqslant \rho/2$, we have $\frac{1}{k}(\frac{r}{\rho})^{\frac{1}{k}} \to 0$ when $k \to 0$. In particular, and this is all we will use in the following, the latter quantity is bounded when k approaches zero.

Proof. The function $v(z) := u(\rho^{1/k}z)$ is defined for $|z| \leqslant 1$ and satisfies $\Delta_{\mathrm{eucl}} v(z) = \rho^2 \frac{k^2}{|z|^{2(1-k)}} f(\rho^{1/k}z)$. Classically, we have

$$v(z) = h(z) + \underbrace{\rho^2 \int_{|w|<1} \frac{k^2}{|w|^{2(1-k)}} f(\rho^{1/k}w) \log\left|\frac{z-w}{1-\bar{w}z}\right| |dw|^2}_{=:I(z)}, \tag{6.3}$$

where h is the harmonic function on the unit disk $\mathbb{D} \subset \mathbb{C}$ whose boundary values are $v|_{\partial\mathbb{D}}$. In particular, there exists a universal constant C_1 such that

$$\sup_{|z|\leqslant\frac{1}{2}} |\nabla^{\mathrm{eucl}} h(z)| \leqslant C_1 \sup_{|z|=1} |v(z)| = C_1 \sup_{B_0(\rho)} |u|.$$

In particular, we get for $|\rho^{-1/k}z| \leqslant 1/2$ (or, equivalently, $r \leqslant 2^{-k}\rho$):

$$|\nabla^{\widehat{g}_k} h(\rho^{-1/k}z)| \leqslant \frac{C_1}{k}\left(\frac{r}{\rho}\right)^{\frac{1}{k}} \cdot \frac{1}{r} \sup_{B_0(\rho)} |u|, \tag{6.4}$$

which takes care of the first part in the RHS of (6.3). To take care of the integral summand $I(z)$, we assume that $|z| \leqslant 1/2$ so that $\nabla^{\mathrm{eucl}} I(z)$ is controlled by $\rho^2 k^2 \sup|f| \cdot \int_{|w|<1} \frac{|dw|^2}{|w|^{2(1-k)}\cdot|z-w|}$. Performing the change of variable $x := w/z$ in the integral, we get $|z|^{2k-1} \int_{|x|<1/|z|} \frac{|dx|^2}{|x|^{2(1-k)}\cdot|x-1|}$. There are three zones: around $x = 0$ the integral is equivalent to $1/k$, around $x = 1$ we have uniform integrability while around $|x| = 1/|z|$ it is equivalent to $|z|^{1-2k}$ hence it is uniformly integrable too. All in all, we find that $|\nabla^{\mathrm{eucl}} I(z)| \lesssim k\rho^2|z|^{2k-1} \cdot \sup|f|$. In terms of $\bar{g}_k$-gradient, this means that

$$|\nabla^{\widehat{g}_k} I(\rho^{-1/k}z)| \lesssim \frac{1}{k} r^{\frac{1}{k}-1}\rho^{-1/k} \cdot k\rho^2 |\rho^{-1/k}z|^{2k-1} \cdot \sup|f| = r \cdot \sup|f|.$$

Combined with (6.4), this yields the desired gradient estimate for u. $\square$

We will also need the following gradient estimate in any dimension for harmonic functions:

Lemma 6.4. *Let $u \in L^\infty(B_p(\rho))$ be a harmonic function, i.e. $\Delta_{\bar{g}_k} u = 0$. Assume that either $p = 0$ or $\rho < r(p)$. Then there exists a universal constant $C = C(n) > 0$ such that*

$$\sup_{B_p(\rho/2)} |\nabla^{\bar{g}_k} u| \leqslant \frac{C}{\rho} \sup_{B_p(\rho)} |u|.$$

In particular, for any integer $\ell > 0$ that

$$\sup_{B_p(\rho/2)} |(D'')^\ell u| \leqslant \frac{C(n,\ell)}{\rho^\ell} \sup_{B_p(\rho)} |u|; \quad \sup_{B_p(\rho/2)} |(D'')^\ell D' u| \leqslant \frac{C(n,\ell)}{\rho^{\ell+1}} \sup_{B_0(\rho)} |u|$$

as well as, if p is not on the divisor

$$\left|\nabla^{\bar{g}_k}\big[\partial_r (D'')^\ell u\big](z)\right| + \left|\nabla^{\bar{g}_k}\big[\frac{1}{kr}\partial_\theta (D'')^\ell u\big](z)\right| \leqslant C(1 + \frac{r(p)}{r}) \cdot \frac{\|u\|_{L^\infty(B_p(\rho))}}{\rho^{\ell+2}}. \tag{6.5}$$

Proof. Following [GS16, Lemma 2.4 & Proposition 2.2], one can approximate $\bar{g}_k$ by smooth metrics $\bar{g}_{k,\varepsilon}$ with non-negative Ricci curvature and use Cheng-Yau's gradient estimate [CY75] to get the first two sets of inequalities. For the last estimate, set $v := (D'')^\ell u$ and observe that $\frac{1}{k}\partial_\theta v$ is harmonic on $B_p(\rho)$ and therefore

$$\left|\frac{1}{k}\partial_\theta v\right| \leqslant C\frac{r\|u\|_{L^\infty(B_p(\rho))}}{\rho^{\ell+1}} \quad \text{on } B_p(\rho/2)$$

thanks to the previous gradient estimate. Here, $C = C(n,\ell)$ and may change from line to line. Iterating that argument, we get

$$\left|\nabla^{\bar{g}_k}\frac{1}{k}\partial_\theta v\right| \leqslant C\frac{r(p)\|u\|_{L^\infty(B_p(\rho))}}{\rho^{\ell+2}} \quad \text{on } B_p(\rho/2). \tag{6.6}$$

Since $|\nabla^{\bar{g}_k}\frac{1}{kr}\partial_\theta v| \leqslant \frac{1}{r^2}\left|\frac{1}{k}\partial_\theta v\right| + \frac{2}{r}\left|\nabla^{\bar{g}_k}\frac{1}{k}\partial_\theta v\right|$, we get

$$|\nabla^{\bar{g}_k}\frac{1}{kr}\partial_\theta v| \leqslant C(\frac{\rho}{r}+\frac{r(p)}{r})\cdot\frac{\|u\|_{L^\infty(B_p(\rho))}}{\rho^{\ell+2}}$$

which provides half of the desired inequality. For the second half, observe that $\nabla^{\bar{g}_k}\partial_r v$ involves the following terms: $\partial_r^2 v, \frac{1}{r}\partial_r(\frac{1}{k}\partial_\theta v)$ and $\nabla''\partial_r v$. The last term is controlled by the gradient estimate already established for the harmonic function $D''v$ and the second one is controlled by (6.6). Finally, the first one can be written $\partial_r^2 v = \underbrace{\Delta_{\bar{g}_k} v}_{=0} - \frac{1}{r}\partial_r v - \frac{1}{k^2r^2}\partial_\theta^2 v - \Delta_{\mathbb{C}^{n-1}} v$ and the estimate follows from the previous ones. □

In Lemma 6.4 above, one can adapt the proof of the estimate (6.5) in the case where $p = 0$ is centered on the divisor, and the RHS becomes $C\frac{\|u\|_{L^\infty(B_p(\rho))}}{r\rho^{\ell+1}}$, which turns out to be too coarse for our later purposes. Instead, we will use the input of Lemma 6.2 to obtain the following estimate, valid for balls centered on the divisor.

Lemma 6.5. *Let $u \in L^\infty(B_0(\rho))$ be a harmonic function, i.e. $\Delta_{\bar{g}_k} u = 0$. There exists $C = C(n,\ell)$ such that for all $z = (r,\theta,z')$ in $B_0(\rho/4)$, one has*

$$\left|\nabla^{\widehat{g}_k}\left[(D'')^\ell u\right](z)\right| + \left|\nabla^{\widehat{g}_k}\left[\frac{1}{k}\partial_\theta(D'')^\ell u\right](z)\right| \leqslant C\frac{\|u\|_{L^\infty(B_0(\rho))}}{\rho^{\ell+1}}\cdot\left(\frac{1}{k}\left(\frac{r}{\rho}\right)^{\frac{1}{k}-1}+\frac{r}{\rho}\right)$$

as tas

$$\left|\nabla^{\widehat{g}_k}\left[\partial_r(D'')^\ell u\right](z)\right| \leqslant C\frac{\|u\|_{L^\infty(B_0(\rho))}}{\rho^{\ell+2}}\cdot\left(\frac{1}{k}\left(\frac{r}{\rho}\right)^{\frac{1}{k}-2}+1\right). \tag{6.7}$$

Proof. The two important points are that $\Delta_{\bar{g}_k}$ commutes with both D'' and ∂_θ and that for any $\bar{g}_k$-harmonic function w, one has $\Delta_{\widehat{g}_k} w = -\Delta_{\mathbb{C}^{n-1}} w$. Set $v := (D'')^\ell u$. By Lemma 6.4,

$$\sup_{B_0(\rho/2)}\left[|v| + |\frac{1}{k}\partial_\theta v|\right] \leqslant C\frac{\|u\|_{L^\infty(B_0(\rho))}}{\rho^{\ell+1}}$$

so that the first inequality now easily follows from Lemma 6.2.

The second inequality requires a bit more work. We start by decomposing

$$\nabla^{\tilde{g}_k}\partial_r v = (\partial_r^2 v, \frac{1}{kr}\partial^2_{r\theta}v, D''\partial_r v)$$

and observing that the last two components are controlled on $B_0(\rho/4)$ by

$$\sup_{B_0(\rho/2)}\left[|\nabla^{\widehat{g}_k}D''v| + \frac{1}{r}|\nabla^{\widehat{g}_k}\frac{1}{k}\partial_\theta v|\right],$$

which in turn is controlled by $M := \frac{\|u\|_{L^\infty(B_0(\rho))}}{\rho^{\ell+2}}\cdot(\frac{1}{k}\left(\frac{r}{\rho}\right)^{\frac{1}{k}-2}+1)$ thanks to the first inequality. We are left to estimating $\partial_r^2 v$. We write

$$\begin{aligned}\partial_r^2 v &= \Delta_{\widehat{g}_k}(v) - \frac{1}{r}\partial_r v - \frac{1}{k^2r^2}\partial_\theta^2 v\\ &= -\Delta_{\mathbb{C}^{n-1}}(v) - \frac{1}{r}\partial_r v - \frac{1}{k^2r^2}\partial_\theta^2 v\end{aligned}$$

and observe that each summand is controlled by

$$\sup_{B_0(\rho/2)}\left[|(D'')^2 v| + \frac{1}{r}\left[|\nabla^{\widehat{g}_k}v| + |\nabla^{\widehat{g}_k}\frac{1}{k}\partial_\theta v|\right]\right] \leqslant CM$$

thanks to the first inequality again. The lemma is proved. □

6.3. **Strategy of the proof of Theorem 6.1.** The main idea is to consider, for a given point $p\in B_k$ a sequence of functions (u_κ) defined on neighborhoods U_κ of p getting smaller and smaller when κ increases and such that

$$\begin{cases}\Delta_{\tilde{g}_k}u_\kappa = f(p) & \text{on } U_\kappa\\ u_\kappa|_{\partial U_\kappa} = u|_{\partial U_\kappa}\end{cases}$$

More precisely, let us set $\lambda := \frac{1}{2}$, $r(p) := d(p,D)$ and choose $U_\kappa = B_p(\lambda^\kappa)$ if $r(p) > \lambda^\kappa$ and $U_\kappa := B_{\tilde{p}}(2\lambda^\kappa)$ otherwise, where $\tilde{p}$ is the projection of p onto D under the natural map $\mathbb{C}^n \to \mathbb{C}^{n-1}, (z_1,\dots,z_n)\to(z_2,\dots,z_n)$. Note that if $\sin(k\pi)r(p) < \rho < r(p)$, then the geodesic ball $B_p(\rho)$ is essentially an annulus (times an euclidean ball).

Since $u_\kappa - \frac{1}{n-1}\|z'\|^2$ is harmonic, u_κ enjoys all the regularity properties shared by harmonic functions.

The strategy is, given two indices i,j and two points $p,q\in B_k$, to estimate $D_iD_ju(p) - D_iD_ju(q)$ by the analogous quantity for u_κ, for some $\kappa = \kappa(p,q)$ chosen carefully. More precisely, the choice of κ with be such that $\lambda^\kappa \simeq 8d$ where $d = d(p,q)$. In particular, U_κ will contain $B_p(2d)$ and thus the geodesic joining p to q.

6.4. $\mathcal{C}^2$ **estimates.** It is convenient to write $\omega(r)$ for the modulus of continuity of f. By assumption, one has $\omega(r)=O(r^\alpha)$. By considering $u-u_\kappa \pm \omega(\lambda^\kappa)\cdot \big[(r-r(p))^2+\|z'-z'(p)\|^2\big]$, one easily deduces from the maximum principle that

$$\|u-u_\kappa\|_{L^\infty(U_\kappa)} \leqslant C(n)\lambda^{2\kappa}\omega(\lambda^\kappa). \tag{6.8}$$

By the triangle inequality, these inequality extend to quantify the *harmonic* functions

$$h_\kappa := u_{\kappa+1}-u_\kappa$$

on $U_{\kappa+1}$ along with their derivatives thanks to Lemma 6.4

$$\|D''h_\kappa\|_{L^\infty(U_{\kappa+2})} \leqslant C(n)\lambda^\kappa\omega(\lambda^\kappa); \quad \|(D'')^2h_\kappa\|_{L^\infty(U_{\kappa+2})} \leqslant C(n)\omega(\lambda^\kappa). \tag{6.9}$$

For $\kappa \gg 1$, one can define on U_κ a single-valued branch $w=z_1^k$ realizing an isomorphic biholomorphism between $(U_\kappa, \bar{g}_k)$ and a euclidean ball $(B_{\mathrm{eucl}}(\lambda^\kappa), g_{\mathrm{eucl}})$. Using this, one gets that whenever $p\notin D$,

$$\lim_{\kappa\to+\infty} D''u_\kappa(p)=D''u(p); \quad \lim_{\kappa\to+\infty}(D'')^2u_\kappa(p)=(D'')^2u(p), \tag{6.10}$$

cf [GS16, Lemma 2.8]. Write $(D'')^2u_\kappa = (D'')^2u_1+\sum_{j=1}^{\kappa-1}(D'')^2(u_{j+1}-u_j)$ on U_κ and then combine (6.10), (6.9) and Lemma 6.4 to obtain

$$\|(D'')^2u\|_{L^\infty(B_k')} \leqslant C(n,\alpha)\big[\|u\|_{L^\infty(B_k)}+\|f\|_{\mathcal{C}^\alpha(B_k)}\big]. \tag{6.11}$$

Since $\Delta_{\widehat{g}_k}u=f-\sum_{j=3}^{2n}D_j^2u$, (6.11) provides a bound for $\|\Delta_{\widehat{g}_k}u\|_{L^\infty(B_k')}$ in terms of $\|u\|_{L^\infty(B_k)}$ and $\|f\|_{L^\infty(B_k)}$.

6.5. $\mathcal{C}^\alpha$ **estimates for the tangential derivatives.** Let $p,q\in B_k'^{*}$ and let $d=d_{\bar{g}_k}(p,q)$. By [GS16, Proposition 2.3], we have

$$\big|(D'')^2u(p)-(D'')^2u(q)\big| \leqslant C(n,\alpha)\Big[d\|u\|_{L^\infty(B_k)}+d^\alpha\|f\|_{\mathcal{C}^\alpha(B_k)}\Big]. \tag{6.12}$$

For the reader's convenience, we recall the main steps. We introduce the functions v_κ playing the role of u_κ but for the point q instead of p. Choose κ such that $d\simeq\lambda^{\kappa+3}$ and assume $r(p)=\min(r(p),r(q))\leqslant 2d$ for simplicity. We have essentially three terms to treat

$$\underbrace{(D'')^2u(p)-(D'')^2u_\kappa(p)}_{=:(\mathrm{I})};\underbrace{(D'')^2u_\kappa(p)-(D'')^2u_\kappa(q)}_{=:(\mathrm{II})};\underbrace{(D'')^2u_\kappa(q)-(D'')^2v_\kappa(q)}_{=:(\mathrm{III})}.$$

The first term is easy to handle:

$$|(\mathrm{I})| = \lim_{N\to+\infty}\left|\sum_{j=\kappa}^N (D'')^2h_j(p)\right| \leqslant C(n)\sum_{j=\kappa}^{+\infty}\omega(\lambda^j) \leqslant C(n,\alpha)d^\alpha\|f\|_{\mathcal{C}^\alpha(B_k)}.$$

For the second term, we use the gradient estimate (6.4) for the harmonic function $(D'')^2h_j$ ($2\leqslant j\leqslant \kappa-1$):

$$\sup_{B_{\bar{p}}(\lambda^j)}|\nabla^{\bar{g}_k}(D'')^2h_j| \leqslant C(n)\lambda^{-j}\omega(\lambda^j)$$

and after integration along the geodesic joining p anq q (which lies in $B_{\tilde{p}}(\lambda^\kappa)^*$)
$$\left|(D'')^2 u_{j+1}(p) - (D'')^2 u_{j+1}(q)\right| \leqslant \left|(D'')^2 u_j(p) - (D'')^2 u_j(q)\right| + C(n) d\lambda^{-j}\omega(\lambda^j)$$
and by iterating
$$|(\mathrm{II})| \leqslant \left|(D'')^2 u_2(p) - (D'')^2 u_2(q)\right| + C(n) d^\alpha \|f\|_{\mathcal{C}^\alpha(B_k)}.$$
The first term in the RHS is almost harmonic on a ball of definite size, so by using the gradient estimate (6.4), on can dominate it by $C(n)d\|u\|_{L^\infty(B_k)}$. As for the third term, the function $u_\kappa - v_\kappa$ is well-defined on $B_{\tilde{q}}(\lambda^\kappa)$, it is almost harmonic and its sup-norm on that ball is of order $\lambda^{2\kappa}\omega(\lambda^\kappa)$ by (6.8). The gradient estimate for harmonic functions (Lemma 6.4) then provides the desired estimate.

6.6. $\mathcal{C}^\alpha$ **estimates for the normal-tangential derivatives.** In this paragraph, we explain the following estimate, cf [GS16, Propositions 2.4&2.5]. The argument is somehow simplified here because we can choose an angle $2\pi k < \pi$; this will simplify the application of Lemma 6.5. Let $p, q \in B_k'^*$ and let $d = d_{\tilde{g}_k}(p,q)$; then

$$\sum_{i=1}^{2}\sum_{j=3}^{2n} \left|D_i D_j u(p) - D_i D_j u(q)\right| \leqslant C(n,\alpha)\Big[d\|u\|_{L^\infty(B_k)} + d^\alpha \|f\|_{\mathcal{C}^\alpha(B_k)}\Big]. \tag{6.13}$$

Again, we will only highlight the main steps, and focus on the $i = 1$ case; i.e. we estimate the Hölder constant of $\partial_r D_j u$ for any $j \geqslant 3$. The case $i = 2$, i.e. estimating $\frac{1}{kr}\partial_\theta D_j u$ is very similar. Borrowing the notation from § 6.5, Lemma 6.4 shows that the harmonic function h_κ satisfies
$$\sup_{U_{\kappa+2}} |\nabla^{\tilde{g}_k} D'' h_\kappa| \leqslant C(n)\omega(\lambda^\kappa). \tag{6.14}$$
Similarly to (6.10), we have
$$\lim_{\kappa\to+\infty} \partial_r u_\kappa(p) = \partial_r u(p); \quad \lim_{\kappa\to+\infty} \partial_r D'' u_\kappa(p) = \partial_r D'' u(p), \tag{6.15}$$
cf [GS16, Lemma 2.10]. To estimate $\partial_r D'' u(p) - \partial_r D'' u(q)$, we fix the integer κ so that $d \simeq \lambda^{\kappa+3}$ and we need to analyze the analogous terms (I)$' := \partial_r D'' u(p) - \partial_r D'' u_\kappa(p)$,
$$(\mathrm{II})' := \partial_r D'' u_\kappa(p) - \partial_r D'' u_\kappa(q),$$
and (III)$' := \partial_r D'' u_\kappa(q) - \partial_r D'' v_\kappa(q)$. The first term is dealt with just as in § 6.5 and the third one relies on the same arguments as before along with (6.14)-(6.15), cf [GS16, Lemma 2.11]. In the following, we thus focus on (II)$'$. As for its analog (II), the key point is to estimate
$$(\mathrm{II})'' := \partial_r D'' h_j(p) - \partial_r D'' h_j(q),$$
for any $2 \leqslant j \leqslant \kappa - 1$ since u_2 is almost harmonic on $B_{\tilde{p}}(\lambda)$ and Lemma 6.5 shows that the $\tilde{g}_k$-gradient of $\partial_r D'' u_2$ is bounded on that ball. Set $w_j := \partial_r D'' h_j$, defined on U_{j+1}. We distinguish two cases.

- *Case 1.* U_κ is centered on the divisor.

In particular, any U_j $(2 \leqslant j \leqslant \kappa - 1)$ is centered on the divisor as well. The estimate (6.7) in Lemma 6.5 shows that for k small enough ($k < \frac{1}{2}$ would suffice), $\nabla^{\bar{g}_k} w_j$ is bounded on $B_{\tilde{p}}(\frac{3}{2}\lambda^j) \supset B_p(2d)$ by $C(n)\lambda^{-j}\omega(\lambda^j)$.

- *Case 2.* U_κ is centered at p.

Necessarily, we have $r(p) \geqslant \lambda^\kappa \simeq 8d$. Along a geodesic $\gamma(t)$ joining p to q, we have the inequality $r(\gamma(t)) \geqslant \frac{r(p)}{2}$ since the distance from p to a point p' with $r(p') \leqslant r(p)/2$ is at least $r(p)/2 \geqslant 2d$. The geodesic γ lies in $B_p(\lambda^\kappa)$ hence equation (6.5) in Lemma 6.4 shows that for any j, $\nabla^{\bar{g}_k} w_j$ is bounded by $C(n)\lambda^{-j}\omega(\lambda^j)$ along γ.

The case by case analysis above has therefore shown that $|w_j(p) - w_j(q)| \leqslant C(n)\lambda^{-j}\omega(\lambda^j) \cdot d$ and we can conclude as in § 6.5.

6.7. **Strong Schauder estimate.** In this section, we intend to improve Theorem 6.1 by controlling the $\mathcal{C}^\alpha$ norm of the non-mixed derivatives of order two of a solution u of the equation $\Delta_{\bar{g}_k} u = f$, that is to get an estimate of $\|\nabla^2_{\bar{g}_k} u\|_{\mathcal{C}^\alpha}$. As we will later see, it all comes down to the following one-dimensional problem.

Proposition 6.6. *Assume that $n = 1$. Let $u \in L^\infty(B_k)$ solve $\Delta_{\bar{g}_k} u = f$ for some $f \in \mathcal{C}^\alpha(B_k)$. Then $u \in \mathcal{C}^{2,\alpha}(B_k)$ and there exists a constant $C = C(n,\alpha)$ such that for all $k \in (0, \frac{1}{4}]$ one has*

$$\|u\|_{\mathcal{C}^{2,\alpha}(B_k')} \leqslant C\left(\|f\|_{\mathcal{C}^\alpha(B_k)} + \|u\|_{\mathcal{C}^0(B_k)}\right). \tag{6.16}$$

Corollary 6.7 (Strong Schauder estimate)**.** *The full Schauder estimate* (6.16) *holds in any dimension.*

Here, the $\mathcal{C}^{2,\alpha}$ norm $D \subset B_k$ is defined by

$$\|u\|_{\mathcal{C}^{2,\alpha}} = \sup_{B_k} \sum_{0 \leq j \leq 2} |\nabla^j u|_{\bar{g}_k} + [\nabla^2 u]_\alpha$$

where

$$[v]_\alpha = \sup_{x,y \in B_k;\ |r(x) - r(y)| < \frac{r(x)}{2}} \frac{|v(x) - v(y)|}{d_{\bar{g}_k}(x,y)^\alpha}. \tag{6.17}$$

This Hölder semi-norm is quite convenient to manipulate as we will see later, and it is well-known that it is equivalent to the usual Hölder semi-norm $\|v\|_\alpha := \sup_{x,y} \frac{|v(x)-v(y)|}{d_{\bar{g}_k}(x,y)^\alpha}$.

Remark 6.8. By classical arguments (see e.g. [Don12, § 4.2], [GS18, § 3.5]), the result of Corollary 6.7 for the flat cone metric extends to perturbations $\Delta_{\bar{g}_k} + a_k \cdot \nabla^2 + b_k \cdot \nabla$, provided the tensors a_k and b_k are small enough in C^α norm. It is easy to check that the uniformity with respect to k is preserved provided a_k and b_k are uniformly small, more precisely for some $\varepsilon = \varepsilon(n,\alpha) > 0$ small enough, one has for all $k \in (0, \frac{1}{4}]$

$$\|a_k\|_{C^\alpha(\bar{g}_k)} + \|b_k\|_{C^\alpha(\bar{g}_k)} < \varepsilon.$$

Proof of Proposition 6.6. The proof of Proposition 6.6 consists in three steps. First, we show that is is enough to prove the estimate for functions u which vanish on ∂B_k and whose integral on every circle $\{r = \text{cst}\}$ is zero. Next, we show that for such functions u, the norm $\|u/r^{2+\alpha}\|_{\mathcal{C}^0}$ is controlled by $\|u\|_{\mathcal{C}^0} + \|f\|_{\mathcal{C}^\alpha}$. Finally, we combine the previous results and Schauder's estimate for the cylindrical metric to conclude.

• *Step 1. The reduction step.* First, we decompose $u = h + v$ where h is harmonic on B_k with $h|_{\partial B_k} = u|_{\partial B_k}$ and v solving $\Delta_{\tilde{g}_k} v = f$, $v|_{\partial B_k} \equiv 0$. All the usual derivatives of h are controlled by it sup norm, itself controlled by its boundary value, hence by $\|u\|_{\mathcal{C}^0}$. Moreover, the formulas (6.1)-(6.2) show that $\|\nabla^3_{\tilde{g}_k} h\|_{\mathcal{C}^0}$ is controlled by $k^{-3} r^{1/k-3} \|\nabla^3_{\text{eucl}} h\|_{\mathcal{C}^0}$. Therefore, Schauder's estimate for v implies that for u.

Next, we write $\mathbb{C}^* = \mathbb{R}^*_+ \times S^1$ and we expand v in Fourier series $v = v_0(r) + \tilde{v}$ where $\tilde{v} = \sum_{n \geqslant 1} v_n(r) e^{in\theta}$. The function $\tilde{v}$ has integral zero on each circle, hence $\tilde{v}|_{\partial B_k} \equiv 0$. As $v_0(r) = \frac{1}{2\pi r} \int_{|z|=r} v$, both $\|v_0\|_{\mathcal{C}^0}$ and $\|\tilde{v}\|_{\mathcal{C}^0}$ are controlled by $\|v\|_{\mathcal{C}^0}$. Since $\Delta_{\tilde{g}_k}$ respects the decomposition, the Fourier series expansion of $f = f_0(r) + \tilde{f}$ is given by $\Delta_{\tilde{g}_k} v_0 + \Delta_{\tilde{g}_k} \tilde{v}$. It is easy to check that $\|f_0\|_{\mathcal{C}^\alpha} \leqslant \|f\|_{\mathcal{C}^\alpha}$. From this one infers two things: first, $\|v_0\|_{\mathcal{C}^{2,\alpha}}$ is under control (e.g. by explicitly solving the ODE $(\partial_r^2 + r^{-1}\partial_r) v_0 = f_0$) and next, $\tilde{f}$ is $\mathcal{C}^\alpha$ and $\|\tilde{f}\|_{\mathcal{C}^\alpha}$ is under control as well.

Therefore, Schauder's estimate for $\tilde{v}$ implies Schauder's estimate for v, hence for u as well. This shows that it is enough to restrict ourselves to functions u which vanish on ∂B_k and whose integral on each circle $\{|z| = r\}$ is zero.

• *Step 2. The improved uniform estimate.* In this step, we show that for u satisfying $u|_{\partial B_k} \equiv 0$ and $\int_{|z|=r} u = 0$ for any r, then there exists a constant $C > 0$ independent of k such that

$$\|u/r^{2+\alpha}\|_{\mathcal{C}^0(B'_k)} \leqslant C \|f\|_{\mathcal{C}^\alpha(B_k)}. \tag{6.18}$$

Set $\rho := \frac{1}{2}$. From (6.3), we have

$$u(z) = \rho^2 \int_{|w|<1} \frac{k^2}{|w|^{2(1-k)}} f(\rho^{1/k} w) \log \left| \frac{\rho^{-1/k} z - w}{1 - \rho^{-1/k} \bar{z}} \right| |dw|^2. \tag{6.19}$$

The integral of u along any circle $\{|w| = s\}$ is zero, so the same is true for f. This implies that

$$\begin{aligned} u(z) &= \rho^2 \int_{|w|<1} \frac{k^2}{|w|^{2(1-k)}} f(\rho^{1/k} w) \log \left| \frac{\rho^{-1/k} z/w - 1}{1 - \rho^{-1/k} \bar{z}\bar{w}} \right| |dw|^2 \\ &= k^2 |z|^{2k} \int_{|t| > \rho^{-1/k}|z|} f(\rho^{1/k} w) \log \left| \frac{t-1}{1 - \rho^{-2/k}|z|^2/t} \right| \frac{|dt|^2}{|t|^{2+2k}} \end{aligned}$$

after performing the change of variables $t := \rho^{-1/k}z/w$. Since $f(0) = 0$, we have $|f| \leqslant C_\alpha \|f\|_{\mathcal{C}^\alpha} \cdot r^\alpha$ and therefore

$$\left|\frac{u(z)}{r^{2+\alpha}}\right| \leqslant C_\alpha \|f\|_{\mathcal{C}^\alpha} \cdot \underbrace{k^2 \int_{|t|>\rho^{-1/k}|z|} \log\left|\frac{t-1}{1-\rho^{-2/k}|z|^2/t}\right| \frac{|dt|^2}{|t|^{2+2k}}}_{=:I(z)}.$$

We are left to bounding the integral $I(z)$ uniformly for all k and all $z \in B'_k$. When $|t|$ is very small, say $|t| \leqslant \varepsilon$, then $\rho^{-1/k}|z| < \varepsilon$ and the log term is a $O(t) + O(\rho^{-1/k}|z|) = O(t)$ hence this portion of the integral is dominated by $k^2 \int_{s=0}^{\varepsilon} \frac{ds}{s^{2k}} = O(1)$. For the rest of the integral, we first observe that for $z \in B'_k$, we have $|\rho^{-2/k}|z|^2/t| \leqslant \rho^{1/k}$ hence

$$k^2 \int_{|t|>\varepsilon} -\log|1-\rho^{-2/k}|z|^2/t| \frac{|dt|^2}{|t|^{2+2k}} \leqslant k^2 \int_{s=\varepsilon}^{+\infty} \frac{ds}{s^{1+2k}} = O(1).$$

We are left to estimating $k^2 \int_{|t|>\varepsilon} \log|t-1| \frac{|dt|^2}{|t|^{2+2k}}$. The region $\varepsilon \leqslant |t| \leqslant 2$ is trivially dealt with, while the remaining region is estimated by

$$\int_{s=2}^{+\infty} \frac{\log s}{s^{1+2k}} ds = \frac{1}{\gamma k}\Big[s^{-\gamma k}(\log s - \frac{1}{\gamma k})\Big]_2^{+\infty} = O(k^{-2})$$

where $\gamma = (2+\alpha)$. The estimate (6.18) is now proved.

• *Step 3. Schauder estimates for the cylinder.* Set $t := \log r$ so that $\bar{g}_k = r^2 g_c$ where $g_c := dt^2 + k^2 d\theta^2$ to be the cylindrical metric on $\mathbb{R} \times S^1$, where the circle has length $2\pi k$. It is complete with bounded curvature hence it satisfies uniform Schauder estimates independent of k and the chosen ball of a given radius (small balls may not be simply connected but one can pass to the universal cover).

Let us pick an arbitrary point $x_0 \in B'_k$ and set $r_0 := r(x_0)$. We define the regions $D := \{|t-t_0| < 2\}$ and $D' := \{|t-t_0| < 1\}$; these depend on the base point x_0. On D, the function $\frac{r}{r_0} = e^{t-t_0}$ has bounded g_c-derivatives at every order (and the same holds for its inverse), and these bounds are independent of x_0. On D, we have $\|\nabla^j_{g_c} u\|_{\mathcal{C}^0} \sim r_0^j \|\nabla^j_{\bar{g}_k} u\|_{\mathcal{C}^0}$ and $\|v\|_{\mathcal{C}^\alpha_{g_c}(D)} \sim r_0^\alpha \|v\|_{\mathcal{C}^\alpha_{\bar{g}_k}(D)}$ by the definition of the Hölder norm for $\bar{g}_k$, cf (6.17). By the same token, $\|r^2 f\|_{\mathcal{C}^\alpha_{g_c}(D)} \sim r_0^{2+\alpha}\|f\|_{\mathcal{C}^\alpha_{\bar{g}_k}(D)}$.

This implies that

$$\begin{aligned} \|u\|_{\mathcal{C}^{2,\alpha}_{\bar{g}_k}(D')} &\lesssim r_0^{-2-\alpha}\|u\|_{\mathcal{C}^{2,\alpha}_{g_c}(D')} \\ &\leqslant C(\|u/r_0^{2+\alpha}\|_{\mathcal{C}^0(D)} + \|f\|_{\mathcal{C}^\alpha_{\bar{g}_k}(D)}) \end{aligned} \tag{6.20}$$

where the last inequality follows from Schauder estimates for the cylindrical metric, since $\Delta_{g_c} u = r^2 f$. Putting (6.18) and (6.20) together, we conclude that $\|u\|_{\mathcal{C}^{2,\alpha}_{\bar{g}_k}(D')} \leqslant C\|f\|_{\mathcal{C}^\alpha_{\bar{g}_k}(D)}$ for some constant C independent of k and x_0. By varying the point x_0 across B'_k and using the first reduction step, we obtain the proposition. □

Proof of Corollary 6.7. We are left to showing that the $\mathcal{C}^\alpha$ norms of $\partial_r^2 u$ and $\frac{1}{kr^2}\partial_\theta^2 u$ are controlled by $\|u\|_{\mathcal{C}^0} + \|f\|_{\mathcal{C}^\alpha}$. We will treat the term $v := \partial_r^2 u$, the other one being entirely similar. Let $x, y \in B'_k$ which we write (z_1, z'_1) and (z_2, z'_2) where $z'_i \in \mathbb{C}^{n-1}$ for $i = 1, 2$. We set $t = (z_2, z'_1)$ and decompose the difference $v(x) - v(y) = (v(x) - v(t)) + (v(t) - v(y))$. On the slice $S_{z'_1} := \mathbb{C}^* \times \{z'_1\}$, the function u satisfies $\Delta_{\hat{g}_k} u = f - \Delta_{\mathbb{C}^{n-1}} u$ and the $\mathcal{C}^\alpha$ norm of the RHS is controlled by Theorem 6.1. By Proposition 6.6, we get that

$$|v(x) - v(t)| \leqslant C d_{\hat{g}_k}(z_1, z_2)^\alpha \leqslant C d_{\tilde{g}_k}(x, y)^\alpha$$

where C is some uniform multiple of $\|u\|_{\mathcal{C}^0} + \|f\|_{\mathcal{C}^\alpha}$.

We now have to take care of $v(t) - v(y)$. The geodesic from t to y lies in the "euclidean slice" $\tilde{S}_{z_2} = \{z_2\} \times \mathbb{C}^{n-1}$ so we only have to estimate $D''v = \partial_r^2 D'' u$ along $\tilde{S}_{z_2}$. But this is entirely similar to what has been done in § 6.6 by using almost harmonic approximations and relying on (6.5)-(6.7) depending on whether $d_{\tilde{g}_k}(t, y) = d_{\mathrm{eucl}}(z'_1, z'_2)$ dominates or not $r(t) = r(y) = |z_2|^k$. □

7. Schauder estimate for collapsed metrics

Notation. Since we have a lot of constants appearing in our estimates, from now we simplify the notation by introducing the relations

$$A \lessapprox B \quad \text{resp.} \quad A \approx B$$

defined by the fact that there is some constant $\mathfrak{C}_n$ depending only on the dimension n (and certainly not on k) such that

$$A \leq \mathfrak{C}_n B \quad \text{resp.} \quad \mathfrak{C}_n^{-1} B \leq A \leq \mathfrak{C}_n B.$$

7.1. **Functional spaces.** We now define the functional spaces in which we will solve the equation. These are weighted Hölder spaces.

Here it is more convenient to use the variable $u = kt$, so that the gluing region is $-2k^\mu \leq u \leq -\frac{1}{2}k^\mu$. We have the form (2.15) for the metric, which we rewrite here:

$$g_{k,L} = 2\varphi_1''(u)\left(\tfrac{1}{4}du^2 + k^2\eta^2\right) - k\varphi_1'(u) g_D. \tag{7.1}$$

We extend the variable u to the whole X by setting $\tilde{u} = k\tilde{t}$, so that $\tilde{u} = k$ in $X \setminus U_L$.

We define a weight on X by

$$w_k = \chi(-\tilde{u}) - (1 - \chi(-\tilde{u}))\tilde{u} \tag{7.2}$$

so that $w_k = 1$ for $u \leq -2$ and $w_k = \tilde{u}$ for $u \geq -\frac{1}{2}$.

Fix a real number δ. For any section f of a tensor bundle, we define the weighted norm (which depends on k):

$$\|f\|_{C^{\ell,\alpha}_\delta} = \sup \sum_{0 \leq j \leq \ell} w_k^{\delta + \frac{j}{2}(1+\frac{1}{n})} |\nabla^j f|_{g_k} + [w_k^{\delta + \frac{\ell}{2}(1+\frac{1}{n})} \nabla^\ell f]_\alpha \tag{7.3}$$

where the semi-norm $[f]_\alpha$ is also weighted:

$$[f]_\alpha = \sup_{d_{g_k}(x,y) < \rho(k w_k^{\frac{1}{n}}(x))^{\frac{1}{2}}} (k w_k^{\frac{1}{n}})^{\frac{\alpha}{2}} \frac{|f(x) - f(y)|}{d_{g_k}(x,y)^\alpha}. \tag{7.4}$$

One can be surprised by these definitions, since (7.3) and (7.4) do not correspond to the same weight: roughly speaking the norm $\|f\|_{C^0} + [f]_\alpha$ is a C^α norm with respect to the metric $k^{-1}w^{-\frac{1}{n}}g_k$, a metric which looks like:

- $k^{-1}g_k$ on $u \leq -1$: this has bounded curvature by Lemma 3.1;
- $k^{-1-\frac{1}{n}}g_k$, that is g_{TY} on the compact part of $X \setminus U_L$;

in both cases the point is that the geometry is controlled so that there are uniform Schauder estimates.

On the other hand, the C^ℓ norm defined by (7.3) has a different weight, motivated by the fact that $|\nabla^j w_k^\beta|_{g_k} = O(w_k^{\beta - \frac{j}{2}(1+\frac{1}{n})})$, so it is well-adapted to functions depending on u only: we will see that these are the functions on which we have the worst estimates, because the other directions collapse.

Remark 7.1. Despite the presence of the weight, one still has the usual estimate for products in Hölder spaces: $\|fg\|_{C^\alpha} \leq \|f\|_{C^\alpha}\|g\|_{C^\alpha}$. This is because for the standard Hölder norms one has actually $[fg]_\alpha \leq \|f\|_{C^0}[g]_\alpha + [f]_\alpha\|g\|_{C^0}$, so adding our weight $kw_k^{1/n}$ does not change the estimate.

7.2. **1-collapsed Schauder estimate.** From Lemma 3.1 we have the following bound for the curvature of $g_{k,L}$:

$$|K(g_{k,L})| \lesssim \Big(\frac{1}{(-\varphi_1'(u))^{n+1}} + \frac{1}{-k\varphi_1'(u)}\Big). \tag{7.5}$$

Since $\varphi_1'(u) \sim c_n(-u)^{\frac{1}{n}}$ when $u \to 0$, then for $u < -\frac{k}{A}$ for some $A > 0$ which will be fixed below, we have for $B = \mathrm{cst.}(1+A)$,

$$|K(g_{k,L})| \leq \frac{B}{-k\varphi_1'(u)}. \tag{7.6}$$

Since $u = kt$, the region $u < -\frac{k}{A}$ corresponds to the region $t < -\frac{1}{A}$ in the Tian-Yau space, that is the exterior of a compact region. We choose that compact region large enough, that is $A > 0$ small enough, so that the asymptotics of the Tian-Yau metric written in section 5.1 are valid. From the estimate (5.9) on the difference $g_k - g_{k,L}$ we see that (7.6) remains true for g_k. For simplicity we write the sequel for $g_{k,L}$ but these bounds imply that our estimates will remain true for the small perturbation g_k.

Since $\varphi_1''(u) \approx e^u$ by (2.17), the distance r_D from the divisor is of order $r_D \approx e^{\frac{u}{2}}$. Therefore a region $r_D > \varepsilon$ corresponds to $u > 2\log\varepsilon + \mathrm{cst}$.

We will also use the function $r_0(u)$ which is the distance to the point $u = 0$, so that when $u \to 0$ one has

$$r_0(u) \approx |u|^{\frac{1}{2}+\frac{1}{2n}}.$$

Note ρ the injectivity radius of the metric g_D, and fix a finite number of balls of radius ρ covering D. Near some $u_0 < -\frac{k}{A}$ we can consider the rescaled metric $h = \frac{g_{k,L}}{-k\varphi_1'(u_0)}$. This is the metric where only the S^1 fibres are collapsed, at speed $\sqrt{k}$ ('1-collapse'). For $u \approx u_0$ the curvature of h is uniformly bounded and $h \simeq \frac{\varphi_1''(u)}{-4k\varphi_1'(u_0)}(2du^2 + k^2\eta^2) + g_D$. Because of the S^1-bundle over D, small balls of radius ρ for h are not simply connected but we can use Schauder estimates in local universal coverings. We define the domain

$$D_{u_0}(\tau) = \left\{ |r_0(u) - r_0(u_0)| \leq \tau \sqrt{k w_k(u_0)^{\frac{1}{n}}} \right\}$$

and we assume the extra condition $r_D(u_0) > \varepsilon\sqrt{k}$, that is $u_0 > \log k + 2\log\varepsilon +$ cst. Said otherwise, we only look at points at a fixed, positive distance to the divisor D with respect to h. We can then check that $D_{u_0}(\rho) \subset B_h(u_0, C\rho)$ and that on $D_{u_0}(\rho)$, we have $\frac{u_0}{u} \lessapprox 1$ (this can be done easily by treating each case $u_0 \approx -\infty, -1, 0$ separately and using the assumption $r_D(u_0) > \varepsilon\sqrt{k}$.) These considerations lead to the following Schauder estimate for the metric h outside the divisor:

$$\|f\|_{C_h^{2,\alpha}(D_{u_0}(\frac{1}{2}\rho))} \lessapprox \left(\|f\|_{C^0(D_{u_0}(\rho))} + \|\Delta_h f\|_{C_h^\alpha(D_{u_0}(\rho))} \right) \tag{7.7}$$

which we rewrite in terms of g_k using the weighted norm (7.4):

$$\|\nabla^2 f\|_{C_0^\alpha(D_{u_0}(\frac{1}{2}\rho))} \lessapprox \frac{1}{k w_k(u_0)^{\frac{1}{n}}} \|f\|_{C^0(D_{u_0}(\rho))} + \|\Delta_{g_k} f\|_{C_0^\alpha(D_{u_0}(\rho))}. \tag{7.8}$$

Note that this estimate extends everywhere:

- Near the divisor we have the same by taking balls centered on the divisor and applying the uniform Schauder estimate established in section 6 to the metric $h = \frac{g_k}{k}$. More precisely, if $u_0 = -\infty$ and $\rho > 0$ is small enough one has $D_{-\infty}(\rho) = \{r < \rho\sqrt{k}\}$ which is topologically a disk in $\mathbb{C}$ times D. The change of variable $R = r/\sqrt{k}$ in (2.19) gives
$$h = \frac{g_k}{k} = \frac{2}{n+1}\left(dR^2 + k^2R^2\eta^2\right) + g_D + O(kR^2).$$
Maybe up to scaling again by a large constant, we see that we have locally a uniformly small (in the sense of Remark 6.8) perturbation of the product of the cone metric $dR^2 + k^2R^2 d\theta^2$ with the flat metric, and therefore we can apply Corollary 6.7 to get
$$\|f\|_{C_h^{2,\alpha}(D_{-\infty}(\frac{1}{2}\rho))} \lessapprox \left(\|f\|_{C^0(D_{-\infty}(\rho))} + \|\Delta_h f\|_{C_h^\alpha(D_{-\infty}(\rho))} \right)$$
and the corresponding inequality (7.8) in terms of g_k (with the weighted norm - in this region we have $w_k \equiv 1$) is also valid for $D_{-\infty}(\rho)$. Therefore, it holds any $u_0 < -\frac{k}{A}$.
- On the Tian-Yau part $X \setminus U_L$, this is the standard Schauder estimate, since the scaling factor with the Tian-Yau metric is $k w_k(u_0)^{\frac{1}{n}} = k^{1+1/n}$.

7.3. **2-collapsed Schauder estimate.** Our aim now is to give an estimate more suitable for the scale of g_k, where the the divisor D is also collapsed at speed $\sqrt{k}$, and the circle at speed k ('2-collapse'). More precisely, we want to replace by a better coefficient the factor $\frac{1}{k w_k(u_0)^{1/n}}$ in (7.8). For u_0 bounded away from zero, this is just the scaling factor $\frac{1}{k}$. We do this in two steps: decomposing in Fourier series along the circle, a direct application of the maximum principle gives the required estimates on nonzero modes. For zero modes, the argument is more complicated: instead of estimates on balls of radius $\frac{1}{\sqrt{k}}$ as above, we would like estimates on balls of fixed radius, say ρ if u is far from 0 or $-\infty$. In the rescaled metric $\frac{1}{k}g_{k,L}$ this corresponds to a cylinder of length approximately $\frac{\rho}{\sqrt{k}}$, converging to an infinite cylinder. The estimates on this limit will provide the estimates we need.

For small values of ε and u_0 away from the divisor, we define the region

$$E_{u_0}(\varepsilon) = \{|r_0(u) - r_0(u_0)| \leq \varepsilon r_0(u_0)\}. \tag{7.9}$$

If $u_0 = -\infty$ that is we consider the divisor D, we use

$$E_{-\infty}(\varepsilon) = \{|r_D(u)| \leq \varepsilon\}. \tag{7.10}$$

Notice that for $\varepsilon > 0$ and u_0 small, $\varphi_1'(u)$ and $\varphi_1''(u)$ do not vary much in $E_{u_0}(\varepsilon)$, that is remain comparable to their value at u_0.

The regions E_{u_0} correspond to the scale of the geometry of g_k, and are therefore very large for the geometry of the previously used $h = \frac{g_{k,L}}{-k\varphi_1'(u_0)}$. We then obtain the better estimate:

Proposition 7.2.

$$\|\nabla^2 f\|_{C_0^\alpha(E_{u_0}(\frac{1}{2}\varepsilon))} \lessapprox \|\Delta_{g_k} f\|_{C_0^\alpha(E_{u_0}(\varepsilon))} + w_k(u_0)^{-1-\frac{1}{n}}\|f\|_{C^0(E_{u_0}(\varepsilon))}. \tag{7.11}$$

Remark that on the Tian-Yau part $X \setminus U_L$ the function w_k takes the value k so this is the same estimate as in (7.8). But the important point is that on $\{u \leqslant -1\}$ the factor w_k does not depend on k, contrarily to the initial estimate (7.8). We deduce the following corollary:

Corollary 7.3. *One has the following uniform estimate, for all functions f on X:*

$$\|f\|_{C_\delta^{2,\alpha}} \lessapprox \|f\|_{C_\delta^0} + \|\Delta_{g_k} f\|_{C^\alpha_{\delta+1+\frac{1}{n}}}.$$

□

Remark 7.4. The estimate $\|f\|_{C_\delta^{2,\alpha}} \lessapprox \|f\|_{C_\delta^0} + \|L_k f\|_{C^\alpha_{\delta+1+\frac{1}{n}}}$ follows for any operator of the shape say $L_k = \Delta_{g_k} + c$ where c is a constant, as one sees immediately using the interpolation estimate $\|f\|_{C^\alpha} \lessapprox \varepsilon^{-\alpha}\|f\|_{C^0} + \varepsilon^\alpha\|f\|_{C^1}$ for any fixed $0 < \varepsilon \ll 1$.

The rest of this section is devoted to the proof of the estimate (7.11). Again from the bounds on $g_k - g_{k,L}$ it is sufficient to prove the estimate on $g_{k,L}$.

First step. We first decompose along each circle into Fourier series $f = \sum_{\mathbb{Z}} f_\ell$ and control nonconstant modes. Here, more precisely, we see L as a S^1 bundle over $\mathbb{R}_- \times D$, and f_ℓ is induced from a section F_ℓ of $p^*L^{-\ell}$ over $\mathbb{R}_- \times D$ by the formula $f_\ell(x) = \langle F_\ell(p(x)), x^{\otimes \ell}\rangle$. On $p^*L^{-\ell}$ over D we have the rough Laplacian $-\Delta_D := \nabla^*\nabla$ constructed from the given connection on L and the metric g_D.

The Laplacian of the metric $g_{k,L}$ preserves the Fourier decomposition and acts on F_ℓ by

$$\Delta_{g_{k,L}} F_\ell = \frac{2}{\varphi_1''(u)}\big(\partial_u^2 F_\ell - \frac{\ell^2}{4k^2}F_\ell\big) + \frac{2(n-1)}{\varphi_1'(u)}\partial_u F_\ell + \frac{1}{-k\varphi_1'(u)}\Delta_D F_\ell \tag{7.12}$$
$$= \Delta_{\mathbb{R}_-\times D} F_\ell - \frac{\ell^2}{2\varphi_1''(u)k^2}F_\ell.$$

Lemma 7.5. *Fix $\varepsilon > 0$ small enough and $\ell \neq 0$. We have the estimate*

$$\sup_{E_{u_0}(\frac12\varepsilon)} |f_\ell| \lessapprox \frac{k}{\ell^2} \sup_{E_{u_0}(\varepsilon)} \big(w_k(u_0)^{\frac1n}|\Delta f_\ell| + w_k(u_0)^{-1}|f_\ell|\big). \tag{7.13}$$

It follows that

$$\sup_{E_{u_0}(\frac12\varepsilon)} |f - f_0| \lessapprox k \sup_{E_{u_0}(\varepsilon)} \big(w_k(u_0)^{\frac1n}|\Delta(f - f_0)| + w_k(u_0)^{-1}|f - f_0|\big). \tag{7.14}$$

Proof. We use the inequality

$$\frac12 \Delta_{\mathbb{R}_-\times D}|F_\ell|^2 \geq \langle \Delta_{\mathbb{R}_-\times D} F_\ell, F_\ell\rangle = \langle \Delta F_\ell, F_\ell\rangle + \frac{\ell^2}{2\varphi_1''(u)k^2}|F_\ell|^2 \tag{7.15}$$

twice.

First, if f_ℓ vanishes at $\partial E_{u_0}(\varepsilon)$, then from the maximum principle and the fact that $\varphi_1''(u) \lessapprox \varphi_1''(u_0)$ on $E_{u_0}(\varepsilon)$ we have

$$\sup_{E_{u_0}(\varepsilon)} |F_\ell| \lessapprox \frac{k^2\varphi_1''(u_0)}{\ell^2} \sup_{E_{u_0}(\varepsilon)} |\Delta F_\ell|$$

and the result follows since $\frac{k}{|u_0|} \leq A$ and $\varphi_1''(u) \approx u^{-1+1/n}$ when $u \to 0$.

Therefore it is sufficient to consider the case where $\Delta f_\ell = 0$. Let us consider the comparison function $g(u) = a\cosh(b(u - u_0))$. Then

$$\frac12 \Delta g = \frac{\partial_u^2 g}{\varphi_1''(u)} + (n-1)\frac{\partial_u g}{\varphi_1'(u)} \leq \Big(\frac{b^2}{\varphi_1''(u)} - \frac{(n-1)b}{\varphi_1'(u)}\Big)g$$
$$\leq \frac{\ell^2}{2k^2\varphi_1''(u)}g$$

if we take $b = \gamma\frac{\ell}{k}$ for some small constant $\gamma > 0$, thanks to the condition $u \leqslant -\frac{k}{A}$ for A small enough. Combining with (7.15) we obtain

$$\Delta(|F_\ell|^2 - g) \geq \frac{\ell^2}{\varphi_1''(u)k^2}(|F_\ell|^2 - g). \tag{7.16}$$

Choose a large enough so that $|F_\ell|^2 \leq g$ at $\partial E_{u_0}(\varepsilon)$, then it follows from (7.16) that $|F_\ell|^2 \leq g$ on $E_{u_0}(\varepsilon)$, which in particular on $E_{u_0}(\frac{\varepsilon}{2})$ we obtain

$$\sup_{E_{u_0}(\frac{\varepsilon}{2})} |F_\ell|^2 \lessapprox \frac{\cosh(\frac12\varepsilon\gamma\frac{\ell}{k}u_0)}{\cosh(\varepsilon\gamma\frac{\ell}{k}u_0)} \sup_{E_{u_0}(\varepsilon)} |F_\ell|^2.$$

Since $x := \frac{|u_0|}{k} \geq \frac{1}{A}$, we certainly have

$$\frac{\cosh(\frac{1}{2}\varepsilon\gamma\frac{\ell}{k}u_0)}{\cosh(\varepsilon\gamma\frac{\ell}{k}u_0)} \lessapprox \frac{1}{(x\ell)^4} \leq \frac{A^2}{(x\ell^2)^2}$$

which gives (7.13).

The case where $u_0 = -\infty$ (that is centered on the divisor) is similar, so we only highlight the modifications to perform. To obtain the inequality

$$\sup_{E_{-\infty}(\varepsilon)} |F_\ell| \lessapprox \frac{k^2}{\ell^2} \sup_{E_{-\infty}(\varepsilon)} |\Delta F_\ell| \tag{7.17}$$

we can modify $E_{-\infty}(\varepsilon)$ and assume that its boundary is $\{u = 2\log\varepsilon\}$. Let us set $u_\varepsilon := u - 2\log\varepsilon$ so that $\Delta u_\varepsilon = \frac{2(n-1)}{\varphi_1'(u)} = O(1)$ and fix $\delta > 0$. Then, we apply the maximum principle to $|F_\ell|^2 + \delta u_\varepsilon$ to obtain, for any fixed x:

$$|F_\ell(x)|^2 \leqslant \frac{k^2}{\ell^2} \sup_{E_{-\infty}(\varepsilon)} |\Delta F_\ell| \cdot \sup_{E_{-\infty}(\varepsilon)} |F_\ell| - \delta u_\varepsilon(x) + O(\delta)$$

and we get (7.17) by taking $\delta \to 0$ and passing to the supremum over $x \in E_{-\infty}(\varepsilon)$. The next step is very similar to the case $u_0 \neq -\infty$ but one chooses instead $g(u) = ae^{\frac{\ell u}{k}}$, satisfying $\Delta g \leqslant \frac{2\ell^2}{k^2\varphi_1''(u)}g$. Then we apply the maximum principle using the same barrier function as before, to obtain $\sup_{E_{-\infty}(\frac{\varepsilon}{2})} |F_\ell| \leqslant e^{-\frac{\ell}{k}\log 2} \sup_{E_{-\infty}(\varepsilon)} |F_\ell|$. The conclusion follows from the inequality $e^{-\frac{\ell}{k}\log 2} \lessapprox \frac{k^2}{\ell^2}$. □

Combining (7.14) with the local Schauder estimate (7.8) for $g = f - f_0$, we obtain

$$\|\nabla^2 g\|_{C_0^\alpha(E_{u_0}(\frac{1}{2}\varepsilon))} \lessapprox \|\Delta_{g_{k,L}} g\|_{C_0^\alpha(E_{u_0}(\varepsilon))} + w_k(u_0)^{-1-\frac{1}{n}} \|g\|_{C^0(E_{u_0}(\varepsilon))}, \tag{7.18}$$

which is the estimate stated in Proposition 7.2.

Second step. We can now restrict the proof of Proposition 7.2 to the case when f is circle invariant. In that case (7.12) reduces to

$$\begin{aligned} \Delta_{g_{k,L}} f &= 2\frac{\partial_u^2 f_\ell}{\varphi_1''(u)} + 2(n-1)\frac{\partial_u f}{\varphi_1'(u)} + \frac{1}{-k\varphi_1'(u)}\Delta_D f \\ &= \frac{1}{-k\varphi_1'(u)}(\partial_v^2 f + \Delta_D f), \end{aligned}$$

where $dv^2 = \frac{\varphi_1''(u)}{-2k\varphi_1'(u)}du^2$ and we choose $v(u_0) = 0$. We now restrict to the case where for every u one has

$$\int_{\{u\}\times D} f = 0. \tag{7.19}$$

We also suppose that we are not close to the divisor D, that is $r_D(u_0) > \varepsilon$. On each slice $\{u\} \times D$ the function f is therefore orthogonal to the kernel of Δ_D, which corresponds to erasing the critical weight 0 of the cylindrical Laplacien $\partial_v^2 + \Delta_D$. It follows that this Laplacian is an isomorphism $C^{2,\alpha}(\mathbb{R} \times D) \to$

$C^\alpha(\mathbb{R}\times D)$, see for example [LM85]. In particular we have, still under condition (7.19),

$$\sup_{\mathbb{R}\times D}|f| \lessapprox \sup_{\mathbb{R}\times D}|(\partial_v^2+\Delta_D)f|. \tag{7.20}$$

Of course the region $E_{u_0}(\varepsilon)$ gives only a bounded region in $\mathbb{R}\times D$, of diameter $2d$ with respect to the variable v. Using a cut-off function for $\frac{d}{2}\leq|v|\leq d$, we deduce from (7.20) combined with the interpolation inequality $\|\partial_v f\|_\infty^2 \lessapprox \|\partial_v^2 f\|_\infty\cdot\|f\|_\infty$ the following estimate

$$\sup_{[-\frac{d}{2},\frac{d}{2}]\times D}|f| \lessapprox \sup_{[-d,d]\times D}\Big[|(\partial_v^2+\Delta_D)f|+\frac{1}{d^2}|f|\Big]. \tag{7.21}$$

Now we come back to the variable u: we have $d^2\approx\frac{\varphi_1''(u_0)}{-k\varphi_1'(u_0)}w_k(u_0)^2$, so we obtain (remember $-\varphi_1'(u_0)\approx w_k(u_0)^{1/n}$):

$$\sup_{E_{u_0}(\frac{\varepsilon}{2})}|f| \lessapprox kw_k(u_0)^{\frac{1}{n}}\sup_{E_{u_0}(\varepsilon)}|\Delta_{g_{k,L}}f|+\frac{1}{\varphi_1''(u_0)w_k(u_0)^2}|f|. \tag{7.22}$$

Combining with (7.8) we finally get

$$\|\nabla^2 f\|_{C_0^\alpha(E_{u_0}(\frac{1}{2}\varepsilon))} \lessapprox \|\Delta_{g_k}f\|_{C_0^\alpha(E_{u_0}(\varepsilon))}+\frac{1}{\varphi_1''(u_0)w_k(u_0)^2}\|f\|_{C^0(E_{u_0}(\varepsilon))}. \tag{7.23}$$

Since $\varphi_1''(u)\approx w_k(u)^{-1+1/n}$ if $r_D(u)>\varepsilon$, the estimate (7.11) follows in that case.

For the case where $u_0=-\infty$, that is $E_{u_0}(\varepsilon)$ is centered on the divisor D, we proceed similarly, except that the limit after rescaling is a half-cylinder instead of a cylinder. The same estimate (7.11) follows.

Third step. There remains only the case of a function $f(u)$ of the variable u alone. But it is then almost obvious that the weight $w_k(u)^{1+1/n}$, which equals $|u|^{1+1/n}$ for u small enough, is the correct weight for the operator $\Delta_{g_{k,L}}$.

8. Convergence in the positive case: proof of Theorem B

8.1. **Bound on the inverse of the linearisation.**

We first give a bound on the inverse of the linearisation $L_k=\frac{1}{2}\Delta_{g_k}+1$ of the operator P_k defined in (5.13). We first prove:

Proposition 8.1. *Fix $\delta\in(0,1)$. There exists a constant c such that for any function f on X such that $\int_X f\,d\,\mathrm{vol}_{g_k}=0$ one has*

$$\|f\|_{C_\delta^{2,\alpha}}\leq c\|L_k f\|_{C^\alpha_{\delta+1+\frac{1}{n}}}. \tag{8.1}$$

We deduce:

Corollary 8.2. *Fix $\delta\in(0,1)$. The operator $L_k: C_\delta^{2,\alpha}\to C^\alpha_{\delta+1+1/n}$ satisfies*

$$\|L_k^{-1}\|\lessapprox k^{-\frac{1}{n}-\delta}. \tag{8.2}$$

Proof of Corollary 8.2. The constant function 1 satisfies $\|1\|_{C_\delta} = 1$ as soon as $\delta \geq 0$. Since $L_k(1) = 1$, the estimate (8.1) is also satisfied on constants.

Set $\delta' := \delta + 1 + \frac{1}{n}$. Given $f \in C^\alpha_{\delta'}$ we decompose $f = \bar{f} + f_1$ with $\bar{f}$ constant and $\int_X f_1 d\,\mathrm{vol}_{g_k} = 0$. We have $L_k^{-1} f = L_k^{-1} f_1 + \bar{f}$ so by (8.1), we have $\|L_k^{-1} f\|_{C^\alpha_\delta} \lessapprox \|f\|_{C^\alpha_{\delta'}} + |\bar{f}|$ and we are reduced to showing that

$$|\bar{f}| \lessapprox k^{-\frac{1}{n}-\delta} \|f\|_{C^0_{\delta'}}. \tag{8.3}$$

Now it follows from (5.9) that on $\{u \leqslant -k^\mu/2\}$, we have $d\,\mathrm{vol}_{g_k} \approx k^n d\,\mathrm{vol}_{g_X}$ for some fixed Riemannian metric g_X on X while on $\{\tilde{u} \geqslant -k^\mu/2\}$, we have $g_k = k^{1+\frac{1}{n}} g_{TY}$ and, in particular, $\int_{\{\tilde{u} \geqslant -k^\mu/2\}} d\,\mathrm{vol}_{g_k} = O(k^{n+1-\mu})$. It follows that $\mathrm{Vol}(g_k) \approx k^n$.

Now, since $|f| \leq \|f\|_{C^0_{\delta'}} w_k^{-\delta'}$ we have,

$$\begin{aligned} |\bar{f}| &\leq \frac{1}{\mathrm{Vol}(g_k)} \int_x |f| d\,\mathrm{vol}_{g_k} \\ &\lessapprox \frac{\|f\|_{C^0_{\delta'}}}{k^n} \int_X w_k^{-\delta'} d\,\mathrm{vol}_{g_k}. \end{aligned}$$

One checks that since $\delta' > 1$, the main contribution of the last integral is on the Tian-Yau part $X \setminus U_L$, and is of order $k^{n+1-\delta'}$. We therefore get $|\bar{f}| \lessapprox k^{1-\delta'} \|f\|_{C^0_{\delta'}}$ and (8.3) is proved. This ends the proof of the corollary. ☐

The rest of this section is devoted to the proof of Proposition 8.1.

Suppose (8.1) is not true. By Corollary 7.3 and Remark 7.4 there exist functions f_k such that $\int_X f_k d\,\mathrm{vol}_{g_k} = 0$ and

$$\|f_k\|_{C^0_\delta} = 1, \qquad \|L_k f_k\|_{C^\alpha_{\delta+1+\frac{1}{n}}} \to 0.$$

Fix $x_k \in X$ such that $w_k(x_k)^\delta |f_k(x_k)| = 1$. We then analyze the various cases depending on the limit of x_k.

First case. The x_k's converge to D and we can extract a limit in the normal bundle L: $u(x_k) < \eta < 0$. A subsequence x_k then converges to x such that $u(x) < 0$. Since $\|f_k\|_{C^{2,\alpha}_\delta}$ remains bounded, we can extract a limit $f_k \to f$. It turns out that f depends on u only because the norm $\|df_k\|_{C^0_{\delta+1}}$ involves a factor k^{-1} in the circle direction or $k^{-1/2}$ in the divisor D direction. Therefore we have a limit $f(u)$ which is nonzero at x and satisfies

$$\sup |\tilde{u}|^\delta |f| = 1, \tag{8.4}$$

$$\frac{f''(u)}{\varphi_1''(u)} + (n-1)\frac{f'(u)}{\varphi_1'(u)} + f(u) = 0. \tag{8.5}$$

Moreover

$$\frac{1}{k^n}\left|\int_{\tilde{u} \geq -k} f_k d\,\mathrm{vol}_{g_k}\right| \leq k^{1-\delta} \|f_k\|_{C^0_\delta}$$

so if $\delta < 1$ we have that

$$\frac{1}{k^n}\int_X f_k d\,\mathrm{vol}_{g_k} \to \int f(u)\varphi_1''(u)\varphi_1'(u)^{n-1}du$$

so the limit $f(u)$ satisfies

$$\int_{-\infty}^{0} f(u)\varphi_1''(u)\varphi_1'(u)^{n-1}du = 0. \tag{8.6}$$

The function $\varphi_1'(u)$ is an obvious solution of (8.5), it corresponds to the dilation vector field in the bundle L. It satisfies $\varphi_1'(u) \to -1$ when $u \to -\infty$, while the other solution is $f = g\varphi_1'$ with $g = \int \frac{du}{(1-e^{-\varphi_1})}$. By (2.10), $f(u) \approx u$ near $-\infty$ (it corresponds to the Green function near D). But this is ruled out by (8.4). Therefore we see that up to a constant we must have $f(u) = \varphi_1'(u)$, which gives a contradiction with (8.6).

Second case. The x_k's still converge to D but we can extract a limit only in the intermediate region between the normal bundle L and the Tian-Yau metric on $X \setminus D$: this is the case where $u(x_k) \to 0$ but $\frac{1}{k}u(x_k) \to +\infty$. It is similar to the previous one, the rescaled functions $u(x_k)^{-\delta} f_k$ converge to a nonzero function f on the Calabi metric on L given by

$$\tfrac{1}{n}|u|^{-1+\frac{1}{n}}(\tfrac{1}{2}du^2 + 2\eta^2) + |u|^{\frac{1}{n}} g_D. \tag{8.7}$$

The function f is harmonic for (8.7) and satisfies $|f| \le |u|^{-\delta}$. But again, because of our Hölder estimates, the limit f is a function of u only, so f is a linear combination of the harmonic functions 1 and $|u|^{\frac{1}{n}}$, which gives a contradiction when $u \to -\infty$.

Third case. The x_k's converge in the Tian-Yau part: $u(x_k) = O(k)$. This is similar: after rescaling, we get a nonzero limit f which is a nonzero harmonic function on the Tian-Yau space $X \setminus D$ with $|f| \le |\tilde{t}|^{-\delta}$. This implies $f = 0$, which is a contradiction.

8.2. **Resolution of the Kähler-Einstein equation.** We first control the quadratic terms of the equation:

Lemma 8.3. *For $\delta \ge 0$ and any function φ we have*

$$\|(\partial\overline{\partial}\varphi)^2\|_{C^\alpha_\delta} \lesssim k^{-\delta}\|\partial\overline{\partial}\varphi\|^2_{C^\alpha_\delta} \tag{8.8}$$

Proof. From Remark 7.1 we see that for $\delta = 0$ we have the estimate. The weight w_k^δ introduces a coefficient $w_k^{-\delta}$ in the estimates, which is maximal when the weight w_k is minimal, that is on the Tian-Yau part where the value of w_k is k. The lemma follows. □

End of the proof of Theorem B. Decompose P_k into an affine part and a higher order term part:

$$P_k(\varphi) = P_k(0) + L_k(\varphi) + Q_k(\varphi).$$

It follows from the lemma that if $\|\partial\bar\partial\varphi\|_{C^\delta_\alpha} < \varepsilon k^{1+1/n+\delta}$ for a small enough $\varepsilon > 0$, then

$$\|Q_k(\varphi) - Q_k(\psi)\|_{C^\alpha_{\delta+1+\frac{1}{n}}} \lessapprox k^{-1-\frac{1}{n}-\delta}\|\varphi-\psi\|_{C^{2,\alpha}_\delta}\left(\|\varphi\|_{C^{2,\alpha}_\delta} + \|\psi\|_{C^{2,\alpha}_\delta}\right). \tag{8.9}$$

On the other hand we have from (5.14)

$$\|P_k(0)\|_{C^\alpha_{\delta+1+\frac{1}{n}}} \lessapprox k^{2(1+\frac{1}{n}+\delta)}. \tag{8.10}$$

Fix δ small enough (one needs actually $\delta < 1-\frac{1}{n}$). Given the bound (8.2), together with (8.9) and (8.10), standard fixed point arguments (see for example [BM11, Lemma 1.3]) now imply that the equation $P_k(\varphi) = 0$ has a unique solution $\varphi(k)$ in a ball of radius $\varepsilon k^{1+\frac{2}{n}+2\delta}$ in $C^{2,\alpha}_\delta$, and this solution actually satisfies

$$\|\varphi(k)\|_{C^{2,\alpha}_\delta} \lessapprox k^{2+1/n+\delta}. \tag{8.11}$$

By [Ber15, Theorem 7.3], the Kähler-Einstein metric $\widehat{\omega}_k := \omega_k + i\partial\bar\partial\varphi(k)$ is the unique Kähler-Einstein metric with cone angle $2\pi k$ along D, i.e. satisfying $\operatorname{Ric}\widehat{\omega}_k = \widehat{\omega}_k + (1-k)[D]$. Given any compact set $M \Subset X\setminus D$, we have for $k \ll 1$:

$$w_k|_M \equiv k, \quad \omega_k|_M = k^{1+\frac{1}{n}}\omega_{TY}, \quad \|k^{-1-\frac{1}{n}}\partial\bar\partial\varphi(k)\|_{\omega_{TY}} \lessapprox k$$

where the last estimate follows by (8.11). In particular, $\|k^{-1-\frac{1}{n}}\widehat{\omega}_k - \omega_{TY}\|_{\omega_{TY}} = O(k)$ on M, and we are done with the first part of Theorem B. The asymptotics of $\widehat{\omega}_k$ are the same as those of ω_k thanks to (8.11), and near D, they are given by $\omega_{k,L}$ thanks to Lemma 5.2. Finally, since the $\operatorname{diam}_{g_{TY}}(\{t \geqslant -a\} \approx a^{\frac{1}{2}+\frac{1}{2n}}$ when $a \to +\infty$, we have $\operatorname{diam}_{k^{1+\frac{1}{n}}g_{TY}}(\{u \geqslant -k^\mu\}) \approx k^{\mu(\frac{1}{2}+\frac{1}{2n})}$ and therefore the Tian-Yau region $\{u \geqslant -k^\mu\}$ is collapsed onto a point when $k \to 0$. The remaining statements in Theorem B now follow from (2.18). ☐

References

[AMRT10] Avner Ash, David Mumford, Michael Rapoport, and Yung-Sheng Tai. *Smooth compactifications of locally symmetric varieties*. Cambridge Mathematical Library. Cambridge University Press, Cambridge, second edition, 2010. With the collaboration of Peter Scholze.

[And06] Michael T. Anderson. Dehn filling and Einstein metrics in higher dimensions. *J. Differ. Geom.*, 73(2):219–261, 2006.

[Ber13] Robert J. Berman. A thermodynamical formalism for Monge-Ampère equations, Moser-Trudinger inequalities and Kähler-Einstein metrics. *Adv. Math.*, 248:1254–1297, 2013.

[Ber15] Bo Berndtsson. A Brunn-Minkowski type inequality for Fano manifolds and some uniqueness theorems in Kähler geometry. *Invent. Math.*, 200(1):149–200, 2015.

[BG14] Robert J. Berman and Henri Guenancia. Kähler-Einstein metrics on stable varieties and log canonical pairs. *Geometric and Function Analysis*, 24(6):1683–1730, 2014.

[Biq02] Olivier Biquard. Métriques autoduales sur la boule. *Invent. math.*, 148(3):545–607, 2002.

[BM11] Olivier Biquard and Vincent Minerbe. A Kummer construction for gravitational instantons. *Commun. Math. Phys.*, 308(3):773–794, 2011.

[Bre13] S. Brendle. Ricci flat Kähler metrics with edge singularities . *International Mathematics Research Notices*, 24:5727–5766, 2013.

[Cal79] E. Calabi. Métriques kählériennes et fibrés holomorphes. *Ann. Sci. École Norm. Sup. (4)*, 12(2):269–294, 1979.

[CGP13] Frédéric Campana, Henri Guenancia, and Mihai Păun. Metrics with cone singularities along normal crossing divisors and holomorphic tensor fields. *Ann. Sci. Éc. Norm. Supér. (4)*, 46(6):879–916, 2013.

[CKZ11] U. Cegrell, S. Kołodziej, and A. Zeriahi. Maximal subextensions of plurisubharmonic functions. *Ann. Fac. Sci. Toulouse Math. (6)*, 20(Fascicule Spécial):101–122, 2011.

[CR15] Ivan A. Cheltsov and Yanir A. Rubinstein. Asymptotically log Fano varieties. *Adv. Math.*, 285:1241–1300, 2015.

[CY75] S. Y. Cheng and S. T. Yau. Differential equations on Riemannian manifolds and their geometric applications. *Comm. Pure Appl. Math.*, 28(3):333–354, 1975.

[dBE20] Martin de Borbon and Gregory Edwards. Schauder estimates on products of cones. Preprint arXiv:2002.07668, 2020.

[DGG20] Eleonora DiNezza, Vincent Guedj, and Henri Guenancia. Families of singular Kähler-Einstein metrics. Preprint arXiv:2003.08178, to appear in J. Eur. Math. Soc., 2020.

[Don12] Simon Donaldson. Kähler metrics with cone singularities along a divisor. In *Essays in mathematics and its applications*, pages 49–79. Springer, Heidelberg, 2012.

[GP16] Henri Guenancia and Mihai Păun. Conic singularities metrics with prescribed Ricci curvature: the case of general cone angles along normal crossing divisors. *J. Differential Geom.*, 103(1):15–57, 2016.

[GS16] Bin Guo and Jian Song. Schauder estimates for equations with cone metrics. Preprint arXiv:1612.00075, 2016.

[GS18] Bin Guo and Jian Song. Schauder estimates for equations with cone metrics, II. Preprint arXiv:1809.03116, 2018.

[Gue20] Henri Guenancia. Kähler–Einstein metrics: From cones to cusps. *J. Reine Angew. Math.*, 759:1–27, 2020.

[GY18] Yoating Gui and Hao Yin. Schauder estimates on smooth and singular spaces. Preprint arXiv:1803.05081, to appear in Ann. Glob. Anal. Geom., 2018.

[Hei12] Hans-Joachim Hein. Gravitational instantons from rational elliptic surfaces. *J. Am. Math. Soc.*, 25(2):355–393, 2012.

[HSVZ] Hans-Joachim Hein, Song Sun, Jeff Viaclovsky, and Ruobing Zhang. Nilpotent structures and collapsing Ricci-flat metrics on K3 surfaces. To appear in JAMS. `https://doi.org/10.1090/jams/978`.

[JMR16] Thalia Jeffres, Rafe Mazzeo, and Yanir A. Rubinstein. Kähler-Einstein metrics with edge singularities. *Ann. of Math. (2)*, 183(1):95–176, 2016. with an Appendix by C. Li and Y. Rubinstein.

[Kob84] R. Kobayashi. Kähler-Einstein metric on an open algebraic manifolds. *Osaka 1. Math.*, 21:399–418, 1984.

[Koł98] Sławomir Kołodziej. The complex Monge-Ampère operator. *Acta Math.*, 180(1):69–117, 1998.

[LM85] Robert B. Lockhart and Robert C. McOwen. Elliptic differential operators on noncompact manifolds. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 12(3):409–447, 1985.

[Mok12] Ngaiming Mok. Projective algebraicity of minimal compactifications of complex-hyperbolic space forms of finite volume. In *Perspectives in analysis, geometry, and topology*, volume 296 of *Progr. Math.*, pages 331–354. Birkhäuser/Springer, New York, 2012.

[Mum77] D. Mumford. Hirzebruch's proportionality theorem in the non-compact case. *Invent. Math.*, 42:239–272, 1977.

[Oda20] Yuji Odaka. Polystable log Calabi-Yau varieties and gravitational instantons. Preprint arXiv:2009.13876, 2020.

[Rub14] Yanir A. Rubinstein. Smooth and singular Kähler-Einstein metrics. In *Geometric and spectral analysis*, volume 630 of *Contemp. Math.*, pages 45–138. Amer. Math. Soc., Providence, RI, 2014.

[RZ20] Yanir A. Rubinstein and Kewei Zhang. Small angle limits of Hamilton's footballs. *Bull. Lond. Math. Soc.*, 52(1):189–199, 2020.

[SW16] Jian Song and Xiaowei Wang. The greatest Ricci lower bound, conical Einstein metrics and Chern number inequality. *Geom. Topol.*, 20(1):49–102, 2016.

[SZ19] Song Sun and Ruobing Zhang. Complex structure degenerations and collapsing of Calabi-Yau metrics. Preprint arXiv:1906.03368, 2019.

[TY87] Gang Tian and Shing-Tung Yau. Existence of Kähler-Einstein metrics on complete Kähler manifolds and their applications to algebraic geometry. *Adv. Ser. Math. Phys. 1*, 1:574–628, 1987. Mathematical aspects of string theory (San Diego, Calif., 1986).

[TY90] Gang Tian and Shing-Tung Yau. Complete Kähler manifolds with zero Ricci curvature. I. *J. Amer. Math. Soc.*, 3(3):579–609, 1990.

[Wan06] Xu-Jia Wang. Schauder estimates for elliptic and parabolic equations. *Chinese Ann. Math. Ser. B*, 27(6):637–642, 2006.

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