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INTEGRAL p-ADIC HODGE THEORY OF FORMAL SCHEMES IN LOW RAMIFICATION

YU MIN

Abstract. We prove that for any proper smooth formal scheme $\mathfrak{X}$ over $\mathcal{O}_K$, where $\mathcal{O}_K$ is the ring of integers in a complete discretely valued non-archimedean extension K of $\mathbb{Q}_p$ with perfect residue field k and ramification degree e, the i-th Breuil–Kisin cohomology group and its Hodge–Tate specialization admit nice decompositions when $ie < p-1$. Thanks to the comparison theorems in the recent works of Bhatt, Morrow and Scholze [BMS18], [BMS19], we can then get an integral comparison theorem for formal schemes when the cohomological degree i satisfies $ie < p-1$, which generalises the case of schemes under the condition $(i+1)e < p-1$ proven by Fontaine and Messing in [FM87] and Caruso in [Car08].

Contents

0. Introduction 1
Acknowledgements 5
Notations 5
1. Recollections on A_{inf}-cohomology 5
1.1. Pro-étale sheaves 6
1.2. The $L\eta$-functor 6
1.3. The A_{inf}-cohomology 7
2. Lemmas in commutative algebra 9
3. Hodge–Tate cohomology 11
4. The unramified case: comparison theorem 14
4.1. Decomposition of Hodge–Tate cohomology groups 14
4.2. Hodge-to-de Rham spectral sequence 16
4.3. Degeneration of the Hodge-to-de Rham spectral sequence 19
5. The ramified case: comparison theorem 22
5.1. Torsion in Breuil–Kisin cohomology groups 22
5.2. Integral comparison theorem 26
6. Categories of Breuil–Kisin modules 27
References 31

0. Introduction

In this paper, we study the A_{inf}-cohomology theory and the Breuil–Kisin cohomology theory constructed respectively in [BMS18], [BMS19], now unified as prismatic cohomology in [BS19].

Let $\mathcal{O}_K$ always be the ring of integers in a complete discretely valued non-archimedean extension K of $\mathbb{Q}_p$ with perfect residue field k and ramification degree e. Our first main result is the following:

Theorem 0.1 (Theorem 3.8, Theorem 5.11)**.** *Let $\mathfrak{X}$ be a proper smooth formal scheme over $\mathcal{O}_K$. Let $\mathcal{O}_C$ be the ring of integers in a complete algebraically closed non-archimedean extension C of K and X be the adic generic fibre of $\bar{\mathfrak{X}} := \mathfrak{X} \times_{\mathrm{Spf}(\mathcal{O}_K)} \mathrm{Spf}(\mathcal{O}_C)$. Assume $ie < p-1$. Then there is an isomorphism of $\mathfrak{S} = W(k)[[u]]$-modules*

$$H^i_{\mathfrak{S}}(\mathfrak{X}) \cong H^i_{\text{ét}}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathfrak{S}$$

where $H^i_{\mathfrak{S}}(\mathfrak{X}) := H^i(R\Gamma_{\mathfrak{S}}(\mathfrak{X}))$ is the Breuil–Kisin cohomology of $\mathfrak{X}$. Consequently, we also have

$$H^i_{A_{\mathrm{inf}}}(\bar{\mathfrak{X}}) \cong H^i_{\text{ét}}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} A_{\mathrm{inf}},$$

where $H^i_{A_{\mathrm{inf}}}(\bar{\mathfrak{X}}) := H^i(R\Gamma_{A_{\mathrm{inf}}}(\bar{\mathfrak{X}}))$ is the A_{inf}-cohomology of $\bar{\mathfrak{X}}$. Similarly under the same assumption $ie < p-1$, there is an isomorphism of $\mathcal{O}_K$-modules

$$H^i_{\mathrm{HT}}(\mathfrak{X}) \cong H^i_{\text{ét}}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathcal{O}_K,$$

and an isomorphism of $\mathcal{O}_C$-modules

$$H^i_{\mathrm{HT}}(\bar{\mathfrak{X}}) \cong H^i_{\text{ét}}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathcal{O}_C,$$

where $H^i_{\mathrm{HT}}(\mathfrak{X}) := H^i(R\Gamma_{\mathrm{HT}}(\mathfrak{X}))$(resp. $H^i_{\mathrm{HT}}(\bar{\mathfrak{X}}) := H^i(R\Gamma_{\mathrm{HT}}(\bar{\mathfrak{X}}))$) is the Hodge–Tate cohomology[1] of $\mathfrak{X}$ (resp. $\bar{\mathfrak{X}}$).

Remark 0.2. Note that the definition of Breuil–Kisin modules (see Definition 1.12) in [BMS18], [BMS19] is slightly more general than the original definition given by Kisin in [Kis06]. The difference lies in the existence of u-torsion (note that $\mathfrak{S} = W(k)[[u]]$ is a two dimensional regular local ring). However, the theorem above shows that the Breuil–Kisin cohomology theory constructed by Bhatt, Morrow and Schloze does take values in the category of Breuil–Kisin modules in a traditional sense, at least when $ie < p-1$.

Unfortunately, we can not give any canonical isomorphisms between these modules. Our method only enables us to compare the module structure. The proof of this theorem relies essentially on the existence of the Breuil–Kisin cohomology and the construction of the A_{inf}-cohomology in [BMS18] by using the $L\eta$-functor and the pro-étale site, which presents a close relation between A_{inf}-cohomology and p-adic étale cohomology. In fact, the $L\eta$-functor provides us with two morphisms between $H^i_{A_{\mathrm{inf}}}(\bar{\mathfrak{X}})$ (resp. $H^i_{\mathrm{HT}}(\bar{\mathfrak{X}})$) and $H^i_{\text{ét}}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} A_{\mathrm{inf}}$ (resp. $H^i_{\text{ét}}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathcal{O}_C$), whose composition in both direction is μ^i (resp. $(\zeta_p - 1)^i$). For the definitions of μ and ζ_p, see Definition 1.5.

Note that $H^i_{\mathrm{HT}}(\bar{\mathfrak{X}})$ is just the base change of $H^i_{\mathrm{HT}}(\mathfrak{X})$ along the natural injection $\mathcal{O}_K \to \mathcal{O}_C$. We can then directly verify the statement about the Hodge–Tate cohomology groups in Theorem 0.1 by studying the two morphisms provided by the $L\eta$-functor.

For the part concerning the Breuil–Kisin cohomology groups, we need to prove some torsion-free results. Namely, when $ie < p-1$, the Breuil–Kisin cohomology group $H^{i+1}_{\mathfrak{S}}(\mathfrak{X})$ is $E(u)$-torsion-free (equivalently, u-torsion-free), where $E(u) \in \mathfrak{S}$ is the Eisenstein polynomial for a fixed uniformizer π in $\mathcal{O}_K$. Moreover for any positive integer n, we have $H^i_{\mathfrak{S}}(\mathfrak{X})/p^n$ is also $E(u)$-torsion-free.

[1]The Hodge–Tate cohomology of $\bar{\mathfrak{X}}$ satisfies: $R\Gamma_{\mathrm{HT}}(\bar{\mathfrak{X}}) \simeq R\Gamma_{A_{\mathrm{inf}}}(\bar{\mathfrak{X}}) \otimes^{\mathbb{L}}_{A_{\mathrm{inf}},\theta} \mathcal{O}_C$. We also call $R\Gamma_{\mathrm{HT}}(\mathfrak{X}) := R\Gamma_{\mathfrak{S}}(\mathfrak{X}) \otimes^{\mathbb{L}}_{\mathfrak{S}} \mathcal{O}_K$ the Hodge–Tate cohomology of $\mathfrak{X}$, which may not be a standard notion.

As a consequence of Theorem 0.1, we can get an integral comparison theorem about p-adic étale cohomology and crystalline cohomology both in the unramified case and ramified case, which generalises the case of schemes studied by Fontaine and Messing in [FM87] and Caruso in [Car08]. This is actually the main motivation of this work. Before we state our result, we give some background about integral comparison theorems.

Integral p-adic Hodge theory. For a proper smooth (formal) scheme $\mathfrak{X}$ over $\mathcal{O}_K$, we can consider the de Rham cohomology $H^i_{\mathrm{dR}}(\mathfrak{X}/\mathcal{O}_K)$ of $\mathfrak{X}$, the p-adic étale cohomology $H^i_{\text{ét}}(X, \mathbb{Z}_p)$ of the geometric (adic) generic fiber X and the crystalline cohomology $H^i_{\mathrm{crys}}(\mathfrak{X}_k/W(k))$ of the special fiber $\mathfrak{X}_k$. Integral p-adic Hodge theory then studies the relations of these cohomology theories.

The first result concerning integral comparison was given by Fontaine and Messing in [FM87].

Theorem 0.3 ([FM87])**.** *Let X be a proper smooth scheme over $W(k)$ and $X_n = X \times_{\mathrm{Spec}(W(k))} \mathrm{Spec}(W_n(k))$, where k is a perfect field of characteristic p. Let G_{K_0} denote the absolute Galois group of $K_0 = W(k)[\frac{1}{p}]$. Then for any integer i such that $0 \leq i \leq p-2$, there exists a natural isomorphism of G_{K_0}-modules*

$$T_{\mathrm{crys}}(H^i_{\mathrm{dR}}(X_n)) \simeq H^i_{\text{ét}}(X_{\bar{K}_0}, \mathbb{Z}_p/p^n)$$

where T_{crys} is a functor from the category of torsion Fontane–Laffaille modules to the category of $\mathbb{Z}_p[G_{K_0}]$-modules, which preserves invariant factors.

Note that $H^i_{\mathrm{dR}}(X_n) \cong H^i_{\mathrm{crys}}(X_k/W_n(k))$. Here we have used implicitly that $H^i_{\mathrm{dR}}(X_n)$ is in the category of torsion Fontane–Laffaille modules, which is actually one of the main difficulties. The proof of Fontaine–Messing's theorem relies on syntomic cohomology which acts as a bridge connecting p-adic étale cohomology and crystalline cohomology.

Recall that rational p-adic Hodge theory provides an equivalence between the category of crystalline representations and the category of (weakly) admissible filtered φ-modules. The idea of Fontane–Laffaille's theory is to try to classify G_{K_0}-stable $\mathbb{Z}_p$-lattices in a crystalline representation V by φ-stable $W(k)$-lattices in D satisfying some conditions, where D is the corresponding admissible filtered φ-module.

To generalize Fontane–Laffaille's theory to the semi-stable case, Breuil introduced the ring S and related categories of S-modules in order to add a monodromy operator. He has also obtained an integral comparison result in the unramified case when $i < p-1$ in [Bre98a]. Later, this result was generalized to the case that $e(i+1) < p-1$ by Caruso in [Car08].

Theorem 0.4 ([Bre98a] [Car08])**.** *Let X be a proper and semi-stable scheme over $\mathcal{O}_K$. Let X_n be $X \times_{\mathrm{Spec}(\mathcal{O}_K)} \mathrm{Spec}(\mathcal{O}_K/p^n)$. Fix a non-negative integer r such that $er < p-1$. Then there exists a canonical isomorphism of Galois modules*

$$H^i_{\text{ét}}(X_{\bar{K}}, \mathbb{Z}/p^n\mathbb{Z})(r) \cong T_{\mathrm{st}*}(H^i_{\log-\mathrm{crys}}(X_n/(S/p^nS)))$$

for any $i < r$.

$T_{\mathrm{st}*}$ is a functor from the category $\mathrm{Mod}^{r,\varphi}_{/S_\infty}$ (see Definition 6.10) to the category of $\mathbb{Z}_p[G_K]$-modules, which preserves invariant factors. The proof also relies on the use of syntomic cohomology. One of the main difficulties in their proof is to show

that $H^i_{\log-\mathrm{crys}}(X_n/(S/p^nS))$ is in the category $\mathrm{Mod}^{r,\varphi}_{/S_\infty}$, in particular, to show that $H^i_{\log-\mathrm{crys}}(X_1/(S/pS))$ is finite free over S/pS.

Remark 0.5. One crucial point of Breuil's theory is that it highly depends on the restriction $r \leq p-1$ which is rooted in the fact that the inclusion $\varphi(\mathrm{Fil}^r S) \subset p^r S$ is true only when $r \leq p-1$. One way to remove this restriction is to consider Breuil–Kisin modules. In fact, one of the main motivations of A_{inf}-cohomology theory is to give a cohomological construction of Breuil–Kisin modules. The techniques in [BMS18] can not directly give the desired Breuil–Kisin cohomology. However, this goal is achieved in [BMS19] by using topological cyclic homology and in [BS19] by defining prismatic site in a more general setting.

Recently, Bhatt, Morrow and Scholze have obtained a more general result about the relation between p-adic étale cohomology and crystalline cohomology in [BMS18] by using A_{inf}-cohomology. Their result does not impose any restriction on the ramification degree, roughly saying that the torsion in the crystalline cohomology gives an upper bound for the torsion in the p-adic étale cohomology.

As we have said, by studying A_{inf}-cohomology and its descent Breuil–Kisin cohomology, we can generalize the results of Fontaine–Messing, Breuil and Caruso to the case of formal schemes, at least in the good reduction case.

Theorem 0.6 (Theorem 4.9, Theorem 5.13)**.** *Let $\mathfrak{X}$ be a proper smooth formal scheme over $\mathcal{O}_K$. Let C be a complete algebraically closed non-archimedean extension of K and $\bar{\mathfrak{X}} := \mathfrak{X} \times_{\mathrm{Spf}(\mathcal{O}_K)} \mathrm{Spf}(\mathcal{O}_C)$. Write X for the adic generic fiber of $\bar{\mathfrak{X}}$. Then when ie $< p-1$, there is an isomorphism of $W(k)$-modules $H^i_{\text{ét}}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} W(k) \cong H^i_{\mathrm{crys}}(\mathfrak{X}_k/W(k))$.*

We will study the unramified case and the ramified case in different ways. For the proof in the unramified case, we need the following theorem:

Theorem 0.7 (Theorem 4.8)**.** *With the same assumptions as the theorem above, when $e = 1$, we have*

$$\mathrm{length}_{\mathbb{Z}_p}(H^i_{\text{ét}}(X, \mathbb{Z}_p)_{\mathrm{tor}}/p^m)) \geq \mathrm{length}_{W(k)}(H^i_{\mathrm{crys}}(\mathfrak{X}_k/W(k))_{\mathrm{tor}}/p^m)$$

for any $i < p-1$ and any positive integer m.

In fact, we first compare Hodge-Tate cohomology to Hodge cohomology by proving that the truncated Hodge-Tate complex of sheaves $\tau^{\leq p-1}\widetilde{\Omega}_{\bar{\mathfrak{X}}}$ is formal in this case, i.e. there is an isomorphism $\tau^{\leq p-1}\widetilde{\Omega}_{\bar{\mathfrak{X}}} \simeq \bigoplus_{i=0}^{p-1} \mathcal{H}^i(\widetilde{\Omega}_{\bar{\mathfrak{X}}})[-i]$. We then study the Hodge-to-de Rham spectral sequence to relate Hodge cohomology to de Rham cohomology. By Theorem 0.1, we can finally relate de Rham (or crystalline) cohomology to p-adic étale cohomology. Note that the theorem above gives a converse to Theorem 1.11 in [BMS18], which implies that $H^n_{\text{ét}}(X, \mathbb{Z}_p)$ and $H^n_{\mathrm{crys}}(\mathfrak{X}_k/W(k))$ have the same invariant factors.

In the ramified case, the integral comparison theorem follows directly from Theorem 0.1 and Theorem 1.9.

Remark 0.8. The A_{inf}-cohomology theory in the semi-stable case has been studied in [CK19]. The Breuil–Kisin cohomology might be also generalised to the semi-stable case by using the prismatic site. Then one could also hope to generalize Theorem 0.1 and Theorem 0.6 to the semi-stable case.

We also remark that although the result in the ramified case can recover that in the unramified case, the method used in the unramified case can lead to the following theorem concerning the Hodge-to-de Rham spectral sequence and integral comparison result for all cohomological degrees.

Theorem 0.9 (Theorem 4.12, Corollary 4.13)**.** *Let $\mathfrak{X}$ be a proper smooth formal scheme over $W(k)$, where k is a perfect field of characteristic p. Let C be a complete algebraically closed non-archimedean extension of $W(k)[1/p]$ and $\mathcal{O}_C$ be its ring of integers. Let $\bar{\mathfrak{X}} = \mathfrak{X} \times_{\mathrm{Spf}(W(k))} \mathrm{Spf}(\mathcal{O}_C)$ and write X for the adic generic fiber of $\bar{\mathfrak{X}}$. Assume the relative dimension of $\mathfrak{X}$ satisfies $\dim\mathfrak{X} < p-1$. Then we have the following results:*

(i) *There is an isomorphism of $W(k)$-modules for all i*

$$H^i_{\text{ét}}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} W(k) \cong H^i_{\text{crys}}(\mathfrak{X}_k/W(k)).$$

(ii) *The (integral) Hodge-to-de Rham spectral sequence degenerates at E_1-page.*

When $\mathfrak{X}$ is a scheme, Theorem 0.9 can be deduced from [FM87] together with Poincaré duality. When $\mathfrak{X}$ is a formal scheme, the comparison isomorphism in Theorem 0.9 can not be deduced from Theorem 0.6 since there is still no Poincaré duality for étale cohomology of rigid analytic varieties over C with coefficient in $\mathbb{Z}/p^n$. We also want to remark that Fontaine and Messing have proved the integral Hodge-to-de Rham spectral sequence degenerates at E_1-page when the special fiber of the proper smooth formal scheme $\mathfrak{X}$ has dimension strictly less than p (cf. Remark 4.14).

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Notations. Throughout this paper, let $\mathfrak{X}$ be a proper smooth formal scheme over $\mathcal{O}_K$, which is the ring of integers in a complete discretely valued non-archimedean extension K of $\mathbb{Q}_p$ with perfect residue field k and ramification degree e.

Fix C a complete algebraically closed non-archimedean extension of K with $\mathcal{O}_C$ its ring of integers. Let $\bar{\mathfrak{X}} = \mathfrak{X} \times_{\mathrm{Spf}(\mathcal{O}_K)} \mathrm{Spf}(\mathcal{O}_C)$ and write X for the adic generic fiber of $\bar{\mathfrak{X}}$.

Let $\mathfrak{X}_k$ denote the special fiber of $\mathfrak{X}$ and $\mathfrak{X}_{\bar{k}}$ denote its base change to $\bar{k}$ which is the residue field of $\mathcal{O}_C$ (note that $\bar{k}$ is not necessarily the algebraic closure of k).

Define $\mathfrak{S} := W(k)[[u]]$. Fix a uniformizer π in $\mathcal{O}_K$. We denote by β the $W(k)$-linear map $\mathfrak{S} \to \mathcal{O}_K$ sending u to π, whose kernel is generated by a fixed Eisenstein polynomial $E = E(u)$ for π.

1. Recollections on A_{inf}-cohomology

In this section, we will simply recall the necessary ingredients for defining the A_{inf}-cohomology theory in [BMS18]. In fact, we will stick to the method using the pro-étale site and the décalage functor $L\eta$, which will provide us with some useful morphisms between A_{inf}-cohomology groups and p-adic étale cohomology groups.

1.1. **Pro-étale sheaves.** We first define some sheaves on the pro-étale site $X_{\text{proét}}$. Recall that there is a natural projection map of sites

$$\omega : X_{\text{proét}} \to X_{\text{ét}}$$

which is defined by pulling back $U \in X_{\text{ét}}$ to the constant tower $(\cdots \to U \to U \to X)$ in $X_{\text{proét}}$. This just reflects the fact that an étale morphism is pro-étale.

Definition 1.1 ([Sch13] Section 6)**.** *Consider the following sheaves on $X_{\text{proét}}$.*

(i) *The integral structure sheaf $\mathcal{O}_X^+ := \omega^*\mathcal{O}_{X_{\text{ét}}}^+$.*
(ii) *The structure sheaf $\mathcal{O}_X := \omega^*\mathcal{O}_{X_{\text{ét}}}$.*
(iii) *The completed integral structure sheaf $\widehat{\mathcal{O}}_X^+ := \varprojlim_n \mathcal{O}_X^+/p^n$.*
(iv) *The completed structure sheaf $\widehat{\mathcal{O}}_X := \widehat{\mathcal{O}}_X^+[\frac{1}{p}]$.*
(v) *The tilted completed integral structure sheaf $\widehat{\mathcal{O}}_{X^\flat}^+ := \varprojlim_\varphi \mathcal{O}_X^+/p$.*
(vi) *Fontaine's period sheaf $\mathbb{A}_{\text{inf},X}$, which is the derived p-adic completion of $W(\widehat{\mathcal{O}}_{X^\flat}^+)$.*

Remark 1.2. In [BMS18, Remark 5.5], it has been pointed out that it is not clear whether $W(\widehat{\mathcal{O}}_{X^\flat}^+)$ is derived p-adic complete. So in order to make the A_{inf}-cohomology theory work well, we need to define $\mathbb{A}_{\text{inf},X}$ as the derived p-adic completion of $W(\widehat{\mathcal{O}}_{X^\flat}^+)$, which is actually a complex of sheaves.

1.2. **The $L\eta$-functor.** The other important ingredient for defining the A_{inf}-cohomology is the décalage functor, which functions as a tool to get rid of "junk torsion". The "junk torsion" exists already in Faltings' approach to p-adic Hodge theory in [Fal88]. The introduction of the décalage functor is actually the main novelty of [BMS18] to deal with this "junk torsion".

Definition 1.3 (The $L\eta$-functor, [BMS18] Section 6)**.** *Let $(T, \mathcal{O}_T)$ be a ringed topos and $\mathcal{I} \subset \mathcal{O}_T$ be an invertible ideal sheaf. For any $\mathcal{I}$-torsion-free complex $C^\bullet \in K(\mathcal{O}_T)$, we can define a new complex $\eta_{\mathcal{I}}C^\bullet = (\eta_{\mathcal{I}}C)^\bullet \in K(\mathcal{O}_T)$ with terms*

$$(\eta_{\mathcal{I}}C)^i := \{x \in C^i | dx \in \mathcal{I} \cdot C^{i+1}\} \otimes_{\mathcal{O}_T} \mathcal{I}^{\otimes i}$$

For every complex $D^\bullet \in K(\mathcal{O}_T)$, there exists a strongly K-flat complex $C^\bullet \in K(\mathcal{O}_T)$ and a quasi-isomorphism $C^\bullet \to D^\bullet$. By saying strongly K-flat, we mean that each C^i is a flat $\mathcal{O}_T$-module and for every acyclic complex $P^\bullet \in K(\mathcal{O}_T)$, the total complex $\mathrm{Tot}(C^\bullet \otimes P^\bullet)$ is acyclic. In particular, $C^\bullet$ is $\mathcal{I}$-torsion free. Then we can define

$$L\eta_{\mathcal{I}} : D(\mathcal{O}_T) \to D(\mathcal{O}_T)$$
$$L\eta_{\mathcal{I}}(D^\bullet) := \eta_{\mathcal{I}}(C^\bullet)$$

A concrete example is to consider a ring A and a non-zero-divisor $a \in A$. If C is a cochain complex of a-torsion free A-modules, we can define the subcomplex $\eta_a C$ of $C[\frac{1}{a}]$ as

$$(\eta_a C)^i := \{x \in a^i C^i : dx \in a^{i+1} \cdot C^{i+1}\}$$

and this induces the corresponding functor $L\eta_a : D(A) \to D(A)$.

Remark 1.4. (i) The $L\eta$-functor is not an exact functor between derived categories. For example, consider the distinguished triangle $\mathbb{Z}/p \to \mathbb{Z}/p^2 \to \mathbb{Z}/p$ where the first map is induced by multiplication by p on $\mathbb{Z}$ and the second map is modulo p. It is easy to see that $L\eta_p(\mathbb{Z}/p) = 0$ and $L\eta_p(\mathbb{Z}/p^2) \neq 0$.
(ii) By [BMS18, Proposition 6.7], the $L\eta$-functor is lax symmetric monoidal.

1.3. **The A_{inf}-cohomology.** We recall some basic definitions in p-adic Hodge theory.

Definition 1.5 ([Fon94]). (i) *Define $\mathcal{O}_C^\flat := \varprojlim_{x\to x^p} \mathcal{O}_C/p$ which is called the tilt of $\mathcal{O}_C$ and $A_{\mathrm{inf}} := W(\mathcal{O}_C^\flat)$, the Witt vector ring of $\mathcal{O}_C^\flat$. Note that $\mathcal{O}_C^\flat$ is a perfect ring of characteristic p and A_{inf} is equipped with a natural Frobenius map φ, which is an isomorphism of rings.*

(ii) *Fix a compatible system of primitive p-power roots of unity $\{\zeta_{p^n}\}_{n\in\mathbb{N}}$ such that $\zeta_{p^{n+1}}^p = \zeta_{p^n}$. Under the isomorphism of multiplicative monoids $\mathcal{O}_C^\flat \cong \varprojlim_{x\to x^p} \mathcal{O}_C$, we define $\epsilon := (1, \zeta_p, \zeta_{p^2}, \cdots, \zeta_{p^n}, \cdots) \in \mathcal{O}_C^\flat$ and $\mu := [\epsilon] - 1 \in A_{\mathrm{inf}}$.*

(iii) *There is a map $\theta : A_{\mathrm{inf}} \to \mathcal{O}_C$ defined by Fontaine. The map θ is surjective and $\ker(\theta)$ is generated by $\xi = \mu/\varphi^{-1}(\mu)$. After twisting with the Frobenius map, we get $\widetilde{\theta} := \theta \circ \varphi^{-1} : A_{\mathrm{inf}} \to \mathcal{O}_C$, whose kernel is generated by $\widetilde{\xi} := \varphi(\xi) = \varphi(\mu)/\mu$.*

Now we are ready to define the A_{inf}-cohomology theory. We consider the natural projection $\nu : X_{\text{proét}} \to \bar{\mathfrak{X}}_{\mathrm{zar}}$, which is actually the composite $X_{\text{proét}} \xrightarrow{\omega} X_{\text{ét}} \to \bar{\mathfrak{X}}_{\text{ét}} \to \bar{\mathfrak{X}}_{\mathrm{zar}}$.

Definition 1.6 ([BMS18] Definition 9.1). *Define $A\Omega_{\bar{\mathfrak{X}}} := L\eta_\mu R\nu_*(\mathbb{A}_{\mathrm{inf},X})$ and $\widetilde{\Omega}_{\bar{\mathfrak{X}}} := L\eta_{\zeta_{p-1}} R\nu_*(\widehat{\mathcal{O}}_X^+)$. The A_{inf}-cohomology is defined to be the Zariski hypercohomology of the complex of sheaves $A\Omega_{\bar{\mathfrak{X}}}$, i.e. $R\Gamma_{A_{\mathrm{inf}}}(\bar{\mathfrak{X}}) := R\Gamma_{\mathrm{zar}}(\bar{\mathfrak{X}}, A\Omega_{\bar{\mathfrak{X}}})$. We can also define the Hodge–Tate cohomology $R\Gamma_{\mathrm{HT}}(\bar{\mathfrak{X}}) := R\Gamma_{\mathrm{zar}}(\bar{\mathfrak{X}}, \widetilde{\Omega}_{\bar{\mathfrak{X}}})$.*

Remark 1.7. As both $R\nu_*$ and the $L\eta$-functor are lax symmetric monoidal, the complex $\widetilde{\Omega}_{\bar{\mathfrak{X}}}$ is a commutative $\mathcal{O}_{\overline{\mathfrak{X}}}$-algebra object in the derived category of $\mathcal{O}_{\overline{\mathfrak{X}}}$-modules $D(\mathcal{O}_{\overline{\mathfrak{X}}})$. For the same reason, the complex $A\Omega_{\bar{\mathfrak{X}}}$ is a commutative ring in the derived category $D(\overline{\mathfrak{X}}_{\mathrm{zar}}, \mathbb{Z})$ of abelian sheaves.

The A_{inf}-cohomology takes values in the category of what we call Breuil–Kisin-Fargues modules.

Definition 1.8 ([BMS18] Definition 4.22). *A Breuil–Kisin-Fargues module is a finitely presented A_{inf}-module M which becomes free over $A_{\mathrm{inf}}[\frac{1}{p}]$ after inverting p and is equipped with an isomorphism*

$$\varphi_M : M \otimes_{A_{\mathrm{inf}},\varphi} A_{\mathrm{inf}}[\frac{1}{\widetilde{\xi}}] \xrightarrow{\simeq} M[\frac{1}{\widetilde{\xi}}].$$

The main theorem about the A_{inf}-cohomology theory is the following:

Theorem 1.9 ([BMS18] Theorem 14.3). *The complex $R\Gamma_{A_{\mathrm{inf}}}(\bar{\mathfrak{X}})$ is a perfect complex of A_{inf}-modules with a φ-linear map $\varphi : R\Gamma_{A_{\mathrm{inf}}}(\bar{\mathfrak{X}}) \to R\Gamma_{A_{\mathrm{inf}}}(\bar{\mathfrak{X}})$ which becomes an isomorphism after inverting ξ resp. $\widetilde{\xi}$. The cohomology groups $H^i_{A_{\mathrm{inf}}}(\bar{\mathfrak{X}}) := H^i(R\Gamma_{A_{\mathrm{inf}}}(\bar{\mathfrak{X}}))$ are Breuil–Kisin-Fargues modules. Moreover, there are several comparison results:*

(i) *With étale cohomology: $R\Gamma_{A_{\mathrm{inf}}}(\bar{\mathfrak{X}}) \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} A_{\mathrm{inf}}[1/\mu] \simeq R\Gamma_{\text{ét}}(X, \mathbb{Z}_p) \otimes^{\mathbb{L}}_{\mathbb{Z}_p} A_{\mathrm{inf}}[1/\mu]$.*

(ii) *With crystalline cohomology: $R\Gamma_{A_{\mathrm{inf}}}(\bar{\mathfrak{X}}) \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} W(\bar{k}) \simeq R\Gamma_{\mathrm{crys}}(\mathfrak{X}_{\bar{k}}/W(\bar{k}))$, where the map $A_{\mathrm{inf}} = W(\mathcal{O}_C^\flat) \to W(\bar{k})$ is induced by the natural projection $\mathcal{O}_C^\flat \to \bar{k}$ (in fact, $\mathcal{O}_C^\flat$ is a valuation ring with residue field $\bar{k}$).*

(iii) *With de Rham cohomology: $R\Gamma_{A_{\mathrm{inf}}}(\bar{\mathfrak{X}}) \otimes^{\mathbb{L}}_{A_{\mathrm{inf}},\theta} \mathcal{O}_C \simeq R\Gamma_{\mathrm{dR}}(\bar{\mathfrak{X}}/\mathcal{O}_C)$.*

(iv) *With Hodge–Tate cohomology: $\widetilde{\Omega}_{\bar{\mathfrak{X}}} \simeq A\Omega_{\bar{\mathfrak{X}}} \otimes^{\mathbb{L}}_{A_{\mathrm{inf}},\widetilde{\theta}} \mathcal{O}_C$ and $R\Gamma_{A_{\mathrm{inf}}}(\bar{\mathfrak{X}}) \otimes^{\mathbb{L}}_{A_{\mathrm{inf}},\widetilde{\theta}} \mathcal{O}_C \simeq R\Gamma_{\mathrm{HT}}(\bar{\mathfrak{X}})$.*

Corollary 1.10. *For all $i \geq 0$, we have isomorphisms and short exact sequences*

(i) $H^i_{A_{\mathrm{inf}}}(\bar{\mathfrak{X}}) \otimes_{A_{\mathrm{inf}}} A_{\mathrm{inf}}[1/\mu] \cong H^i_{\text{ét}}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} A_{\mathrm{inf}}[1/\mu]$.

(ii) $0 \to H^i_{A_{\mathrm{inf}}}(\bar{\mathfrak{X}}) \otimes_{A_{\mathrm{inf}}} W(\bar{k}) \to H^i_{\mathrm{crys}}(\mathfrak{X}_{\bar{k}}/W(\bar{k})) \to Tor_1^{A_{\mathrm{inf}}}(H^{i+1}_{A_{\mathrm{inf}}}(\bar{\mathfrak{X}}), W(\bar{k})) \to 0$.

(iii) $0 \to H^i_{A_{\mathrm{inf}}}(\bar{\mathfrak{X}}) \otimes_{A_{\mathrm{inf}},\theta} \mathcal{O}_C \to H^i_{\mathrm{dR}}(\bar{\mathfrak{X}}/\mathcal{O}_C) \to H^{i+1}_{A_{\mathrm{inf}}}(\bar{\mathfrak{X}})[\xi] \to 0$.

(iv) $0 \to H^i_{A_{\mathrm{inf}}}(\bar{\mathfrak{X}}) \otimes_{A_{\mathrm{inf}},\widetilde{\theta}} \mathcal{O}_C \to H^i_{\mathrm{HT}}(\bar{\mathfrak{X}}) \to H^{i+1}_{A_{\mathrm{inf}}}(\bar{\mathfrak{X}})[\widetilde{\xi}] \to 0$.

One of the most important applications of the A_{inf}-cohomology theory is to enable us to compare étale cohomology to crystalline cohomology integrally without any restrictions on the degree of cohomology groups and the ramification degree of the base field. More precisely, it can be showed that the torsion in the crystalline cohomology gives an upper bound for the torsion in the étale cohomology.

Theorem 1.11 ([BMS18] Theorem 14.5)**.** *For any $n, i \geq 0$, we have the inequality*

$$\mathrm{length}_{W(k)}(H^i_{\mathrm{crys}}(\mathfrak{X}_k/W(k))_{\mathrm{tor}}/p^n) \geq \mathrm{length}_{\mathbb{Z}_p}(H^i_{\text{ét}}(X, \mathbb{Z}_p)_{\mathrm{tor}}/p^n)$$

where $H^i_{\mathrm{crys}}(\mathfrak{X}_k/W(k))_{\mathrm{tor}}$ is the torsion submodule of $H^i_{\mathrm{crys}}(\mathfrak{X}_k/W(k))$ and $H^i_{\text{ét}}(X, \mathbb{Z}_p)_{\mathrm{tor}}$ is the torsion submodule of $H^i_{\text{ét}}(X, \mathbb{Z}_p)$.

As we have mentioned, there is a refinement of the A_{inf}-cohomology, i.e. the Breuil–Kisin cohomology, which is an $\mathfrak{S}$-linear cohomology and recovers the A_{inf}-cohomology after base change along a faithfully flat map[2] $\alpha : \mathfrak{S} \to A_{\mathrm{inf}}$ (in particular, we have $(\alpha(E)) = (\widetilde{\xi})$). The Breuil–Kisin cohomology gives a cohomological construction of Breuil–Kisin modules, which plays a very important role in integral p-adic Hodge theory.

The first construction of the Breuil–Kisin cohomology is given in [BMS19] by using topological cyclic homology. Another construction is given in [BS19] by using the prismatic site. We will not say anything about the construction of the Breuil–Kisin cohomology theory here but choose to state a similar comparison theorem as in the A_{inf} case.

Let $R\Gamma_{\mathfrak{S}}(\mathfrak{X})$ denote the Breuil-Kisin cohomology attached to $\mathfrak{X}$. We first recall the definition of Breuil–Kisin module which is slightly more general than the original definition due to Kisin.

Definition 1.12 ([BMS18] Definition 4.1)**.** *A Breuil–Kisin module is a finitely generated $\mathfrak{S}$-module M together with an isomorphism*

$$\varphi_M : M \otimes_{\mathfrak{S},\varphi} \mathfrak{S}[\frac{1}{E}] \to M[\frac{1}{E}].$$

Theorem 1.13 ([BMS19] Theorem 1.2)**.** *The Breuil–Kisin cohomology $R\Gamma_{\mathfrak{S}}(\mathfrak{X})$ of $\mathfrak{X}$ is a perfect complex of $\mathfrak{S}$-modules. Moreover, it is equipped with a φ-linear map $\varphi : R\Gamma_{\mathfrak{S}}(\mathfrak{X}) \to R\Gamma_{\mathfrak{S}}(\mathfrak{X})$ which induces an isomorphism*

$$R\Gamma_{\mathfrak{S}}(\mathfrak{X}) \otimes^{\mathbb{L}}_{\mathfrak{S},\varphi} \mathfrak{S}[\frac{1}{E}] \simeq R\Gamma_{\mathfrak{S}}(\mathfrak{X})[\frac{1}{E}]$$

The cohomology groups $H^i_{\mathfrak{S}}(\mathfrak{X}) := H^i(R\Gamma_{\mathfrak{S}}(\mathfrak{X}))$ are Breuil–Kisin modules. There are several specializations that recover other p-adic cohomology theories:

(i) *With A_{inf}-cohomology: after base change along $\alpha : \mathfrak{S} \to A_{\mathrm{inf}}$, it recovers A_{inf}-cohomology : $R\Gamma_{\mathfrak{S}}(\mathfrak{X}) \otimes^{\mathbb{L}}_{\mathfrak{S},\alpha} A_{\mathrm{inf}} \simeq R\Gamma_{A_{\mathrm{inf}}}(\bar{\mathfrak{X}})$.*

[2]To define the map α, we fix a compatible system of p-power roots $\pi^{1/p^n} \in C$, which defines an element $\pi^\flat = (\pi, \pi^{1/p}, \pi^{1/p^2}, \cdots) \in \varprojlim_{x \mapsto x^p} \mathcal{O}_C \cong \mathcal{O}_C^\flat$. Then α is defined to send u to $[\pi^\flat]^p$ and be the Frobenius on $W(k)$.

(ii) *With étale cohomology:* $R\Gamma_{\mathfrak{S}}(\mathfrak{X}) \otimes^{\mathbb{L}}_{\mathfrak{S},\widetilde{\alpha}} A_{\mathrm{inf}}[1/\mu] \simeq R\Gamma_{\text{ét}}(X, \mathbb{Z}_p) \otimes^{\mathbb{L}}_{\mathbb{Z}_p} A_{\mathrm{inf}}[1/\mu]$, *where* $\widetilde{\alpha}$ *is the composite* $\mathfrak{S} \xrightarrow{\alpha} A_{\mathrm{inf}} \hookrightarrow A_{\mathrm{inf}}[1/\mu]$.

(iii) *With de Rham cohomology:* $R\Gamma_{\mathfrak{S}}(\mathfrak{X}) \otimes^{\mathbb{L}}_{\mathfrak{S},\widetilde{\beta}} \mathcal{O}_K \simeq R\Gamma_{\mathrm{dR}}(\mathfrak{X}/\mathcal{O}_K)$, *where* $\widetilde{\beta} := \beta \circ \varphi : \mathfrak{S} \to \mathcal{O}_K$.

(iv) *With crystalline cohomology: after base change along the map* $\mathfrak{S} \to W(k)$ *which is the Frobenius on* $W(k)$ *and sends* u *to 0, it recovers the crystalline cohomology of the special fiber:* $R\Gamma_{\mathfrak{S}}(\mathfrak{X}) \otimes^{\mathbb{L}}_{\mathfrak{S}} W(k) \simeq R\Gamma_{\mathrm{crys}}(\mathfrak{X}_k/W(k))$.

For later convenience, we define $R\Gamma_{\mathrm{HT}}(\mathfrak{X}) := R\Gamma_{\mathfrak{S}}(\mathfrak{X}) \otimes^{\mathbb{L}}_{\mathfrak{S},\beta} \mathcal{O}_K$ and call it the Hodge–Tate cohomology of $\mathfrak{X}$. Note that there is an isomorphism: $R\Gamma_{\mathrm{HT}}(\mathfrak{X}) \otimes^{\mathbb{L}}_{\mathcal{O}_K} \mathcal{O}_C \simeq R\Gamma_{\mathrm{HT}}(\overline{\mathfrak{X}})$.

Remark 1.14. Note that there is a Frobenius twist appearing in the specializations above. As explained in [BMS19, Remark 1.4], this is not artificial but contains some information about the torsion in the de Rham cohomology.

2. Lemmas in commutative algebra

In this section, we will recollect some results on finitely presented modules over $\mathcal{O}_C$ and prove some key lemmas that are frequently used later.

We begin with the following lemma:

Lemma 2.1 ([Sta19]Lemma 0ASN)**.** *Let* R *be a ring. The following are equivalent:*

(i) *For* $a, b \in R$, *either* a *divides* b *or* b *divides* a.

(ii) *Every finitely generated ideal is principal and* R *is local.*

(iii) *The set of ideals of* R *are linearly ordered by inclusion.*

In particular, all valuation rings satisfy these equivalent conditions. The module structure of finitely presented modules over valuation rings is similar to that of finitely generated modules over principal ideal domains as the following lemma shows.

Lemma 2.2 ([Sta19]Lemma 0ASP)**.** *Let* R *be a ring satisfying the equivalent conditions above, then every finitely presented* R*-module is isomorphic to a finite direct sum of modules of the form* R/aR *where* $a \in R$.

Corollary 2.3. *Any finitely presented module over* $\mathcal{O}_C$ *is of the form* $\bigoplus_{i=1}^n \mathcal{O}_C/\pi_i$ *for some* $\pi_i \in \mathcal{O}_C$.

We will need to study finitely presented torsion $\mathcal{O}_C$-modules later. The main tool to deal with these modules is the length function $l_{\mathcal{O}_C}$, as used in [CK19], see also [Bha17]. In particular, this length function behaves additively under short exact sequences. Usually, we use the normalized length function, i.e. $l_{\mathcal{O}_C}(\mathcal{O}_C/p) = 1$.

Lemma 2.4. *Let* A *and* B *be base changes to* $\mathcal{O}_C$ *of finitely presented torsion* $W(k)$*-modules. If for each* $m > 0$, *we have*

$$l_{\mathcal{O}_C}(A/p^m) = l_{\mathcal{O}_C}(B/p^m),$$

then A *is isomorphic to* B *as* $\mathcal{O}_C$*-modules.*

Proof. Note that $2l_{\mathcal{O}_C}(A/p) - l_{\mathcal{O}_C}(A/p^2)$ is the number of the invariant factor p in A. This implies that the number of the invariant factor p of A is equal to that of B. By induction on m, it is easy to prove $A \cong B$ as $\mathcal{O}_C$-modules.

□

Next, we want to prove the following key lemma which will be used in the comparison of Hodge–Tate cohomology and p-adic étale cohomology.

Lemma 2.5. *Let $M = \bigoplus_{i=1}^m \mathcal{O}_C/\beta^{m_i}$ and $N = \bigoplus_{j=1}^n \mathcal{O}_C/\beta^{n_j}$, where $\beta \neq 0$ is in the maximal ideal $\mathfrak{m}$ of $\mathcal{O}_C$ and all m_i, n_j are positive integers. Suppose there are two $\mathcal{O}_C$-linear morphisms $f : M \to N$ and $g : N \to M$ such that $g \circ f = \alpha$ and $f \circ g = \alpha$, where $\alpha \in \mathcal{O}_C$ and $v(\beta) > v(\alpha)$. Then $m = n$ and the multi-sets $\{m_i\}$ and $\{n_j\}$ are the same, i.e., $M \cong N$.*

In order to prove this lemma, we consider all finitely presented torsion modules over $\mathcal{O}_C$. As we have mentioned, any such module looks like $\bigoplus_{i=1}^n \mathcal{O}_C/\pi_i$ for some non-zero $\pi_i \in \mathfrak{m}$. We call $\mathrm{trk}(N) := n$ the torsion-rank of N. Note that the torsion-rank of N is equal to the dimension of N base changed to the residue field of $\mathcal{O}_C$. So it is well-defined. We will also use the normalized length function $l_{\mathcal{O}_C}$ for finitely presented torsion $\mathcal{O}_C$-modules.

Now we prove a lemma concerning the torsion-rank:

Lemma 2.6. *Let $N \hookrightarrow M$ be an injection of finitely presented torsion $\mathcal{O}_C$-modules. Then $\mathrm{trk}(N) \leq \mathrm{trk}(M)$. Dually if $N \twoheadrightarrow M$ is a surjection of finitely presented torsion $\mathcal{O}_C$-modules, then $\mathrm{trk}(N) \geq \mathrm{trk}(M)$.*

Proof. Write $N = \bigoplus_{i=1}^n \mathcal{O}_C/\pi_i$ and $M = \bigoplus_{i=1}^m \mathcal{O}_C/\varpi_i$. Let π be the smallest of the π_i (i.e., the one with the smallest valuation), and let ϖ be the largest of ϖ_i. Then

$$(\mathcal{O}_C/\pi)^n \subset N \hookrightarrow M \subset (\mathcal{O}_C/\varpi)^m,$$

which shows that $\varpi \in \pi\mathcal{O}_C$; write $\varpi = \pi x$. The composition of these maps lands in the π-torsion of $(\mathcal{O}_C/\varpi)^m$, which is isomorphic to $(x\mathcal{O}_C/\varpi\mathcal{O}_C)^m \cong (\mathcal{O}_C/\pi\mathcal{O}_C)^m$. So now we have an injection $(\mathcal{O}_C/\pi)^n \hookrightarrow (\mathcal{O}_C/\pi)^m$. Taking length shows that $n \leq m$.

If $N \twoheadrightarrow M$ is a surjection of finitely presented torsion $\mathcal{O}_C$-modules, we can consider the injection $\mathrm{Hom}(M, \mathcal{O}_C/t) \hookrightarrow \mathrm{Hom}(N, \mathcal{O}_C/t)$ where t is any non-zero element in $\mathfrak{m}$. Then we have $\mathrm{trk}(M) = \mathrm{trk}(\mathrm{Hom}(M, \mathcal{O}_C/t)) \leq \mathrm{trk}(\mathrm{Hom}(N, \mathcal{O}_C/t)) = \mathrm{trk}(N)$. □

Lemma 2.7. *Let $g : N \to M$ be a morphism of finitely presented torsion $\mathcal{O}_C$-modules; write $N = \bigoplus_{i=1}^n \mathcal{O}_C/\pi_i$ and $M = \bigoplus_{i=1}^m \mathcal{O}_C/\varpi_i$. Assume that $\ker(g)$ is killed by some element $\alpha \in \mathcal{O}_C$ whose valuation is strictly smaller that all of the π_i. Then $\mathrm{trk}(N) \leq \mathrm{trk}(M)$.*

Proof. By assumption $\ker(g)$ is contained in the α-torsion $N[\alpha]$ of N, which is given by $N[\alpha] \cong \bigoplus_{i=1}^n \frac{\pi_i}{\alpha}\mathcal{O}_C/\pi_i\mathcal{O}_C$. So

$$N \twoheadrightarrow N/\ker(g) \twoheadrightarrow N/N[\alpha] \cong \bigoplus_{i=1}^n \frac{\pi_i}{\alpha}\mathcal{O}_C/\pi_i\mathcal{O}_C.$$

Taking torsion-ranks, Lemma 2.6 for surjections shows that $\mathrm{trk}(N/\ker(g)) = \mathrm{trk}(N)$. But $N/\ker(g) \hookrightarrow M$, so Lemma 2.6 also shows that $\mathrm{trk}(N/\ker(g)) \leq \mathrm{trk}(M)$. □

Now we are ready to prove Lemma 2.5.

Proof of Lemma 2.5. Note that the number of invariant factor β^k in M is equal to $\mathrm{trk}(\beta^{k-1}M) - \mathrm{trk}(\beta^k M)$. By Lemma 2.7, we have $\mathrm{trk}(\beta^k M) = \mathrm{trk}(\beta^k N)$ for any k. This means that the number of invariant factor β^k in M and that in N are equal for any k. So we must have $M \cong N$. □

Lemma 2.8. *Let $M = \mathcal{O}_C^r \oplus (\bigoplus_{i=1}^m \mathcal{O}_C/\beta^{m_i})$ and $N = \mathcal{O}_C^s \oplus (\bigoplus_{j=1}^n \mathcal{O}_C/\beta^{n_j})$. Suppose there are two $\mathcal{O}_C$-linear morphisms $f : M \to N$ and $g : N \to M$ such that $g \circ f = \alpha$ and $f \circ g = \alpha$, where $\alpha \in \mathcal{O}_C$ and $v(\beta) > v(\alpha)$. Then $M \cong N$. In particular, if $M = 0$, then $N = 0$.*

Proof. According to Lemma 2.5, M/β^k and N/β^k are isomorphic for all k. For large enough k, this means the torsion submodule M_{tor} of M is isomorphic to the torsion submodule N_{tor} of N and also the rank of the free part of M is equal to that of N, i.e. $r = s$. We are done. □

3. Hodge–Tate cohomology

In this section, we study the Hodge–Tate specialization of the Breuil–Kisin cohomology and prove the isomorphism concerning Hodge–Tate cohomology groups in Theorem 0.1 under the restriction $ie < p - 1$.

Our strategy is to first study the Hodge–Tate specialization of the A_{inf}-cohomology of $\bar{\mathfrak{X}}$. We can take advantage of the $L\eta$-construction of A_{inf}-cohomology and its Hodge–Tate specilizaton. This will provide us with two morphisms which enable us to use Lemma 2.8. In order to make this more precise, we need to introduce the framework of almost mathematics (derived category version) following [Bha18].

Definition 3.1 (The pair $(\mathcal{O}_C, \mathfrak{m})$)**.** *Let $\mathfrak{m}$ denote the maximal ideal of $\mathcal{O}_C$. We say an $\mathcal{O}_C$-module M is almost zero if $\mathfrak{m} \cdot M = 0$. A map $f : K \to L$ in $D(\mathcal{O}_C)$ is an almost isomorphism if the cohomology groups of the mapping cone of f are almost zero.*

Now we consider the almost derived category of $\mathcal{O}_C$-modules. Precisely, there are two functors:

$$D(\mathcal{O}_C) \xrightarrow{()^a} D(\mathcal{O}_C)^a := D(\mathcal{O}_C)/D(k), \qquad K \mapsto K^a$$

$$D(\mathcal{O}_C)^a \xrightarrow{()_*} D(\mathcal{O}_C), \qquad K^a \mapsto (K^a)_* := \mathrm{RHom}_{\mathcal{O}_C}(\mathfrak{m}, K)$$

where the Verdier quotient $D(\mathcal{O}_C)/D(k)$ is actually the localization of $D(\mathcal{O}_C)$ with respect to almost isomorphisms. The functor $()_*$ is right adjoint to the quotient functor $()^a$.

Lemma 3.2. *If C is spherically complete, i.e. any decreaing sequence of discs in C has nonempty intersection, we have $K \simeq (K^a)_*$ for any perfect complex $K \in D(\mathcal{O}_C)$.*

Proof. See [Bha18, Lemma 3.4]. □

There are similar constructions and results in the setting of A_{inf}-modules.

Definition 3.3 (The pair $(A_{\text{inf}}, W(\mathfrak{m}^\flat))$)**.** *An A_{inf}-module M is called almost zero if $W(\mathfrak{m}^\flat) \cdot M = 0$, where $W(\mathfrak{m}^\flat) = \mathrm{Ker}(A_{\text{inf}} \to W(\bar{k}))$. A map $f : K \to L$ in $D(A_{\text{inf}})$ is called an almost isomorphism if the cohomology groups of the mapping cone of f are almost zero.*

Similarly, we consider the almost derived category of A_{inf}-modules. Let $D_{\text{comp}}(A_{\text{inf}}) \subset D(A_{\text{inf}})$ be the full subcategory of all derived p-adically complete complexes. There are two functors:

$$D_{\text{comp}}(A_{\text{inf}}) \xrightarrow{()^a} D_{\text{comp}}(A_{\text{inf}})^a := D_{\text{comp}}(A_{\text{inf}})/D_{\text{comp}}(W(k)), \quad K \mapsto K^a$$

$$D_{\text{comp}}(A_{\text{inf}})^a \xrightarrow{()_*} D_{\text{comp}}(A_{\text{inf}}), \quad K^a \to (K^a)_* := \mathrm{RHom}_{A_{\text{inf}}}(W(\mathfrak{m}^\flat), K)$$

where the Verdier quotient $D_{\mathrm{comp}}(A_{\mathrm{inf}})^a := D_{\mathrm{comp}}(A_{\mathrm{inf}})/D_{\mathrm{comp}}(W(k))$ is actually the localization of $D_{\mathrm{comp}}(A_{\mathrm{inf}})$ with respect to almost isomorphisms. The functor $()_*$ is also right adjoint to $()^a$.

Lemma 3.4. *If C is spherically complete, we have $K \simeq (K^a)_*$ for any perfect complex $K \in D_{\mathrm{comp}}(A_{\mathrm{inf}})$.*

Proof. See [Bha18, Lemma 3.10]. □

Now we are ready to study the structure of the Hodge–Tate cohomology groups. We first state a lemma about the $L\eta$-functor, which will give us two important maps connecting Hodge–Tate cohomology and p-adic étale cohomology.

Lemma 3.5. *Let A be a commutative ring and $a \in A$ be a non-zero divisor. Assume $K \in D^{[0,s]}(A)$ with $H^0(K)$ being a-torsion free. Then there are natural maps $L\eta_a(K) \to K$ and $K \to L\eta_a(K)$ whose composition in either direction is a^s.*

Proof. This is [BMS18, Lemma 6.9]. We give the proof here.

Firstly, we choose a representative L of K such that L is a-torsion free. Then we apply the truncation functor $\tau^{\leq s}$ and $\tau^{\geq 0}$ to L, i.e. $\tau^{\leq s}\tau^{\geq 0}L = (\cdots \to 0 \to L^0/\mathrm{Im}(d^{-1}) \to L^1 \to \cdots \to L^{s-1} \to \ker(d^n) \to 0 \cdots)$. Since $K \in D^{[0,s]}(A)$, $\tau^{\leq s}\tau^{\geq 0}L$ is still isomorphic to K. Moreover $\tau^{\leq s}\tau^{\geq 0}L$ is still a-torsion free. It is easy to see that $\ker(d^n)$ is a-torsion free. For $L^0/\mathrm{Im}(d^{-1})$, suppose $\bar{x} \in L^0/\mathrm{Im}(d^{-1})$ is killed by a, then $ax \in \mathrm{Im}(d^{-1})$ for any lifting $x \in L^0$ of $\bar{x}$ and $d^0(ax) = ad^0(x) = 0$. As L^0 is a-torsion free, $d^0(x)$ must be 0, which implies that $x \in \ker(d^0)$. But this also means that $H^0(L) = H^0(K)$ has a-torsion. So $\tau^{\leq s}\tau^{\geq 0}L$ is still a-torsion free and we can apply η-functor to it.

There is a natural inclusion $\eta_a(\tau^{\leq s}\tau^{\geq 0}L) \to \tau^{\leq s}\tau^{\geq 0}L$. We can define another map $\tau^{\leq s}\tau^{\geq 0}L \to \eta_a(\tau^{\leq s}\tau^{\geq 0}L)$ by multiplying by a^s. Then the composition of these two maps in either direction is a^s. □

We may apply Lemma 3.5 to $A = \mathcal{O}_{\bar{\mathfrak{X}}}$, $a = \zeta_p - 1$ and $K = \tau^{\leq i}R\nu_*\widehat{\mathcal{O}}_X^+$. In fact $\tau^{\leq i}R\nu_*\widehat{\mathcal{O}}_X^+$ is in $D^{[0,i]}(\mathcal{O}_{\bar{\mathfrak{X}}})$ with $H^0(\tau^{\leq i}R\nu_*\widehat{\mathcal{O}}_X^+)$ being $(\zeta_p - 1)$-torsion-free. By the same argument in the proof of Lemma 3.5 we can always find a representative L of $\tau^{\leq i}R\nu_*\widehat{\mathcal{O}}_X^+$ such that L is $(\zeta_p - 1)$-torsion-free and $L^s = 0$ for any $s \notin [0, i]$. Then there are two natural maps which we denote by f and g,

$$f : L\eta_{\zeta_p - 1}(\tau^{\leq i}R\nu_*\widehat{\mathcal{O}}_X^+) \simeq \tau^{\leq i}\widetilde{\Omega}_{\bar{\mathfrak{X}}} \to \tau^{\leq i}R\nu_*\widehat{\mathcal{O}}_X^+$$

$$g : \tau^{\leq i}R\nu_*\widehat{\mathcal{O}}_X^+ \to \tau^{\leq i}\widetilde{\Omega}_{\bar{\mathfrak{X}}}$$

whose composition in either direction is $(\zeta_p - 1)^i$. The isomorphism $L\eta_{\zeta_p-1}(\tau^{\leq i}R\nu_*\widehat{\mathcal{O}}_X^+) \simeq \tau^{\leq i}\widetilde{\Omega}_{\bar{\mathfrak{X}}}$ is due to the commutativity of the $L\eta$ functor and the canonical truncation functor $\tau^{\leq i}$ (see [BMS18, Corollary 6.5]). Recall that for any $K \in D(\mathcal{O}_{\bar{\mathfrak{X}}})$, $\tau^{\leq i}K := (\cdots \to K^{i-1} \xrightarrow{d^{i-1}} \ker(d^i) \to 0 \to \cdots)$.

Passing to sheaf cohomology, we get two natural maps

$$f : \tau^{\leq i}R\Gamma_{\mathrm{zar}}(\bar{\mathfrak{X}}, \tau^{\leq i}\widetilde{\Omega}_{\bar{\mathfrak{X}}}) \to \tau^{\leq i}R\Gamma_{\mathrm{zar}}(\bar{\mathfrak{X}}, \tau^{\leq i}R\nu_*\widehat{\mathcal{O}}_X^+)$$

$$g : \tau^{\leq i}R\Gamma_{\mathrm{zar}}(\bar{\mathfrak{X}}, \tau^{\leq i}R\nu_*\widehat{\mathcal{O}}_X^+) \to \tau^{\leq i}R\Gamma_{\mathrm{zar}}(\bar{\mathfrak{X}}, \tau^{\leq i}\widetilde{\Omega}_{\bar{\mathfrak{X}}})$$

whose composition in either direction is $(\zeta_p - 1)^i$. Since there is an isomorphsim

$$\tau^{\leq i}R\Gamma_{\mathrm{zar}}(\bar{\mathfrak{X}}, \tau^{\leq i}\widetilde{\Omega}_{\bar{\mathfrak{X}}}) \simeq \tau^{\leq i}R\Gamma_{\mathrm{zar}}(\bar{\mathfrak{X}}, \widetilde{\Omega}_{\mathfrak{X}})$$

which is induced by the natural morphism $\tau^{\leq i}\widetilde{\Omega}_{\bar{\mathfrak{X}}} \to \widetilde{\Omega}_{\bar{\mathfrak{X}}}$, we get two maps

$$f : \tau^{\leq i} R\Gamma_{\mathrm{zar}}(\bar{\mathfrak{X}}, \widetilde{\Omega}_{\bar{\mathfrak{X}}}) \to \tau^{\leq i} R\Gamma_{\mathrm{zar}}(\bar{\mathfrak{X}}, R\nu_* \widehat{\mathcal{O}}_X^+)$$
$$g : \tau^{\leq i} R\Gamma_{\mathrm{zar}}(\bar{\mathfrak{X}}, R\nu_* \widehat{\mathcal{O}}_X^+) \to \tau^{\leq i} R\Gamma_{\mathrm{zar}}(\bar{\mathfrak{X}}, \widetilde{\Omega}_{\bar{\mathfrak{X}}})$$

whose composition in either direction is $(\zeta_p - 1)^i$.

Note that there is an isomorphism $R\Gamma_{\mathrm{zar}}(\bar{\mathfrak{X}}, R\nu_* \widehat{\mathcal{O}}_X^+) \simeq R\Gamma_{\mathrm{proét}}(X, \widehat{\mathcal{O}}_X^+)$. What we want to study at the end is not the pro-étale cohomology but the p-adic étale cohomology. Actually we get almost what we want. Recall the primitive comparison theorem due to Scholze.

Theorem 3.6 ([Sch13, Theorem 8.4])**.** *For any proper smooth adic space Y over C, there are natural almost isomorphisms*

$$R\Gamma_{\text{ét}}(Y, \mathbb{Z}_p) \otimes^{\mathbb{L}}_{\mathbb{Z}_p} \mathcal{O}_C \simeq R\Gamma_{\mathrm{proét}}(Y, \widehat{\mathcal{O}}_Y^+)$$

and

$$R\Gamma_{\text{ét}}(Y, \mathbb{Z}_p) \otimes^{\mathbb{L}}_{\mathbb{Z}_p} A_{\mathrm{inf}} \simeq R\Gamma_{\mathrm{proét}}(Y, \mathbb{A}_{\mathrm{inf},Y}).$$

Then by passing to the world of almost mathematics, we get two natural maps in $D(\mathcal{O}_C)^a$:

$$f^a : (\tau^{\leq i} R\Gamma_{\mathrm{zar}}(\bar{\mathfrak{X}}, \widetilde{\Omega}_{\bar{\mathfrak{X}}}))^a \to (\tau^{\leq i} R\Gamma_{\mathrm{zar}}(\bar{\mathfrak{X}}, R\nu_* \widehat{\mathcal{O}}_X^+))^a \simeq (\tau^{\leq i} R\Gamma_{\text{ét}}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathcal{O}_C)^a$$
$$g^a : (\tau^{\leq i} R\Gamma_{\mathrm{zar}}(\bar{\mathfrak{X}}, R\nu_* \widehat{\mathcal{O}}_X^+))^a \simeq (\tau^{\leq i} R\Gamma_{\text{ét}}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathcal{O}_C)^a \to (\tau^{\leq i} R\Gamma_{\mathrm{zar}}(\bar{\mathfrak{X}}, \tau^{\leq i}\widetilde{\Omega}_{\bar{\mathfrak{X}}}))^a$$

Lemma 3.7. *The complex $\tau^{\leq i} R\Gamma_{\mathrm{HT}}(\bar{\mathfrak{X}}) = \tau^{\leq i} R\Gamma_{\mathrm{zar}}(\bar{\mathfrak{X}}, \widetilde{\Omega}_{\bar{\mathfrak{X}}})$ (resp. $\tau^{\leq i} R\Gamma_{A_{\mathrm{inf}}}(\bar{\mathfrak{X}})$) is a perfect complex of $\mathcal{O}_C$-modules (resp. A_{inf}-modules).*

Proof. Recall that we have

$$R\Gamma_{\mathrm{HT}}(\bar{\mathfrak{X}}) \simeq R\Gamma_{\mathfrak{S}}(\mathfrak{X}) \otimes^{\mathbb{L}}_{\mathfrak{S},\alpha} A_{\mathrm{inf}} \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} A_{\mathrm{inf}}/\widetilde{\xi} \simeq R\Gamma_{\mathfrak{S}}(\mathfrak{X}) \otimes^{\mathbb{L}}_{\mathfrak{S},\beta} \mathcal{O}_K \otimes^{\mathbb{L}}_{\mathcal{O}_K} \mathcal{O}_C.$$

Since $R\Gamma_{\mathfrak{S}}(\mathfrak{X})$ is a perfect complex of $\mathfrak{S}$-modules and $R\Gamma_{\mathrm{HT}}(\mathfrak{X}) := R\Gamma_{\mathfrak{S}}(\mathfrak{X}) \otimes_{\mathfrak{S},\beta} \mathcal{O}_K$, the Hodge–Tate cohomology $R\Gamma_{\mathrm{HT}}(\mathfrak{X})$ of $\mathfrak{X}$ is a perfect complex of $\mathcal{O}_K$-modules by [Sta19, Lemma 066W]. Moreover as $\mathcal{O}_K$ is a Notherian local ring, the cohomology groups $H^i_{\mathrm{HT}}(\mathfrak{X})$ are finitely generated $\mathcal{O}_K$-modules and so finitely presented $\mathcal{O}_K$-modules. So we see that every Hodge–Tate cohomology group $H^i_{\mathrm{HT}}(\bar{\mathfrak{X}})$ is also finitely presented over $\mathcal{O}_C$. By Lemma 2.3, this means $H^i_{\mathrm{HT}}(\bar{\mathfrak{X}}) \cong \bigoplus_{j=1}^n \mathcal{O}_C/\pi_j$ for some $\pi_j \in \mathcal{O}_C$. So $H^i_{\mathrm{HT}}(\bar{\mathfrak{X}})$ is perfect. The lemma hence follows from [Sta19, Lemma 066U]. For $\tau^{\leq i} R\Gamma_{A_{\mathrm{inf}}}(\bar{\mathfrak{X}})$, this follows from [BMS18, Lemma 4.9] stating that each $H^i_{A_{\mathrm{inf}}}(\bar{\mathfrak{X}})$ is perfect. □

As $\tau^{\leq i} R\Gamma_{\text{ét}}(X, \mathbb{Z}_p)$ and $\tau^{\leq i} R\Gamma_{\mathrm{zar}}(\bar{\mathfrak{X}}, \widetilde{\Omega}_{\bar{\mathfrak{X}}})$ are perfect complexes, Lemma 3.2 shows that if C is spherically complete, we then have $(\tau^{\leq i} R\Gamma(\bar{\mathfrak{X}}, \widetilde{\Omega}_{\bar{\mathfrak{X}}}))^a_* \simeq \tau^{\leq i} R\Gamma(\mathfrak{X}, \widetilde{\Omega}_{\mathfrak{X}})$ and $(\tau^{\leq i} R\Gamma_{\text{ét}}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathcal{O}_C)^a_* \simeq \tau^{\leq i} R\Gamma_{\text{ét}}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathcal{O}_C$. By moving back to the real world, we have two maps

$$(f^a)_* : \tau^{\leq i} R\Gamma_{\mathrm{zar}}(\bar{\mathfrak{X}}, \widetilde{\Omega}_{\bar{\mathfrak{X}}}) \to \tau^{\leq i} R\Gamma_{\text{ét}}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathcal{O}_C$$
$$(g^a)_* : \tau^{\leq i} R\Gamma_{\text{ét}}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathcal{O}_C \to \tau^{\leq i} R\Gamma_{\mathrm{zar}}(\bar{\mathfrak{X}}, \widetilde{\Omega}_{\bar{\mathfrak{X}}})$$

whose composition in either direction is $(\zeta_p - 1)^i$. These two maps induce maps between cohomology groups for any $n \leq i$.

$$f : H^n(\bar{\mathfrak{X}}, \widetilde{\Omega}_{\bar{\mathfrak{X}}}) \to H^n_{\text{ét}}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathcal{O}_C$$
$$g : H^n_{\text{ét}}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathcal{O}_C \to H^n(\bar{\mathfrak{X}}, \widetilde{\Omega}_{\bar{\mathfrak{X}}})$$

whose composition in either direction is $(\zeta_p - 1)^i$.

Now we come to the following key theorem:

Theorem 3.8. *Let $\mathfrak{X}$ be a proper smooth formal scheme over $\mathcal{O}_K$, where $\mathcal{O}_K$ is the ring of integers in a complete discretely valued non-archimedean extension K of $\mathbb{Q}_p$ with perfect residue field k and ramification degree e. Let $\mathcal{O}_C$ be the ring of integers in a complete and algebraically closed extension C of K and X be the adic generic fibre of $\bar{\mathfrak{X}} := \mathfrak{X} \times_{\mathrm{Spf}(\mathcal{O}_K)} \mathrm{Spf}(\mathcal{O}_C)$. Assuming $ie < p-1$, there is an isomorphism of $\mathcal{O}_C$-modules between the Hodge–Tate cohomology group and the p-adic étale cohomology group*

$$H^i_{\mathrm{HT}}(\bar{\mathfrak{X}}) := H^i(\bar{\mathfrak{X}}, \widetilde{\Omega}_{\mathfrak{X}}) \cong H^i_{\text{ét}}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathcal{O}_C.$$

Proof. Note that replacing C by its spherical completion C' will not make any difference to this theorem. The spherical completion always exists (cf. [Rob13, Chapter 3]), which is still complete and algebraically closed. On one hand, p-adic étale cohomology is insensitive to such extensions in the rigid-analytic setting (cf. [Hub13, Section 0.3.2]). On the other hand, by the base change of prismatic cohomology (cf. [BS19, Theorem 1.8]), we have $H^i_{\mathrm{HT}}(\overline{\mathfrak{X}} \otimes_{\mathcal{O}_C} \mathcal{O}_{C'}) \cong H^i_{\mathrm{HT}}(\overline{\mathfrak{X}}) \otimes_{\mathcal{O}_C} \mathcal{O}_{C'}$ and the natural injection $\mathcal{O}_C \to \mathcal{O}_{C'}$ is flat.

So now we assume C is spherically complete. Using the flat base change along $\alpha : \mathfrak{S} \to A_{\mathrm{inf}}$ from the Breuil–Kisin cohomology to the A_{inf}-cohomology, we see that $H^i(\bar{\mathfrak{X}}, \widetilde{\Omega}_{\bar{\mathfrak{X}}})$ has a decomposition as $\mathcal{O}_C^n \oplus (\bigoplus_{j=1}^m \mathcal{O}_C/\pi^{m_j})$. By requiring $ie < p-1$, we have $v((\zeta_p - 1)^i) < v(\pi)$ in $\mathcal{O}_C$ as $v((\zeta_p - 1)^{p-1}) = v(p)$ and $v(p) = v(\pi^e)$. Now the theorem follows from Lemma 2.8 and the existence of maps

$$f : H^i(\bar{\mathfrak{X}}, \widetilde{\Omega}_{\bar{\mathfrak{X}}}) \to H^i_{\text{ét}}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathcal{O}_C$$
$$g : H^i_{\text{ét}}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathcal{O}_C \to H^i(\bar{\mathfrak{X}}, \widetilde{\Omega}_{\bar{\mathfrak{X}}}).$$

□

4. The unramified case: comparison theorem

In this section, let $\mathcal{O}_K = W(k)$, i.e. the ramification degree $e = 1$. We will study the relation between the p-adic étale cohomology group $H^i_{\text{ét}}(X, \mathbb{Z}_p)$ and the crystalline cohomology group $H^i_{\mathrm{crys}}(\mathfrak{X}_k/W(k))$ in the unramifed case. Note that in the unramified case, the crystalline cohomology $R\Gamma_{\mathrm{crys}}(\mathfrak{X}_k/W(k))$ is canonically isomorphic to the de Rham cohomology $R\Gamma_{\mathrm{dR}}(\mathfrak{X}/\mathcal{O}_K)$.

In order to prove the integral comparison theorem, we first relate Hodge–Tate cohomology to Hodge cohomology. And then we can use Theorem 3.8 to get a link between Hodge cohomology and p-adic étale cohomology. The last step is to study the Hodge-to-de Rham spectral sequence and we can prove the converse to [BMS18, Theorem 14.5], which results in the final comparison theorem.

4.1. Decomposition of Hodge–Tate cohomology groups.

In this subsection, we explain how to relate Hodge–Tate cohomology to Hodge cohomology. In fact, we can show that the complex of sheaves $\tau^{\leq p-1}\widetilde{\Omega}_{\bar{\mathfrak{X}}}$ is formal in the unramified case.

Theorem 4.1. *The complex of sheaves $\tau^{\leq p-1}\widetilde{\Omega}_{\bar{\mathfrak{X}}}$ is formal, i.e. there is an isomorphism*

$$\gamma : \bigoplus_{i=0}^{p-1} \Omega^i_{\bar{\mathfrak{X}}}\{-i\}[-i] \simeq \tau^{\leq p-1}\widetilde{\Omega}_{\bar{\mathfrak{X}}},$$

where $\Omega^i_{\bar{\mathfrak{X}}} := \varprojlim \Omega^i_{(\bar{\mathfrak{X}}/p^n)/(\mathcal{O}_C/p^n)}$ *is the* $\mathcal{O}_{\bar{\mathfrak{X}}}$*-module of continuous differentials and* $\Omega^i_{\bar{\mathfrak{X}}}\{-i\}$ *is the Breuil–Kisin twist of* $\Omega^i_{\bar{\mathfrak{X}}}$*.*

Proof. We proceed by first showing that $\tau^{\leq 1}\widetilde{\Omega}_{\bar{\mathfrak{X}}}$ is formal and then constructing the general isomorphism in the statement. In this proof, $\mathbb{L}_{\bar{\mathfrak{X}}/\mathbb{Z}_p}$ and $\mathbb{L}_{\bar{\mathfrak{X}}/W(k)}$ always mean the derived p-adic complete cotangent complex.

By [BMS18, Proposition 8.15], there is an isomorphism $\tau^{\leq 1}\widetilde{\Omega}_{\bar{\mathfrak{X}}} \simeq \mathbb{L}_{\bar{\mathfrak{X}}/\mathbb{Z}_p}\{-1\}[-1]$. Considering the sequence of sheaves $\mathbb{Z}_p \to W(k) \to \mathcal{O}_{\bar{\mathfrak{X}}}$, there is an associated distinguished triangle

$$\widehat{\mathbb{L}_{W(k)/\mathbb{Z}_p}} \otimes^{\mathbb{L}}_{W(k)} \mathcal{O}_{\bar{\mathfrak{X}}} \to \mathbb{L}_{\bar{\mathfrak{X}}/\mathbb{Z}_p} \to \mathbb{L}_{\bar{\mathfrak{X}}/W(k)}.$$

By derived Nakayama lemma, we know that $\widehat{\mathbb{L}_{W(k)/\mathbb{Z}_p}}$ vanishes as $\mathbb{L}_{k/\mathbb{F}_p}$ vanishes. Therefore, we have

$$\mathbb{L}_{\bar{\mathfrak{X}}/\mathbb{Z}_p}\{-1\}[-1] \simeq \mathbb{L}_{\bar{\mathfrak{X}}/W(k)}\{-1\}[-1].$$

For any affine open $\mathrm{Spf}(R) \subset \mathfrak{X}$, write $\bar{R}$ for the base change $R \otimes_{W(k)} \mathcal{O}_C$ and $\widehat{\bar{R}}$ for its p-adic completion. Then we have $\widehat{\mathbb{L}_{\widehat{\bar{R}}/W(k)}} \simeq \widehat{\mathbb{L}_{\bar{R}/W(k)}}$.

By the Künneth property of cotangent complex (cf. [Ill06]), we get

$$\mathbb{L}_{\bar{R}/W(k)} \simeq (\mathbb{L}_{\mathcal{O}_C/W(k)} \otimes^{\mathbb{L}}_{W(k)} R) \oplus (\mathbb{L}_{R/W(k)} \otimes^{\mathbb{L}}_{W(k)} \mathcal{O}_C).$$

Applying the derived p-adic completion functor which is exact, we see

$$\widehat{\mathbb{L}_{R\otimes_{W(k)}\mathcal{O}_C/W(k)}} \simeq (\widehat{\mathbb{L}_{\mathcal{O}_C/W(k)} \otimes^{\mathbb{L}}_{W(k)} R}) \oplus (\widehat{\mathbb{L}_{R/W(k)} \otimes^{\mathbb{L}}_{W(k)} \mathcal{O}_C}).$$

On one hand, we have

$$\widehat{\mathbb{L}_{\mathcal{O}_C/W(k)} \otimes_{W(k)} R} \simeq \widehat{\mathbb{L}_{\mathcal{O}_C/W(k)} \otimes_{W(k)} R} \simeq \widehat{\bar{R}\{1\}[1]}.$$

As $\widehat{\bar{R}}$ coincides with the derived p-adic completion of $\bar{R}$ (cf. [Sta19, Example 0BKG]), we have $\widehat{\mathbb{L}_{\mathcal{O}_C/W(k)} \otimes_{W(k)} R} \simeq \widehat{\bar{R}}\{1\}[1]$. On the other hand, by the base change property of cotangent complex (cf. [Ill06]), we get $\mathbb{L}_{R/W(k)} \otimes_{W(k)} \mathcal{O}_C \simeq \mathbb{L}_{\bar{R}/\mathcal{O}_C}$. The derived p-adic completion $\widehat{\mathbb{L}_{\bar{R}/\mathcal{O}_C}}$ is isomorphic to $\varprojlim_n \Omega^1_{(\bar{R}/p^n)/(\mathcal{O}_C/p^n)}$. In fact as $\bar{R}/p^n$ is a smooth $\mathcal{O}_C/p^n$-algebra for all n, we have

$$\widehat{\mathbb{L}_{\bar{R}/\mathcal{O}_C}} \simeq \mathrm{Rlim}(\mathbb{L}_{\bar{R}/\mathcal{O}_C} \otimes^{\mathbb{L}}_{\mathbb{Z}_p} \mathbb{Z}_p/p^n) \simeq \mathrm{Rlim}(\mathbb{L}_{(\bar{R}/p^n)/(\mathcal{O}_C/p^n)}) \simeq \varprojlim_n \Omega^1_{(\bar{R}/p^n)/(\mathcal{O}_C/p^n)}.$$

So finally there is an isomorphism

$$\mathbb{L}_{\bar{\mathfrak{X}}/W(k)} \simeq \mathcal{O}_{\bar{\mathfrak{X}}}\{1\}[1] \oplus \Omega^1_{\bar{\mathfrak{X}}}$$

and we get a decomposition $\tau^{\leq 1}\widetilde{\Omega}_{\bar{\mathfrak{X}}} \simeq \mathcal{O}_{\bar{\mathfrak{X}}} \oplus \Omega^1_{\bar{\mathfrak{X}}}\{-1\}[-1]$. In particular, we have a map $\gamma_1 : \Omega^1_{\bar{\mathfrak{X}}}\{-1\}[-1] \to \widetilde{\Omega}_{\bar{\mathfrak{X}}}$ which gives the Hodge–Tate isomorphism $C^{-1} : \Omega^1_{\bar{\mathfrak{X}}}\{-1\} \to \mathcal{H}^1(\widetilde{\Omega}_{\bar{\mathfrak{X}}})$ (cf. [BMS18, Theorem 8.3]).

Now we consider the map for any $i \leq p-1$ given by

$$(\Omega^1_{\bar{\mathfrak{X}}})^{\otimes i} \to \Omega^i_{\bar{\mathfrak{X}}}, \quad \omega_1 \otimes \cdots \otimes \omega_i \mapsto \omega_1 \wedge \cdots \wedge \omega_i$$

It has an anti-symmetrization section a as shown in [DI87], given by

$$a(\omega_1 \wedge \cdots \wedge \omega_i) = (1/i!) \sum_{s \in Sym_i} \mathrm{sgn}(s)\omega_{s(1)} \otimes \cdots \otimes \omega_{s(i)}$$

Then we define γ_i as the composite

$$\Omega^i_{\bar{\mathfrak{X}}}\{-i\}[-i] \xrightarrow{a} (\Omega^1_{\bar{\mathfrak{X}}}\{-1\})^{\otimes i}[-i] \simeq (\Omega^1_{\bar{\mathfrak{X}}}\{-1\}[-1])^{\otimes^{\mathbb{L}} i} \xrightarrow{\gamma_1^{\otimes^{\mathbb{L}} i}} (\widetilde{\Omega}_{\bar{\mathfrak{X}}})^{\otimes^{\mathbb{L}} i} \xrightarrow{\text{multi}} \widetilde{\Omega}_{\bar{\mathfrak{X}}}$$

Note that $\widetilde{\Omega}_{\bar{\mathfrak{X}}}$ is a commutative $\mathcal{O}_{\bar{\mathfrak{X}}}$-algebra object in $D(\mathcal{O}_{\bar{\mathfrak{X}}})$ (see Remark 1.7). By applying $\mathcal{H}^i$, we have

$$\Omega^i_{\bar{\mathfrak{X}}}\{-i\} \xrightarrow{a} \mathcal{H}^i((\Omega^1_{\bar{\mathfrak{X}}}\{-1\}[-1])^{\otimes^{\mathbb{L}} i}) \cong (\mathcal{H}^1(\Omega^1_{\bar{\mathfrak{X}}}\{-1\}[-1]))^{\otimes i} \xrightarrow{\gamma_1^{\otimes^{\mathbb{L}} i}} (\mathcal{H}^1(\widetilde{\Omega}_{\bar{\mathfrak{X}}}))^{\otimes i} \to \mathcal{H}^i((\widetilde{\Omega}_{\bar{\mathfrak{X}}})^{\otimes^{\mathbb{L}} i}) \xrightarrow{\text{multi}} \mathcal{H}^i(\widetilde{\Omega}_{\bar{\mathfrak{X}}})$$

Since the Hodge–Tate isomorphism is compatible with multiplication (cf. [BMS18, Corollary 8.13]), this composite is exactly the Hodge–Tate isomorphism $C^{-1} : \Omega^i_{\bar{\mathfrak{X}}}\{-i\} \simeq \mathcal{H}^i(\widetilde{\Omega}_{\bar{\mathfrak{X}}})$. So we get the desired isomorphism $\gamma = \bigoplus_{i=0}^{p-1} \gamma_i : \bigoplus_{i=0}^{p-1} \Omega^i_{\bar{\mathfrak{X}}}\{-i\}[-i] \simeq \tau^{\leq p-1}\widetilde{\Omega}_{\bar{\mathfrak{X}}}$. □

Remark 4.2. Note that the key input in the proof above is the Hodge–Tate isomorphism $C^{-1} : \Omega^i_{\bar{\mathfrak{X}}}\{-i\} \to \mathcal{H}^i(\widetilde{\Omega}_{\bar{\mathfrak{X}}})$. In general, there is a Hodge–Tate isomorphism for any bounded prism (A, I) (cf. [BS19, Theorem 4.10]) and also a generalization of the isomorphism $\tau^{\leq 1}\widetilde{\Omega}_{\bar{\mathfrak{X}}} \simeq \mathbb{L}_{\bar{\mathfrak{X}}/\mathbb{Z}_p}\{-1\}[-1]$.

The map $\mathcal{O}_{\bar{\mathfrak{X}}} \to \tau^{\leq 1}\widetilde{\Omega}_{\bar{\mathfrak{X}}}$ splits as an $\mathcal{O}_{\bar{\mathfrak{X}}}$-module map if and only if $\bar{\mathfrak{X}}$ lifts to $A_{\inf}/\widetilde{\xi}^2$ (cf. [BMS18, Remark 8.4]). In the ramified case, this seems to be hardly satisfied due to the non-vanishing of the cotangent complex $\mathbb{L}_{\mathcal{O}_K/W(k)}$. Note that $H^0(\mathbb{L}_{\mathcal{O}_K/W(k)}) \simeq \Omega^1_{\mathcal{O}_K/W(k)}$ is generated by one element (cf. [Ser13, Chapter III, Proposition 14]).

Corollary 4.3. *There is a natural decomposition for any $n \leq p-1$,*

$$H^n_{\mathrm{HT}}(\bar{\mathfrak{X}}) = H^n(\bar{\mathfrak{X}}, \widetilde{\Omega}_{\bar{\mathfrak{X}}}) \cong \bigoplus_{i=0}^{n} H^{n-i}(\bar{\mathfrak{X}}, \Omega^i_{\bar{\mathfrak{X}}}\{-i\}).$$

4.2. **Hodge-to-de Rham spectral sequence.** In this subsection, we study the Hodge-to-de Rham spectral sequence and finish the proof of the integral comparison theorem in the unramified case. More precisely, we will prove the converse to Theorem 1.11 by analyzing the length of the torsion submodule of de Rham cohomology groups and that of p-adic étale cohomology groups.

Note that we have the Hodge-to-de Rham spectral sequence

$$E_1^{i,j} = H^j(\bar{\mathfrak{X}}, \Omega^i_{\bar{\mathfrak{X}}}) \Longrightarrow H^{i+j}(\bar{\mathfrak{X}}, \Omega^\bullet_{\bar{\mathfrak{X}}}) = H^{i+j}_{\mathrm{dR}}(\bar{\mathfrak{X}}/\mathcal{O}_C)$$

As $\bar{\mathfrak{X}} = \mathfrak{X} \times_{\mathrm{Spf}(W(k))} \mathrm{Spf}(\mathcal{O}_C)$, this spectral sequence can be seen as the flat base change to $\mathcal{O}_C$ of the Hodge-to-de Rham spectral sequence of $\mathfrak{X}$ over $W(k)$. This tells us $E_\infty^{i,j}$ is a finitely presented $\mathcal{O}_C$-module (note that $E_\infty^{i,j}$ is also a subquotient of $H^j(\bar{\mathfrak{X}}, \Omega^i_{\bar{\mathfrak{X}}})$).

For any integers i and n such that $0 \leq i \leq n$, we have the abutment filtration

$$0 = F^{n+1} \subset F^n \subset \cdots \subset F^0 = H^n_{\mathrm{dR}}(\bar{\mathfrak{X}}/\mathcal{O}_C)$$

and the short exact sequences

$$0 \to F^{i+1} \to F^i \to E_\infty^{i,n-i} \to 0.$$

Now we consider the normalized length $l_{\mathcal{O}_C}$ for finitely presented torsion $\mathcal{O}_C$-modules. Recall that this length behaves additively under short exact sequences and $l_{\mathcal{O}_C}(\mathcal{O}_C/p) = 1$. For any finitely presented $\mathcal{O}_C$-module M, one can deduce from Lemma 2.3 that M_{tor}

is also a finitely presented $\mathcal{O}_C$-module and so is M_{tor}/p^m for any $m > 0$. Then we have the following lemma:

Lemma 4.4. *For any short exact sequence of finitely presented $\mathcal{O}_C$-modules*

$$0 \to A \to B \to C \to 0$$

we have $l_{\mathcal{O}_C}(B_{\mathrm{tor}}) \leq l_{\mathcal{O}_C}(A_{\mathrm{tor}}) + l_{\mathcal{O}_C}(C_{\mathrm{tor}})$ and $l_{\mathcal{O}_C}(B_{\mathrm{tor}}/p^m) \leq l_{\mathcal{O}_C}(A_{\mathrm{tor}}/p^m) + l_{\mathcal{O}_C}(C_{\mathrm{tor}}/p^m)$ for any $m > 0$.

Proof. For the first statement, it is easy to see that $M = B_{\mathrm{tor}}/A_{\mathrm{tor}}$ is a submodule of C_{tor}, so we have $l_{\mathcal{O}_C}(M) = l_{\mathcal{O}_C}(B_{\mathrm{tor}}) - l_{\mathcal{O}_C}(A_{\mathrm{tor}}) \leq l_{\mathcal{O}_C}(C_{\mathrm{tor}})$ by the additivity of the length. For the second one, we have an exact sequence

$$M[p^m] \to A_{\mathrm{tor}}/p^m \to B_{\mathrm{tor}}/p^m \to M/p^m \to 0$$

So we get $l_{\mathcal{O}_C}(B_{\mathrm{tor}}/p^m) \leq l_{\mathcal{O}_C}(A_{\mathrm{tor}}/p^m) + l_{\mathcal{O}_C}(M/p^m)$. Then we need to prove that $l_{\mathcal{O}_C}(M/p^m) \leq l_{\mathcal{O}_C}(C_{\mathrm{tor}}/p^m)$. More generally, given two finitely presented torsion $\mathcal{O}_C$ modules $N_1 \subset N_2$, there is an exact sequence

$$N[p^m] \to N_1/p^m \to N_2/p^m \to N/p^m \to 0$$

where $N = N_2/N_1$. Note that $l_{\mathcal{O}_C}(N[p^m]) = l_{\mathcal{O}_C}(N/p^m)$. In fact, this follows from the exact sequence

$$0 \to N[p^m] \to N \xrightarrow{p^m} N \to N/p^m \to 0$$

Hence $l_{\mathcal{O}_C}(N_2/p^m) \geq l_{\mathcal{O}_C}(N/p^m) + l_{\mathcal{O}_C}(N_1/p^m) - l_{\mathcal{O}_C}(N[p^m]) = l_{\mathcal{O}_C}(N_1/p^m)$. So finally we get $l_{\mathcal{O}_C}(B_{\mathrm{tor}}/p^m) \leq l_{\mathcal{O}_C}(A_{\mathrm{tor}}/p^m) + l_{\mathcal{O}_C}(C_{\mathrm{tor}}/p^m)$. □

Corollary 4.5. *For any integers i and n such that $0 \leq i \leq n$ and any positive integer m, we have $l_{\mathcal{O}_C}(F^i_{\mathrm{tor}}/p^m) \leq l_{\mathcal{O}_C}(F^{i+1}_{\mathrm{tor}}/p^m) + l_{\mathcal{O}_C}({E_\infty^{i,n-i}}_{\mathrm{tor}}/p^m)$. In particular, $l_{\mathcal{O}_C}(H^n_{\mathrm{dR}}(\bar{\mathfrak{X}}/\mathcal{O}_C)_{\mathrm{tor}}/p^m) \leq \sum_{i=0}^n l_{\mathcal{O}_C}({E_\infty^{i,n-i}}_{\mathrm{tor}}/p^m)$.*

Recall that the rational Hodge-to-de Rham spectral sequence degenerates at E_1 page:

Theorem 4.6 ([Sch13, Corollary 1.8])**.** *For any proper smooth rigid analytic space Y over C, the Hodge-to-de Rham spectral sequence*

$$E_1^{i,j} = H^j(Y, \Omega^i_Y) \Longrightarrow H^{i+j}_{\mathrm{dR}}(Y/C)$$

degenerates at E_1. Moreover, for all $i \geq 0$,

$$\sum_{j=0}^{i} \dim_C H^{i-j}(Y, \Omega^j_Y) = \dim_C H^i_{\mathrm{dR}}(Y/C) = \dim_{\mathbb{Q}_p} H^i_{\text{ét}}(Y, \mathbb{Q}_p).$$

As a consequence, we have the following lemma:

Lemma 4.7. *For any $m > 0$, we have*

$$l_{\mathcal{O}_C}({E_\infty^{i,n-i}}_{\mathrm{tor}}/p^m) \leq l_{\mathcal{O}_C}(H^{n-i}(\bar{\mathfrak{X}}, \Omega^i_{\bar{\mathfrak{X}}})_{\mathrm{tor}}/p^m).$$

Proof. Theorem 4.14 tells us that the integral Hodge-to-de Rham spectral sequence degenerates at E_1 after inverting p. This means that the coboundaries $B_\infty^{i,n-i}$ must be a finitely presented torsion $\mathcal{O}_C$-module. Consider the short exact sequence

$$0 \to B_\infty^{i,n-i} \to Z_\infty^{i,n-i} \to E_\infty^{i,n-i} \to 0.$$

For any $x \in {E_\infty^{i,n-i}}_{\mathrm{tor}}$, there exists $\widehat{x} \in Z_\infty^{i,n-i}$ whose image in $E_\infty^{i,n-i}$ is x. As $E_\infty^{i,n-i}$ is killed by p^N for some large enough N, we can see that $p^N\widehat{x}$ is in $B_\infty^{i,n-i} \subset {Z_\infty^{i,n-i}}_{\mathrm{tor}}$. So we have another short exact sequence

$$0 \to B_\infty^{i,n-i} \to {Z_\infty^{i,n-i}}_{\mathrm{tor}} \to {E_\infty^{i,n-i}}_{\mathrm{tor}} \to 0.$$

Then by the additivity of the length, we get that

$$l_{\mathcal{O}_C}({E_\infty^{i,n-i}}_{\mathrm{tor}}/p^m) \leq l_{\mathcal{O}_C}({Z_\infty^{i,n-i}}_{\mathrm{tor}}/p^m),$$

and

$$l_{\mathcal{O}_C}({Z_\infty^{i,n-i}}_{\mathrm{tor}}/p^m) = l_{\mathcal{O}_C}({Z_\infty^{i,n-i}}_{\mathrm{tor}}[p^m]) \leq l_{\mathcal{O}_C}(H^{n-i}(\bar{\mathfrak{X}}, \Omega^i_{\bar{\mathfrak{X}}})_{\mathrm{tor}}[p^m]) = l_{\mathcal{O}_C}(H^{n-i}(\bar{\mathfrak{X}}, \Omega^i_{\bar{\mathfrak{X}}})_{\mathrm{tor}}/p^m).$$

So we have $l_{\mathcal{O}_C}({E_\infty^{i,n-i}}_{\mathrm{tor}}/p^m) \leq l_{\mathcal{O}_C}(H^{n-i}(\bar{\mathfrak{X}}, \Omega^i_{\bar{\mathfrak{X}}})_{\mathrm{tor}}/p^m)$. □

Now we prove the converse to Theorem 1.11.

Theorem 4.8. *For any positive integer m and any integer n such that $0 \leq n < p-1$, we have*

$$l_{\mathcal{O}_C}(H^n_{\mathrm{dR}}(\bar{\mathfrak{X}}/\mathcal{O}_C)_{\mathrm{tor}}/p^m) \leq l_{\mathcal{O}_C}(H^n_{\text{ét}}(X, \mathbb{Z}_p)_{\mathrm{tor}} \otimes_{\mathbb{Z}_p} \mathcal{O}_C/p^m).$$

Proof. By Theorem 3.8 and Theorem 4.3, we have

$$H^n_{\text{ét}}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathcal{O}_C \cong H^n_{\mathrm{HT}}(\bar{\mathfrak{X}}) \cong \bigoplus_{i=0}^{n} H^{n-i}(\bar{\mathfrak{X}}, \Omega^i_{\bar{\mathfrak{X}}}).$$

This implies that

$$\sum_{i=0}^{n} l_{\mathcal{O}_C}(H^{n-i}(\bar{\mathfrak{X}}, \Omega^i_{\bar{\mathfrak{X}}})_{\mathrm{tor}}/p^m) = l_{\mathcal{O}_C}(H^n_{\text{ét}}(X, \mathbb{Z}_p)_{\mathrm{tor}} \otimes_{\mathbb{Z}_p} \mathcal{O}_C/p^m)$$

Moreover, by Corollary 4.5 and Lemma 4.7, we have

$$l_{\mathcal{O}_C}(H^n_{\mathrm{dR}}(\bar{\mathfrak{X}}/\mathcal{O}_C)_{\mathrm{tor}}/p^m) \leq \sum_{i=0}^{n} l_{\mathcal{O}_C}({E_\infty^{i,n-i}}_{\mathrm{tor}}/p^m) \leq \sum_{i=0}^{n} l_{\mathcal{O}_C}(H^{n-i}(\bar{\mathfrak{X}}, \Omega^i_{\bar{\mathfrak{X}}})_{\mathrm{tor}}/p^m).$$

So we get that

$$l_{\mathcal{O}_C}(H^n_{\mathrm{dR}}(\bar{\mathfrak{X}}/\mathcal{O}_C)_{\mathrm{tor}}/p^m) \leq l_{\mathcal{O}_C}(H^n_{\text{ét}}(X, \mathbb{Z}_p)_{\mathrm{tor}} \otimes_{\mathbb{Z}_p} \mathcal{O}_C/p^m)$$

□

Theorem 4.9. *For any $n < p-1$, there is an isomorphism of $W(k)$-modules*

$$H^n_{\mathrm{crys}}(\mathfrak{X}_k/W(k)) \cong H^n_{\text{ét}}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} W(k).$$

Proof. We first prove that there is an isomorphism of $\mathcal{O}_C$-modules

$$H^n_{\mathrm{dR}}(\bar{\mathfrak{X}}/\mathcal{O}_C) \cong H^n_{\text{ét}}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathcal{O}_C.$$

Note that Theorem 1.11 tells us that for any positive integer m,

$$l_{\mathcal{O}_C}(H^n_{\text{ét}}(X, \mathbb{Z}_p)_{\mathrm{tor}} \otimes_{\mathbb{Z}_p} \mathcal{O}_C/p^m) \leq l_{\mathcal{O}_C}(H^n_{\mathrm{dR}}(\bar{\mathfrak{X}}/\mathcal{O}_C)_{\mathrm{tor}}/p^m)$$

So they must be equal by Theorem 4.8. This means that $H^n_{\text{ét}}(X, \mathbb{Z}_p)_{\mathrm{tor}} \otimes_{\mathbb{Z}_p} \mathcal{O}_C \cong H^n_{\mathrm{dR}}(\bar{\mathfrak{X}}/\mathcal{O}_C)_{\mathrm{tor}}$ by Lemma 2.4. Furthermore by [BMS18, Theorem 1.1], the $\mathcal{O}_C$-modules $H^n_{\mathrm{dR}}(\bar{\mathfrak{X}}/\mathcal{O}_C)$ and $H^n_{\text{ét}}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathcal{O}_C$ have the same rank. So we have $H^n_{\mathrm{dR}}(\bar{\mathfrak{X}}/\mathcal{O}_C) \cong H^n_{\text{ét}}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathcal{O}_C$.

On the other hand, there is an isomorphism between de Rham cohomology and crystalline cohomology in the unramified case (cf. [Ber06])

$$H^n_{\mathrm{dR}}(\mathfrak{X}/W(k)) \cong H^n_{\mathrm{crys}}(\mathfrak{X}_k/W(k)).$$

We also have

$$H^n_{\mathrm{dR}}(\mathfrak{X}/W(k)) \otimes_{W(k)} \mathcal{O}_C \cong H^n_{\mathrm{dR}}(\bar{\mathfrak{X}}/\mathcal{O}_C)$$

by base change of de Rham cohomology. So finally we get the isomorphism of $W(k)$-modules

$$H^n_{\mathrm{crys}}(\mathfrak{X}_k/W(k)) \cong H^n_{\text{ét}}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} W(k).$$

□

4.3. **Degeneration of the Hodge-to-de Rham spectral sequence.** In this subsection, we assume $d = \dim\mathfrak{X} < p-1$, where $\dim\mathfrak{X}$ means the relative dimension of $\mathfrak{X}$. We will improve Theorem 4.9 by considering all cohomological degrees and study the degeneration of the Hodge-to-de Rham spectral sequence. These will follow from improvements of Theorem 3.8 and Corollary 4.3.

We begin with an improvement of Corollary 4.3.

Lemma 4.10. *When $d = \dim\mathfrak{X} < p-1$, we have*

$$H^n_{\mathrm{HT}}(\bar{\mathfrak{X}}) = H^n(\bar{\mathfrak{X}}, \widetilde{\Omega}_{\bar{\mathfrak{X}}}) \cong \bigoplus_{i=0}^{n} H^{n-i}(\bar{\mathfrak{X}}, \Omega^i_{\bar{\mathfrak{X}}}\{-i\}).$$

for all n.

Proof. Recall the Hodge–Tate isomorphism: $H^i(\widetilde{\Omega}_{\overline{\mathfrak{X}}}) \cong \Omega^i_{\overline{\mathfrak{X}}}$ (cf. [BMS18, Theorem 8.3]). When $i \geq p-1 > d$, we have $\Omega^i_{\overline{\mathfrak{X}}} = 0$. This implies $\tau^{\leq p-2}\widetilde{\Omega}_{\overline{\mathfrak{X}}} \simeq \widetilde{\Omega}_{\overline{\mathfrak{X}}}$. In particular, the whole complex $\widetilde{\Omega}_{\overline{\mathfrak{X}}}$ is formal by Theorem 4.1, from which this lemma follows.

□

Next we study the comparison between Hodge–Tate cohomology and p-adic étale cohomology. Recall that we have the following two maps

$$f : \tau^{\leq d}\widetilde{\Omega}_{\overline{\mathfrak{X}}} \to \tau^{\leq d} R\nu_* \widehat{\mathcal{O}}^+_X$$

$$g : \tau^{\leq d} R\nu_* \widehat{\mathcal{O}}^+_X \to \tau^{\leq d}\widetilde{\Omega}_{\overline{\mathfrak{X}}}$$

whose composition in either direction is $(\zeta_p - 1)^d$.

We claim that $R\nu_*\widehat{\mathcal{O}}^+_X$ is almost supported in degrees $\leq d$, i.e. there is an almost isomorphism $\tau^{\leq d} R\nu_*\widehat{\mathcal{O}}^+_X \simeq R\nu_*\widehat{\mathcal{O}}^+_X$. We will check this locally.

Recall that an $\mathcal{O}_C$-algebra R is called formally smooth (as in [BMS18]) if it is a p-adically complete flat $\mathcal{O}_C$-algebra such that R/p is a smooth $\mathcal{O}_C/p$-algebra. And a formally smooth $\mathcal{O}_C$-algebra R is called small (cf. [BMS18, Definition 8.5]) if there is an étale map

$$\square : \mathrm{Spf} R \to \mathrm{Spf}\mathcal{O}_C\langle T_1^{\pm 1}, \cdots, T_d^{\pm 1}\rangle.$$

We call such étale map a framing. Given a framing, we can define

$$R_\infty := R\widehat{\otimes}_{\mathcal{O}_C\langle T_1^{\pm 1}, \cdots, T_d^{\pm 1}\rangle}\mathcal{O}_C\langle T_1^{\pm 1/p^\infty}, \cdots, T_d^{\pm 1/p^\infty}\rangle$$

which is an integral perfectoid ring. And there is an action of $\Gamma = \mathbb{Z}_p(1)^d$ on it. More precisely, choose a compatible system (ζ_{p^m}) of p-power roots of unity and let γ_i, $i =$

$1, \cdots, d$ be generators of Γ. Then γ_i acts by sending T_i^{1/p^m} to $\zeta_{p^m} T_i^{1/p^m}$ and sending T_j^{1/p^m} to T_j^{1/p^m} for $j \neq i$.

By Faltings' almost purity theorem (cf. [Fal88, Chapter 1, Section 3 and 4]) and [Sch13, Proposition 3.5, Proposition 3.7, Corollary 6.6], there is an almost isomorphism of complexes of $\mathcal{O}_C$-modules

$$R\Gamma(\Gamma, R_\infty) \to R\Gamma(Y_{\text{proét}}, \widehat{\mathcal{O}}_Y^+),$$

where $Y = \mathrm{Spa}(R[1/p], R)$. Moreover the continuous group cohomology on the left hand side can be calculated by the Koszul complex $K_{R_\infty}(\gamma_1 - 1, \cdots, \gamma_d - 1)$ by [BMS18, Lemma 7.3], which can be defined as

$$K_{R_\infty}(\gamma_1 - 1, \cdots, \gamma_d - 1) = R_\infty \otimes_{\mathbb{Z}[\gamma_1, \cdots, \gamma_d]} (\bigotimes_{i=1}^{d} (\mathbb{Z}[\gamma_1, \cdots, \gamma_d] \xrightarrow{\gamma_i - 1} \mathbb{Z}[\gamma_1, \cdots, \gamma_d])).$$

This complex sits in non-negative cohomological degrees $[0, d]$. On the other hand, since $\overline{\mathfrak{X}}$ is a proper smooth formal scheme over $\mathcal{O}_C$, there exists a basis of small affine opens (cf. [Ked03, Theorem 2], [Bha18, Lemma 4.9]). So when $i > d$, we get that $R^i \nu_* \widehat{\mathcal{O}}_X^+$ is almost zero.

So now we have an almost isomorphism: $\tau^{\leq d} R\nu_* \widehat{\mathcal{O}}_X^+ \to R\nu_* \widehat{\mathcal{O}}_X^+$. Taking cohomology, we then get an almost isomorphism: $R\Gamma(\overline{\mathfrak{X}}, \tau^{\leq d} R\nu_* \widehat{\mathcal{O}}_X^+) \to R\Gamma(\overline{\mathfrak{X}}, R\nu_* \widehat{\mathcal{O}}_X^+)$. Again by Theorem 3.6, we get two maps in almost derived category $D(\mathcal{O}_C)^a$:

$$f : (R\Gamma(\overline{\mathfrak{X}}, \tau^{\leq d} \widetilde{\Omega}_{\overline{\mathfrak{X}}}))^a \to (R\Gamma_{\text{ét}}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathcal{O}_C)^a$$

$$g : (R\Gamma_{\text{ét}}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathcal{O}_C)^a \to (R\Gamma(\overline{\mathfrak{X}}, \tau^{\leq d} \widetilde{\Omega}_{\overline{\mathfrak{X}}}))^a$$

whose composition in either direction is $(\zeta_p - 1)^d$. Since both sides are perfect complexes of $\mathcal{O}_C$-modules, we get two maps in the derived category $D(\mathcal{O}_C)$:

$$f : R\Gamma(\overline{\mathfrak{X}}, \tau^{\leq d} \widetilde{\Omega}_{\overline{\mathfrak{X}}}) \to R\Gamma_{\text{ét}}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathcal{O}_C$$

$$g : R\Gamma_{\text{ét}}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathcal{O}_C \to R\Gamma(\overline{\mathfrak{X}}, \tau^{\leq d} \widetilde{\Omega}_{\overline{\mathfrak{X}}})$$

whose composition in either direction is $(\zeta_p - 1)^d$.

Now as $\tau^{\leq d} \widetilde{\Omega}_{\overline{\mathfrak{X}}} \simeq \widetilde{\Omega}_{\overline{\mathfrak{X}}}$, we have $R\Gamma(\overline{\mathfrak{X}}, \tau^{\leq d} \widetilde{\Omega}_{\overline{\mathfrak{X}}}) \simeq R\Gamma(\overline{\mathfrak{X}}, \widetilde{\Omega}_{\overline{\mathfrak{X}}}) = R\Gamma_{\mathrm{HT}}(\overline{\mathfrak{X}})$. So we get two maps

$$f : R\Gamma_{\mathrm{HT}}(\overline{\mathfrak{X}}) \to R\Gamma_{\text{ét}}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathcal{O}_C$$

$$g : R\Gamma_{\text{ét}}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathcal{O}_C \to R\Gamma_{\mathrm{HT}}(\overline{\mathfrak{X}})$$

whose composition in either direction is $(\zeta_p - 1)^d$.

Theorem 4.11. *There is an isomorphism of $\mathcal{O}_C$-modules for all n*

$$H^n_{\mathrm{HT}}(\overline{\mathfrak{X}}) \cong H^n_{\text{ét}}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathcal{O}_C.$$

Proof. This follows from Lemma 2.8. □

Theorem 4.12. *Assume $d = \dim \mathfrak{X} < p - 1$. Then there is an isomorphism of $W(k)$-modules for all n*

$$H^n_{\mathrm{crys}}(\mathfrak{X}_k / W(k)) \cong H^n_{\text{ét}}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} W(k).$$

Proof. Note that if Theorem 4.8 is true for all n, then Theorem 4.9 is true for all n. And if Theorem 3.8 and Theorem 4.3 are true for all cohomological degrees, then Theorem 4.9 is true for all cohomological degrees. So this theorem follows from Theorem 4.10 and Theorem 4.11.

□

Corollary 4.13. *If $d = \dim(\mathfrak{X}) < p-1$, the coboundaries $B_\infty^{i,n-i}$ vanish for all n. In particular the Hodge-to-de Rham spectral sequence degenerates at E_1-page.*

Proof. By Theorem 4.10 and Theorem 4.11, we see that

$$\sum_{i=0}^{n} l_{\mathcal{O}_C}(H^{n-i}(\bar{\mathfrak{X}}, \Omega^i_{\bar{\mathfrak{X}}})_{\mathrm{tor}}/p^m) = l_{\mathcal{O}_C}(H^n_{\text{ét}}(X, \mathbb{Z}_p)_{\mathrm{tor}} \otimes_{\mathbb{Z}_p} \mathcal{O}_C/p^m)$$

is true for all n.

Theorem 4.12 shows that for all n we have

$$l_{\mathcal{O}_C}(H^n_{\mathrm{dR}}(\bar{\mathfrak{X}}/\mathcal{O}_C)_{\mathrm{tor}}/p^m) = l_{\mathcal{O}_C}(H^n_{\text{ét}}(X, \mathbb{Z}_p)_{\mathrm{tor}} \otimes_{\mathbb{Z}_p} \mathcal{O}_C/p^m).$$

So we conclude that

$$l_{\mathcal{O}_C}(H^n_{\mathrm{dR}}(\bar{\mathfrak{X}}/\mathcal{O}_C)_{\mathrm{tor}}/p^m) = \sum_{i=0}^{n} l_{\mathcal{O}_C}(H^{n-i}(\bar{\mathfrak{X}}, \Omega^i_{\bar{\mathfrak{X}}})_{\mathrm{tor}}/p^m)$$

holds for all n.

As we have seen in the proof of Lemma 4.7, there are inequalities for all n

$$l_{\mathcal{O}_C}({E_\infty^{i,n-i}}_{\mathrm{tor}}/p^m) \leq l_{\mathcal{O}_C}({Z_\infty^{i,n-i}}_{\mathrm{tor}}/p^m) \leq l_{\mathcal{O}_C}(H^{n-i}(\bar{\mathfrak{X}}, \Omega^i_{\bar{\mathfrak{X}}})_{\mathrm{tor}}/p^m).$$

Also by using the same argument as in the proof of Theorem 4.8, we have

$$l_{\mathcal{O}_C}(H^n_{\mathrm{dR}}(\bar{\mathfrak{X}}/\mathcal{O}_C)_{\mathrm{tor}}/p^m) \leq \sum_{i=0}^{n} l_{\mathcal{O}_C}({E_\infty^{i,n-i}}_{\mathrm{tor}}/p^m) \leq \sum_{i=0}^{n} l_{\mathcal{O}_C}(H^{n-i}(\bar{\mathfrak{X}}, \Omega^i_{\bar{\mathfrak{X}}})_{\mathrm{tor}}/p^m).$$

holds for all n. But these inequalities are in fact equalities. This means that

$$l_{\mathcal{O}_C}({E_\infty^{i,n-i}}_{\mathrm{tor}}/p^m) = l_{\mathcal{O}_C}({Z_\infty^{i,n-i}}_{\mathrm{tor}}/p^m) = l_{\mathcal{O}_C}(H^{n-i}(\bar{\mathfrak{X}}, \Omega^i_{\bar{\mathfrak{X}}})_{\mathrm{tor}}/p^m).$$

In other words, the coboundaries $B_\infty^{i,n-i}$ vanish as we have $l_{\mathcal{O}_C}(B_\infty^{i,n-i}) = l_{\mathcal{O}_C}({Z_\infty^{i,n-i}}_{\mathrm{tor}}) - l_{\mathcal{O}_C}({E_\infty^{i,n-i}}_{\mathrm{tor}}) = 0$. So the Hodge-to-de Rham spectral sequence degenerates at E_1-page. □

Remark 4.14. We collect some other results about the degeneration of the (integral) Hodge-to-de Rham spectral sequence.

(i) In [FM87, Corollary 2.7], Fontaine and Messing have proved that for any proper smooth (formal) scheme $\mathfrak{Y}$ whose special fiber has dimension strictly less than p, the Hodge-to-de Rham spectral sequence degenerates at E_1-page. Their proof makes use of the syntomic cohomology.

(ii) For any projective smooth scheme $\mathfrak{Y}$ over $W(k)$ where k is a perfect field of characteristic p, Kazuya Kato has proved that if $\dim(\mathfrak{Y}) \leq p$, the Hodge-to-de Rham spectral sequence degenerates at E_1-page and the de Rham cohomology groups are Fontane–Laffaille modules (cf. [K$^+$87, chapter II, Proposition 2.5]).

(iii) For any proper smooth formal scheme $\mathfrak{Y}$ over $\mathcal{O}_K$, where $\mathcal{O}_K$ is the ring of integers of a complete discretely valued non-archimedean extension K of $\mathbb{Q}_p$ with perfect residue field k and ramification degree e. Let $\mathfrak{S}$ be $W(k)[[u]]$ and E be an Eisenstein polynomial for a uniformizer π of $\mathcal{O}_K$. Shizhang Li has proved that if $\mathfrak{Y}$ can be lifted to $\mathfrak{S}/(E^2)$ and $\dim(\mathfrak{Y}) \cdot e < p-1$, then the Hodge-to-de Rham spectral sequence is split degenerate (cf. [Li20, Theorem 1.1]). His proof uses Theorem 0.1.

5. The ramified case: comparison theorem

In this section, we will get some properties about the torsion in the Breuil–Kisin cohomology groups $H^{i+1}_{\mathfrak{S}}(\mathfrak{X})$ when $ie < p-1$ and obtain an integral comparison theorem comparing the p-adic étale cohomology groups and the crystalline cohomology groups.

5.1. Torsion in Breuil–Kisin cohomology groups.

Note that the ring $\mathfrak{S} = W(k)[[u]]$ is a two-dimensional regular local ring. The structure of $\mathfrak{S}$-modules is subtle in general (see Remark 5.12). In particular, it is difficult to study the u-torsion. But in our case, it turns out to be simpler.

Recall that we can define $A_{\mathrm{inf}} := W(\mathcal{O}_C^\flat)$ as in Definition 1.5. We start by studying the A_{inf}-cohomology groups of $\overline{\mathfrak{X}}$.

Lemma 5.1. *The A_{inf}-cohomology group $H^{i+1}_{A_{\mathrm{inf}}}(\bar{\mathfrak{X}}) := H^{i+1}(\bar{\mathfrak{X}}, A\Omega_{\bar{\mathfrak{X}}})$ is $\widetilde{\xi}$-torsion-free for any i such that $ie < p-1$.*

Proof. We assume that C is spherically complete. As in the proof of Theorem 3.8, we see that the spherical completion of C exists and is still complete and algebraically closed. Moreover since $R\Gamma_{A_{\mathrm{inf}}}(\overline{\mathfrak{X}}) \simeq R\Gamma_{\mathfrak{S}}(\mathfrak{X}) \otimes^{\mathbb{L}}_{\mathfrak{S},\alpha} A_{\mathrm{inf}}$ where $\alpha : \mathfrak{S} \to A_{\mathrm{inf}}$ is the faithfully flat map taking (E) to $(\widetilde{\xi})$, we have $H^{i+1}_{A_{\mathrm{inf}}}(\bar{\mathfrak{X}}) \cong H^{i+1}_{\mathfrak{S}}(\mathfrak{X}) \otimes_{\mathfrak{S},\alpha} A_{\mathrm{inf}}$, in particular $H^{i+1}_{A_{\mathrm{inf}}}(\bar{\mathfrak{X}})$ is $\widetilde{\xi}$-torsion-free if and only if $H^{i+1}_{\mathfrak{S}}(\mathfrak{X})$ is E-torsion-free as $(\alpha(E)) = (\widetilde{\xi})$. So it does not matter whether C is spherically complete or not.

As in Chapter 3, we apply Lemma 3.5 to the complex of sheaves of A_{inf}-modules $\tau^{\leq i} R\nu_* \mathbb{A}_{\mathrm{inf},X}$ and the element $\mu \in A_{\mathrm{inf}}$. Precisely, in the category $D^{[0,i]}(\overline{\mathfrak{X}}, A_{\mathrm{inf}})$, we get two natural maps

$$f : \tau^{\leq i} R\nu_* \mathbb{A}_{\mathrm{inf},X} \to L\eta_\mu \tau^{\leq i} R\nu_* \mathbb{A}_{\mathrm{inf},X} \simeq \tau^{\leq i} A\Omega_{\overline{\mathfrak{X}}}$$

$$g : \tau^{\leq i} A\Omega_{\overline{\mathfrak{X}}} \simeq L\eta_\mu \tau^{\leq i} R\nu_* \mathbb{A}_{\mathrm{inf},X} \to \tau^{\leq i} R\nu_* \mathbb{A}_{\mathrm{inf},X}$$

whose composition in either direction is μ^i.

We consider the the complex of sheaves $\tau^{\leq i} R\nu_* \widehat{\mathcal{O}}_X^+$ as in the category $D(\bar{\mathfrak{X}}, A_{\mathrm{inf}})$ via the map $A_{\mathrm{inf}} \xrightarrow{\widetilde{\theta}} \mathcal{O}_C \to \mathcal{O}_{\bar{\mathfrak{X}}}$. Moreover it is in the category $D^{[0,i]}(\bar{\mathfrak{X}}, A_{\mathrm{inf}})$.

There is a map $\tau^{\leq i} R\nu_* \mathbb{A}_{\mathrm{inf,X}} \to \tau^{\leq i} R\nu_* \widehat{\mathcal{O}}_X^+$ induced by $\widetilde{\theta} : \mathbb{A}_{\mathrm{inf,X}} \to \widehat{\mathcal{O}}_X^+$. So we can get a commutative diagram

$$\begin{tikzcd}
L\eta_\mu \tau^{\leq i} R\nu_* \mathbb{A}_{\mathrm{inf},X} \arrow[r, "s_1"] \arrow[d, shift left, "f_1"] & L\eta_\mu \tau^{\leq i} R\nu_* \widehat{\mathcal{O}}_X^+ \arrow[d, shift left, "f_2"] \\
\tau^{\leq i} R\nu_* \mathbb{A}_{\mathrm{inf},X} \arrow[r, "s_2"] \arrow[u, shift left, "g_1"] & \tau^{\leq i} R\nu_* \widehat{\mathcal{O}}_X^+ \arrow[u, shift left, "g_2"]
\end{tikzcd}$$

where the composition of f_j with g_j in either direction is μ^i for $j = 1, 2$. Note that $L\eta_{\zeta_p-1}\tau^{\leq i}R\nu_*\widehat{\mathcal{O}}_X^+$ is isomorphic to $L\eta_\mu\tau^{\leq i}R\nu_*\widehat{\mathcal{O}}_X^+$ in $D(\bar{\mathfrak{X}}, A_{\mathrm{inf}})$.

Recall that $\tau^{\leq i}R\Gamma_{A_{\mathrm{inf}}}(\bar{\mathfrak{X}})$ is a perfect complex of A_{inf}-modules according to Lemma 3.7. Then by the second almost isomorphism in Theorem 3.6 and Lemma 3.4, we can get two maps

$$f : \tau^{\leq i}R\Gamma_{A_{\mathrm{inf}}}(\bar{\mathfrak{X}}) \to \tau^{\leq i}R\Gamma_{\text{ét}}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} A_{\mathrm{inf}}.$$

$$g : \tau^{\leq i}R\Gamma_{\text{ét}}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} A_{\mathrm{inf}} \to \tau^{\leq i}R\Gamma_{A_{\mathrm{inf}}}(\bar{\mathfrak{X}})$$

whose composition in either direction is μ^i.

By taking cohomology, we can obtain another commutative diagram

$$\begin{tikzcd}
H^i_{A_{\mathrm{inf}}}(\bar{\mathfrak{X}}) \arrow[r, "s_1"] \arrow[d, shift left, "f_1"] & H^i_{\mathrm{HT}}(\bar{\mathfrak{X}}) \arrow[d, shift left, "f_2"] \\
H^i_{\text{ét}}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} A_{\mathrm{inf}} \arrow[r, "s_2"] \arrow[u, shift left, "g_1"] & H^i_{\text{ét}}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathcal{O}_C \arrow[u, shift left, "g_2"]
\end{tikzcd}$$

Note that $\mathrm{Coker}(s_1)$ is in fact $H^{i+1}_{A_{\mathrm{inf}}}(\bar{\mathfrak{X}})[\widetilde{\xi}]$ and $\mathrm{Coker}(s_2) = 0$.

Therefore we get two induced maps

$$H^{i+1}_{A_{\mathrm{inf}}}(\bar{\mathfrak{X}})[\widetilde{\xi}] \underset{g_3}{\overset{f_3}{\rightleftarrows}} 0$$

where the composition of f_3 and g_3 in either direction is μ^i. Since $H^{i+1}_{A_{\mathrm{inf}}}(\bar{\mathfrak{X}})[\widetilde{\xi}] \simeq H^{i+1}_{\mathfrak{S}}(\mathfrak{X})[E] \otimes_{\mathcal{O}_K} \mathcal{O}_C$ as $\mathcal{O}_C$-modules, it has a decomposition as $\mathcal{O}_C^m \oplus (\bigoplus_{s=1}^n \mathcal{O}_C/\pi^{n_s})$. Note that the image of μ under the reduction $A_{\mathrm{inf}} \to A_{\mathrm{inf}}/\widetilde{\xi}$ is $\zeta_p - 1$ and $v((\zeta_p - 1)^i) < v(\pi)$ when $ie < p - 1$. We then can get $H^{i+1}_{A_{\mathrm{inf}}}(\bar{\mathfrak{X}})[\widetilde{\xi}] = 0$ by Lemma 2.8.

□

Remark 5.2. The previous version of this lemma covers the cohomological degree i such that $ie < p - 1$. We want to thank Shizhang Li for pointing out that the previous proof can be improved slightly to include the cohomological degree $i + 1$ such that $ie < p - 1$.

In the next lemma, we give an equivalent statement to the $\widetilde{\xi}$-torsion-freeness for some special A_{inf}-modules.

Lemma 5.3. *Let M be a finitely presented A_{inf}-module such that $M[\frac{1}{p}]$ is finite projective over $A_{\mathrm{inf}}[\frac{1}{p}]$, and let $x \in \mathfrak{m}\backslash(p)$ where $\mathfrak{m}$ is the maximal ideal of A_{inf}. Then M is $\widetilde{\xi}$-torsion-free if and only if it is x-torison-free.*

Proof. Note that the radical ideal of (p, x) is the maximal ideal. If there exists $a \in M$ such that $xa = 0$, then for any other $y \in m\backslash(p)$, we have $y^n a = 0$ for any sufficiently large n. This is because all torsion in M is killed by some power of p. Then this lemma follows. □

Corollary 5.4. *When $ie < p - 1$, the A_{inf}-cohomology group $H^{i+1}_{A_{\mathrm{inf}}}(\bar{\mathfrak{X}})$ is ξ-torsion-free and the Breuil–Kisin cohomology group $H^{i+1}_{\mathfrak{S}}(\mathfrak{X})$ is both E-torsion-free and u-torsion-free.*

Recall that for any finitely presented A_{inf}-module M such that $M[\frac{1}{p}]$ is finite projective over $A_{\mathrm{inf}}[\frac{1}{p}]$, we have the following proposition:

Proposition 5.5 ([BMS18] Proposition 4.13)**.** *Let M be a finitely presented A_{inf}-module such that $M[\frac{1}{p}]$ is finite projective over $A_{\mathrm{inf}}[\frac{1}{p}]$. Then there is a functorial exact sequence*

$$0 \to M_{\mathrm{tor}} \to M \to M_{\mathrm{free}} \to \overline{M} \to 0$$

satisfying:

(i) *M_{tor}, the torsion submodule of M, is finitely presented and perfect as an A_{inf}-module, and is killed by p^n for $n \gg 0$.*
(ii) *M_{free} is a finite free A_{inf}-module.*
(iii) *$\overline{M}$ is finitely presented and perfect as an A_{inf}-module, and is supported at the closed point $s \in \mathrm{Spec}(A_{\mathrm{inf}})$.*

Here we recall the construction of the free module M_{free}. Since M/M_{tor} is torsion-free, the quasi-coherent sheaf associated to it restricts to a vector bundle on $\mathrm{Spec}(A_{\mathrm{inf}})\backslash\{s\}$ by [BMS18, Lemma 4.10]. By [BMS18, Lemma 4.6], the global section of this vector bundle is a finite free A_{inf}-module, which gives M_{free}. In particular, if M/M_{tor} is free itself, then $M/M_{\mathrm{tor}} = M_{\mathrm{free}}$. For more details, see the proof of [BMS18, Proposition 4.13].

By applying this proposition to $H^i_{A_{\mathrm{inf}}}(\bar{\mathfrak{X}})$, we can obtain the following lemma saying that $H^i_{A_{\mathrm{inf}}}(\bar{\mathfrak{X}})$ is a direct sum of its torsion submodule and a free A_{inf}-module.

Lemma 5.6. *For any i such that $ie < p-1$, the term $\overline{M}$ in the functorial exact sequence*

$$0 \to M_{\mathrm{tor}} \to M = H^i_{A_{\mathrm{inf}}}(\bar{\mathfrak{X}}) \to M_{\mathrm{free}} \to \overline{M} \to 0$$

vanishes.

Proof. Let $N = H^i_{\text{ét}}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} A_{\mathrm{inf}}$, we have two maps $f : M \to N$ and $g : N \to M$, whose composition in either direction is μ^i. Then we have a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M_{\mathrm{tor}} & \longrightarrow & M & \longrightarrow & M_{\mathrm{free}} & \longrightarrow & \overline{M} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & N_{\mathrm{tor}} & \longrightarrow & N & \longrightarrow & N_{\mathrm{free}} & \longrightarrow & 0 & \longrightarrow & 0
\end{array}$$

by functoriality. All the vertical maps have inverses up to μ^i.

On the other hand, the exact sequence associated to $H^i_{A_{\mathrm{inf}}}(\bar{\mathfrak{X}})$ is the flat base change of the canonical exact sequence associated to $H^i_{\mathfrak{S}}(\mathfrak{X})$ (see [BMS18, Proposition 4.3 and 4.13]). Hence $\overline{M} \cong \overline{H^i_{\mathfrak{S}}(\mathfrak{X})} \otimes_{\mathfrak{S}} A_{\mathrm{inf}}$ and $\overline{M}/\widetilde{\xi} \cong (\overline{H^i_{\mathfrak{S}}(\mathfrak{X})}/E) \otimes_{\mathfrak{S}} A_{\mathrm{inf}}$ where $\overline{H^i_{\mathfrak{S}}(\mathfrak{X})}$ is a torsion $\mathfrak{S}$-module and is killed by some power of (p, u). Again, by using the decomposition of $\overline{H^i_{\mathfrak{S}}(\mathfrak{X})}/E$ and the fact that $v((\zeta_p - 1)^i) < v(\pi)$ when $ie < p-1$, we get $\overline{H^i_{\mathfrak{S}}(\mathfrak{X})}/E = 0$ and $\overline{M}/\widetilde{\xi} = 0$ by Lemma 2.8. Then $\overline{M} = 0$ follows from Nakayama lemma.

□

Corollary 5.7. *For any i such that $ie < p-1$, the A_{inf}-cohomology group $H^i_{A_{\mathrm{inf}}}(\overline{\mathfrak{X}})$ is a direct sum of a free A_{inf}-module and its torsion submodule. Also, the Breuil–Kisin cohomology group $H^i_{\mathfrak{S}}(\mathfrak{X})$ is a direct sum of a free $\mathfrak{S}$-module and its torsion submodule.*

In the following part, we consider the torsion submodules of the cohomology groups $H^i_{A_{\mathrm{inf}}}(\bar{\mathfrak{X}})$ and $H^i_{\mathfrak{S}}(\mathfrak{X})$, and let $H^i_{A-\mathrm{tor}}$, $H^i_{\mathfrak{S}-\mathrm{tor}}$ denote them respectively.

We first prove a key lemma which enables us to study the structure of $H^n_{\mathfrak{S}-\mathrm{tor}}$.

Lemma 5.8. *For any i such that $ie < p-1$, the modules $(p^s H^i_{A-\mathrm{tor}})/p^m$ (resp. $(p^s H^i_{\mathfrak{S}-\mathrm{tor}})/p^m$) are $\widetilde{\xi}$-torsion-free (resp. E-torsion-free) for all non-negative integers m, s.*

Proof. Recall that we have two injective map $f : H^i_{A-\mathrm{tor}} \to H^i_{\text{ét}-\mathrm{tor}} \otimes_{\mathbb{Z}_p} A_{\mathrm{inf}}$ and $g : H^i_{\text{ét}-\mathrm{tor}} \otimes_{\mathbb{Z}_p} A_{\mathrm{inf}} \to H^i_{A-\mathrm{tor}}$ whose composition in either direction is μ^i. These induce two new maps (we still denote f and g) between $((p^s H^i_{A-\mathrm{tor}})/p^m)[\widetilde{\xi}]$ and $((p^s H^i_{\text{ét}-\mathrm{tor}}) \otimes_{\mathbb{Z}_p} A_{\mathrm{inf}}/p^m)[\widetilde{\xi}]$ whose composition in either direction is μ^i. Note that $((p^s H^i_{\text{ét}-\mathrm{tor}}) \otimes_{\mathbb{Z}_p} A_{\mathrm{inf}}/p^m)[\widetilde{\xi}] = 0$. This means $((p^s H^i_{A-\mathrm{tor}})/p^m)[\widetilde{\xi}]$ is killed by μ^i. As $((p^s H^i_{A-\mathrm{tor}})/p^m)[\widetilde{\xi}] \cong ((p^s H^i_{\mathfrak{S}-\mathrm{tor}})/p^m)[E] \otimes_{\mathfrak{S}} A_{\mathrm{inf}}$ admits a decomposition as $\bigoplus_{t=1}^n \mathcal{O}_C/\pi^{n_t}$ and $v((\zeta_p-1)^i) < v(\pi)$, the module $((p^s H^i_{A-\mathrm{tor}})/p^m)[\widetilde{\xi}]$ must be 0 by Lemma 2.8. Since $((p^s H^i_{A-\mathrm{tor}})/p^m)[\widetilde{\xi}] \cong (p^s H^i_{\mathfrak{S}-\mathrm{tor}})/p^m)[E] \otimes_{\mathfrak{S},\alpha} A_{\mathrm{inf}}$ and the map $\alpha : \mathfrak{S} \to A_{\mathrm{inf}}$ is faithfully flat, we also have $(p^s H^i_{\mathfrak{S}-\mathrm{tor}})/p^m)$ is E-torsion-free. □

In order to determine the module structure of $H^i_{\mathfrak{S}}(\mathfrak{X})$, we need the following lemma.

Lemma 5.9. *Let M be a finitely presented torsion $\mathfrak{S}$-module. If $M/p \cong (\mathfrak{S}/p)^n$ and $pM \cong \bigoplus_{i=1}^r \mathfrak{S}/p^{n_i}$, we have an isomorphism of $\mathfrak{S}$-modules: $M \cong \bigoplus_{i=1}^n \mathfrak{S}/p^{m_i}$.*

Proof. The proof is just that of [Bre98b, Lemma 2.3.1.1], simply by replacing S by $\mathfrak{S}$. For readers' convenience, we give the proof here.

Choose $m \geq 0$ such that $p^m M = 0$. Let $(e_1, e_2, \cdots, e_n)$ be a basis of M/pM over $\mathfrak{S}/p$ and we choose their liftings $\widehat{e}_1, \widehat{e}_2, \cdots, \widehat{e}_n$ in M. By Nakayama lemma, we see that M is generated by $(\widehat{e}_1, \widehat{e}_2, \cdots, \widehat{e}_n)$ as a $\mathfrak{S}/p^m$-module. So $(p\widehat{e}_1, p\widehat{e}_2, \cdots, p\widehat{e}_n)$ generate the $\mathfrak{S}/p^m$-module pM.

After renumbering $(\widehat{e}_i)$, we can suppose that the images of $p\widehat{e}_1, p\widehat{e}_2, \cdots, p\widehat{e}_r$ in $pM \otimes_{\mathfrak{S}/p^m} k$ form a basis over k. Choose $f_1, \cdots, f_r \in pM$ such that $pM \cong \bigoplus_{i=1}^r \mathfrak{S}/p^{n_i}\mathfrak{S} \cdot f_i$. Then there exists a $r \times r$-matrix $A \in M_r(\mathfrak{S}/p^m\mathfrak{S})$ such that $(f_1, f_2, \cdots, f_r)A = (p\widehat{e}_1, p\widehat{e}_2, \cdots, p\widehat{e}_r)$. Since $A \mod (p, u) \in GL_r(k)$, we know that A is in $GL_r(\mathfrak{S}/p^m\mathfrak{S})$. So we can replace $(\widehat{e}_1, \widehat{e}_2, \cdots, \widehat{e}_r)$ by $(\widehat{e}_1, \widehat{e}_2, \cdots, \widehat{e}_r)A^{-1}$ and suppose $p\widehat{e}_i = f_i$ for $1 \leq i \leq r$.

For $r+1 \leq j \leq n$, there exist $a_{ij} \in \mathfrak{S}/p^m\mathfrak{S}$ for $1 \leq i \leq r$ such that $p\widehat{e}_j = \sum_{i=1}^r a_{ij} f_i = \sum_{i=1}^r a_{ij} p\widehat{e}_i$. Again, we can replace $\widehat{e}_j$ by $\widehat{e}_j - \sum_{i=1}^r a_{ij}\widehat{e}_i$ for $r+1 \leq j \leq n$. That means we can suppose $p\widehat{e}_j = 0$ for $r+1 \leq j \leq n$.

Finally, we can construct a surjective morphism of $\mathfrak{S}/p^m\mathfrak{S}$-module:

$$h : M' = (\bigoplus_{i=1}^r \mathfrak{S}/p^{n_i+1}\mathfrak{S} \times g_i) \bigoplus (\bigoplus_{i=r+1}^n \mathfrak{S}/p\mathfrak{S} \times g_i) \to M$$

$$g_i \mapsto \widehat{e}_i$$

Note that the morphism $h : M' \to M$ induces two isomorphisms: $h_1 : pM' \xrightarrow{\sim} pM$ and $h_2 : M'/pM' \xrightarrow{\sim} M/pM$ under the choice of $\widehat{e}_i$, $1 \leq i \leq n$. For any x such that $h(x) = 0$, if $x \in pM'$, then $x = 0$ since $h_2(x) = h(x) = 0$. If $x \notin pM'$, then $h_2(\bar{x}) = 0$ implies that $x \in pM'$ where $\bar{x}$ is the image of x in M'/pM'. So $h : M' \to M$ must be an isomorphism. We are done. □

Corollary 5.10. *Let M be a finitely presented torsion $\mathfrak{S}$-module which is killed by some power of p. If $(p^s M)/p$ is u-torsion-free for all $s \geq 0$, the module M admits a decomposition as $M \cong \bigoplus_{i=1}^n \mathfrak{S}/p^{m_i}$.*

Proof. To prove this corollary, we want to apply Lemma 5.9 to M. Note that M/p is u-torsion-free by our assumption, therefore finite free as a $\mathfrak{S}/p = k[[u]]$-module. So we need to prove that pM admits a nice decomposition as in Lemma 5.9. Since the module $(pM)/p$ is also u-torsion-free by our assumption, we only need to prove that p^2M admits a nice decomposition as in Lemma 5.9. We can continue this process until that we need to prove p^mM admits a nice decomposition as in Lemma 5.9 for some m such that M is killed by p^{m+1}. As $p(p^mM) = 0$ and $(p^mM)/p = p^mM$ has no u-torsion, we see that p^mM is a free $\mathfrak{S}/p$-module by Lemma 5.9. So we are done.

□

5.2. **Integral comparison theorem.** Now we state our main theorem of this section comparing the module structure of Breuil–Kisin cohomology groups to that of p-adic étale cohomology groups.

Theorem 5.11. *Let $\mathfrak{X}$ be a proper smooth formal scheme over $\mathcal{O}_K$, where $\mathcal{O}_K$ is the ring of integers in a complete discretely valued non-archimedean extension K of $\mathbb{Q}_p$ with perfect residue field k and ramification degree e. Let $\mathcal{O}_C$ be the ring of integers in a complete algebraically closed non-archimedean extension C of K and X be the adic generic fibre of $\bar{\mathfrak{X}} := \mathfrak{X} \times_{\mathrm{Spf}(\mathcal{O}_K)} \mathrm{Spf}(\mathcal{O}_C)$. Assuming $ie < p-1$, there is an isomorphism of $\mathfrak{S}$-modules*

$$H^i_{\mathfrak{S}}(\mathfrak{X}) \cong H^i_{\text{ét}}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathfrak{S}.$$

In particular, we also have an isomorphism of A_{inf}-modules

$$H^i_{A_{\mathrm{inf}}}(\overline{\mathfrak{X}}) \cong H^i_{\text{ét}}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} A_{\mathrm{inf}}.$$

Proof. Note that the torsion submodule $H^i_{\mathfrak{S}-\mathrm{tor}}$ of $H^i_{\mathfrak{S}}(\mathfrak{X})$ is killed by some power of p. Then by Lemma 5.8 and Lemma 5.10, we get a decomposition $H^i_{\mathfrak{S}-\mathrm{tor}} \cong \bigoplus_{t=1}^n \mathfrak{S}/p^{m_t}$.

Since $H^i_{\mathfrak{S}}(\mathfrak{X})$ is a direct sum of a free $\mathfrak{S}$-module and $H^i_{\mathfrak{S}-\mathrm{tor}}$ by Corollary 5.7, this theorem then follows from the étale specialization of the Breuil–Kisin cohomology groups (see Theorem 1.13)

$$H^n_{\mathfrak{S}}(\mathfrak{X}) \otimes_{\mathfrak{S}} A_{\mathrm{inf}}[1/\mu] \cong H^n_{\text{ét}}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} A_{\mathrm{inf}}[1/\mu].$$

where the map $\mathfrak{S} \to A_{\mathrm{inf}}[1/\mu]$ is the composition of the faithfully flat map $\alpha : \mathfrak{S} \to A_{\mathrm{inf}}$ and the natural injection $A_{\mathrm{inf}} = W(\mathcal{O}_C^\flat) \to A_{\mathrm{inf}}[1/\mu]$.

□

Remark 5.12. In general, for any finitely generated module M over $\mathfrak{S}$ (or any other two dimensional regular local ring), there is a pseudo-isomorphism between M and $\mathfrak{S}^r \oplus (\bigoplus_{i=1}^n \mathfrak{S}/\mathcal{P}_i)$ where each $\mathcal{P}_i$ is a prime ideal of height 1. Pseudo-isomorphism means its localization at all prime ideals of height 1 is in fact an isomorphism. Within the range $ie < p-1$, the theorem above tells us that the classical p-adic cohomology theories provide enough information to determine the structure of Breuil–Kisin cohomology groups. But beyond this range, the situation gets subtle.

Now we come to prove the integral comparison theorem in the ramified case.

Theorem 5.13. *Let $\mathfrak{X}$ be a proper smooth formal scheme over $\mathcal{O}_K$, where $\mathcal{O}_K$ is the ring of integers in a complete discretely valued non-archimedean extension K of $\mathbb{Q}_p$ with perfect residue field k and ramification degree e. Let $\mathcal{O}_C$ be the ring of integers in a complete algebraically closed non-archimedean extension C of K with residue field $\bar{k}$. Let*

X be the adic generic fibre of $\bar{\mathfrak{X}} := \mathfrak{X} \times_{\mathrm{Spf}(\mathcal{O}_K)} \mathrm{Spf}(\mathcal{O}_C)$ and $\mathfrak{X}_k$ be the special fiber of $\mathfrak{X}$. If $ie < p-1$, then there is an isomorphism of $W(k)$-modules

$$H^i_{\text{ét}}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} W(k) \cong H^i_{\text{crys}}(\mathfrak{X}_k/W(k)).$$

Proof. Assume $ie < p-1$. By Corollary 1.10 and Corollary 5.4, we have an isomorphism of $\mathcal{O}_C$-modules

$$H^i_{A_{\text{inf}}}(\bar{\mathfrak{X}})/\xi \cong H^i_{\text{dR}}(\bar{\mathfrak{X}}/\mathcal{O}_C).$$

Since we also have $H^i_{A_{\text{inf}}}(\bar{\mathfrak{X}}) \cong H^i_{\text{ét}}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} A_{\text{inf}}$ by Theorem 5.11, we get an isomorphism of $\mathcal{O}_C$-modules

$$H^i_{\text{dR}}(\bar{\mathfrak{X}}/\mathcal{O}_C) \cong H^i_{\text{ét}}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathcal{O}_C.$$

Note that when $e < p$, we have an integral comparison isomorphism between de Rham cohomology and crystalline cohomology (cf. [Ber06])

$$H^i_{\text{dR}}(\bar{\mathfrak{X}}/\mathcal{O}_C) \cong H^i_{\text{crys}}(\mathfrak{X}_{\bar{k}}/W(\bar{k})) \otimes_{W(\bar{k})} \mathcal{O}_C$$

where $\mathfrak{X}_{\bar{k}} := \mathfrak{X}_k \otimes_k \bar{k}$.

So finally, we get the isomorphism

$$H^i_{\text{ét}}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} W(\bar{k}) \cong H^i_{\text{crys}}(\mathfrak{X}_{\bar{k}}/W(\bar{k})).$$

By virtue of the base change of crystalline cohomology

$$H^i_{\text{crys}}(\mathfrak{X}_{\bar{k}}/W(\bar{k})) \cong H^i_{\text{crys}}(\mathfrak{X}_k/W(k)) \otimes_{W(k)} W(\bar{k}),$$

we also have

$$H^i_{\text{ét}}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} W(k) \cong H^i_{\text{crys}}(\mathfrak{X}_k/W(k)).$$

□

Remark 5.14. When $(i+1)e < p-1$, the proof of the integral comparison isomorphism for schemes in [Car08] depends on the fact that the crystalline cohomology groups $H^i_{\text{crys}}(\mathfrak{X}_{\mathcal{O}_K/p}/S)$ admits a decomposition as $H^i_{\text{crys}}(\mathfrak{X}_{\mathcal{O}_K/p}/S) \cong S^n \oplus (\bigoplus_{j=1}^m S/p^{a_j})$. This can also be deduced from Theorem 5.11 and the base change of prismatic cohomology along the map of prisms $(\mathfrak{S}, (E)) \to (S, (p))$, which is the composition of the Frobenius map $\mathfrak{S} \to \mathfrak{S}$ and the natural injection $\mathfrak{S} \hookrightarrow S$.

6. Categories of Breuil–Kisin modules

In this section, we want to give a slightly more general result about the structure of torsion Breuil–Kisin modules of height r, under the restriction $er < p-1$. Namely, all torsion Breuil–Kisin modules in this case are isomorphic to $\bigoplus_{i=1}^n \mathfrak{S}/p^{a_i}$. As a result, this gives another proof of Theorem 5.11 without using Lemma 5.8.

Recall that there is a natural $W(k)$-linear surjection $\beta : \mathfrak{S} = W(k)[[u]] \to \mathcal{O}_K$ sending u to π. The kernel of this map is generated by an Eisenstein polynomial $E = E(u)$ for π. Fix a non-negative integer r. We first need to define some categories that we will study.

Definition 6.1 ($'\mathrm{Mod}^{r,\varphi}_{/\mathfrak{S}}$, [CL09])**.** *The objects of category $'\mathrm{Mod}^{r,\varphi}_{/\mathfrak{S}}$ are defined to be $\mathfrak{S}$-modules $\mathfrak{M}$ equipped with a φ-linear endomorphism $\varphi : \mathfrak{M} \to \mathfrak{M}$ such that the cokernel of $id \otimes \varphi : \varphi^*\mathfrak{M} := \mathfrak{S} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M} \to \mathfrak{M}$ is killed by E^r. Morphisms are homomorphisms of $\mathfrak{S}$-modules compatible with φ. We say that a short sequence $0 \to \mathfrak{M}_1 \to \mathfrak{M}_2 \to \mathfrak{M}_3 \to 0$ is exact if it is exact in the abelian category of $\mathfrak{S}$-modules.*

Definition 6.2 ($\mathrm{Mod}^{r,\varphi}_{/\mathfrak{S}_1}$, [CL09])**.** *The category* $\mathrm{Mod}^{r,\varphi}_{/\mathfrak{S}_1}$ *is the full subcategory of* $'\mathrm{Mod}^{r,\varphi}_{/\mathfrak{S}}$ *spanned by the objects which are finite free over* $\mathfrak{S}_1 := \mathfrak{S}/p = k[[u]]$.

Definition 6.3 ($\mathrm{Mod}^{r,\varphi}_{/\mathfrak{S}_\infty}$, [CL09])**.** *We define* $\mathrm{Mod}^{r,\varphi}_{/\mathfrak{S}_\infty}$ *to be the smallest full subcategory of* $'\mathrm{Mod}^{r,\varphi}_{/\mathfrak{S}}$ *which contains* $\mathrm{Mod}^{r,\varphi}_{/\mathfrak{S}_1}$ *and is stable under extensions.*

Remark 6.4. The category $\mathrm{Mod}^{1,\varphi}_{/\mathfrak{S}_1}$ first appeared in [Bre]. And the category $\mathrm{Mod}^{1,\varphi}_{/\mathfrak{S}_\infty}$ is just the category $\mathrm{Mod}/\mathfrak{S}$ defined by Kisin in [Kis06].

The following lemma gives us some important descriptions of objects in $\mathrm{Mod}^{r,\varphi}_{/\mathfrak{S}_\infty}$.

Lemma 6.5. (i) *For any* $\mathfrak{M}$ *in* $\mathrm{Mod}^{r,\varphi}_{/\mathfrak{S}_\infty}$*, the morphism* $id \otimes \varphi : \varphi^*\mathfrak{M} \to \mathfrak{M}$ *is injective.*

(ii) *An object* $\mathfrak{M}$ *in* $'\mathrm{Mod}^{r,\varphi}_{/\mathfrak{S}}$ *is in* $\mathrm{Mod}^{r,\varphi}_{/\mathfrak{S}_\infty}$ *if and only if it is of finite type over* $\mathfrak{S}$*, it has no* u*-torsion and it is killed by some power of* p.

Proof. See [Liu07, section 2.3]. □

Corollary 6.6. *The torsion submodule* $H^i_{\mathfrak{S}-\mathrm{tor}}$ *of the Breuil–Kisin cohomology groups of a proper smooth formal scheme over* $\mathcal{O}_K$ *is in the category* $\mathrm{Mod}^{r,\varphi}_{/\mathfrak{S}_\infty}$ *when* $i \le r < \frac{p-1}{e}$.

Proof. This follow from Corollary 5.4 and [BS19, Theorem 1.8 (6)] □

Next we introduce Breuil's ring S and define some related categories analogous to those associated with the ring $\mathfrak{S}$.

Definition 6.7 (Breuil's ring)**.** *Let* S *be the* p*-adic completion of the PD-envelope of* $W(k)[u]$ *with respect to the ideal* $(E) \subset W(k)[u]$*. The ring* S *is endowed with several additional structures:*

(i) *a canonical (PD-)filtration:* $\mathrm{Fil}^{\mathrm{i}}S$ *is the* p*-adic completion of the ideal generated by elements* $(\frac{E^m}{m!})_{m\ge i}$.

(ii) *a Frobenius* φ*: it is the unique continuous map which is Frobenius semi-linear over* $W(k)$ *and sends* u *to* u^p.

For $r < p-1$, we have $\varphi(\mathrm{Fil}^{\mathrm{r}}S) \subset p^rS$ and we can define $\varphi_r = \frac{\varphi}{p^r} : \mathrm{Fil}^{\mathrm{r}}S \to S$. Set $S_n := S/p^n$.

Definition 6.8 ($'\mathrm{Mod}^{r,\varphi}_{/S}$, [CL09])**.** *The objects of* $'\mathrm{Mod}^{r,\varphi}_{/S}$ *are the following data:*

(i) *an* S*-module;*

(ii) *a submodule* $\mathrm{Fil}^{\mathrm{r}}M \subset M$ *such that* $\mathrm{Fil}^{\mathrm{r}}S \cdot M \subset \mathrm{Fil}^{\mathrm{r}}M$*;*

(iii) *a* φ*-linear map* $\varphi_r : \mathrm{Fil}^{\mathrm{r}}M \to M$ *such that for all* $s \in \mathrm{Fil}^{\mathrm{r}}S$ *and* $x \in M$ *we have* $\varphi_r(sx) = c^{-r}\varphi_r(s)\varphi_r(E^rx)$*, where* $c = \varphi_1(E)$.

The morphisms are homomorphisms of S*-modules compatible with additional structures. We say a short sequence* $0 \to M_1 \to M_2 \to M_3 \to 0$ *in* $'\mathrm{Mod}^{r,\varphi}_{/S}$ *is exact if both sequences* $0 \to M_1 \to M_2 \to M_3 \to 0$ *and* $0 \to \mathrm{Fil}^{\mathrm{r}}M_1 \to \mathrm{Fil}^{\mathrm{r}}M_2 \to \mathrm{Fil}^{\mathrm{r}}M_3 \to 0$ *are exact in the abelian category of* S*-modules.*

Definition 6.9 ($\mathrm{Mod}^{r,\varphi}_{/S_1}$, [CL09])**.** *The objects of* $\mathrm{Mod}^{r,\varphi}_{/S_1}$ *are* M *in* $'\mathrm{Mod}^{r,\varphi}_{/S}$ *such that* M *is finite free over* S_1 *and the image of* φ_r *generates* M *as an* S*-module.*

Definition 6.10 ($\mathrm{Mod}^{r,\varphi}_{/S_\infty}$, [CL09])**.** *The category* $\mathrm{Mod}^{r,\varphi}_{/S_\infty}$ *is the smallest subcategory of* $'\mathrm{Mod}^{r,\varphi}_{/S}$ *containing* $\mathrm{Mod}^{r,\varphi}_{/S_1}$ *and is stable under extensions.*

For any $r < p-1$, one can define a functor $M_{\mathfrak{S}_\infty} : \mathrm{Mod}^{r,\varphi}_{/\mathfrak{S}_\infty} \to {}'\mathrm{Mod}^{r,\varphi}_{/S}$ as follows:

(i) $M_{\mathfrak{S}_\infty}(\mathfrak{M}) = S \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$. Here $\varphi : \mathfrak{S} \to S$ is the composite $\mathfrak{S} \to \mathfrak{S} \to S$ where the first map is the Frobenius on $\mathfrak{S}$ and the second map is the canonical injection.

(ii) Submodule: The Frobenius on $\mathfrak{M}$ induces a S-linear map $id \otimes \varphi : S \otimes_{\varphi,\mathfrak{S}} \mathfrak{M} \to S \otimes_{\mathfrak{S}} \mathfrak{M}$. The submodule $\mathrm{Fil}^{\mathrm{r}} M_{\mathfrak{S}_\infty}(\mathfrak{M})$ is then defined by the following formula:

$$\mathrm{Fil}^{\mathrm{r}} M_{\mathfrak{S}_\infty}(\mathfrak{M}) := \{x \in M_{\mathfrak{S}_\infty}(\mathfrak{M}) \mid (id \otimes \varphi)(x) \in \mathrm{Fil}^{\mathrm{r}} S \otimes_{\mathfrak{S}} \mathfrak{M} \subset S \otimes_{\mathfrak{S}} \mathfrak{M})\}$$

(iii) Frobenius: the map φ_r is the following composite:

$$\mathrm{Fil}^{\mathrm{r}} M_{\mathfrak{S}_\infty}(\mathfrak{M}) \xrightarrow{id\otimes\varphi} \mathrm{Fil}^{\mathrm{r}} S \otimes_{\mathfrak{S}} \mathfrak{M} \xrightarrow{\varphi_r\otimes id} M_{\mathfrak{S}_\infty}(\mathfrak{M}).$$

We state a theorem describing the functor $M_{\mathfrak{S}_\infty}$.

Theorem 6.11. *For any $r < p-1$, the functor $M_{\mathfrak{S}_\infty}$ takes value in $\mathrm{Mod}^{r,\varphi}_{/S_\infty}$. The induced functor $M_{\mathfrak{S}_\infty} : \mathrm{Mod}^{r,\varphi}_{/\mathfrak{S}_\infty} \to \mathrm{Mod}^{r,\varphi}_{/S_\infty}$ is exact and it is an equivalence of categories. Moreover, if we choose M_{S_∞} a quasi-inverse of $M_{\mathfrak{S}_\infty}$, then the functor M_{S_∞} is also exact.*

Proof. See [CL09, Proposition 2.1.2, Theorem 2.3.1, Proposition 2.3.2]. □

Theorem 6.12. *Assuming $er < p-1$, the category $\mathrm{Mod}^{r,\varphi}_{/S_\infty}$ is an abelian category and every object is of the form $\bigoplus_{i=1}^n S/p^{a_i}$. For any morphism $f : (M_1, \mathrm{Fil}^{\mathrm{r}} M_1, \varphi_r) \to (M_2, \mathrm{Fil}^{\mathrm{r}} M_2, \varphi_r)$ in $\mathrm{Mod}^{r,\varphi}_{/S_\infty}$, the underlying module of $\mathrm{Ker}(f)$ is the kernel of the morphism $f : M_1 \to M_2$ in the category of S-modules and the underlying module of $\mathrm{Fil}^{\mathrm{r}}\mathrm{Ker}(f)$ is the kernel of the morphism $f : \mathrm{Fil}^{\mathrm{r}} M_1 \to \mathrm{Fil}^{\mathrm{r}} M_2$ in the category of S-modules. A Similar statement is true for $\mathrm{Coker}(f)$.*

Proof. See [Car06, Section 3]. We remark that the category which Caruso used is different from ours but they can be proved to be equivalent by using a generalization of [Bre98a, Proposition 2.3.1.2], as mentioned in the proof of [Car08, Theorem 4.2.1]. □

Remark 6.13. This theorem is false without the restriction $er < p-1$.

From now on, we fix a non-negative integer r such that $er < p-1$. Then $\mathrm{Mod}^{r,\varphi}_{/\mathfrak{S}_\infty}$ is an abelian category.

Lemma 6.14. *For any morphism $f : \mathfrak{M}_1 \to \mathfrak{M}_2$ in $\mathrm{Mod}^{r,\varphi}_{/\mathfrak{S}_\infty}$, the underlying module of $\mathrm{Ker}(f)$ is the kernel of the morphism $f : \mathfrak{M}_1 \to \mathfrak{M}_2$ in the category of $\mathfrak{S}$-modules. A Similar statement is true for $\mathrm{Coker}(f)$.*

Proof. By Lemma 6.5, the kernel and the image of the underlying morphism $f : \mathfrak{M}_1 \to \mathfrak{M}_2$ in the category of $\mathfrak{S}$-modules together with the induced Frobenius maps are objects of $\mathrm{Mod}^{r,\varphi}_{/\mathfrak{S}_\infty}$. It is easy to see that the kernel equipped with the induced Frobenius map is indeed $\mathrm{Ker}(f)$ in the category $\mathrm{Mod}^{r,\varphi}_{/\mathfrak{S}_\infty}$. So we can assume $f : \mathfrak{M}_1 \to \mathfrak{M}_2$ is injective. Then $M_{\mathfrak{S}_\infty}(f)$ is also injective. In fact, let L be the kernel of $M_{\mathfrak{S}_\infty}(f)$ and we choose a quasi-inverse functor M_{S_∞} of $M_{\mathfrak{S}_\infty}$. Let $h : \mathfrak{L} \to \mathfrak{M}_1$ be the image of the inclusion $L \to M_{\mathfrak{S}_\infty}(\mathfrak{M}_1)$ under M_{S_∞}. Then $f \circ h = 0$, which implies $h = 0$. In consequence, we have $L = 0$. Put $M = \mathrm{Coker}(f)$. By Theorem 6.11 and Theorem 6.12, we get an exact sequence $0 \to \mathfrak{M}_1 \to \mathfrak{M}_2 \to M_{S_\infty}(M) \to 0$ in the exact category $\mathrm{Mod}^{r,\varphi}_{/\mathfrak{S}_\infty}$ (where the class of the exact sequences is as defined in Definition 6.1). So we have $M_{S_\infty}(M)$ is isomorphic to $\mathfrak{M}_2/\mathfrak{M}_1$ as $\mathfrak{S}$-modules. In particular $\mathfrak{M}_2/\mathfrak{M}_1$ has no u-torsion. By

Lemma 6.5, the module $\mathfrak{M}_2/\mathfrak{M}_1$ equipped with the induced Frobenius map is an object of $\mathrm{Mod}^{r,\varphi}_{/\mathfrak{S}_\infty}$. It is easy to check that $\mathrm{Coker}(f)$ is isomorphic to $\mathfrak{M}_2/\mathfrak{M}_1$ equipped with the induced Frobenius map. □

Corollary 6.15. *The full subcategory* $\mathrm{Mod}^{r,\varphi}_{/\mathfrak{S}_1}$ *of* $\mathrm{Mod}^{r,\varphi}_{/\mathfrak{S}_\infty}$ *is an abelian category.*

Proof. For any morphism $f:\mathfrak{M}_1\to\mathfrak{M}_2$ in $\mathrm{Mod}^{r,\varphi}_{/\mathfrak{S}_1}$, $\mathrm{Ker}(f)$ and $\mathrm{Coker}(f)$ are then both killed by p. By Lemma 6.5, they are u-torsion free. So $\mathrm{Ker}(f)$ and $\mathrm{Coker}(f)$ are in the category $\mathrm{Mod}^{r,\varphi}_{/\mathfrak{S}_1}$. □

Let $\mathrm{ModFI}^{r,\varphi}_{/\mathfrak{S}_\infty}$ denote the full subcategory of $\mathrm{Mod}^{r,\varphi}_{/\mathfrak{S}_\infty}$ spanned by the objects that are isomorphic to $\bigoplus_{i=1}^n \mathfrak{S}/p^{a_i}$ as $\mathfrak{S}$-modules. In particular, $\mathrm{ModFI}^{r,\varphi}_{/\mathfrak{S}_\infty}$ contains $\mathrm{Mod}^{r,\varphi}_{/\mathfrak{S}_1}$.

Lemma 6.16. *For any* $\mathfrak{M}\in\mathrm{Mod}^{r,\varphi}_{/\mathfrak{S}_\infty}$*, the quotient* $\mathfrak{M}/p$ *is in* $\mathrm{Mod}^{r,\varphi}_{/\mathfrak{S}_1}$.

Proof. Consider the morphism $\mathfrak{M}\xrightarrow{\times p}\mathfrak{M}$ in $\mathrm{Mod}^{r,\varphi}_{/\mathfrak{S}_\infty}$. Since $\mathrm{Mod}^{r,\varphi}_{/\mathfrak{S}_\infty}$ is an abelian category, we know that $\mathfrak{M}/p$ is also in $\mathrm{Mod}^{r,\varphi}_{/\mathfrak{S}_\infty}$. It is killed by p and has no u-torsion by Lemma 6.5, therefore $\mathfrak{M}/p$ is in $\mathrm{Mod}^{r,\varphi}_{/\mathfrak{S}_1}$. □

We now reformulate Lemma 5.9 by using the categories we have defined.

Lemma 6.17. *Let* $\mathfrak{M}$ *be in* $\mathrm{Mod}^{r,\varphi}_{/\mathfrak{S}_\infty}$*. If* $p\mathfrak{M}$ *is in* $\mathrm{ModFI}^{r,\varphi}_{/\mathfrak{S}_\infty}$*, so is* $\mathfrak{M}$.

Proof. By Lemma 6.16, we have $\mathfrak{M}/p\in\mathrm{Mod}^{r,\varphi}_{/\mathfrak{S}_\infty}$. Then this lemma follows from Lemma 5.9. □

Lemma 6.18. *Let* $\mathfrak{L}\hookrightarrow\mathfrak{M}$ *be an injection in* $\mathrm{Mod}^{r,\varphi}_{/\mathfrak{S}_\infty}$*. If* $\mathfrak{M}$ *is in* $\mathrm{ModFI}^{r,\varphi}_{/\mathfrak{S}_\infty}$*, so is* $\mathfrak{L}$.

Proof. We show that $p\mathfrak{L}$ is in $\mathrm{ModFI}^{r,\varphi}_{/\mathfrak{S}_\infty}$, then this lemma follows from Lemma 6.17. Consider the map $p\mathfrak{L}\hookrightarrow p\mathfrak{M}$. We proceed by induction on the minimal integer such that $p^n\mathfrak{M}=0$. If $n=1$, this is easy. Assume that when $n<m$ this lemma is true. Then when $n=m$, $p\mathfrak{L}$ is also in $\mathrm{ModFI}^{r,\varphi}_{/\mathfrak{S}_\infty}$ as $p^{m-1}(p\mathfrak{M})=0$. We are done. □

Theorem 6.19. *The category* $\mathrm{ModFI}^{r,\varphi}_{/\mathfrak{S}_\infty}$ *is an abelian category.*

Proof. For any morphism $f:\mathfrak{M}_1\to\mathfrak{M}_2$ in $\mathrm{ModFI}^{r,\varphi}_{/\mathfrak{S}_\infty}$, we need to show $\mathfrak{L}=\mathrm{Ker}(f)$ and $\mathfrak{C}=\mathrm{Coker}(f)$ are also in the category $\mathrm{ModFI}^{r,\varphi}_{/\mathfrak{S}_\infty}$. For the kernel $\mathfrak{L}$, this follows from Lemma 6.18. For the cokernel $\mathfrak{C}$, we proceed by induction on the minimal integer n such that $p^n\mathfrak{M}_2=0$. Without loss of generality, we can assume f is an injection.

When $n=1$, we have $\mathfrak{M}_1,\mathfrak{M}_2$ are both in $\mathrm{Mod}^{r,\varphi}_{/\mathfrak{S}_1}$. Then by Corollary 6.15, we see that $\mathfrak{C}$ is also in $\mathrm{Mod}^{r,\varphi}_{/\mathfrak{S}_1}\subset\mathrm{ModFI}^{r,\varphi}_{/\mathfrak{S}_\infty}$. Now suppose the statement is true when $n<m$. When $n=m$, consider the sequence $p\mathfrak{M}_1\to p\mathfrak{M}_2\to p\mathfrak{C}$. Then there is a short exact sequence $0\to\mathfrak{L}'\to p\mathfrak{M}_2/p\mathfrak{M}_1\to p\mathfrak{C}\to 0$. Since $p^{m-1}(p\mathfrak{M}_2/p\mathfrak{M}_1)=0$, by the assumption, we get $p\mathfrak{C}$ is in $\mathrm{ModFI}^{r,\varphi}_{/\mathfrak{S}_\infty}$. Then by Lemma 6.17, we see that $\mathfrak{C}$ is also in $\mathrm{ModFI}^{r,\varphi}_{/\mathfrak{S}_\infty}$. This finishes the proof. □

Theorem 6.20. *There is an equivalence of categories:* $\mathrm{ModFI}^{r,\varphi}_{/\mathfrak{S}_\infty}$=$\mathrm{Mod}^{r,\varphi}_{/\mathfrak{S}_\infty}$.

Proof. We just need to prove that every object $\mathfrak{M}$ in $\mathrm{Mod}^{r,\varphi}_{/\mathfrak{S}_\infty}$ is also in $\mathrm{ModFI}^{r,\varphi}_{/\mathfrak{S}_\infty}$. To see this, we proceed by induction on the minimal integer n such that $p^n\mathfrak{M}=0$.

When $n=1$, this follows from Lemma 6.17. Now suppose the statement is true when $n<m$. Then when $n=m$, we know that $p\mathfrak{M}$ is killed by p^{m-1}. So by the assumption,

we have $p\mathfrak{M} \in \mathrm{ModFI}^{r,\varphi}_{/\mathfrak{S}_\infty}$. By Lemma 6.17, we can obtain that $\mathfrak{M} \in \mathrm{ModFI}^{r,\varphi}_{/\mathfrak{S}_\infty}$. We are done. □

So Theorem 6.20 provides another proof of Theorem 5.11.

Theorem 6.21. *For any $i \leq r < \frac{p-1}{e}$, we have $H^i_{\mathfrak{S}-\mathrm{tor}}$, the torsion submodule of the Breuil–Kisin cohomology group of a proper smooth formal scheme over $\mathcal{O}_K$, is in the category* $\mathrm{ModFI}^{r,\varphi}_{/\mathfrak{S}_\infty}$*, i.e. $H^i_{\mathfrak{S}-\mathrm{tor}} \cong \bigoplus_{i=1}^n \mathfrak{S}/p^{a_i}$.*

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