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ABSTRACT. Consider a smooth, geometrically irreducible, projective curve of genus $g \geq 2$ defined over a number field of degree $d \geq 1$. It has at most finitely many rational points by the Mordell Conjecture, a theorem of Faltings. We show that the number of rational points is bounded only in terms of g, d, and the Mordell–Weil rank of the curve's Jacobian, thereby answering in the affirmative a question of Mazur. In addition we obtain uniform bounds, in g and d, for the number of geometric torsion points of the Jacobian which lie in the image of an Abel–Jacobi map. Both estimates generalize our previous work for 1-parameter families. Our proof uses Vojta's approach to the Mordell Conjecture, and the key new ingredient is the generalization of a height inequality due to the second- and third-named authors.

CONTENTS

1. INTRODUCTION

Let F be a field. By a curve defined over F we mean a geometrically irreducible, projective variety of dimension 1 defined over F. Let C be a smooth curve of genus at least 2 defined over a number field F. As was conjectured by Mordell and proved by Faltings [Fal83], $C(F)$, the set of F-rational points of C, is finite.

We let $\operatorname{Jac}(C)$ denote the Jacobian of C. Recall that $\operatorname{Jac}(C)(F)$ is a finitely generated abelian group by the Mordell–Weil Theorem.

The aim of this paper is to bound $\#C(F)$ from above. Here is our first result.

Theorem 1.1. *Let $g \geq 2$ and $d \geq 1$ be integers. Then there exists a constant $c = c(g,d) \geq 1$ with the following property. If C is a smooth curve of genus g defined over a*

number field F with $[F : \mathbb{Q}] \leq d$, then

(1.1) $$\#C(F) \leq c^{1+\rho}$$

where ρ is the rank of $\mathrm{Jac}(C)(F)$.

This theorem gives an affirmative answer to a question posed by Mazur [Maz00, Page 223]. See also [Maz86, top of page 234] for an earlier question. Before this, Lang formulated a related conjecture [Lan78, page 140] on the number of integral points of elliptic curves.

The method of our theorem builds up on the work of many others. At the core we follow Vojta's proof [Voj91] of the Mordell Conjecture. Vojta's proof was later simplified by Bombieri [Bom90] and further developed by Faltings [Fal91]. Silverman [Sil93] proved a bound of the quality (1.1) if C ranges over twists of a given smooth curve. The bound by de Diego [dD97] is of the form $c(g)7^\rho$, where $c(g) > 0$ depends only on g; the value 7 had already arisen in Bombieri's work. But she only counts points whose height is large in terms of a height of C. Work of David–Philippon [DP02] and Rémond [Rém00a] led to explicit estimates. Recently, Alpoge [Alp18] [Alp20, Theorem 6.1.1] improved 7 to 1.872 and, for g large enough, even to 1.311.

On combining the Vojta and Mumford Inequalities one gets an upper bound for the number of *large points* in $C(F)$; these are points whose height is sufficiently large relative to a suitable height of C. A lower bound for the Néron–Tate height, such as proved by David–Philippon [DP02], can be used to count the number of remaining points which we sometimes call *small points*. Indeed, Rémond [Rém00a] made the Vojta and Mumford Inequalities explicit and obtained explicit upper bounds for the number of rational points on curves embedded in abelian varieties. The resulting cardinality bounds depend on a suitable notion of height of C, an artifact of the lower bounds for the Néron–Tate height. Later, David–Philippon [DP07] proved stronger height lower bounds in a power of an elliptic curve. They then obtained uniform estimates of the quality (1.1) for a curve in a power of elliptic curves, thus providing evidence that Mazur's Question had a positive answer, see also David–Nakamaye–Philippon's work [DNP07].

We give an overview of the general method in more detail in §1.1 below.

The main innovation of this paper is to prove a lower bound for the Néron–Tate height that is sufficiently strong to eliminate the dependency on the height of C. This leads to a uniform estimate as in Theorem 1.1. In prior work [DGH19] we applied the earlier height lower bound [GH19] to recover a variant of Theorem 1.1 in a one-parameter family of smooth curves.

We now explain some further results that follow from the approach described above. For an integer $g \geq 1$, let $\mathbb{A}_{g,1}$ denote the coarse moduli space of principally polarized abelian varieties of dimension g. This is an irreducible quasi-projective variety which we can take as defined over $\overline{\mathbb{Q}}$, the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$. Suppose we are presented with an immersion $\iota\colon \mathbb{A}_{g,1} \to \mathbb{P}^m_{\overline{\mathbb{Q}}}$ into projective space. Let $h\colon \mathbb{P}^m_{\overline{\mathbb{Q}}}(\overline{\mathbb{Q}}) \to \mathbb{R}$ denote the absolute logarithmic Weil height, *cf.* [BG06, §1.5.1]. For brevity, we sometimes call h the Weil height. If C is a smooth curve of genus $g \geq 2$ defined over $\overline{\mathbb{Q}}$ and if $P_0 \in C(\overline{\mathbb{Q}})$, then we can consider $C - P_0$ as a curve in $\mathrm{Jac}(C)$ via the Abel–Jacobi map. We use $[\mathrm{Jac}(C)]$ to denote the point in $\mathbb{A}_{g,1}(\overline{\mathbb{Q}})$ parametrizing $\mathrm{Jac}(C)$.

An abelian group Γ is said to have finite rank if $\Gamma \otimes \mathbb{Q}$ is a finite dimensional $\mathbb{Q}$-vector space. In this case $\dim \Gamma \otimes \mathbb{Q}$ is the rank of Γ. Consider an abelian variety A defined over $\mathbb{C}$ and let Γ be a finite rank subgroup of $A(\mathbb{C})$. Lang [Lan65] conjectured that a curve in A intersects Γ in a finite set unless the curve is smooth of genus 1. The Conjecture follows from Faltings's Theorem [Fal83] and work of Raynaud [Ray83].

The following theorem is more in the spirit of [Maz86, top of page 234].

Theorem 1.2. *Let $g \geq 2$ and let ι be as above. Then there exist two constants $c_1 = c_1(g, \iota) \geq 0$ and $c_2 = c_2(g, \iota) \geq 1$ with the following property. Let C be a smooth curve of genus g defined over $\overline{\mathbb{Q}}$, let $P_0 \in C(\overline{\mathbb{Q}})$, and let Γ be a subgroup of $\mathrm{Jac}(C)(\overline{\mathbb{Q}})$ of finite rank $\rho \geq 0$. If $h(\iota([\mathrm{Jac}(C)])) \geq c_1$, then*

$$\#(C(\overline{\mathbb{Q}}) - P_0) \cap \Gamma \leq c_2^{1+\rho}.$$

The following corollary follows from Theorem 1.2 applied to $\Gamma = \mathrm{Jac}(C)(\overline{\mathbb{Q}})_{\mathrm{tors}}$, which has rank 0.

Corollary 1.3. *Let $g \geq 2$ and let ι be as above. Then there exist two constants $c_1 = c_1(g, \iota) \geq 0$ and $c_2 = c_2(g, \iota) \geq 1$ with the following property. Let C be a smooth curve of genus g defined over $\overline{\mathbb{Q}}$ and let $P_0 \in C(\overline{\mathbb{Q}})$. If $h(\iota([\mathrm{Jac}(C)])) \geq c_1$, then*

$$\#(C(\overline{\mathbb{Q}}) - P_0) \cap \mathrm{Jac}(C)(\overline{\mathbb{Q}})_{\mathrm{tors}} \leq c_2.$$

As in Theorem 1.1 we can drop the condition on the height of the Jacobian by working over a number field of bounded degree.

Theorem 1.4. *Let $g \geq 2$ and $d \geq 1$ be integers. Then there exists a constant $c = c(g, d) \geq 1$ with the following property. Let C be a smooth curve of genus g defined over a number field $F \subseteq \overline{\mathbb{Q}}$ with $[F : \mathbb{Q}] \leq d$ and let $P_0 \in C(\overline{\mathbb{Q}})$, then*

$$\#(C(\overline{\mathbb{Q}}) - P_0) \cap \mathrm{Jac}(C)(\overline{\mathbb{Q}})_{\mathrm{tors}} \leq c.$$

Let us recall some previous results towards Mazur's Question for rational points, *i.e.*, towards Theorem 1.1. Based on the method of Vojta, Alpoge [Alp18] proved that the average number of rational points on a curve of genus 2 with a marked Weierstrass point is bounded. Let C be a smooth curve of genus $g \geq 2$ defined over a number field $F \subseteq \overline{\mathbb{Q}}$. The Chabauty–Coleman approach [Cha41, Col85] yields estimates under an additional hypothesis on the rank of Mordell–Weil group. For example, if $\mathrm{Jac}(C)(F)$ has rank at most $g - 3$, Stoll [Sto19] showed that $\#C(F)$ is bounded solely in terms of $[F : \mathbb{Q}]$ and g if C is hyperelliptic; Katz–Rabinoff–Zureick-Brown [KRZB16] later, under the same rank hypothesis, removed the hyperelliptic hypothesis. Checcoli, Veneziano, and Viada [CVV17] obtain an effective height bound under a restriction on the Mordell–Weil rank.

As for algebraic torsion points, *i.e.*, in the direction of Theorem 1.4, DeMarco–Krieger–Ye [DKY20] proved a bound on the cardinality of torsion points on any genus 2 curve that admits a degree-two map to an elliptic curve when the Abel–Jacobi map is based at a Weierstrass point. Moreover, their bound is independent of $[F : \mathbb{Q}]$.

1.1. **Néron–Tate distance of algebraic points on curves.**

Let C be a smooth curve defined over $\overline{\mathbb{Q}}$ of genus $g \geq 2$, let $P_0 \in C(\overline{\mathbb{Q}})$, and let Γ be a subgroup of $\mathrm{Jac}(C)(\overline{\mathbb{Q}})$

of finite rank ρ. For simplicity we identify C with its image under the Abel–Jacobi embedding $C \to \mathrm{Jac}(C)$ via P_0.

Our proof of Theorem 1.2 follows the spirit of the method of Vojta, later generalized by Faltings. Let $\hat{h} \colon \mathrm{Jac}(C)(\overline{\mathbb{Q}}) \to \mathbb{R}$ denote the Néron–Tate height attached to a symmetric and ample line bundle on $\mathrm{Jac}(C)$. We divide $C(\overline{\mathbb{Q}}) \cap \Gamma$ into two parts:

- Small points $\left\{ P \in C(\overline{\mathbb{Q}}) \cap \Gamma : \hat{h}(P) \leq B(C) \right\}$;
- Large points $\left\{ P \in C(\overline{\mathbb{Q}}) \cap \Gamma : \hat{h}(P) > B(C) \right\}$

where $B(C)$ is allowed to depend on a suitable height of C. It turns out that we can take $B(C) = c_0 \max\{1, h(\iota([\mathrm{Jac}(C)]))\}$ for some $c_0 = c_0(g, \iota) > 0$. The constant c_0 is chosen in a way that accommodates both the *Mumford inequality* and the *Vojta inequality*. Combining these two inequalities yields an upper bound on the number of large points by $c_1(g)^{1+\rho}$, see for example Vojta's [Voj91, Theorem 6.1] in the important case where Γ is the group of points of $\mathrm{Jac}(C)$ rational over a number field or more generally in the work of David–Philippon [DP02, DP07] and Rémond [Rém00a].

Thus in order to prove Theorem 1.2, it suffices to bound the number of small points. In this paper we find such a bound by studying the Néron–Tate distance of points in $C(\overline{\mathbb{Q}})$.

Roughly speaking, we find positive constants c_1, c_2, c_3, and c_4 that depend on g and ι, but not on C, such that if $h(\iota([\mathrm{Jac}(C)])) \geq c_1$ then for all $P \in C(\overline{\mathbb{Q}})$ we have the following alternative.

- Either P lies in a subset of $C(\overline{\mathbb{Q}})$ of cardinality at most c_2,
- or $\left\{ Q \in C(\overline{\mathbb{Q}}) : \hat{h}(Q - P) \leq h(\iota([\mathrm{Jac}(C)]))/c_3 \right\} < c_4$.

This dichotomy is stated in Proposition 7.1. In this paper, we make the statement precise by referring to the universal family of genus g smooth curves with suitable level structure, and the Néron–Tate height on each Jacobian attached to the tautological line bundle. The setup is done in §6.

This proposition can be seen as a relative version of the Bogomolov conjecture for abelian varieties with large height. It has the following upshot: If $h(\iota([\mathrm{Jac}(C)])) \geq c_1$, then the small points in $C(\overline{\mathbb{Q}}) \cap \Gamma$ lie in a set of uniformly bounded cardinality, or are contained in $(1 + c_0 c_3)^\rho$ balls in the Néron–Tate metric, with each ball containing at most c_4 points. This will yield the desired bound in Theorem 1.2, as executed in §8.

1.2. Height inequality and non-degeneracy.

We follow the framework presented in our previous work [DGH19]. In *loc. cit.* we proved the result for 1-parameter families, as an application of the second- and third-named authors' height inequality [GH19, Theorem 1.4]. Passing to general cases requires generalizing this height inequality to higher dimensional bases. The generalization has two parts: generalizing the inequality itself under the non-degeneracy condition and generalizing the criterion of non-degenerate subvarieties. We execute the first part in the current paper while the second part was done by the second-named author in [Gao20a]. Let us explain the setup.

Let k be an algebraically closed subfield of $\mathbb{C}$. Let S be a regular, irreducible, quasi-projective variety defined over k that is Zariski open in an irreducible projective variety $\overline{S} \subseteq \mathbb{P}_k^m$. Let $\pi \colon \mathcal{A} \to S$ be an abelian scheme of relative dimension $g \geq 1$. We suppose that we are presented with a closed immersion $\mathcal{A} \to \mathbb{P}_k^n \times S$ over S. On the generic fiber

of π we assume that this immersion comes from a basis of the global sections of the l-th power of a symmetric and ample line bundle with $l \geq 4$. If $k = \overline{\mathbb{Q}}$ and as described in §3.1 we obtain two height functions, the restriction of the Weil height $h \colon \overline{S}(\overline{\mathbb{Q}}) \to \mathbb{R}$ and the Néron–Tate height $\hat{h}_{\mathcal{A}} \colon \mathcal{A}(\overline{\mathbb{Q}}) \to \mathbb{R}$.

Let $\ell \geq 3$ be an integer. Throughout the whole paper, by *level-ℓ-structure* we always mean *symplectic level-ℓ-structure*. For the purpose of our main applications, including Theorems 1.1 and 1.2, it suffices to work under the following hypothesis.

(Hyp): $\mathcal{A} \to S$ carries a principal polarization and has level-ℓ-structure for some $\ell \geq 3$. So in the main body of the paper, we will focus on the case (Hyp). The general case where (Hyp) is not assumed will be handled in Appendix B.

The non-degenerate subvarieties of $\mathcal{A}$ are defined using the *Betti map* which we briefly describe here; the precise definition will be given by Proposition B.2 and in Proposition 2.1 under (Hyp).

For any $s \in S(\mathbb{C})$, there exists an open neighborhood $\Delta \subseteq S^{\mathrm{an}}$ of s which we may assume is simply-connected. Then one can define a basis $\omega_1(s), \ldots, \omega_{2g}(s)$ of the period lattice of each fiber $s \in \Delta$ as holomorphic functions of s. Now each fiber $\mathcal{A}_s = \pi^{-1}(s)$ can be identified with the complex torus $\mathbb{C}^g / (\mathbb{Z}\omega_1(s) \oplus \cdots \oplus \mathbb{Z}\omega_{2g}(s))$, and each point $x \in \mathcal{A}_s(\mathbb{C})$ can be expressed as the class of $\sum_{i=1}^{2g} b_i(x)\omega_i(s)$ for real numbers $b_1(x), \ldots, b_{2g}(x)$. Then $b_\Delta(x)$ is defined to be the class of the $2g$-tuple $(b_1(x), \ldots, b_{2g}(x)) \in \mathbb{R}^{2g}$ modulo $\mathbb{Z}^{2g}$. We obtain with a real-analytic map

$$b_\Delta \colon \mathcal{A}_\Delta = \pi^{-1}(\Delta) \to \mathbb{T}^{2g},$$

which is fiberwise a group isomorphism and where $\mathbb{T}^{2g}$ is the real torus of dimension $2g$.

Definition 1.5. *An irreducible subvariety X of $\mathcal{A}$ is said to be non-degenerate if there exists an open non-empty subset Δ of S^{an}, with the Betti map $b_\Delta \colon \mathcal{A}_\Delta := \pi^{-1}(\Delta) \to \mathbb{T}^{2g}$, such that*

$$(1.2) \qquad \max_{x \in X^{\mathrm{sm,an}} \cap \mathcal{A}_\Delta} \mathrm{rank}_{\mathbb{R}}(\mathrm{d}b_\Delta|_{X^{\mathrm{sm,an}}})_x = 2 \dim X$$

where $\mathrm{d}b_\Delta$ denotes the differential and $X^{\mathrm{sm,an}}$ is the analytification of the regular locus of X.

As the inequality $\leq$ in (1.2) trivially holds true, (1.2) is equivalent to: there exists $x \in X^{\mathrm{sm,an}} \cap \mathcal{A}_\Delta$ such that $\mathrm{rank}_{\mathbb{R}}(\mathrm{d}b_\Delta|_{X^{\mathrm{sm,an}}})_x = 2 \dim X$.

In Proposition 2.2(iii) we give another characterization of non-degenerate subvarieties. We can now formulate the height inequality.

Theorem 1.6. *Suppose that $\mathcal{A}$ and S are as above with $k = \overline{\mathbb{Q}}$; in particular, $\mathcal{A}$ satisfies* (Hyp). *Let X be a closed irreducible subvariety of $\mathcal{A}$ defined over $\overline{\mathbb{Q}}$ that dominates S. Suppose X is non-degenerate, as defined in Definition 1.5. Then there exist constants $c_1 > 0$ and $c_2 \geq 0$ and a Zariski open dense subset U of X with*

$$\hat{h}_{\mathcal{A}}(P) \geq c_1 h(\pi(P)) - c_2 \quad \text{for all} \quad P \in U(\overline{\mathbb{Q}}).$$

Note that [GH19, Theorem 1.4] is, up to some minor reduction, precisely Theorem 1.6 for $\dim S = 1$ together with the criterion for X to be non-degenerate when $\dim S = 1$. In general, the degeneracy behavior of X is fully studied in [Gao20a]. See [Gao20a, Theorem 1.1] for the criterion. However in practice, we sometimes still want to understand

the height comparison on some degenerate X. One way to achieve this is by applying [Gao20a, Theorem 1.3], which asserts the following statement: If X satisfies some reasonable properties, then we can apply Theorem 1.6 after doing some simple operations with X.

For the purpose of proving Proposition 7.1 and furthermore Theorem 1.2, we work in the following situation.

Let $\mathbb{A}_{g,\ell}$ denote the moduli space of principally polarized g-dimensional abelian varieties with level-ℓ-structure. It is a classical fact that $\mathbb{A}_{g,\ell}$ is represented by an irreducible, regular, quasi-projective variety defined over a number field, see [MFK94, Theorem 7.9 and below] or [OS80, Theorem 1.9], so it is a fine moduli space. Let $\mathbb{M}_{g,\ell}$ be the fine moduli space of smooth curves of genus g whose Jacobian is equipped with level-ℓ-structure; see [DM69, (5.14)] or [OS80, Theorem 1.8] for the existence. Then $\mathbb{M}_{g,\ell}$ is an irreducible, regular, quasi-projective variety defined over a number field.

To avoid confusion on different notations in different references, we make the following convention throughout the paper. We will take $\mathbb{A}_{g,\ell}$ and $\mathbb{M}_{g,\ell}$ as geometrically irreducible varieties. Some authors define $\mathbb{A}_{g,\ell}$ over $\mathbb{Z}[1/\ell]$ (or over $\mathbb{Z}$) and then consider it over $\overline{\mathbb{Q}}$ by base change. The $\overline{\mathbb{Q}}$-variety thus obtained may not be irreducible, and each irreducible component is defined over $\mathbb{Q}(\zeta_\ell)$ for some root of unity ζ_ℓ of order ℓ. Choosing a geometrically irreducible component of $\mathbb{A}_{g,\ell}$ amounts to fixing a complex root of unity of order ℓ. We fix such a choice once and for all and consider $\mathbb{A}_{g,\ell}$ as an irreducible variety defined over $\overline{\mathbb{Q}}$. The same holds for $\mathbb{M}_{g,\ell}$. We will usually fix ℓ and abbreviate $\mathbb{A}_{g,\ell}$ (resp. $\mathbb{M}_{g,\ell}$) by $\mathbb{A}_g$ (resp. $\mathbb{M}_g$). It is often convenient to consider $\mathbb{A}_g$ and $\mathbb{M}_g$ as over $\overline{\mathbb{Q}}$, but sometimes we will recall that both arise from varieties defined over the number field $\mathbb{Q}(\zeta_\ell)$. We denote the coarse moduli space of smooth curves of genus g with $\mathbb{M}_{g,1}$.

Furthermore, let $\mathfrak{C}_g \to \mathbb{M}_g$ be the universal curve and $\mathfrak{A}_g \to \mathbb{A}_g$ be the universal abelian variety. Taking the Jacobian of a smooth curve leads to the Torelli morphism $\mathbb{M}_g \to \mathbb{A}_g$ which is finite-to-1 (but not injective as we have level structure). Moreover, for $M \geq 2$ let $\mathscr{D}_M$ denote the M-th Faltings–Zhang morphism fiberwise defined by sending

$$(1.3) \qquad\qquad (P_0, P_1, \ldots, P_M) \mapsto (P_1 - P_0, \ldots, P_M - P_0);$$

we give a precise definition of this morphism in §6.1. Roughly speaking, we will apply Theorem 1.6 to

$$X := \mathscr{D}_M(\underbrace{\mathfrak{C}_g \times_{\mathbb{M}_g} \cdots \times_{\mathbb{M}_g} \mathfrak{C}_g}_{(M+1)\text{-copies}}) \subseteq \underbrace{\mathfrak{A}_g \times_{\mathbb{M}_g} \cdots \times_{\mathbb{M}_g} \mathfrak{A}_g}_{M\text{-copies}}$$

for a suitable M. To verify non-degeneracy we will refer to the second-named author's work [Gao20a, Theorem 1.2'] which applies if M is large in terms of g. So we can apply Theorem 1.6 to such X. This will eventually lead to Proposition 7.1.

The morphism and its variants are powerful tools in diophantine geometry, see [Fal91, Lemma 4.1]. It is closely connected to problems involving small Néron–Tate height, see [Zha98, Lemma 3.1]. Stoll [Sto19] used a variant of (1.3) to show that a conjecture of Pink [Pin05b] on unlikely intersections implies Theorem 1.2 with the condition $h(\iota([\mathrm{Jac}(C)])) \geq c_1$ removed and with C allowed to be defined over $\mathbb{C}$.

At this stage it is worth outlining the main steps of the proof of [Gao20a, Theorem 1.2'], or the more general [Gao20a, Theorem 1.3], due to its importance to the current paper. The major step is to establish a criterion, *in simple geometric terms*, for an irreducible

subvariety X of the universal abelian variety $\mathfrak{A}_g$ to be degenerate. Roughly speaking, the proof of the desired criterion is divided into two steps. Step 1 transfers the degeneracy property to an *unlikely intersection problem* in $\mathfrak{A}_g$ by invoking the *mixed Ax–Schanuel theorem* for $\mathfrak{A}_g$ [Gao20b, Theorem 1.1]. More precisely we show that X is degenerate if and only if X is the union of subvarieties satisfying an appropriate unlikely intersection property. Step 2 solves this unlikely intersection problem, and the key point is to use [Gao20b, Theorem 1.4] to prove that the union mentioned above is a finite union. In this step the notion of *weakly optimal subvarieties* introduced by the third-named author and Pila [HP16] is involved.

1.3. **General notation.** We collect here an overview of notation used throughout the text.

Let S be an irreducible, quasi-projective variety defined over an algebraically closed field k. Then S^{sm} denotes the regular locus of X. If $\pi\colon \mathcal{A} \to S$ is an abelian scheme then $[N]\colon \mathcal{A} \to \mathcal{A}$ is the multiplication-by-N morphism for all $N \in \mathbb{N} = \{1, 2, 3, \ldots\}$, and if $s \in S(k)$, the fiber $\mathcal{A}_s = \pi^{-1}(s)$ is an abelian variety defined over k. If $k \subseteq \mathbb{C}$, then S^{an} denotes the analytification of S; it carries a natural topology that is Hausdorff.

We write $\mathbb{T}$ for the circle group $\{z \in \mathbb{C} : |z| = 1\}$.

2. Betti map and Betti form

The goals of this section are to revisit the Betti map, the Betti form and make a link between them. In this paper we construct the Betti map using the universal family of principally polarized abelian varieties with level-ℓ-structure and bypass the ad-hoc construction found in [GH19].

In this section we will make the following assumptions. All varieties are defined over the field $\mathbb{C}$. Let S be an irreducible, regular, quasi-projective variety over $\mathbb{C}$. Let $\pi\colon \mathcal{A} \to S$ be an abelian scheme of relative dimension g, that carries a principal

polarization, and such that $\mathcal{A}$ is equipped with level-ℓ-structure, for some $\ell \geq 3$, *i.e.*, (Hyp) is satisfied.

Proposition 2.1. *Let $s_0 \in S(\mathbb{C})$. Then there exist an open neighborhood Δ of s_0 in S^{an}, and a map $b_\Delta \colon \mathcal{A}_\Delta := \pi^{-1}(\Delta) \to \mathbb{T}^{2g}$, called the Betti map, with the following properties.*

(i) *For each $s \in \Delta$ the restriction $b_\Delta|_{\mathcal{A}_s(\mathbb{C})} \colon \mathcal{A}_s(\mathbb{C}) \to \mathbb{T}^{2g}$ is a group isomorphism.*
(ii) *For each $\xi \in \mathbb{T}^{2g}$ the preimage $b_\Delta^{-1}(\xi)$ is a complex analytic subset of $\mathcal{A}_\Delta$.*
(iii) *The product $(b_\Delta, \pi) \colon \mathcal{A}_\Delta \to \mathbb{T}^{2g} \times \Delta$ is a real analytic isomorphism.*

The properties (i) – (iii) do not uniquely determine b_Δ. Indeed, composing b_Δ with an automorphism of the topological group $\mathbb{T}^{2g}$, *i.e.*, an element of $\mathrm{GL}_{2g}(\mathbb{Z})$, leads to a new Betti map satisfying (i) – (iii). After shrinking Δ we may assume that it is connected. In this case, an application of the Baire Category Theorem shows that b_Δ is uniquely determined by (i) and (iii) up to composition with a unique element of $\mathrm{GL}_{2g}(\mathbb{Z})$.

André, Corvaja, and Zannier [ACZ20] recently began the study of the maximal rank of the Betti map, especially the submersivity, using a slightly different definition. A full study of this maximal rank was realized in [Gao20a]. Closely related to the Betti map is the Betti form, a semi-positive $(1,1)$-form on $\mathcal{A}^{\mathrm{an}}$, which was first introduced in Mok [Mok91].

Proposition 2.2. *There exists a closed $(1,1)$-form ω on $\mathcal{A}^{\mathrm{an}}$, called the Betti form, such that the following properties hold.*

(i) *The $(1,1)$-form ω is semi-positive, i.e., at each point the associated Hermitian form is positive semi-definite.*
(ii) *For all $N \in \mathbb{Z}$ we have $[N]^* \omega = N^2 \omega$.*
(iii) *If X is an irreducible subvariety of $\mathcal{A}$ of dimension d and $\Delta \subseteq S^{\mathrm{an}}$ is open with $X^{\mathrm{sm,an}} \cap \mathcal{A}_\Delta \neq \emptyset$, then*

$$\omega|_{X^{\mathrm{sm,an}}}^{\wedge d} \not\equiv 0 \quad \text{if and only if} \quad \max_{x \in X^{\mathrm{sm,an}} \cap \mathcal{A}_\Delta} \mathrm{rank}_{\mathbb{R}}(\mathrm{d}b_\Delta|_{X^{\mathrm{sm,an}}})_x = 2d.$$

We will prove both propositions during the course of this section using the universal abelian variety. A dynamical approach can be found in [CGHX21, §2].

2.1. Betti map for the universal abelian variety.

Our proof of Proposition 2.1 follows the construction in [Gao20a, §3-§4]. We divide it into several steps.

We start to prove Proposition 2.1 for $S = \mathbb{A}_g$, the moduli space of principally polarized abelian variety of dimension g with level-ℓ-structure; it is a fine moduli space. Let $\pi^{\mathrm{univ}} \colon \mathfrak{A}_g \to \mathbb{A}_g$ be the universal abelian variety.

The universal covering $\mathfrak{H}_g \to \mathbb{A}_g^{\mathrm{an}}$, where $\mathfrak{H}_g$ is the Siegel upper half space, gives a polarized family of abelian varieties $\mathcal{A}_{\mathfrak{H}_g} \to \mathfrak{H}_g$ fitting into the diagram

$$
\begin{array}{ccc}
\mathcal{A}_{\mathfrak{H}_g} := \mathfrak{A}_g \times_{\mathbb{A}_g^{\mathrm{an}}} \mathfrak{H}_g & \xrightarrow{\ u_B\ } & \mathfrak{A}_g^{\mathrm{an}} \\
\downarrow & & \downarrow{\scriptstyle \pi^{\mathrm{univ}}} \\
\mathfrak{H}_g & \longrightarrow & \mathbb{A}_g^{\mathrm{an}}.
\end{array}
$$

For the universal covering $u \colon \mathbb{C}^g \times \mathfrak{H}_g \to \mathcal{A}_{\mathfrak{H}_g}$ and for each $Z \in \mathfrak{H}_g$, the kernel of $u|_{\mathbb{C}^g \times \{Z\}}$ is $\mathbb{Z}^g + Z\mathbb{Z}^g$. Thus the map $\mathbb{C}^g \times \mathfrak{H}_g \to \mathbb{R}^g \times \mathbb{R}^g \times \mathfrak{H}_g \to \mathbb{R}^{2g}$, where the first map is the

inverse of $(a, b, Z) \mapsto (a + Zb, Z)$ and the second map is the natural projection, descends to a *real* analytic map

$$b^{\mathrm{univ}} \colon \mathcal{A}_{\mathfrak{H}_g} \to \mathbb{T}^{2g}.$$

Now for each $s_0 \in \mathbb{A}_g(\mathbb{C})$, there exists a contractible, relatively compact, open neighborhood Δ of s_0 in $\mathbb{A}_g^{\mathrm{an}}$ such that $\mathfrak{A}_{g,\Delta} := (\pi^{\mathrm{univ}})^{-1}(\Delta)$ can be identified with $\mathcal{A}_{\mathfrak{H}_g,\Delta'}$ for some open subset Δ' of $\mathfrak{H}_g$. The composite $b_\Delta \colon \mathfrak{A}_{g,\Delta} \cong \mathcal{A}_{\mathfrak{H}_g,\Delta'} \to \mathbb{T}^{2g}$ is real analytic and satisfies the three properties listed in Proposition 2.1. Thus b_Δ is the desired Betti map in this case. Note that for a fixed (small enough) Δ, there are infinitely choices of Δ'; but for Δ small enough, if Δ'_1 and Δ'_2 are two such choices, then $\Delta'_2 = \alpha \cdot \Delta'_1$ for some $\alpha \in \mathrm{Sp}_{2g}(\mathbb{Z}) \subseteq \mathrm{SL}_{2g}(\mathbb{Z})$. Thus we have proved Proposition 2.1 for $\mathfrak{A}_g \to \mathbb{A}_g$.

2.2. Betti form for the universal abelian variety.

For the universal covering $\mathbf{u} = u_B \circ u \colon \mathbb{C}^g \times \mathfrak{H}_g \to \mathfrak{A}_g^{\mathrm{an}}$, we will use (w, Z) to denote the coordinates on $\mathbb{C}^g \times \mathfrak{H}_g$. Below Im denotes imaginary part.

Lemma 2.3. *Define*

$$\hat{\omega}^{\mathrm{univ}} := \sqrt{-1}\partial\overline{\partial}\left(2(\mathrm{Im}w)^{\mathsf{T}}(\mathrm{Im}Z)^{-1}(\mathrm{Im}w)\right).$$

Then $\hat{\omega}^{\mathrm{univ}}$ is a closed semi-positive $(1,1)$-form on $\mathbb{C}^g \times \mathfrak{H}_g$ satisfying

$$(2.1) \qquad \hat{\omega}^{\mathrm{univ}} = \sqrt{-1}\left(\mathrm{d}ZY^{-1}\mathrm{Im}(w) - \mathrm{d}w\right)^{\mathsf{T}} \wedge Y^{-1}\left(\mathrm{d}\overline{Z}Y^{-1}\mathrm{Im}(w) - \mathrm{d}\overline{w}\right)$$

with $Y = \mathrm{Im}(Z)$; here and below the symbol $\wedge$ is used as a combination of wedge product and matrix multiplication when appropriate. Moreover, if $N \in \mathbb{Z}$ and if we denote by $\widetilde{N} \colon \mathbb{C}^g \times \mathfrak{H}_g \to \mathbb{C}^g \times \mathfrak{H}_g$ the map $(w, Z) \mapsto (Nw, Z)$, then $\widetilde{N}^*\hat{\omega}^{\mathrm{univ}} = N^2\hat{\omega}^{\mathrm{univ}}$.

Proof. The $(1,1)$-form $\hat{\omega}^{\mathrm{univ}}$ is closed since $\mathrm{d} = \partial + \overline{\partial}$. We will prove the semi-positivity by direct computation.

We have the following formulae for partial derivatives

$$\overline{\partial}\mathrm{Im}w = \frac{\sqrt{-1}}{2}\mathrm{d}\overline{w}, \qquad \overline{\partial}(Y^{-1}) = -\frac{\sqrt{-1}}{2}Y^{-1}\mathrm{d}\overline{Z}Y^{-1},$$

$$\partial\mathrm{Im}w = -\frac{\sqrt{-1}}{2}\mathrm{d}w, \qquad \partial(Y^{-1}) = \frac{\sqrt{-1}}{2}Y^{-1}\mathrm{d}ZY^{-1}.$$

Let us prove the formulae on the right. We hereby do it for $\partial(Y^{-1}) = \frac{\sqrt{-1}}{2}Y^{-1}\mathrm{d}ZY^{-1}$ and the other one is similar. Taking partial derivatives on both sides of $YY^{-1} = I$, we get $(\partial Y)Y^{-1} + Y\partial(Y^{-1}) = 0$. So $\partial(Y^{-1}) = -Y^{-1}(\partial Y)Y^{-1}$. But $\partial Y = \partial\mathrm{Im}Z = -\frac{\sqrt{-1}}{2}\mathrm{d}Z$. Hence we get the desired formula for $\partial(Y^{-1})$.

Using these formulae and the Leibniz rule (note that $Z = Z^{\mathsf{T}}$ and hence $\mathrm{d}Z = \mathrm{d}Z^{\mathsf{T}}$), we get

$$\hat{\omega}^{\mathrm{univ}} = \sqrt{-1}\left((\mathrm{d}w)^{\mathsf{T}}Y^{-1} \wedge \mathrm{d}\overline{w} + (\mathrm{Im}w)^{\mathsf{T}}Y^{-1}\mathrm{d}Z \wedge Y^{-1}\mathrm{d}\overline{Z}Y^{-1}(\mathrm{Im}w)\right.$$
$$\left. - (\mathrm{Im}w)^{\mathsf{T}}Y^{-1}\mathrm{d}ZY^{-1} \wedge \mathrm{d}\overline{w} - (\mathrm{d}w)^{\mathsf{T}} \wedge Y^{-1}\mathrm{d}\overline{Z}Y^{-1}(\mathrm{Im}w)\right).$$

Rearranging yields the desired equality (2.1). The associated form is

$$H \colon \left((\xi_w, \xi_Z), (\eta_w, \eta_Z)\right) \mapsto \left(\xi_Z Y^{-1}\mathrm{Im}(w) - \xi_w\right)^{\mathsf{T}} Y^{-1} \left(\overline{\eta_Z}Y^{-1}\mathrm{Im}(w) - \overline{\eta_w}\right),$$

for $\xi_w, \eta_w \in \mathbb{C}^g$ and $\xi_Z, \eta_Z \in \mathrm{Mat}_g(\mathbb{C})$ symmetric, is Hermitian and so $\hat{\omega}^{\mathrm{univ}}$ is real. Moreover,

$$H\big((\xi_w, \xi_Z), (\xi_w, \xi_Z)\big) = v^{\mathsf{T}} Y^{-1} \overline{v} \quad \text{with} \quad v = \xi_Z Y^{-1} \mathrm{Im}(w) - \xi_w.$$

But Y is positive definite as a real symmetric matrix and thus positive definite as a Hermitian matrix. We see that H is positive semi-definite and this implies that $\hat{\omega}^{\mathrm{univ}}$ is positive semi-definite.

The "moreover" part of the lemma is clear. $\qquad\square$

Next we want to show that $\hat{\omega}^{\mathrm{univ}}$ descends to a $(1,1)$-form on $\mathfrak{A}_g^{\mathrm{an}}$. To do this, we first show that $\hat{\omega}^{\mathrm{univ}}$ can be written in a simple form under an appropriate change of coordinates.

Define the complex space $\mathcal{X}_{2g,\mathrm{a}}$, which is the universal covering of $\mathfrak{A}_g^{\mathrm{an}}$, as follows:

- As a real algebraic space, $\mathcal{X}_{2g,\mathrm{a}} := \mathbb{R}^{2g} \times \mathfrak{H}_g$.
- The complex structure on $\mathcal{X}_{2g,\mathrm{a}}$ is given by

$$(2.2) \qquad \mathbb{R}^{2g} \times \mathfrak{H}_g = \mathbb{R}^g \times \mathbb{R}^g \times \mathfrak{H}_g \cong \mathbb{C}^g \times \mathfrak{H}_g, \qquad (a, b, Z) \mapsto (a + Zb, Z).$$

Lemma 2.4. *Let $\hat{\omega}^{\mathrm{univ}}$ be as in Lemma 2.3. Then under the change of coordinates (2.2), we have $\hat{\omega}^{\mathrm{univ}} = 2(\mathrm{d}a)^{\mathsf{T}} \wedge \mathrm{d}b$.*

Proof. For the moment we write $Z = X + \sqrt{-1}Y$ with X and Y the real and imaginary part of $Z \in \mathfrak{H}_g$, respectively. Note that $w = a + Zb = (a + Xb) + \sqrt{-1}Yb$. Hence $Y^{-1}(\mathrm{Im}\,w) = b$ and $\mathrm{d}w = \mathrm{d}a + Z\mathrm{d}b + \mathrm{d}Zb$. Using this and noting that Z is symmetric, we have that (2.1) becomes

$$\hat{\omega}^{\mathrm{univ}} = \sqrt{-1}\left(\sqrt{-1}(\mathrm{d}b)^{\mathsf{T}}Y + (\mathrm{d}b)^{\mathsf{T}}X + (\mathrm{d}a)^{\mathsf{T}}\right) \wedge Y^{-1}\left(\mathrm{d}a + X\mathrm{d}b - \sqrt{-1}Y\mathrm{d}b\right)$$

$$= \sqrt{-1}\big(\sqrt{-1}(\mathrm{d}b)^{\mathsf{T}} \wedge \mathrm{d}a + (\mathrm{d}b)^{\mathsf{T}} \wedge Y\mathrm{d}b + (\mathrm{d}b)^{\mathsf{T}}X \wedge Y^{-1}\mathrm{d}a + (\mathrm{d}b)^{\mathsf{T}}X \wedge Y^{-1}X\mathrm{d}b +$$
$$(\mathrm{d}a)^{\mathsf{T}} \wedge Y^{-1}\mathrm{d}a + (\mathrm{d}a)^{\mathsf{T}} \wedge Y^{-1}X\mathrm{d}b - \sqrt{-1}(\mathrm{d}a)^{\mathsf{T}} \wedge \mathrm{d}b\big).$$

Many terms will vanish. Indeed, if M is a matrix, then $(\mathrm{d}b)^{\mathsf{T}} \wedge M\mathrm{d}a = -(\mathrm{d}a)^{\mathsf{T}} \wedge M^{\mathsf{T}}\mathrm{d}b$. As $(XY^{-1})^{\mathsf{T}} = Y^{-1}X$ and as $(\mathrm{d}b)^{\mathsf{T}}X \wedge Y^{-1}\mathrm{d}a = (\mathrm{d}b)^{\mathsf{T}} \wedge XY^{-1}\mathrm{d}a$ we find $(\mathrm{d}b)^{\mathsf{T}}X \wedge Y^{-1}\mathrm{d}a + (\mathrm{d}a)^{\mathsf{T}} \wedge Y^{-1}X\mathrm{d}b = 0$. Observe that Y is symmetric, and so $(\mathrm{d}b)^{\mathsf{T}} \wedge Y\mathrm{d}b = -(\mathrm{d}b)^{\mathsf{T}} \wedge Y\mathrm{d}b$ vanishes. Arguing along the same line and using that Y^{-1} and $XY^{-1}X$ are symmetric we find $(\mathrm{d}a)^{\mathsf{T}} \wedge Y^{-1}\mathrm{d}a = 0$ and $(\mathrm{d}b)^{\mathsf{T}}X \wedge Y^{-1}X = (\mathrm{d}b)^{\mathsf{T}} \wedge XY^{-1}X\mathrm{d}b = 0$. We are left with $\hat{\omega}^{\mathrm{univ}} = 2(\mathrm{d}a)^{\mathsf{T}} \wedge \mathrm{d}b$. $\qquad\square$

Corollary 2.5. *Let $\hat{C}$ be an irreducible, 1-dimensional, complex analytic subset of an open subset of $\mathcal{X}_{2g,\mathrm{a}} = \mathbb{R}^{2g} \times \mathfrak{H}_g$ and $\hat{C}^{\mathrm{sm}}$ its smooth locus. Then $\hat{\omega}^{\mathrm{univ}}$ restricted to $\hat{C}^{\mathrm{sm}}$ is trivial if and only if $\hat{C} \subseteq \{r\} \times \mathfrak{H}_g$ for some $r \in \mathbb{R}^{2g}$.*

Proof. First, assume that the coordinates (a, b) of $\mathbb{R}^{2g}$ are constant on $\hat{C}$. Then $\hat{\omega}^{\mathrm{univ}}$, which is simply $2(\mathrm{d}a)^{\mathsf{T}} \wedge \mathrm{d}b$ by Lemma 2.4, vanishes on $\hat{C}^{\mathrm{sm}}$.

Conversely, suppose that $\hat{\omega}^{\mathrm{univ}}$ vanishes identically on $\hat{C}^{\mathrm{sm}}$. This time we use (2.1) from Lemma 2.3. As Y^{-1} is positive definite we find $\mathrm{d}ZY^{-1}\mathrm{Im}(w) = \mathrm{d}w$ on $\hat{C}^{\mathrm{sm}}$. Using the change of coordinates $w = a + Zb$ we deduce $\mathrm{Im}(w) = Yb$ and $\mathrm{d}w = \mathrm{d}a + \mathrm{d}Zb + Z\mathrm{d}b$. So $\mathrm{d}Zb = \mathrm{d}ZY^{-1}\mathrm{Im}(w) = \mathrm{d}w = \mathrm{d}a + \mathrm{d}Zb + Z\mathrm{d}b$ on $\hat{C}^{\mathrm{sm}}$. This equality simplifies to

$\mathrm{d}a + Z\mathrm{d}b = 0$ on $\hat{C}^{\mathrm{sm}}$. As a and b are real valued and as $Z \in \mathfrak{H}_g$ we conclude $\mathrm{d}a = \mathrm{d}b = 0$ on $\hat{C}^{\mathrm{sm}}$. So a and b are constant on $\hat{C}$. $\qquad\square$

Lemma 2.6. *Let $\hat{\omega}^{\mathrm{univ}}$ be as in Lemma 2.3. Then $\hat{\omega}^{\mathrm{univ}}$ descends to a semi-positive $(1,1)$-form ω^{univ} on $\mathfrak{A}_g$. Moreover, for $N \in \mathbb{Z}$ we have $[N]^* \omega^{\mathrm{univ}} = N^2 \omega^{\mathrm{univ}}$.*

Proof. Let Sp_{2g} be the symplectic group defined over $\mathbb{Q}$, and let V_{2g} be the vector group over $\mathbb{Q}$ of dimension $2g$. Then the natural action of Sp_{2g} on V_{2g} defines a group $P_{2g,\mathrm{a}} := V_{2g} \rtimes \mathrm{Sp}_{2g}$.

We use the classical action of $\mathrm{Sp}_{2g}(\mathbb{R})$ on $\mathfrak{H}_g$, it is transitive. The real coordinate on $\mathcal{X}_{2g,\mathrm{a}}$ on the left hand side of (2.2) has the following advantage. The group $P_{2g,\mathrm{a}}(\mathbb{R})$ acts transitively on $\mathcal{X}_{2g,\mathrm{a}}$ by the formula

$$(v, h) \cdot (v', Z) := (v + hv', hZ)$$

for $(v, h) \in P_{2g,\mathrm{a}}(\mathbb{R})$ and $(v', Z) \in \mathbb{R}^{2g} \times \mathfrak{H}_g = \mathcal{X}_{2g,\mathrm{a}}$. The space $\mathfrak{A}_g^{\mathrm{an}}$ is then obtained as the quotient of $\mathcal{X}_{2g,\mathrm{a}}$ by a congruence subgroup of $P_{2g,\mathrm{a}}(\mathbb{Q})$. We refer to [Pin89, 10.5–10.9] or [Pin05a, Construction 2.9 and Example 2.12] for these facts.

It is clear that both $V_{2g}(\mathbb{R})$ and $\mathrm{Sp}_{2g}(\mathbb{R})$ preserve $2(\mathrm{d}a)^{\mathsf{T}} \wedge \mathrm{d}b$. Thus this 2-form is invariant under the action of $P_{2g,\mathrm{a}}(\mathbb{R})$ on $\mathcal{X}_{2g,\mathrm{a}}$.

So by Lemma 2.4, the previous two paragraphs imply that $\hat{\omega}^{\mathrm{univ}}$ descends to a $(1,1)$-form ω^{univ} on $\mathfrak{A}_g$. The semi-positivity of ω^{univ} follows from Lemma 2.3.

The property $[N]^* \omega^{\mathrm{univ}} = N^2 \omega^{\mathrm{univ}}$ follows from the "moreover" part of Lemma 2.3 and the following commutative diagram

$$
\begin{array}{ccc}
\mathbb{C}^g \times \mathfrak{H}_g & \xrightarrow{\widetilde{N}} & \mathbb{C}^g \times \mathfrak{H}_g \\
\downarrow & & \downarrow \\
\mathfrak{A}_g^{\mathrm{an}} & \xrightarrow{[N]} & \mathfrak{A}_g^{\mathrm{an}}.
\end{array}
$$
$\qquad\square$

This semi-positive $(1,1)$-form ω^{univ} will be the Betti form for $\mathfrak{A}_g \to \mathbb{A}_g$, as desired in Proposition 2.2. To show this, it suffices to establish property (iii) of Proposition 2.2. Hence it suffices to prove the following proposition.

Proposition 2.7. *Assume $\mathcal{A} \to S$ is $\mathfrak{A}_g \to \mathbb{A}_g$. Let X be an irreducible subvariety of $\mathfrak{A}_g$ of dimension d and let Δ be an open subset of S^{an} with $X^{\mathrm{sm,an}} \cap \mathcal{A}_\Delta \neq \emptyset$. Then*

$$(2.3) \qquad \omega^{\mathrm{univ}}|_{X^{\mathrm{sm,an}}}^{\wedge d} \not\equiv 0 \quad \text{if and only if} \quad \max_{x \in X^{\mathrm{sm,an}} \cap \mathcal{A}_\Delta} \mathrm{rank}_{\mathbb{R}}(\mathrm{d}b_\Delta|_{X^{\mathrm{sm,an}}})_x = 2d.$$

Proof. We begin by reformulating Corollary 2.5. If C is an irreducible, 1-dimensional, complex analytic subset of an open subset of $\mathcal{A}_\Delta$, then

$$(2.4) \qquad \omega^{\mathrm{univ}}|_{C^{\mathrm{sm}}} = 0 \quad \text{if and only if} \quad b_\Delta(C) \text{ is a point};$$

indeed, this claim is local and it follows using the universal covering $\mathbf{u} \colon \mathbb{C}^g \times \mathfrak{H}_g \to \mathfrak{A}_g$.

We assume first that the right side of (2.3) is false, *i.e.*, the maximal rank is strictly less than $2d = 2\dim X$. So every $x \in X^{\mathrm{sm,an}} \cap \mathcal{A}_\Delta$ is a non-isolated point of $b_\Delta^{-1}(r) \cap X^{\mathrm{sm,an}}$ where $r = b_\Delta(x)$. Because $b_\Delta^{-1}(r)$ is a complex analytic subset of $\mathcal{A}_\Delta$ (by Proposition 2.1.(ii) for $\mathfrak{A}_g \to \mathbb{A}_g$) and $X^{\mathrm{sm,an}}$ is complex analytic in a neighborhood of x in $\mathcal{A}^{\mathrm{an}}$, there exists an irreducible complex analytic curve C in $b_\Delta^{-1}(r) \cap X^{\mathrm{sm,an}}$ passing through x. In particular, $b_\Delta(C)$ is a point and so $\omega^{\mathrm{univ}}|_{C^{\mathrm{sm}}} \equiv 0$ by (2.4).

The upshot of the previous paragraph is that the Hermitian form attached to the semi-positive $(1,1)$-form $\omega^{\mathrm{univ}}|_{X^{\mathrm{sm,an}}}$ vanishes along the tangent space of C^{sm}; it is degenerate. We can complete a tangent vector of C^{sm} to a basis of the tangent space of $X^{\mathrm{sm,an}}$. Considering holomorphic local coordinates we find $\omega^{\mathrm{univ}}|_{X^{\mathrm{sm,an}}}^{\wedge d} = 0$ at every point of C^{sm}. By continuity it also vanishes at $x \in C$. Since $x \in X^{\mathrm{sm,an}} \cap \mathcal{A}_\Delta$ was arbitrary, we conclude $\omega^{\mathrm{univ}}|_{X^{\mathrm{sm,an}}}^{\wedge d} \equiv 0$.

For the converse we assume $\omega^{\mathrm{univ}}|_{X^{\mathrm{sm,an}}}^{\wedge d} \equiv 0$. So the Hermitian form attached to this semi-positive $(1,1)$-form is degenerate. Thus for each $x \in X^{\mathrm{sm,an}}$, using holomorphic local coordinates we find an irreducible, 1-dimensional, complex analytic subset C_x which passes through x and is contained in $X^{\mathrm{sm,an}}$ such that $\omega^{\mathrm{univ}}|_{X^{\mathrm{sm,an}}}$ vanishes along the tangent space of C_x^{sm}. So $\omega^{\mathrm{univ}}|_{C_x^{\mathrm{sm}}} \equiv 0$, and hence $b_\Delta(C_x)$ is a point by (2.4). Letting the point x run over $X^{\mathrm{sm,an}}$, we conclude that the rank on the right side of (2.3) is strictly less than $2d$. $\qquad\square$

2.3. General case.

We now prove Propositions 2.1 and 2.2 for $\pi\colon \mathcal{A} \to S$ as near the beginning of this section. In particular, we assume (Hyp). With the construction in §2.1, the rest of the proof of Proposition 2.1 follows the construction in [Gao20a, §4].

As $\mathbb{A}_g$ is a fine moduli space there exists a Cartesian diagram

$$
\begin{array}{ccc}
\mathcal{A} & \overset{\iota}{\longrightarrow} & \mathfrak{A}_g \\
{\scriptstyle \pi}\downarrow & \lrcorner & \downarrow \\
S & \overset{\iota_S}{\longrightarrow} & \mathbb{A}_g.
\end{array}
$$

Now let $s_0 \in S(\mathbb{C})$. Applying Proposition 2.1 to the universal abelian variety $\mathfrak{A}_g \to \mathbb{A}_g$ and $\iota_S(s_0) \in \mathbb{A}_g(\mathbb{C})$, we obtain an open neighborhood Δ_0 of $\iota_S(s_0)$ in $\mathbb{A}_g^{\mathrm{an}}$ and a map

$$
b_{\Delta_0}\colon \mathfrak{A}_g|_{\Delta_0} \to \mathbb{T}^{2g}
$$

satisfying the properties listed in Proposition 2.1.

Now let $\Delta = \iota_S^{-1}(\Delta_0)$. Then Δ is an open neighborhood of s in S^{an}. Denote by $\mathcal{A}_\Delta = \pi^{-1}(\Delta)$ and define

$$
b_\Delta = b_{\Delta_0} \circ \iota\colon \mathcal{A}_\Delta \to \mathbb{T}^{2g}.
$$

Then b_Δ satisfies the properties listed in Proposition 2.1 for $\mathcal{A} \to S$. Hence b_Δ is our desired Betti map.

Next let us turn to the Betti form. Let ω^{univ} be the semi-positive $(1,1)$-form on $\mathfrak{A}_g$ as in Lemma 2.6. Define $\omega := \iota^* \omega^{\mathrm{univ}}$. We will show that ω satisfies the properties listed in Proposition 2.2.

The $(1,1)$-form ω is semi-positive as it is the pull-back of the semi-positive form ω^{univ}. Moreover, it satisfies $[N]^*\omega = N^2\omega$ since ω^{univ} has this property. Hence we have established properties (i) and (ii) of Proposition 2.2.

Let us verify (iii) of Proposition 2.2. Suppose X is an irreducible subvariety of $\mathcal{A}$ of dimension d. Let Δ be an open subset of S^{an} with $X^{\mathrm{sm,an}} \cap \mathcal{A}_\Delta \neq \emptyset$; we may shrink Δ subject to this condition. Let $Z = \overline{\iota(X)}$ and observe $\dim Z \leq d$.

Since $\omega = \iota^* \omega^{\mathrm{univ}}$, we have

$$
(2.5) \qquad \omega|_{X^{\mathrm{sm,an}}}^{\wedge d} \not\equiv 0 \quad \text{if and only if} \quad \omega^{\mathrm{univ}}|_{Z^{\mathrm{sm,an}}}^{\wedge d} \not\equiv 0.
$$

Next by definition of b_Δ, we have the following property: For suitable non-empty open subsets Δ of S^{an} and Δ_0 of $\mathbb{A}_g^{\mathrm{an}}$ such that $\iota_S(\Delta) \subseteq \Delta_0$, we have
$$(2.6)$$
$$\max_{x \in X^{\mathrm{sm,an}} \cap \mathcal{A}_\Delta} \mathrm{rank}_\mathbb{R}(\mathrm{d}b_\Delta|_{X^{\mathrm{sm,an}}})_x \leq \max_{x \in Z^{\mathrm{sm,an}} \cap \mathfrak{A}_{g,\Delta_0}} \mathrm{rank}_\mathbb{R}(\mathrm{d}b_{\Delta_0}|_{Z^{\mathrm{sm,an}}})_x \leq 2\dim Z \leq 2d.$$

Suppose first that $\omega|_{X^{\mathrm{sm,an}}}^{\wedge d} \not\equiv 0$, then (2.5) implies $\omega^{\mathrm{univ}}|_{Z^{\mathrm{sm,an}}}^{\wedge d} \not\equiv 0$ and in particular $d = \dim Z$. We can apply Proposition 2.2(iii) to Z and obtain $\max_{x \in Z^{\mathrm{sm,an}} \cap \mathfrak{A}_{g,\Delta_0}} \mathrm{rank}_\mathbb{R}(\mathrm{d}b_{\Delta_0}|_{Z^{\mathrm{sm,an}}})_x = 2d$. Now $\iota|_X \colon X \to Z$ is generically finite as $\dim X = \dim Z$, so the first inequality in (2.6) is an equality. We conclude
$$(2.7) \qquad \max_{x \in X^{\mathrm{sm,an}} \cap \mathcal{A}_\Delta} \mathrm{rank}_\mathbb{R}(\mathrm{d}b_\Delta|_{X^{\mathrm{sm,an}}})_x = 2d.$$

Conversely, assume (2.7) holds true. Then we have equalities throughout in (2.6). By Proposition 2.2(iii) applied to Z and by (2.5) we get $\omega|_{X^{\mathrm{sm,an}}}^{\wedge d} \not\equiv 0$.

3. Setup and notation for the height inequality

In the next few sections we will prove Theorem 1.6. Let us first fix the setting.

All varieties are over an algebraically closed subfield k of $\mathbb{C}$. The ambient data is given as above Theorem 1.6. We repeat it here.

- Let S be a regular, irreducible, quasi-projective variety over k that is Zariski open in an irreducible projective variety $\overline{S} \subseteq \mathbb{P}_k^m$.
- Let $\pi \colon \mathcal{A} \to S$ be an abelian scheme presented by a closed immersion $\mathcal{A} \to \mathbb{P}_k^n \times S$ over S.
- From the previous point, we get a closed immersion of the generic fiber A of $\mathcal{A} \to S$ into $\mathbb{P}_{k(S)}^n$. We assume that $A \to \mathbb{P}_{k(S)}^n$ arises from a basis of the global sections of the l-th power L of a symmetric ample line bundle with $l \geq 4$.
- Finally, we assume (Hyp) as on page 5.

From the third bullet power, we see that the image of A is projectively normal in $\mathbb{P}_{k(S)}^n$, cf. [Mum70, Theorem 9]. By the fourth bullet point, Proposition 2.2 provides the Betti form ω on $\mathcal{A}^{\mathrm{an}}$.

For $s \in S(k)$ we write $\mathcal{A}_s$ for the abelian variety $\pi^{-1}(s)$.

Remark 3.1. *Let S be as in the first bullet point. Let $\pi \colon \mathcal{A} \to S$ be an abelian scheme. Suppose L_0 is a symmetric and ample line bundle on A, the generic fiber of π. An immersion of $\mathcal{A}$ as in the second bullet point can be obtained as follows. By [Ray70, Théorème XI 1.13] there exists an S-ample line bundle $\mathcal{L}$ on $\mathcal{A}$ whose restriction to the generic fiber of $\mathcal{A} \to S$ is isomorphic to $L_0^{\otimes l}$ for some integer $l \geq 4$. We may assume in addition that $\mathcal{L}$ satisfies $[-1]^* \mathcal{L} \cong \mathcal{L}$ and even that $\mathcal{L}$ becomes trivial when pulled back under the zero section $S \to \mathcal{A}$, see [Ray70, Remarque XI 1.3a]. After replacing $\mathcal{L}$ by a sufficiently high power, we may assume that $\mathcal{L}$ is very ample over S. We fix a basis of global sections of $L_0^{\otimes l}$ and, as $l \geq 4$, thereby realize the generic fiber of π as a projectively normal subvariety of $\mathbb{P}_{k(S)}^n$. Now we can take L in the third bullet point to be $L_0^{\otimes l}$, which is the restriction of $\mathcal{L}$ to A. A closed immersion $\mathcal{A} \to \mathbb{P}_k^n \times S$ as in the second bullet point arises from $\mathcal{L} \otimes \pi^* \mathcal{M}$ for some very ample line bundle $\mathcal{M}$ on S; see [Gro61, Proposition 4.4.10.(ii) and Proposition 4.1.4]. On restricting to a fiber of $\mathcal{A} \to S$ the induced closed immersion $\mathcal{A}_s \to \mathbb{P}_k^n$ comes from the restriction $\mathcal{L}|_{\mathcal{A}_s}$.*

Write $\overline{\mathcal{A}}$ for the Zariski closure of $\mathcal{A}$ in $\mathbb{P}^n_k \times \overline{S}$. Then $\overline{\mathcal{A}}$ is irreducible but not necessarily regular. On any product of r projective spaces and if $a_1, \ldots, a_r \in \mathbb{Z}$, we let $\mathcal{O}(a_1, \ldots, a_r)$ denote the tensor product over all $i \in \{1, \ldots, r\}$ of the pull-back under the i-th projection of $\mathcal{O}(a_i)$. We write $\mathcal{L}$ for the restriction of $\mathcal{O}(1,1)$ to $\overline{\mathcal{A}}$.

3.1. **Height functions on $\mathcal{A}$.** If $k = \overline{\mathbb{Q}}$ we have several height functions on $\mathcal{A}(\overline{\mathbb{Q}})$.

For any $n \in \mathbb{N}$, we always consider the absolute logarithmic Weil height function $\mathbb{P}^n_{\overline{\mathbb{Q}}}(\overline{\mathbb{Q}}) \to \mathbb{R}$, or just Weil height, defined as in [BG06, §1.5.1].

Now say $P \in \mathcal{A}(\overline{\mathbb{Q}})$, we write $P = (P', \pi(P))$ with $P' \in \mathbb{P}^n_{\overline{\mathbb{Q}}}(\overline{\mathbb{Q}})$ and $\pi(P) \in \mathbb{P}^m_{\overline{\mathbb{Q}}}(\overline{\mathbb{Q}})$. The sum of Weil heights

$$(3.1) \qquad\qquad h(P) = h(P') + h(\pi(P))$$

defines our first height $\mathcal{A}(\overline{\mathbb{Q}}) \to [0, \infty)$ which we call the *naive height* on $\mathcal{A}$. It depends on the fixed immersion of $\mathcal{A}$.

The line bundle $[-1]^*\mathcal{L}|_{\mathcal{A}} \otimes \mathcal{L}|_{\mathcal{A}}^{\otimes -1}$ of $\mathcal{A}$ restricted to the generic fiber A of $\mathcal{A} \to S$ equals $[-1]^*L \otimes L^{\otimes -1}$. By the third bullet point above this line bundle is trivial. So it equals $\pi^*\mathcal{K}$ for some line bundle $\mathcal{K}$ of S by [Gro67, Corollaire 21.4.13 (pp. 361 of EGA IV-4, in Errata et Addenda, liste 3)]. We conclude $[-1]^*\mathcal{L}|_{\mathcal{A}_s} \cong \mathcal{L}|_{\mathcal{A}_s}$ for all $s \in S(\overline{\mathbb{Q}})$. So the function (3.1) represents the height function, defined up-to $O_s(1)$, given by the Height Machine, *cf.* [BG06, Theorem 2.3.8], applied to $(\mathcal{A}_s, \mathcal{L}_s)$. As $\mathcal{L}_s$ is symmetric, the *fiberwise Néron–Tate* or *canonical height* $\hat{h}_{\mathcal{A}}: \mathcal{A}(\overline{\mathbb{Q}}) \to [0, \infty)$, defined by the convergent limit

$$(3.2) \qquad\qquad \hat{h}_{\mathcal{A}}(P) = \lim_{N \to \infty} \frac{h([N](P))}{N^2},$$

is a quadratic form on $\mathcal{A}_s(\overline{\mathbb{Q}})$. In the notation [BG06, Chapter 9] the height (3.2) is $\hat{h}_{\mathcal{A}_s, \mathcal{L}_s}$ where $s = \pi(P)$.

Remark 3.2. *We use here the notation of Remark 3.1. So that the immersion $\mathcal{A} \to \mathbb{P}^n_S$ arises via $L_0^{\otimes l}$. To normalize, we divide (3.2) by l and obtain the Néron–Tate height $\hat{h}_{\mathcal{A}, L_0}: \mathcal{A}(\overline{\mathbb{Q}}) \to [0, \infty)$.*

Let us verify that $\hat{h}_{\mathcal{A}, L_0}$ depends only on L_0. Suppose $\mathcal{L}'$ is another line bundle on $\mathcal{A}$ that restricts to $L_0^{\otimes l}$, then $\mathcal{L}' \otimes \mathcal{L}^{\otimes -1}$ is trivial on A. By [Gro67, Corollaire 21.4.13 (pp. 361 of EGA IV-4, in Errata et Addenda, liste 3)], this difference is the pull-back of some line bundle on S under $\mathcal{A} \to S$. So the restriction of $\mathcal{L}' \otimes \mathcal{L}^{\otimes -1}$ to $\mathcal{A}_s$ for each $s \in S(\mathbb{C})$ is trivial. Thus $\mathcal{L}|_{\mathcal{A}_s}$ and $\mathcal{L}'|_{\mathcal{A}_s}$ induce the same Néron–Tate height on $\mathcal{A}_s(\overline{\mathbb{Q}})$, see [BG06, §9.2].

3.2. **Integration against the Betti form.** Let $\mathcal{A}$ and S be as in the beginning of this section, so they are defined over an algebraically closed subfield k of $\mathbb{C}$. Recall that ω is the Betti form on $\mathcal{A}^{\mathrm{an}}$ as provided by Proposition 2.2. In particular, it is a semi-positive $(1,1)$-form on $\mathcal{A}^{\mathrm{an}}$ such that $[N]^*\omega = N^2\omega$ for all $N \in \mathbb{Z}$. We discuss here a modification of the Betti form that has compact support.

Fix X to be an irreducible closed subvariety of $\mathcal{A}$ of dimension d, such that $\pi|_X: X \to S$ is dominant.

We are not allowed to integrate $\omega^{\wedge d}$ over $X^{\mathrm{sm, an}}$ as $\omega^{\wedge d}$ may not have compact support. So we modify ω in the following way.

Suppose we are provided with a base point $s_0 \in S^{\mathrm{an}}$. Let furthermore Δ be a relatively compact, contractible, open neighborhood of s_0 in S^{an}. Denote by $\mathcal{A}_\Delta$ the open subset $\pi^{-1}(\Delta)$ of $\mathcal{A}^{\mathrm{an}}$. Fix a smooth bump function $\vartheta \colon S^{\mathrm{an}} \to [0,1]$ with compact support $K \subseteq \Delta$ such that $\vartheta(s_0) = 1$. Finally, we define $\theta = \vartheta \circ \pi \colon \mathcal{A}^{\mathrm{an}} \to [0,1]$. Then $\theta\omega$ is a semi-positive smooth $(1,1)$-form on $\mathcal{A}^{\mathrm{an}}$; unlike the Betti form, it may not be closed. By construction, the support of $\theta\omega$ lies in $\pi^{-1}(K)$ which is compact as π is proper and K is compact.

Remark 3.3. *Suppose X is non-degenerate, namely X satisfies one of the two equivalent conditions in property (iii) of Proposition 2.2. Then X^{an} contains a smooth point P_0 at which $\omega|_{X^{\mathrm{sm,an}}}^{\wedge d} > 0$. Then we will take $s_0 = \pi(P_0)$.*

3.3. The graph construction.

Let $N \in \mathbb{Z}$. The multiplication-by-N morphism $[N] \colon \mathcal{A} \to \mathcal{A}$ may not extend to a morphism $\overline{\mathcal{A}} \to \overline{\mathcal{A}}$. We overcome this by using the graph construction.

Recall that we have identified $\mathcal{A} \subseteq \overline{\mathcal{A}} \subseteq \mathbb{P}_k^n \times \overline{S} \subseteq \mathbb{P}_k^n \times \mathbb{P}_k^m$.

We write $\rho_1, \rho_2 \colon \mathbb{P}_k^n \times \mathbb{P}_k^n \times \mathbb{P}_k^m \to \mathbb{P}_k^n \times \mathbb{P}_k^m$ for the two projections $\rho_1(P, Q, s) = (P, s)$ and $\rho_2(P, Q, s) = (Q, s)$.

Consider Γ_N the graph of $[N]$, determined by

$$\Gamma_N = \{(P, [N](P)) : P \in \mathcal{A}(k)\}.$$

We consider it as an irreducible closed subvariety of $\mathcal{A} \times_S \mathcal{A}$.

Let X be an irreducible closed subvariety of $\mathcal{A}$ of dimension d. The graph X_N of $[N]$ restricted to X is an irreducible closed subvariety of Γ_N determined by

$$\{(P, [N]P) : P \in X(k)\}.$$

Observe that $\rho_1|_{\Gamma_N} \colon \Gamma_N \to \mathcal{A}$ is an isomorphism; it maps $(P, [N](P))$ to P. So we can use $\rho_1|_{\Gamma_N}^{-1}$ to identify X with X_N.

Moreover, $\rho_2|_{\Gamma_N}$ maps $(P, [N](P))$ to $[N](P)$. Therefore

$$(3.3) \qquad\qquad \rho_2|_{\Gamma_N} \circ \rho_1|_{\Gamma_N}^{-1} = [N].$$

Let $\overline{X_N}$ be the Zariski closure of X_N in $\overline{\mathcal{A}} \times_{\overline{S}} \overline{\mathcal{A}} \subseteq \mathbb{P}_{\overline{S}}^n \times_{\overline{S}} \mathbb{P}_{\overline{S}}^n = \mathbb{P}_k^n \times \mathbb{P}_k^n \times \overline{S} \subseteq \mathbb{P}_k^n \times \mathbb{P}_k^n \times \mathbb{P}_k^m$. Then $\overline{X_N}$ is an irreducible projective variety (which is not necessarily regular) with $\dim \overline{X_N} = \dim X_N = \dim X$.

In the next section, we will use the following line bundles on $\overline{X_N}$. Define

$$(3.4) \qquad\qquad \mathcal{F} = \rho_2^* \mathcal{O}(1,1)|_{\overline{X_N}} = \mathcal{O}(0,1,1)|_{\overline{X_N}}$$

and

$$(3.5) \qquad\qquad \mathcal{M} = \mathcal{O}(0,0,1)|_{\overline{X_N}}.$$

Let us close this subsection by relating the height functions defined by $\mathcal{F}$ and $\mathcal{M}$ with the ones in §3.1. Assume $k = \overline{\mathbb{Q}}$. Let $P \in X(\overline{\mathbb{Q}})$. Write $P = (P', \pi(P))$ with $P' \in \mathbb{P}_{\overline{\mathbb{Q}}}^n(\overline{\mathbb{Q}})$ and $\pi(P) \in \mathbb{P}_{\overline{\mathbb{Q}}}^m(\overline{\mathbb{Q}})$. We have $[N](P) = (P_N', \pi(P))$ for some $P_N' \in \mathbb{P}_{\overline{\mathbb{Q}}}^n(\overline{\mathbb{Q}})$.

Under the immersion $\overline{X_N} \subseteq \mathbb{P}_{\overline{\mathbb{Q}}}^n \times \mathbb{P}_{\overline{\mathbb{Q}}}^n \times \mathbb{P}_{\overline{\mathbb{Q}}}^m$, the point $(P, [N]P)$ in $\overline{X_N}(\overline{\mathbb{Q}})$ becomes $P_N = (P', P_N', \pi(P)) \in (\mathbb{P}_{\overline{\mathbb{Q}}}^n \times \mathbb{P}_{\overline{\mathbb{Q}}}^n \times \mathbb{P}_{\overline{\mathbb{Q}}}^m)(\overline{\mathbb{Q}})$. The function $P_N \mapsto h([N](P)) = h(P_N') + h(\pi(P))$ defined in (3.1) represents the height attached by the Height Machine to $(\overline{X_N}, \mathcal{F})$ and $P_N \mapsto h(\pi(P))$ represents the height attached to $(\overline{X_N}, \mathcal{M})$.

4. INTERSECTION THEORY AND HEIGHT INEQUALITY ON THE TOTAL SPACE

We keep the notation of §3. So we have a closed immersion $\mathcal{A} \to \mathbb{P}_k^n \times S$ over S satisfying the properties stated near the beginning of §3. Moreover, S is a Zariski open subset of an irreducible projective variety $\overline{S} \subseteq \mathbb{P}_k^m$. We assume in addition $k = \overline{\mathbb{Q}}$. Let X be a closed irreducible subvariety of $\mathcal{A}$ of dimension d defined over $\overline{\mathbb{Q}}$, such that $\pi|_X \colon X \to S$ is dominant. Let ω be the Betti form on $\mathcal{A}$ as defined in Proposition 2.2.

Proposition 4.1. *We keep the notation from above and suppose that X^{an} contains a smooth point at which $\omega|_{X^{\mathrm{sm,an}}}^{\wedge d} > 0$. Then there exists a constant $c_1 > 0$ satisfying the following property. Let $N \in \mathbb{N}$ be a power of 2, there exist a Zariski open dense subset U_N of X defined over $\overline{\mathbb{Q}}$ and a constant $c_2(N)$ such that*

$$h([N]P) \geq c_1 N^2 h(\pi(P)) - c_2(N) \qquad \text{for all } P \in U_N(\overline{\mathbb{Q}}).$$

The goal of this section is to prove Proposition 4.1. The key idea is to apply a theorem of Siu [Laz04, Theorem 2.2.15]. Let us briefly explain the main points before moving on to the proof.

Let X be as in Proposition 4.1, and let $P \in X(\overline{\mathbb{Q}})$. For each $N \in \mathbb{N}$, we work with $X_N \subseteq \mathcal{A} \times_S \mathcal{A}$, the graph of $[N] \colon \mathcal{A} \to \mathcal{A}$, and its Zariski closure $\overline{X_N}$ in $\overline{\mathcal{A}} \times_{\overline{S}} \overline{\mathcal{A}} \subseteq \mathbb{P}_k^n \times \mathbb{P}_k^n \times \mathbb{P}_k^m$. The point P gives rise to a point $P_N \in \overline{X_N}$; see §3.3. Consider the line bundles $\mathcal{F} = \mathcal{O}(0,1,1)|_{\overline{X_N}}$ and $\mathcal{M} = \mathcal{O}(0,0,1)|_{\overline{X_N}}$. Choosing representatives as in last paragraph of §3.3 our height inequality in Proposition 4.1 is equivalent to

$$h_{\overline{X_N}, \mathcal{F}}(P_N) \geq c_1 N^2 h_{\overline{X_N}, \mathcal{M}}(P_N) - c_2'(N)$$

for some $c_2'(N)$ independent of P (which may be different from $c_2(N)$). By the Height Machine it suffices to find positive integers p and q, independent of N, such that $\mathcal{F}^{\otimes q} \otimes \mathcal{M}^{\otimes -pN^2}$ is a big line bundle on $\overline{X_N}$; we can then take $c_1 = p/q$.

Both $\mathcal{F}$ and $\mathcal{M}$ are nef line bundles. Thus a criterion of bigness by Siu [Laz04, Theorem 2.2.15], states that $\mathcal{F}^{\otimes q} \otimes \mathcal{M}^{\otimes -pN^2}$ is big if $(\mathcal{F}^{\cdot d}) > d c_1 (\mathcal{M}^{\otimes N^2} \cdot \mathcal{F}^{\cdot(d-1)})$. Note that $(\mathcal{M}^{\otimes N^2} \cdot \mathcal{F}^{\cdot(d-1)}) = N^2 (\mathcal{M} \cdot \mathcal{F}^{\cdot(d-1)})$ by multi-linearity of intersection numbers. Thus our task becomes comparing two intersection numbers. Our application continues to work if the numerical factor $d = \dim X$ is replaced by any positive factor that depends only on the dimension. So it remains to prove an appropriate lower bound for $(\mathcal{F}^{\cdot d})$ and an appropriate upper bound for $(\mathcal{M} \cdot \mathcal{F}^{\cdot(d-1)})$.

The proof of Proposition 4.1 will be organized as follows in this section. We first prove the appropriate lower bound for $(\mathcal{F}^{\cdot d})$ in Proposition 4.2. This is where we use the hypothesis that $\omega|_{X^{\mathrm{sm,an}}}^{\wedge d} > 0$ at some smooth point of X^{an}. Next we prove the appropriate lower bound for $(\mathcal{M} \cdot \mathcal{F}^{\cdot(d-1)})$ in Proposition 4.3. At this step the assumption of N being a power of 2 is used. Then we finish the proof of Proposition 4.1 by applying Siu's theorem in §4.3.

4.1. **Bounding an intersection number from below.** Let X be as in Proposition 4.1. For each $N \in \mathbb{N}$, let $\overline{X_N} \subseteq \mathbb{P}_k^n \times \mathbb{P}_k^n \times \mathbb{P}_k^m$ be as in §3.3. In particular, $\dim \overline{X_N} = d$. Let $\mathcal{F} = \mathcal{O}(0,1,1)|_{\overline{X_N}}$ be as in (3.4). The top self-intersection of $\mathcal{F}$ on $\overline{X_N}$ is bounded from below in the following proposition. To prove it, we may replace X by its base change to $\mathbb{C}$.

Proposition 4.2. *Suppose X^{an} contains a smooth point at which $\omega|_{X^{\mathrm{sm,an}}}^{\wedge d} > 0$. Then there exists a constant $\kappa > 0$, independent of N, such that $(\mathcal{F}^{\cdot d}) \geq \kappa N^{2d}$ for all $N \in \mathbb{N}$.*

Proof. We fix a point $P_0 \in X^{\mathrm{sm,an}}$ at which $\omega|_{X^{\mathrm{sm,an}}}^{\wedge d}$ is positive and let $s_0 = \pi(P_0), \Delta, \vartheta, \theta$, and K be as in §3.2, see Remark 3.3. In particular $\vartheta(P_0) = \theta \circ \pi(P_0) = 1$. We extend ϑ to a smooth function on $(\mathbb{P}_{\mathbb{C}}^m)^{\mathrm{an}}$ by setting it 0 outside of the compact set $K \subseteq S^{\mathrm{an}}$. This extends $\theta = \vartheta \circ \pi$ to all of $(\mathbb{P}_{\mathbb{C}}^n \times \mathbb{P}_{\mathbb{C}}^m)^{\mathrm{an}}$.

Let α be the pull-back of the Fubini–Study form under the analytification of the Segre morphism $\mathbb{P}_{\mathbb{C}}^n \times \mathbb{P}_{\mathbb{C}}^m \to \mathbb{P}_{\mathbb{C}}^{(n+1)(m+1)-1}$. We replace α by its restriction to $\overline{\mathcal{A}}^{\mathrm{an}}$. Thus α represents the Chern class of $\mathcal{O}(1,1) \in \mathrm{Pic}(\mathbb{P}_{\mathbb{C}}^n \times \mathbb{P}_{\mathbb{C}}^m)$ restricted to $\overline{\mathcal{A}}^{\mathrm{an}}$, using common notation.

Note that α is strictly positive on all of $\mathcal{A}^{\mathrm{an}}$. Since Δ is relatively compact we can find a constant $C > 0$ with

$$(4.1) \qquad C\alpha|_{\mathcal{A}_\Delta} - \omega|_{\mathcal{A}_\Delta} \geq 0.$$

As the smooth and non-negative function $\theta = \vartheta \circ \pi$ on $\mathcal{A}^{\mathrm{an}}$ has support in $\pi^{-1}(K) \subseteq \pi^{-1}(\Delta) = \mathcal{A}_\Delta$ we have

$$C\theta\alpha - \theta\omega \geq 0.$$

We pull this $(1,1)$-form back under the holomorphic map $[N]\colon \mathcal{A}^{\mathrm{an}} \to \mathcal{A}^{\mathrm{an}}$ and get

$$(4.2) \qquad C[N]^*(\theta\alpha) - N^2\theta\omega = C[N]^*(\theta\alpha) - [N]^*(\theta\omega) \geq 0$$

where we used $[N]^*\omega = N^2\omega$ and $[N]^*\theta = \theta$; the former is a property of the Betti form, see Proposition 2.2(ii) and the latter holds as θ is the pull-back from the base of ϑ.

We define

$$\beta = C[N]^*(\theta\alpha) - N^2\theta\omega,$$

which is a $(1,1)$-form on $\mathcal{A}^{\mathrm{an}}$. It is semi-positive by (4.2). The support of θ is contained in $\pi^{-1}(K)$, which we have identified as compact at the end of §3.2. So $C[N]^*(\theta\alpha)$ and $N^2\theta\omega$ have compact support on $\mathcal{A}^{\mathrm{an}}$.

We claim that $\int_{X^{\mathrm{sm,an}}}(C[N]^*(\theta\alpha))^{\wedge d} \geq \int_{X^{\mathrm{sm,an}}}(N^2\theta\omega)^{\wedge d}$.

First observe that both integrals are well-defined as both $[N]^*(\theta\alpha)$ and $N^2\theta\omega$ have compact support on $\mathcal{A}^{\mathrm{an}}$; this follows from work of Lelong [Lel57] which we use freely below. A textbook proof can be found in [Voi02, Theorem 11.21] and [Dem12, §III.2.B]. To prove the inequality let us write $\beta = \gamma - \delta$ with $\gamma = C[N]^*(\theta\alpha)$ and $\delta = N^2\theta\omega$. Then
(4.3)

$$\int_{X^{\mathrm{sm,an}}} \gamma^{\wedge d} - \int_{X^{\mathrm{sm,an}}} \delta^{\wedge d} = \int_{X^{\mathrm{sm,an}}} (\delta + \beta)^{\wedge d} - \int_{X^{\mathrm{sm,an}}} \delta^{\wedge d} = \sum_{i=0}^{d-1} \binom{d}{i} \int_{X^{\mathrm{sm,an}}} \delta^{\wedge i} \wedge \beta^{\wedge(d-i)}$$

as the exterior product is commutative on even degree forms. We know that $\beta \geq 0$ on $\mathcal{A}^{\mathrm{an}}$ and it is also crucial that $\delta \geq 0$ on $\mathcal{A}^{\mathrm{an}}$, the latter follows from $\omega \geq 0$, property (i) of Proposition 2.2, and from $\theta \geq 0$. Then $\delta^{\wedge i} \wedge \beta^{\wedge(d-i)}$ is semi-positive on $\mathcal{A}^{\mathrm{an}}$; see [Dem12, Proposition III.1.11][1]. Thus the right-hand side of (4.3) is non-negative, see [Dem12, Theorem III.2.7], and our claim is settled.

---

[1]As our convention is somewhat different from Demailly's, let us explain how to apply [Dem12, Proposition III.1.11]. Our definition of semi-positive $(1,1)$-form coincides with that of *positive* $(1,1)$-form of [Dem12, Chapter III] by Corollary 1.7 of *loc.cit.*, and thus are precisely the strongly positive $(1,1)$-forms of [Dem12, Chapter III] by Corollary 1.9 of *loc.cit.* Therefore we can apply the cited proposition.

The claim implies

$$(4.4) \qquad C^d \int_{X^{\mathrm{sm,an}}} [N]^*(\theta\alpha)^{\wedge d} \geq \kappa' N^{2d} \quad \text{where} \quad \kappa' = \int_{X^{\mathrm{sm,an}}} (\theta\omega)^{\wedge d}.$$

We have $\kappa' > 0$. Indeed, $(\theta\omega)^{\wedge d}$ is semi-positive on $\mathcal{A}^{\mathrm{an}}$ because $\omega \geq 0$ (Proposition 2.2(i)) and $\theta \geq 0$ (by construction). But $\omega|_{X^{\mathrm{sm,an}}}^{\wedge d}$ is positive at $P_0 \in X^{\mathrm{sm,an}}$ by choice of P_0 and $\theta \circ \pi(P_0) = 1$ by choice of θ. So $(\theta\omega)|_{X^{\mathrm{sm,an}}}^{\wedge d}$ is positive at $P_0 \in X^{\mathrm{sm,an}}$. Thus $\kappa' > 0$.

Next we want to relate the integral on the left in (4.4) with an intersection number. First we recall that $[N]$ is given in terms of the graph construction, *cf.* (3.3). So we may rewrite

$$(4.5) \quad \int_{X^{\mathrm{sm,an}}} [N]^*(\theta\alpha)^{\wedge d} = \int_{X^{\mathrm{sm,an}}} (\rho_2|_{\Gamma_N} \circ \rho_1|_{\Gamma_N}^{-1})^*(\theta\alpha)^{\wedge d} = \int_{X^{\mathrm{sm,an}}} (\rho_1|_{\Gamma_N}^{-1})^* \rho_2|_{\Gamma_N}^*(\theta\alpha)^{\wedge d},$$

here Γ_N, ρ_1, and ρ_2 are as defined in §3.3.

Because $\rho_1|_{\Gamma_N^{\mathrm{an}}} : \Gamma_N^{\mathrm{an}} \to \mathcal{A}^{\mathrm{an}}$ is biholomorphic we can change coordinates and integrate over X_N, which is a complex analytic subset of the graph Γ_N, itself a complex manifold. More precisely, we have

$$(4.6) \qquad \int_{X^{\mathrm{sm,an}}} (\rho_1|_{\Gamma_N}^{-1})^* \rho_2|_{\Gamma_N}^*(\theta\alpha)^{\wedge d} = \int_{X_N^{\mathrm{sm,an}}} \rho_2|_{\Gamma_N}^*(\theta\alpha)^{\wedge d}.$$

Recall that α is the restriction to $\overline{\mathcal{A}}^{\mathrm{an}}$ of a $(1,1)$-form on $(\mathbb{P}_\mathbb{C}^n \times \mathbb{P}_\mathbb{C}^m)^{\mathrm{an}}$. Moreover, ρ_2 is also defined on all of $\mathbb{P}_\mathbb{C}^n \times \mathbb{P}_\mathbb{C}^n \times \mathbb{P}_\mathbb{C}^m$. So $\rho_2|_{\Gamma_N}^*(\theta\alpha)$ is the restriction to Γ_N of a $(1,1)$-form defined on $(\mathbb{P}_\mathbb{C}^n \times \mathbb{P}_\mathbb{C}^n \times \mathbb{P}_\mathbb{C}^m)^{\mathrm{an}}$. Observe that $X_N^{\mathrm{sm,an}} \subseteq \overline{X_N}^{\mathrm{an}}$ and the difference has dimension strictly less than $d = \dim X_N$. This justifies

$$(4.7) \qquad \int_{X_N^{\mathrm{sm,an}}} \rho_2|_{\Gamma_N}^*(\theta\alpha)^{\wedge d} = \int_{\overline{X_N}^{\mathrm{an}}} \rho_2^*(\theta\alpha)^{\wedge d}$$

where we take $\overline{X_N}^{\mathrm{an}}$ as a complex analytic subset of the analytification of $\mathbb{P}_\mathbb{C}^n \times \mathbb{P}_\mathbb{C}^n \times \mathbb{P}_\mathbb{C}^m$ and $\rho_2^*(\theta\alpha)$ as a $(1,1)$-form on this ambient space. Now θ takes values in $[0,1]$ and so

$$(4.8) \qquad \int_{\overline{X_N}^{\mathrm{an}}} \rho_2^*(\theta\alpha)^{\wedge d} \leq \int_{\overline{X_N}^{\mathrm{an}}} (\rho_2^*\alpha)^{\wedge d}.$$

The pull-back $\rho_2^*\alpha$ represents $\rho_2^*\mathcal{O}(1,1) \in \mathrm{Pic}(\mathbb{P}_\mathbb{C}^n \times \mathbb{P}_\mathbb{C}^n \times \mathbb{P}_\mathbb{C}^m)$ in the Picard group and has compact support as $(\mathbb{P}_\mathbb{C}^n \times \mathbb{P}_\mathbb{C}^n \times \mathbb{P}_\mathbb{C}^m)^{\mathrm{an}}$ is compact. But integration coincides with the intersection pairing in the compact case; see [Voi02, Theorem 11.21]. In particular, we have

$$(4.9) \qquad \int_{\overline{X_N}^{\mathrm{an}}} (\rho_2^*\alpha)^{\wedge d} = (\rho_2^*\mathcal{O}(1,1)^{\cdot d}[\overline{X_N}])$$

where the intersection takes place in $\mathbb{P}_\mathbb{C}^n \times \mathbb{P}_\mathbb{C}^n \times \mathbb{P}_\mathbb{C}^m$. We recall (3.4) and apply the projection formula to obtain

$$(4.10) \qquad (\rho_2^*\mathcal{O}(1,1)^{\cdot d}[\overline{X_N}]) = (\mathcal{O}(0,1,1)^{\cdot d}[\overline{X_N}]) = (\mathcal{F}^{\cdot d}).$$

The (in)equalities (4.5), (4.6), (4.7), (4.8), (4.9), and (4.10) yield

$$\int_{X^{\mathrm{sm,an}}} [N]^*(\theta\alpha)^{\wedge d} \leq (\mathcal{F}^{\cdot d}).$$

We recall the lower bound (4.4) to obtain $(\mathcal{F}^{\cdot d}) \geq (\kappa'/C^d)N^{2d}$ where C comes from (4.1) and $\kappa' > 0$ comes from (4.4). The proposition follows with $\kappa = \kappa'/C^d$. $\qquad\square$

4.2. **Bounding an intersection number from above.** We keep the notation from the last subsection with $k = \overline{\mathbb{Q}}$. So X is as above Proposition 4.1 with $\dim X = d$ For each $N \in \mathbb{N}$, let $\overline{X_N} \subseteq \mathbb{P}_k^n \times \mathbb{P}_k^n \times \mathbb{P}_k^m$ be the graph construction as in §3.3. In particular, $\dim \overline{X_N} = d$. Here we need $\mathcal{F} = \mathcal{O}(0,1,1)|_{\overline{X_N}}$ as defined in (3.4) and also $\mathcal{M} = \mathcal{O}(0,0,1)|_{\overline{X_N}}$ as defined in (3.5).

Proposition 4.3. *Assume $d \geq 1$. There exists a constant $c > 0$ depending on the data introduced above with the following property. Say $N \geq 1$ is a power of 2, then*

$$(\mathcal{M} \cdot \mathcal{F}^{\cdot(d-1)}) \leq cN^{2(d-1)}.$$

Let us make some preliminary remarks before the proof. A similar upper bound for the intersection number was derived by the third-named author in [Hab09, Hab13] using Philippon's version [Phi86] of Bézout's Theorem for multiprojective space. The approach here is similar but does not refer to Philippon's result. Rather, we rely on the following well-known positivity property of the intersection theory of multiprojective space: any effective Weil divisor on a multiprojective space is nef. This approach was motivated by Kühne's [Küh20] work on semiabelian varieties.

Proof of Proposition 4.3. Recall that $[2]\colon A \to A$ is the multiplication-by-2 morphism on A. For the symmetric and ample line bundle L on A, we have $[2]^*L \cong L^{\otimes 4}$. Recall that A is projectively normal in $\mathbb{P}_{k(S)}^n$. By a result of Serre, [Wal87, Corollaire 2, Appendix II], the morphism $[2]$ is represented by homogeneous polynomials $f_0, \dots, f_n$ in the $n+1$ projective coordinates of $\mathbb{P}^n$ of degree 4, with coefficients in $k(S)$ and with no common zeros in A.

Recall that the family $\mathcal{A}$ is embedded in $\mathbb{P}_k^n \times S \subseteq \mathbb{P}_k^n \times \mathbb{P}_k^m$. We can spread out the $f_0, \dots, f_n$. More precisely, there exist a Zariski closed, proper subset $Z \subsetneq \mathcal{A}$ and polynomials $f_0, \dots, f_n \in k[\mathbf{X}, \mathbf{S}]$ that are bihomogeneous of degree $(4, D')$ in the $(n+1)$-tuple of projective coordinates $\mathbf{X}$ of $\mathbb{P}_k^n$ and the $(m+1)$-tuple of projective coordinates $\mathbf{S}$ of $\mathbb{P}_k^m$, with the following properties:

(i) the polynomials $f_0, \dots, f_n$ have no common zeros on $(\mathcal{A} \setminus Z)(k)$, and
(ii) if $(P, s) \in (\mathcal{A} \setminus Z)(k)$, then $[2](P, s) = \big([f_0(P, s) : \dots : f_n(P, s)], s\big)$.

Moreover, as $f_0, \dots, f_n$ have no common zero on the generic fiber, we may assume that $\pi(Z)$ is Zariski closed and proper in S. So we may assume that $Z = \pi^{-1}(\pi(Z)) \subsetneq \mathcal{A}$ and in particular, $[2]$ maps $\mathcal{A} \setminus Z$ to itself.

The 4 in the bidegree $(4, D')$ comes from $2^2 = 4$. The degree D' with respect to the base coordinates $\mathbf{S}$ is more mysterious. However, by successively iterating we will get it under control.

For each integer $l \geq 1$ we require polynomials $f_0^{(l)}, \dots, f_n^{(l)}$ to describe multiplication-by-2^l, *cf.* [GH19, §9]. In order to obtain information on the degree with respect to $\mathbf{S}$ we construct them by iterating the $f_0^{(1)} = f_0, \dots, f_n^{(1)} = f_n$. For all $i \in \{0, \dots, n\}$ we set

$$f_i^{(l+1)}(\mathbf{X}, \mathbf{S}) = f_i\left(\big(f_0^{(l)}(\mathbf{X}, \mathbf{S}), \dots, f_n^{(l)}(\mathbf{X}, \mathbf{S})\big), \mathbf{S}\right)$$

for all i; it is bihomogeneous in $\mathbf{X}$ and $\mathbf{S}$. So for all $l \geq 1$

(i) the polynomials $f_0^{(l)}, \ldots, f_n^{(l)}$ have no common zeros on $(\mathcal{A} \setminus Z)(k)$, and

(ii) if $(P, s) \in (\mathcal{A} \setminus Z)(k)$, then $[2^l](P, s) = \left([f_0^{(l)}(P, s) : \cdots : f_n^{(l)}(P, s)], s\right)$.

If for all i the polynomials $f_i^{(l)}$ are bihomogeneous of degree (D_l, D_l'), then all $f_i^{(l+1)}$ are bihomogeneous of degree $(4D_l, D' + 4D_l')$. Recall that $(D_1, D_1') = (4, D')$, thus the recurrence relations

$$D_{l+1} = 4D_l \quad \text{and} \quad D_{l+1}' = D' + 4D_l'$$

imply

$$(4.11) \qquad D_l = 4^l \quad \text{and} \quad D_l' = \frac{4^l - 1}{3}D' \leq 4^l D'$$

for all $l \geq 1$. Up-to the constant linear factor D' the bidegrees both grow like 4^l.

We proceed as follows to cut out the graph $\overline{X}_N$ where $N = 2^l$. We start out with $\overline{X} \subseteq \mathbb{P}_k^n \times \mathbb{P}_k^m$. As X dominates S but Z does not, there is an i such that $f_i^{(l)}$ does not vanish identically on X, without loss of generality we assume $i = 0$.

Then as i varies over $\{1, \ldots, n\}$ we obtain n trihomogeneous polynomials

$$g_i := Y_i f_0^{(l)}(\mathbf{X}, \mathbf{S}) - Y_0 f_i^{(l)}(\mathbf{X}, \mathbf{S})$$

where $Y_0, \ldots, Y_n$ are the projective coordinates on the middle factor of $\mathbb{P}_k^n \times \mathbb{P}_k^n \times \mathbb{P}_k^m$. The tridegree of these polynomials is $(D_l, 1, D_l')$. Their zero locus on $\overline{X} \times \mathbb{P}_k^n$ has the graph $\overline{X}_N$ as an irreducible component; by permuting coordinates we consider $\overline{X} \times \mathbb{P}_k^n$ as a subvariety of $\mathbb{P}_k^n \times \mathbb{P}_k^n \times \mathbb{P}_k^m$. We will see below that this is a proper component of the said intersection. However, there may be further irreducible components in this intersection, some could even have dimension greater than $\dim \overline{X}_N$.

This issue is clarified by the positivity result [Ful98, Corollary 12.2.(a)]. We apply it to the ambient variety $\mathbb{P}_k^n \times \mathbb{P}_k^n \times \mathbb{P}_k^m$, which becomes X in Fulton's notation; observe that the tangent bundle of a product of projective spaces is generated by its global sections, *cf.* [Ful98, Examples 12.2.1.(a) and (c)]. For $i \in \{1, \ldots, n\}$, the V_i in Fulton's notation is the zero set of g_i, and V_{n+1} is $\overline{X} \times \mathbb{P}_k^n$. So $r = n+1$ and $V_1, \ldots, V_{n+1}$ are equidimensional. Observe that

$$\sum_{i=1}^{r} \dim V_i - (r-1)\dim \mathbb{P}_k^n \times \mathbb{P}_k^n \times \mathbb{P}_k^m = (2n+m-1)(r-1) + \dim \overline{X} \times \mathbb{P}_k^n - (r-1)(2n+m)$$

$$= \dim \overline{X} = \dim \overline{X}_N,$$

so, and as announced above, $\overline{X}_N$ is a proper component in the intersection of $V_1, \ldots, V_n$, and $\overline{X} \times \mathbb{P}_k^n$. By Fulton's [Ful98, Corollary 12.2.(a)] the cycle class attached to the intersection of $\overline{X} \times \mathbb{P}_k^n$ with the zero locus of $g_1, \ldots, g_n$ is represented by a positive cycle on $\mathbb{P}_k^n \times \mathbb{P}_k^n \times \mathbb{P}_k^m$, one of whose components is $\overline{X}_N$. As $\mathcal{O}(0,0,1)$ and $\mathcal{O}(0,1,1)$ are numerically effective we conclude

(4.12)

$$\left(\mathcal{O}(0,0,1)\mathcal{O}(0,1,1)^{\cdot(d-1)}[\overline{X}_N]\right) \leq \left(\mathcal{O}(0,0,1)\mathcal{O}(0,1,1)^{\cdot(d-1)}\mathcal{O}(D_l,1,D_l')^{\cdot n}[\overline{X} \times \mathbb{P}_k^n]\right).$$

The cycle $[\overline{X} \times \mathbb{P}_k^n]$ is linearly equivalent to $\sum_{i+p=n+m-d} a_{ip} H_1^{\cdot i} H_2^{\cdot p}$, with H_1 and H_2 hyperplane pullbacks of the factors $\mathbb{P}_k^n \times \mathbb{P}_k^m \supseteq \overline{X}$, respectively, and with a_{ip} non-negative

integers that depend only on $\overline{X}$. Thus the left-hand side of (4.12) is at most

$$\sum_{i+p=n+m-d} a_{ip}\left(\mathcal{O}(0,0,1)\mathcal{O}(0,1,1)^{\cdot(d-1)}\mathcal{O}(D_l,1,D_l')^{\cdot n}\mathcal{O}(1,0,0)^{\cdot i}\mathcal{O}(0,0,1)^{\cdot p}\right).$$

We can expand the sum using linearly of intersection numbers to find that it equals

$$\sum_{\substack{i+p=n+m-d \\ j'+p'=d-1 \\ i''+j''+p''=n}} a_{ip}\binom{d-1}{j',p'}\binom{n}{i'',j'',p''}D_l^{i''}D_l'^{p''}\left(\mathcal{O}(1,0,0)^{\cdot(i+i'')}\mathcal{O}(0,1,0)^{\cdot(j'+j'')}\mathcal{O}(0,0,1)^{\cdot(1+p+p'+p'')}\right)$$

Only terms with $i+i'' \le n$ and $j'+j'' \le n$ and $1+p+p'+p'' \le m$ contribute to the sum. On the other hand, any term in the sum satisfies $i+i''+j'+j''+1+p+p'+p''=2n+m$. So we can assume $i+i''=n$ and $j'+j''=n$ and $1+p+p'+p''=m$ in the sum which thus simplifies to

$$\sum_{\substack{i+p=n+m-d,i+i''=n \\ j'+p'=d-1,j'+j''=n \\ i''+j''+p''=n,p+p'+p''=m-1}} a_{ip}\binom{d-1}{j',p'}\binom{n}{i'',j'',p''}D_l^{i''}D_l'^{p''}.$$

Note $i''+p''=n-j''=j'=d-1-p' \le d-1$. We recall (4.11) and conclude that the left-hand side of (4.12) is at most

$$(4.13) \quad (4^lD')^{d-1}\sum_{\substack{i+p=n+m-d \\ j'+p'=d-1 \\ i''+j''+p''=n}} a_{ip}\binom{d-1}{j',p'}\binom{n}{i'',j'',p''} \le (4^lD')^{d-1}2^{d-1}3^n\sum_{i+p=n+m-d} a_{ip}.$$

We recall $N = 2^l$ and use the projection formula with the estimates above to find

$$(\mathcal{O}(0,0,1)|_{\overline{X_N}}\mathcal{O}(0,1,1)|_{\overline{X_N}}^{\cdot(d-1)}) \le cN^{2(d-1)}$$

where $c > 0$ depends only on X. Recall our definition $\mathcal{F} = \mathcal{O}(0,1,1)|_{\overline{X_N}}$ and $\mathcal{M} = \mathcal{O}(0,0,1)|_{\overline{X_N}}$. So we get $(\mathcal{M}\cdot\mathcal{F}^{\cdot(d-1)}) \le cN^{2(d-1)}$, as desired. $\square$

4.3. Proof of Proposition 4.1.

Now let us prove Proposition 4.1 by comparing the intersection number inequalities in Propositions 4.2 and 4.3.

Let X be of dimension d as in Proposition 4.1. The case $d = 0$ is trivial. So we assume $d \ge 1$. We may assume $N = 2^l$ with $l \in \mathbb{N}$. Let $\overline{X_N} \subseteq \mathbb{P}_k^n \times \mathbb{P}_k^n \times \mathbb{P}_k^m$ be as in §3.3. In particular $\dim \overline{X_N} = d$.

Let $\kappa > 0$ be as in Proposition 4.2. Then $(\mathcal{F}^{\cdot d}) \ge \kappa N^{2d}$. Let $c > 0$ be as in Proposition 4.3. Then $(\mathcal{M}\cdot\mathcal{F}^{\cdot(d-1)}) \le cN^{2(d-1)}$. We have indicated how to obtain κ and c at the end of the proof of each one of the corresponding propositions.

Fix a rational number c_1 such that

$$(4.14) \qquad\qquad 0 < c_1cd < \kappa.$$

Let q be a multiple of the denominator of c_1. Using the bounds above and linearity of intersection numbers we get

$$d(\mathcal{M}^{\otimes qc_1N^2}\cdot(\mathcal{F}^{\otimes q})^{\cdot(d-1)}) = dq^dc_1N^2(\mathcal{M}\cdot\mathcal{F}^{\cdot(d-1)}) \le dq^dc_1N^2cN^{2(d-1)} < \kappa q^dN^{2d} \le ((\mathcal{F}^{\otimes q})^{\cdot d}).$$

Then $\mathcal{F}^{\otimes q} \otimes \mathcal{M}^{\otimes -qc_1N^2}$ is a big line bundle on $\overline{X_N}$ by a theorem of Siu [Laz04, Theorem 2.2.15]. After possibly replacing q by a multiple the line bundle $\mathcal{F}^{\otimes q} \otimes \mathcal{M}^{\otimes -qc_1N^2}$ admits a non-zero global section. Say $h_{\overline{X_N}, \mathcal{F}}$ and $h_{\overline{X_N}, \mathcal{M}}$ are representatives of heights on $\overline{X_N}$ attached by the Height Machine to $\mathcal{F}$ and $\mathcal{M}$, respectively. After canceling q we conclude that $h_{\overline{X_N}, \mathcal{F}} - c_1 N^2 h_{\overline{X_N}, \mathcal{M}}$ is bounded from below on a Zariski open and dense subset of $\overline{X_N}$. The image of this subset under the projection ρ_1 contains a Zariski open and dense subset U_N of X. It follows from the end of §3.3 that there exists $c_2(N)$ such that

$$h([N](P)) \geq c_1 N^2 h(\pi(P)) - c_2(N) \quad \text{for all } P \in U_N(\overline{\mathbb{Q}}). \qquad \square$$

5. Proof of the height inequality Theorem 1.6

We keep the notation of §3. In particular, S is a regular, irreducible, quasi-projective variety over $\overline{\mathbb{Q}}$ and $\pi \colon \mathcal{A} \to S$ is an abelian scheme of relative dimension $g \geq 1$. Moreover, we have immersions as in §3 and we assume (Hyp) from page 5. We use the heights introduced in §3.1. Let X be an irreducible closed subvariety of $\mathcal{A}$ defined over $\overline{\mathbb{Q}}$. We assume that X dominates S and is non-degenerate as defined in Definition 1.5

The upshot of (Hyp) is that we obtain from Proposition 2.2 the Betti form ω on $\mathcal{A}^{\mathrm{an}}$. Moreover, part (i) and (iii) of Proposition 2.2 implies that, for $d = \dim X$,

$$(5.1) \qquad \omega|_{X^{\mathrm{sm,an}}}^{\wedge d} > 0 \quad \text{at some smooth point of } X^{\mathrm{an}}.$$

Our assumption (5.1) allows us to apply Proposition 4.1 to X. There exists a constant $c_1 > 0$ as in (4.14) such that the following holds. Let $N \in \mathbb{N}$ be a power of 2, there exists a Zariski open dense subset $U_N \subseteq X$ and a constant $c_2(N) \geq 0$ such that

$$(5.2) \qquad h([N]P) \geq c_1 N^2 h(\pi(P)) - c_2(N)$$

for all $P \in U_N(\overline{\mathbb{Q}})$; we stress that U_N and $c_2 \geq 0$ may depend on N in addition to $X, \mathcal{A}$, and the various immersions such as $\mathcal{A} \subseteq \mathbb{P}^n_{\overline{\mathbb{Q}}} \times \mathbb{P}^m_{\overline{\mathbb{Q}}}$.

By the Theorem of Silverman-Tate, see [Sil83, Theorem A] and Theorem A.1, there exist a constant $c_0 \geq 0$ such that

$$(5.3) \qquad |\hat{h}_{\mathcal{A}}(P) - h(P)| \leq c_0 \max\{1, h(\pi(P))\} \leq c_0 \big(1 + h(\pi(P))\big)$$

for all $P \in \mathcal{A}(\overline{\mathbb{Q}})$.

Next we kill Zimmer constants as in Masser's [Zan12, Appendix C]. For any $P \in U_N(\overline{\mathbb{Q}})$, we have

$$\begin{aligned} \hat{h}_{\mathcal{A}}([N](P)) &\geq h([N](P)) - c_0\big(1 + h(\pi([N](P)))\big) && \text{(by (5.3))} \\ &= h([N](P)) - c_0\big(1 + h(\pi(P))\big) && \text{(as } \pi([N](P)) = \pi(P)) \\ &\geq c_1 N^2 h(\pi(P)) - c_2(N) - c_0\big(1 + h(\pi(P))\big) && \text{(by (5.2)).} \end{aligned}$$

We use $\hat{h}_{\mathcal{A}}([N]P) = N^2 \hat{h}_{\mathcal{A}}(P)$, divide by N^2, and rearrange to get

$$\hat{h}_{\mathcal{A}}(P) \geq \Big(c_1 - \frac{c_0}{N^2}\Big) h(\pi(P)) - \frac{c_2(N) + c_0}{N^2}$$

for all $N \in \mathbb{N}$ that are powers of 2 and all $P \in U_N(\overline{\mathbb{Q}})$.

Recall that c_0 and c_1 are independent of N. We fix $N \in \mathbb{N}$ to be the least power of 2 such that $N^2 \geq 2c_0/c_1$. As $h(\pi(P))$ is non-negative we get

$$\hat{h}_{\mathcal{A}}(P) \geq \frac{c_1}{2} h(\pi(P)) - \frac{c_2(N) + c_0}{N^2}$$

for all $P \in U_N(\overline{\mathbb{Q}})$. Since N is now fixed, the Zariski open dense subset U_N of X is also fixed. The theorem follows after adjusting c_1 and c_2. $\square$

Remark 5.1. *In the proof of Theorem 1.6 we can keep track of the process to compute the constant $c_1 > 0$. Use the notation in §3. In particular ω is the Betti form on $\mathcal{A}$, we have an immersion $\mathcal{A} \subseteq \mathbb{P}^n_{\mathbb{C}} \times \mathbb{P}^m_{\mathbb{C}}$, α is a $(1,1)$-form on $\mathbb{P}^n_{\mathbb{C}} \times \mathbb{P}^m_{\mathbb{C}}$ representing the Chern class of $\mathcal{O}(1,1)$, $\Delta \subseteq S^{\mathrm{an}}$ is open and relative compact, and $\theta \colon \mathcal{A}^{\mathrm{an}} \to [0,1]$ (which factors through S^{an}) is a smooth function with compact support contained in $\mathcal{A}_\Delta := \pi^{-1}(\Delta)$. The function θ should furthermore satisfy $\theta(P_0) = 1$ for some $P_0 \in X^{\mathrm{sm}}(\mathbb{C})$ such that $(\omega|_{X^{\mathrm{sm}}})^{\wedge d}$ is positive at P_0, where $d = \dim X$.*

Assume $d \geq 1$. The proof of Theorem 1.6 tells us that one half of any rational number satisfying the inequality (4.14) can be taken as c_1. So the constant $c_1 > 0$ can be taken to be any rational number in $(0, \kappa/(2cd))$, such that:

- *$\kappa = \kappa'/C^d$, where $\kappa' = \int_{X^{\mathrm{sm,an}}} (\theta\omega)^{\wedge d}$, as in (4.4), and C satisfies $C\alpha|_{\mathcal{A}_\Delta} - \omega|_{\mathcal{A}_\Delta} \geq 0$, as in (4.1),*
- *c is a constant depending on a certain degree of X and coming from (4.13).*

6. Preparation for counting points

6.1. The universal family and non-degeneracy.
In this section, we fix the basic setup to prove Proposition 7.1, described as the alternative on page 4, and our main results.

Fix an integer $g \geq 2$. Recall from §1.2 that $\mathbb{M}_g$ denotes the fine moduli space of smooth curves of genus g, with level-ℓ-structure where $\ell \geq 3$ is fixed, *cf.* [ACG11, Chapter XVI, Theorem 2.11 (or above Proposition 2.8)], [DM69, (5.14)], or [OS80, Theorem 1.8]. It is known that $\mathbb{M}_g$ is a regular, quasi-projective variety of dimension $3g - 3$. We regard it over $\overline{\mathbb{Q}}$; it is irreducible according to our convention introduced below Theorem 1.6. There exists a universal curve $\mathfrak{C}_g$ over $\mathbb{M}_g$, it is smooth and proper over $\mathbb{M}_g$ and its fibers are smooth curves of genus g. Moreover, $\mathfrak{C}_g \to \mathbb{M}_g$ is projective, *cf.* [DM69, Corollary to Theorem 1.2] or [BLR90, Remark 2, §9.3].

Denote by $\mathrm{Jac}(\mathfrak{C}_g)$ the relative Jacobian of $\mathfrak{C}_g \to \mathbb{M}_g$. It is an abelian scheme coming with a natural principal polarization and equipped with level-ℓ-structure, see [MFK94, Proposition 6.9].

Recall from §1.2 that $\mathbb{A}_g$ denotes the fine moduli space of principally polarized abelian varieties of dimension g, with level-ℓ-structure. Moreover, $\mathbb{A}_g$ is regular and quasi-projective; see [MFK94, Theorem 7.9 and below] or [OS80, Theorem 1.9]. We regard it as defined over $\overline{\mathbb{Q}}$; it is irreducible according to our convention. Let $\pi \colon \mathfrak{A}_g \to \mathbb{A}_g$ be the universal abelian variety; it is an abelian scheme. Note that π is projective; we refer to Remark 3.1 for this and other details.

As $\mathbb{A}_g$ is a fine moduli space we have the following Cartesian diagram

(6.1)

$$
\begin{array}{ccc}
\mathrm{Jac}(\mathfrak{C}_g) & \longrightarrow & \mathfrak{A}_g \\
\downarrow & \lrcorner & \downarrow \pi \\
\mathbb{M}_g & \xrightarrow{\;\tau\;} & \mathbb{A}_g
\end{array}
$$

the bottom arrow is the Torelli morphism. As we have level structure, the Torelli morphism need not be injective on $\mathbb{C}$-points, but it is finite-to-one on such points, *cf.* [OS80, Lemma 1.11].

We also fix an ample line bundle $\mathcal{M}$ on $\overline{\mathbb{A}_g}$, where $\overline{\mathbb{A}_g}$ is a, possibly non-regular, projective variety containing $\mathbb{A}_g$ as a Zariski open and dense subset. The Height Machine provides an equivalence class of height functions of which we fix a representative $h_{\overline{\mathbb{A}_g},\mathcal{M}} \colon \overline{\mathbb{A}_g}(\overline{\mathbb{Q}}) \to \mathbb{R}$.

Next we fix a projective embedding of $\mathfrak{A}_g$ over $\mathbb{A}_g$. There is a relatively ample line bundle $\mathcal{L}$ on $\mathfrak{A}_g/\mathbb{A}_g$ with $[-1]^*\mathcal{L} = \mathcal{L}$; see [Ray70, Théorème XI 1.4]. After replacing $\mathcal{L}$ by $\mathcal{L}^{\otimes N}$, with $N \geq 4$ large enough, we can assume that $\mathcal{L}$ is very ample relative over $\mathbb{A}_g$ and $[-1]^*\mathcal{L} = \mathcal{L}$. By [Gro61, Proposition 4.4.10(ii) and Proposition 4.1.4], we then have a closed immersion $\mathfrak{A}_g \to \mathbb{P}^n_{\overline{\mathbb{Q}}} \times \mathbb{A}_g$ over $\mathbb{A}_g$ arising from $\mathcal{L} \otimes \pi^*(\mathcal{M}|_{\mathbb{A}_g}^{\otimes p})$ for some integer $p \geq 1$, note that $\mathcal{M}|_{\mathbb{A}_g}$ is ample. For each $s \in \mathbb{A}_g(\overline{\mathbb{Q}})$, the fiber $\mathfrak{A}_{g,s} = \pi^{-1}(s)$ is realized as a projective subvariety of $\mathbb{P}^n_{\overline{\mathbb{Q}}}$ and the induced closed immersion $\mathfrak{A}_{g,s} \to \mathbb{P}^n_{\overline{\mathbb{Q}}}$ comes from the restriction $\mathcal{L}|_{\mathfrak{A}_{g,s}}$ which is ample. Flatness of $\mathfrak{A}_g \to \mathbb{A}_g$ implies that $\dim H^0(\mathfrak{A}_{g,s}, \mathcal{L}_{\mathfrak{A}_{g.s}})$ is independent of s. So $\mathfrak{A}_{g,s}$ is projectively normal inside $\mathbb{P}^n_{\overline{\mathbb{Q}}}$, a property that will play a role later on.

Recall that $\mathcal{L}$ is symmetric and very ample on each fiber of $\mathfrak{A}_g$. By Tate's Limit Argument we obtain the fiberwise Néron–Tate height, *cf.* (3.2),

(6.2)
$$
\hat{h} \colon \mathfrak{A}_g(\overline{\mathbb{Q}}) \to [0,\infty).
$$

Let $M \geq 1$ be an integer. We write $\mathfrak{A}_g^{[M]}$ for the M-fold fibered power $\mathfrak{A}_g \times_{\mathbb{A}_g} \cdots \times_{\mathbb{A}_g} \mathfrak{A}_g$ over $\mathbb{A}_g$. Then $\mathfrak{A}_g^{[M]} \to \mathbb{A}_g$ is an abelian scheme.

By taking the product we obtain closed immersions $\mathfrak{A}_g^{[M]} \to (\mathbb{P}^n_{\overline{\mathbb{Q}}})^M \times \mathbb{A}_g$. The fiber of $\mathfrak{A}_g^{[M]} \to \mathbb{A}_g$ above $s \in \mathbb{A}_g(\overline{\mathbb{Q}})$ is the M-fold power of $\mathfrak{A}_{g,s}$. The associated fiberwise Néron–Tate height $\hat{h} \colon \mathfrak{A}_g^{[M]}(\overline{\mathbb{Q}}) \to [0,\infty)$ is the sum of the Néron–Tate heights, as in (6.2), of the M coordinates.

Let us now define the Faltings–Zhang morphism. In our setting the relative Picard scheme $\mathrm{Pic}(\mathfrak{C}_g/\mathbb{M}_g)$ exists as a group scheme over $\mathbb{M}_g$. It is a union over all $p \in \mathbb{Z}$ of open and closed subschemes $\mathrm{Pic}^p(\mathfrak{C}_g/\mathbb{M}_g)$, where p indicates the degree of a line bundle. By definition we have $\mathrm{Jac}(\mathfrak{C}_g) = \mathrm{Pic}^0(\mathfrak{C}_g/\mathbb{M}_g)$. We cannot expect to have a section of $\mathfrak{C}_g \to \mathbb{M}_g$, so we cannot expect to find an immersion of $\mathfrak{C}_g$ into $\mathrm{Jac}(\mathfrak{C}_g/\mathbb{M}_g)$. As constructed in the proof of [MFK94, Proposition 6.9] we do have a morphism $\mathfrak{C}_g \to \mathrm{Pic}^1(\mathfrak{C}_g/\mathbb{M}_g)$ over $\mathbb{M}_g$. Let $\mathfrak{C}_g^{[M]}$ and $\mathrm{Pic}^p(\mathfrak{C}_g/\mathbb{M}_g)^{[M]}$ denote the respective M-th fibered powers over $\mathbb{M}_g$. The difference morphism coming from the group scheme law $\mathrm{Pic}(\mathfrak{C}_g/\mathbb{M}_g) \times_{\mathbb{M}_g} \mathrm{Pic}(\mathfrak{C}_g/\mathbb{M}_g) \to \mathrm{Pic}(\mathfrak{C}_g/\mathbb{M}_g)$ restricts to a morphism $\mathrm{Pic}^1(\mathfrak{C}_g/\mathbb{M}_g) \times_{\mathbb{M}_g} \mathrm{Pic}^1(\mathfrak{C}_g/\mathbb{M}_g) \to \mathrm{Jac}(\mathfrak{C}_g/\mathbb{M}_g)$ of schemes over $\mathbb{M}_g$. We take the appropriate product morphism over $\mathbb{M}_g$
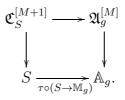
to get a morphism

$$(6.3) \qquad \mathfrak{C}_g^{[M+1]} \to \mathrm{Jac}(\mathfrak{C}_g/\mathbb{M}_g)^{[M]}$$

over $\mathbb{M}_g$. The choice of product is modeled after (1.3). More precisely, consider the situation above a k-point of $\mathbb{M}_g$, where k is an algebraically closed field. The fiber of $\mathfrak{C}_g \to \mathbb{M}_g$ above this point is a smooth curve C defined over k of genus g. For $P_0, \ldots, P_M \in C(k)$ the morphism (6.3) maps $(P_0, P_1, \ldots, P_M) \mapsto (P_1 - P_0, P_2 - P_0, \ldots, P_M - P_0)$ where the difference takes place in the Jacobian of C.

Recall (6.1). We take the M-fold product and compose with (6.3) to obtain a commutative diagram of morphisms of schemes

$$(6.4) \qquad \begin{array}{ccc} \mathfrak{C}_g^{[M+1]} & \longrightarrow & \mathfrak{A}_g^{[M]} \\ \downarrow & & \downarrow \\ \mathbb{M}_g & \xrightarrow{\ \tau\ } & \mathbb{A}_g. \end{array}$$

If $S \to \mathbb{M}_g$ is a morphism of schemes then we define $\mathfrak{C}_S = \mathfrak{C}_g \times_{\mathbb{M}_g} S$ and $\mathfrak{C}_S^{[M]} = \mathfrak{C}_g^{[M]} \times_{\mathbb{M}_g} S$. If S is irreducible, then so is $\mathfrak{C}_S^{[M]}$ by induction on M and a topological argument using that $\mathfrak{C}_S \to S$ is smooth and hence open. Taking the fibered product with S and composing with (6.4) yields a commutative diagram of morphisms of schemes

$$\begin{array}{ccc} \mathfrak{C}_S^{[M+1]} & \longrightarrow & \mathfrak{A}_g^{[M]} \\ \downarrow & & \downarrow \\ S & \xrightarrow[\tau \circ (S \to \mathbb{M}_g)]{} & \mathbb{A}_g. \end{array}$$

By the universal property of the fibered product we get a morphism of schemes

$$(6.5) \qquad \begin{array}{ccc} \mathfrak{C}_S^{[M+1]} & \xrightarrow{\ \mathscr{D}_M\ } & \mathfrak{A}_g^{[M]} \times_{\mathbb{A}_g} S \\ & \searrow & \downarrow \\ & & S \end{array}$$

over S. We call $\mathscr{D}_M$ the *Faltings–Zhang morphism* (over S). Then $\mathscr{D}_M$ is proper since the diagonal arrow in (6.5) is proper.

Let for the moment $S \to \mathbb{M}_g$ be the identity. If $s \in \mathbb{M}_g(\overline{\mathbb{Q}})$, then $\mathfrak{C}_s$ is the curve parametrized by s, and $\mathfrak{A}_{g,\tau(s)}$ is its Jacobian. To embed $\mathfrak{C}_s$ into $\mathfrak{A}_{g,\tau(s)}$ we must work with a base point $P \in \mathfrak{C}_s(\overline{\mathbb{Q}})$. Then $\mathfrak{C}_s - P = \mathscr{D}_1(\{P\} \times \mathfrak{C}_s)$ is an irreducible curve inside $\mathfrak{A}_g$ lying above $\tau(s)$. Hence it provides a closed immersion $\mathfrak{C}_s - P \subseteq \mathbb{P}_{\overline{\mathbb{Q}}}^n$.

Let $\deg X$ denote the degree of an irreducible closed subvariety X of $\mathbb{P}_{\overline{\mathbb{Q}}}^n$ and let $h(X)$ denote its height, *cf.* [BGS94].

Lemma 6.1. *There exists a constant c such that the following two properties hold for all $s \in \mathbb{M}_g(\overline{\mathbb{Q}})$.*

   (i) *We have $\deg(\mathfrak{C}_s - P) \le c$ for all $P \in \mathfrak{A}_{g,\tau(s)}(\overline{\mathbb{Q}})$.*

   (ii) *There exists $P_s \in \mathfrak{C}_s(\overline{\mathbb{Q}})$ such that $h(\mathfrak{C}_s - P_s) \le c \max\{1, h_{\overline{\mathbb{A}_g}, \mathcal{M}}(\tau(s))\}$.*

Proof. We need a quasi-section of $\mathfrak{C}_g \to \mathbb{M}_g$ as provided by [Gro67, Corollaire 17.16.3(ii)]. So there is an affine scheme S and a morphism $S \to \mathfrak{C}_g$ that factors through a surjective, quasi-finite, étale morphism $S \to \mathbb{M}_g$. We consider the product $\mathfrak{C}_g \times_{\mathbb{M}_g} S \to \mathfrak{C}_g^{[2]}$ composed with the Faltings–Zhang morphism $\mathscr{D}_1 \colon \mathfrak{C}_g^{[2]} \to \mathfrak{A}_g \times_{\mathbb{A}_g} \mathbb{M}_g$ over $\mathbb{M}_g$ and then the projection of $\mathfrak{A}_g$. This is a morphism of schemes $\mathfrak{C}_g \times_{\mathbb{M}_g} S \to \mathfrak{A}_g$. Its image is a constructible subset of $\mathfrak{A}_g$. So it is a union of finitely many irreducible Zariski locally closed subsets $\{X_i\}_i$ of $\mathfrak{A}_g$. We have the following property.

Given a point $s \in \mathbb{M}_g(\overline{\mathbb{Q}})$, there is an i such that the fiber of $\pi|_{X_i} \colon X_i \to \mathbb{A}_g$ above $\tau(s)$ is a finite union of irreducible curves, up-to finitely many points one of these curves is $\mathfrak{C}_s - P_s$ with $P_s \in \mathfrak{C}_s(\overline{\mathbb{Q}})$.

We have a closed immersion $\mathfrak{A}_g \to \mathbb{P}_{\overline{\mathbb{Q}}}^n \times \mathbb{A}_g$. Moreover, a sufficiently large positive power of $\mathcal{M}$ induces a closed immersion of $\overline{\mathbb{A}_g} \to \mathbb{P}_{\overline{\mathbb{Q}}}^m$ for some m. Thus, we consider $\mathbb{A}_g$ as a Zariski locally closed subset $\mathbb{P}_{\overline{\mathbb{Q}}}^m$. We identify each X_i with its image in $\mathbb{P}_{\overline{\mathbb{Q}}}^n \times \mathbb{P}_{\overline{\mathbb{Q}}}^m$, an irreducible Zariski locally closed set. Then $\mathfrak{C}_s - P_s \subseteq \mathbb{P}_{\overline{\mathbb{Q}}}^n$ arises as an irreducible component of the intersection of some Zariski closure $\overline{X_i}$ with $\mathbb{P}_{\overline{\mathbb{Q}}}^n \times \{\tau(s)\}$.

We use the Segre embedding $\mathbb{P}_{\overline{\mathbb{Q}}}^n \times \mathbb{P}_{\overline{\mathbb{Q}}}^m \to \mathbb{P}_{\overline{\mathbb{Q}}}^{(n+1)(m+1)-1}$ to embed our situation into projective space. By Bézout's Theorem [Ful98, Example 8.4.6], $\deg(\mathfrak{C}_s - P_s)$ is bounded from above uniformly in s. Translating a curve inside $\mathfrak{A}_{g,\tau(s)}$ by a point of $\mathfrak{A}_{g,\tau(s)}(\overline{\mathbb{Q}})$ does not change its degree. So if $P \in \mathfrak{A}_{g,\tau(s)}$, then $\deg(\mathfrak{C}_s - P) = \deg(\mathfrak{C}_s - P_s)$. This yields (i).

Part (ii) follows as (i) but this time we use the Arithmetic Bézout Theorem, still executing the intersection after applying the Segre embedding. Indeed, recall that $\mathfrak{C}_s - P_s$ as an irreducible of the intersection of some $\overline{X_i}$ with $\mathbb{P}_{\overline{\mathbb{Q}}}^n \times \{\tau(s)\}$. The height and degree of $\overline{X_i}$ are bounded from above independently of s; the same holds for the degree of $\mathbb{P}_{\overline{\mathbb{Q}}}^n \times \{\tau(s)\}$. The height of $\mathbb{P}_{\overline{\mathbb{Q}}}^n \times \{\tau(s)\}$ is bounded from above linearly in terms of $h(\tau(s))$. Finally, we can apply [Phi95, Théorème 3]. Finally, note that by the Height Machine the absolute logarithmic Weil height $h(\tau(s))$, where $\tau(s)$ is understood as an element of $\mathbb{P}_{\overline{\mathbb{Q}}}^m(\overline{\mathbb{Q}})$, is bounded from above linearly in terms of $h_{\overline{\mathbb{A}_g}, \mathcal{M}}(\tau(s))$. $\qquad\square$

6.2. Non-degeneracy of $\mathscr{D}_M(\mathfrak{C}_S^{[M+1]})$ for large M.

In this subsection all varieties are defined over the field $\mathbb{C}$. We keep the notation of the previous subsection and let S be an irreducible variety with a quasi-finite morphism $S \to \mathbb{M}_g$. Note that $\mathscr{D}_M(\mathfrak{C}_S^{[M+1]})$ is Zariski closed in $\mathfrak{A}_g^{[M]} \times_{\mathbb{A}_g} S$ because $\mathscr{D}_M$ is proper. We endow this image with the reduced induced scheme structure.

The following non-degeneracy theorem proved by the second-named author is crucial to prove our main result. It confirms that Theorem 1.6 can be applied to $\mathscr{D}_M(\mathfrak{C}_S^{[M+1]})$ for $M \geq 3g - 2$. We obtain a height inequality on a Zariski open dense subset.

Theorem 6.2 ([Gao20a, Theorem 1.2']). *Let S be an irreducible variety with a (not necessarily dominant) quasi-finite morphism $S \to \mathbb{M}_g$. Assume $g \geq 2$ and $M \geq 3g - 2$. Then $\mathscr{D}_M(\mathfrak{C}_S^{[M+1]})$, which is a closed irreducible subvariety of $\mathfrak{A}_g^{[M]} \times_{\mathbb{A}_g} S$, is non-degenerate in the sense of Definition 1.5.*

The fibered product in the theorem involves $S \to \mathbb{M}_g \xrightarrow{\tau} \mathbb{A}_g$.
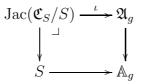
More precisely, the meaning of the conclusion of the theorem is as follows. For the abelian scheme $\pi\colon \mathcal{A} = \mathfrak{A}_g^{[M]} \times_{\mathbb{A}_g} S \to S$ and for the irreducible subvariety $X := \mathscr{D}_M(\mathfrak{C}_S^{[M+1]})$ of $\mathcal{A}$, there exists a open non-empty subset Δ of S^{an}, with the Betti map $b_\Delta\colon \mathcal{A}_\Delta = \pi^{-1}(\Delta) \to \mathbb{T}^{2g}$, such that

$$(6.6) \qquad \mathrm{rank}_\mathbb{R}(\mathrm{d}b_\Delta|_{X^{\mathrm{sm,an}}})_x = 2\dim X \quad \text{for some } x \in X^{\mathrm{sm,an}} \cap \mathcal{A}_\Delta,$$

when $g \geq 2$ and $M \geq 3g - 2$.

Proof. This theorem, essentially [Gao20a, Theorem 1.2'], is a consequence of Theorem 1.3 of *loc.cit.* Because of its importance to the current paper, we hereby give more details of the deduction.

We start by showing the result for the case where $\mathfrak{C}_S \to S$ admits a section ϵ. In this case ϵ induces an Abel–Jacobi embedding $j_\epsilon\colon \mathfrak{C}_S \to \mathrm{Jac}(\mathfrak{C}_S/S)$, which is a closed immersion of S-schemes. The modular map is the Cartesian diagram

$$
\begin{array}{ccc}
\mathrm{Jac}(\mathfrak{C}_S/S) & \xrightarrow{\ \iota\ } & \mathfrak{A}_g \\
\downarrow & \lrcorner & \downarrow \\
S & \longrightarrow & \mathbb{A}_g
\end{array}
$$

with the bottom morphism being the composite of the given $S \to \mathbb{M}_g$ with the Torelli map $\tau\colon \mathbb{M}_g \to \mathbb{A}_g$. The Torelli map τ is quasi-finite; see [OS80, Lemma 1.11]. Thus the bottom morphism is quasi-finite. Hence ι is quasi-finite.

We wish to apply [Gao20a, Theorem 1.3] to the subvariety $j_\epsilon(\mathfrak{C}_S)$ of the abelian scheme $\mathrm{Jac}(\mathfrak{C}_S/S) \to S$. We need to verify the hypotheses. First of all $\iota|_{j_\epsilon(\mathfrak{C}_S)}$ is generically finite because ι is quasi-finite. Hypothesis (a) is satisfied since $\dim j_\epsilon(\mathfrak{C}_S) = \dim S + 1 > \dim S$. For hypotheses (b) and (c), note that for any $s \in S(\mathbb{C})$, the fiber $j_\epsilon(\mathfrak{C}_S)_s$ is the Abel–Jacobi embedding of $\mathfrak{C}_s$ in its Jacobian via the point $\epsilon(s)$. Thus hypothesis (b) is satisfied because each curve generates its Jacobian, and hypothesis (c) holds true since $g \geq 2$.

Thus we can apply [Gao20a, Theorem 1.3.(ii)] and obtain that $\mathscr{D}_M(\mathfrak{C}_S^{[M+1]})$ is non-degenerate[2] if $M \geq j_\epsilon(\mathfrak{C}_S) = \dim S + 1$. But $\dim S \leq \dim \mathbb{M}_g = 3g - 3$. Hence $\mathscr{D}_M(\mathfrak{C}_S^{[M+1]})$ is non-degenerate if $M \geq 3g - 2$.

For an arbitrary S, the generic fiber of $\mathfrak{C}_S \to S$ has a rational point over some finite extension of $K(S)$, the function field of S. Thus there exists a quasi-finite étale dominant (not necessarily surjective) morphism $\rho\colon S' \to S$, with S' irreducible, such that $\mathfrak{C}_{S'} = \mathfrak{C}_S \times_S S' \to S'$ admits a section. Thus $X' := \mathscr{D}_M(\mathfrak{C}_{S'}^{[M+1]})$, as a subvariety of $\mathcal{A}' := \mathfrak{A}_g^{[M]} \times_{\mathbb{A}_g} S'$, is non-degenerate by the previous case. So there exists a connected, open non-empty subset Δ' of S'^{an}, with the Betti map $b_{\Delta'}\colon \mathcal{A}'_{\Delta'} \to \mathbb{T}^{2g}$, such that for some $x' \in X'^{\mathrm{sm,an}} \cap \mathcal{A}'_{\Delta'}$ we have

$$\mathrm{rank}_\mathbb{R}(\mathrm{d}b_{\Delta'}|_{X'^{\mathrm{sm,an}}})_{x'} = 2\dim X'.$$

We may furthermore shrink Δ' so that $\rho|_{\Delta'}$ is a diffeomorphism. In particular $\Delta := \rho(\Delta')$ is open in S^{an}.

---

[2]Observe that $\mathscr{D}_M(\mathfrak{C}_S^{[M+1]}) = \mathscr{D}_M^\mathcal{A}(j_\epsilon(\mathfrak{C}_S^{[M+1]}))$, with $\mathscr{D}_M^\mathcal{A}$ be as in [Gao20a, Theorem 1.3.(ii)] with $\mathcal{A} = \mathfrak{A}_g \times_{\mathbb{A}_g} S \cong \mathrm{Jac}(\mathfrak{C}_S/S)$. See below (6.3).

Denote by $\rho'_{\mathcal{A}} \colon \mathcal{A}' = \mathcal{A} \times_S S' \to \mathcal{A}$ the projection to the first factor. Then $\rho'_{\mathcal{A}}|_{\mathcal{A}'_{\Delta'}}$ is a diffeomorphism. Both $\mathcal{A} \to S$ and $\mathcal{A}' \to S'$ carry level-ℓ-structures. By construction and uniqueness properties of the Betti map, we may assume that $b_\Delta \colon \mathcal{A}_\Delta \to \mathbb{T}^{2g}$ equals $b_{\Delta'} \circ (\rho'_{\mathcal{A}}|_{\mathcal{A}'_{\Delta'}})^{-1}$. Thus

$$\mathrm{rank}_{\mathbb{R}}(\mathrm{d}b_\Delta|_{X^{\mathrm{sm,an}}})_x = 2 \dim X'$$

with $x = \rho_{\mathcal{A}}(x')$. So (6.6) holds true because $\dim X' = \dim X$. Hence we are done. $\square$

6.3. **Technical lemmas.** The following lemma will be useful in the proofs of the desired bounds. Let for the moment k be an algebraically closed field and $M \geq 1, n \geq 1$ integers. If Z is a Zariski closed subset of $(\mathbb{P}^n_k)^M$ we let $\deg Z$ denote the sum of the degrees of all irreducible components of Z with respect to $\mathcal{O}(1, \ldots, 1)$.

Lemma 6.3. *Let $C \subseteq \mathbb{P}^n_k$ be an irreducible curve defined over k and let $Z \subseteq (\mathbb{P}^n_k)^M$ be a Zariski closed subset of $(\mathbb{P}^n_k)^M$ such that $C^M = C \times \cdots \times C \not\subseteq Z$. Then there exists a number B, depending only on $M, \deg C$, and $\deg Z$, satisfying the following property. If $\Sigma \subseteq C(k)$ has cardinality $\geq B$, then $\Sigma^M = \Sigma \times \cdots \times \Sigma \not\subseteq Z(k)$.*

Proof. Let us prove this lemma by induction on M. The case $M = 1$ follows easily from Bézout's Theorem.

Assume the lemma is proved for $1, \ldots, M - 1$. Let $q \colon (\mathbb{P}^n_k)^M \to \mathbb{P}^n_k$ be the projection to the first factor.

The number of irreducible components of $Z \cap C^M$ and their degrees are bounded from above in terms of $M, \deg C$, and $\deg Z$ by Bézout's Theorem applied to the Segre embedding. Let Z' be the union of all irreducible components Y of $Z \cap C^M$ with $\dim q(Y) \geq 1$, let Z'' be the union of all other irreducible components.

Note that $q(Z') \subseteq C$. For all $P \in C(k)$ the fiber $q|_{Z'}^{-1}(P) = Z' \cap (\{P\} \times (\mathbb{P}^n_k)^{M-1})$ has dimension at most $\dim Z' - 1 \leq M - 2$. So the projection of $q|_{Z'}^{-1}(P)$ to the final factors $(\mathbb{P}^n_k)^{M-1}$ does not contain C^{M-1}. By Bézout's Theorem the degree of this projection is bounded in terms of $M, \deg C$, and $\deg Z$. We apply the induction hypothesis to the projection of $q|_Y^{-1}(P)$ to $(\mathbb{P}^n_k)^{M-1}$ and obtain a number B', depending only on $M, \deg C$, and $\deg Z$ satisfying the following property. If $\Sigma \subseteq C(k)$ has cardinality $\geq B'$, then $\{P\} \times \Sigma^{M-1} \not\subseteq Z'(k)$ for all $P \in C(k)$.

Now $\dim q(Z'') = 0$, so $q(Z'')$ is a finite set of cardinality at most B'', the number of irreducible components of $Z \cap C^M$.

The lemma follows with $B = \max\{B', B'' + 1\}$. $\square$

In the next lemma we use the Faltings–Zhang morphism in a single abelian variety A, i.e., $\mathscr{D}_M \colon A^{M+1} \to A^M$ defined by $(P_0, \ldots, P_M) \mapsto (P_1 - P_0, \ldots, P_M - P_0)$.

Lemma 6.4. *Let A be an abelian variety defined over $\overline{\mathbb{Q}}$ and suppose C is a smooth curve of genus $g \geq 2$ contained in A. If Z is an irreducible Zariski closed and proper subset of $\mathscr{D}_M(C^{M+1})$, then*

$$\#\{P \in C(\overline{\mathbb{Q}}) : (C - P)^M \subseteq Z\} \leq 84(g - 1).$$

Proof. For simplicity denote by $\Xi = \{P \in C(\overline{\mathbb{Q}}) : (C - P)^M \subseteq Z\}$. Fix $P_0 \in \Xi$. It suffices to prove that there are only $84(g - 1)$ possibilities for $P_1 - P_0$ when P_1 runs over Ξ.

Say $P_1 \in \Xi$ and let $i \in \{0, 1\}$. Note that $Z \subsetneq \mathscr{D}_M(C^{M+1})$ and so $\dim Z < \dim \mathscr{D}_M(C^{M+1}) \leq M+1$, as $\mathscr{D}_M(C^{M+1})$ irreducible. Note that $(C - P_i)^M \subseteq Z$, and so both have dimension M. As Z is irreducible we find $(C - P_i)^M = Z$ for $i = 0$ and $i = 1$.

Applying the first projection $A^M \to A$ yields $C - P_1 = C - P_0$. In other words, $P_1 - P_0$ stabilizes C. By Hurwitz's Theorem [Hur92], a smooth curve over $\overline{\mathbb{Q}}$ of genus $g \geq 2$ has at most $84(g-1)$ automorphisms. Hence we are done. $\qquad\square$

7. Néron–Tate distance between points on curves

The goal of this section is to prove Proposition 7.1, below. Namely we will show that $\overline{\mathbb{Q}}$-points on smooth curves are rather "sparse", in the sense that the Néron–Tate distance between two $\overline{\mathbb{Q}}$-points on a smooth curve C is in general large compared with the Weil height of C.

We use the notation from §6.1. Recall that we have fixed a projective compactification $\overline{\mathbb{A}_g}$ of $\mathbb{A}_g$ over $\overline{\mathbb{Q}}$, an ample line bundle $\mathcal{M}$, and a height function $h_{\overline{\mathbb{A}_g}, \mathcal{M}} \colon \mathbb{A}_g(\overline{\mathbb{Q}}) \to \mathbb{R}$ attached to this pair. We also fixed a closed immersion $\mathfrak{A}_g$ into $\mathbb{P}^n_{\overline{\mathbb{Q}}} \times \mathbb{A}_g$ over $\mathbb{A}_g$ and let $\tau \colon \mathbb{M}_g \to \mathbb{A}_g$ denote the Torelli morphism. If $s \in \mathbb{M}_g(\overline{\mathbb{Q}})$ then $\mathfrak{C}_s$ is a smooth curve of genus g defined over $\overline{\mathbb{Q}}$. Moreover, if $P, Q \in \mathfrak{C}_s(\overline{\mathbb{Q}})$, then $P - Q$ is a well-defined element of $\mathfrak{A}_g(\overline{\mathbb{Q}})$ and so is its Néron–Tate height $\hat{h}(P - Q)$, see (6.2).

Proposition 7.1. *Let S be an irreducible closed subvariety of $\mathbb{M}_g$ defined over $\overline{\mathbb{Q}}$. There exist positive constants c_1, c_2, c_3, c_4 depending on the choices made above and on S with the following property. Let $s \in S(\overline{\mathbb{Q}})$ with $h_{\overline{\mathbb{A}_g}, \mathcal{M}}(\tau(s)) \geq c_1$. There exists a subset $\Xi_s \subseteq \mathfrak{C}_s(\overline{\mathbb{Q}})$ with $\#\Xi_s \leq c_2$ such that any $P \in \mathfrak{C}_s(\overline{\mathbb{Q}})$ satisfies the following alternative.*

  (i) *Either $P \in \Xi_s$;*
  (ii) *or $\#\{Q \in \mathfrak{C}_s(\overline{\mathbb{Q}}) : \hat{h}(Q - P) \leq h_{\overline{\mathbb{A}_g}, \mathcal{M}}(\tau(s))/c_3\} < c_4$.*

Proof. We fix an immersion of $\mathbb{M}_g$ into some projective space and let $\overline{\mathbb{M}_g}$ denote its Zariski closure. By a standard triangle inequality estimate, there exist constants $c'' > 0$ and $c''' \geq 0$ such that

$$(7.1) \qquad\qquad h(s) \geq c'' h_{\overline{\mathbb{A}_g}, \mathcal{M}}(\tau(s)) - c'''$$

for all $s \in \mathbb{M}_g(\overline{\mathbb{Q}})$; see [Sil11, Lemma 4], the left-hand side is the Weil height and represents a height function coming from an ample line bundle on $\overline{\mathbb{M}_g}$. If $h_{\overline{\mathbb{A}_g}, \mathcal{M}}(\tau(s)) \geq c_1$ with $c_1 \geq 2c'''/c''$ then $h(s) \geq c'' h_{\overline{\mathbb{A}_g}, \mathcal{M}}(\tau(s))/2$. We find that it suffices to prove the alternative with $h_{\overline{\mathbb{A}_g}, \mathcal{M}}(\tau(s))$ replaced by $h(s)$ in (ii) and adjusting c_3. Our proof is by induction on $\dim S$.

If $\dim S = 0$, then the proposition follows by enlarging c_1.

If $\dim S \geq 1$, we fix $M = 3g - 2$. Applying Theorem 6.2 to the immersion $S^{\mathrm{sm}} \hookrightarrow \mathbb{M}_g$, we conclude that the closed irreducible subvariety $X := \mathscr{D}_M(\mathfrak{C}_{S^{\mathrm{sm}}}^{[M+1]})$ of the abelian scheme $\mathcal{A} = \mathfrak{A}_g^{[M]} \times_{\mathbb{A}_g} S^{\mathrm{sm}} \to S^{\mathrm{sm}}$ is non-degenerate. Hence we can apply Theorem 1.6 to $\mathcal{A}$ and X (and the compactification $\overline{S}$ is the Zariski closure of S in $\overline{\mathbb{M}_g}$). So, combined with (7.1), there exist constants $c > 0$ and c' as well as a Zariski open dense subset U of X, satisfying the following property. For all $s \in S(\overline{\mathbb{Q}})$ and all $P, Q_1, \ldots, Q_M \in \mathfrak{C}_s(\overline{\mathbb{Q}})$,

we have

(7.2) $\quad ch(s) \leq \hat{h}(Q_1 - P) + \cdots + \hat{h}(Q_M - P) + c'$ if $\quad (Q_1 - P, \ldots, Q_M - P) \in U(\overline{\mathbb{Q}}).$

Observe that $\pi(X) = S^{\mathrm{sm}}$, where $\pi \colon \mathcal{A} \to S^{\mathrm{sm}}$ is the structure morphism. Therefore, $S \setminus \pi(U)$ is not Zariski dense in S. Let $S_1, \ldots, S_r$ be the irreducible components of the Zariski closure of $S \setminus \pi(U)$ in S. Then $\dim S_j \leq \dim S - 1$ for all j.

By the induction hypothesis, this proposition holds for all S_j. Thus it remains to prove the conclusion of this proposition for curves above

(7.3) $$s \in S(\overline{\mathbb{Q}}) \setminus \bigcup_{j=1}^{r} S_j(\overline{\mathbb{Q}}) \subseteq \pi(U(\overline{\mathbb{Q}})).$$

First we construct Ξ_s and then we will show that we are in one of the two alternatives.

It is convenient to fix a base point $P_s \in \mathfrak{C}_s(\overline{\mathbb{Q}})$ and consider $(\mathfrak{C}_s - P_s)^M$ as a subvariety of $\mathcal{A}_s = \pi^{-1}(s)$.

Let us set $W = X \setminus U$, it is a Zariski closed and proper subset of X. By (7.3) we find $W_s \subsetneq X_s = \mathscr{D}_M(\mathfrak{C}_s^{[M+1]})$.

Let Z be an irreducible component of W_s. Consider the set

$$\Xi_Z := \{P \in \mathfrak{C}_s(\overline{\mathbb{Q}}) : (\mathfrak{C}_s - P)^M \subseteq Z\}.$$

Apply Lemma 6.4 to $A = (\mathfrak{A}_g)_{\tau(s)}, C = \mathfrak{C}_s - P_s \subseteq A$, and Z. As $Z \subsetneq \mathscr{D}_M(\mathfrak{C}_s^{[M+1]})$ we have $\#\Xi_Z \leq 84(g-1)$.

Let $\Xi_s = \bigcup_Z \Xi_Z$ where Z runs over all irreducible components of W_s. The number of irreducible components is bounded from above in an algebraic family. So the number of irreducible components of W_s is bounded from above by a number that is independent of s; but it may depend on W. We take c_2 to be such a number multiplied with $84(g-1)$. Thus $\#\Xi_s \leq c_2$ if (7.3) and with c_2 independent of s and P.

Say $P \in \mathfrak{C}_s(\overline{\mathbb{Q}})$ and $P \notin \Xi_s$. So we are not in case (i) of the proposition. Then $(\mathfrak{C}_s - P)^M \not\subseteq W_s$. We want to apply Lemma 6.3 to $\mathfrak{C}_s - P$ and W_s.

Recall that the abelian scheme $\mathfrak{A}_g$ is embedded in $\mathbb{P}_{\overline{\mathbb{Q}}}^n \times \mathbb{A}_g$ over $\mathbb{A}_g$, $cf.$ §6.1. So $\mathcal{A}$ is embedded in $(\mathbb{P}_{\overline{\mathbb{Q}}}^n)^M \times S^{\mathrm{sm}}$ over S^{sm}. We may identify $\mathfrak{C}_s - P$ with a smooth curve in $\mathbb{P}_{\overline{\mathbb{Q}}}^n$. The degree of $\mathfrak{C}_s - P$ as a subvariety of $\mathbb{P}_{\overline{\mathbb{Q}}}^n$ is bounded independently of s by Lemma 6.1(i); applying the Torelli morphism τ does not affect the degree. Moreover, W_s is Zariski closed in $X_s \subseteq \mathcal{A}_s$. Still holding s fixed we may take W_s as a Zariski closed subset of $(\mathbb{P}_{\overline{\mathbb{Q}}}^n)^M$. Being the fiber above s of a subvariety of $(\mathbb{P}_{\overline{\mathbb{Q}}}^n)^M \times S^{\mathrm{sm}}$, we find that the degree of W_s is bounded from above independently of s. From Lemma 6.3 we thus obtain a number c_4, depending only on these bounds and with the following property. Any subset $\Sigma \subseteq \mathfrak{C}_s(\overline{\mathbb{Q}})$ with cardinality $\geq c_4$ satisfies $(\Sigma - P)^M \not\subseteq W_s$. It is crucial that c_4 is independent of s.

We work with $\Sigma = \{Q \in \mathfrak{C}_s(\overline{\mathbb{Q}}) : \hat{h}(Q - P) \leq h(s)/c_3\}$ with $c_3 = 2M/c$. If $\#\Sigma < c_4$, then we are in alternative (ii) of the proposition.

Finally, let us assume $\#\Sigma \geq c_4$. The discussion above implies that there exist $Q_1, \ldots, Q_M \in \Sigma$ such that $(Q_1 - P, \ldots, Q_M - P) \notin W_s(\overline{\mathbb{Q}})$, $i.e.$, $(Q_1 - P, \ldots, Q_M - P) \in U(\overline{\mathbb{Q}})$. Thus we can apply (7.2) and obtain

$$h(s) \leq \frac{1}{c}\left(M\frac{ch(s)}{2M} + c'\right) = \frac{1}{2}h(s) + \frac{c'}{c}.$$

Hence $h(s) \leq 2c'/c$. Now (7.1) implies $h_{\overline{\mathbb{A}_g},\mathcal{M}}(\tau(s)) < c_1$ if $c_1 > (2c'/c + c''')/c''$. So this case cannot occur if $h_{\overline{\mathbb{A}_g},\mathcal{M}}(\tau(s)) \geq c_1$ and c_1 is sufficiently large. $\qquad\square$

8. Proof of Theorems 1.1, 1.2, and 1.4

The goal of this section is to prove the theorems and the corollary in the introduction. To this end let $g \geq 2$; we retain the notation of §6.1. In particular, $\pi \colon \mathfrak{A}_g \to \mathbb{A}_g$ is the universal family of principally polarized abelian varieties of dimension g with level-ℓ-structure where $\ell \geq 3$ and $\tau \colon \mathbb{M}_g \to \mathbb{A}_g$ is the Torelli morphism.

Proposition 8.1. *The exist constants $c_1 \geq 0, c_2 \geq 1$ depending on the choices made above with the following property. Let $s \in \mathbb{M}_g(\overline{\mathbb{Q}})$ with $h_{\overline{\mathbb{A}_g},\mathcal{M}}(\tau(s)) \geq c_1$. Suppose Γ is a finite rank subgroup of $\mathfrak{A}_{g,\tau(s)}(\overline{\mathbb{Q}})$ with rank $\rho \geq 0$. If $P_0 \in \mathfrak{C}_s(\overline{\mathbb{Q}})$, then*

$$\#(\mathfrak{C}_s(\overline{\mathbb{Q}}) - P_0) \cap \Gamma \leq c_2^{1+\rho}.$$

The proof combines Vojta's approach to the Mordell Conjecture with the results obtained in §7. We will use Rémond's quantitative version [Rém00a, Rém00b] of Vojta's method. A similar approach was used in the authors's earlier work [DGH19] which also contains a review of Vojta's method in §2. Let us recall the fundamental facts before proving Proposition 8.1.

Suppose we are given an abelian variety A of dimension g that is defined over $\overline{\mathbb{Q}}$ and is presented with a symmetric and very ample line bundle L. We assume also that we have a closed immersion of A into some projective space $\mathbb{P}^n_{\mathbb{Q}}$ determined by a basis of the global sections of L. We assume that A becomes a projectively normal subvariety of $\mathbb{P}^n_{\mathbb{Q}}$. This is the case if L is an at least fourth power of a symmetric and ample line bundle.

Suppose C is an irreducible curve in A. Then let $\deg C$ denote the degree of C considered as subvariety of $A \subseteq \mathbb{P}^n_{\mathbb{Q}}$, *i.e.*, $\deg C = (C.L)$. Moreover, let $h(C)$ denote the height of C.

On the ambient projective space we have the Weil height $h \colon \mathbb{P}^n_{\mathbb{Q}}(\overline{\mathbb{Q}}) \to [0, \infty)$. Tate's Limit Argument, compare (3.2), applied to h yields the Néron–Tate height $\hat{h}_L \colon A(\overline{\mathbb{Q}}) \to [0, \infty)$. It vanishes precisely on the points of finite order. Moreover, it follows from Tate's construction that there exists a constant $c_{\mathrm{NT}} \geq 0$, which depends on A, such that

$$(8.1) \qquad\qquad |\hat{h}_L(P) - h(P)| \leq c_{\mathrm{NT}}$$

for all $P \in A(\overline{\mathbb{Q}})$.

Finally, we need a measure for the heights of homogeneous polynomials that define the addition and substraction on A, as required in Rémond's [Rém00b]. Consider the $n+1$ global sections of $\mathcal{O}(1)$ corresponding to the projective coordinates of $\mathbb{P}^n_{\mathbb{Q}}$. They restrict to global sections $\xi_0, \ldots, \xi_n$ of L on A. Let $f \colon A \times A \to A \times A$ denote the morphism induced by $(P, Q) \mapsto (P + Q, P - Q)$, and let $p_1, p_2 \colon A \times A \to A$ be the first and section projection, respectively. For all $i, j \in \{0, \ldots, n\}$ there are $P_{ij} \in \overline{\mathbb{Q}}[\mathbf{X}, \mathbf{X}']$ with

$$(8.2) \qquad f^*(p_1^*\xi_i \otimes p_2^*\xi_j) = P_{ij}((p_1^*\xi_0, \ldots, p_1^*\xi_n), (p_2^*\xi_0, \ldots, p_2^*\xi_n))$$

and where P_{ij} is bihomogeneous of bidegree $(2,2)$ in $\mathbf{X} = (X_0, \ldots, X_n)$ and $\mathbf{X}' = (X_0', \ldots, X_n')$; see [Rém00b, Proposition 5.2] with $a = b = 1$ for the existence of the

P_{ij}. Here we require that $\xi_0, \ldots, \xi_n$ constitute a basis of $H^0(A, L)$. Let h_1 denote the Weil height of the point in projective space whose coordinates are all coefficients of all P_{ij}.

We point out a minor omission in [DGH19, §2]: h_1 there must also involve both addition and subtraction on A, and not just the addition.

The lemma below is [DGH19, Corollary 2.3] which is itself a standard application of Rémond's explicit formulation of the Vojta and Mumford inequalities. We thus obtain a bound that is exponential in the rank of the subgroup Γ for points of sufficiently large Néron–Tate height.

Lemma 8.2. *Let C be an irreducible curve in A. There exists a constant $c = c(n, \deg C) \geq 1$ depending only on n and $\deg C$ with the following property. Suppose Γ is a subgroup of $A(\overline{\mathbb{Q}})$ of finite rank $\rho \geq 0$. If C is not the translate of an algebraic subgroup of A, then*

$$\# \left\{ P \in C(\overline{\mathbb{Q}}) \cap \Gamma : \hat{h}_L(P) > c \max\{1, h(C), c_{\mathrm{NT}}, h_1\} \right\} \leq c^\rho.$$

Proof of Proposition 8.1. As in §6.1 we have a closed immersion $\mathfrak{A}_g \to \mathbb{P}^n_{\overline{\mathbb{Q}}} \times \mathbb{A}_g$ over $\mathbb{A}_g$. Let $s \in \mathbb{M}_g(\overline{\mathbb{Q}})$ with

$$(8.3) \qquad h_{\overline{\mathbb{A}_g}, \mathcal{M}}(\tau(s)) \geq \max\{1, c_1\},$$

where c_1 comes from Proposition 7.1 applied to $S = \mathbb{M}_g$.

We now bound two quantities attached to the abelian variety $A = \mathfrak{A}_{g, \tau(s)}$ taken with its closed immersion into $\mathbb{P}^n_{\overline{\mathbb{Q}}}$. Observe that this closed immersion satisfies the condition imposed at the beginning of this section with $L = \mathcal{L}|_A$ where $\mathcal{L}$ is as in §6.1. These quantities may depend on s. Below, $c > 0$ denotes a constant that depends on the fixed data such as g, n, and the ambient objects such as $\mathfrak{A}_g$ but not on s. We will increase c freely and without notice.

Bounding c_{NT}. For this we require the Silverman–Tate Theorem, Theorem A.1, applied to $\pi \colon \mathfrak{A}_g \to \mathbb{A}_g$. Recall that h is the Weil height on $\mathbb{P}^n_{\overline{\mathbb{Q}}}(\overline{\mathbb{Q}})$. For all $P \in A(\overline{\mathbb{Q}})$ we have $|h(P) - \hat{h}(P)| \leq c \max\{1, h_{\overline{\mathbb{A}_g}, \mathcal{M}}(\tau(s))\}$; note that we can bound $h_{\overline{s}}(\pi(P))$ from above linearly in terms of $h_{\overline{\mathbb{A}_g}, \mathcal{M}}(\tau(s))$ by the Height Machine. So we may take

$$(8.4) \qquad c_{\mathrm{NT}} = c \max\{1, h_{\overline{\mathbb{A}_g}, \mathcal{M}}(\tau(s))\}.$$

Bounding h_1. Recall that $f \colon A^2 \to A^2$ sends (P, Q) to $(P + Q, P - Q)$. We know that P_{ij} as above exist. Here we will construct such a family with controlled height. To this end we consider points $P = [\zeta_0 : \cdots : \zeta_n], Q = [\eta_0 : \cdots : \eta_n] \in A(\overline{\mathbb{Q}})$. Then $f(P, Q) = ([\nu_0^+ : \cdots : \nu_n^+], [\nu_0^- : \cdots : \nu_n^-])$. Recall that $A = \mathfrak{A}_{g, \tau(s)}$ is presented as a projectively normal subvariety of $\mathbb{P}^n_{\overline{\mathbb{Q}}}$ by the construction in §6.1. By (8.2) there is for each $i, j \in \{0, \ldots, n\}$ a bihomogeneous polynomial P_{ij} of bidegree $(2, 2)$ that is independent of P and Q, with

$$(8.5) \qquad \nu_i^+ \nu_j^- = \lambda P_{ij}((\zeta_0, \ldots, \zeta_n), (\eta_0, \ldots, \eta_n))$$

for some non-zero $\lambda \in \overline{\mathbb{Q}}$ that may depend on (P, Q). We eliminate λ and consider

$$(8.6) \quad \nu_i^+ \nu_j^- P_{i'j'}((\zeta_0, \ldots, \zeta_n), (\eta_0, \ldots, \eta_n)) - \nu_{i'}^+ \nu_{j'}^- P_{ij}((\zeta_0, \ldots, \zeta_n), (\eta_0, \ldots, \eta_n)) = 0$$

as a system of homogeneous linear equations parametrized by $(i, j), (i', j') \in \{0, \ldots, n\}^2$, the unknowns are the coefficients of the P_{ij}. As each P_{ij} is bihomogeneous of bidegree $(2, 2)$, the number of unknowns is $N = (n + 1)^2 \binom{n+2}{2}^2$ which is independent of s.

Each pair of points $(P, Q) \in A(\overline{\mathbb{Q}})^2$ yields one system of linear equations. We know that there is a non-trivial solution $(P_{ij})_{ij}$ that solves for all (P, Q) simultaneously and such that some P_{ij} does not vanish identically on $A \times A$. Our goal is to find such a common solution of controlled height.

First, observe that a common solution for when (P, Q) runs over all torsion points of $A^2(\overline{\mathbb{Q}})$ is a common solution for all pairs (P, Q). Indeed, this follows as torsion points of $A^2(\overline{\mathbb{Q}})$ lie Zariski dense in A^2. Second, observe that the full system has finite rank $M < N$ so it suffices to consider only finite many torsion points (P, Q).

Our task is thus to find a common solution to (8.6) for all $(i, j), (i', j')$, where some P_{ij} does not vanish identically on $A \times A$, and where $[\zeta_0 : \cdots : \zeta_n], [\eta_0 : \cdots : \eta_n]$, and $[\nu_0^{\pm} : \cdots : \nu_n^{\pm}]$ are certain torsion points on $A(\overline{\mathbb{Q}})$. We may assume that some ζ_i is 1 and similarly for η_i and $\nu_i^{\pm}$. So all coordinates are in $\overline{\mathbb{Q}}$. Moreover, the height of each torsion point is at most c_{NT} by (8.1). The resulting system of linear equations is represented by an $M \times N$ matrix with algebraic coefficients. By elementary properties of the height, each coefficient in the system has affine Weil height $c \cdot c_{\mathrm{NT}}$, for c large enough. It is tempting, but unnecessary, to invoke Siegel's Lemma to find a non-trivial solution. As M, N are bounded in terms of n, Cramer's Rule establishes the existence of a basis of non-zero solution such that the Weil height of the coefficient vector is at most $c \cdot c_{\mathrm{NT}}$. Among this basis there is one solution where one P_{ij} does not vanish identically on $A \times A$. By (8.4) we find

$$(8.7) \qquad h_1 \leq c \max\{1, h_{\overline{\mathbb{A}_g}, \mathcal{M}}(\tau(s))\}$$

for the projective height of the tuple $(P_{ij})_{ij}$. Thus (8.6) holds and so we get (8.5) on all of $A^2(\overline{\mathbb{Q}})$, at least with λ a rational function on $A \times A$ that is not identically zero. But λ cannot vanish anywhere on $A \times A$, as otherwise the left-hand side of (8.5) would vanish at some point of $A \times A$ for all (i, j). Hence λ is a non-zero constant. Replacing P_{ij} by λP_{ij} does not change the projective height; we get (8.2) with the desired bound for h_1.

Bounding height and degree of a curve. By Lemma 6.1 we have

$$(8.8) \qquad \deg(\mathfrak{C}_s - P_s) \leq c \quad \text{and} \quad h(\mathfrak{C}_s - P_s) \leq c \max\{1, h_{\overline{\mathbb{A}_g}, \mathcal{M}}(\tau(s))\}$$

for some $P_s \in \mathfrak{C}_s(\overline{\mathbb{Q}})$.

We now follow the argumentation in [DGH19]. Let Γ be a subgroup of $\mathfrak{A}_{g,\tau(s)}(\overline{\mathbb{Q}})$ for finite rank ρ. We first prove the proposition in the case $P_0 = P_s$. We apply Lemma 8.2 to the curve $C = \mathfrak{C}_s - P_s \subseteq \mathfrak{A}_{g,\tau(s)} = A$ and use the bounds (8.4), (8.7), and (8.8). Note that C is a smooth curve of genus $g \geq 2$. So it cannot be the translate of an algebraic subgroup of A. It follows that the number of points $P \in \mathfrak{C}_s(\overline{\mathbb{Q}})$ with $P - P_s \in \Gamma$ and $\hat{h}(P - P_s) > R^2$ where

$$(8.9) \qquad R = (c \max\{1, h_{\overline{\mathbb{A}_g}, \mathcal{M}}(\tau(s))\})^{1/2}$$

is at most $c^{\rho} \leq c^{1+\rho}$.

The burden of this paper is to find a bound of the same quality for the number of pairwise distinct points $P_1, P_2, P_3, \ldots$ in $\mathfrak{C}_s(\overline{\mathbb{Q}})$ with $\hat{h}(P_i - P_s) \leq R^2$. This is where Proposition 7.1 enters. Recall our assumption (8.3) on s. Let c_2' be the constant c_2 from Proposition 7.1; it is independent of s. As $\#\Xi_s \leq c_2'$ we may assume $P_i \notin \Xi_s$ for all i. So we may assume that each P_i is in the second alternative of Proposition 7.1.

As in [DGH19] we use the Euclidean norm $|\cdot|$ defined by $\hat{h}^{1/2}$ on the ρ-dimensional $\mathbb{R}$-vector space $\Gamma \otimes \mathbb{R}$. Let $r \in (0, R]$. By an elementary ball packing argument, any subset of $\Gamma \otimes \mathbb{R}$ contained in a closed ball of radius R is covered by at most $(1 + 2R/r)^\rho$ closed balls of radius r centered at elements of the given subset; see [Rém00a, Lemme 6.1]. We apply this geometric argument to R as in (8.9) and to r, the positive square-root of $h_{\overline{\mathbb{A}_g}, \mathcal{M}}(\tau(s))/c_3 = \max\{1, h_{\overline{\mathbb{A}_g}, \mathcal{M}}(\tau(s))\}/c_3$. The contribution of the height $h_{\overline{\mathbb{A}_g}, \mathcal{M}}(\tau(s))$ cancels in the quotient R/r. We find $R/r \leq c$. So the number of balls in the covering is at most $c^{1+\rho}$.

By Proposition 7.1(ii) the number of the P_i's that map to a single closed ball of radius r is at most c_4. Thus after increasing c we find that $\#\{P_i \in \mathfrak{C}_s(\overline{\mathbb{Q}}) \cap \Gamma : \hat{h}(P_i - P_s) \leq R^2\} \leq c_4 c^{1+\rho}$, as desired. This completes the proof of the proposition in the case $P_0 = P_s$ for sufficiently large c_2.

The case of a general base point follows easily as our estimates depend only on the rank ρ of Γ. Indeed, let $P_0 \in \mathfrak{C}_s(\overline{\mathbb{Q}})$ be an arbitrary point and let Γ' be the subgroup of $\mathfrak{A}_{g,\tau(s)}(\overline{\mathbb{Q}})$ generated by Γ and $P_0 - P_s$. Its rank is at most $\rho + 1$.

Now if $Q \in \mathfrak{C}_s(\overline{\mathbb{Q}}) - P_0$ lies in Γ, then $Q + P_0 - P_s \in \mathfrak{C}_s(\overline{\mathbb{Q}}) - P_s$ lies in Γ'. The number of such Q is at most $c_2^{2+\rho}$ by what we already proved. The proposition follows as $c_2^{2+\rho} \leq (c_2^2)^{1+\rho}$ and since we may replace c_2 by c_2^2. $\square$

Proof of Theorem 1.1. It is possible to deduce Theorem 1.1 from Theorem 1.2, which we prove below. However in view of the importance of Theorem 1.1, we hereby give it a complete proof.

This proof works for any level $\ell \geq 3$, but we may fix $\ell = 3$ for definiteness. Let $\mathbb{A}_g, \overline{\mathbb{A}_g}, \mathcal{M}$, and $h_{\overline{\mathbb{A}_g}, \mathcal{M}}$ be as in §6.1.

Our curve C corresponds to an F-rational point s_F of $\mathbb{M}_{g,1}$, the coarse moduli space of smooth genus g curves without level structure.

The fine moduli space $\mathbb{M}_g$ of smooth genus g curves with level-ℓ-structure is a finite cover of $\mathbb{M}_{g,1}$. For this proof it is convenient to recall that $\mathbb{M}_g$ is defined over the cyclotomic field generated by a third root of unity; recall the convention that we fixed a third root of unity and that $\mathbb{M}_g$ is geometrically irreducible. Say $s \in \mathbb{M}_g(\overline{\mathbb{Q}})$ maps to s_F. Then $F' = F(s)$ is a number field and $[F' : F]$ is bounded above only in terms of g and ℓ. We may identify $C_{F'} = C \otimes_F F'$ with $\mathfrak{C}_s$, the fiber of $\mathfrak{C}_g \to \mathbb{M}_g$ above s.

Constructing the Jacobian commutes with finite field extension. We thus view $\Gamma = \mathrm{Jac}(C)(F)$ as a subgroup of $\mathrm{Jac}(C)(\overline{\mathbb{Q}}) = \mathrm{Jac}(C_{F'})(\overline{\mathbb{Q}})$.

To prove the theorem we may assume $C(F) \neq \emptyset$. So fix $P_0 \in C(F)$. We consider the Abel–Jacobi embedding $C - P_0 \subseteq \mathrm{Jac}(C)$ defined over F. Then $\#C(F) \leq \#(C_{F'}(\overline{\mathbb{Q}}) - P_0) \cap \Gamma = \#(\mathfrak{C}_s(\overline{\mathbb{Q}}) - P_0) \cap \Gamma$. If $h_{\overline{\mathbb{A}_g}, \mathcal{M}}(\tau(s)) \geq c_1$, the theorem follows from Proposition 8.1. Note that in this case, the constant c in (1.1) is independent of d.

So we may assume that the height of $\tau(s)$ is less than c_1. As $[F' : \mathbb{Q}] \leq [F' : F][F : \mathbb{Q}] \leq [F' : F]d$, Northcott's Theorem implies that $\tau(s)$ comes from a finite set in $\mathbb{A}_g(\overline{\mathbb{Q}})$

that depends only on g, d, and ℓ. The same holds for s and thus $F' = F(s)$ since the Torelli morphism τ is finite-to-1 and thus has fibers of bounded cardinality. This means that the remaining C are twists in finitely many F'-isomorphism classes. But then it suffices to apply Rémond's estimate [DP02, page 643] to a single $C_{F'}$ and use $\#C(F) \leq \#(C_{F'}(\overline{\mathbb{Q}}) - P_0) \cap \Gamma$ to conclude the theorem.

Silverman's older result [Sil93, Theorem 1] also handles uniformity among twists. $\square$

Let us explain how to obtain some extra uniformity in the second case of the proof. More precisely, we show that the constant $c(g, d)$ in (1.1) grows polynomially in d. We retain the above proof's notation.

Denote by $\rho \colon \mathbb{A}_g = \mathbb{A}_{g,\ell} \to \mathbb{A}_{g,1}$ the natural morphism to the coarse moduli space which forgets the level structure. We recall that $\overline{\mathbb{A}_g}$ is presented as a closed subvariety of projective space induced by a basis of global sections of a positive powers of ample line bundle $\mathcal{M}$. Let $\iota \colon \mathbb{A}_{g,1} \to \mathbb{P}_{\mathbb{Q}}^m$ be an immersion, as before Theorem 1.2. By [Sil11, Lemma 4], there exist $c' > 0$ and $c'' \geq 0$ depending on the immersions such that $h(\iota(\rho(t))) \leq c' h_{\overline{\mathbb{A}_g}, \mathcal{M}}(t) + c''$ for all $t \in \mathbb{A}_g(\overline{\mathbb{Q}})$. So $h(\iota(\rho(\tau(s)))) \leq c' c_1 + c''$ is bounded uniformly in the second case.

By fundamental work of Faltings [Fal83, §3 including the proof of Lemma 3], see also [FC90, the remarks below Proposition V.4.4 and Proposition V.4.5], the *stable Faltings height* of $\mathfrak{A}_{g,\tau(s)}$ is bounded from above in terms of $c' c_1 + c''$ and g only. The height $h_{\mathrm{DP}}(\mathfrak{A}_{g,\tau(s)})$ used by David and Philippon is bounded similarly by work of Bost and David, see [DP02, Corollaire 6.9] and [Paz12].

In Rémond's bound [DP02, page 643] for $\#(C_{F'}(\overline{\mathbb{Q}}) - P_0) \cap \Gamma$, the base in the exponential depends polynomially on $D \max\{1, h_{\mathrm{DP}}(\mathfrak{A}_{g,\tau(s)})\}$, where D is the degree over $\mathbb{Q}$ of a suitable field of definition of $\mathfrak{A}_{g,\tau(s)}$. As this abelian variety can be defined over F' we may assume $D \leq [F' : F]d$ is bounded linearly in d. Recall that $\deg(C_{F'} - P_0)$ is bounded from above uniformly. So Rémond's bound implies that $c(g, d)$ in (1.1) can be chosen to grow at most polynomially in d.

The definition of $h_{\mathrm{DP}}(\mathfrak{A}_{g,\tau(s)})$ involves theta functions and a different kind of level structure. Using standard results on heights and by going down and up in the level structure it is likely that one can bound $h_{\mathrm{DP}}(\mathfrak{A}_{g,\tau(s)})$ from above directly in terms of $h_{\overline{\mathbb{A}_g}, \mathcal{M}}(\tau(s))$. For this one would need to work with a different level ℓ in the proof of Theorem 1.1.

Proof of Theorem 1.2. We keep the same notation as in the proof of Theorem 1.1. So $\ell = 3$ and $\mathbb{A}_g, \overline{\mathbb{A}_g}, \mathcal{M}$, and $h_{\overline{\mathbb{A}_g}, \mathcal{M}}$ are as in §6.1.

Let C be a smooth curve of genus $g \geq 2$ defined over $\overline{\mathbb{Q}}$, and let Γ be a finite rank subgroup of $\mathrm{Jac}(C)(\overline{\mathbb{Q}})$. Let $P_0 \in C(\overline{\mathbb{Q}})$.

The curve C corresponds to a $\overline{\mathbb{Q}}$-point s_{c} of $\mathbb{M}_{g,1}$.

The fine moduli space $\mathbb{M}_g$ of smooth genus g curves with level-ℓ-structure is a finite covering of $\mathbb{M}_{g,1}$. So there exists an $s \in \mathbb{M}_g(\overline{\mathbb{Q}})$ that maps to s_{c}. Thus C is isomorphic, over $\overline{\mathbb{Q}}$, to the fiber $\mathfrak{C}_s$ of the universal family $\mathfrak{C}_g \to \mathbb{M}_g$. We thus view Γ as a finite rank subgroup of $\mathrm{Jac}(\mathfrak{C}_s)(\overline{\mathbb{Q}})$, and $P_0 \in \mathfrak{C}_s(\overline{\mathbb{Q}})$.

Consider the Abel–Jacobi embedding $C - P_0 \subseteq \mathrm{Jac}(C)$. Then $\#(C(\overline{\mathbb{Q}}) - P_0) \cap \Gamma = \#(\mathfrak{C}_s(\overline{\mathbb{Q}}) - P_0) \cap \Gamma$. If $h_{\overline{\mathbb{A}_g}, \mathcal{M}}(\tau(s)) \geq c_1$, then $\#(C(\overline{\mathbb{Q}}) - P_0) \cap \Gamma \leq c_2^{1+\rho}$ by Proposition 8.1.

Thus it suffices to find a constant $c_1' \geq 0$ that is independent of C and such that $h(\iota([\mathrm{Jac}(C)])) \geq c_1'$ implies $h_{\overline{\mathbb{A}_g,\mathcal{M}}}(\tau(s)) \geq c_1$.

As after the proof of Theorem 1.1 denote by $\rho\colon \mathbb{A}_g = \mathbb{A}_{g,\ell} \to \mathbb{A}_{g,1}$ the natural morphism. As in the proof of Theorem 1.2 we use $h(\iota(\rho(t))) \leq c' h_{\overline{\mathbb{A}_g,\mathcal{M}}}(t) + c''$ for all $t \in \mathbb{A}_g(\overline{\mathbb{Q}})$. The theorem follows since $\rho(\tau(s)) = [\mathrm{Jac}(C)]$. $\qquad\square$

Remark 8.3. *It is possible to prove Theorem 1.1 (without the dependency claims on $c(g,d)$) using Theorem 1.2. Let C be a smooth curve of genus $g \geq 2$ defined over a number field $F \subseteq \overline{\mathbb{Q}}$. Then by taking $\Gamma = \mathrm{Jac}(C)(F)$ in Theorem 1.2, we can conclude Theorem 1.1 if $h(\iota([\mathrm{Jac}(C)])) \geq c_1$. The case $h(\iota([\mathrm{Jac}(C)])) < c_1$ can be handled as in the proof of Theorem 1.1, and one can furthermore obtain extra uniformity for c_2 in Theorem 1.2 by applying Rémond's bound [DP02, page 643] as after the proof of Theorem 1.1.*

Proof of Theorem 1.4. Let C be a smooth curve of genus $g \geq 2$ defined over a number field $F \subseteq \overline{\mathbb{Q}}$.

Apply Theorem 1.2 to $C_{\overline{\mathbb{Q}}}$, $P_0 \in C(\overline{\mathbb{Q}})$ and $\Gamma = \mathrm{Jac}(C)(\overline{\mathbb{Q}})_{\mathrm{tors}}$, whose rank is 0. Then we obtain $c_1 \geq 0$ and $c_2 \geq 1$ such that

$$\#(C(\overline{\mathbb{Q}}) - P_0) \cap \mathrm{Jac}(C)(\overline{\mathbb{Q}})_{\mathrm{tors}} \leq c_2$$

if $h(\iota([\mathrm{Jac}(C_{\overline{\mathbb{Q}}})])) \geq c_1$.

By the Northcott property and Torelli's Theorem, there are up-to $\overline{\mathbb{Q}}$-isomophism only finitely many $C_{\overline{\mathbb{Q}}}$'s defined over a number field F with $[F : \mathbb{Q}] \leq d$ such that $h(\iota([\mathrm{Jac}(C_{\overline{\mathbb{Q}}})])) < c_1$. By applying Raynaud's result on the Manin–Mumford Conjecture to each one of these finitely many curves separately, we obtain Theorem 1.4. $\qquad\square$

APPENDIX A. THE SILVERMAN–TATE THEOREM REVISITED

Our goal in this appendix is to present a treatment of the Silverman–Tate Theorem, [Sil83, Theorem A], using the language of Cartier divisors. Using Cartier divisors as opposed to Weil divisors allows us to relax the flatness hypotheses imposed on π in the notation of [Sil83, §3]. Apart from this minor tweak we closely follow the original argument presented by Silverman.

Suppose S is a regular, irreducible, quasi-projective variety over $\overline{\mathbb{Q}}$. Let $\pi\colon \mathcal{A} \to S$ be an abelian scheme. We write η for the generic point of S and $\mathcal{A}_\eta$ for the generic fiber of π. Then $\mathcal{A}_\eta$ is an abelian variety defined over $\overline{\mathbb{Q}}(\eta)$.

Suppose we are presented with a closed immersion $\mathcal{A} \to \mathbb{P}^n_{\overline{\mathbb{Q}}} \times S$ over S and with a projective variety $\overline{S}$ containing S as a Zariski open and dense subset. We will assume that $\overline{S}$ is embedded into $\mathbb{P}^m_{\overline{\mathbb{Q}}}$. We do not assume that $\overline{S}$ is regular.

We identify $\mathcal{A}$ with a subvariety of $\mathbb{P}^n_{\overline{\mathbb{Q}}} \times S$. Moreover, let $\overline{\mathcal{A}}$ denote the Zariski closure of $\mathcal{A}$ in $\mathbb{P}^n_S = \mathbb{P}^n_{\overline{\mathbb{Q}}} \times \overline{S} \subseteq \mathbb{P}^n_{\overline{\mathbb{Q}}} \times \mathbb{P}^m_{\overline{\mathbb{Q}}}$.

We set $\overline{\mathcal{L}} = \mathcal{O}(1,1)|_{\overline{\mathcal{A}}}$ and $\mathcal{L} = \overline{\mathcal{L}}|_{\mathcal{A}}$. We will assume in addition that $[-1]^*\mathcal{L}_\eta \cong \mathcal{L}_\eta$ where $\mathcal{L}_\eta$ is the restriction of $\mathcal{L}$ to $\mathcal{A}_\eta$. This implies $[2]^*\mathcal{L}_\eta \cong \mathcal{L}_\eta^{\otimes 4}$.

Given these immersions, we have several height functions. For $(P,s) \in \overline{\mathcal{A}}(\overline{\mathbb{Q}}) \subseteq \mathbb{P}^n_{\overline{\mathbb{Q}}}(\overline{\mathbb{Q}}) \times \mathbb{P}^m_{\overline{\mathbb{Q}}}(\overline{\mathbb{Q}})$ we define $h(P,s) = h(P) + h(s)$ using the Weil height. Moreover, for

$s \in \overline{S}(\overline{\mathbb{Q}}) \subseteq \mathbb{P}^m_{\mathbb{Q}}(\overline{\mathbb{Q}})$ we define $h_{\overline{S}}(s) = h(s)$. Finally, for all $P \in \mathcal{A}(\overline{\mathbb{Q}})$ we denote by

$$\hat{h}_{\mathcal{A}}(P) = \lim_{N \to \infty} \frac{h([N](P))}{N^2}$$

the Néron–Tate height with respect to $\mathcal{L}$; it is well-known that the limit converges, *cf.* the reference around (3.2).

We will prove the following variant of the Silverman–Tate Theorem.

Theorem A.1. *There exists a constant $c > 0$ such that for all $P \in \mathcal{A}(\overline{\mathbb{Q}})$ we have*

$$\left| \hat{h}_{\mathcal{A}}(P) - h(P) \right| \leq c \max\{1, h_{\overline{S}}(\pi(P))\}.$$

The constant c depends on $\mathcal{A}$ and on the various immersions but not on P. The proof is distributed over the next subsections.

A.1. **Extending multiplication-by-2.** We keep the notation from the previous subsection. We have constructed a (very naive) projective model $\overline{\mathcal{A}}$ of $\mathcal{A}$. Note that $\overline{\mathcal{A}}$ and $\overline{S}$ may fail to be regular. Moreover, the natural morphism $\overline{\mathcal{A}} \to \overline{S}$, which we also denote by π, may fail to be smooth or even flat.

Multiplication-by-2 is a morphism $[2]: \mathcal{A} \to \mathcal{A}$ that extends to a rational map $\overline{\mathcal{A}} \dashrightarrow \overline{\mathcal{A}}$. We consider the graph of $[2]$ on $\mathcal{A}$ as a subvariety of $\mathcal{A} \times_S \mathcal{A}$. Let $\overline{\mathcal{A}}'$ be the Zariski closure of this graph inside $\overline{\mathcal{A}} \times_{\overline{S}} \overline{\mathcal{A}}$. Write $\rho: \overline{\mathcal{A}}' \to \overline{\mathcal{A}}$ for the restriction of the projection onto the first factor and $[2]$ for the restriction onto the second factor. We may identify $\mathcal{A}$ with a Zariski open subset of $\overline{\mathcal{A}}'$. Under this identification, ρ restricts to the identity on $\mathcal{A}$ and $[2]$ restricts to multiplication-by-2 on $\mathcal{A}$.

The following diagram commutes

$$
\begin{array}{ccccc}
\overline{\mathcal{A}} & \xleftarrow{\;\rho\;} & \overline{\mathcal{A}}' & \xrightarrow{\;[2]\;} & \overline{\mathcal{A}} \\
\cup| & & \cup| & & \cup| \\
\mathcal{A} & =\!=\!= & \mathcal{A} & \xrightarrow{\;[2]\;} & \mathcal{A}
\end{array}
$$

where the first and third inclusions are equal and the middle one comes from the identification involved in the graph construction.

A.2. **Proof of the Silverman–Tate Theorem.** We keep the notation from the previous subsection.

Proposition A.2. *There exists a constant $c_1 > 0$ such that*

(A.1) $$|h([2](P)) - 4h(P)| \leq c_1 \max\{1, h_{\overline{S}}(\pi(P))\}$$

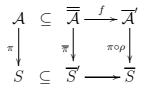*holds for all $P \in \mathcal{A}(\overline{\mathbb{Q}})$.*

Proof. We define

(A.2) $$\mathcal{F}' = [2]^* \overline{\mathcal{L}} \otimes \rho^* \overline{\mathcal{L}}^{\otimes(-4)} \in \operatorname{Pic}(\overline{\mathcal{A}}').$$

Recall that we have identified $\mathcal{A}$ with a Zariski open subset of $\overline{\mathcal{A}}'$. The restriction of $[2]^* \overline{\mathcal{L}}$ to the generic fiber $\mathcal{A}_\eta \subseteq \mathcal{A} \subseteq \overline{\mathcal{A}}'$ coincides with $[2]^* \mathcal{L}_\eta$ and the restriction of $\rho^* \overline{\mathcal{L}}$ to $\mathcal{A}_\eta$ is identified with $\mathcal{L}_\eta$. Using our assumption $[2]^* \mathcal{L}_\eta \cong \mathcal{L}_\eta^{\otimes 4}$ on the generic fiber $\mathcal{A}_\eta$ we see that $\mathcal{F}'$ is trivial on $\mathcal{A}_\eta$.

By [Gro67, Corollaire 21.4.13 (pp. 361 of EGA IV-4, in Errata et Addenda, liste 3)] applied to $\mathcal{A} \to S$ there exists a line bundle $\mathcal{M}$ on S such that $\pi|_{\mathcal{A}}^* \mathcal{M} \cong \mathcal{F}'|_{\mathcal{A}}$.

Let us first desingularize the compactified base $\overline{S}$ by applying Hironaka's Theorem. Thus there is a proper, birational morphism $b \colon \overline{S}' \to \overline{S}$ that is an isomorphism above S such that $\overline{S}'$ is regular. We consider S as Zariski open in $\overline{S}'$. Note that b is even projective and $\overline{S}'$ is integral. So $\overline{S}'$ is an irreducible, regular, projective variety.

Now consider the base change $\overline{\mathcal{A}}' \times_{\overline{S}} \overline{S}'$. This new scheme may fail to be irreducible or even reduced. However, recall that b is an isomorphism above the regular $S \subseteq \overline{S}$. So $(\overline{\mathcal{A}}' \times_{\overline{S}} \overline{S}')_S = \overline{\mathcal{A}}' \times_{\overline{S}} S$ is isomorphic to $\mathcal{A}$ and thus integral. We may consider $\mathcal{A}$ as an open subscheme of $\overline{\mathcal{A}}' \times_{\overline{S}} \overline{S}'$. It must be contained in an irreducible component of $\overline{\mathcal{A}}' \times_{\overline{S}} \overline{S}'$. We endow this irreducible component with the reduced induced structure and obtain an integral, closed subscheme $\overline{\overline{\mathcal{A}}} \subseteq \overline{\mathcal{A}}' \times_{\overline{S}} \overline{S}'$. We get a commutative diagram

$$
\begin{array}{ccccc}
\mathcal{A} & \subseteq & \overline{\overline{\mathcal{A}}} & \xrightarrow{\ f\ } & \overline{\mathcal{A}}' \\
{\scriptstyle \pi} \downarrow & & {\scriptstyle \overline{\pi}} \downarrow & & \downarrow {\scriptstyle \pi \circ \rho} \\
S & \subseteq & \overline{S}' & \longrightarrow & \overline{S}
\end{array}
$$

The horizontal morphisms compose to the identity on the domain.

We consider S as a Zariski open subset of $\overline{S}'$. As $\overline{S}'$ is regular, we can extend $\mathcal{M}$ to a line bundle on the regular $\overline{S}'$, cf. [GW10, Corollary 11.41]. The pull-back $f^* \mathcal{F}' \otimes \overline{\pi}^* \mathcal{M}^{\otimes(-1)}$ is trivial on $\mathcal{A} \subseteq \overline{\overline{\mathcal{A}}}$.

By Hironaka's Theorem there is a proper, birational morphism $\beta \colon \widetilde{\mathcal{A}} \to \overline{\overline{\mathcal{A}}}$ that is an isomorphism above $\mathcal{A}$ (which is regular) such that $\widetilde{\mathcal{A}}$ is regular. We may identify $\mathcal{A}$ with a Zariski open subset of $\widetilde{\mathcal{A}}$.

Now we pull everything back to the regular $\widetilde{\mathcal{A}}$. More precisely, we set $\mathcal{F} = \beta^* f^* \mathcal{F}'$. Then $\mathcal{F} \otimes \beta^* \overline{\pi}^* \mathcal{M}^{\otimes(-1)}$ is trivial when restricted to $\mathcal{A}$.

To a Cartier divisor D we attach its line bundle $\mathcal{O}(D)$. As $\widetilde{\mathcal{A}}$ is integral we may fix a Cartier divisor D on $\widetilde{\mathcal{A}}$ with $\mathcal{O}(D) \cong \mathcal{F} \otimes \beta^* \overline{\pi}^* \mathcal{M}^{\otimes(-1)}$. Let $\mathrm{cyc}(D)$ denote the Weil divisor of $\widetilde{\mathcal{A}}$ attached to D. The linear equivalence class of $\mathrm{cyc}(D)$ restricted to $\mathcal{A}$ is trivial. By [GW10, Proposition 11.40] $\mathrm{cyc}(D)$ is linearly equivalent to a Weil divisor $\sum_{i=1}^r n_i Z_i$ with $Z_i \subseteq \widetilde{\mathcal{A}} \setminus \mathcal{A}$ irreducible and of codimension 1 in $\widetilde{\mathcal{A}}$.

We let $\widetilde{\pi}$ denote the composition $\widetilde{\mathcal{A}} \to \overline{\overline{\mathcal{A}}} \to \overline{S}'$. Let us consider $\widetilde{\pi}(Z_i) = Y_i$. As $\widetilde{\pi}$ is proper, each Y_i is an irreducible closed subvariety of $\overline{S}'$. Moreover, $Y_i \subseteq \widetilde{\pi}(\widetilde{\mathcal{A}} \setminus \mathcal{A}) \subseteq \overline{S}' \setminus S$. So Y_i has dimension at most $\dim \overline{S}' - 1$. But Y_i could have codimension at least 2 and thus fail to be the support of a Weil divisor. On the regular $\overline{S}'$ a Cartier divisor is the same thing as a Weil divisor; see [GW10, Theorem 11.38(2)]. For each i we fix a Cartier divisor E_i of $\overline{S}'$ such that $\mathrm{cyc}(E_i)$ equals a prime Weil divisor supported on an irreducible subvariety containing Y_i. Since $\mathrm{cyc}(E_i)$ is effective, we find that E_i is effective, see [GW10, Theorem 11.38(1)] and its proof. An effective Cartier divisor and its image under the cycle map $\mathrm{cyc}(\cdot)$ have equal support. So the subscheme of $\overline{S}'$ attached to E_i contains Y_i.

The pull-back $\widetilde{\pi}^*E_i$ is well-defined as a Cartier divisor, we do not require that π is flat, *cf.* [GW10, Proposition 11.48(b)]. By [GW10, Corollary 11.49] the inverse image $\widetilde{\pi}^{-1}(E_i)$, taken as a subscheme of $\widetilde{\mathcal{A}}$ is the subscheme attached to $\widetilde{\pi}^*E_i$ and $\widetilde{\pi}^*E_i$ is effective.

Note that $\widetilde{\pi}^{-1}(E_i) \supseteq \widetilde{\pi}^{-1}(Y_i) \supseteq Z_i$. The support satisfies $\mathrm{Supp}(\widetilde{\pi}^*E_i) \supseteq Z_i$. Moreover, as $\widetilde{\pi}^*E_i$ is effective, $\mathrm{cyc}(\widetilde{\pi}^*E_i)$ is effective and $\mathrm{Supp}(\mathrm{cyc}(\widetilde{\pi}^*E_i)) = \mathrm{Supp}(\widetilde{\pi}^*E_i)$. Thus

$$\pm \sum_{i=1}^{r} n_i Z_i \leq \mathrm{cyc}\left(\widetilde{\pi}^* \sum_{i=1}^{r} |n_i| E_i\right).$$

Recall that $\mathrm{cyc}(D) = \mathrm{cyc}(\mathrm{div}\phi) + \sum_{i=1}^{r} n_i Z_i$ for some rational function ϕ on $\widetilde{\mathcal{A}}$. Therefore,

$$0 \leq \mathrm{cyc}\left(\pm(D - \mathrm{div}\phi) + \widetilde{\pi}^* \sum_{i=1}^{r} |n_i| E_i\right).$$

Since $\widetilde{\mathcal{A}}$ is regular and in particular normal, we find that

(A.3) $$\pm(D - \mathrm{div}\phi) + \widetilde{\pi}^* \sum_{i=1}^{r} |n_i| E_i$$

is an effective Cartier divisor for both signs; see [GW10, Theorem 11.38(1)] and its proof. Moreover, its support equals the support of

$$0 \leq \mathrm{cyc}\left(\pm(D - \mathrm{div}\phi) + \widetilde{\pi}^* \sum_{i=1}^{r} |n_i| E_i\right) = \pm\mathrm{cyc}(D - \mathrm{div}\phi) + \sum_{i=1}^{r} |n_i|\mathrm{cyc}(\widetilde{\pi}^*E_i).$$

Thus the support of (A.3) lies in $\bigcup_{i=1}^{r} \mathrm{Supp}(\widetilde{\pi}^*E_i)$.

We apply $\mathcal{O}(\cdot)$ and pass again to line bundles. Let us denote $\mathcal{E} = \mathcal{O}(\sum_{i=1}^{r} |n_i| E_i)$, a line bundle on $\overline{S}'$. The line bundle attached to (A.3) is $(\mathcal{F} \otimes \beta^*\overline{\pi}^*\mathcal{M}^{\otimes(-1)})^{\otimes(\pm 1)} \otimes \widetilde{\pi}^*\mathcal{E}$. Since (A.3) is effective, both $(\mathcal{F} \otimes \beta^*\overline{\pi}^*\mathcal{M}^{\otimes(-1)})^{\otimes(\pm 1)} \otimes \widetilde{\pi}^*\mathcal{E}$ have a non-zero global section.

By the Height Machine this translates to

$$h_{\widetilde{\mathcal{A}}, (\mathcal{F} \otimes \beta^*\overline{\pi}^*\mathcal{M}^{\otimes(-1)})^{\otimes(\pm 1)} \otimes \widetilde{\pi}^*\mathcal{E}}(P) \geq O(1)$$

for all $\widetilde{P} \in \widetilde{\mathcal{A}}(\overline{\mathbb{Q}})$ with $\widetilde{\pi}(\widetilde{P}) \notin \bigcup_i \mathrm{Supp}(E_i)$. By functoriality properties of the Height Machine we obtain

$$|h_{\overline{\mathcal{A}}', \mathcal{F}'}(f(\beta(\widetilde{P})))| \leq h_{\overline{S}', \mathcal{E}}(\widetilde{\pi}(\widetilde{P})) + |h_{\overline{S}', \mathcal{M}}(\overline{\pi}(\beta(\widetilde{P})))| + O(1)$$

for the same $\widetilde{P}$. We recall (A.2) and again use the Height Machine to find

$$|h([2](P')) - 4h(\rho(P'))| \leq h_{\overline{S}', \mathcal{E}}(\widetilde{\pi}(\widetilde{P})) + |h_{\overline{S}', \mathcal{M}}(\widetilde{\pi}(\widetilde{P}))| + O(1)$$

where $P' = f(\beta(\widetilde{P}))$. Observe that all points of $\mathcal{A}(\overline{\mathbb{Q}})$ are in the image of $f \circ \beta$.

We recall that the desingularization morphism $\overline{S}' \to \overline{S}$ is an isomorphism above S and that we have identified $\mathcal{A}$ with a Zariski open subset of $\overline{\mathcal{A}}'$ and of $\overline{\mathcal{A}}$. Under these identifications and if P' corresponds to $P \in \mathcal{A}(\overline{\mathbb{Q}})$, then $[2](P')$ is the duplicate of P, $\rho(P') = P$, and $\widetilde{\pi}(\widetilde{P}) = \pi(\rho(P')) = \pi(P)$. We apply the Height Machine a final time and use that $h_{\overline{S}}$ arises from the Weil height restricted to $\overline{S}(\overline{\mathbb{Q}})$. We find

$$|h([2](P)) - 4h(P)| \leq c_1 \max\{1, h_{\overline{S}}(\pi(P))\}$$

for all $P \in \mathcal{A}(\overline{\mathbb{Q}})$ with $\widetilde{\pi}(\widetilde{P}) \notin \bigcup_i \mathrm{Supp}(E_i)$, under the identifications above.

Let $P \in \mathcal{A}(\overline{\mathbb{Q}})$. As the Y_i lie in $\widetilde{\pi}(\widetilde{\mathcal{A}} \setminus \mathcal{A})$ we can choose all E_i above to avoid $\pi(P)$. After doing this finitely often (using noetherian induction) and replacing the E_i from before and adjusting c_1, we find

$$|h([2](P)) - 4h(P)| \leq c_1 \max\{1, h_{\overline{S}}(\pi(P))\}$$

for all $P \in \mathcal{A}(\overline{\mathbb{Q}})$ where $c_1 > 0$ is independent of P. □

Proof of Theorem A.1. Having (A.1) at our disposal the proof follows a well-known argument. Indeed, say $l \geq k \geq 0$ are integers. Then applying the triangle inequality to the appropriate telescoping sum yields

$$\left| \frac{h([2^l](P))}{4^l} - \frac{h([2^k](P))}{4^k} \right| \leq \sum_{m=k}^{l-1} \left| \frac{h([2^{m+1}](P))}{4^{m+1}} - \frac{h([2^m](P))}{4^m} \right|$$

$$\leq \sum_{m=k}^{l-1} 4^{-(m+1)} \left| h([2^{m+1}](P)) - 4h([2^m](P)) \right|.$$

We apply (A.1) to $[2^m](P)$ and find that the sum is bounded by $c_1 x \sum_{m=k}^{l-1} 4^{-(m+1)} \leq c_1 x 4^{-k}$ where $x = \max\{1, h_{\overline{S}}(\pi(P))\}$. So $\left(h([2^l](P))/4^l\right)_{l \geq 1}$ is a Cauchy sequence with limit $\hat{h}_{\overline{\mathcal{A}}}(P)$. Taking $k = 0$ and $l \to \infty$ we obtain from the estimates above that $|\hat{h}_{\overline{\mathcal{A}}}(P) - h(P)| \leq c_1 x$, as desired. □

APPENDIX B. FULL VERSION OF THEOREM 1.6

The goal of this section is to prove the full version of Theorem 1.6, *i.e.*, without assuming (Hyp). Let S be an irreducible quasi-projective variety defined over $\overline{\mathbb{Q}}$ and let $\pi \colon \mathcal{A} \to S$ be an abelian scheme of relative dimension $g \geq 1$.

Let $\mathcal{L}$ be a relative ample line bundle on $\mathcal{A} \to S$ with $[-1]^*\mathcal{L} = \mathcal{L}$, and let $\mathcal{M}$ be an ample line bundle on a compactification $\overline{S}$ of S. All data above are assumed to be defined over $\overline{\mathbb{Q}}$. Set $\hat{h}_{\mathcal{A}, \mathcal{L}} \colon \mathcal{A}(\overline{\mathbb{Q}}) \to \mathbb{R}$ to be the fiberwise Néron–Tate height $\hat{h}_{\mathcal{A}, \mathcal{L}}(P) = \hat{h}_{\mathcal{A}_s, \mathcal{L}_s}(P)$ with $s = \pi(P)$, and $h_{\overline{S}, \mathcal{M}} \colon \overline{S}(\overline{\mathbb{Q}}) \to \mathbb{R}$ to be a representative of the height provided by the Height Machine; *cf.* [BG06, Chapter 2 and 9].

The main result of this appendix is the following theorem.

Theorem B.1. *Let X be an irreducible subvariety of $\mathcal{A}$ defined over $\overline{\mathbb{Q}}$. Suppose X is non-degenerate, as defined in Definition B.4. Then there exist constants $c_1 > 0$ and $c_2 \geq 0$ and a Zariski open dense subset U of X with*

$$(\text{B.1}) \qquad \hat{h}_{\mathcal{A}, \mathcal{L}}(P) \geq c_1 h_{\overline{S}, \mathcal{M}}(\pi(P)) - c_2 \quad \text{for all} \quad P \in U(\overline{\mathbb{Q}}).$$

Compared to Theorem 1.6, $\mathcal{A} \to S$ is no longer required to satisfy (Hyp). Other minor improvements are that S is not required to be regular and X is not required to be closed.

In Definition 1.5, we defined non-degenerate subvarieties using the generic rank of the Betti map if $\mathcal{A} \to S$ satisfies (Hyp). For an arbitrary $\mathcal{A} \to S$, the definition is similar. But we need to first of all extend our construction of the Betti map, Proposition 2.1, to an arbitrary $\mathcal{A} \to S$.

B.1. **Betti map.** In this subsection, we extend the construction of the Betti map, *i.e.*, Proposition 2.1, to an arbitrary $\mathcal{A} \to S$ with S regular. In this subsection, let S be an irreducible, regular, quasi-projective variety over $\mathbb{C}$ and let $\pi \colon \mathcal{A} \to S$ be an abelian scheme of relative dimension $g \geq 1$.

Proposition B.2. *Let $s_0 \in S(\mathbb{C})$. Then there exist an open neighborhood Δ of s_0 in S^{an}, and a map $b_\Delta \colon \mathcal{A}_\Delta := \pi^{-1}(\Delta) \to \mathbb{T}^{2g}$, called the Betti map, with the following properties.*

(i) *For each $s \in \Delta$ the restriction $b_\Delta|_{\mathcal{A}_s(\mathbb{C})} \colon \mathcal{A}_s(\mathbb{C}) \to \mathbb{T}^{2g}$ is a group isomorphism.*
(ii) *For each $\xi \in \mathbb{T}^{2g}$ the preimage $b_\Delta^{-1}(\xi)$ is a complex analytic subset of $\mathcal{A}_\Delta$.*
(iii) *The product $(b_\Delta, \pi) \colon \mathcal{A}_\Delta \to \mathbb{T}^{2g} \times \Delta$ is a real analytic isomorphism.*

Just as in the case of Proposition 2.1, the Betti map is uniquely determined by properties (i) and (iii) up-to the action of $\mathrm{GL}_{2g}(\mathbb{Z})$ if Δ is connected. Composing with an $\alpha \in \mathrm{GL}_{2g}(\mathbb{Z})$ does not change the rank. So by the discussion on the uniqueness above, any map $\mathcal{A}_\Delta \to \mathbb{T}^{2g}$ satisfying the three properties listed in Proposition B.2 will be called *Betti map*.

Proof. Our proof of Proposition B.2 follows the construction in [Gao20a, §3–§4]. We divide it into several steps.

By [GN09, §2.1], $\mathcal{A} \to S$ carries a polarization of type $D = \mathrm{diag}(d_1, \ldots, d_g)$ for some positive integers $d_1 | d_2 | \cdots | d_g$.

Case: Moduli space with level structure. Fix $\ell \geq 3$ with $(\ell, d_g) = 1$. We start by proving Proposition B.2 for $S = \mathbb{A}_{g,D,\ell}$, the moduli space of abelian varieties of dimension g polarized of type D with level-ℓ-structure. It is a fine moduli space; see [GN09, Theorem 2.3.1]. Let $\pi_D^{\mathrm{univ}} \colon \mathfrak{A}_{g,D,\ell} \to \mathbb{A}_{g,D,\ell}$ be the universal abelian variety.

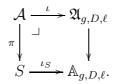The universal covering $\mathfrak{H}_g \to \mathbb{A}_{g,D,\ell}^{\mathrm{an}}$ [GN09, Proposition 1.3.2], where $\mathfrak{H}_g$ is the Siegel upper half space, gives a family of abelian varieties $\mathcal{A}_{\mathfrak{H}_g, D} \to \mathfrak{H}_g$ fitting into the diagram

$$
\begin{array}{ccc}
\mathcal{A}_{\mathfrak{H}_g, D} := \mathfrak{A}_{g,D,\ell} \times_{\mathbb{A}_{g,D,\ell}} \mathfrak{H}_g & \longrightarrow & \mathfrak{A}_{g,D,\ell}^{\mathrm{an}} \\
\downarrow & & \downarrow{\scriptstyle \pi_D^{\mathrm{univ}}} \\
\mathfrak{H}_g & \longrightarrow & \mathbb{A}_{g,D,\ell}^{\mathrm{an}}.
\end{array}
$$

The family $\mathcal{A}_{\mathfrak{H}_g, D} \to \mathfrak{H}_g$ is polarized of type D. For the universal covering $u \colon \mathbb{C}^g \times \mathfrak{H}_g \to \mathcal{A}_{\mathfrak{H}_g, D}$ and for each $Z \in \mathfrak{H}_g$, the kernel of $u|_{\mathbb{C}^g \times \{Z\}}$ is $D\mathbb{Z}^g + Z\mathbb{Z}^g$. Thus the map $\mathbb{C}^g \times \mathfrak{H}_g \to \mathbb{R}^g \times \mathbb{R}^g \times \mathfrak{H}_g \to \mathbb{R}^{2g}$, where the first map is the inverse of $(a, b, Z) \mapsto (Da + Zb, Z)$ and the second map is the natural projection, descends to a real analytic map

$$b^{\mathrm{univ}} \colon \mathcal{A}_{\mathfrak{H}_g, D} \to \mathbb{T}^{2g}.$$

Now for each $s_0 \in \mathbb{A}_{g,D,\ell}(\mathbb{C})$, there exists a contractible, relatively compact, open neighborhood Δ of s_0 in $\mathbb{A}_{g,D,\ell}^{\mathrm{an}}$ such that $\mathfrak{A}_{g,D,\ell,\Delta} := (\pi_D^{\mathrm{univ}})^{-1}(\Delta)$ can be identified with $\mathcal{A}_{\mathfrak{H}_g, \Delta'}$ for some open subset Δ' of $\mathfrak{H}_g$. The composite $b_\Delta \colon \mathfrak{A}_{g,D,\ell,\Delta} \cong \mathcal{A}_{\mathfrak{H}_g, D, \Delta'} \to \mathbb{T}^{2g}$ clearly satisfies the three properties listed in Proposition B.2. Thus b_Δ is the desired Betti map in this case.

Case: With level structure. Assume that $\mathcal{A} \to S$ carries level-ℓ-structure for some $\ell \geq 3$ with $(\ell, d_g) = 1$. As $\mathbb{A}_{g,D,\ell}$ is a fine moduli space there exists a Cartesian diagram

$$
\begin{array}{ccc}
\mathcal{A} & \xrightarrow{\iota} & \mathfrak{A}_{g,D,\ell} \\
\pi \downarrow & \lrcorner & \downarrow \\
S & \xrightarrow{\iota_S} & \mathbb{A}_{g,D,\ell}.
\end{array}
$$

Now let $s_0 \in S(\mathbb{C})$. Applying Proposition B.2 to the universal abelian variety $\mathfrak{A}_{g,D,\ell} \to \mathbb{A}_{g,D,\ell}$ and $\iota_S(s_0) \in \mathbb{A}_{g,D,\ell}(\mathbb{C})$, we obtain an open neighborhood Δ_0 of $\iota_S(s_0)$ in $\mathbb{A}_{g,D,\ell}^{\mathrm{an}}$ and a real analytic map

$$
b_{\Delta_0} : \mathfrak{A}_{g,\Delta_0} \to \mathbb{T}^{2g}
$$

satisfying the properties listed in Proposition B.2.

Now let $\Delta = \iota_S^{-1}(\Delta_0)$. Then Δ is an open neighborhood of s_0 in S^{an}. Denote by $\mathcal{A}_\Delta = \pi^{-1}(\Delta)$ and define

$$
b_\Delta = b_{\Delta_0} \circ \iota : \mathcal{A}_\Delta \to \mathbb{T}^{2g}.
$$

Then b_Δ satisfies the properties listed in Proposition B.2 for $\mathcal{A} \to S$. Hence b_Δ is our desired Betti map.

Case: General case. Let $s_0 \in S(\mathbb{C})$ and $\ell \geq 3$ be a prime with $(\ell, d_g) = 1$.

Fix any irreducible component S_0 of the kernel $\ker[\ell]$ of $[\ell] : \mathcal{A} \to \mathcal{A}$. It is Zariski open in $\ker[\ell]$ as S is regular, so we consider it with its natural open subscheme structure. Then $S_0 \to \ker[\ell]$ is both a closed and open immersion. So $S_0 \to S$, the composition with the finite étale morphism $\ker[\ell] \to S$, is finite and étale. The upshot is that the base change of $\mathcal{A} \to S$ by $S_0 \to S$ admits an ℓ-torsion section. After repeating this finitely many times we obtain a finite and étale morphism $\rho : S' \to S$ where S' is irreducible and such that $\mathcal{A}' := \mathcal{A} \times_S S' \to S'$ has level-ℓ-structure. Note that S' is regular as S is regular and regularity ascends along étale morphisms. Moreover, $\mathcal{A}' \to S'$ is still polarized of type D.

Let $s_0' \in \rho^{-1}(s_0)$. Applying Proposition B.2 to $\mathcal{A}' \to S'$ and $s_0' \in S'(\mathbb{C})$, we obtain an open neighborhood Δ' of s_0' in $(S')^{\mathrm{an}}$ and a map $b_{\Delta'} : \mathcal{A}'_{\Delta'} \to \mathbb{T}^{2g}$ satisfying the properties listed in Proposition B.2.

Let $\Delta = \rho(\Delta')$. Up to shrinking Δ', we may assume that $\rho|_{\Delta'} : \Delta' \to \Delta$ is a homeomorphism and that Δ is an open neighborhood of s_0 in S^{an}. Thus $\mathcal{A}'_{\Delta'} \cong \mathcal{A}_\Delta$. Now define

$$
b_\Delta : \mathcal{A}_\Delta \to \mathbb{T}^{2g}
$$

to be the composite of the inverse of $\mathcal{A}'_{\Delta'} \cong \mathcal{A}_\Delta$ and $b_{\Delta'}$. Then b_Δ is our desired Betti map. $\qquad\square$

Here is an easy property of the generic rank of the Betti map.

Lemma B.3. *Let $b_\Delta : \mathcal{A}_\Delta \to \mathbb{T}^{2g}$ be a Betti map as in Proposition B.2. Let X be an irreducible subvariety of $\mathcal{A}$ with $X^{\mathrm{an}} \cap \mathcal{A}_\Delta \neq \emptyset$. Let U be a Zariski open dense subset of X. Then*

$$
(B.2) \qquad \max_{x \in X^{\mathrm{sm}}(\mathbb{C}) \cap \mathcal{A}_\Delta} \mathrm{rank}_\mathbb{R}(\mathrm{d}b_\Delta|_{X^{\mathrm{sm,an}}})_x = \max_{u \in U^{\mathrm{sm}}(\mathbb{C}) \cap \mathcal{A}_\Delta} \mathrm{rank}_\mathbb{R}(\mathrm{d}b_\Delta|_{U^{\mathrm{sm,an}}})_u.
$$

Proof. The statement (B.2) is true on replacing "=" by "≥", as $X^{\mathrm{sm,an}} \supseteq U^{\mathrm{sm,an}}$.

For the converse inequality we set $\max_{x \in X^{\mathrm{sm,an}} \cap \mathcal{A}_\Delta} \mathrm{rank}_{\mathbb{R}}(\mathrm{d}b_\Delta|_{X^{\mathrm{sm,an}}})_x = r$ and pick $x \in X^{\mathrm{sm,an}} \cap \mathcal{A}_\Delta$ satisfying $\mathrm{rank}_{\mathbb{R}}(\mathrm{d}b_\Delta|_{X^{\mathrm{sm,an}}})_x = r$. Then there exists an open neighborhood V of x in $X^{\mathrm{sm,an}}$ such that $\mathrm{rank}_{\mathbb{R}}(\mathrm{d}b_\Delta|_{X^{\mathrm{sm,an}}})_u = r$ for all $u \in V$. But $U^{\mathrm{sm}}(\mathbb{C}) \cap V \neq \emptyset$ since $U^{\mathrm{sm}} \neq \emptyset$ is Zariski open in X and V is Zariski dense in X. Thus there exists a $u \in U^{\mathrm{sm}}(\mathbb{C}) \cap V$. Then we must have $\mathrm{rank}_{\mathbb{R}}(\mathrm{d}b_\Delta|_{U^{\mathrm{sm,an}}})_u = r$ and the lemma follows. $\square$

B.2. Non-degenerate subvariety and Theorem 1.6.

We keep the notation as in the beginning of this appendix.

Definition B.4. *An irreducible subvariety X of $\mathcal{A}$ is said to be non-degenerate if there exists an open non-empty subset Δ of $S^{\mathrm{sm,an}}$, with the Betti map $b_\Delta \colon \mathcal{A}_\Delta := \pi^{-1}(\Delta) \to \mathbb{T}^{2g}$ as in Proposition B.2, such that $X^{\mathrm{sm,an}} \cap \mathcal{A}_\Delta \neq \emptyset$ and*

$$\max_{x \in X^{\mathrm{sm}}(\mathbb{C}) \cap \mathcal{A}_\Delta} \mathrm{rank}_{\mathbb{R}}(\mathrm{d}b_\Delta|_{X^{\mathrm{sm,an}}})_x = 2 \dim X.$$

Now we are ready to prove Theorem B.1.

Proof of Theorem B.1. Let $\ell \geq 3$ be a prime. We will reduce the current theorem to Theorem 1.6 by successively assuming, in addition to the hypothesis of Theorem B.1, that

   (i) X is Zariski closed in $\mathcal{A}$,
   (ii) $\pi|_X \colon X \to S$ is dominant,
   (iii) S is regular,
   (iv) $\mathcal{A} \to S$ is S-isogenous to an abelian scheme which carries a principal polarization,
   (v) $\mathcal{A} \to S$ carries a principal polarization,
   (vi) $\mathcal{A}$ carries a level ℓ-structure, and
   (vii) we have the same hypothesis as Theorem 1.6.

We will proceed the proof with six dévissage steps. In dévissage step n we will deduce the theorem under the hypotheses (i),...,$(n-1)$ from the theorem under the hypotheses (i),...,(n).

First dévissage: reduction to the case where X is Zariski closed in $\mathcal{A}$.

Let $\overline{X}$ denote the Zariski closure of X in $\mathcal{A}$. Then X is a Zariski open dense subset of $\overline{X}$ and $\dim X = \dim \overline{X}$. Therefore, $\overline{X}$ is non-degenerate if X is non-degenerate. Now if (B.1) holds true on a Zariski open dense subset U of $\overline{X}$, then (B.1) clearly holds true on $U \cap X$, which is Zariski open and dense in X. Thus it suffices to prove (B.1) with X replaced by $\overline{X}$.

Second dévissage: reduction to the case where $\pi|_X \colon X \to S$ is dominant.

As X is non-degenerate, there exists a non-empty open subset Δ of $S^{\mathrm{sm,an}}$, with Betti map b_Δ, such that $\mathrm{rank}_{\mathbb{R}}(\mathrm{d}b_\Delta|_{X^{\mathrm{sm,an}}})_x = 2 \dim X$ for some $x \in X^{\mathrm{sm}}(\mathbb{C}) \cap \mathcal{A}_\Delta$.

Endow the Zariski closed set $S' = \pi(X)$ with the reduced induced subscheme structure and set $\mathcal{A}' = \mathcal{A} \times_S S' = \pi^{-1}(S')$. Then $X \times_S S'$ identifies with X via the natural projection $\mathcal{A}' \to \mathcal{A}$. Hence there exists a non-empty open subset Δ' of $(S')^{\mathrm{sm,an}}$ with $\pi(x) \in \Delta' \subseteq \Delta$ and

$$\mathrm{rank}_{\mathbb{R}}(\mathrm{d}b_{\Delta'}|_{X^{\mathrm{sm,an}}})_x = 2 \dim X.$$

Thus X is a non-degenerate subvariety of $\mathcal{A}'$. On the other hand, the conclusion of Theorem B.1 does not change with $\mathcal{A} \to S$ replaced by $\mathcal{A}' \to S'$, $\mathcal{L}$ replaced by $\mathcal{L}|_{\mathcal{A}'}$

and $\mathcal{M}$ replaced by $\mathcal{M}|_{\overline{S'}}$, where $\overline{S'}$ is the Zariski closure of S' in $\overline{S}$. Hence it suffices to prove Theorem B.1 after these replacements and thus we may assume that X dominates S.

Third dévissage: reduction to the case where S is regular.

Recall that S^{sm} is the regular locus of S. Now $\pi|_X \colon X \to S$ is dominant, so $X' = X \cap \pi^{-1}(S^{\mathrm{sm}})$ is Zariski open and dense in X. Since X is non-degenerate it follows by definition that X' is non-degenerate. Moreover, the conclusion of Theorem B.1 does not change if we replace $\mathcal{A} \to S$ by $\mathcal{A}' = \pi^{-1}(S^{\mathrm{sm}}) \to S^{\mathrm{sm}}$, $\mathcal{L}$ by $\mathcal{L}|_{\mathcal{A}'}$, and X by X'. Finally, observe that X' is Zariski closed in $\mathcal{A}'$ and $\pi(X') = \pi(X) \cap S^{\mathrm{reg}}$, so $\pi|_{X'} \colon X' \to S^{\mathrm{reg}}$ is dominant.

Fourth dévissage: reduction to the case where $\pi \colon \mathcal{A} \to S$ is S-isogenous to an abelian scheme which carries a principal polarization.

By [Mum74, §23, Corollary 1], each abelian variety over an algebraic closed field is isogenous to a principally polarized one. Applying this to the geometric generic fiber of $\mathcal{A} \to S$, we obtain a quasi-finite étale dominant morphism $\rho \colon S' \to S$ with S' irreducible and the following property: There exists a principally polarized A'_0 that is isogenous over $\overline{\mathbb{Q}}(S')$ to the generic fiber A' of $\mathcal{A}' := \mathcal{A} \times_S S' \to S'$. Up to replacing S' by an open dense subscheme, we may furthermore assume that A'_0 extends to an abelian scheme $\mathcal{A}'_0 \to S'$. Denote by $\rho_{\mathcal{A}} \colon \mathcal{A}' = \mathcal{A} \times_S S' \to \mathcal{A}$ the natural projection; it is a quasi-finite étale dominant morphism.

As regularity ascends along étale morphisms and as S is regular we conclude that S' is regular. Thus $\mathcal{A}'_0 \to S'$ carries a principal polarization by [Ray70, Théorème XI 1.4], and the isogeny $A'_0 \to A'$ extends to an S'-isogeny $\mathcal{A}'_0 \to \mathcal{A}'$ by [Ray70, Lemme XI 1.15].

There is an irreducible component X' of $\rho_{\mathcal{A}}^{-1}(X)$ with $\dim X' = \dim X$. Then X' is Zariski closed in $\mathcal{A}'$, the image $\rho_{\mathcal{A}}(X')$ is Zariski dense in X, and thus X' dominates S' (it even surjects to S' since $\mathcal{A}' \to S'$ is proper and X' is closed). We claim that X', as a subvariety of $\mathcal{A}'$, is non-degenerate. Indeed, $\rho_{\mathcal{A}}(X')$ contains a Zariski open dense subset U of X. Since X is a non-degenerate subvariety of $\mathcal{A}$, so is U by Lemma B.3. So there exists an open subset Δ of S^{an} with the Betti map $b_\Delta \colon \mathcal{A}_\Delta \to \mathbb{T}^{2g}$ such that $\mathrm{rank}_{\mathbb{R}}(db_\Delta|_{U^{\mathrm{an,sm}}})_u = 2 \dim U = 2 \dim X$ for all u from a non-empty open subset of U^{an}. Take Δ' to be a connected component of $\rho^{-1}(\Delta)$ such that $X' \cap (\pi')^{-1}(\Delta') \neq \emptyset$. Set $\mathcal{A}'_{\Delta'} = (\pi')^{-1}(\Delta')$, and replace Δ by $\rho(\Delta')$. Note that $\rho|_{\Delta'} \colon \Delta' \cong \Delta$ is then bianalytic after possibly shrinking Δ' (and so is $\rho_{\mathcal{A}} \colon \mathcal{A}'_{\Delta'} \cong \mathcal{A}_\Delta$). Now $b_\Delta \circ \rho_{\mathcal{A}}|_{\mathcal{A}'_{\Delta'}} \colon \mathcal{A}'_{\Delta'} \to \mathbb{T}^{2g}$ satisfies the three properties listed in Proposition B.2. So $b_\Delta \circ \rho_{\mathcal{A}}|_{\mathcal{A}'_{\Delta'}}$ is the Betti map, which we denote for simplicity by $b_{\Delta'}$; see below Proposition B.2. For $u' \in (\rho_{\mathcal{A}}|_{\mathcal{A}'_{\Delta'}})^{-1}(u) \cap X'^{\mathrm{an}}$ and for sufficiently general u, we have $\mathrm{rank}_{\mathbb{R}}(db_{\Delta'}|_{X'^{\mathrm{an,sm}}})_{u'} = 2 \dim X$. So X', as a subvariety of the abelian scheme $\mathcal{A}'$ over S', is non-degenerate.

Now we have a non-degenerate subvariety X' of the abelian scheme $\pi' \colon \mathcal{A}' \to S'$. The line bundle $\rho_{\mathcal{A}}^* \mathcal{L}$ on $\mathcal{A}'$ is relatively ample. Suppose that $\mathcal{M}'$ is an ample line bundle on some compactification $\overline{S'}$ of S'.

Assume that Theorem B.1 holds for $\pi' \colon \mathcal{A}' \to S'$, $\rho_{\mathcal{A}}^* \mathcal{L}$, $\mathcal{M}'$, and X'. Thus there exist constants $c'_1 > 0$, $c'_2 \geq 0$ and a Zariski open non-empty subset U' of X' with

$$\hat{h}_{\mathcal{A}', \rho_{\mathcal{A}}^* \mathcal{L}}(P') \geq c'_1 h_{\overline{S'}, \mathcal{M}'}(\pi'(P')) - c'_2 \quad \text{for all} \quad P' \in U'(\overline{\mathbb{Q}}).$$

Denote by $P = \rho_{\mathcal{A}}(P')$. By the Height Machine we have $\hat{h}_{\mathcal{A}', \rho_{\mathcal{A}}^* \mathcal{L}}(P') = \hat{h}_{\mathcal{A}, \mathcal{L}}(P)$.

By [Sil11, Lemma 4] applied to $\rho\colon S' \to S$ and the line bundles $\mathcal{M}'$ and $\mathcal{M}$, there exist $c' = c'(\rho, \mathcal{M}', \mathcal{M}) > 0$ and $c'' = c''(\rho, \mathcal{M}', \mathcal{M}) \geq 0$ such that $h_{\overline{S}', \mathcal{M}'}(\pi'(P')) \geq c'h_{\overline{S}, \mathcal{M}}(\rho(\pi'(P'))) - c'' = c'h_{\overline{S}, \mathcal{M}}(\pi(P)) - c''$ for all $P' \in \mathcal{A}'(\overline{\mathbb{Q}})$. Hence the height inequality above implies

$$\hat{h}_{\mathcal{A}, \mathcal{L}}(P) \geq c_1'c'h_{\overline{S}, \mathcal{M}}(\pi(P)) - (c_1'c'' + c_2') \quad \text{for all} \quad P \in \rho_{\mathcal{A}}(U')(\overline{\mathbb{Q}}).$$

Now that $\rho_{\mathcal{A}}(U')$ contains a Zariski open non-empty (hence dense) subset U of X by Chevalley's Theorem. Thus Theorem B.1 also holds true for $\pi\colon \mathcal{A} \to S$, $\mathcal{L}$, $\mathcal{M}$, and X.

In summary, we have shown that it suffices to prove Theorem B.1 for $\pi'\colon \mathcal{A}' \to S'$, $\rho_{\mathcal{A}}^*\mathcal{L}$, $\mathcal{M}'$ and X'. Thus we are reduced to the case where the generic fiber of $\mathcal{A} \to S$ is isogenous to a principally polarized abelian variety.

Fifth dévissage: reduction to the case where $\pi\colon \mathcal{A} \to S$ carries a principal polarization.

From the previous dévissage, there exists a principally polarized abelian scheme $\pi_0\colon \mathcal{A}_0 \to S$ with an S-isogeny $\lambda\colon \mathcal{A}_0 \to \mathcal{A}$. Note that λ is a finite étale morphism. The line bundle $\lambda^*\mathcal{L}$ on $\mathcal{A}_0$ is relatively ample. By the Height Machine we have $\hat{h}_{\mathcal{A}_0, \lambda^*\mathcal{L}}(P') = \hat{h}_{\mathcal{A}, \mathcal{L}}(\lambda(P'))$ for all $P' \in \mathcal{A}_0(\overline{\mathbb{Q}})$.

There is an irreducible component X_0 of $\lambda^{-1}(X)$ with $\dim X_0 = \dim X$. Then X_0 is Zariski closed in $\mathcal{A}_0$ and thus X_0 dominates S (it even surjects to S since $X = \lambda(X_0)$). We claim that X_0, as a subvariety of $\mathcal{A}_0$, is non-degenerate. Assume this. Then it suffices to prove the height inequality (B.1) with $\mathcal{A} \to S$ replaced by $\mathcal{A}_0 \to S$, X replaced by X_0, and $\mathcal{L}$ replaced by $\lambda^*\mathcal{L}$.

It remains to prove that X_0 is a non-degenerate subvariety of $\mathcal{A}_0$. To do this, we need some preparation on Betti maps. Let Δ be an open subset of S^{an} with the Betti map $b_\Delta\colon \mathcal{A}_\Delta \to \mathbb{T}^{2g}$. Set $\mathcal{A}_{0,\Delta} = \pi_0^{-1}(\Delta)$, and denote by λ_Δ the restriction of $\lambda\colon \mathcal{A}_0 \to \mathcal{A}$ to $\mathcal{A}_{0,\Delta}$. Up to shrinking Δ we have a Betti map $b_{0,\Delta}\colon \mathcal{A}_{0,\Delta} \to \mathbb{T}^{2g}$. By property (iii) of Proposition B.2, we have two real analytic isomorphisms $(b_{0,\Delta}, \pi_0)\colon \mathcal{A}_{0,\Delta} \cong \mathbb{T}^{2g} \times \Delta$ and $(b_\Delta, \pi)\colon \mathcal{A}_\Delta \cong \mathbb{T}^{2g} \times \Delta$. Thus there exists a real analytic map $\lambda'\colon \mathbb{T}^{2g} \times \Delta \to \mathbb{T}^{2g} \times \Delta$ such that the following diagram commutes

$$
\begin{array}{ccc}
\mathcal{A}_{0,\Delta} & \xrightarrow[\sim]{(b_{0,\Delta}, \pi_0)} & \mathbb{T}^{2g} \times \Delta \\
\lambda_\Delta \downarrow & & \downarrow \lambda' \\
\mathcal{A}_\Delta & \xrightarrow[\sim]{(b_\Delta, \pi)} & \mathbb{T}^{2g} \times \Delta.
\end{array}
$$

As λ is a finite map, $(\lambda')^{-1}(r)$ is a finite set for each $r \in \mathbb{T}^{2g} \times \Delta$. As λ is an S-morphism, for each $s \in \Delta$ we have $\lambda'(\mathbb{T}^{2g} \times \{s\}) \subseteq \mathbb{T}^{2g} \times \{s\}$.

By property (i) of Proposition B.2, for each $s \in \Delta$ the restriction $\lambda'|_{\mathbb{T}^{2g} \times \{s\}}$ is a group homomorphism $\mathbb{T}^{2g} \to \mathbb{T}^{2g}$. Thus $\ker(\lambda'|_{\mathbb{T}^{2g} \times \{s\}})$ is a finite, hence discrete, subgroup of $\mathbb{T}^{2g}$. In particular, $\ker(\lambda'|_{\mathbb{T}^{2g} \times \{s\}})$ is locally constant. Up to shrinking Δ, we may assume $\ker(\lambda'|_{\mathbb{T}^{2g} \times \{s\}}) = H$ for each $s \in \Delta$. Set $\lambda_{\mathbb{T}}\colon \mathbb{T}^{2g} \to \mathbb{T}^{2g}$ the quotient by the finite subgroup H. Then the diagram above induces a commutative diagram

$$
\text{(B.3)} \qquad
\begin{array}{ccc}
\mathcal{A}_{0,\Delta} & \xrightarrow{b_{0,\Delta}} & \mathbb{T}^{2g} \\
\lambda_\Delta \downarrow & & \downarrow \lambda_{\mathbb{T}} \\
\mathcal{A}_\Delta & \xrightarrow{b_\Delta} & \mathbb{T}^{2g}.
\end{array}
$$

Note that $\lambda_{\mathbb{T}}$ is a local homeomorphism.

Now we turn to proving that X_0 is non-degenerate in $\mathcal{A}_0$. Indeed, as X is a non-degenerate subvariety of $\mathcal{A}$, there exists an open subset Δ of S^{an} with the Betti map $b_\Delta\colon \mathcal{A}_\Delta \to \mathbb{T}^{2g}$ such that $\mathrm{rank}_{\mathbb{R}}(db_\Delta|_{X^{\mathrm{an,sm}}})_x = 2\dim X$ for all x from a non-empty open subset of X^{an}. For $x_0 \in \lambda^{-1}(x) \cap X_0^{\mathrm{an}}$ and for sufficiently general x, we have $\mathrm{rank}_{\mathbb{R}}(d(\lambda_{\mathbb{T}} \circ b_{0,\Delta})|_{X_0^{\mathrm{an,sm}}})_{x_0} = 2\dim X = 2\dim X_0$. But $\lambda_{\mathbb{T}}$ is a local homeomorphism, so $\mathrm{rank}_{\mathbb{R}}(db_{0,\Delta}|_{X_0^{\mathrm{an,sm}}})_{x_0} = 2\dim X_0$. Thus X_0 is non-degenerate.

Sixth dévissage: reduction to the case where $\mathcal{A}/S$ carries level ℓ structure.

As in the treatment of the general case in the proof of Proposition B.2 there exists a finite and étale morphism $S' \to S$ where S' is regular and irreducible such that $\mathcal{A}' := \mathcal{A} \times_S S'$ carries level ℓ-structure.

Denote by $\rho_{\mathcal{A}}\colon \mathcal{A}' \to \mathcal{A}$ the natural projection. By a similar argument as the fourth dévissage step, it suffices to prove the height inequality (B.1) with $\mathcal{A} \to S$ replaced by $\mathcal{A}' \to S'$, X replaced by an irreducible component of $\rho_{\mathcal{A}}^{-1}(X)$ with $\dim X' = \dim X$, and $\mathcal{L}$ replaced by $\rho_{\mathcal{A}}^*\mathcal{L}$, and $\mathcal{M}$ replaced by an ample line bundle on some compactification of S'. As in the fourth dévissage X' dominates S'. Finally, $\mathcal{A}'/S'$ still carries a principal polarization.

Seventh dévissage: reduction to Theorem 1.6.

It remains to prove the height inequality (B.1) with the extra hypotheses (i) - (v) listed above using Theorem 1.6. In this theorem we assumed in addition that the fiberwise Néron–Tate height on $\mathcal{A}(\overline{\mathbb{Q}})$ is induced by a closed immersion $\mathcal{A} \to \mathbb{P}^n_{\overline{\mathbb{Q}}} \times S$ satisfying the second and third bullet at the beginning of §3 and that the height on $\overline{S}(\overline{\mathbb{Q}})$ is the restriction of the absolute logarthmic Weil height coming from a closed immersion $\overline{S} \to \mathbb{P}^m_{\overline{\mathbb{Q}}}$.

A basis of the global sections of the line bundle $\mathcal{M}^{\otimes p}$, for some p large enough, gives rise to a closed immersion $\overline{S} \subseteq \mathbb{P}^m_{\overline{\mathbb{Q}}}$. This gives the first bullet point at the beginning of §3. Note that the Weil height h on $\mathbb{P}^m_{\overline{\mathbb{Q}}}(\overline{\mathbb{Q}})$ restricted to $\overline{S}(\overline{\mathbb{Q}})$ via this immersion differs from $p h_{\overline{S},\mathcal{M}}$ by a bounded function.

For the line bundle $\mathcal{L}$ on $\mathcal{A}$, which is ample relative over S, we have that $\mathcal{L}^{\otimes 4}$ is relatively very ample on $\mathcal{A}/S$. Thus by [Gro61, Proposition 4.4.10.(ii) and Proposition 4.1.4], there is a closed immersion $\mathcal{A} \to \mathbb{P}^n_S = \mathbb{P}^n \times S$ given by global sections of $\mathcal{L}^{\otimes 4} \otimes \pi^*\mathcal{M}^{\otimes q}$ for some large q. When restricted to the generic fiber A of $\mathcal{A} \to S$, we get a closed immersion $A \to \mathbb{P}^n_{k(S)}$ which arises from a basis of the global sections of $L^{\otimes 4}$, where L is the restriction of $\mathcal{L}$ over the generic fiber A. Moreover L is ample since $\mathcal{L}$ is relatively ample, and L is symmetric since $[-1]^*\mathcal{L} = \mathcal{L}$. Thus we also have the second and third bullet points at the beginning of §3.

Note that the height function $\hat{h}_{\mathcal{A}}$ defined in (3.2) is then

$$\hat{h}_{\mathcal{A}}\colon \mathcal{A}(\overline{\mathbb{Q}}) \to [0,\infty), \quad P \to \hat{h}_{\mathcal{A}_s,\mathcal{L}_s^{\otimes 4}}(P)$$

where $s = \pi(P)$. So $\hat{h}_{\mathcal{A},\mathcal{L}} = (1/4)\hat{h}_{\mathcal{A}}$.

The full hypothesis of Theorem 1.6 is now satisfied for $\mathcal{A}$ and X, $e.g.$, (Hyp) is just (iv) and (v). We get constants $c_1 > 0$ and c_2 and a Zariski open dense subset U of X such that

$$\hat{h}_{\mathcal{A}}(P) \geq c_1 h(\pi(P)) - c_2 \quad \text{for all} \quad P \in U(\overline{\mathbb{Q}}).$$

Thus (B.1) holds true with c_1 replaced by $(c_1 p)/4$ and c_2 replaced by $c_2/4 + O_S(1)$, where $O_S(1)$ is a bounded function on $S(\overline{\mathbb{Q}})$. So we are done. $\qquad\square$

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