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INVARiance OF THE GERSTENHABER ALGEBRA STRUCTure 
oN TATE-HOCHSCHILD COHOMOLOGY

ZHENGFANG WANG

Abstract. Keller proved in 1999 that the Gerstenhaber algebra structure on the Hochschild cohomology of an algebra is an invariant of the derived category. In this paper, we adapt his approach to show that the Gerstenhaber algebra structure on the Tate-Hochschild cohomology of an algebra is preserved under singular equivalences of Morita type with level, a notion introduced by the author in previous work.

Keywords. Gerstenhaber algebra, Singularity category, Tate-Hochschild cohomology.

1. Introduction

In [Wan15a, Wan18], we constructed a Gerstenhaber algebra structure on the Tate-Hochschild cohomology ring $\text{HH}^*_\text{sg}(A, A)$ implicit in Buchweitz’ work [Buc] for an algebra $A$ projective over a commutative ring $k$ and such that $A$ and the enveloping algebra $A \otimes_k A^{\text{op}}$ are Noetherian. The cup product is given by the Yoneda product in the singularity category of the enveloping algebra $A \otimes_k A^{\text{op}}$. Recall that the singularity category $D^\text{sg}(A)$ (cf. [Buc, Orl]) of a Noetherian algebra $A$ is defined as the Verdier quotient of the bounded derived category $D^b(A)$ of finitely generated (left) $A$-modules by the full subcategory $\text{Perf}(A)$ consisting of complexes quasi-isomorphic to bounded complexes of finitely generated projective $A$-modules. The Lie bracket on $\text{HH}^*_\text{sg}(A, A)$ was defined in [Wan15a, Wan18] as the graded commutator of a certain circle product $\circ$ extending naturally the Gerstenhaber circle product on Hochschild cohomology. In particular, for a self-injective algebra, in positive degrees, this Lie bracket coincides with the Gerstenhaber bracket in Hochschild cohomology. In [Wan15a, Wan18], we also proved that the natural morphism, induced by the quotient functor from the bounded derived category to the singularity category of $A \otimes_k A^{\text{op}}$, from the Hochschild cohomology ring $\text{HH}^*(A, A)$ to $\text{HH}^*_\text{sg}(A, A)$ is a morphism of Gerstenhaber algebras. By the very recent work of Keller [Kel18], the Tate-Hochschild cohomology of an algebra $A$ is isomorphic, as graded algebras, to the Hochschild cohomology of the dg singularity category (i.e. the canonical dg enhancement of the singularity category) of $A$. This yields a second Gerstenhaber algebra structure on Tate-Hochschild cohomology, which is conjectured to coincide with the one introduced in [Wan15a, Wan18]. For more details, we refer to Keller’s conjecture [Kel18, Conjecture 1.2].

Keller proved in [Kel99] that the Gerstenhaber algebra structures on Hochschild cohomology rings are preserved under derived equivalences of standard type. That is, let $X$ be a complex of $A$-$B$-bimodules such that the total derived tensor product by $X$ is an equivalence between the derived categories of two $k$-algebras $A$ and $B$. Then $X$ yields a natural isomorphism of Gerstenhaber algebras from $\text{HH}^*(A, A)$ to $\text{HH}^*(B, B)$. In this paper, we will show that the Gerstenhaber algebra structure on the Tate-Hochschild cohomology ring is also preserved under derived equivalences of standard type. In fact, we will prove a stronger result. Namely, the Gerstenhaber algebra structure on the Tate-Hochschild cohomology ring is preserved under singular equivalences of Morita type with level (cf. [Wan15b] and Section 6 below). Recall that a derived equivalence of standard type induces a singular equivalence of Morita type with level (cf. [Wan15b]).
The paper is organized as follows. In Section 2, we recall the construction of the normalized bar resolution of an algebra \( A \) and provide some natural liftings of elements in \( \text{HH}^*_{\text{sg}}(A, A) \) along the normalized bar resolution. In Section 3, we introduce the bullet product \( \bullet \) and the circle product \( \circ \). Using these two products, we construct two dg modules \( C^L(f, g) \) and \( C^R(f, g) \) associated to the cohomology classes \( f \) and \( g \) in \( \text{HH}^*(A, \Omega^*_{\text{sg}}(A)) \). These two dg modules play a crucial role in the proof of our main result.

In Section 4, we recall the notions of \( R \)-relative derived categories and \( R \)-relative derived tensor products. In Section 5, we develop the singular infinitesimal deformation theory of the identity bimodule in analogy with the infinitesimal deformation theory of [KeVo, Ric, Wei, Zim]. As a result, we give an interpretation of the Gerstenhaber bracket on the Tate-Hochschild cohomology ring from the point of view of the singular infinitesimal deformation theory. In Section 6, we prove our main result.

**Theorem 1.1** (=Theorem 6.2 and Corollary 6.3). Let \( k \) be a field. Let \( A \) and \( B \) be two Noetherian (not necessarily commutative) \( k \)-algebras such that the enveloping algebras \( A \otimes A^{\text{op}} \) and \( B \otimes B^{\text{op}} \) are Noetherian. Suppose that \((A M_{B, B} N_A)\) defines a singular equivalence of Morita type with level \( l \in \mathbb{Z}_{\geq 0} \). Then the functor

\[
\Sigma^l(M \otimes_B - \otimes_B N) : \text{D}_{\text{sg}}(B \otimes_k B^{\text{op}}) \to \text{D}_{\text{sg}}(A \otimes_k A^{\text{op}})
\]

induces an isomorphism of Gerstenhaber algebras between the Tate-Hochschild cohomology rings \( \text{HH}^*_{\text{sg}}(A, A) \) and \( \text{HH}^*_{\text{sg}}(B, B) \). In particular, the Gerstenhaber algebra structure on the Tate-Hochschild cohomology ring is invariant under derived equivalences.

**Remark 1.2.** Let \( k \) be an algebraically closed field. Let \( A \) and \( B \) be two (finite dimensional) symmetric \( k \)-algebras which are related by a stable equivalence of Morita type. Then the authors in [KLZ, Theorem 10.7] proved that there is an isomorphism of Gerstenhaber algebras (more generally, BV algebras) between \( \text{HH}^0_{\text{sg}}(A, A) \) and \( \text{HH}^0_{\text{sg}}(B, B) \).

Throughout this paper, we fix a field \( k \). The unadorned tensor product \( \otimes \) and \( \text{Hom} \) represent the tensor product \( \otimes_k \) and \( \text{Hom}_k \) over the field \( k \), respectively. We write the composition \( g \circ f \) of two maps \( f : X \to Y \) and \( g : Y \to Z \) as \( gf \). We write the identity map \( \text{Id}_X : X \to X \) simply as \( \text{Id} \) when no confusion can arise. We will follow the Koszul sign rule for the tensor product: \( (f \otimes g)(x \otimes y) = (-1)^{|g||x|}f(x) \otimes g(y) \) where \( |g| \) is the degree of the homogeneous map \( g \) and \( |x| \) is the degree of the element \( x \in X \).

The notions of differential graded (dg) algebras and relative tensor products are frequently used in this paper. For more details, we refer to [Ke99, Ke98, BeLu], and to [KeVo, Ric, Wei, Zim] for the notions of triangulated categories and derived categories.

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2. Normalized bar resolution
2.1. The definition. Let $A$ be an associative algebra over a field $k$. The normalized bar resolution (cf. e.g. [Lod]) is defined as the dg $A$-$A$-bimodule $\text{Bar}_r(A) := \bigoplus_{p \geq 0} \text{Bar}_p(A)$, with $\text{Bar}_p(A) := A \otimes (\Sigma A)^{\otimes p} \otimes A (p \geq 0)$ in degree $p$ and the differential of degree $-1$

$$d_p(a_0 \otimes \overline{a}_{1,p} \otimes a_{p+1}) = a_0 a_1 \otimes \overline{a}_{2,p} \otimes a_{p+1} +$$

$$\sum_{i=1}^{p-1} (-1)^i a_0 \otimes \overline{a}_{1,i-1} \otimes \overline{a}_{i} a_{i+1} \otimes \overline{a}_{i+2,p} \otimes a_{p+1} + (-1)^p a_0 \otimes \overline{a}_{1,p-1} \otimes a_p a_{p+1}.$$

Let us explain the notations appeared above: We denote by $\overline{\Sigma A}$ the graded $k$-module concentrated in degree $1$ with $(\overline{\Sigma A})_1 = A/(k \cdot 1)$; Let $\pi : A \to (\Sigma A)$ be the natural projection of degree $1$. Then we denote $\overline{\pi} = \pi(a)$ for any $a \in A$. The degree of $\overline{\pi}$ is $|\overline{\pi}| = 1$; We simply write $\overline{a} \otimes \overline{a}_{i+1} \otimes \cdots \otimes \overline{a}_{j} \in (\Sigma A)^{\otimes (j-i+1)}$ as $\overline{a}_{i,j}$. It is well-known that $\text{Bar}_r(A)$ is a projective bimodule resolution of $A$ with the augmentation map $\tau_0 = d_0 : A \otimes A \to A, a \otimes b \to ab$. For convenience, we set $\text{Bar}_{-1}(A) = A$.

For any $p \in \mathbb{Z}_{\geq 0}$, we denote the kernel of the differential $d_{p-1} : \text{Bar}_{p-1}(A) \to \text{Bar}_{p-2}(A)$ by $\Omega^p_{\text{sy}}(A)$. In particular, we set $\Omega^0_{\text{sy}}(A) = A$. It is clear that $\Omega^p_{\text{sy}}(A)$ is an $A$-$A$-bimodule. For convenience, we view $\Omega^p_{\text{sy}}(A)$ as a dg bimodule concentrated in degree $1$. For $p \geq 0$, we denote by $\text{Bar}_{> p}(A)$ the ‘$p$-truncated’ dg $A$-$A$-bimodule with $\text{Bar}_{> p}(A)_i = \text{Bar}(A)$ if $i \geq p$ and $\text{Bar}_{> p}(A)_i = 0$ if $i < p$. Recall that, for a chain complex $(X,d)$ and $p \in \mathbb{Z}$, the $p$-shifted complex $(\Sigma^p X, \Sigma^p d)$ is defined as $(\Sigma^p X)_n = X_{n-p}$ with the differential $(\Sigma^p d)_n = (-1)^p d_{n-p}$ for any $n \in \mathbb{Z}$.

**Remark 2.1.** Note that the ‘$p$-truncated’ augmented normalized bar resolution

$$\text{Bar}_{\geq p}(A) : \cdots \to \text{Bar}_{p+1}(A) \xrightarrow{d_{p+1}} \text{Bar}_p(A) \xrightarrow{\tau_p = d_p} \Sigma^{-1} \Omega^p_{\text{sy}}(A) \to 0$$

is exact for any fixed $p \in \mathbb{Z}_{\geq 0}$. For this, we define a $k$-linear map for any $r \geq 0$,

$$s^L_r : \text{Bar}_r(A) \to \text{Bar}_{r+1}(A), \quad a_0 \otimes \overline{a}_{1,r} \otimes a_{r+1} \mapsto (-1)^{r+1} a_0 \otimes \overline{a}_{1,r+1} \otimes 1.
$$

$$\cdots \to \text{Bar}_{p+1}(A) \xrightarrow{d_{p+1}} \text{Bar}_p(A) \xrightarrow{d_p} \Sigma^{-1} \Omega^p_{\text{sy}}(A) \xrightarrow{\tau_p = d_p} \Sigma^{-1} \Omega^p_{\text{sy}}(A) \xrightarrow{\tau_p = d_p} 0$$

$$\cdots \to \text{Bar}_p(A) \xrightarrow{d_p} \Sigma^{-1} \Omega^p_{\text{sy}}(A) \xrightarrow{\tau_p = d_p} \Sigma^{-1} \Omega^p_{\text{sy}}(A) \xrightarrow{\tau_p = d_p} 0$$

It is straightforward to verify that $s^L d + ds^L = \text{Id}_{\text{Bar}_{\geq p}(A)}$. This yields the exactness of $\text{Bar}_{\geq p}(A)$. Note that $s^L$ is a morphism of left graded $A$-modules (but not a morphism of graded $A \otimes A^{op}$-modules). Similarly, if we define

$$s^R_r : \text{Bar}_r(A) \to \text{Bar}_{r+1}(A), \quad a_0 \otimes \overline{a}_{1,r} \otimes a_{r+1} \mapsto 1 \otimes \overline{a}_{0,r} \otimes a_{r+1},$$

then we have that $s^R d + ds^R = \text{Id}_{\text{Bar}_{\geq p}(A)}$ and $s^R$ is a morphism of right graded $A$-modules.

For any $p,q \in \mathbb{Z}_{\geq 0}$, we will construct a morphism of dg $A$-$A$-bimodules between $\text{Bar}_{p+q}(A)$ and $\text{Bar}_{p}(A) \otimes_A \text{Bar}_{q}(A)$. We define

$$\Delta_{p,q} : \text{Bar}_{p+q}(A) \to \text{Bar}_p(A) \otimes_A \text{Bar}_q(A)$$

as follows. For $a_0 \otimes \overline{a}_{1,p+q+r} \otimes a_{p+q+r+1} \in \text{Bar}_{p+q+r}(A)$, where $r \geq 0$,

$$\Delta_{p,q}(a_0 \otimes \overline{a}_{1,p+q+r} \otimes a_{p+q+r+1}) = \sum_{i=0}^{r} (a_0 \otimes \overline{a}_{1,p+i} \otimes 1) \otimes_A (1 \otimes \overline{a}_{p+i+1,p+q+r} \otimes a_{p+q+r+1}).$$
It is a routine computation to verify that $\Delta_{p,q}$ is a morphism of dg $A$-$A$-bimodules.

**Lemma 2.2.** For any $p, q \in \mathbb{Z}_{\geq 0}$, $\Delta_{p,q}$ is an isomorphism in the homotopy category $\mathcal{K}(A \otimes A^\text{op}-\text{Mod})$ of dg $A$-$A$-bimodules.

**Proof.** For $p = 0$ or $q = 0$, we have a natural isomorphism $\mu_{p,q} : \Omega^p_{\text{sy}}(A) \otimes_A \Omega^q_{\text{sy}}(A) \cong \Omega^{p+q}_{\text{sy}}(A)$ since $\Omega^0_{\text{sy}}(A) = A$. For $p, q > 0$, consider the following composition of maps

$$\mu_{p,q} : \Omega^p_{\text{sy}}(A) \otimes_A \Omega^q_{\text{sy}}(A) \hookrightarrow A \otimes (\Sigma A)^{\otimes p-1} \otimes A \otimes (\Sigma A)^{\otimes q-1} \otimes A \xrightarrow{\text{Id}^{p-1} \otimes \pi \otimes \text{Id}^{q-1}} A \otimes (\Sigma A)^{\otimes p+q-1} \otimes A,$$

where the first map is given by the tensor product of the natural inclusions

$$\Omega^p_{\text{sy}}(A) \hookrightarrow \text{Bar}_{p-1}(A), \quad \Omega^q_{\text{sy}}(A) \hookrightarrow \text{Bar}_{q-1}(A),$$

and where $\pi : A \to \Sigma A$ is the natural projection of degree 1. More concretely, let $x := \sum_i a_i^0 \otimes a_{i,1,p-1} \otimes a_{i,p}^1 \in \Omega^p_{\text{sy}}(A)$ and $y := \sum_j b_j^0 \otimes b_{j,1,q-1} \otimes b_{j,q}^1 \in \Omega^q_{\text{sy}}(A)$. Then

$$\mu_{p,q}(x \otimes_A y) = \sum_{i,j} a_i^0 \otimes a_{i,1,p-1}^1 \otimes a_{i,q}^1 b_j^0 \otimes b_{j,1,q-1}^1 \otimes b_{j,q}^1.$$

Notice that the image of $\mu_{p,q}$ lies in $\Omega^{p+q}_{\text{sy}}(A)$ since $d_{p+q-1} \mu_{p,q}(x \otimes_A y) = 0$. This induces an $A$-$A$-bimodule homomorphism $\mu_{p,q} : \Omega^p_{\text{sy}}(A) \otimes_A \Omega^q_{\text{sy}}(A) \to \Omega^{p+q}_{\text{sy}}(A)$. We claim that $\mu_{p,q}$ is a bijection and its inverse $\mu_{p,q}^{-1} : \Omega^{p+q}_{\text{sy}}(A) \otimes_A \Omega^p_{\text{sy}}(A) \otimes_A \Omega^q_{\text{sy}}(A)$ sends an element $x := \sum_i a_i^0 \otimes a_{i,1,p+q-1}^1 \otimes a_{i,p+q}^q \in \Omega^{p+q}_{\text{sy}}(A)$ to

$$\mu_{p,q}^{-1}(x) = (-1)^{p+q} \sum_i d_p (a_i^0 \otimes a_{i,1,p}^1 \otimes 1) \otimes_A d_q (1 \otimes a_{i,p+1,p+q}^q \otimes 1).$$

Indeed, we have

$$\mu_{p,q} \mu_{p,q}^{-1}(x) = (-1)^{p+q} \sum_i a_i^0 a_i^1 \otimes \overline{a}_{i,2,p+q}^1 \otimes 1 +$$

$$\sum_{i,j} \sum_{j=1}^{p+q-1} (-1)^{p+q+j} a_i^0 \otimes a_{i,1,j-1}^1 \otimes a_{i,j+1}^j \otimes a_{i,j+2,p+q}^1 \otimes 1 +$$

$$\sum_i a_i^0 \otimes a_{i,1,p+q-1}^1 \otimes a_{i,p+q}^q = (\text{Id} \otimes^{p+q-2} \otimes \pi)(dx) \otimes 1 + x = x,$$

where the third identity comes from the identity $dx = 0$ (since $x \in \Omega^{p+q}_{\text{sy}}(A)$). Similarly, for $x := \sum_i a_i^0 \otimes a_{i,1,p-1}^1 \otimes a_{i,p}^q \in \Omega^p_{\text{sy}}(A)$ and $y := \sum_j b_j^0 \otimes b_{j,1,q-1}^1 \otimes b_{j,q}^q \in \Omega^q_{\text{sy}}(A)$, we have

$$\mu_{p,q}^{-1} \mu_{p,q}(x \otimes_A y) = \mu_{p,q}^{-1} \left( \sum_{i,j} a_i^0 \otimes a_{i,1,p-1}^1 \otimes a_{i,q}^q b_j^0 \otimes b_{j,1,q-1}^1 \otimes b_{j,q}^q \right)$$

$$= (-1)^{p+q} \sum_{i,j} d_p (a_i^0 \otimes a_{i,1,p-1}^1 \otimes a_{i,q}^q b_j^0 \otimes 1) \otimes_A d_q (1 \otimes b_{j,1,q}^q \otimes 1)$$

$$= x \otimes_A y,$$

where the third identity comes from the identities $dx = 0$ and $dy = 0$. This proves the claim. It is clear that $\mu_{p,q}$ is a morphism of $A$-$A$-bimodules. Hence so is $\mu_{p,q}^{-1}$. Since $(\tau_p \otimes_A \tau_q) \Delta_{p,q} = \mu_{p,q}^{-1} \tau_{p+q}$, we get that $\Delta_{p,q}$ is a lifting of the isomorphism $\mu_{p,q}^{-1}$ between the resolutions $\text{Bar}_{p+q}(A)$ and $\text{Bar}_{p}(A) \otimes_A \text{Bar}_{q}(A)$. Hence it is an isomorphism in the homotopy category $\mathcal{K}(A \otimes A^\text{op}-\text{Mod})$ of dg $A$-$A$-bimodules. \qed
For \( p \geq 0 \), we define the dg \( A-A \)-bimodule of left noncommutative differential \( p \)-forms as \( \Omega_{\text{nc}}^{L,p}(A) = A \otimes (\Sigma A)^{\otimes p} \). Clearly, \( \Omega_{\text{nc}}^{L,p}(A) \) is concentrated in degree \( p \). The bimodule structure is given by
\[
a(a_0 \otimes \overline{a}_{1,p}) \bullet b = -(\text{Id}^{\otimes p} \otimes \pi)d_p(aa_0 \otimes \overline{a}_{1,p} \otimes b)
\]
for \( a, b \in A \) and \( a_0 \otimes \overline{a}_{1,p} \in A \otimes (\Sigma A)^{\otimes p} \). Here when \((\text{Id}^{\otimes p} \otimes \pi)\) is applied to the element \( d_p(aa_0 \otimes \overline{a}_{1,p} \otimes b)\), additional signs will appear because of the Koszul sign rule since \( \pi \) is a map of degree 1. More explicitly, we have
\[
a(a_0 \otimes \overline{a}_{1,p}) \bullet b = (-1)^p aa_0 a_1 \otimes \overline{a}_{2,p} \otimes b + \sum_{i=1}^{p-1} (-1)^{p+i} aa_0 \otimes \overline{a}_{i-1,i} \otimes \overline{a}_{i+1,2} \otimes \overline{a}_{i+2} \otimes b + aa_0 \otimes \overline{a}_{1,p-1} \otimes a_p b.
\]
Similarly, the dg \( A-A \)-bimodule of right noncommutative differential \( p \)-forms is defined as \( \Omega_{\text{nc}}^{R,p}(A) = (\Sigma A)^{\otimes p} \otimes A \). The bimodule structure is given by
\[
a \triangleright (\overline{a}_{1,p} \otimes a_{p+1})b = (\pi \otimes \text{Id}^{\otimes p})d_p(a \otimes \overline{a}_{1,p} \otimes a_{p+1} b).
\]

The following lemma is very useful throughout the present paper.

**Lemma 2.3.** We have two isomorphisms of dg \( A-A \)-bimodules
\[
\alpha_p^L : \Omega_{\text{nc}}^{L,p}(A) \xrightarrow{\cong} \Omega_{\text{sy}}^{p}(A), \quad a_0 \otimes \overline{a}_{1,p} \mapsto -d_p(a_0 \otimes \overline{a}_{1,p} \otimes 1);
\]
\[
\alpha_p^R : \Omega_{\text{nc}}^{R,p}(A) \xrightarrow{\cong} \Omega_{\text{sy}}^{p}(A), \quad \overline{a}_{1,p} \otimes a_0 \mapsto d_p(1 \otimes \overline{a}_{1,p} \otimes a_0).
\]

**Proof.** First, we claim that both \( \alpha_p^L \) and \( \alpha_p^R \) are bijective. Indeed, the inverse of \( \alpha_p^L \) is given by
\[
(\alpha_p^L)^{-1}(x) = (-1)^{p-1} \sum_i a_0^i \otimes a_{1,p}^i
\]
for \( x := \sum_i a_0^i \otimes a_{1,p-1}^i \otimes a_p^i \in \Omega_{\text{sy}}^{p}(A) \). That is, \((\alpha_p^L)^{-1}\) is the composition of maps
\[
(\alpha_p^L)^{-1} : \Omega_{\text{sy}}^{p}(A) \hookrightarrow A \otimes (\Sigma A)^{\otimes p-1} \otimes A \xrightarrow{\text{Id}^{\otimes p} \otimes \pi} \Omega_{\text{nc}}^{L,p}(A).
\]
Here the sign \((-1)^{p-1}\) is hidden in the Koszul sign rule. From a straightforward computation, we get that \( \alpha_p^L(\alpha_p^L)^{-1} = \text{Id} \) and \((\alpha_p^L)^{-1} \alpha_p^L = \text{Id} \). Similarly, the inverse of \( \alpha_p^R \) is given by
\[
(\alpha_p^R)^{-1}(x) = \sum_i a_{0,p-1}^i \otimes a_p^i
\]
for \( x := \sum_i a_{0}^i \otimes a_{1,p-1}^i \otimes a_p^i \in \Omega_{\text{sy}}^{p}(A) \). That is, \((\alpha_p^R)^{-1}\) is the composition of maps
\[
(\alpha_p^R)^{-1} : \Omega_{\text{sy}}^{p}(A) \hookrightarrow A \otimes (\Sigma A)^{\otimes p-1} \otimes A \xrightarrow{\pi \otimes \text{Id}^{\otimes p}} \Omega_{\text{nc}}^{R,p}(A).
\]
This proves the claim. It remains to check that \( \alpha_p^L \) and \( \alpha_p^R \) are morphisms of \( A-A \)-bimodules. For this, given \( a_0 \otimes \overline{a}_{1,p} \in A \otimes (\Sigma A)^{\otimes p} \), we have
\[
\alpha_p^L(a(a_0 \otimes \overline{a}_{1,p}) \bullet b) = -d_p((\text{Id}^{\otimes p} \otimes \pi)d_p(aa_0 \otimes \overline{a}_{1,p} \otimes b) \otimes 1)
\]
\[
= d_p(aa_0 \otimes \overline{a}_{1,p} \otimes b)
\]
\[
= a \alpha_p^L(a(a_0 \otimes \overline{a}_{1,p})b),
\]
where the second identity follows from \( d_p d_{p+1}(aa_0 \otimes \overline{a}_{1,p} \otimes b \otimes 1) = 0 \). This shows that \( \alpha_p^L \) is a morphism of \( A-A \)-bimodules. By a similar computation, we get that \( \alpha_p^R \) is a morphism of \( A-A \)-bimodules. This proves the lemma. \( \square \)
2.2. Two liftings. Let $M$ be a graded $A$-$A$-bimodule. Recall that the Hochschild cohomology $HH^*(A, M)$ with coefficients in $M$ is computed by the Hochschild cochain complex $(C^*(A, M), \delta)$ with

$$C^m(A, M) = \prod_{i \geq 0} \mathcal{H}om^{-m}((\Sigma A)^{\otimes i}, M), \quad \text{for } m \in \mathbb{Z},$$

where $(\Sigma A)^{\otimes 0} = k$ and $\mathcal{H}om^{-m}((\Sigma A)^{\otimes i}, M)$ is the set of $k$-linear maps of degree $-m$ from chain complexes $(\Sigma A)^{\otimes i}$ to $M$. Recall that a $k$-linear map $f : X \to Y$ between two chain complexes $X$ and $Y$ is of degree $m$ if $f(X_i) \subset Y_{i+m}$ for any $i \in \mathbb{Z}$. The differential $\delta$ (of degree one) is given by, for $f \in C^m(A, M)$,

$$\delta^m(f)(\bar{a}_{i+1}) = a_1 f(\bar{a}_{2,i+1}) + \sum_{j=1}^i (-1)^j f(\bar{a}_{i+1,j-1} \otimes \overline{a}_{jkat} \otimes \bar{a}_{j+2,i+1}) + (-1)^{i+1} f(\bar{a}_{i,1}) a_{i+1}.$$

Let $m, p \in \mathbb{Z}_{\geq 0}$ and $f \in HH^{m-p}(A, \Omega^p_{sy}(A))$. Recall that $\Omega^p_{sy}(A)$ is a graded $A$-$A$-bimodule concentrated in degree $p$. Then $f$ can be represented by an element $f \in C^{m-p}(A, \Omega^p_{sy}(A)) = Hom((\Sigma A)^{\otimes m}, \Omega^p_{sy}(A))$ such that $\delta(f) = 0$. Denote

$$f^L : (\Sigma A)^{\otimes m} \xrightarrow{f} \Omega^p_{sy}(A) \xrightarrow{(\alpha^L_p)^{-1}} \Omega^L_{ne}(A) = A \otimes (\Sigma A)^{\otimes p},$$

$$f^R : (\Sigma A)^{\otimes m} \xrightarrow{f} \Omega^p_{sy}(A) \xrightarrow{(\alpha^R_p)^{-1}} \Omega^R_{ne}(A) = (\Sigma A)^{\otimes p} \otimes A,$$

where $(\alpha^L_p)^{-1}$ and $(\alpha^R_p)^{-1}$ are defined in Lemma 2.3. These two maps induce two liftings

$$\vartheta^L(f), \vartheta^R(f) : Bar_*(A) \to \Sigma^{m-p} Bar_{\geq p}(A)$$

in the following way. Let $x := a_0 \otimes \bar{a}_{1,r} \otimes a_{r+1} \in Bar_r(A)$. If $r < m$, we define

$$\vartheta^L(f)(x) = \vartheta^R(f)(x) = 0.$$

If $r \geq m$, we define

$$\vartheta^L(f)(x) = a_0 f^L(\bar{a}_{1,m}) \otimes \bar{a}_{m+1,r} \otimes a_{r+1},$$

$$\vartheta^R(f)(x) = (-1)^{(m-p)(r-m)} a_0 \otimes \bar{a}_{1,r-m} \otimes f^R(\bar{a}_{r-m+1,r}) a_{r+1}.$$

It follows from $\delta(f) = 0$ that $\vartheta^L(f)$ and $\vartheta^R(f)$ are indeed morphisms of dg $A$-$A$-bimodules. It is well-known from homological algebra (cf. e.g. [Wei, Comparison Theorem 2.2.6]) that $\vartheta^L(f)$ is homotopy equivalent to $\vartheta^R(f)$. In fact, there exists a specific chain homotopy

$$h(f) : Bar_*(A) \to \Sigma^{m-p-1} Bar_{\geq p}(A)$$

from $\vartheta^L(f)$ to $\vartheta^R(f)$ defined as follows. For any $r \in \mathbb{Z}_{\geq 0}$,

$$h_r(f)(a_0 \otimes \bar{a}_{1,r} \otimes a_{r+1}) = \begin{cases} 0 & \text{for } r \leq m - 1, \\ \sum_{i=0}^{r-m} (-1)^{(m-p-1)i} a_0 \otimes \bar{a}_{1,i} \otimes \overline{f}(\bar{a}_{i+1,i+m}) \otimes \bar{a}_{i+m+1,r} \otimes a_{r+1} & \text{for } r \geq m, \end{cases}$$

where

$$\overline{f} : (\Sigma A)^{\otimes m} \xrightarrow{f} \Omega^p_{sy}(A) \xrightarrow{\pi \otimes \bar{I} \otimes \pi} (\Sigma A)^{p+1}.$$

Indeed, it is easy to verify that $\vartheta^L(f) - \vartheta^R(f) = h(f)d + dh(f)$. Notice that $h(f)$ is a morphism of graded $A$-$A$-bimodules. It follows that $\vartheta^L(f)$ is isomorphic to $\vartheta^R(f)$ in the homotopy category $\mathcal{K}^-(A \otimes A^{op}$-$\text{Mod})$. Therefore, both $\vartheta^L(f)$ and $\vartheta^R(f)$ are representatives of $f \in Hom_{\mathcal{D}^{b}(A \otimes A^{op})}(A, \Sigma^{m-p} \Omega^p_{sy}(A))$ in $\mathcal{K}^-(A \otimes A^{op}$-$\text{Mod})$. 
From $f \in \text{HH}^{m-p}(A, \Omega_{sy}^p(A))$, we may get an element $\Omega_{sy}^r(f) \in \text{HH}^{m-p}(A, \Omega_{sy}^{r+p}(A))$ for any $r \geq 0$, which is represented by the element

$$
\Omega_{sy}^r(f) : \text{Bar}_{m+r}(A) \rightarrow \sum^{m-p}\Omega_{sy}^{p+r}(A)
$$

$$
a_0 \otimes \mathfrak{a}_{1,m+r} \otimes a_{m+r+1} \mapsto d_{p+r}(a_0 f^L(\mathfrak{a}_{1,m}) \otimes \mathfrak{a}_{m+1,m+r} \otimes a_{m+r+1}).
$$

Similarly, $\Omega_{sy}^r(f)$ may also be represented by the element

$$
\Omega_{sy}^r(f) : \text{Bar}_{m+r}(A) \rightarrow \sum^{m-p}\Omega_{sy}^{p+r}(A)
$$

$$
a_0 \otimes \mathfrak{a}_{1,m+r} \otimes a_{m+r+1} \mapsto (-1)^{(m-p)r}d_{p+r}(a_0 \otimes \mathfrak{a}_{1,r} \otimes f^R(\mathfrak{a}_{1+r,m+r})a_{m+r+1}).
$$

**Remark 2.4.** The above homotopy $h(f)$ induces a homotopy $h^{L,R}_r(f) := d_{p+r}h_{m+r-1}(f)$ such that $h^{L,R}_r(f)d_{m+r} = \Omega_{sy}^{L,r}(f) - \Omega_{sy}^{R,r}(f)$. For any $f \in C^{m-p}(A, \Omega_{sy}^p(A))$ such that $\delta(f) = 0$, we have the following identities

$$
\mu_{r,p+s}(d_r \otimes A \Omega_{sy}^{R,s}(f))\Delta_{r,m+s} = \Omega_{sy}^{R,s}(f),
$$

$$
\mu_{p+s,r}(\Omega_{sy}^{L,s}(f) \otimes A d_r)\Delta_{m+s,r} = \Omega_{sy}^{L,s}(f),
$$

which can be verified by straightforward computation.

Therefore, we have a map for any $r > 0$,

$$
\Omega_{sy}^r : \text{HH}^{m-p}(A, \Omega_{sy}^p(A)) \rightarrow \text{HH}^{m-p}(A, \Omega_{sy}^{p+r}(A)), \quad f \mapsto \Omega_{sy}^{L,r}(f) = \Omega_{sy}^{R,r}(f).
$$

Notice that $\Omega_{sy}^r(\Omega_{sy}^s(f)) = \Omega_{sy}^{r+s}(f)$ for $r, s \geq 0$ since $\Omega_{sy}^{L,s}(\Omega_{sy}^{R,s}(f)) = \Omega_{sy}^{R,s}(f)$. This induces an inductive system

$$
\cdots \rightarrow \text{HH}^{m-p}(A, \Omega_{sy}^p(A)) \rightarrow \text{HH}^{m-p}(A, \Omega_{sy}^{p+1}(A)) \rightarrow \cdots \rightarrow \text{HH}^{m-p}(A, \Omega_{sy}^{p+r}(A)) \rightarrow \cdots.
$$

It follows from [Van15a Proposition 3.1] that if $A$ is a Noetherian algebra over a field $k$ such that the enveloping algebra $A \otimes A^{op}$ is Noetherian, then the colimit of the above inductive system is isomorphic to the $(m-p)$-th Tate-Hochschild cohomology group

$$
\text{HH}_{sg}^{m-p}(A, A) := \text{Hom}_{\mathcal{D}_{sg}(A \otimes A^{op})}(A, \Sigma^{m-p}A), \quad m - p \in \mathbb{Z},
$$

where $\mathcal{D}_{sg}(A \otimes A^{op})$ is the singularity category of the enveloping algebra $A \otimes A^{op}$. Recall that the *singularity category* $\mathcal{D}_{sg}(A)$ (cf. [Buc, Orl]) of a Noetherian algebra $A$ is defined as the Verdier quotient of the bounded derived category $\mathcal{D}^b(A\text{-mod})$ of finitely generated (left) $A$-modules by the full subcategory $\text{Perf}(A)$ consisting of complexes quasi-isomorphic to bounded complexes of finitely generated projective $A$-modules.

3. **DG $k[\epsilon_i]/(\epsilon_i^2)$-modules.**

3.1. **A construction of dg $k[\epsilon_i]/(\epsilon_i^2)$-modules.** Let $A$ be a Noetherian algebra over a field $k$ such that the enveloping algebra $A \otimes A^{op}$ is Noetherian. For $i \in \mathbb{Z}$, we denote by $R_i$ the commutative dg algebra $k[\epsilon_i]/(\epsilon_i^2)$ with trivial differential, where $\epsilon_i$ is of degree $i$. With a slight abuse of notation, we denote by $\epsilon_i$ the kernel of the augmentation $R_i \rightarrow k$.

Clearly, $\epsilon_i$ is the one-dimensional graded $k$-vector space concentrated in degree $i$. For a chain complex $X$ of (left) $A$-modules, there is a natural isomorphism of chain complexes between $\Sigma^i X$ and the tensor product $\epsilon_i \otimes X$. In what follows, we will not distinguish between them.

Let $\alpha : X \rightarrow Y$ be a morphism (of degree zero) of chain complexes. Recall that the mapping cone of $\alpha$ is defined as the chain complex $\text{Cone}(\alpha) = \Sigma X \oplus Y$ with differential
Lemma 3.1. Let $\beta : X \to Y$ be a morphism of dg $A$-modules. Let $\alpha : X \to \Sigma^{i+1}Y$ be another morphism of dg $A$-modules. Then there is a dg $R_i \otimes A$-module structure on $C(\alpha)$ induced by $\alpha$ and $\beta$.

Proof. By definition, the complex $C(\alpha)$ is equal to $(X \oplus \Sigma^i Y; (\begin{pmatrix} d & 0 \\ \alpha & \Sigma^{-1}d \end{pmatrix})_\alpha)$. The graded $R_i$-module structure on $C(\alpha)$ is given as follows: For $x + \Sigma^i y \in X \oplus \Sigma^i Y$, the action of $\lambda + \mu \epsilon_i \in R_i$ $(\lambda, \mu \in k)$ is

\[(\lambda + \mu \epsilon_i)(x + \Sigma^i y) = \lambda x + \Sigma^i(\lambda y + \mu \beta(x)).\]

It is clear that this action is compatible with the differential $(\begin{pmatrix} d & 0 \\ \alpha & \Sigma^{-1}d \end{pmatrix})_\alpha$. This proves the lemma. \hfill \Box

Let $m, p \in \mathbb{Z}_{\geq 0}$ and $f \in \text{HH}^{m-p}(A, \Omega^r_{sy}(A))$. In Section 2.2 we have defined two liftings $\vartheta^L(f)$ and $\vartheta^R(f)$ associated to $f$. It follows from Lemma 3.1 that $C(\vartheta^L(f))$ and $C(\vartheta^R(f))$ are dg $R_{m-p-1} \otimes A \otimes A^{_{op}}$-modules. To see this, we take the map $\beta$ in Lemma 3.1 to be the natural projection $\text{Bar}_r(A) \to \text{Bar}_{\geq p}(A)$ and $\alpha = \vartheta^L(f)$ (resp. $\alpha = \vartheta^R(f)$). In particular, as graded $R_{m-p-1} \otimes A \otimes A^{_{op}}$-modules, we have

\[C(\vartheta^L(f)) \cong \bigoplus_{i=0}^{p-1} (k \otimes \text{Bar}_i(A)) \bigoplus R_{m-p-1} \otimes \text{Bar}_{\geq p}(A) \cong C(\vartheta^R(f)),\]

where $k$ is viewed as the $R_{m-p-1}$-module concentrated in degree zero and thus $k \otimes \text{Bar}_i(A)$ is an $R_{m-p-1} \otimes A \otimes A^{_{op}}$-module concentrated in degree $i$. In Section 2.2 we have also defined two cocycles $\Omega^L_{sy}(f)$ and $\Omega^R_{sy}(f)$ representing the element $\Omega^r_{sy}(f) \in \text{HH}^{m-p}(A, \Omega^r_{sy}(A))$ for $r \geq 0$. We note that both $C(\Omega^L_{sy}(f))$ and $C(\Omega^R_{sy}(f))$ are dg $R_{m-p-1} \otimes A \otimes A^{_{op}}$-modules. For this, we take the map $\beta$ in Lemma 3.1 to be the projection $\text{Bar}_r(A) \to \Omega^r_{sy}(A)$ induced by the natural map $\text{Bar}_{p+r}(A) \to \Omega^r_{sy}(A)$ and $\alpha = \Omega^L_{sy}(f)$ (resp. $\alpha = \Omega^R_{sy}(f)$). In particular, as graded $R_{m-p-1} \otimes A \otimes A^{_{op}}$-modules, we have

\[C(\Omega^L_{sy}(f)) \cong \bigoplus_{i \neq p+r} (k \otimes \text{Bar}_i(A)) \bigoplus (\text{Bar}_{p+r}(A) \otimes \Sigma^{m-p-1} \Omega^r_{sy}(A)) \cong C(\Omega^R_{sy}(f)),\]

where $(\text{Bar}_{p+r}(A) \otimes \Sigma^{m-p-1} \Omega^r_{sy}(A))$ is the graded $R_{m-p-1} \otimes A \otimes A^{_{op}}$-module determined by the action

\[\epsilon_{m-p-1} \cdot x := (-1)^{m-p-1} d_{p+r}(x) \in \Sigma^{m-p-1} \Omega^r_{sy}(A)\]

for any $x \in \text{Bar}_{p+r}(A)$. When $r = 0$, we get that $C(f)$ is a dg $R_{m-p-1} \otimes A \otimes A^{_{op}}$-module.

Remark 3.2. Let $f_1$ and $f_2$ be two different cocycles representing $f \in \text{HH}^{m-p}(A, \Omega^r_{sy}(A))$. Then there exists $\alpha \in \text{Hom}(\Sigma^A \otimes A_{m-1}, \Omega^r_{sy}(A))$ such that $f_1 - f_2 = \delta(\alpha)$. Define a map

\[\vartheta^L(\alpha) : \text{Bar}_r(A) \to \Sigma^{m-p-1} \text{Bar}_{\geq p}(A)\]

as follows. Let $x = a_0 \otimes \overline{a}_{1,r} \otimes a_{r+1} \in \text{Bar}_r(A)$. If $r < m-1$, we define $\vartheta^L(\alpha)(x) = 0$. If $r \geq m-1$, we define

\[\vartheta^L(\alpha)(x) = a_0 \alpha^L(\overline{a}_{1,m-1}) \otimes \overline{a}_{m,r} \otimes a_{r+1},\]
where $\alpha^L$ is defined as in Section 2.2. Notice that the identity $f_1 - f_2 = \delta(\alpha)$ yields $\vartheta^L(\alpha)d + d\vartheta^L(\alpha) = \vartheta^L(f_1) - \vartheta^L(f_2)$. Thus the map \[
abla^{\vartheta^L}_{(0)} : C(\vartheta^L(f_2)) \to C(\vartheta^L(f_1)) \]
is an isomorphism of dg $R_{m-p-1} \otimes A \otimes A^{op}$-modules with inverse \[
abla^{\vartheta^L}_{(0)} : C(\vartheta^L(f_1)) \to C(\vartheta^L(f_2)). \] This shows that $C(\vartheta^L(f))$ does not depend, up to isomorphism of dg $R_{m-p-1} \otimes A \otimes A^{op}$-modules, on the choice of the representatives of $f$. Similar arguments are used to prove that $C(\vartheta^R(f)), C(\Omega^R_{\text{sy}}(f))$ and $C(\Omega^{\text{sy}}_{\vartheta^R}(f))$ are independent of the choice of the representatives of $f$.

**Lemma 3.3.** Let $m \in \mathbb{Z}_{>0}$ and $p \in \mathbb{Z}_{\geq 0}$. For $f \in \text{HH}^{m-p}(A, \Omega^p_{\text{sy}}(A))$, the following assertions hold.

(i) $C(\vartheta^L(f))$ is isomorphic to $C(\vartheta^R(f))$ as dg $R_{m-p-1} \otimes A \otimes A^{op}$-modules.

(ii) The morphism of dg $R_{m-p-1} \otimes A \otimes A^{op}$-modules \[
abla_p = \left( \begin{array}{cc} 1 & 0 \\ 0 & \sigma_p \end{array} \right) : C(\vartheta^L(f)) \to C(f) \]
is an isomorphism in the homotopy category $\mathcal{K}(R_{m-p-1} \otimes A)$ and in $\mathcal{K}(R_{m-p-1} \otimes A^{op})$, where $\sigma_p : \epsilon_{m-p-1} \otimes \text{Bar}_p(A) \to \epsilon_{m-p} \otimes \Omega^p_{\text{sy}}(A)$ is the surjection induced by the augmentation $\tau_p : \text{Bar}_p(A) \to \Omega^p_{\text{sy}}(A)$.

**Proof.** Let us prove assertion (i). Consider the morphism of chain complexes \[
abla^{\vartheta^L}_{(0)} : C(\vartheta^L(f)) \to C(\vartheta^R(f)), \]
where $h(f)$ is the chain homotopy defined in \(1\). Note that \[
abla^{\vartheta^L}_{(0)} \]
is a morphism of dg $R_{m-p-1} \otimes A \otimes A^{op}$-modules since $h(f)$ is a morphism of dg $A \otimes A^{op}$-modules and is compatible with the action of $\epsilon_{m-p-1}$. In fact, it is an isomorphism with inverse \[
abla^{\vartheta^L}_{(0)} : C(\vartheta^R(f)) \to C(\vartheta^L(f)). \] This proves assertion (i).

Let us prove assertion (ii). We claim that $C(\vartheta^L(f))$ is isomorphic to $C(\vartheta^L(0)) = \text{Bar}_s(A) \oplus \epsilon_{m-p-1} \otimes \text{Bar}_{\geq p}(A)$ as dg $R_{m-p-1} \otimes A$-modules. Indeed, we define a morphism of graded $A$-modules \[
abla \] as dg $R_{m-p-1} \otimes A$-modules.

Notice that we have $\vartheta^L(f) = df - fd$. This yields a morphism of dg $R_{m-p-1} \otimes A$-modules \[
abla(f) = \left( \begin{array}{cc} 1 & 0 \\ f & 1 \end{array} \right) : C(\vartheta^L(f)) \to C(\vartheta^L(0)) \]
since $\vartheta(f)$ is compatible with the action of $\epsilon_{m-p-1}$ and we have \[
abla(f) = \left( \begin{array}{cc} 1 & 0 \\ f & 1 \end{array} \right) \left( \begin{array}{cc} 1 & 0 \\ \vartheta(f) & 1 \end{array} \right) = \left( \begin{array}{cc} 1 & 0 \\ f & 1 \end{array} \right) \left( \begin{array}{cc} 1 & 0 \\ \vartheta(f) & 1 \end{array} \right). \]

It is clear that $\vartheta(f)$ is an isomorphism with inverse \[
abla(\vartheta(f)) : C(\vartheta^L(0)) \to C(\vartheta^L(f)). \]
This proves the claim. Similarly, we have an isomorphism of dg $R_{m-p-1} \otimes A$-modules \[
abla(f) = \left( \begin{array}{cc} 1 & 0 \\ \sigma_p & 1 \end{array} \right) : C(f) \to C(0) = \text{Bar}_s(A) \oplus \epsilon_{m-p-1} \otimes \Omega^p_{\text{sy}}(A), \]
where $\sigma_p \vartheta$ is the following composition of maps \[
abla \text{Bar}_s(A) \xrightarrow{\vartheta} \epsilon_{m-p-1} \otimes \text{Bar}_{\geq p}(A) \xrightarrow{\sigma_p} \epsilon_{m-p-1} \otimes \Omega^p_{\text{sy}}(A). \]
Note that the following diagram commutes
\[
\begin{array}{ccc}
C(\partial^L(f)) & \xrightarrow{\sigma_p} & C(f) \\
\phi(f) & \cong & \cong \\
C(\partial^L(0)) & \xrightarrow{\sigma_p} & C(0).
\end{array}
\]

To prove that \(\sigma_p : C(\partial^L(f)) \to C(f)\) is an isomorphism in \(\mathcal{K}(R_{m-p-1} \otimes A)\), it is equivalent to prove that \(\sigma_p : C(\partial^L(0)) \to C(0)\) is an isomorphism in \(\mathcal{K}(R_{m-p-1} \otimes A)\). We have a commutative diagram of distinguished triangles in \(\mathcal{K}(R_{m-p-1} \otimes A)\)
\[
\begin{array}{ccc}
\text{Bar}_{<p}(A) & \xrightarrow{=} & \text{Bar}_{\geq p}(A) \\
\text{Bar}_{<p}(A) & \xrightarrow{=} & \text{Bar}_{\geq p}(A) \\
\text{Bar}_{\geq p}(A) & \xrightarrow{=} & \text{Bar}_{\geq p}(A) \\
\text{Bar}_{\geq p}(A) & \xrightarrow{=} & \text{Bar}_{\geq p}(A) \\
\end{array}
\]

It is clear that \((\text{id}_0 0 \sigma_p)\) is an isomorphism in \(\mathcal{K}(R_{m-p-1} \otimes A)\) since we have the following commutative diagram
\[
\begin{array}{ccc}
\text{Bar}_{\geq p}(A) \oplus \epsilon_{m-p-1} \otimes \text{Bar}_{\geq p}(A) & \xrightarrow{(\text{id}_0 0 \sigma_p)} & \text{Bar}_{\geq p}(A) \oplus \epsilon_{m-p-1} \otimes \Omega_{sy}^p(A) \\
\end{array}
\]

where \((\tau_p 0 0)\) and \((\tau_p 0 0)\) are isomorphisms in \(\mathcal{K}(R_{m-p-1} \otimes A)\). This implies that \(\sigma_p\) is an isomorphism in \(\mathcal{K}(R_{m-p-1} \otimes A)\). By a similar argument, we can prove that \(\sigma_p\) is an isomorphism in \(\mathcal{K}(R_{m-p-1} \otimes A^{op})\). This proves assertion (ii). The proof is complete. \(\Box\)

3.2. Dg modules arising from the bullet and circle products. From Section 2.2, we have a map \(\Omega_{sy}^{p} : HH^r(A, \Omega_{sy}^{p}(A)) \to HH^r(A, \Omega_{sy}^{p+r}(A))\) for \(p, r \geq 0\). Recall that the Tate-Hochschild cohomology \(HH^*_sg(A, A)\) is isomorphic to the colimit of the inductive system
\[
\begin{array}{ccc}
HH^*(A, A) & \xrightarrow{\Omega_{sy}^1} & HH^*(A, \Omega_{sy}^1(A)) \\
& \xrightarrow{\Omega_{sy}^1} & \cdots \xrightarrow{\Omega_{sy}^1} HH^*(A, \Omega_{sy}^p(A)) \to \cdots
\end{array}
\]

Let us recall the Lie bracket \([\cdot, \cdot]\) on \(HH^*_sg(A, A)\) constructed in \[\text{Wan15a, Wan18}\]. The notations in the present paper are slightly different from those in \[\text{Wan18}\] since we are using the dg bimodules \(\Omega_{nc}^r(A)\) instead of \(\Omega_{nc}^r(A)\). For \(f \in C^{m-p}(A, \Omega_{sy}^p(A))\) and \(g \in C^{n-q}(A, \Omega_{sy}^q(A))\), set
\[
f \bullet g := \begin{cases} 
(Id^{\otimes q} \otimes f^R)(Id^{\otimes i-1} \otimes \overline{f} \otimes Id^{\otimes m-i}) & \text{if } 1 \leq i \leq m, \\
(Id^{\otimes q+i} \otimes \overline{f} \otimes Id^{\otimes i})(Id^{\otimes m-i} \otimes g^R) & \text{if } -q \leq i \leq -1,
\end{cases}
\]

where \(\overline{f}, f^L\) and \(f^R\) are defined as in Section 2.2, namely
\[
\overline{f} : (\Sigma A)^{\otimes m} \xrightarrow{f} \Omega_{sy}^p(A) \xrightarrow{\pi \otimes Id^{\otimes p-1} \otimes \pi} (\Sigma A)^{p+1};
\]
\[
f^L : (\Sigma A)^{\otimes m} \xrightarrow{f} \Omega_{sy}^p(A) \xrightarrow{(\alpha_p)^{i-1}} \Omega_{nc}^{L_p}(A) = A \otimes (\Sigma A)^{\otimes p};
\]
\[
f^R : (\Sigma A)^{\otimes m} \xrightarrow{f} \Omega_{sy}^p(A) \xrightarrow{(\alpha_p)^{i-1}} \Omega_{nc}^{R_p}(A) = (\Sigma A)^{\otimes p} \otimes A.
\]
Clearly, we have $f \bullet g \in C^{m+n-p-q-1}(A, \Omega_{nc}^{R,p+q}(A))$. For instance, $f \bullet g \ (i > 0)$ is the composition of maps

$$(\Sigma A)^{\otimes m+n-1} \xrightarrow{\text{Id} \otimes \ldots \otimes \text{Id} \otimes f} (\Sigma A)^{\otimes m+q} \xrightarrow{\text{Id} \otimes f^R} (\Sigma A)^{\otimes p+q} \otimes A = \Omega_{nc}^{R,p+q}(A).$$

Since the isomorphism $\alpha^R_p : \Omega_{nc}^{R,p}(A) \cong \Omega_{sy}^{p}(A)$ (cf. Lemma 2.3) induces an isomorphism $\alpha^R_p : C^*(A, \Omega_{nc}^{R,p}(A)) \cong C^*(A, \Omega_{sy}^{p}(A))$, we have $\alpha^R_p(f \bullet g) \in C^{m+n-p-q-1}(A, \Omega_{sy}^{p+q}(A))$. We define

$$f \circ g := \sum_{i=1}^m \alpha^R_{p+q}(f \bullet g) - (-1)^{(m-p-1)(n-q-1)} \sum_{i=1}^p \alpha^R_{p+q}(g \bullet i f);$$

$$f \bullet g := \sum_{i=1}^m \alpha^R_{p+q}(f \bullet g) + \sum_{i=1}^q \alpha^R_{p+q}(f \bullet i g);$$

$$[f, g] := f \bullet g - (-1)^{(m-p-1)(n-q-1)} g \bullet f = f \circ g - (-1)^{(m-p-1)(n-q-1)} g \circ f.$$

We remark that when these formulas are applied to elements, additional signs will appear because of the Koszul sign rule. When $p = q = 0$, $f \circ g = f \bullet g$ is the usual Gerstenhaber circle product and $[\cdot, \cdot]$ is the usual Gerstenhaber bracket on $C^*(A, A)$. Then from [Wan18, Section 4.2], we get that $[\cdot, \cdot]$ respects the map $\Omega_{sy}^p : \text{HH}^*(A, \Omega_{sy}^p(A)) \to \text{HH}^*(A, \Omega_{sy}^{p+q}(A))$. Thus it induces a well-defined Lie bracket (still denoted by $[\cdot, \cdot]$) on $\text{HH}^*_{sy}(A, A)$. We have the following very important observation.

**Lemma 3.4.** For two cocycles $f \in C^{m-p}(A, \Omega_{sy}^p(A))$ and $g \in C^{n-q}(A, \Omega_{sy}^q(A))$, the following identities hold in $C^{m+n-p-q-1}(A, \Omega_{sy}^{p+q}(A))$

$$g \bullet f = \Omega_{sy}^{R,p}(g)h(f) - h^L_{p+R}(g)\partial^R(f),$$

$$g \circ f = \Omega_{sy}^{R,p}(g)h(f) + h^L_{q+R}(f)\partial^R(g),$$

where $h(f)$ is defined in [4] and $h^L_{q+R}(f)$ is defined in Remark 2.4.

**Proof.** This follows from the following identities

$$\Omega_{sy}^{R,p}(g)h(f) = \sum_{i=1}^n \alpha^R_{p+q}(g \bullet i f), \quad h^L_{p+R}(g)\partial^R(f) = \sum_{i=1}^p \alpha^R_{p+q}(g \bullet i f).$$

Let us verify these two identities. For this, we have

$$\Omega_{sy}^{R,p}(g)h(f)(\overline{a}_{i,m+n-1})$$

$$= \sum_{i=1}^n (-1)^{(m-p-1)(i-1)} \Omega_{sy}^{R,p}(g)(\overline{a}_{i,j-1} \otimes \overline{f}(\overline{a}_{i,j+m-1} \otimes \overline{a}_{i+m,m+n-1}))$$

$$= \sum_{i=1}^n (-1)^{(m-p-1)(i-1)} \alpha^R_{p+q}(\text{Id} \otimes g^R)(\overline{a}_{i,j-1} \otimes \overline{f}(\overline{a}_{i,j+m-1} \otimes \overline{a}_{i+m,m+n-1}))$$

$$= \sum_{i=1}^n \alpha^R_{p+q}(g \bullet i f)(\overline{a}_{i,m+n-1}).$$
Similarly, we have
\[ h^{L,R}_p(g)g^R(f)(\overline{a}_{1,m+n-1}) = (-1)^{(m-p)(n-q)}h^{L,R}_p(g)(\overline{a}_{1,n-1} \otimes f^R(\overline{a}_{n,m+n-1})) \]
\[ = (-1)^{(m-p)(n-1)} \sum_{i=1}^p \alpha^{R}_{p+q}(\text{Id} \otimes g \otimes \text{Id} \otimes \text{Id})((\overline{a}_{1,n-1} \otimes f^R(\overline{a}_{n,m+n-1})) \]
\[ = \sum_{i=1}^p \alpha^{R}_{p+q}(g \bullet_i f)(\overline{a}_{1,m+n-1}). \]

This proves the lemma. \( \square \)

The Yoneda product
\[ \cup' : \HH^{m-p}(A, \Omega^n_{\text{sy}}(A)) \otimes \HH^{n-q}(A, \Omega^p_{\text{sy}}(A)) \to \HH^{m+n-p-q}(A, \Omega^{p+q}_{\text{sy}}(A)) \]
is given by the composition
\[ \HH^{m-p}(A, \Omega^n_{\text{sy}}) \otimes \HH^{n-q}(A, \Omega^p_{\text{sy}}) \to \HH^{m+n-p-q}(A, \Omega^n_{\text{sy}} \otimes \Omega^p_{\text{sy}}) \to \HH^{m+n-p-q}(A, \Omega^{p+q}_{\text{sy}}), \]
where we simply write \( \Omega^p_{\text{sy}} \) for \( \Omega^n_{\text{sy}}(A) \); and the second morphism is the isomorphism induced by \( \mu_{p,q} : \Omega^p_{\text{sy}}(A) \otimes \Omega^q_{\text{sy}}(A) \to \Omega^{p+q}_{\text{sy}}(A) \) (cf. the proof of Lemma 2.2). At the complex level, \( \cup' \) is given as follows. For \( f \in C^{m-p}(A, \Omega^p_{\text{sy}}(A)) \) and \( g \in C^{n-q}(A, \Omega^q_{\text{sy}}(A)), \)
\[ f \cup' g(\overline{a}_{1,m+n}) = \mu_{p,q}(f(\overline{a}_{1,m}) \otimes A g(\overline{a}_{m+1,m+n})). \]

We defined another cup product \( \cup \) in \([\text{Wan18}], \text{Section 4}\):
\[ f \cup g = \alpha^{R}_{p+q}(\text{Id} \otimes \mu)(\text{Id} \otimes f^R \otimes \text{Id})(\text{Id} \otimes g \otimes g^R). \]

More precisely, \( f \cup g \) is the composition of maps
\[ A^{m+n} \otimes \text{Id}^m \otimes g^R \to \overline{A}^{m+q} \otimes A \to \overline{A}^{p+q} \otimes A \otimes A \to \Omega_{nc}^{p+q}(A) \to \Omega_{\text{sy}}^{p+q}(A), \]
where we simply write \( \overline{A} \) for \( \Sigma A \). At the cohomology level, the cup product \( \cup' \) is equal to \( \cup \) (cf. \([\text{Wan18}], \text{Section 4}\)). We note that \( \cup \) is compatible with the map \( \Omega^p_{\text{sy}} \). Thus, it induces a well-defined cup product \( \cup' = \cup : \HH^p_{\text{sy}}(A, A) \otimes \HH^q_{\text{sy}}(A, A) \to \HH^{p+q}_{\text{sy}}(A, A) \).

**Remark 3.5.** It is clear that the two products \( \cup \) and \( \cup' \) at the complex level are not (graded-)commutative. But we have the following identity
\[ f \cup' g - (-1)^{(m-p)(n-q)} g \cup' f = (-1)^{m-p} \delta(g \bullet f), \]
\[ f \cup g - (-1)^{(m-p)(n-q)} g \cup f = (-1)^{m-p} \delta(g \circ f), \]
for any \( f \in C^{m-p}(A, \Omega^p_{\text{sy}}(A)) \) and \( g \in C^{n-q}(A, \Omega^q_{\text{sy}}(A)) \) such that \( \delta(f) = 0 = \delta(g) \) (cf. \([\text{Wan18}], \text{Proposition 4.4}\)). This shows that \( \cup' = \cup \) is graded-commutative at the cohomology level.

In the following, we will use the identities in \([2]\) to construct two dg \( R_{m-p-1} \otimes R_{n-q-1} \otimes A \otimes A^{op} \)-modules \( C^L(f, g) \) and \( C^R(f, g) \) (see below), which are independent (up to isomorphism) of the choice of representatives in the cohomology classes of \( f \) and \( g \) (cf. Lemma 3.8.5). We stress that these two dg modules play a crucial role in the proof of Proposition 5.9, a key step in proving our main Theorem 6.2.

Let \( f \in \HH^{m-p}(A, \Omega^p_{\text{sy}}(A)) \) and \( g \in \HH^{n-q}(A, \Omega^q_{\text{sy}}(A)), \) which are represented by the cocycles \( f \in C^{m-p}(A, \Omega^p_{\text{sy}}(A)) \) and \( g \in C^{n-q}(A, \Omega^q_{\text{sy}}(A)) \) respectively. Let us consider the
following three chain complexes associated to \( f \) and \( g \). For simplicity, we set \( r := m - p - 1 \) and \( s := n - q - 1 \). The first complex is \( C(f, g) \) defined as

\[
\text{Bar}_s(A) \oplus \epsilon_r \otimes \text{Bar}_{\geq p}(A) \oplus \epsilon_s \otimes \text{Bar}_{\geq q}(A) \oplus \epsilon_{r+s} \otimes \Omega_{syt}^{p+q}(A).
\]

The identity \( \Omega_{syt}^{L,q}(f)\vartheta_R(g) = \Omega_{syt}^{R,p}(g)\vartheta_L(f) \) implies that \( C(f, g) \) is a well-defined complex. The second one is \( C^L(f, g) \) defined as

\[
\text{Bar}_s(A) \oplus \epsilon_r \otimes \text{Bar}_{\geq p}(A) \oplus \epsilon_s \otimes \text{Bar}_{\geq q}(A) \oplus \epsilon_{r+s} \otimes \Omega_{syt}^{p+q}(A).
\]

The first identity in \([2]\) ensures that \( C^L(f, g) \) is a complex. The third one is \( C^R(f, g) \) defined as

\[
\text{Bar}_s(A) \oplus \epsilon_r \otimes \text{Bar}_{\geq p}(A) \oplus \epsilon_s \otimes \text{Bar}_{\geq q}(A) \oplus \epsilon_{r+s} \otimes \Omega_{syt}^{p+q}(A).
\]

The second identity in \([2]\) yields that \( C^R(f, g) \) is a complex.

**Lemma 3.6.** For any \( f \in \text{HH}^{m-p}(A, \Omega_{syt}^p(A)) \) and \( g \in \text{HH}^{n-q}(A, \Omega_{syt}^q(A)) \), we have isomorphisms of complexes

\[ C(f, g) \cong C^L(f, g) \cong C^R(f, g). \]

**Proof.** Let us define a map \( s^L(f, g) : C^L(f, g) \to C(f, g) \) as

\[
s^L(f, g) := \begin{pmatrix}
1_d & 0 & 0 & 0 \\
\vartheta_L(f) & 1_d & 0 & 0 \\
0 & 0 & 1_d & 0 \\
0 & h_p^L, R(g) & 0 & 1_d
\end{pmatrix}
\]

where \( h_p^L, R(g) \) is defined in Remark \([2, 4]\) and \( h(f) \) is defined in \([1]\). We have the following identity

\[
\begin{pmatrix}
1_d & 0 & 0 & 0 \\
\vartheta_L(f) & 1_d & 0 & 0 \\
0 & 0 & 1_d & 0 \\
0 & h_p^L, R(g) & 0 & 1_d
\end{pmatrix}
\begin{pmatrix}
1_d & 0 & 0 & 0 \\
\vartheta_R(f) & d & 0 & 0 \\
0 & d & 0 & 0 \\
0 & \Omega_{syt}^{R,p}(g) & \Omega_{syt}^{R,q}(f) & 0
\end{pmatrix}
= \begin{pmatrix}
1_d & 0 & 0 & 0 \\
\vartheta_R(f) & d & 0 & 0 \\
0 & d & 0 & 0 \\
0 & \Omega_{syt}^{R,p}(g) & \Omega_{syt}^{R,q}(f) & 0
\end{pmatrix}
\begin{pmatrix}
1_d & 0 & 0 & 0 \\
\vartheta_L(f) & 1_d & 0 & 0 \\
0 & 0 & 1_d & 0 \\
0 & h_p^L, R(g) & 0 & 1_d
\end{pmatrix}.
\]
since $g \bullet f = \Omega^{R,p}_{sy}(g) h(f) - h^{L,R}_{p}(g) \vartheta^{R}(f)$ (cf. Lemma 3.4). Here, we simply write $\text{Id}_{r} \otimes \Omega^{R,p}_{sy}(g)$ (resp. $\text{Id}_{s} \otimes \Omega^{R,q}_{sy}(f)$) as $\Omega^{R,p}_{sy}(g)$ (resp. $\Omega^{R,q}_{sy}(f)$). It follows that $s^{L}(f, g)$ is a morphism of complexes. Note that the following identity holds

$$s^{L}(f, g)^{-1} = \begin{pmatrix} \text{Id} & 0 & 0 & 0 \\ -h(f) & \text{Id} & 0 & 0 \\ h^{L,R}_{p}(g) h(f) & -h^{L,R}_{p}(g) & \text{Id} & 0 \\ h^{L,R}_{p}(g) & h^{L,R}_{p}(f) & \text{Id} & 0 \end{pmatrix}.$$

Let us prove $C^{L}(f, g) \cong C^{R}(f, g)$. Consider the following map $s'(f, g) : C^{L}(f, g) \rightarrow C^{R}(f, g)$ given by

$$s'(f, g) := \begin{pmatrix} \text{Id} & 0 & 0 & 0 \\ 0 & \text{Id} & 0 & 0 \\ 0 & 0 & \text{Id} & 0 \\ 0 & 0 & \text{Id} & \text{Id} \end{pmatrix}.$$

Note that the following identity holds

$$s'(f, g) \begin{pmatrix} g^{R}(f) & 0 & 0 & 0 \\ g^{R}_{p}(f) & 0 & 0 & 0 \\ g^{R}_{q}(f) & 0 & 0 & 0 \\ g \bullet f & \Omega^{R,p}_{sy}(g) & \Omega^{R,q}_{sy}(f) & 0 \end{pmatrix} = \begin{pmatrix} d & 0 & 0 & 0 \\ \vartheta^{R}(f) & d & 0 & 0 \\ \vartheta^{R}_{p}(g) & d & 0 & 0 \\ \vartheta^{R}_{q}(f) & \Omega^{R,p}_{sy}(g) & \Omega^{R,q}_{sy}(f) & 0 \end{pmatrix} s'(f, g),$$

since by Lemma 3.4 we have $g \bullet f + h^{L,R}_{p}(g) \vartheta^{R}(f) + h^{L,R}_{q}(g) \vartheta^{R}(g) = g \circ f$ and by Remark 3.4 we have

$$h^{L,R}_{p}(f) d = \Omega^{L,q}_{sy}(f) - \Omega^{R,q}_{sy}(f), \quad h^{L,R}_{q}(g) d = \Omega^{L,p}_{sy}(g) - \Omega^{R,p}_{sy}(g).$$

This implies that $s'(f, g)$ is a morphism of complexes. It is clear $s'(f, g)$ is an isomorphism with inverse

$$s'(f, g)^{-1} = \begin{pmatrix} \text{Id} & 0 & 0 & 0 \\ 0 & \text{Id} & 0 & 0 \\ 0 & 0 & \text{Id} & 0 \\ 0 & -h^{L,R}_{p}(g) & -h^{L,R}_{q}(f) & \text{Id} \end{pmatrix}.$$

This proves the lemma.}

**Remark 3.7.** It is clear that $C(f, g)$ has a natural dg $R \otimes A \otimes A^{op}$-module structure, where $R := R_{m-p-1} \otimes R_{n-q-1}$ is the tensor product of the dg algebras $R_{m-p-1}$ and $R_{n-q-1}$. Then, via the above isomorphisms in Lemma 3.8, the complexes $C^{L}(f, g)$ and $C^{R}(f, g)$ inherit the structure of a dg $R \otimes A \otimes A^{op}$-module from $C(f, g)$. Hence all the three dg $R \otimes A \otimes A^{op}$-modules are isomorphic. The tensor product $C(f) \otimes_{A} C(g)$ is endowed with a natural dg $R \otimes A \otimes A^{op}$-module structure.

**Lemma 3.8.** Let $m, n \in \mathbb{Z}_{>0}$. For any $f \in \text{HH}^{m-p}(A, \Omega^{p}_{sy}(A))$ and $g \in \text{HH}^{n-q}(A, \Omega^{q}_{sy}(A))$, we have a morphism of dg $R \otimes A \otimes A^{op}$-modules

$$\Phi(f, g) : C(f, g) \rightarrow C(f) \otimes_{A} C(g)$$

such that $\Phi(f, g)$ is an isomorphism in $\mathcal{K}(R \otimes A)$ and in $\mathcal{K}(R \otimes A^{op})$.

**Proof.** Set $r := m - p - 1$ and $s := n - q - 1$. Let us write down the complex $C(f) \otimes_{A} C(g)$.

Recall that $C(f)$ is the following complex

$$C(f) = \text{Bar}_{*}(A) \oplus \Sigma^{r} \Omega^{p}_{sy}(A).$$
Similarly, we have an isomorphism of dg $A \otimes A^{op}$-modules

$$\tilde{\mu}_p : \Sigma^r \Omega^p_{sy}(A) \otimes_A \text{Bar}_*(A) \xrightarrow{\sim} \Sigma^r \text{Bar}_{\geq p}(A)$$

defined as the composition of maps

$$\Sigma^r \Omega^p_{sy}(A) \otimes_A \text{Bar}_*(A) \xrightarrow{\Sigma^r \alpha_p^{-1} \otimes_A \text{Id}} \Sigma^r \Omega^{L,p}_{nc}(A) \otimes_A \text{Bar}_*(A) \xrightarrow{\sim} \Sigma^r \text{Bar}_{\geq p}(A),$$

where $(\alpha_p^{-1})$ is defined in Lemma 2.2 and the second isomorphism is given by

$$\Sigma^r \Omega^{L,p}_{nc}(A) \otimes_A (A \otimes (\Sigma A)^{op} \otimes A) \xrightarrow{\sim} \Sigma^r A \otimes (\Sigma A)^{op+i} \otimes A \xrightarrow{\sim} \Sigma^r \text{Bar}_{p+i}(A).$$

Similarly, we have an isomorphism of dg $A \otimes A^{op}$-modules

$$\tilde{\mu}_q : \text{Bar}_*(A) \otimes_A \Sigma^r \Omega^q_{sy}(A) \xrightarrow{\sim} \Sigma^r \text{Bar}_{\geq q}(A).$$

Recall that we also have an isomorphism $\mu_{p,q} : \Omega^p_{sy}(A) \otimes_A \Omega^q_{sy}(A) \xrightarrow{\sim} \Omega^{p+q}_{sy}(A)$ from Lemma 2.2. Using the above isomorphisms, we get that $C(f) \otimes_A C(g)$ is isomorphic to the following complex (denoted by $\tilde{C}_1(f, g)$)

$$\text{Bar}_*(A) \otimes_A \text{Bar}_*(A) \oplus \Sigma^r \text{Bar}_{\geq p}(A) \oplus \Sigma^r \text{Bar}_{\geq q}(A) \oplus \Sigma^{r+s} \Omega^{p+q}_{sy}(A).$$

Here the isomorphism $C(f) \otimes_A C(g) \xrightarrow{\sim} \tilde{C}_1(f, g)$ is given by $t_1(f, g) := \left(\begin{array}{cccc} \text{Id} & 0 & 0 & 0 \\ 0 & \tilde{\mu}_p & 0 & 0 \\ 0 & 0 & \tilde{\mu}_q & 0 \\ 0 & 0 & 0 & \mu_{p,q} \end{array}\right)$.

Via this isomorphism, the complex $\tilde{C}_1(f, g)$ inherits the structure of a dg $R \otimes A \otimes A^{op}$-module from $C(f) \otimes_A C(g)$. We construct a morphism of graded $R \otimes A \otimes A^{op}$-modules

$$t(f, g) = \left(\begin{array}{cccc} \Delta_{0,0} & 0 & 0 & 0 \\ 0 & \text{Id} & 0 & 0 \\ 0 & 0 & \tilde{\mu}_p & 0 \\ 0 & 0 & 0 & \mu_{p,q} \end{array}\right) : C(f, g) \rightarrow \tilde{C}_1(f, g),$$

where $\Delta_{0,0}$ is defined in Section 2.1. We claim that $t(f, g)$ commutes with differentials. Indeed, it is sufficient to verify the following identity

$$t(f, g) \begin{pmatrix} d & 0 & 0 & 0 \\ 0 & d & 0 & 0 \\ 0 & 0 & d & 0 \\ 0 & 0 & 0 & d \end{pmatrix} \begin{pmatrix} \vartheta^L(f) \\ \vartheta^R(g) \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \mu_{p,q} & 0 & 0 & 0 \\ 0 & \tilde{\mu}_p & 0 & 0 \\ 0 & 0 & \tilde{\mu}_q & 0 \end{pmatrix} t(f, g).$$

The above identity holds since we have

$$\vartheta^L(f) = \tilde{\mu}_p(f \otimes_A \text{Id}) \Delta_{0,0}, \quad \vartheta^R(g) = \tilde{\mu}_q(\text{Id} \otimes_A g) \Delta_{0,0}.$$
Following diagram commutes in the category of dg $R$-modules:

$$\Omega^{R,p}_{sy}(g)\tilde{\mu}_p = \mu_{p,q}(1d \otimes_A g), \quad \Omega^{L,q}_{sy}(f)\tilde{\mu}_q = \mu_{p,q}(f \otimes_A 1d).$$

This proves the claim. Therefore, we get a morphism of dg $R \otimes A \otimes A^{op}$-modules

$$\Phi(f, g) = t_1(f, g)^{-1}t(f, g) : C(f, g) \to C(f) \otimes_A C(g).$$

It remains to show that $\Phi(f, g)$ is an isomorphism in $\mathcal{K}(R \otimes A)$ and in $\mathcal{K}(R \otimes A^{op})$. For this, it follows from the proof of Lemma 3.3 that there is an isomorphism of dg $R \otimes A$-modules

$$\begin{pmatrix}
\tilde{f} & 0 \\
\tilde{g} & 1d \\
d_{p+q}^{R,p}_{sy}(\tilde{g}) & d_{p+q}^{R,p}_{sy}(\tilde{f}) & 1d
\end{pmatrix} : (f, g) \to (0, 0),$$

where $\tilde{g}(x) = \partial^R(g)(x \otimes 1)$ and $\tilde{f}(x) = \partial^L(f)(x \otimes 1)$. Here we leave it to the reader to check that the above map is indeed a morphism of dg $R \otimes A$-modules. Similarly, we have an isomorphism of dg $R \otimes A$-modules $C(f) \otimes_A C(g) \cong C(0) \otimes_A C(0)$. Moreover, the following diagram commutes in the category of dg $R \otimes A$-modules

$$\begin{array}{ccc}
C(f, g) & \overset{\Phi(f,g)}{\longrightarrow} & C(f) \otimes_A C(g) \\
\downarrow_{\cong} & & \downarrow_{\cong} \\
C(0, 0) & \overset{\Phi(0,0)}{\longrightarrow} & C(0) \otimes_A C(0).
\end{array}$$

Thus, to prove that $\Phi(f, g)$ is an isomorphism in $\mathcal{K}(R \otimes A)$, it is equivalent to prove that $\Phi(0, 0)$ is an isomorphism in $\mathcal{K}(R \otimes A)$. For this, consider the distinguished triangle

$$B_{<p} \to C(0) \to R_r \otimes \Omega_{sy}^p(A) \to \Sigma B_{<p}$$
in $\mathcal{K}(R_r \otimes A)$. Applying the tensor functor $- \otimes_A C(0)$, we get the triangle

$$B_{<p} \otimes_A C(0) \to C(0) \otimes_A C(0) \to R_r \otimes (B_{\geq p}(A) \oplus \Sigma^s \Omega_{sy}^{p+q}(A)) \to \Sigma B_{<p}$$
in $\mathcal{K}(R \otimes A)$. Moreover, we have the following commutative diagram

$$\begin{array}{cccc}
B_{<p} \otimes_A B_{s} \oplus \Sigma^s B_{<p+q} & \longrightarrow & C(0) \otimes_A C(0) & \longrightarrow & R_r \otimes (B_{\geq p}(A) \oplus \Sigma^s \Omega_{sy}^{p+q}(A)) \\
(\Delta_{0} \ 0 \ 1d) & \cong & \Phi(0,0) & = & \\
B_{<p} \otimes \Sigma B_{<p+q} & \longrightarrow & C(0, 0) & \longrightarrow & R_r \otimes (B_{\geq p}(A) \oplus \Sigma^s \Omega_{sy}^{p+q}(A))
\end{array}.$$ 

Notice that the morphism $(\Delta_{0} \ 0 \ 1d)$ is an isomorphism in $\mathcal{K}(R \otimes A)$, it follows that $\Phi(0,0)$ is an isomorphism and thus $\Phi(f, g)$ is an isomorphism in $\mathcal{K}(R \otimes A)$. Similarly, $\Phi(f, g)$ is an isomorphism in $\mathcal{K}(R \otimes A^{op})$. This proves the lemma.

4. $R$-RELATIVE DERIVED TENSOR PRODUCT

Let us start with the general setting. Let $k$ be a field. Let $R$ be a commutative dg $k$-algebra and $E$ be a dg $R$-algebra. The $R$-relative (unbounded) derived category $\mathcal{D}_R(E)$ is a k-linear category with objects being dg $E$-modules. The morphisms of $\mathcal{D}_R(E)$ are obtained from morphisms of dg $E$-modules by the localization with respect to all $R$-relative quasi-isomorphisms, i.e. all morphisms $s : L \to M$ of dg $E$-modules whose restriction to $R$ is a homotopy equivalence. For instance, the $k$-relative derived category $\mathcal{D}_k(E)$ of the dg $k$-algebra $E$ coincides with the usual derived category $\mathcal{D}(E)$. The $R$-relative derived category $\mathcal{D}_R(R)$ is the homotopy category $\mathcal{K}(R)$ of $R$. We also consider the $R$-relative bounded derived category $\mathcal{D}_R^{b}(E)$, which is by definition the full subcategory of $\mathcal{D}_R(E)$ consisting of those objects $X$ such that there are only finitely many integers $i$ such that $H_i(X) \neq 0$. For more details on $R$-relative derived categories, we refer to [Kel98, Kel99].
Let $A$ and $B$ be two associative $k$-algebras. Let $X$ be a dg $R \otimes A \otimes \text{B}^{\text{op}}$-module. Then we have the $R$-relative derived tensor product induced by $X$, in the sense of [Del].

\[ X \otimes^{L,R}_{R \otimes B} - : \mathcal{D}_{R}(R \otimes B) \rightarrow \mathcal{D}_{R}(R \otimes A). \]

**Remark 4.1.** From [Kel98, Section 7], it follows that $X \otimes^{L,R}_{R \otimes B} - \cong \text{p}_{\text{rel}}X \otimes_{R \otimes B} -$ , where the dg $R \otimes A \otimes \text{B}^{\text{op}}$-module $\text{p}_{\text{rel}}X$ is $R$-relatively quasi-isomorphic to $X$ and $R$-relatively closed as a dg $R \otimes \text{B}^{\text{op}}$-module, i.e. $\text{Hom}_{\mathcal{K}(R \otimes \text{B}^{\text{op}})}(\text{p}_{\text{rel}}X, M) \cong \text{Hom}_{\mathcal{D}(R \otimes \text{B}^{\text{op}})}(\text{p}_{\text{rel}}X, M)$ for any dg $R \otimes \text{B}^{\text{op}}$-module $M$. For instance, we have the isomorphism $\text{Hom}_{\mathcal{K}(R \otimes \text{B}^{\text{op}})}(R \otimes B, M) \cong \text{Hom}_{\mathcal{D}(R \otimes \text{B}^{\text{op}})}(R \otimes B, M)$ for any dg $R \otimes \text{B}^{\text{op}}$-module $M$, and hence $(R \otimes B) \otimes^{L,R}_{R \otimes B} - \cong (R \otimes B) \otimes_{R \otimes B} -$ (cf. [Kel98, Section 7.4]).

**Lemma 4.2.** Let $X$ be a dg $R$-module and $P$ be a ($k$-relatively) closed dg $\text{B}^{\text{op}}$-module. Then $X \otimes P$ is $R$-relatively closed as a dg $R \otimes \text{B}^{\text{op}}$-module, namely, we have

\[ \text{Hom}_{\mathcal{K}(R \otimes \text{B}^{\text{op}})}(X \otimes P, M) \cong \text{Hom}_{\mathcal{D}(R \otimes \text{B}^{\text{op}})}(X \otimes P, M), \]

for any dg $R \otimes \text{B}^{\text{op}}$-module $M$. As a consequence, $(X \otimes P) \otimes^{L,R}_{B \otimes k} R - \cong (X \otimes P) \otimes_{B \otimes k} R -$.

**Proof.** It suffices to show that $\text{Hom}_{\mathcal{K}(R \otimes \text{B}^{\text{op}})}(X \otimes P, M) = 0$ when $M$ is an $\text{R}^{\text{op}}$-contractible (i.e. $M \cong 0$ in $\mathcal{K}(\text{R}^{\text{op}})$) dg $R \otimes \text{B}^{\text{op}}$-module. For this, let us write $\text{Hom}$ for the Hom-complexes. Equivalently, we need to show that the complex $\text{Hom}_{\mathcal{D}(R \otimes \text{B}^{\text{op}})}(X \otimes P, M)$ is acyclic when $M$ is $R$-contractible. Since we have

\[ \text{Hom}_{\mathcal{D}(R \otimes \text{B}^{\text{op}})}(X \otimes P, M) \cong \text{Hom}_{\mathcal{R}}(X, \text{Hom}_{\mathcal{B}^{\text{op}}}(P, M)), \]

to prove that $\text{Hom}_{\mathcal{D}(R \otimes \text{B}^{\text{op}})}(X \otimes P, M)$ is acyclic, it suffices to prove that $\text{Hom}_{\mathcal{B}^{\text{op}}}(P, M)$ is $R$-contractible. Clearly, this holds for $P = B$ and is inherited by shifts and arbitrary coproducts (because products of $R$-contractible dg $R$-modules are still $R$-contractible). This is also inherited by extensions that split in the category of graded $\text{B}^{\text{op}}$-modules. Therefore, it holds for any closed dg $\text{B}^{\text{op}}$-modules. This proves the lemma. \[ \Box \]

**Remark 4.3.** We would like to thank the referee for providing a shorter proof of Lemma 4.2 and thank Keller for pointing out that this lemma holds for any closed dg $\text{B}^{\text{op}}$-module $P$.

**Proposition 4.4.** Let $m > 0$. For a cocycle $f \in C^{m-p}(A, \Omega^p_y(A))$ (i.e. $\delta(f) = 0$), all the three dg modules $C(f), C(\vartheta^L(f))$ and $C(\vartheta^R(f))$ are $R_{m-p-1}$-relatively closed as dg $R_{m-p-1} \otimes A$-modules and as dg $R_{m-p-1} \otimes A^{\text{op}}$-modules.

**Proof.** It follows from Lemma 4.3 that all the three modules are isomorphic in $\mathcal{K}(R_{m-p-1} \otimes A)$ and in $\mathcal{K}(R_{m-p-1} \otimes A^{\text{op}})$. Therefore, it is sufficient to prove that $C(\vartheta^L(f))$ is $R_{m-p-1}$-relatively closed as a dg $R_{m-p-1} \otimes A$-module and as a dg $R_{m-p-1} \otimes A^{\text{op}}$-module. From the proof of Lemma 4.3, we have an isomorphism of dg $R_{m-p-1} \otimes A$-modules

\[ \phi(f) : C(\vartheta^L(f)) \cong C(\vartheta^L(0)) = \text{Bar}_*(A) \oplus \epsilon_{m-p-1} \otimes \text{Bar}_{\geq p}(A). \]

Let us prove that $C(\vartheta^L(0))$ is $R_{m-p-1}$-relatively closed in $\mathcal{K}(R_{m-p-1} \otimes A)$. For this, we have a distinguished triangle in $\mathcal{K}(R_{m-p-1} \otimes A)$

\[ \text{Bar}_{0,p-1}(A) \rightarrow C(\vartheta^L(0)) \rightarrow R_{m-p-1} \otimes \text{Bar}_{\geq p}(A) \rightarrow \Sigma \text{Bar}_{0,p-1}(A). \]

By Lemma 4.2 both $\text{Bar}_{0,p-1}(A)$ and $R_{m-p-1} \otimes \text{Bar}_{\geq p}(A)$ are $R_{m-p-1}$-relatively closed in $\mathcal{K}(R_{m-p-1} \otimes A)$. Hence so is $C(\vartheta^L(0))$. This proves that $C(\vartheta^L(f))$ is $R_{m-p-1}$-relatively closed in $\mathcal{K}(R_{m-p-1} \otimes A)$. Similarly, we can prove that it is also $R_{m-p-1}$-relatively closed in $\mathcal{K}(R_{m-p-1} \otimes A^{\text{op}})$. This proves the proposition. \[ \Box \]
Proposition 4.5. Let $m > 0$ and $p \geq 0$. Let $f, g \in \HH^{m-p}(A, \Omega^p_{\text{sy}}(A))$. Then we have a morphism of dg $R_{m-1} \otimes A \otimes A^{\text{op}}$-modules

$$
\Psi(f, g) : C(f) \otimes_{R_{m-1} \otimes A} C(g) \xrightarrow{\sim} C(\Omega^p_{\text{sy}}(f + g))
$$

such that $\Psi(f, g)$ is an isomorphism in $\mathcal{K}(R_{m-1} \otimes A)$ and in $\mathcal{K}(R_{m-1} \otimes A^{\text{op}})$. Here the map $\Omega^p_{\text{sy}} : \HH^{m-p}(A, \Omega^p_{\text{sy}}(A)) \to \HH^{m-p}(A, \Omega^p_{\text{sy}}(A))$ is defined in Section 2.2.

Proof. For simplicity, we write $Bar_*(A)$ as $B_*$ throughout this proof. Then the dg module $C(f) \otimes_{R_{m-1} \otimes A} C(g)$ is illustrated as follows

$$
(B_* \oplus \Sigma^{m-p-1}\Omega^p_{\text{sy}}(A)) \otimes_{R_{m-1} \otimes A} (B_* \oplus \Sigma^{m-p-1}\Omega^p_{\text{sy}}(A)).
$$

We claim that $C(f) \otimes_{R_{m-1} \otimes A} C(g)$ is isomorphic to the following dg $R_{m-1} \otimes A \otimes A^{\text{op}}$-module $C_1(f, g)$

$$
B_* \otimes_A B_* \oplus \Sigma^{m-p-1}\Omega^p_{\text{sy}}(A) \otimes_A \Omega^p_{\text{sy}}(A).
$$

Indeed, as graded $R_{m-1} \otimes A \otimes A^{\text{op}}$-modules, we have

$$
C(f) \cong \bigoplus_{i \neq p} (k \otimes B_i) \bigoplus (B_p \oplus \Sigma^{m-p-1}\Omega^p_{\text{sy}}(A)).
$$

Thus, as graded $R_{m-1} \otimes A \otimes A^{\text{op}}$-modules, we have

$$
C(f) \otimes_{R_{m-1} \otimes A} C(g)
\cong \left( \bigoplus_{i \neq p} (k \otimes B_i) \bigoplus (B_p \oplus \Sigma^{m-p-1}\Omega^p_{\text{sy}}(A)) \right) \otimes_{R_{m-1} \otimes A} \left( \bigoplus_{i \neq p} (k \otimes B_i) \bigoplus (B_p \oplus \Sigma^{m-p-1}\Omega^p_{\text{sy}}(A)) \right)
\cong \left( B_i \otimes_A B_i \bigoplus B_i \otimes A B_i \right) \bigoplus \left( B_i \otimes A B_i \right)
\cong B_* \otimes_A B_* \oplus \Sigma^{m-p-1}\Omega^p_{\text{sy}}(A) \otimes_A \Omega^p_{\text{sy}}(A)
\cong C_1(f, g),
$$

where the second isomorphism comes from the following isomorphisms

$$(k \otimes B_i) \otimes_{R_{m-1} \otimes A} (k \otimes B_j) \cong B_i \otimes_A B_j;
$$

(4) $X_p \otimes_{R_{m-1} \otimes A} X_p \cong B_p \otimes_A B_p \oplus \Sigma^{m-p-1}\Omega^p_{\text{sy}}(A) \otimes_A \Omega^p_{\text{sy}}(A);\n$ $X_p \otimes_{R_{m-1} \otimes A} (k \otimes B_i) \cong B_p \otimes_A B_i, \quad (k \otimes B_i) \otimes_{R_{m-1} \otimes A} X_p \cong B_i \otimes_A B_p;
$$

where $X_p := B_p \oplus \Sigma^{m-p-1}\Omega^p_{\text{sy}}(A)$. The first isomorphism in (4) is due to the fact that $k \otimes_{R_{m-1}} k \cong k$. Let us prove the second isomorphism in (4). For this, we have a short
exact sequence of dg $R_{m-p-1} \otimes A \otimes A^{op}$-modules
\begin{equation}
0 \rightarrow \epsilon_{m-p-1} \otimes \Sigma^{-1} \Omega^{p+1}_{sy}(A) \rightarrow R_{m-p-1} \otimes B_p \rightarrow X_p \rightarrow 0.
\end{equation}
Applying the tensor functor $- \otimes_{R_{m-p-1} \otimes A} X_p$ to (5), we obtain the following exact sequence
\begin{equation} 
0 \rightarrow \Sigma^{-1} \Omega^{p+1}_{sy}(A) \otimes A \epsilon_{m-p-1} X_p \rightarrow B_p \otimes_A X_p \rightarrow X_p \otimes_{R_{m-p-1} \otimes A} X_p \rightarrow 0.
\end{equation}
This implies that
\begin{displaymath}
X_p \otimes_{R_{m-p-1} \otimes A} X_p \cong (B_p \otimes_A X_p) / ((\Sigma^{-1} \Omega^{p+1}_{sy}(A) \otimes A \epsilon_{m-p-1} X_p)
\cong (B_p \otimes_A X_p) / ((\Sigma^{-1} \Omega^{p+1}_{sy}(A) \otimes A \Sigma^{m-p-1} \Omega^{p}_{sy}(A)))
\cong B_p \otimes_A B_p \otimes \Sigma^{m-p-1} \Omega^{2p}_{sy}(A) \otimes A \Omega^{p}_{sy}(A).
\end{displaymath}
This shows the second isomorphism in (4). Similarly, applying the functor $- \otimes_{R_{m-p-1} \otimes A} (k \otimes B_1)$ (resp. $(k \otimes B_1) \otimes_{R_{m-p-1} \otimes A} -$) to (5), we may get the third isomorphism (resp. the forth isomorphism) in (4). Then, from the construction of the tensor product of dg modules, we see that the differential is exactly given by that of $C_1(f, g)$. This proves the claim.

We identity $C(f) \otimes_{R_{m-p-1} \otimes A} C(g)$ with $C_1(f, g)$ via the above isomorphism. Let us denote by $C_1'(f, g)$ the following dg $R_{m-p-1} \otimes A \otimes A^{op}$-module
\begin{displaymath}
\mu_{p,q}(f \otimes_A d_p + d_p \otimes_A g) \Delta_{0,0} \\
B_* \oplus \Sigma^{m-p-1} \Omega^{2p}_{sy}(A),
\end{displaymath}
where the coproduct $\Delta_{0,0} : B_* \rightarrow B_* \otimes_A B_*$ and the isomorphism $\mu_{p,q} : \Omega^p_{sy}(A) \otimes A \Omega^q_{sy}(A) \cong \Omega^{p+q}_{sy}(A)$ are defined in Section 2.3. Note that we have a morphism of dg $R_{m-p-1} \otimes A \otimes A^{op}$-modules
\begin{displaymath}
\Psi' = \begin{pmatrix} \Delta_{0,0} & 0 \\ 0 & \mu_{p,q} \end{pmatrix} : C_1'(f, g) \rightarrow C_1(f, g) = C(f) \otimes_{R_{m-p-1} \otimes A} C(g).
\end{displaymath}

By Remark 2.3 we have that
\begin{displaymath}
\mu_{p,p}(f \otimes_A d_p + d_p \otimes_A g) \Delta_{0,0} = \Omega^L_{sy}(f) + \Omega^R_{sy}(g) = \Omega^L_{sy}(f) + \Omega^R_{sy}(g) = \Omega^p_{sy}(f + g)
\end{displaymath}
in $HH^{m-p}(A, \Omega^{2p}_{sy}(A))$. Thus $C_1'(f, g)$ is isomorphic to $C(\Omega^p_{sy}(f + g))$ since $C(\Omega^p_{sy}(f + g))$ does not depend on the choice of the representatives (cf. Remark 3.2). This yields a morphism of dg $R_{m-p-1} \otimes A \otimes A^{op}$-modules
\begin{displaymath}
\Psi(f, g) : C(\Omega^p_{sy}(f + g)) \rightarrow C(f) \otimes_{R_{m-p-1} \otimes A} C(g).
\end{displaymath}
It remains to prove that $\Psi$ is an isomorphism in $\mathcal{K}(R_{m-p-1} \otimes A)$ and in $\mathcal{K}(R_{m-p-1} \otimes A^{op})$. From the proof of Lemma 3.5 it follows that we have the following commutative diagram
\begin{displaymath}
\begin{array}{cccc}
C(\Omega^p_{sy}(f + g)) & \xrightarrow{\Psi(f, g)} & C(f) \otimes_{R_{m-p-1} \otimes A} C(g) \\
\psi(\Omega^p_{sy})(f) & \cong & \psi(\Omega^p_{sy})(f) & \rightarrow C(f) \otimes_{R_{m-p-1} \otimes A} C(0) \rightarrow C_1(f, 0) \\
\psi(\Omega^p_{sy})(g) & \xrightarrow{\Psi(0, 0)} & \psi(0, 0) & \xrightarrow{\psi(f) \otimes \text{Id} \otimes \text{Id} \otimes 0 \text{Id}} C_1(0, 0)
\end{array}
\end{displaymath}
in $\mathcal{K}(R_{m-p-1} \otimes A)$. Thus, to prove that $\Psi(f, g)$ is an isomorphism in $\mathcal{K}(R_{m-p-1} \otimes A)$, it is equivalent to prove that $\Psi(0, 0)$ is an isomorphism. Note that the latter follows from
the following commutative diagram of distinguished triangles in \( \mathcal{K}(R_{m-p-1} \otimes A) \)

\[
\begin{array}{ccc}
B_{<2p} & \rightarrow & C(0) \\
\cong & \Delta & \cong (\Delta 0 I_d) \\
(B_\ast \otimes B_\ast)_{<2p} & \rightarrow & (B_\ast \otimes B_\ast)_{\geq 2p} + \Sigma^{m-p-1} \Omega_{A_\ast}^2(A) \\
\end{array}
\]

Similarly, we can prove that \( \Psi(f, g) \) is an isomorphism in \( \mathcal{K}(R_{m-p-1} \otimes A_{\text{op}}) \). This proves the proposition.

5. SINGULAR INFINITESIMAL DEFORMATION THEORY

In this section, we follow [Kel99] and develop the singular infinitesimal deformation theory of the identity bimodule. Let \( k \) be a field and \( R \) be an augmented commutative dg \( k \)-algebra. We denote by \( n \) the kernel of the augmentation \( R \rightarrow k \). We always suppose that \( \text{dim}_k n < \infty \).

Let \( A \) be a Noetherian \( k \)-algebra such that the enveloping algebra \( A \otimes A_{\text{op}} \) is Noetherian. Let \( D^b(\text{Proj}) (A \otimes A_{\text{op}}) \) be the full subcategory of \( D^b(A \otimes A_{\text{op}}) \) formed by all the complexes quasi-isomorphic to bounded complexes \( X \) of (not necessarily finitely generated) \( A \)-bimodules such that each component \( X_i \) of \( X \) is projective as a left \( A \)-module and as a right \( A \)-module. For instance, \( A \in D^b(\text{Proj}) (A \otimes A_{\text{op}}) \).

Let \( D_{R}(R \otimes A \otimes A_{\text{op}}) \) be the \( R \)-relative right bounded derived category of \( \text{dg } R \otimes A \otimes A_{\text{op}} \)-modules (cf. Section 3). We consider its full subcategory \( D_{R,cl}(R \otimes A \otimes A_{\text{op}}) \) formed by all the objects \( X \in D_{R}(R \otimes A \otimes A_{\text{op}}) \) satisfying the following two conditions

(i) \( X \) is \( R \)-relatively closed as a \( \text{dg } R \otimes A \)-module and as a \( \text{dg } R \otimes A_{\text{op}} \)-module,
(ii) \( k \otimes_R X \in D_{R,cl}(A \otimes A_{\text{op}}) \).

Denote by \( \mathcal{P}(R \otimes A \otimes A_{\text{op}}) \) the thick triangulated subcategory of \( D_{R,cl}(R \otimes A \otimes A_{\text{op}}) \) generated by all the objects \( P \) such that \( k \otimes_R P \) is quasi-isomorphic to bounded complexes of projective \( A \otimes A_{\text{op}} \)-modules. We define the \( R \)-relative monoidal singularity category of \( A \) as

\[
D_{sg,R}(A \otimes A_{\text{op}}) := \frac{D_{R,cl}(R \otimes A \otimes A_{\text{op}})}{\mathcal{P}(R \otimes A \otimes A_{\text{op}})}.
\]

In particular, the \( k \)-relative monoidal singularity category \( D_{sg,k}(A \otimes A_{\text{op}}) \) coincides with \( \frac{D_{R}(A \otimes A_{\text{op}})}{\mathcal{P}(A \otimes A_{\text{op}})} \), where \( D_{cl}(A \otimes A_{\text{op}}) \) is the full subcategory of \( D^{-}(A \otimes A_{\text{op}}) \) formed by all the objects \( X \) such that each component \( X_i \) is projective as a left \( A \)-module and as a right \( A \)-module. Thus it is a full triangulated subcategory of the (partially) completed singularity category \( \mathcal{S}_R[A \otimes A_{\text{op}}] := D^{-}(A \otimes A_{\text{op}})/\mathcal{P}(A \otimes A_{\text{op}}) \) defined in [Kel18] Section 2.1. It follows from Lemma 3.2.2 in loc. cit. that the singularity category \( D_{sg}(A \otimes A_{\text{op}}) \) (in the sense of Buchweitz and Orlov) is a full subcategory of \( \mathcal{S}_{R}[A \otimes A_{\text{op}}] \) and is also a full subcategory of \( D_{sg,k}(A \otimes A_{\text{op}}) \). As a consequence, we have

\[
\text{HH}_i^{s}(A, A) := \text{Hom}_{D_{sg,R}(A \otimes A_{\text{op}})}(A, \Sigma^i A) \cong \text{Hom}_{D_{sg,k}(A \otimes A_{\text{op}})}(A, \Sigma^i A), \quad \text{for any } i \in \mathbb{Z}.
\]

**Lemma 5.1.** The \( R \)-relative monoidal singularity category \( D_{sg,R}(A \otimes A_{\text{op}}) \) endowed with the \( R \)-relative tensor product \( \otimes^{L}_{R \otimes A} \) is a monoidal category with the unit object \( R \otimes A \).

**Proof.** Let us first prove that \( D_{R,cl}(R \otimes A \otimes A_{\text{op}}) \) is a monoidal category. Since all the objects in \( D_{R,cl}(R \otimes A \otimes A_{\text{op}}) \) are \( R \)-relatively closed as \( \text{dg } R \otimes A \)-modules and as \( \text{dg } R \otimes A_{\text{op}} \)-modules, we have \( \otimes^{L}_{R \otimes A} \cong \otimes_{R \otimes A} \). Let \( X \) and \( Y \) be two objects in \( D_{R,cl}(R \otimes A \otimes A_{\text{op}}) \). Then we claim that the object \( X \otimes_{R \otimes A} Y \) is in \( D_{R,cl}(R \otimes A \otimes A_{\text{op}}) \). Indeed, \( X \otimes_{R \otimes A} Y \) satisfies the condition (i). That is, \( X \otimes_{R \otimes A} Y \) is \( R \)-relatively closed as a \( \text{dg } R \otimes A \)-module...
and as a dg $R \otimes A^{\text{op}}$-module. The reason is as follows. Recall from [Ko98, Section 7] that $Y$ is $R$-relatively closed as a dg $R \otimes A$-module if and only if $Y$ admits a filtration of dg $R \otimes A$-modules

$$0 = Y_{-1} \subset Y_0 \subset Y_1 \subset \cdots \subset Y_p \subset \cdots \subset Y$$

such that

- $Y$ is the union of the $Y_p$, $p \in \mathbb{Z}_{\geq 0}$,
- the inclusion $Y_p \subset Y_{p+1}$ splits in the category of graded $R \otimes A$-modules, $p \in \mathbb{Z}_{\geq 0}$,
- each quotient $Y_p/Y_{p-1}$ is isomorphic as dg $R \otimes A$-modules to a direct summand of a direct sum of dg modules $M \otimes A$, where $M$ is dg $R$-module.

This induces a filtration of $X \otimes_{R \otimes A} Y$ in the category of dg $R \otimes A$-modules

$$0 = X \otimes_{R \otimes A} Y_{-1} \subset X \otimes_{R \otimes A} Y_0 \subset \cdots \subset X \otimes_{R \otimes A} Y_p \subset \cdots \subset X \otimes_{R \otimes A} Y.$$

It follows that $X \otimes_{R \otimes A} Y$ is $R$-relatively closed as a dg $R \otimes A$-module, since each quotient

$$(X \otimes_{R \otimes A} Y_p)/(X \otimes_{R \otimes A} Y_{p-1}) \cong X \otimes_{R \otimes A} Y_p/Y_{p-1}$$

is $R$-relatively closed as a dg $R \otimes A$-module. The same argument shows that $X \otimes_{R \otimes A} Y$ is also $R$-relatively closed as a dg $R \otimes A^{\text{op}}$-module. This proves that $X \otimes_{R \otimes A} Y$ satisfies the condition (i). It remains to prove that $X \otimes_{R \otimes A} Y$ satisfies the condition (ii). This follows from the following isomorphisms

$$k \otimes_R (X \otimes_{R \otimes A} Y) \cong (k \otimes_R k) \otimes_R (X \otimes_{R \otimes A} Y) \cong (k \otimes_R k) X \otimes_A (k \otimes_R Y).$$

The proof of the claim is complete. Therefore, $D_{R,cl}(R \otimes A \otimes A^{\text{op}})$ is a monoidal category. The above isomorphisms (6) also implies that $P(R \otimes A \otimes A^{\text{op}})$ is a tensor ideal, thus we have that $\otimes_{L,R}^{R \otimes A} = \otimes_{R \otimes A}$ is well-defined in $D_{sg,R}(A \otimes A^{\text{op}})$. This proves the lemma. \hfill \square

**Remark 5.2.** Let $f : R \rightarrow S$ be a morphism of commutative dg $k$-algebras. Then we have a well-defined functor $S \otimes_R - : D_{sg,R}(A \otimes A^{\text{op}}) \rightarrow D_{sg,S}(A \otimes A^{\text{op}})$ since $S \otimes_R X \in P(S \otimes A \otimes A^{\text{op}})$ for any $X \in P(R \otimes A \otimes A^{\text{op}})$.

### 5.1. Singular infinitesimal deformations

Let $k$ be a field and $A$ be a Noetherian $k$-algebra such that the enveloping algebra $A \otimes A^{\text{op}}$ is Noetherian. Let $R$ be an augmented commutative dg $k$-algebra such that $\dim_k R < \infty$. Let $n$ be the kernel of the augmentation $R \rightarrow k$ with $n^2 = 0$. For example, $R = R_m := k[e_m]/e_m^2$.

Define a singular infinitesimal deformations of $A$ as the pair $(L, u)$, where $L$ is an object in $D_{R,cl}(A \otimes A^{\text{op}})$ such that the canonical projection $\mathfrak{n} \otimes_R L \rightarrow \mathfrak{n}L$ is an isomorphism in $D_{sg,k}(A \otimes A^{\text{op}})$; and $u : k \otimes_R L \rightarrow A$ is an isomorphism in $D_{sg,k}(A \otimes A^{\text{op}})$. We also define a category $\mathcal{F}$ as the category whose objects are the singular infinitesimal deformations $(L, u)$ of $A$ and morphisms from $(L, u)$ to $(L', u')$ are given by morphisms $v : L \rightarrow L'$ of $D_{sg,R}(A \otimes A^{\text{op}})$ such that $u' \circ (\text{Id}_k \otimes_R v) = u$. That is, the following diagram

$$\begin{array}{ccc}
 k \otimes_R L & \xrightarrow{u} & A \\
 \downarrow \text{Id}_k \otimes_R v & & \\
 k \otimes_R L' & \xrightarrow{u'} & \\
\end{array}$$

commutes in $D_{sg,k}(A \otimes A^{\text{op}})$. We denote by $\text{sgDefo}(A, R \rightarrow k)$ the set of isomorphism classes of objects of $\mathcal{F}$ and denote by $\text{sgDefo}'(A, R \rightarrow k)$ the set of isomorphism classes of weak singular deformations of $A$, i.e. objects $L$ in $D_{sg,R}(A \otimes A^{\text{op}})$ such that $\mathfrak{n} \otimes_R L \cong \mathfrak{n}L$ and $k \otimes_R L \cong A$ in $D_{sg,k}(A \otimes A^{\text{op}})$.

Let $(L, u)$ be an object of $\mathcal{F}$. Since $L$ is a dg $R \otimes A \otimes A^{\text{op}}$-module, we have the exact sequence $0 \rightarrow \mathfrak{n}L \rightarrow L \rightarrow k \otimes_R L \rightarrow 0$ of dg $A \otimes A^{\text{op}}$-modules, which splits as a sequence of dg $k$-modules (since $k$ is a field). Thus, it gives rise to a distinguished triangle.
Let $nL 	o L 	o k \otimes_R L \to \Sigma nL$ in $D^b(A \otimes A^{op})$. Since $n \otimes_R L \cong nL$ in $D_{sg,k}(A \otimes A^{op})$, we have the distinguished triangle $n \otimes_R L \to L \to k \otimes_R L \to \Sigma n \otimes_R L$ in $D_{sg,k}(A \otimes A^{op})$. From $n^2 = 0$, it follows that $n \otimes_R L \cong n \otimes (k \otimes_R L)$ as dg modules. We define a morphism $\epsilon(L,u) : A \to \Sigma n \otimes A$ of $D_{sg,k}(A \otimes A^{op})$ by the following diagram

$$
\begin{array}{cccc}
 k \otimes_R L & \xrightarrow{\epsilon'} & \Sigma n \otimes_R L & \xrightarrow{\cong} & \Sigma n \otimes (k \otimes_R L) \\
 \cong & u & \cong & \Sigma n \otimes u & \\
 A & \xrightarrow{\epsilon(L,u)} & \Sigma n \otimes A \\
\end{array}
$$

We claim that the morphism $\epsilon(L,u)$ only depends on the isomorphism class of $(L,u)$ in the category $F$. Indeed, let $(L',u') \in F$ be such that there exists an isomorphism $v : (L,u) \to (L',u')$ in $F$. To simplify the notational burden, we denote by $k \otimes v$ the morphism $Id_k \otimes v$, etc. Then we have the following commutative diagram in $D_{sg,k}(A \otimes A^{op})$.

$$
\begin{array}{cccc}
 k \otimes_R L & \xrightarrow{\epsilon'} & \Sigma n \otimes_R L & \xrightarrow{\cong} & \Sigma n \otimes (k \otimes_R L) \\
 \cong & u & \cong & \Sigma n \otimes u & \\
 A & \xrightarrow{\epsilon(L,u)} & \Sigma n \otimes A \\
\end{array}
$$

We claim that the morphism $k \otimes_R v : k \otimes_R L \to k \otimes_R L'$ is an isomorphism in $D_{sg,k}(A \otimes A^{op})$. Indeed, it suffices to prove that $\text{Cone}(k \otimes_R v) = 0$ in $D_{sg,k}(A \otimes A^{op})$. Since $v : L \to L'$ is an isomorphism in $D_{sg,k}(A \otimes A^{op})$, we have that $\text{Cone}(v)$ is in $P(R \otimes A \otimes A^{op})$. Thus, $\text{Cone}(k \otimes_R v) \cong k \otimes_R \text{Cone}(v) \in P(A \otimes A^{op})$. This proves the claim. The above commutative diagram implies that $\epsilon(L,u) = \epsilon(L',u')$. Therefore we obtain a map $\Phi : \text{sgDefo}(A, R \to k) \to \text{Hom}_{D_{sg,k}(A \otimes A^{op})}(A, \Sigma n \otimes A)$ which sends $(L,u)$ to $\epsilon(L,u)$.

We will construct the map $\Psi : \text{Hom}_{D_{sg,k}(A \otimes A^{op})}(A, \Sigma m \otimes A) \to \text{sgDefo}(A, R_m \to k)$ in the case of $R = R_m$ for $m \in \mathbb{Z}$ as follows. Let $f : A \to \Sigma m \otimes A$ be a morphism in $D_{sg,k}(A \otimes A^{op})$. Take a representative $f' \in \text{HH}^{m+1}(A, \Omega^{op}_{sg}(A))$. It follows from Proposition 4.3 that the dg $R_m \otimes A \otimes A^{op}$-module $C(f')$ lies in $D_{R_m,cl}(R_m \otimes A \otimes A^{op})$. We claim that the canonical morphism $\epsilon_m \otimes R_m C(f') \to \epsilon_m C(f')$ is an isomorphism in $D_{sg,k}(A \otimes A^{op})$. Indeed, since $\epsilon_m^2 = 0$, we have that

$$
\epsilon_m \otimes R_m C(f') \cong \epsilon_m \otimes (k \otimes R_m C(f')) \cong \epsilon_m \otimes \text{Bar}_s(A),
$$

where the second isomorphism comes from the fact that $k \otimes R_m M \cong M/\epsilon_m M$ for any dg $R_m$-module $M$. Then the canonical morphism $\epsilon_m \otimes R_m C(f') \to \epsilon_m C(f')$ is given by the following commutative diagram in $D_{sg,k}(A \otimes A^{op})$:

$$
\begin{array}{cccc}
 \epsilon_m \otimes R_m C(f') & \xrightarrow{\cong} & \epsilon_m C(f') \\
 \xi & & \xi & \\
 \epsilon_m \otimes \text{Bar}_s(A) & \xrightarrow{\rho} & \epsilon_m \otimes \Omega^{op}_{sg}(A),
\end{array}
$$

where $\rho$ is induced by the natural projection $d_P : \text{Bar}_p(A) \to \Omega^{op}_{sg}(A)$ which is an isomorphism in $D_{sg,k}(A \otimes A^{op})$. Hence the canonical morphism $\epsilon_m \otimes R_m C(f') \to \epsilon_m C(f')$ is an isomorphism in $D_{sg,k}(A \otimes A^{op})$. This proves the claim. Consider the canonical morphism of complexes of $A$-$A$-bimodules $u' : k \otimes R_m C(f') \xrightarrow{\cong} \text{Bar}_s(A) \xrightarrow{d_0} A$. Clearly, it is an isomorphism in $D_{sg,k}(A \otimes A^{op})$. By definition, we obtain that $(C(f'), u') \in F$. 
Let us define $\Psi(f) = (C(f'), u')$. The following claim ensures that $\Psi(f)$ is well-defined.

**Claim 5.3.** $\Psi(f)$ is independent of the choice of the representatives of $f \in \HH_{sg}^{m+1}(A, A)$.

**Proof.** Indeed, let $f'' \in C^{m+1}(A, \Omega_{sg}^q(A))$ be another representative of $f$. Without loss of

the generality, we may assume $q \geq p$. Since both $f'$ and $f''$ represent the same element $f$, we have $\Omega_{sg}^{q-p}(f') = f''$ in $\HH_{sg}^{q-p}(A, \Omega_{sg}^q(A))$, where the map $\Omega_{sg}^{q-p} : \HH_{sg}^q(A, \Omega_{sg}^q(A)) \to \HH_{sg}^{q-p}(A, \Omega_{sg}^q(A))$ is defined in Section (2.2). Equivalently, there exists $h \in C_m(A, \Omega_{sg}^q(A))$ such that $\Omega_{sg}^{q-p}(f') - f'' = \delta(h)$. Now we prove that

$$(C(f'), u') = (C(f''), u'') \quad \text{in} \sgDeo(A, R_m \to k).$$

For this, let us define a morphism $\rho : C(f') \to C(f'')$ of $\mathcal{D}_{sg, R_m}(A \otimes A^{op})$ as follows

$$
\begin{align*}
\sigma_{p-1}^{-1} &\cong C(\varphi^L(f')) \\
\phi &\cong C(\varphi^L(\Omega_{sy}^{q-p}(f'))) = C(\varphi^L(f' + \delta(h))) \\
&\cong C(\varphi^L(f'')).
\end{align*}
$$

Let us explain the above morphisms: The morphism $\phi : C(\varphi^L(f')) \to C(\varphi^L(f''))$ defined in Lemma (3.3) is an isomorphism in $\mathcal{D}^b(R_m \otimes A \otimes A^{op})$, thus it is an isomorphism in $\mathcal{D}_{sg, R_m}(A \otimes A^{op})$. The morphism $\varphi$ is induced by the natural projection $\text{Bar}_{\varphi}(A) \to \text{Bar}_{\varphi}(A)$; The morphism $\left(\frac{\text{id}}{-\varphi^{L}(h)} \frac{0}{\text{id}}\right)$ is an isomorphism with inverse $\left(\frac{\text{id}}{\varphi^{L}(h)} \frac{0}{\text{id}}\right)$, where $\varphi^{L}(h)$ is defined in Remark (3.2). Let us prove that $\phi$ is an isomorphism in $\mathcal{D}_{sg, R_m}(A \otimes A^{op})$.

For this, it suffices to prove that $\text{Cone}(\phi) \in \mathcal{P}(R_m \otimes A \otimes A^{op})$, namely both $\text{Cone}(\phi)$ and $k \otimes R \text{Cone}(\phi)$ are quasi-isomorphic to bounded complexes of projective $A \otimes A^{op}$-modules. Since $\phi$ is surjective, there is a short exact sequence of $\text{dg} R_m \otimes A \otimes A^{op}$-modules

$$0 \to B_{p,q-1} \to C(\varphi^L(f')) \xrightarrow{\phi} C(\varphi^L(\Omega_{sy}^{q-p}(f'))) \to 0,$$

where $B_{p,q-1}$ is the truncated complex

$$
\begin{align*}
B_{p,q-1} : 0 &\to \text{Bar}_{q-1}(A) \\
&\xrightarrow{d_{q-2}} \text{Bar}_{q-1}(A) \\
&\xrightarrow{d_{q-1}} \cdots \\
&\xrightarrow{d_{p+1}} \text{Bar}_{p}(A) \\
&\to 0.
\end{align*}
$$

This induces a distinguished triangle in $\mathcal{D}(A \otimes A^{op})$

$$
B_{p,q-1} \to C(\varphi^L(f')) \xrightarrow{\phi} C(\varphi^L(\Omega_{sy}^{q-p}(f'))) \to \Sigma B_{p,q-1}.
$$

It follows that $\text{Cone}(\phi) \cong \Sigma B_{p,q-1}$ in $\mathcal{D}(A \otimes A^{op})$ and thus $\text{Cone}(\phi)$ is quasi-isomorphic to a bounded complex of projective $A \otimes A^{op}$-modules. Since

$$
k \otimes R \text{Cone}(\phi) \cong \text{Cone}(\phi)/\epsilon_{m-1} \cong \text{Cone}(\text{id} : \text{Bar}_{s}(A) \to \text{Bar}_{s}(A)),$$

we get that $k \otimes R \text{Cone}(\phi) \cong 0$ in $\mathcal{D}(A \otimes A^{op})$. This prove that $\text{Cone}(\phi) \in \mathcal{P}(R_m \otimes A \otimes A^{op})$. Thus the morphism $\rho : C(f') \to C(f'')$ is an isomorphism of $\mathcal{D}_{sg, R_m}(A \otimes A^{op})$.

Applying the tensor functor $k \otimes R_m$ to the above diagram (7), we have the following commutative diagram in $\mathcal{D}_{sg, k}(A \otimes A^{op})$

$$
\begin{align*}
k \otimes R_m C(f') &\xrightarrow{\text{id} \otimes \rho} k \otimes R_m C(f'') \\
\downarrow u' &\quad \downarrow u''
\end{align*}
$$

since $u'$ is the following composition of maps

$$u' : k \otimes R_m C(f') \to \text{Bar}_{s}(A) \xrightarrow{d_{a}} A.$$
This yields \((C(f'), u') = (C(f''), u'')\) in \(\text{sgDefo}(A, R_m \to k)\). Therefore, \(\Psi(f)\) is independent of the choice of the representative of \(f\). This proves the claim.

As a consequence, we get a map \(\Psi : \text{Hom}_{\mathcal{D}_{sg}(A \otimes A^{op})}(A, \Sigma \epsilon_m \otimes A) \to \text{sgDefo}(A, R_m \to k)\).

**Proposition 5.4.** The map \(\Psi : \text{Hom}_{\mathcal{D}_{sg}(A \otimes A^{op})}(A, \Sigma \epsilon_m \otimes A) \to \text{sgDefo}(A, R_m \to k)\) is injective for any \(m \in \mathbb{Z}\).

**Proof.** It is sufficient to prove that \(\Phi \Psi = \text{Id}\). For this, let \(f' \in \text{HH}_{m+1}(A, \Omega^2_{sy}(A))\) be a representative of \(f \in \text{HH}_{m+1}(A, A)\). Then we have \(\Phi \Psi(f) = \Phi(C(f'), u') = f'\). This proves the proposition.

Note that the group \(\text{Aut}_{\mathcal{D}_{sg}(A \otimes A^{op})}(A)\) of automorphisms of \(A\) in \(\mathcal{D}_{sg}(A \otimes A^{op})\) acts on \(\text{Hom}_{\mathcal{D}_{sg}(A \otimes A^{op})}(A, \Sigma \epsilon_m \otimes A)\) via

\[ s \cdot f := (\Sigma \epsilon_m \otimes s^{-1})fs \]

for \(s \in \text{Aut}_{\mathcal{D}_{sg}(A \otimes A^{op})}(A)\) and \(f \in \text{Hom}_{\mathcal{D}_{sg}(A \otimes A^{op})}(A, \Sigma \epsilon_m \otimes A)\). The group \(\text{Aut}_{\mathcal{D}_{sg}(A \otimes A^{op})}(A)\) acts on \(\text{sgDefo}(A, R_m \to k)\) via

\[ s \cdot (L, u) := (L, su). \]

Clearly, the forgetful map induces a bijection

\[ \text{sgDefo}(A, R_m \to k)/ \text{Aut}_{\mathcal{D}_{sg}(A \otimes A^{op})}(A) \cong \text{sgDefo}'(A, R_m \to k). \]

Recall that \(\text{sgDefo}'(A, R_m \to k)\) is the set of isomorphism classes of weak singular deformations (cf. the second paragraph of Section 5.1). The map \(\Psi\) induces an injection

\[ \Psi : \text{Hom}_{\mathcal{D}_{sg}(A \otimes A^{op})}(A, \Sigma \epsilon_m \otimes A)/ \text{Aut}_{\mathcal{D}_{sg}(A \otimes A^{op})}(A) \hookrightarrow \text{sgDefo}'(A, R_m \to k). \]

**Lemma 5.5.** For any \(m \in \mathbb{Z}\), \(\text{Aut}_{\mathcal{D}_{sg}(A \otimes A^{op})}(A)\) acts trivially on \(\text{Hom}_{\mathcal{D}_{sg}(A \otimes A^{op})}(A, \Sigma \epsilon_m \otimes A)\). As a consequence, the following natural map is bijective.

\[ \text{Hom}_{\mathcal{D}_{sg}(A \otimes A^{op})}(A, \Sigma \epsilon_m \otimes A) \to \text{Hom}_{\mathcal{D}_{sg}(A \otimes A^{op})}(A, \Sigma \epsilon_m \otimes A)/ \text{Aut}_{\mathcal{D}_{sg}(A \otimes A^{op})}(A). \]

**Proof.** For \(s \in \text{Aut}_{\mathcal{D}_{sg}(A \otimes A^{op})}(A)\) and \(f \in \text{Hom}_{\mathcal{D}_{sg}(A \otimes A^{op})}(A, \Sigma \epsilon_m \otimes A)\), we need to show that \((\Sigma \epsilon_m \otimes s^{-1})fs = f\). Since the Yoneda product in \(\mathcal{D}_{sg}(A \otimes A^{op})\) corresponds to the cup product in \(\text{HH}_{m+1}(A)\), we have

\[ (\Sigma \epsilon_m \otimes s^{-1})fs = s^{-1}f \cup s = f \cup s^{-1} \cup s = f, \]

where the second identity comes from the fact that the cup product is graded commutative. This shows that the action of \(\text{Aut}_{\mathcal{D}_{sg}(A \otimes A^{op})}(A)\) is trivial. Hence the map \(\text{Hom}_{\mathcal{D}_{sg}(A \otimes A^{op})}(A, \Sigma \epsilon_m \otimes A) \to \text{Hom}_{\mathcal{D}_{sg}(A \otimes A^{op})}(A, \Sigma \epsilon_m \otimes A)/ \text{Aut}_{\mathcal{D}_{sg}(A \otimes A^{op})}(A)\) is bijective.

**Remark 5.6.** By Proposition 5.4 and Lemma 5.5, we obtain a natural embedding \(\Psi' : \text{Hom}_{\mathcal{D}_{sg}(A \otimes A^{op})}(A, \Sigma \epsilon_m \otimes A) \hookrightarrow \text{sgDefo}'(A, R_m \to k)\) for any \(m \in \mathbb{Z}\). We set

\[ G_A(\epsilon_m) = \text{Im}(\Psi' : \text{Hom}_{\mathcal{D}_{sg}(A \otimes A^{op})}(A, \Sigma \epsilon_m \otimes A) \hookrightarrow \text{sgDefo}'(A, R_m \to k)). \]

**Lemma 5.7.** The isomorphism \(\Psi' : \text{Hom}_{\mathcal{D}_{sg}(A \otimes A^{op})}(A, \Sigma \epsilon_m \otimes A) \cong G_A(\epsilon_m)\) is a monoid isomorphism, where the monoid structure on \(\text{Hom}_{\mathcal{D}_{sg}(A \otimes A^{op})}(A, \Sigma \epsilon_m \otimes A)\) is the additive structure; and the monoid structure on \(G_A(\epsilon_m)\) is given by \(\otimes_{R_m \otimes A}\).

**Proof.** Let \(f, g \in \text{Hom}_{\mathcal{D}_{sg}(A \otimes A^{op})}(A, \Sigma \epsilon_m \otimes A)\), which are represented by two elements \(f_1, g_1 \in \text{HH}_{m+1}(A, \Omega^2_{sy}(A))\) respectively. From Proposition 5.5, it follows that

\[ \Psi'(f) \otimes_{R_m \otimes A} \Psi'(g) \cong C(f_1) \otimes_{R_m \otimes A} C(g_1) \cong C(\Omega^{2p}_{sy}(f_1 + g_1)). \]

Since \(C(\Omega^{2p}_{sy}(f_1 + g_1)) = C(f_1 + g_1) = \Psi'(f + g)\) in \(\text{sgDefo}'(A, R_m \to k)\) (cf. Claim 5.3), we get that \(\Psi'\) is a monoid morphism. This proves the lemma.
5.2. The generalized Lie algebra of a group-valued functor. Let $k$ be a field. Let $A$ be a Noetherian $k$-algebra such that the enveloping algebra $A \otimes A^{op}$ is Noetherian. Denote by $cdg_k$, the category of finite-dimensional augmented commutative dg $k$-algebras and by $\text{grp}$ the category of groups. We define the functor $\text{sgDPic}_A : cdg_k \to \text{grp}$ sending $R \in cdg_k$ to the $R$-relative singular derived Picard group

$$\text{sgDPic}_A(R) := \{ L \in D_{sg,R}(A \otimes A^{op}) \mid \text{there exists } L' \in D_{sg,R}(A \otimes A^{op}) \text{ such that} \}

where $\sim$ means isomorphisms in $D_{sg,R}(A \otimes A^{op})$. A morphism $f : R \to S$ in $cdg_k$ induces the group homomorphism $\text{sgDPic}_A(f) : \text{sgDPic}_A(R) \to \text{sgDPic}_A(S)$ sending $L \in \text{sgDPic}_A(R)$ to $S \otimes_R L \in \text{sgDPic}_A(S)$ (cf. Remark 5.4). Then the generalized Lie algebra $\text{Lie}\text{sgDPic}_A$ associated to the group-valued functor $\text{sgDPic}_A$ is given by $(m \in \mathbb{Z})$

$$\text{Lie}\text{sgDPic}_A^m := \text{Ker}(\text{sgDPic}_A(R_m) \to \text{sgDPic}_A(k)) = \{ L \in \text{sgDPic}_A(R_m) \mid k \otimes_{R_m} L \cong A \text{ in } D_{sg,k}(A \otimes A^{op}) \}. \tag{5.9}$$

Remark 5.8. Recall from Remark 5.6 that we denote

$$G_A(\epsilon_m) = \text{Im}(\Psi' : \text{Hom}_{D_{sg}(A \otimes A^{op})}(A, \Sigma \epsilon_m \otimes A) \to \text{sgDefo}'(A, R_m \to k)). \tag{5.10}$$

By Proposition 5.4 we have $G_A(\epsilon_m) \hookrightarrow \text{Lie}\text{sgDPic}_A^m$ for any $m \in \mathbb{Z}$ since

$$C(f) \otimes_{R_m \otimes A} C(-A) \cong C(-A) \otimes_{R_m \otimes A} C(A) \cong R_m \otimes A \tag{5.11}$$

in $D_{sg,R}(A \otimes A^{op})$ and $k \otimes_{R_m} C(f) \cong A$ in $D_{sg,k}(A \otimes A^{op})$.

It follows from Lemma 5.7 that $\Psi'$ is a monoid isomorphism. Hence $G_A(\epsilon_m)$ has a $k$-vector space structure inherited from that of $\text{Hom}_{D_{sg}(A \otimes A^{op})}(A, \Sigma \epsilon_m \otimes A)$. We will define a Lie bracket on $G_A := \bigoplus_{m \in \mathbb{Z}} G_A(\epsilon_m)$ as follows. Let $L_1$ and $L_2$ represent elements of $G_A(\epsilon_m)$ and $G_A(\epsilon_n)$, respectively. Let $U_i$ be the image of $L_i$ in $D_{sg,R}(A \otimes A^{op})$ where $R = R_m \otimes R_n$ ($i = 1, 2$). Note that $U_i$ are invertible objects of the monoidal category $D_{sg,R}(A \otimes A^{op})$ (cf. Lemma 5.7), namely $U_i \in \text{sgDPic}_A(R)$. Let $V$ be the commutator of $U_1$ with $U_2$, namely $V = U_1U_2^{-1}U_1^{-1} \in \text{sgDPic}_A(R)$. Then Proposition 5.9 below shows that $V$ lies in $G_A(\epsilon_{m+n})$ under the morphism $\text{sgDPic}_A(R_{m+n}) \to \text{sgDPic}_A(R)$ induced by the natural embedding $R_{m+n} \hookrightarrow R$. Let us define $[L_1, L_2] := V \in G_A(\epsilon_{m+n})$.

Proposition 5.9. Let $f \in \text{HH}_{sg}^{m+1}(A, A)$ and $g \in \text{HH}_{sg}^{n+1}(A, A)$. Then the commutator $[\Psi'(f), \Psi'(g)] := \Psi'(f) \otimes_{R_m \otimes A} \Psi'(g) \otimes_{R_n \otimes A} \Psi'(-f) \otimes_{R_m \otimes A} \Psi'(-g)$ equals to $\Psi'([f, g])$ in $G_A(\epsilon_m \otimes \epsilon_n)$, where we write

$$\Psi'(f) := R_n \otimes \Psi'(f), \quad \Psi'(g) := R_m \otimes \Psi'(g),

\Psi'([f, g]) := (R_m \otimes R_n) \otimes_{R_{m+n}} \Psi'([f, g]). \tag{5.12}$$

Here $[f, g]$ is the Lie bracket in $\text{HH}_{*}(A, A)$ (cf. Section 7.2).

Proof. Note that $\Psi'(f) = C(f)$. Then from Lemma 5.7 it follows that to verify the identity $[\Psi'(f), \Psi'(g)] = \Psi'(([f, g])$ in $\text{sgDefo}'(A, R_{m+n} \to k)$ is equivalent to verify the following isomorphism in $D_{sg,R_{m+n} \otimes R_k}(A \otimes A^{op})$

$$\Psi'(f) \otimes_{R_m \otimes A} \Psi'(g) \cong \Psi'([f, g]) \otimes_{R_m \otimes A} \Psi'([f, g]) \otimes_{R_{m+n} \otimes A} \Psi'(f). \tag{5.13}$$
By Lemma 3.8, the left hand side of (9) is isomorphic to $C^R(f, g)$. The right hand side is

$$\text{RHS} \cong (\langle R_m \otimes R_n \rangle \otimes_{R_{m+n}} C([f, g])) \otimes_{R_m \otimes R_n} C^L(g, f)$$

$$\cong (\langle R_m \otimes R_n \rangle \otimes_{R_{m+n}} C([f, g])) \otimes_{A \otimes R_m \otimes R_n} C^L(g, f)$$

$$\cong C([f, g]) \otimes_{R_{m+n} \otimes A} C^L(g, f),$$

where the first isomorphism follows from Lemma 3.8 and the second one is because of the fact that $(\langle R_m \otimes R_n \rangle \otimes_{R_{m+n}} C([f, g])))$ is $(R_m \otimes R_n)$-relatively closed. Now let us compute $C([f, g]) \otimes_{R_{m+n} \otimes A} C^L(g, f)$ which is illustrated as follows

where, for simplicity, we write $\text{Bar}_* (A)$ as $B_*$. As graded modules, we have

$$C([f, g]) \otimes_{R_{m+n} \otimes A} C^L(g, f)$$

$$\cong B_* \otimes_A (B_* \oplus \Sigma^n B_{\geq q} + \Sigma^m B_{\geq p}) + \Sigma^{m+n} \Omega_{sy}^{p+q} (A) \otimes_A \Omega_{sy}^{p+q} (A)$$

$$\cong B_* \otimes_A B_* + B_* \otimes_A \Sigma^n B_{\geq q} + B_* \otimes_A \Sigma^m B_{\geq p} + \Sigma^{m+n} \Omega_{sy}^{p+q} (A) \otimes_A \Omega_{sy}^{p+q} (A),$$

where in the first identity we use the following two isomorphisms

$$(k \otimes B_i) \otimes_{R_{m+n} \otimes A} M \cong B_i \otimes_A (M / \epsilon_{m+n} M)$$

$$(B_{p+q} \oplus \Sigma^{m+n} \Omega_{sy}^{p+q} (A)) \otimes_{R_{m+n} \otimes A} M \cong (B_{p+q} \otimes_A M) / (\Sigma^{-1} \Omega_{sy}^{p+q+1} (A) \otimes_A \epsilon_{m+n} M)$$

for any dg $R_{m+n} \otimes A$-module $M$. The proofs of the above two isomorphisms are similar to the ones of the isomorphisms in [4]. From the construction of the tensor product of dg modules, we obtain the differential illustrated as follows.
Using the quasi-isomorphism $\Delta = \Delta_{p,q} : B_{\geq p+q} \to B_{\geq p} \otimes A B_{\geq q}$ and the isomorphism $\mu = \mu_{p+q,p+q} : \Omega_{sy}^{p+q}(A) \otimes_A \Omega_{sy}^{p+q}(A) \cong \Omega_{sy}^{2(p+q)}(A)$, the above dg $R_m \otimes R_n \otimes A \otimes A^{op}$-module is $R_m \otimes R_n$-relatively quasi-isomorphic to the following one (denoted by $C_2(f,g)$)

$$\begin{align*}
B_\ast & \oplus \epsilon_n \otimes B_{\geq q} \\
& \oplus \epsilon_m \otimes B_{\geq p} \\
& \oplus \epsilon_{m+n} \otimes \Omega_{sy}^{2(p+q)}(A)
\end{align*}$$

where we take

(i) $\tilde{g} = \mu(d_{p+q} \otimes_A \Omega_{sy}^{L,p+q}(g)) \Delta$;

(ii) $\tilde{f} = \mu(d_{p+q} \otimes_A \Omega_{sy}^{L,p+q}(f)) \Delta$;

(iii) $H = \mu([f,g] \otimes_A d_{p+q} + d_{p+q} \otimes_A (f \cdot g)) \Delta$.

From Remark 2.4 we have

$$H = \Omega_{sy}^{L,p+q}([f,g]) + \Omega_{sy}^{R,p+q}(f \cdot g).$$

We claim that $C_2(f,g)$ is isomorphic to the following dg $R_m \otimes R_n \otimes A \otimes A^{op}$-module (denoted by $C_2'(f,g)$)

$$\begin{align*}
B_\ast & \oplus \epsilon_n \otimes B_{\geq q} \\
& \oplus \epsilon_m \otimes B_{\geq p} \\
& \oplus \epsilon_{m+n} \otimes \Omega_{sy}^{2(p+q)}(A).
\end{align*}$$

Indeed, since both $\tilde{g}$ and $\Omega_{sy}^{R,2p+q}(g)$ represent the cocycle $\Omega_{sy}^{2p+q}(g)$, there is a coboundary $g_1 : Bar_{n+2p+q-1} \to \Omega_{sy}^{2(p+q)}(A)$ such that $g_1 d_{n+2p+q} = \Omega_{sy}^{R,2p+q}(g) - \tilde{g}$. Note that

$$\tilde{g} - \Omega_{sy}^{R,2p+q}(g) = \mu(d_{p+q} \otimes_A (\Omega_{sy}^{L,p+q}(g) - \Omega_{sy}^{R,p+q}(g))) \Delta,$$

it follows from Remark 2.4 that we may choose $g_1 = -\mu(d_{p+q} \otimes_A h_{p+q}^{L,R}(g)) \Delta$. Similarly, we take

$$f_1 = -\mu(d_{p+q} \otimes_A h_{p+q}^{L,R}(f)) \Delta : Bar_{m+2q+p-1}(A) \to \Omega_{sy}^{2(p+q)}(A).$$

Then we have $f_1 d_{m+2q+p} = \Omega_{sy}^{R,2q+p}(f) - \tilde{f}$. Since $[f,g]$ is a cocycle, there is a homotopy

$$h = h_{p+q}^{L,R}([f,g]) : Bar_{m+n+p+q-2}(A) \to \Omega_{sy}^{2(p+q)}(A)$$

such that $hd_{m+n+p+q-1} = \Omega_{sy}^{L,p+q}([f,g]) - \Omega_{sy}^{R,p+q}([f,g])$. Let us construct a morphism of graded $A \otimes A^{op}$-modules

$$\rho = \begin{pmatrix}
I & 0 & 0 & 0 \\
0 & I & 0 & 0 \\
0 & 0 & I & 0 \\
h & f_1 & g_1 & I
\end{pmatrix} : C_2(f,g) \to C_2'(f,g).$$
The following identity holds
\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
h & f & g & 0
\end{pmatrix}
\begin{pmatrix}
d & 0 & 0 & 0 \\
0 & d & 0 & 0 \\
0 & 0 & d & 0 \\
0 & 0 & 0 & d \\
h & f & g & 0
\end{pmatrix}
\begin{pmatrix}
d & 0 & 0 & 0 \\
0 & d & 0 & 0 \\
0 & 0 & d & 0 \\
0 & 0 & 0 & d \\
h & f & g & 0
\end{pmatrix},
\]
since we have
\[
f_1\vartheta^p(f) + g_1\vartheta^p(f) = \mu(d_{p+q} \otimes A \Delta h_{p+q}^R(f) \vartheta^p(f)) + \mu(d_{p+q} \otimes A \Delta h_{p+q}^R(f) \vartheta^p(f)) = \mu(d_{p+q} \otimes A (g \circ f - g \circ f)) + \mu(d_{p+q} \otimes A (g \circ f - g \circ f)),
\]
where the first identity is due to the definition of \(\Delta\); the second identity is because of Lemma 3.4; and the third identity follows from Remark 2.4. This implies that the graded subspace \(G_{\ast}\) is isomorphic to \(C_{\ast}(f, g)\) in \(D_{\ast}(A \otimes A^\text{op})\).

In order to prove (9), we do not know whether the generalized Lie algebra \(\text{Lie}_{sgDPic}\) is indeed a graded Lie algebra. Keller in [Kel99] proved the identity \(\Psi'(\vartheta) \cdot \Psi'(\vartheta) = \Psi'(f, g)\) for any \(f, g \in \text{HH}^+_{sg}(A, A)\) in a quite different way, where he used the intrinsic interpretation of the Gerstenhaber bracket by Stasheff [Sta].
6. The invariance under singular equivalence of Morita type with level

Let $k$ be a field. Let $A$ and $B$ be two Noetherian $k$-algebras such that the enveloping algebras $A \otimes A^{\text{op}}$ and $B \otimes B^{\text{op}}$ are Noetherian. Let $A \Lambda B$ and $B \Lambda A$ be an $A$-$B$-bimodule and a $B$-$A$-bimodule, respectively. Recall from [Wan15b] that $(A \Lambda B, B \Lambda A)$ defines a singular equivalence of Morita type with level $l \in \mathbb{Z}_{\geq 0}$ if the following conditions are satisfied:

1. $M$ is finitely generated projective as a left $A$-module and as a right $B$-module,
2. $N$ is finitely generated projective as a left $B$-module and as a right $A$-module,
3. there exist isomorphisms $M \otimes_B N \cong \Omega^l_{\text{sg}}(A)$ in $(A \otimes A^{\text{op}})$-mod, and $N \otimes_A M \cong \Omega^l_{\text{sg}}(B)$ in $(B \otimes B^{\text{op}})$-mod, where $A \otimes A^{\text{op}}$-mod denotes the stable module category of $A$-$A$-bimodules.

**Remark 6.1.** Note that the tensor product $M \otimes_B - : \mathcal{D}_{\text{sg}}(B) \to \mathcal{D}_{\text{sg}}(A)$ is an equivalence of triangulated categories with the quasi-inverse $\Sigma^l(N \otimes_A -) : \mathcal{D}_{\text{sg}}(A) \to \mathcal{D}_{\text{sg}}(B)$. Similarly, we have the following equivalence of triangulated categories

$$\Sigma^l(M \otimes_B - \otimes_B N) : \mathcal{D}_{\text{sg}}(B \otimes B^{\text{op}}) \to \mathcal{D}_{\text{sg}}(A \otimes A^{\text{op}}).$$

Let us now prove the main result of this paper.

**Theorem 6.2.** Let $A$ and $B$ be two Noetherian algebras over a field $k$ such that the enveloping algebras $A \otimes A^{\text{op}}$ and $B \otimes B^{\text{op}}$ are Noetherian. Suppose that $(A \Lambda B, B \Lambda A)$ defines a singular equivalence of Morita type with level $l \in \mathbb{Z}_{\geq 0}$. Then the functor $\Sigma^l(M \otimes_B - \otimes_B N)$ induces an isomorphism of Gerstenhaber algebras between Tate-Hochschild cohomology rings $\text{HH}_{\text{sg}}^*(A, A)$ and $\text{HH}_{\text{sg}}^*(B, B)$.

**Proof.** First from the facts that the functor $\Sigma^l(M \otimes_B - \otimes_B N)$ induces an equivalence between $\mathcal{D}_{\text{sg}}(B \otimes B^{\text{op}})$ and $\mathcal{D}_{\text{sg}}(A \otimes A^{\text{op}})$ and that the coproduct $\cup$ in $\text{HH}_{\text{sg}}^*(A, A)$ coincides with the Yoneda product in $\mathcal{D}_{\text{sg}}(A \otimes A^{\text{op}})$, it follows that $\Sigma^l(M \otimes_B - \otimes_B N)$ yields an isomorphism of graded-commutative algebras between $\text{HH}^*(B, B)$ and $\text{HH}^*(A, A)$. It remains to prove that $\Sigma^l(M \otimes_B - \otimes_B N)$ induces an isomorphism of graded Lie algebras. For this, $\Sigma^l(M \otimes_B - \otimes_B N)$ induces an isomorphism between $\text{sgDPic}_B$ and $\text{sgDPic}_A$ and thus induces an isomorphism between $\text{Lie} \text{sgDPic}_B$ and $\text{Lie} \text{sgDPic}_A$. In particular, this restricts to an isomorphism of graded Lie algebras $\Sigma^l(M \otimes_B - \otimes_B N) : G_B \to G_A$, where $G_A := \bigoplus_{m \in \mathbb{Z}} G_A(\epsilon_m)$. Consider the following commutative diagram

\[
\begin{array}{ccc}
\mathcal{D}_{\text{sg}}^*(B, B) & \xrightarrow{\Sigma^l(M \otimes_B - \otimes_B N)} & \mathcal{D}_{\text{sg}}^*(A, A) \\
\cong & & \cong \\
\text{HH}_{\text{sg}}^{*+1}(B, B) & \xrightarrow{\Sigma^l(M \otimes_B - \otimes_B N)} & \text{HH}_{\text{sg}}^{*+1}(A, A).
\end{array}
\]

Since it follows from Corollary 5.10 that the vertical morphisms are isomorphisms of graded Lie algebras, the bottom horizontal map induces an isomorphism of Gerstenhaber algebras between $\text{HH}_{\text{sg}}^*(B, B)$ and $\text{HH}_{\text{sg}}^*(A, A)$. This proves the theorem. \hfill \Box

**Corollary 6.3.** Let $A$ and $B$ be two Noetherian $k$-algebras such that the enveloping algebras $A \otimes A^{\text{op}}$ and $B \otimes B^{\text{op}}$ are Noetherian. Assume that the derived categories $\mathcal{D}(A)$ and $\mathcal{D}(B)$ are equivalent as triangulated categories. Then there exists an isomorphism of Gerstenhaber algebras between $\text{HH}_{\text{sg}}^*(A, A)$ and $\text{HH}_{\text{sg}}^*(B, B)$.

**Proof.** This comes from Theorem 6.2 and the fact that two derived equivalent algebras induce a singular equivalence of Morita type with some level $l \in \mathbb{Z}_{\geq 0}$ (cf. [Wan15b]). \hfill \Box
References


