The Navarro conjecture for the alternating groups

Olivier Brunat, Rishi Nath

THE NAVARRO CONJECTURE FOR THE ALTERNATING GROUPS

OLIVIER BRUNAT AND RISHI NATH

Abstract. Recently Navarro proposed a strengthening of the unsolved McKay conjecture using Galois automorphisms. We prove that the Navarro conjecture and its blockwise version hold for the alternating groups.

1. Introduction

Let G be a finite group of order n and p be a prime divisor of n. We denote by $\mathrm{Irr}(G)$ the set of irreducible complex characters of G, and by $\mathrm{Irr}_{p'}(G)$ the subset of irreducible characters with degree prime to p. In 1972, John McKay conjectured that $|\mathrm{Irr}_{p'}(G)| = |\mathrm{Irr}_{p'}(\mathrm{N}_G(P))|$, where P is a Sylow p-subgroup of G. Although the conjecture remains open, there is strong evidence in its favor. In 2007, I. M. Isaacs, G. Malle and G. Navarro [5] reduced the problem to a question on finite simple groups. In particular, they assert that if a set of conditions holds for all non abelian finite simple groups, then the original conjecture holds for all finite groups. Using this strategy, Malle and Späth recently proved [9] that McKay conjecture holds at $p = 2$ for all finite groups.

The McKay conjecture has lead to a family of other conjectures on finite groups. For example, the conjectures of Alperin-McKay, of Dade, of Broué and of Isaacs-Navarro are of a similar flavor. This paper is concerned with a refinement of the McKay conjecture due to Navarro [13], which posits not only a correspondence between the set of global-and-local irreducible characters of p'-degree, but also between their character values.

In order to state the conjecture more precisely, we introduce some notation. Let $\mathbb{Q}_n = \mathbb{Q}(\omega_n)$ be the cyclotomic subfield of $\mathbb{C}$, where $\omega_n = e^{2i\pi/n}$, and $\mathcal{G}_n = \mathrm{Gal}(\mathbb{Q}_n|\mathbb{Q})$. For any $f \in \mathcal{G}_n$, $\chi \in \mathrm{Irr}(G)$ and $g \in G$, we set ${}^f\chi(g) = f(\chi(g))$, inducing an action of $\mathcal{G}_n$ on $\mathrm{Irr}(G)$ and then on $\mathrm{Irr}_{p'}(G)$. Furthermore, if H is a subgroup of G of order d, then d divides n and $\mathbb{Q}_d$ is a subfield of $\mathbb{Q}_n$. Note also that, if $f \in \mathcal{G}_n$, then $f(\omega_d)$ is a primitive d-root of unity, that is, there is some integer r prime to d such that $f(\omega_d) = \omega_d^r$. In particular, $f(\mathbb{Q}_d) = \mathbb{Q}_d$ and $f|_{\mathbb{Q}_d} \in \mathcal{G}_d$. Hence, $\mathcal{G}_n$ acts on $\mathrm{Irr}(H)$ through $\mathcal{G}_n \to \mathcal{G}_d$, $f \mapsto f|_{\mathbb{Q}_d}$.

Even though there cannot exist a bijection $\mathrm{Irr}_{p'}(G) \to \mathrm{Irr}_{p'}(\mathrm{N}_G(P))$ that commutes with $\mathcal{G}_n$, Gabriel Navarro observed in [13] that there should exist a bijection commuting with a special subgroup $\mathcal{H}_n$ of $\mathcal{G}_n$. More precisely, if we write $n = p^\ell m$ with m prime to p, then ω_n can be uniquely writen as a product $\omega\delta$, where ω has order p^ℓ and δ has order m. It follows that $\mathcal{G}_n = \mathcal{K}_n \times \mathcal{J}_n$, where $\mathcal{K}_n$ and $\mathcal{J}_n$ are respectively the subgroups of $\mathcal{G}_n$ fixing δ and ω. Let σ_n be the element of $\mathcal{J}_n$ such that $\sigma_n(\delta) = \delta^p$. If we set $\mathcal{H}_n = \mathcal{K}_n \times \langle \sigma_n \rangle$, then $\mathcal{K}_n$ is isomorphic to $\mathrm{Gal}(\mathbb{Q}_{p^\ell}|\mathbb{Q})$,

2010 *Mathematics Subject Classification.* Primary 20C30; Secondary 20C15.

and $\mathcal{H}_n$ is thus the subgroup of $\mathcal{G}_n$ which acts on the p'-roots of unity of $\mathbb{Q}_n$ by a power of p. In [13, Conjecture A], Navarro conjectured that for any $f \in \mathcal{H}_n$, there are the same number of characters of $\mathrm{Irr}_{p'}(G)$ and of $\mathrm{Irr}_{p'}(\mathrm{N}_G(P))$ fixed by f. In the following, elements of $\mathcal{H}_n$ will be called *Navarro automorphisms*.

While significant progress has been made on the McKay conjecture, evidence of the veracity of the Navarro refinement has been limited to a handful of cases: groups of odd order by Isaacs [6], for solvable groups (E. Dade), for sporadic groups, for symmetric groups (P. Fong), for simple groups of Lie type in defining characteristic (L. Ruhstorfer [16]), and for alternating groups for $p = 2$ (by the second author [11]). A. Turull gave in [18] a conjecture which implies the Navarro conjecture. He proved in [18] his conjecture for the special linear groups in defining characteristic and in [17] for p-solvable groups. Recently, Navarro, Spaeth and Vallejo proved a reduction theorem of Navarro refinement to the quasisimple groups [14].

In this paper, we verify that when p is odd the conjecture holds for an important family of simple groups, the alternating groups. More precisely, we will prove the following general result.

Theorem 1.1. *Let n be a positive integer, and $2 < p \leq n$ be an odd prime number. Fix a Sylow p-subgroup P of $\mathfrak{A}_n$. Then there is a natural $\mathcal{H}_{n!/2}$-equivariant bijection*

$$\Phi : \mathrm{Irr}_{p'}(\mathfrak{A}_n) \to \mathrm{Irr}_{p'}(\mathrm{N}_{\mathfrak{A}_n}(P)).$$

The paper is organized as follows. In Section 2 we discuss the Navarro conjecture for the symmetric groups. It was noted in [13] that this was checked by Fong. However, since the details will be useful in our work, we supply them for the convenience of the reader.

In Section 3, after studying the representation theory of the alternating subgroup of a direct product of groups contained in some symmetric group, we describe the action of automorphisms on $\mathrm{Irr}_{p'}(\mathrm{N}_{\mathfrak{A}_n}(P))$.

In Section 4, we describe the action of Navarro automorphisms on $\mathrm{Irr}_{p'}(\mathfrak{A}_n)$. To this end, we obtain an explicit formula for the diagonal hook lengths of a symmetric partition of n in terms of the diagonal hooks of the p-core and p-quotient. These results are of independent interest : many partition-theoretic questions about Ramanujan-type congruences, monotonicity and the Durfee square can be answered using the relationship between a partition and its p-core and p-quotient (see for example the work of F. Garvan, D. Kim and D. Stanton [3]).

Finally, Section 5 and 6 are devoted to the proof of the Navarro conjecture and its blockwise version for the alternating groups with no condition over the prime p.

2. Verification of the conjecture for the symmetric groups

Let n be a positive integer and p be a prime number. Let P be a Sylow p-subgroup of $\mathfrak{S}_n$, and set $N = \mathrm{N}_{\mathfrak{S}_n}(P)$. First, following [2, §1 and §2] we describe a parametrization of $\mathrm{Irr}_{p'}(\mathfrak{S}_n)$ and of $\mathrm{Irr}_{p'}(N)$. The irreducible characters of $\mathfrak{S}_n$ are naturally labeled by the set of partitions of n. For any such partition λ, we denote by χ_λ the corresponding character of $\mathfrak{S}_n$.

For any partition λ of n, we write $|\lambda| = n$ for the size of λ. We also will denote by $Y(\lambda)$ for the Young diagram of λ. Using matrix notation, we associate to any (i, j)-box of $Y(\lambda)$, an (i, j)-hook with *hook-length* h_{ij}. We denote by $\mathfrak{D}(\lambda)$ the set of *diagonal hooks* of λ, that is, the hooks in positions (i,i). Such a hook will be known as the ith-diagonal hook. We denote by

$$\mathfrak{d}(\lambda) = \{|h| \mid h \in \mathfrak{D}(\lambda)\} \tag{1}$$

the set of hook-lengths of the diagonal hooks of λ.

Recall that λ is *symmetric* if $\lambda = \lambda^*$. When λ is symmetric the ith-diagonal hook is uniquely determined by its hook-length.

Recall that any partition λ is completely determined by its p-core $\mathcal{C}or_p(\lambda)$ and its p-quotient $\mathcal{Q}uo_p(\lambda) = (\lambda_0, \dots, \lambda_{p-1})$; [15, §3]. Write $I = \{0, \dots, p-1\}$ and $I^0 = \emptyset$. Set $\lambda^{\emptyset} = \lambda$. Let k be a non-negative integer, and assume $\lambda^{\underline{j}}$ is constructed for any $\underline{j} \in I^k$. Then we define $\lambda_{\underline{j}} = \mathcal{C}or_p(\lambda^{\underline{j}})$ and for $\underline{j} = (j_1, \dots, j_{k+1}) \in I^{k+1}$, we write $\lambda^{\underline{j}} = \mathcal{Q}uo_p(\lambda^{j_1,\dots,j_k})_{j_{k+1}}$. For any $k \geq 0$, write $\mathcal{C}or_p^{(k)}(\lambda) = \{\lambda_{\underline{j}} \mid \underline{j} \in I^k\}$. The set

$$\mathcal{CT}(\lambda) = \bigcup_{k \geq 0} \mathcal{C}or_p^{(k)}(\lambda) \tag{2}$$

is called the *p-core tower* of λ. For more details, we refer to [15, p. 41].

On the other hand, recall that by [2, Proposition 1.1], $\chi_\lambda \in \mathrm{Irr}_{p'}(\mathfrak{S}_n)$ if and only if $0 \leq c_k(\lambda) \leq p-1$ for all $k \geq 0$, where $c_k(\lambda) = \sum_{\underline{j} \in I^k} |\lambda_{\underline{j}}|$.

Let $n = n_0 + n_1 p + n_2 p^2 + n_3 p^3 + \cdots$ be the p-adic expansion of n. Note that the p'-irreducible characters of N are exactly the ones that have P' in their kernel; that is, the irreducible characters of N which can be lifted from the projection $N \to N/P'$. Furthermore, by [2, §2], one has

$$N/P' \simeq \mathfrak{S}_{n_0} \times \prod_{k \geq 1} Y^k \wr \mathfrak{S}_{n_k}, \tag{3}$$

where X is a Sylow p-subgroup of $\mathfrak{S}_p$ and $Y = \mathrm{N}_{\mathfrak{S}_p}(X)$.

Let $k \geq 1$. Write $N_k = Y^k \wr \mathfrak{S}_{n_k}$. The elements of N_k are denoted by $(y;\sigma)$, where $y = (y_1, \dots, y_{n_k}) \in (Y^k)^{n_k}$ and $\sigma \in \mathfrak{S}_{n_k}$. For any $\sigma \in \mathfrak{S}_{n_k}$, we denote by $C(\sigma)$ the set of cycles of σ with respect to its canonical decomposition into cycles with disjoint supports. For $\tau \in C(\sigma)$, the corresponding "cycle" of N_k is $(y_\tau;\tau)$, where $(y_\tau)_j = y_j$ if $j \in \mathrm{supp}(\tau)$ and $(y_\tau)_j = 1$ otherwise. For any $\tau \in C(\sigma)$, we also define the *cycle product* $\mathfrak{c}((y;\sigma),\tau) = \prod_{j \in \mathrm{supp}(\tau)} y_j$ of $(y;\sigma)$ with respect to τ.

Note that $Y = \langle a \rangle \rtimes \langle b \rangle$ with a and b of order p and $p-1$ respectively. Recall that Y has $p-1$ linear characters obtained by lifting the ones of $\langle b \rangle$ through $Y \to Y/\langle a \rangle \simeq \langle b \rangle$, and one character of degree $p-1$ obtained by inducing any non-trivial characters of $\langle a \rangle$ to Y. Write

$$\mathrm{Irr}(Y) = \{\xi_0, \dots, \xi_{p-1}\}. \tag{4}$$

Then these characters are $\mathbb{Q}(\omega_{p-1})$-valued by construction.

Let $k \geq 1$. For $\underline{j} = (j_1, \dots, j_k) \in I^k$, we set $\xi_{\underline{j}} = \xi_{j_1} \otimes \xi_{j_2} \otimes \cdots \otimes \xi_{j_k}$. Then

$$\mathrm{Irr}(Y^k) = \{\xi_{\underline{j}} \mid \underline{j} \in I^k\}. \tag{5}$$

Let $\mathcal{MP}(p^k, n_k)$ be the set of p^k-multipartitions of n_k that is, multipartitions $\underline{\lambda} = (\lambda_{\underline{j}};\, \underline{j} \in I^k)$ such that $\sum_{\underline{j} \in I^k} |\lambda_{\underline{j}}| = n_k$.

Remark 2.1. In the following, we will always assume that the $\lambda_{\underline{j}}$'s in $\underline{\lambda}$ appear in increasing lexicographic order.

By [8], the irreducible characters of N_k can be labeled by $\mathcal{MP}(p^k, n_k)$ as follows. Let $\underline{\lambda} = (\lambda_{\underline{j}};\, \underline{j} \in I^k)$ be such that $\sum_{\underline{j}\in I^k} |\lambda_{\underline{j}}| = n_k$. Consider the irreducible character

$$\xi_{\underline{\lambda}} = \bigotimes_{\underline{j}\in I^k} \underbrace{\xi_{\underline{j}} \otimes \cdots \otimes \xi_{\underline{j}}}_{|\lambda_{\underline{j}}| \text{ times}}$$

of $(Y^k)^{n_k}$. If we set $\mathfrak{S}_{\underline{\lambda}} = \prod_{\underline{j}\in I^k} \mathfrak{S}_{|\lambda_{\underline{j}}|}$, then the inertial subgroup of $\xi_{\underline{\lambda}}$ in N_k is

$$N_{k,\underline{\lambda}} = (Y^k)^{n_k} \rtimes \mathfrak{S}_{\underline{\lambda}} = \prod_{\underline{j}\in I^k} Y^k \wr \mathfrak{S}_{|\lambda_{\underline{j}}|}.$$

We denote by $E(\xi_{\underline{\lambda}})$ the James-Kerber extension of $\xi_{\underline{\lambda}}$ to $N_{k,\underline{\lambda}}$ described in [8, §4.3]. Note that $E(\xi_{\underline{\lambda}}) = \bigotimes E(\xi_{\underline{j}}^{|\lambda_{\underline{j}}|})$ and [8, Lemma 4.3.9] gives

$$\begin{aligned} E(\xi_{\underline{\lambda}})\left(\prod_{\underline{j}\in I^k} (y_{\underline{j}};\sigma_{\underline{j}})\right) &= \prod_{\underline{j}\in I^k} E(\xi_{\underline{j}}^{|\lambda_{\underline{j}}|})(y_{\underline{j}};\sigma_{\underline{j}}) \\ &= \prod_{\underline{j}\in I^k, |\lambda_{\underline{j}}|\neq 0} \ \prod_{\tau\in C(\sigma_{\underline{j}})} \xi_{\underline{j}}(\mathfrak{c}((y_{\underline{j}};\sigma_{\underline{j}}),\tau)). \end{aligned} \tag{6}$$

Now, write $\chi_{\underline{\lambda}}$ for the characters $\prod_{\underline{j}\in I^k} \chi_{\lambda_{\underline{j}}} \in \mathfrak{S}_{\underline{\lambda}}$ lifted through the canonical projection $N_{k,\underline{\lambda}} \to N_{k,\underline{\lambda}}/(Y^k)^{n_k} \simeq \mathfrak{S}_{\underline{\lambda}}$, and define

$$\psi_{\underline{\lambda},k} = \mathrm{Ind}_{N_{k,\underline{\lambda}}}^{N_k} \left(E(\xi_{\underline{\lambda}})\chi_{\underline{\lambda}}\right). \tag{7}$$

Then

$$\mathrm{Irr}(N_k) = \left\{\psi_{\underline{\lambda},k} \,|\, \underline{\lambda} \in \mathcal{MP}(p^k, n_k)\right\}.$$

The following result will be useful.

Lemma 2.2. *Let G be a finite group of order n and H a subgroup of G. Let $f \in \mathcal{G}_n$. Then for any class function ϕ on H, we have*

$${}^f\,\mathrm{Ind}_H^G(\phi) = \mathrm{Ind}_H^G({}^f\phi).$$

Proposition 2.3. *The Navarro conjecture holds for the symmetric groups.*

Proof. Let n be a positive integer and $p \leq n$ be a prime number. Since the characters of $\mathfrak{S}_n$ are rational-valued, they are fixed by any automorphisms of $\mathcal{H}_{n!}$. It remains to show that any p'-order irreducible characters of N is also fixed. From (3), it is sufficient to show that for $k \geq 1$, the irreducible characters of N_k are fixed under any $f \in \mathcal{H}_{n!}$. If $f \in \mathcal{K}_{n!}$, then f fixes any p'-roots of unity. However, the values of the characters of $\mathrm{Irr}(Y)$ lie in $\mathbb{Q}(\omega_{p-1})$, and are thus fixed by f. Write $\sigma = \sigma_{n!}$. We have $\sigma(x) = x^p$ for any p'-root of unity x. Since ω_{p-1} is a p'-root of unity, we deduce that $\sigma(\omega_{p-1}) = \omega_{p-1}^p = \omega_{p-1}$, and σ fixes the characters of $\mathrm{Irr}(Y)$. In either case, the irreducible characters of Y are fixed by $\mathcal{H}_{n!}$. Let $n = n_0 + n_1 p + \cdots$ be the p-adic expansion of n as above and fix $k \geq 1$. Let $\underline{\lambda} \in \mathcal{MP}(p^k, n_k)$. From (6) and the fact that $\chi_{\underline{\lambda}}$ is rational-valued, we obtain that $E(\xi_{\underline{\lambda}})\chi_{\underline{\lambda}}$ is fixed under any $f \in \mathcal{H}_{n!}$. Finally, we conclude using Lemma 2.2. ∎

3. Alternating groups. The local case

For any subgroup G of $\mathfrak{S}_n$, we set $G^+ = G \cap \mathfrak{A}_n$. In particular, $[G : G^+] \leq 2$ and G^+ is the kernel of the restriction to G of the sign character of $\mathfrak{S}_n$, also denoted by $\operatorname{sgn} : G \to \{-1, 1\}$. Suppose $G^+ \neq G$. For $\chi \in \operatorname{Irr}(G)$, we write $\chi^* = \chi \otimes \operatorname{sgn}$. By Clifford theory, if $\chi \neq \chi^*$, then χ and χ^* restrict to an irreducible character of G^+ also denoted by χ. If $\chi = \chi^*$, then the restriction of χ to G^+ is the sum of two irreducible characters denoted by χ^+ and χ^-. Note that in this case, $\chi(g) = 0$ for all $g \notin G^+$, and $\chi(g) = \chi^+(g) + \chi^-(g)$ for $g \in G^+$. All irreducible characters of G^+ are obtained exactly once by this process. A *split class* c of G is a conjugacy class of G contained in G^+ such that c is the union of two G^+-classes c^+ and c^-.

Let f be a Galois automorphism and $\chi \in \operatorname{Irr}(G)$ be such that $\chi = \chi^*$ and ${}^f\chi = \chi$. Then f acts on $\{\chi^+, \chi^-\}$. We define $\varepsilon(\chi, f) \in \{-1, 1\}$ by setting $\varepsilon(\chi, f) = 1$ if ${}^f\chi^+ = \chi^+$ and $\varepsilon(\chi, f) = -1$ otherwise. In particular, for $\eta \in \{-1, 1\}$ we have

$$ {}^f\chi^\eta = \chi^{\varepsilon(\chi,f)\eta}. \tag{8}$$

3.1. **Reduction of the problem.** Let $G_1, \ldots, G_r$ be subgroups of $\mathfrak{S}_n$ such that $G = G_1 \times \cdots \times G_r \subseteq \mathfrak{S}_n$ is a direct product and assume $G_j^+ \neq G_j$ for all $1 \leq j \leq r$. Fix $\sigma_j \in G_j \backslash G_j^+$. Let $\chi = \chi_1 \otimes \cdots \otimes \chi_r \in \operatorname{Irr}(G)$ be such that $\chi_j \in \operatorname{Irr}(G_j)$ for all $1 \leq j \leq r$. First, we remark that $\operatorname{sgn} = \operatorname{sgn} \otimes \cdots \otimes \operatorname{sgn} \in \operatorname{Irr}(G)$, thus

$$\begin{aligned} \chi^* &= \chi \otimes \operatorname{sgn} \qquad (9)\\ &= (\chi_1 \otimes \cdots \otimes \chi_r) \otimes (\operatorname{sgn} \otimes \cdots \otimes \operatorname{sgn}) \\ &= (\chi_1 \otimes \operatorname{sgn}) \otimes \cdots \otimes (\chi_r \otimes \operatorname{sgn}) \\ &= \chi_1^* \otimes \cdots \otimes \chi_r^*. \end{aligned}$$

In particular

$$\chi = \chi^* \iff \chi_j = \chi_j^* \quad \text{for all } 1 \leq j \leq r. \tag{10}$$

Suppose $\chi = \chi^*$. Write $N = G_1^+ \times \cdots \times G_r^+$. Then N is a normal subgroup of G. For $\underline{\epsilon} = (\epsilon_1, \ldots, \epsilon_r) \in \{-1, 1\}^r$, we set

$$\chi_{\underline{\epsilon}} = \chi_1^{\epsilon_1} \otimes \cdots \otimes \chi_r^{\epsilon_r}.$$

Consider a constituent ϕ of $\operatorname{Res}_N^G(\chi)$. There is $\underline{\alpha} = (\alpha_1, \ldots, \alpha_r) \in \{-1, 1\}^r$ such that $\phi = \chi_{\underline{\alpha}}$. Furthermore,

$${}^x\phi = \chi_1^+ \otimes \chi_2^+ \otimes \cdots \otimes \chi_r^+,$$

where $x = \prod_{\alpha_j = -1} \sigma_j$. It follows that the G-orbit of ϕ is $\mathcal{O} = \{\chi_{\underline{\epsilon}} \mid \underline{\epsilon} \in \{-1, 1\}^r\}$, and Clifford theory gives

$$\operatorname{Res}_N^G(\chi) = \sum_{\underline{\epsilon} \in \{-1,1\}^r} \chi_{\underline{\epsilon}}.$$

On the other hand, N is contained in G^+, thus $\chi_1^+ \otimes \chi_2^+ \otimes \cdots \otimes \chi_r^+$ is a constituent of the restriction to N of either χ^+ or χ^-. Without loss of generality, we choose it to be a constituent of the restriction of χ^+. Now, for $\eta \in \{-1, 1\}$, we set

$$R^\eta = \{(\epsilon_1, \ldots, \epsilon_r) \in \{-1, 1\}^r \mid \epsilon_1 \cdots \epsilon_r = \eta\} \quad \text{and} \quad \mathcal{O}^\eta = \{\chi_{\underline{\epsilon}} \mid \underline{\epsilon} \in R^\eta\}.$$

Let $(\epsilon_1, \ldots, \epsilon_r) \in R^+$. Define $x = \prod_{\epsilon_j = -1} \sigma_j$. Since the number of $1 \leq j \leq r$ with $\epsilon_j = -1$ is even, we deduce that $x \in G^+$ and ${}^x(\chi_1^+ \otimes \chi_2^+ \otimes \cdots \otimes \chi_r^+) = \chi_1^{\epsilon_1} \otimes \cdots \otimes \chi_r^{\epsilon_r}$. In particular, the characters of $\mathcal{O}^+$ lie in the same G^+-orbit. By Clifford theory, $\mathcal{O}$ decomposes into two G^+-orbits of the same size. Since $|\mathcal{O}^+| = |R^+| = |R^-| = |\mathcal{O}^-|$, and $\mathcal{O}^+ \sqcup \mathcal{O}^- = \mathcal{O}$, we deduce that $\mathcal{O}^+$ and $\mathcal{O}^-$ are the two G^+-orbits of $\mathcal{O}$. Again, by Clifford theory, we obtain that

$$\operatorname{Res}_N^{G^+}(\chi^\eta) = \sum_{\underline{\epsilon} \in R^\eta} \chi_{\underline{\epsilon}}. \tag{11}$$

Remark 3.1. Let $g = g_1 \cdots g_r \in G^+$ with $g_j \in G_j$ for $1 \leq j \leq r$. Then g lies in a split class of G if and only if g_j lies in a split class of G_j for all $1 \leq j \leq r$. Indeed, g lies in a split class of G if and only if $\mathrm{C}_G(g) = \mathrm{C}_{G^+}(g)$. Assume some g_j does not belong to a split class of G_j. If $g_j \notin G_j^+$, then $g_j \in \mathrm{C}_G(g) \setminus \mathrm{C}_{G^+}(g)$. If $g_j \in G_j^+$, then there is $x \in \mathrm{C}_{G_j}(g_j) \setminus \mathrm{C}_{G_j^+}(g_j)$, and $x = 1 \cdots 1 x 1 \cdots 1 \in \mathrm{C}_G(g) \setminus \mathrm{C}_{G^+}(g)$. Conversely, suppose that g_j lies in a split class of G_j for all $1 \leq j \leq r$. Then $\mathrm{C}_{G_j}(g_j) = \mathrm{C}_{G_j^+}(g_j)$, so that

$$\mathrm{C}_G(g) = \mathrm{C}_{G_1}(g_1) \cdots \mathrm{C}_{G_r}(g_r) = \mathrm{C}_{G_1^+}(g_1) \cdots \mathrm{C}_{G_r^+}(g_r) = \mathrm{C}_N(g) \leq \mathrm{C}_{G^+}(g) \leq \mathrm{C}_G(g),$$

and $\mathrm{C}_G(g) = \mathrm{C}_{G^+}(g)$, as required.

Proposition 3.2. *Write $G = G_1 \times \cdots \times G_r \subseteq \mathfrak{S}_n$ as above. Let $\chi \in \operatorname{Irr}(G)$ be such that $\chi = \chi^*$, and let χ_j^+ and χ_j^- be as above. For $f \in \mathcal{G}_{|G|}$ such that ${}^f\chi = \chi$, with the notation (8), we have*

$$\varepsilon(\chi, f) = \prod_{j=1}^r \varepsilon(\chi_j, f).$$

Proof. Let $\eta \in \{-1, 1\}$. First, we remark that either ${}^f\chi^\eta = \chi^\eta$ or ${}^f\chi^\eta = \chi^{-\eta}$ because f fixes χ. So, the set $\mathcal{O}$ is f-stable, and f acts on $\{\mathcal{O}^+, \mathcal{O}^-\}$. Furthermore, ${}^f\chi^\eta = \chi^\eta$ if and only if $f(\mathcal{O}^\eta) = \mathcal{O}^\eta$. However, we have $f(\mathcal{O}^+) = \mathcal{O}^+$ if and only if $f(\chi_1^+ \otimes \chi_2^+ \otimes \cdots \otimes \chi_r^+) \in \mathcal{O}^+$ if and only if the number of $1 \leq j \leq r$ such that χ_j^+ are not fixed by f is even. The result follows. ◻

3.2. **Irreducible characters of Y^k and of $(Y^k)^+$.** Write $I = \{0, \ldots, p-1\}$ as above. We now describe how to construct the characters of $\operatorname{Irr}(Y)$ in (4). For $0 \leq j \leq p-2$, define the linear character $\zeta_j : Y \to \mathbb{C}^*$ by setting $\zeta_j(a^u b^v) = \omega_{p-1}^{jv}$, and write ζ for the induced character of any non-trivial character of $\langle a \rangle$ to Y. In particular, $\zeta_j(1) = 1$ for all $0 \leq j \leq p-2$, and $\zeta(1) = p-1$. Set $p^* = (p-1)/2$. Since sgn is the only linear character of Y of order 2, we have $\mathrm{sgn} = \zeta_{p^*}$ and $\zeta_j^* = \zeta_{p^*+j}$. On the other hand, $\{0, \ldots, p-2\} = \{0, \ldots, p^*-1\} \cup \{p^*, p^*+1, \ldots, 2p^*-1\}$. So, in (4) we set $\xi_{p^*} = \zeta$, $\xi_j = \zeta_j$ and $\xi_{p-1-j} = \zeta_{p^*+j}$ for all $j \in \{0, \ldots, p^*-1\}$.

Note that $\langle a \rangle$ and $\langle b^2 \rangle$ are subgroups of Y^+. By an order argument, we obtain that $Y^+ = \langle a \rangle \rtimes \langle b^2 \rangle$. By Clifford theory, the characters ξ_j and ξ_{p-1-j} for $0 \leq j \leq p^*-1$ restrict to the same linear character of Y^+, also denoted by ξ_j, and ξ_{p^*} splits into two irreducible characters $\xi_{p^*}^+$ and $\xi_{p^*}^-$ of degree $(p-1)/2$. Now, we will specify the values of $\xi_{p^*}^+$ and $\xi_{p^*}^-$. For every $0 \leq j \leq p-1$, set $\alpha_j : \langle a \rangle \to \mathbb{C}^*$, $a^k \mapsto \omega_p^{jk}$. Write u for the integer such that $bab^{-1} = a^u$. Since for all $0 \leq j \leq p-1$, $0 \leq k \leq p-1$, and $0 \leq l \leq p-2$

$${}^{b^l}\alpha_j(a^k) = \alpha_j(a^{u^l k}) = \omega_p^{u^l kj} = \alpha_{u^l j}(a^k),$$

we deduce that the $\langle b^2 \rangle$-orbits on $\operatorname{Irr}(\langle a \rangle)$ are

$$\{\alpha_0\}, \quad \{\alpha_j \mid j \in S\} \quad \text{and} \quad \{\alpha_j \mid j \in \overline{S}\},$$

where S is the set of square elements of $(\mathbb{Z}/(p-1)\mathbb{Z})^\times$, and $\overline{S}$ the non-squares. Then by Clifford theory with respect to the normal subgroup $\langle a\rangle$ of Y^+, we can choose the labels such that

$$\operatorname{Res}^{Y^+}_{\langle a\rangle}(\xi^+_{p^*}) = \sum_{j\in S}\alpha_j \quad \text{and} \quad \operatorname{Res}^{Y^+}_{\langle a\rangle}(\xi^-_{p^*}) = \sum_{j\in\overline{S}}\alpha_j, \tag{12}$$

and the inertial subgroup in Y^+ of α_j with $j\neq 0$ is $\langle a\rangle$, hence $\xi^+_{p^*}$ and $\xi^-_{p^*}$ are the induced characters to Y^+ of ξ_j with $j\in S$ and $j\in\overline{S}$, respectively. Thus, $\xi^\pm_{p^*}$ vanishes outside $\langle a\rangle$, and using (12) and [4, Thm. 1 p.75], we obtain

$$(\xi^+_{p^*}-\xi^-_{p^*})(a) = \sum_{j\in S}\alpha_j(a) - \sum_{j\in\underline{S}}\alpha_j(a) = \sum_{j=1}^{p}\left(\frac{j}{p}\right)\omega_p^j = i^{(p-1)/2}\sqrt{p}, \tag{13}$$

where $\left(\frac{j}{p}\right)$ is the Legendre symbol and i is a complex square root of -1. This in particular proves that Y has only one split class, with representative a. We write $a^+=a$, and $a^-\in\langle a\rangle$ for an element conjugate to a in Y but not in Y^+.

Let $k\geq 1$ be an integer. By Remark 3.1, the group Y^k has only one split class with representative $\underline{a}=(a,a,\ldots,a)$. Furthermore, Section §3.1 implies that Y^k has only one sgn-stable character $\xi_{\underline{p}^*(k)}=\xi_{p^*}\otimes\cdots\otimes\xi_{p^*}$, where

$$\underline{p}^*(k) = ((p-1)/2,\ldots,(p-1)/2)\in I^k. \tag{14}$$

Set $\alpha=\xi^+_{p^*}(a)$ and $\beta=\xi^-_{p^*}(a)$, and for $\underline{\epsilon}\in\{-1,1\}^k$, denote by $\mathfrak{n}(\underline{\epsilon})$ the number of $1\leq j\leq k$ such that $\epsilon_j=-1$. Now, Equation (11) gives

$$\begin{aligned}
(\xi^+_{\underline{p}^*(k)}-\xi^-_{\underline{p}^*(k)})(\underline{a}) &= \sum_{\underline{\epsilon}\in R^+}\chi_{\underline{\epsilon}}(a) - \sum_{\underline{\epsilon}\in R^-}\chi_{\underline{\epsilon}}(a)\\
&= \sum_{0\leq 2j\leq k}\sum_{\mathfrak{n}(\underline{\epsilon})=2j}\alpha^{2j}\beta^{k-2j} - \sum_{0<2j+1\leq k}\sum_{\mathfrak{n}(\underline{\epsilon})=2j+1}\alpha^{2j+1}\beta^{k-2j-1}\\
&= \sum_{0\leq 2j\leq k}\binom{k}{2j}(-1)^{2j}\alpha^{2j}\beta^{k-2j} + \sum_{0<2j+1\leq k}\binom{k}{2j+1}(-1)^{2j+1}\alpha^{2j+1}\beta^{k-2j-1}\\
&= \sum_{j=0}^{k}\binom{k}{j}(-1)^j\alpha^j\beta^{k-j}\\
&= (\alpha-\beta)^k
\end{aligned}$$

by Newton's binomial formula. Finally we deduce from (13) that

$$(\xi^+_{\underline{p}^*(k)}-\xi^-_{\underline{p}^*(k)})(\underline{a}) = i^{k(p-1)/2}\sqrt{p^k}. \tag{15}$$

3.3. **Irreducible characters of $(Y^k\wr\mathfrak{S}_w)^+$.** Let k and w be two positive integers. In this section, we set

$$N = Y^k\wr\mathfrak{S}_w \quad \text{and} \quad M=(Y^k)^w.$$

For $\underline{j}=(j_0,\ldots,j_{k-1})\in I^k$, define

$$\pi_p(\underline{j}) = j_{k-1}+j_{k-2}p+\cdots+j_0p^{k-1}.$$

By the uniqueness of the p-adic expansion of a positive integer, we note that the map $\pi_p: I^k\to\{0,\ldots,p^k-1\}$ is a bijection.

We now generalize the Equation (14) defining an involution $*$ on I^k, by setting

$$\underline{j}^* = (p-1-j_0, \dots, p-1-j_{k-1}) \in I^k.$$

Lemma 3.3. *For* $\underline{j} \in I^k$*, one has*

$$\pi_p(\underline{j}^*) = p^k - 1 - \pi_p(\underline{j}) \quad \textit{and} \quad \xi_{\underline{j}}^* = \xi_{\underline{j}^*}.$$

Proof. We have

$$\begin{aligned}\pi_p(\underline{j}^*) &= (p-1-j_{k-1}) + (p-1-j_{k-2})p + \cdots + (p-1-j_0)p^{k-1} \\ &= (p-1)(1+p+\cdots+p^{k-1}) - \pi_p(\underline{j}) \\ &= p^k - 1 - \pi_p(\underline{j}).\end{aligned}$$

□

We follow the convention of Remark 2.1 to label $\operatorname{Irr}(N)$. Moreover, for any $\underline{\lambda} = (\lambda_0, \dots, \lambda_{p^k-1}) \in \mathcal{MP}(p^k, w)$, define

$$\underline{\lambda}^* = (\lambda_{p^k-1}^*, \lambda_{p^k-2}^*, \dots, \lambda_1^*, \lambda_0^*) \in \mathcal{MP}(p^k, w),$$

where λ^* denotes the conjugate partition of λ. To simplify the notation (7), we set $\psi_{\underline{\lambda},k} = \psi_{\underline{\lambda}}$.

Lemma 3.4. *If* $\underline{j} \in I^k$ *and* $\underline{\lambda} = (\lambda_{\underline{j}}; \underline{j} \in I^k) \in \mathcal{MP}(p^k, w)$*, then*

$$\psi_{\underline{\lambda}}^* = \psi_{\underline{\lambda}^*}.$$

Proof. Let $g = \prod_{\underline{j} \in I^k} (y_{\underline{j}}; \sigma_{\underline{j}}) \in N_{k,\underline{\lambda}}$. Then $g = \prod_{\underline{j} \in I^k} \prod_{\tau \in C(\sigma_{\underline{j}})} (y_{\underline{j},\tau}, \tau)$,

$$\operatorname{sgn}(g) = \prod_{\underline{j} \in I^k} \prod_{\tau \in C(\sigma_{\underline{j}})} \operatorname{sgn}(y_{\underline{j},\tau}) \operatorname{sgn}(\tau),$$

because sgn is a group homomorphism, and $\operatorname{sgn}(y_{\underline{j},\tau}) = \operatorname{sgn}(\mathfrak{c}((y_{\underline{j}}; \sigma_{\underline{j}}), \tau))$. Hence

$$\begin{aligned}(E(\xi_{\underline{\lambda}})\chi_{\underline{\lambda}})^*(g) &= \operatorname{sgn}(g) E(\xi_{\underline{\lambda}})(g) \chi_{\underline{\lambda}}(g) \\ &= \operatorname{sgn}(g) \prod_{\underline{j} \in I^k} \prod_{\tau \in C(\sigma_{\underline{j}})} \xi_{\underline{j}}(\mathfrak{c}((y_{\underline{j}}; \sigma_{\underline{j}}), \tau)) \chi_{\underline{\lambda}}(\sigma) \\ &= \operatorname{sgn}(\sigma) \prod_{\underline{j} \in I^k} \prod_{\tau \in C(\sigma_{\underline{j}})} \operatorname{sgn}(\mathfrak{c}((y_{\underline{j}}; \sigma_{\underline{j}}), \tau)) \xi_{\underline{j}}(\mathfrak{c}((y_{\underline{j}}; \sigma_{\underline{j}}), \tau)) \chi_{\underline{\lambda}}(\sigma) \\ &= \operatorname{sgn}(\sigma) \prod_{\underline{j} \in I^k} \prod_{\tau \in C(\sigma_{\underline{j}})} \xi_{\underline{j}}^*(\mathfrak{c}((y_{\underline{j}}; \sigma_{\underline{j}}), \tau)) \chi_{\underline{\lambda}}(\sigma) \\ &= \operatorname{sgn}(\sigma) \prod_{\underline{j} \in I^k} \prod_{\tau \in C(\sigma_{\underline{j}})} \xi_{\underline{j}^*}(\mathfrak{c}((y_{\underline{j}}; \sigma_{\underline{j}}), \tau)) \chi_{\underline{\lambda}}(\sigma) \quad \text{(by Lemma 3.3)} \\ &= \prod_{\underline{j} \in I^k} \prod_{\tau \in C(\sigma_{\underline{j}})} \xi_{\underline{j}^*}(\mathfrak{c}((y_{\underline{j}}; \sigma_{\underline{j}}), \tau)) \chi_{\underline{\lambda}}^*(\sigma),\end{aligned}$$

where $\sigma = \prod \sigma_{\underline{j}}$.

Let $w_{\underline{\lambda}} \in \mathfrak{S}_w$ be the permutation that sends the support of $\lambda_{\underline{j}}$ to that of $\lambda^*_{\underline{j}^*}$. So, $\mathfrak{S}_{\underline{\lambda}^*} = {}^{w_{\underline{\lambda}}}\mathfrak{S}_{\underline{\lambda}}$, and the decomposition of ${}^{w_{\underline{\lambda}}}g$ with respect to $N_{k,\underline{\lambda}^*}$ is

$$ {}^{w_{\underline{\lambda}}}g = \prod_{\underline{j}\in I^k} ({}^{w_{\underline{\lambda}}}y_{\underline{j}}; {}^{w_{\underline{\lambda}}}\tau_{\underline{j}}), $$

and since $\mathfrak{c}((y_{\underline{j}};\sigma_{\underline{j}}),\tau) = \mathfrak{c}(({}^{w_{\underline{\lambda}}}y_{\underline{j}}; {}^{w_{\underline{\lambda}}}\sigma_{\underline{j}}), {}^{w_{\underline{\lambda}}}\tau)$, we deduce that

$$ \begin{aligned} {}^{w_{\underline{\lambda}}}E(\xi_{\underline{\lambda}^*})(g) &= \prod_{\underline{j}\in I^k} \prod_{\tau\in C(\sigma_j)} \xi_{\underline{j}^*}(\mathfrak{c}(({}^{w_{\underline{\lambda}}}y_{\underline{j}}; {}^{w_{\underline{\lambda}}}\sigma_{\underline{j}}), {}^{w_{\underline{\lambda}}}\tau)) \\ &= \prod_{\underline{j}\in I^k} \prod_{\tau\in C(\sigma_j)} \xi_{\underline{j}^*}(\mathfrak{c}((y_{\underline{j}};\sigma_{\underline{j}}),\tau)). \end{aligned} $$

Since ${}^{w_{\underline{\lambda}}}\chi_{\underline{\lambda}^*} = \chi^*_{\underline{\lambda}}$, we obtain

$$ (E(\xi_{\underline{\lambda}})\chi_{\underline{\lambda}})^* = {}^{w_{\underline{\lambda}}}(E(\xi_{\underline{\lambda}^*})\chi_{\underline{\lambda}^*}). \tag{16} $$

It follows that

$$ \begin{aligned} \psi^*_{\underline{\lambda}} &= \operatorname{sgn} \operatorname{Ind}^N_{N_{k,\underline{\lambda}}}(E(\xi_{\underline{\lambda}})\chi_{\underline{\lambda}}) \\ &= \operatorname{Ind}^N_{N_{k,\underline{\lambda}}}(\operatorname{sgn} E(\xi_{\underline{\lambda}})\chi_{\underline{\lambda}}) \\ &= \operatorname{Ind}^N_{N_{k,\underline{\lambda}}}(E(\xi_{\underline{\lambda}})\chi_{\underline{\lambda}})^* \\ &= \operatorname{Ind}^N_{w_{\underline{\lambda}}^{-1}N_{k,\underline{\lambda}^*}w_{\underline{\lambda}}} {}^{w_\lambda}(E(\xi_{\underline{\lambda}^*})\chi_{\underline{\lambda}^*}) \\ &= \operatorname{Ind}^N_{N_{k,\underline{\lambda}^*}}(E(\xi_{\underline{\lambda}^*})\chi_{\underline{\lambda}^*}) \\ &= \psi_{\underline{\lambda}^*}, \end{aligned} $$

as required. ∎

Lemma 3.5. *Let G be a group, and H and K be subgroups of G. Let $x \in G$ be such that x normalizes H and K, and $H \cap \langle x\rangle = K \cap \langle x\rangle = 1$. Let $t \in \mathrm{N}_K(H)$. For every $g \in \langle tx\rangle$, write $g = g_t g_x$ for unique $g_t \in K$ and $g_x \in \langle x\rangle$. Assume there is a representation $\rho : H \to \mathrm{GL}(V)$ that extends to a representation $\widetilde{\rho} : H \rtimes \langle x\rangle \mapsto \mathrm{GL}(V)$. If $\rho({}^{g_t}h) = \rho(h)$ for all $h \in H$ and $g \in \langle tx\rangle$, then the map*

$$ \varphi : H \rtimes \langle tx\rangle \to \mathrm{GL}(V),\ hg \mapsto \widetilde{\rho}(hg_x) $$

is a representation of $H \rtimes \langle tx\rangle$.

Proof. First, we remark that if $g = \langle tx\rangle$, then there is an integer j such that $g = (tx)^j = t\,{}^x t \cdots {}^{x^{j-1}} t x^j$, so $g_t = t\,{}^x t \cdots {}^{x^{j-1}} t \in K$ and $g_x = x^j$ because x normalizes K. Furthermore, this expression is unique because $K \cap \langle x\rangle = 1$. Note also that if $g, g' \in \langle tx\rangle$, then $(gg')_x = g_x g'_x$. Now, for $h, h' \in H$ and $g, g' \in \langle tx\rangle$, we have $hgh'g' = h^g h' gg' = h^g h'(gg')_t g_x g'_x$. Thus,

$$\begin{aligned}\varphi(hgh'g') &= \widetilde{\rho}(h^g h' g_x g'_x) \\ &= \rho(h)\rho({}^{g_t}({}^{g_x}h'))\widetilde{\rho}(g_x g'_x) \\ &= \rho(h)\rho({}^{g_x}h')\rho(g_x g'_x) \\ &= \widetilde{\rho}(h^{g_x} h' g_x g'_x) \\ &= \widetilde{\rho}(hg_x h' g'_x) \\ &= \widetilde{\rho}(hg_x)\widetilde{\rho}(h'g'_x) \\ &= \varphi(hg)\varphi(h'g'),\end{aligned}$$

as required. □

A multipartition $\underline{\lambda} = (\lambda_{\underline{j}};\, j \in I^k) \in \mathcal{MP}(p^k, w)$ is called *symmetric* if $\underline{\lambda}^* = \underline{\lambda}$. We denote by $\mathcal{SP}(p^k, w)$ the set of symmetric multipartitions of $\mathcal{MP}(p^k, w)$. Let $\underline{c} = (c_{\underline{j}} \in \mathbb{N};\, \underline{j} \in I^k)$ be such that $\sum_{\underline{j}} c_{\underline{j}} = w$, and $c_{\underline{j}} = c_{\underline{j}^*}$ for all $\underline{j} \in I^k$. Define

$$\mathcal{P}_{\underline{c}} = \{\underline{\lambda} \in \mathcal{SP}(p^k, w) \mid \ \forall \underline{j} \in I^k, |\lambda_{\underline{j}}| = c_{\underline{j}}\}.$$

For any $\underline{\lambda} \in \mathcal{P}_{\underline{c}}$, the characters $\xi_{\underline{\lambda}}$ and their inertial subgroup $N_{k,\underline{\lambda}}$ depend only on $\underline{c}$. We write $\xi_{\underline{c}}$ and $N_{\underline{c}}$ in the following.

Proposition 3.6. *Let $\underline{\lambda} \in \mathcal{SP}(p^k, w)$ be such that $\lambda_{\underline{p}^*(k)} = \emptyset$. If $f \in \mathcal{K}_{n!/2}$, then $\varepsilon(\psi_{\underline{\lambda}}, f) = 1$. Furthermore,*

$$\varepsilon(\psi_{\underline{\lambda}}, \sigma_{n!/2}) = (-1)^{\frac{(p-1)w}{4}}.$$

Proof. Let $\underline{c} = (c_{\underline{j}};\, \underline{j} \in I^k)$ be such that $c_{\underline{p}^*(k)} = 0$. Furthermore, since $\underline{\lambda}$ is a symmetric multipartition, $c_{\underline{j}} = c_{\underline{j}^*}$ and it follows that

$$w = \sum_{\{\underline{j},\underline{j}^*\},\, \underline{j} \neq \underline{p}^*(k)} (c_{\underline{j}} + c_{\underline{j}^*}) = 2 \sum_{\{\underline{j},\underline{j}^*\},\, \underline{j} \neq \underline{p}^*(k)} c_{\underline{j}},$$

hence w is even. By Clifford theory with respect to the normal subgroup M of N, the characters $\psi_{\underline{\lambda}}$ for $\underline{\lambda} \in \mathcal{P}_{\underline{c}}$ are the constituents of $\mathrm{Ind}_M^N(\xi_{\underline{c}})$. Write $\vartheta_{\underline{c}}$ for the restriction of $\xi_{\underline{c}}$ to M^+. Since $\xi_{\underline{c}}$ is not sgn-stable, we have $\vartheta_{\underline{c}} \in \mathrm{Irr}(M^+)$ by Clifford theory with respect to $M^+ \trianglelefteq M$. Furthermore, Mackey's formula gives

$$\mathrm{Res}_{N^+}^N \mathrm{Ind}_M^N(\xi_{\underline{c}}) = \mathrm{Ind}_{M^+}^{N^+}(\vartheta_{\underline{c}}).$$

Hence, the irreducible characters $\psi_{\underline{\lambda}}^+$ and $\psi_{\underline{\lambda}}^-$ for $\underline{\lambda} \in \mathcal{P}_{\underline{c}}$ appear in the Clifford theory with respect to $M^+ \trianglelefteq N^+$ associated to the character $\vartheta_{\underline{c}}$. Denote by $T_{\underline{c}}$ the inertial subgroup of $\vartheta_{\underline{c}}$ with respect to $M^+ \trianglelefteq N^+$.

Let $\underline{\lambda} \in \mathcal{P}_{\underline{c}}$. The character $\vartheta_{\underline{c}}$ is M-stable, thus

$$\langle \mathrm{Ind}_{M^+}^{G^+}(\vartheta_{\underline{c}}), \psi_{\underline{\lambda}}^+ \rangle = \langle \mathrm{Ind}_{M^+}^{G^+}(\vartheta_{\underline{c}}), \psi_{\underline{\lambda}}^- \rangle.$$

We also have $\operatorname{Ind}_{M^+}^{M}(\vartheta_{\underline{c}}) = \xi_{\underline{c}} + \xi_{\underline{c}}^*$, and the last two characters are N-conjugate, in particular, $\operatorname{Ind}_M^N(\xi_{\underline{c}}) = \operatorname{Ind}_M^N(\xi_{\underline{c}}^*)$. Now, we deduce from Frobenius reciprocity that

$$\begin{aligned}
\langle \operatorname{Ind}_{M^+}^{N^+}(\vartheta_{\underline{c}}), \psi_{\underline{\lambda}}^+\rangle &= \frac{1}{2}\langle \operatorname{Ind}_{M^+}^{N^+}(\vartheta_{\underline{c}}), \psi_{\underline{\lambda}}^+ + \psi_{\underline{\lambda}}^-\rangle \\
&= \frac{1}{2}\langle \operatorname{Ind}_{M^+}^{N^+}(\vartheta_{\underline{c}}), \operatorname{Res}_{N^+}^{N}(\psi_{\underline{\lambda}})\rangle \\
&= \frac{1}{2}\langle \operatorname{Ind}_{M^+}^{N}(\vartheta_{\underline{c}}), \psi_{\underline{\lambda}}\rangle \\
&= \frac{1}{2}\langle \operatorname{Ind}_{M}^{N}\operatorname{Ind}_{M^+}^{M}(\vartheta_{\underline{c}}), \psi_{\underline{\lambda}}\rangle \\
&= \frac{1}{2}\langle \operatorname{Ind}_{M}^{N}(\xi_{\underline{c}} + \xi_{\underline{c}}^*), \psi_{\underline{\lambda}}\rangle \\
&= \frac{1}{2}\langle 2\operatorname{Ind}_{M}^{N}(\xi_{\underline{c}}), \psi_{\underline{\lambda}}\rangle \\
&= \langle \operatorname{Ind}_{M}^{N}(\xi_{\underline{c}}), \psi_{\underline{\lambda}}\rangle.
\end{aligned}$$

Let t and t' be the number of N-conjugate characters of $\xi_{\underline{c}}$ and of N^+-conjugate characters of $\vartheta_{\underline{c}}$, respectively. Then, by Clifford theory, if $e = \langle \operatorname{Ind}_M^N(\xi_{\underline{c}}), \psi_{\underline{c}}\rangle$, then

$$\psi_{\underline{\lambda}}(1) = et\xi_{\underline{c}}(1) \qquad \psi_{\underline{\lambda}}(1)^+ = et'\vartheta_{\underline{c}}(1).$$

Hence, $2t' = t$ because $\vartheta_{\underline{c}}(1) = \xi_{\underline{c}}(1)$ and $2\psi_{\underline{\lambda}}(1)^+ = \psi_{\underline{\lambda}}(1)$.

Note that $N_{\underline{c}}^+ \leq T_{\underline{c}}$ and that $N/N_{\underline{c}} \simeq N^+/N_{\underline{c}}^+$, and

$$t = \frac{|N|}{|N_{\underline{c}}|} = \frac{|N^+|}{|N_{\underline{c}}^+|} \quad \text{and} \quad t' = \frac{|N^+|}{|T_{\underline{c}}|}.$$

Then $T_{\underline{c}}$ is an extension of degree 2 of $N_{\underline{c}}^+$. Since $\underline{\lambda}$ is symmetric and $\lambda_{\underline{p^*(k)}} = \emptyset$, the permutation $w_{\underline{\lambda}}$ defined in the proof of Lemma 3.4 is an involution that exchanges the supports of $\lambda_{\underline{j}}$ and $\lambda_{\underline{j}^*}$ for all $\underline{j} \in I^k$. We remark that $w_{\underline{\lambda}}$ is the same element for any $\underline{\lambda} \in \mathcal{P}_{\underline{c}}$, we will denote it by $w_{\underline{c}}$. Denote by $\theta_{\underline{\lambda}}$ the restriction of $E(\xi_{\underline{\lambda}})\chi_{\underline{\lambda}}$ to $N_{\underline{\lambda},c}^+$ which is irreducible because $E(\xi_{\underline{\lambda}})\chi_{\underline{\lambda}} \neq (E(\xi_{\underline{\lambda}})\chi_{\underline{\lambda}})^*$. Then for all $g \in N^+$,

$$\theta_{\underline{\lambda}}(g) = E(\xi_{\underline{\lambda}})\chi_{\underline{\lambda}}(g) = (E(\xi_{\underline{\lambda}})\chi_{\underline{\lambda}})^*(g) = {}^{w_{\underline{c}}}(E(\xi_{\underline{\lambda}})\chi_{\underline{\lambda}})(g) = {}^{w_{\underline{c}}}\theta_{\underline{\lambda}}(g) \tag{17}$$

by Equation (16). Let $h \in N_{\underline{c}}\backslash N_{\underline{c}}^+$. We set $t_{\underline{c}} = w_{\underline{c}}$ if $w_{\underline{c}} \in N^+$, and $t_{\underline{c}} = hw_{\underline{c}}$ otherwise. We remark that

$$\operatorname{sgn}(w_{\underline{c}}) = (-1)^{w/2}. \tag{18}$$

Now, we define $\underline{\mu}$ as follows. For any $\underline{j} \in I^k$ such that $\pi_p(\underline{j}) < (p^k-1)/2$, set $\mu_{\underline{j}} = (c_{\underline{j}})$ and $\mu_{\underline{j}^*} = (1^{c_{\underline{j}}})$, and $\mu_{\underline{p^*(k)}} = 0$. So, $\mu \in \mathcal{P}_{\underline{c}}$, and $\operatorname{Res}_M^{N_{\underline{c}}}(\theta_{\underline{\mu}}) = \xi_{\underline{c}}$. In particular, Equation (17) gives

$$T_{\underline{c}} = \langle N_{\underline{c}}^+, t_{\underline{c}}\rangle.$$

Since $T_{\underline{c}}$ is a cyclic extension of $N_{\underline{c}}^+$, by [7, 11.22] we can extend $\theta_{\underline{\mu}}$ to a character $\widetilde{\theta}_{\underline{\mu}}$ of $T_{\underline{c}}$. Thus, by Gallagher's theorem (see [7, 6.17]), the constituents of $\operatorname{Ind}_{M^+}^{N^+}(\vartheta_{\underline{c}})$ are

$$\rho_\alpha = \operatorname{Ind}_{T_{\underline{c}}}^{N^+}(\widetilde{\theta}_{\underline{\mu}} \otimes \alpha), \tag{19}$$

where α is any irreducible characters of $T_{\underline{c}}/M^+$ lifted through $T_{\underline{c}} \to T_{\underline{c}}/M^+$.

If we write $H_{\underline{c}} = \langle N_{\underline{c}}, t_{\underline{c}}\rangle$, then $H_{\underline{c}}^+ = T_{\underline{c}}$ and

$$H_{\underline{c}}/M \simeq T_{\underline{c}}/M^+.$$

However, if we choose $h \in M\backslash M^+$, then $t_{\underline{c}}^2 \in M$ and the image of $t_{\underline{c}}$ in $H_{\underline{c}}/M$ has order 2, and can be identified with $w_{\underline{c}}$. It follows that

$$H_{\underline{c}}/M \simeq \mathfrak{S}_{\underline{c}} \rtimes \langle w_{\underline{c}}\rangle.$$

Set $L = \mathfrak{S}_{\underline{c}} \rtimes \langle w_{\underline{c}}\rangle$. We now will prove that the irreducible characters of L are integer-valued. Let $\phi \in \mathrm{Irr}(\mathfrak{S}_{\underline{c}})$. If ϕ is not L-stable, then $\widetilde{\phi} = \mathrm{Ind}_{\mathfrak{S}_{\underline{c}}}^{L}(\phi) \in \mathrm{Irr}(L)$ and $\widetilde{\phi}(g) = \phi(g) + {}^{w_{\overline{c}}}\phi(g) \in \mathbb{Z}$ if $g \in \mathfrak{S}_{\underline{c}}$ and 0 otherwise.

Assume ϕ is L-stable. Then ϕ extends to L (because L is a cyclic extension of $\mathfrak{S}_{\underline{c}}$) and has exactly two extensions $\widetilde{\phi}$ and $\widetilde{\phi}\otimes\varepsilon$, where ε is the lift of the non-trivial character of $\langle w_{\underline{c}}\rangle$. Now, for $\underline{j} \in I^k$ such that $\pi_p(\underline{j}) < (p^k-1)/2$, write $\tau_{\underline{j}}$ for the involution that exchanges the supports of $c_{\underline{j}}$ and $c_{\underline{j}^*}$. One has $w_{\underline{c}} = \prod \tau_{\underline{j}}$. Since $c_{\underline{j}} = c_{\underline{j}^*}$, the group L can be viewed as a subgroup of

$$L' = \prod_{\pi_p(\underline{j})<(p^k-1)/2} \mathfrak{S}_{c_{\underline{j}}} \wr \langle \tau_{\underline{j}}\rangle.$$

Since ϕ is L-stable, we must have $\phi_{\underline{j}} = \phi_{\underline{j}^*}$, and ϕ is $\tau_{\underline{j}}$ stable for all $\underline{j}$. Thus, ϕ is L'-stable and can be extend to L' because L' is a direct product of wreath products isomorphic to $\mathfrak{S}_{c_{\underline{j}}} \wr \mathfrak{S}_2$. Denote by $E(\phi)$ the James-Kerber extension as above. By (6), $E(\phi)$ takes integer values. However, $\mathrm{Res}_L^{L'}(E(\phi))$ is either $\widetilde{\phi}$ or $\widetilde{\phi}\otimes\varepsilon$. Thus, $\widetilde{\phi}$ and $\widetilde{\phi}\otimes\varepsilon$ also take integer values.

The argument above implies that any $\alpha \in \mathrm{Irr}(T_{\underline{c}}/M^+)$ takes integer values. Let $f \in \mathcal{H}_{n!/2}$. By Proposition 2.3, $\theta_{\underline{\mu}}$ is f-fixed. The two extensions of $\theta_{\underline{\mu}}$ to $T_{\underline{c}}$ are $\widetilde{\theta}_{\underline{\mu}}$ and $\widetilde{\theta}_{\underline{\mu}} \otimes \varepsilon$. Thus, either $f(\widetilde{\theta}_{\underline{\mu}}) = \widetilde{\theta}_{\underline{\mu}}$ or $f(\widetilde{\theta}_{\underline{\mu}}) = \widetilde{\theta}_{\underline{\mu}} \otimes \varepsilon$. Then (19) and Lemma 2.2 give $f(\rho_\alpha) = \rho_\alpha$ in the first case, and $f(\rho_\alpha) = \rho_{\alpha\otimes\varepsilon}$ in the second case.

On the other hand, $f(\widetilde{\theta}_{\underline{\mu}}) = \widetilde{\theta}_{\underline{\mu}\otimes\varepsilon}$ if and only if $f(\widetilde{\theta}_{\underline{\mu}}(gt_{\underline{c}})) = -\widetilde{\theta}_{\underline{\mu}}(gt_{\underline{c}})$ for all $g \in N_{\underline{c}}^+$ if and only if there exists $g_0 \in N_{\underline{c}}^+$ such that

$$\widetilde{\theta}_{\underline{\mu}}(g_0 t_{\underline{c}}) \neq 0 \quad \text{and} \quad f(\widetilde{\theta}_{\underline{\mu}}(g_0 t_{\underline{c}})) = -\widetilde{\theta}_{\underline{\mu}}(g_0 t_{\underline{c}}). \tag{20}$$

We will use this criterion to understand the action of f on $\widetilde{\theta}_{\underline{\mu}}$. Set $H = ((Y^k)^+)^w$ and

$$G = H \rtimes \langle w_{\underline{c}}\rangle.$$

For $\underline{j}$ such that $\pi_p(\underline{j}) < (p^k-1)/2$, define $Y_{c_{\underline{j}}} = ((Y^k)^+)^{2c_{\underline{j}}} \leq H$ corresponding to the supports of $\mathfrak{S}_{c_{\underline{j}}}$ and $\mathfrak{S}_{c_{\underline{j}^*}}$. Then G can be viewed as a subgroup of

$$G' = \prod_{\pi_p(\underline{j})<(p^k-1)/2} Y_{c_{\underline{j}}} \wr \langle \tau_{\underline{j}}\rangle,$$

where $\tau_{\underline{j}}$ is defined as before.

The character $\xi_{\underline{c}}$ is not sgn-stable. It takes non-zero values outside M^+, hence outside H, and the restriction $\eta_{\underline{c}}$ of $\xi_{\underline{c}}$ to H is irreducible by Clifford theory with respect to $H \trianglelefteq M^+$. Moreover, if we write $\eta_{\underline{j}} = \mathrm{Res}_{(Y^k)^+}^{Y^k}(\xi_{\underline{j}})$, then $\eta_{\underline{j}^*} = \eta_{\underline{j}}$. In particular,

$$\eta_{\underline{c}} = \prod_{\pi_p(\underline{j})<(p^k-1)/2} \eta_{\underline{j}}^{c_{\underline{j}}} \otimes \eta_{\underline{j}}^{c_{\underline{j}}}.$$

It follows that $\eta_{\underline{c}}$ extends to G^+, and the James-Kerber extension $E(\eta_{\underline{c}})$ has integer values. Hence $\operatorname{Res}_G^{G'}(E(\eta_{\underline{c}}))$ takes integer values, and by Gallagher's theorem, the extension of $\eta_{\underline{c}}$ to G takes a non-zero and integer value on $w_{\underline{c}}$.

Suppose $w \equiv 0 \mod 4$. Then by (18), we take $t_{\underline{c}} = w_{\underline{c}}$. By the previous discussion,

$$\theta_{\underline{c}}(w_{\underline{c}}) = \pm \operatorname{Res}_G^{G'}(E(\eta_{\underline{c}}))(w_{\underline{c}}) \in \mathbb{Z}$$

is a non-zero integer. We deduce from criterion (20) that the characters of N^+ are fixed by all $f \in \mathcal{H}_{n!/2}$.

Suppose $w \equiv 2 \mod 4$. Let $y \in M$. We label the components of y as follows. For $\underline{j} \in I^k$ such that $c_{\underline{j}} \neq 0$, write $y_{\underline{j}} = (y_{\underline{j},1}, \ldots, y_{\underline{j},c_{\underline{j}}}) \in (Y^k)^{c_{\underline{j}}}$, where $y_{\underline{j},i} = (y_{j_1,i}, \ldots, y_{j_k,i}) \in Y^k$ for all $1 \leq i \leq c_{\underline{j}}$. One has

$$\xi_{\underline{c}}(y) = \prod_{\underline{j}} \xi_{\underline{j}}^{c_{\underline{j}}}(y_{\underline{j}}).$$

Let $\underline{u}$ be such that $c_{\underline{u}} \neq 0$. So $\underline{u} \neq \underline{p}^*(k)$, and there is $u_r \neq 0$ with $r \neq (p-1)/2$. Let h be the element of M that is trivial on any component of Y^{kw} except $h_{u_r,1} = b$. Set $h' = {}^{w_{\underline{c}}}h$, which is the element of M all of whose components are trivial except $h'_{u_r^*,1} = b$. Since $h \notin N^+$, by (18) we take $t_{\underline{c}} = hw_{\underline{c}}$. Remark that $w_{\underline{c}}$ normalizes H and M^+, $\langle w_{\underline{c}} \rangle \cap H = \langle w_{\underline{c}} \rangle \cap M^+ = 1$, and $h \in M^+$ normalizes H.

For any $1 \leq j \leq p$, denote by $\mathcal{X}_j$ a representation of Y with character ξ_j. Then

$$R_{\underline{c}} = \prod_{\underline{j}} (\mathcal{X}_{j_1} \otimes \cdots \otimes \mathcal{X}_{j_k})^{c_{\underline{j}}}$$

is a representation of M with character $\xi_{\underline{c}}$. For any positive integer l,

$$t_{\underline{c}}^{2l} = h^l h'^l \quad \text{and} \quad t_{\underline{c}}^{2l+1} = h^{l+1} h'^l w_{\underline{c}}.$$

Then $t_{\underline{c}}$ has order $2(p-1)$ and if $g \in \langle t_{\underline{c}} \rangle$, then g_h (see the notation of Lemma 3.5 with $t = h$) has possibly non zero values only on the components of Y^{kw} labeled by $(u_r, 1)$ and $(u_r^*, 1)$.

However, for any $x \in Y$, we have ${}^x\mathcal{X}_{u_r,1} = \mathcal{X}_{u_r,1}$ and ${}^x\mathcal{X}_{u_r^*,1} = \mathcal{X}_{u_r^*,1}$ because these two representations have dimension 1. Hence, if we denote by $\rho_{\underline{c}}$ the restriction of $R_{\underline{c}}$ to H, then ${}^{g_h}\rho_{\underline{c}} = \rho_{\underline{c}}$ for all $g \in \langle t_{\underline{c}} \rangle$. Thus, by Lemma 3.5, we can extend $\rho_{\underline{c}}$ to $Q = H \rtimes \langle t_{\underline{c}} \rangle$, and the character $\widetilde{\eta}_{\underline{c}}$ of this extension takes the same values as $E(\eta_{\underline{c}})$. Moreover, by Gallagher's theorem, every extension of $\eta_{\underline{c}}$ to Q is of the form $\widetilde{\eta}_{\underline{c}} \otimes \beta$, where β is an irreducible character of $\langle t_{\underline{c}} \rangle$. The irreducible characters of $\operatorname{Irr}(\langle t_{\underline{c}} \rangle)$ are $\beta_j : \langle t_{\underline{c}} \rangle \to \mathbb{C}^*$ for $0 \leq j \leq 2p-3$ defined by $\beta_j(t_{\underline{c}}^l) = \omega_{2(p-1)}^{jl}$. Since $\operatorname{Res}_Q^{T_{\underline{c}}}(\widetilde{\theta}_{\underline{c}})$ is such an extension, there is $0 \leq s \leq 2p-3$ such that

$$\operatorname{Res}_Q^{T_{\underline{c}}}(\widetilde{\theta}_{\underline{c}}) = \widetilde{\rho}_{\underline{c}} \otimes \beta_s. \tag{21}$$

We notice that $\widetilde{\rho}_{\underline{c}}(t_{\underline{c}}^l)$ is equal to $E(\eta_{\underline{c}})(1)$ if l is even, and to $E(\eta_{\underline{c}})(t_{\underline{c}})$ if l is odd. In either case, (6) implies that theses values are positive integers.

Recall that $t_{\underline{c}}^2 = hh'$ is the element whose every component is trivial except those labeled $(u_r, 1)$ and $(u_r^*, 1)$ taking the value b. By (6), we have

$$\widetilde{\theta}_{\underline{c}}(t_{\underline{c}}^2) = \theta_{\underline{c}}(t_{\underline{c}}^2) = -\omega_{p-1}^r(b)^2\theta_{\underline{c}}(1) = -\omega_{p-1}^{2r}\eta_{\underline{c}}(1). \tag{22}$$

Using (21), we also have

$$\widetilde{\theta}_{\underline{c}}(t_{\underline{c}}^2) = \mathrm{Res}_Q^{T_{\underline{c}}}(\widetilde{\theta}_{\underline{c}})(t_{\underline{c}}^2) = \widetilde{\rho}_{\underline{c}}(t_{\underline{c}}^2)\beta_s(t_{\underline{c}}^2) = \omega_{2(p-1)}^{2s}\eta_{\underline{c}}(1) = \omega_{p-1}^s\eta_{\underline{c}}(1).$$

Comparing with (22), we obtain

$$\omega_{p-1}^{s-2r} = -1.$$

However, $-1 \in \langle \omega_{p-1}^2 \rangle = \langle \omega_{(p-1)/2} \rangle$ if and only if $(p-1)/2$ is even. Hence, if $p \equiv 1 \mod 4$, s has to be even, and if $p \equiv 3 \mod 4$, s has to be odd.

On the other hand, (21) gives

$$\widetilde{\theta}_{\underline{c}}(t_{\underline{c}}) = E(\eta_{\underline{c}})(t_{\underline{c}})\beta_s(t_{\underline{c}}) = E(\eta_{\underline{c}})(t_{\underline{c}})\omega_{2(p-1)}^s. \tag{23}$$

Since $E(\eta_{\underline{c}})(t_{\underline{c}})$ is fixed by any $f \in \mathcal{H}_{n!/2}$ because $\eta_{\underline{c}}$ is, and $\omega_{2(p-1)}^s$ is fixed by any $f \in \mathcal{K}_{n!/2}$, we deduce from (20) that the characters $\psi_\lambda^\pm$ are fixed by $f \in \mathcal{K}_{n!/2}$ for all $\underline{\lambda} \in \mathcal{P}_{\underline{c}}$.

Finally, we remark that $\omega_{2(p-1)}^{p-1} = \omega_2 = -1$. Thus, $\omega_{2(p-1)}^p = -\omega_{2(p-1)}$, and $\omega_{2(p-1)}^{2p} = \omega_{2(p-1)}^2$. Then by (23), $\sigma_{n!/2}$ fixes $\widetilde{\theta}_{\underline{c}}(t_{\underline{c}})$ if s is even, that is when $p \equiv 1 \mod 4$ and

$$\sigma_{n!/2}(\widetilde{\theta}_{\underline{c}}(t_{\underline{c}})) = -\widetilde{\theta}_{\underline{c}}(t_{\underline{c}})$$

if s is odd, that is $p \equiv 3 \mod 4$. The result follows from the criterion (20). □

Since $\sqrt{p}$ is a root of the polynomial $x^2 - p \in \mathbb{Q}[x]$, we have $f(\sqrt{p}) = \pm\sqrt{p}$ for $f \in \mathcal{K}_{n!}$. Denote by $\epsilon_f \in \{-1, 1\}$ the sign such that $f(\sqrt{p}) = \epsilon_f\sqrt{p}$.

Proposition 3.7. *Let $\underline{\lambda} \in \mathcal{SP}(p^k, w)$ be such that $\lambda_{\underline{j}} = \emptyset$ for all $\underline{j} \neq \underline{p}^*(k)$. If $f \in \mathcal{K}_{n!}$, then*

$$\varepsilon(\psi_{\underline{\lambda}}, f) = \epsilon_f^{kd} \cdot \varepsilon(\chi_{\lambda_{\underline{p}^*(k)}}, f),$$

where $\varepsilon(\chi_{\lambda_{\underline{p}^*(k)}}, f)$ is defined in (8), and d is the number of diagonal hooks in the Young diagram of $\lambda_{\underline{p}^*(k)}$. Moreover,

$$\varepsilon(\psi_{\underline{\lambda}}, \sigma_{n!/2}) = (-1)^{dk(p-1)/2} \cdot \varepsilon(\chi_{\lambda_{\underline{p}^*(k)}}, \sigma_{n!/2}).$$

Proof. As in the proof of the Proposition 3.7, we consider the group $H = ((Y^k)^+)^w$. Write $\xi = \xi_{\underline{p}^*(k)}^w \in \mathrm{Irr}(M)$. This is the unique split character of M by (10) and §3.2. Denote by ξ^+ the constituent of $\mathrm{Res}_{M^+}^M(\xi)$ such that $(\xi_{\underline{p}^*(k)}^+)^w \in \mathrm{Irr}(H)$ is a constituent of $\mathrm{Res}_H^{M^+}(\xi^+)$. First, we remark that the subgroup $U = M^+ \rtimes \mathfrak{A}_w$ is a normal subgroup of N^+ because it has index 2. Moreover, the inertial subgroup in U of ξ^+ and ξ^- is U. Let $s \in N^+ \backslash U$. Then $s = (h; \tau)$ with $h \in M \backslash M^+$ and $\tau \in \mathfrak{S}_w \backslash \mathfrak{A}_w$, and ${}^s\xi^+ = \xi^-$. It follows that

$${}^s\,\mathrm{Ind}_{M^+}^U(\xi^+) = \mathrm{Ind}_{M^+}^U(\xi^-),$$

because $M^+ \trianglelefteq N^+$ and $U \trianglelefteq N^+$. Furthermore, $\mathrm{Ind}_{M^+}^U(\xi^+)$ and $\mathrm{Ind}_{M^+}^U(\xi^-)$ have no constituents in common by Clifford theory with respect to $M^+ \trianglelefteq U$. It follows that if χ is a constituent of $\mathrm{Ind}_{M^+}^U(\xi^+)$, then ${}^s\chi \neq \chi$. Hence, $\mathrm{Ind}_U^{N^+}(\chi)$ is irreducible by Clifford theory with respect to $U \trianglelefteq N^+$. By the transitivity of induction and Mackey's formula,

$$\mathrm{Res}^N_{N^+}\mathrm{Ind}^N_M(\xi) = \mathrm{Res}^N_{N^+}\mathrm{Ind}^N_M\mathrm{Ind}^M_{M^+}(\xi^+) = \mathrm{Res}^N_{N^+}\mathrm{Ind}^N_{M^+}(\xi^+) = \mathrm{Ind}^{N^+}_{M^+}(\xi^+).$$

Hence, $\psi^+_{\underline{\lambda}}$ and $\psi^-_{\underline{\lambda}}$ restrict to U into two irreducible components. We write $\psi^{\pm}_{\underline{\lambda},\pm}$ for the constituent of $\mathrm{Res}^{N^+}_U(\psi^{\pm}_{\underline{\lambda}})$ which belongs to $\mathrm{Ind}^U_{M^+}(\xi^{\pm})$.

Now we show how to extend ξ^+ and ξ^- to U. Consider the wreath product $V = H \rtimes \mathfrak{A}_w$. Denote by $\nu^+ = (\xi^+_{\underline{p}^*(k)})^w \in \mathrm{Irr}(H)$ and $\nu^- = {}^s\nu^+$. By Clifford theory with respect to $H \trianglelefteq M^+$ and the previous choice of labeling, we have

$$\mathrm{Ind}^{M^+}_H(\nu^+) = \xi^+ \quad \text{and} \quad \mathrm{Ind}^{M^+}_H(\nu^-) = \xi^-.$$

Write $E(\nu^+)$ for the James-Kerber extension of ν^+ to V. Therefore, Mackey's formula gives

$$\mathrm{Res}^U_{M^+}\mathrm{Ind}^U_V(E(\nu^+)) = \mathrm{Ind}^{M^+}_H(\nu^+) = \xi^+.$$

Thus, $\mathcal{V}^+ = \mathrm{Ind}^U_V(E(\nu^+))$ is an extension of ξ^+ to U. By Gallagher's theorem [7, Corollary 6.17], the constituents of $\mathrm{Ind}^U_{M^+}(\xi^+)$ are of the form $\zeta_{\mu,+} = \mathcal{V}^+ \otimes \chi_\mu$ if $\mu \neq \mu^*$ and $\zeta^{\pm}_{\mu,+} = \mathcal{V}^+ \otimes \chi^{\pm}_\mu$ if $\mu = \mu^*$. Here, χ_μ and $\chi^{\pm}_\mu$ are the irreducible characters of $\mathfrak{A}_w$. If we set $\mathcal{V}^- = {}^s\mathcal{V}^+$, then $\mathcal{V}^+ \neq \mathcal{V}^-$ because it is a constituent of $\mathrm{Ind}^U_{M^+}(\xi^-)$. Thus,

$${}^s\psi^{\pm}_{\underline{\lambda},\pm} = {}^s(\mathcal{V}^{\pm} \otimes \chi^{\pm}_{\lambda_{\underline{p}^*(k)}}) = {}^h(\mathcal{V}^{\pm}) \otimes {}^\tau(\chi^{\pm}_{\lambda_{\underline{p}^*(k)}}) = \mathcal{V}^{\mp} \otimes \chi^{\mp}_{\lambda_{\underline{p}^*(k)}} = \psi^{\mp}_{\underline{\lambda},\mp},$$

and

$$\mathrm{Res}^{N^+}_U(\psi^+_{\underline{\lambda}}) = \psi^+_{\underline{\lambda},+} + \psi^-_{\underline{\lambda},-} \quad \text{and} \quad \mathrm{Res}^{N^+}_U(\psi^-_{\underline{\lambda}}) = \psi^-_{\underline{\lambda},+} + \psi^+_{\underline{\lambda},-}. \tag{24}$$

Consider the element $g = (u,\pi)$ such that the cycle lengths of π are the diagonal hook lengths of $\lambda_{\underline{p}^*(k)}$, and u is such that every cycle of g has cyclic product equal to $\underline{a}$. Then $g \in U$ and

$$\mathcal{V}^+(g) = \mathrm{Ind}^U_V(E(\nu^+))(g) = \sum_{\substack{t\in[U/V]\\ {}^tg\in V}} E(\nu^+)({}^tg) = \sum_{\substack{t\in[U/V]\\ {}^tg\in V}} \prod_{\gamma\in C(\pi)} \nu^+(\mathfrak{c}({}^tg,\gamma)).$$

However, $U/V \simeq M^+/H$. Hence, we can take for transversal of U mod V the set

$$[U/V] = \{t_{\underline{\alpha}} = (b^{\alpha_1},\ldots,b^{\alpha_w}) \,|\, \underline{\alpha} \in \{0,1\}^w,\ \alpha_1+\cdots+\alpha_w \equiv 0 \pmod 2\}.$$

Moreover, ${}^{t_\alpha}g \in U$ if and only if $b^{\alpha_j}u_jb^{-\alpha_{\pi^{-1}(j)}} \in (Y^k)^+$ for all $1 \leq j \leq w$, if and only if $b^{\alpha_j}b^{-\alpha_{\pi^{-1}(j)}} \in (Y^k)^+$ (because ${}^{b^{\alpha_j}}u_j \in (Y^k)^+$) if and only if $b^{\alpha_j-\alpha_{\pi^{-1}(j)}} \in (Y^k)^+$, *i.e.* $\alpha_j = \alpha_{\pi^{-1}(j)}$, that is all α_j are equal on the cycles of π. Denote by T the set of elements of $[U/V]$ that satisfy this property. By [8, 4.2.6], for any $\gamma \in C(\pi)$ and $t_{\underline{\alpha}} \in T$,

$$\mathfrak{c}({}^{t_{\underline{\alpha}}}g,\gamma) = {}^{b^{\alpha_\gamma}}\mathfrak{c}(g,\gamma) = {}^{b^{\alpha_\gamma}}\underline{a}.$$

Thus

$$\mathcal{V}^+(g) = \sum_{t_{\underline{\alpha}}\in T}\prod_{\gamma\in C(\pi)} \nu^+({}^{b^{\alpha_\gamma}}\underline{a}).$$

Let γ_0 be the cycle of $C(\pi)$ whose support contains 1, and define $y \in M$ such that $y_i = b$ if $i \in \mathrm{supp}(\gamma_0)$ and 1 otherwise. Since $|\gamma_0|$ is odd, $y \in M\backslash M^+$. Using that $\mathcal{V}^-(g) = \mathcal{V}^+({}^yg)$, the same computation as above shows that

$$\mathcal{V}^-(g) = \sum_{t_{\underline{\alpha}}\in\overline{T}}\prod_{\gamma\in C(\pi)} \nu^+({}^{b^{\alpha_\gamma}}\underline{a}),$$

where $\overline{T}$ is the set of $t_{\underline{\alpha}}$ such that the α_i are constant on the cycle of π and $\alpha_1+\cdots+\alpha_w \equiv 1 \pmod 2$. Since the lengths of the cycles of π are odd, we have

$$\sum_{j=1}^{w} \alpha_j \equiv \sum_{\gamma\in C(\pi)} \alpha_\gamma \pmod 2$$

for every $t_{\underline{\alpha}} \in T \cup \overline{T}$. Therefore, by a computation similar to that proving (15), we obtain

$$(\mathcal{V}^+ - \mathcal{V}^-)(g) = i^{dk(p-1)/2}\sqrt{p^{dk}}, \tag{25}$$

where $d = |C(\pi)|$. By [8, 2.5.13], we also have

$$(\chi^+_{\lambda_{\underline{p}^*(k)}} - \chi^-_{\lambda_{\underline{p}^*(k)}})(\pi) \neq 0. \tag{26}$$

Let $f \in \mathcal{G}_{n!/2}$. By (24), if ${}^f\psi^{\pm}_{\underline{\lambda},\pm} = \psi^{\pm}_{\underline{\lambda},\pm}$ or ${}^f\psi^{\pm}_{\underline{\lambda},\pm} = \psi^{\mp}_{\underline{\lambda},\mp}$, then $\varepsilon(\psi_{\underline{\lambda},f}) = 1$, and if ${}^f\psi^{\pm}_{\underline{\lambda},\pm} = \psi^{\mp}_{\underline{\lambda},\pm}$, then $\varepsilon(\psi_{\underline{\lambda},f}) = -1$. However,

$$\psi^{\pm}_{\underline{\lambda},\pm}(g) = \mathcal{V}^{\pm}(g)\chi^{\pm}_{\lambda_{\underline{p}^*(k)}}(\pi) \tag{27}$$

is non-zero, and $f(\psi^{\pm}_{\underline{\lambda},\pm}(g)) = f(\mathcal{V}^{\pm}(g)) \cdot f(\chi^{\pm}_{\lambda_{\underline{p}^*(k)}})$. Thus, by equalities (25),(26) and (27), we have ${}^f\psi^{\pm}_{\underline{\lambda},\pm} = \psi^{\mp}_{\underline{\lambda},\pm}$ if and only if $f(\mathcal{V}^{\pm}(g)) = \mathcal{V}^{\pm}(g)$ and $f(\chi^{\pm}_{\lambda_{\underline{p}^*(k)}}(\pi)) = \chi^{\mp}_{\lambda_{\underline{p}^*(k)}}(\pi)$ or $f(\mathcal{V}^{\pm}(g)) = \mathcal{V}^{\mp}(g)$ and $f(\chi^{\pm}_{\lambda_{\underline{p}^*(k)}}(\pi)) = \chi^{\pm}_{\lambda_{\underline{p}^*(k)}}(\pi)$.

Now, if $f \in \mathcal{K}_{n!/2}$, then $f(i) = i$. Note also that $\sigma_{n!/2}(i) = (-1)^{(p-1)/2} i$ and that $\sigma_{n!/2}$ fixes $\sqrt{p}$. The result then follows from (25). □

Let $\underline{\lambda} \in \mathcal{SP}(p^k, w)$. Let $w' = |\lambda_{\underline{p}^*(k)}|$ and $w'' = w - w'$. Define $\underline{\lambda}'' \in \mathcal{SP}(p^k, w')$ such that each part is empty except $\lambda''_{\underline{p}^*(k)} = \lambda_{\underline{p}^*(k)}$, and $\underline{\lambda}' \in \mathcal{SP}(p^k, w'')$ such that $\lambda'_{\underline{j}} = \lambda_{\underline{j}}$ when $\underline{p} \neq \underline{p}^*(k)$ and $\lambda_{\underline{p}^*(k)} = \emptyset$. Denote by $\psi_{\underline{\lambda}'}$ and $\psi_{\underline{\lambda}''}$ the corresponding irreducible characters of $N_{k,w'}$ and $N_{k,w''}$, respectively.

Theorem 3.8. *Let $\underline{\lambda} \in \mathcal{SP}(p^k, w)$. Then for any $f \in \mathcal{G}_{n!/2}$,*

$$\varepsilon(\psi_{\underline{\lambda}}, f) = \varepsilon(\psi_{\underline{\lambda}'}, f) \cdot \varepsilon(\psi_{\underline{\lambda}''}, f).$$

Proof. Let $\underline{\lambda} \in \mathcal{SP}(p^k, w)$. Assume $\underline{\lambda}' \neq \emptyset$ and $\underline{\lambda}'' \neq \emptyset$. Set $\underline{c} = (|\lambda_{\underline{j}}|,\ \underline{j} \in I^k)$, $\underline{c}' = (0, \ldots, 0, c_{\underline{p}^*(k)}, 0, \ldots, 0)$ and $\underline{c}''$ such that the coordinates of $\underline{c}$ and $\underline{c}''$ are the same, except $c''_{\underline{p}^*(k)} = 0$. Since $\underline{\lambda}'' \neq \emptyset$, one has $\xi^*_{\underline{c}} \neq \xi_{\underline{c}}$, and the restriction $\vartheta_{\underline{c}}$ of $\xi_{\underline{c}}$ to M^+ is irreducible. By Mackey's formula,

$$\operatorname{Res}^{N}_{N^+} \operatorname{Ind}^{N}_{M}(\xi_{\underline{c}}) = \operatorname{Ind}^{N^+}_{M^+}(\vartheta_{\underline{c}}).$$

Thus, $\psi^+_{\underline{\lambda}}$ and $\psi^-_{\underline{\lambda}}$ appear in the Clifford theory attached to $\vartheta_{\underline{c}}$ with respect to $M^+ \trianglelefteq N^+$. Moreover, by an argument similar to the one in the proof of Proposition 3.6, the inertial group of $\vartheta_{\underline{c}}$ is an extension of degree 2 of $N^+_{\underline{c}}$. Let $t_{\underline{c}''}$ be an element of $N^+_{\underline{c}''}$ as in the proof of Proposition 3.6, and $H_{\underline{c}''} = \langle N_{\underline{c}''}, t_{\underline{c}''}\rangle$. Consider

$$H_{\underline{c}} = N_{w'} \times H_{\underline{c}''}. \tag{28}$$

Then the elements of $H^+_{\underline{c}} = \langle (N_{w'} \times N_{\underline{c}''})^+, t_{\underline{c}''}\rangle$ fix $\vartheta_{\underline{c}}$ and this group is an extension of degree 2 of $(N_{w'} \times N_{\underline{c}''})^+ = N_{\underline{c}}$. Thus, the inertial subgroup of $\vartheta_{\underline{c}}$ is $H^+_{\underline{c}}$.

On the other hand, $E(\xi_{\underline{c}}\chi_{\underline{\lambda}})$ is not $H_{\underline{c}}$-stable. Hence, $\widetilde{\psi}_{\underline{\lambda}} = \mathrm{Ind}_{N_{\underline{c}}}^{H_{\underline{c}}}(E(\xi_{\underline{c}})\chi_{\underline{\lambda}})$ is irreducible and by Mackey's formula

$$\mathrm{Res}_{H_{\underline{c}}^+}^{H_{\underline{c}}}(\widetilde{\psi}_{\underline{\lambda}}) = \mathrm{Ind}_{N_{\underline{c}}^+}^{H_{\underline{c}}^+}(\theta_{\underline{\lambda}}) = \theta_{\underline{\lambda}}^+ + \theta_{\underline{\lambda}}^-,$$

where $\theta_{\underline{\lambda}}$ is the restriction of $E(\xi_{\underline{c}})\chi_{\underline{\lambda}}$ to $N_{\underline{c}}^+$. Again, by Mackey's formula,

$$\begin{aligned}\psi_{\underline{\lambda}}^+ + \psi_{\underline{\lambda}}^- &= \mathrm{Res}_{N^+}^{N}(\psi_{\underline{\lambda}}) \\ &= \mathrm{Res}_{N^+}^{N}\mathrm{Ind}_{N_{\underline{c}}}^{N}(E(\xi_{\underline{c}})\chi_{\underline{\lambda}}) \\ &= \mathrm{Ind}_{N_{\underline{c}}^+}^{N^+}(\theta_{\underline{\lambda}}) \\ &= \mathrm{Ind}_{N_{\underline{c}}^+}^{N^+}(\theta_{\underline{\lambda}}^+) + \mathrm{Ind}_{N_{\underline{c}}^+}^{N^+}(\theta_{\underline{\lambda}}^-).\end{aligned} \tag{29}$$

In particular, we can choose the label such that $\psi_{\underline{\lambda}}^{\eta} = \mathrm{Ind}_{N_{\underline{c}}^+}^{N^+}(\theta_{\underline{\lambda}}^{\eta})$ for $\eta \in \{-1, 1\}$.

Let $f \in \mathcal{H}_{n!/2}$. By Lemma 2.2, one has

$$\varepsilon(\psi_{\underline{\lambda}}, f) = \varepsilon(\widetilde{\psi}_{\underline{\lambda}}, f). \tag{30}$$

Note that $E(\xi_{\underline{c}})\chi_{\underline{\lambda}} = E(\xi_{\underline{c}'})\chi_{\underline{\lambda}'} \otimes E(\xi_{\underline{c}''})\chi_{\underline{\lambda}''}$, hence

$$\widetilde{\psi}_{\underline{\lambda}} = \psi_{\underline{\lambda}'} \otimes \widetilde{\psi}_{\underline{\lambda}''}, \tag{31}$$

where $\widetilde{\psi}_{\underline{\lambda}''} = \mathrm{Ind}_{N_{\underline{c}''}}^{H_{\underline{c}''}}(E(\xi_{\underline{c}''})\chi_{\underline{\lambda}''}) \in \mathrm{Irr}(H_{\underline{c}''})$. We remark that the computations (29) and (30) applied to $N_{k,w''}$ give

$$\varepsilon(\psi_{\underline{\lambda}''}, f) = \varepsilon(\widetilde{\psi}_{\underline{\lambda}''}, f). \tag{32}$$

Now, $E(\xi_{\underline{c}})\chi_{\underline{\lambda}}$ is f-stable, thus $\widetilde{\psi}_{\underline{\lambda}}$ also is by Lemma 2.2. Applying Proposition 3.2 with respect to the direct product (28), and using (31) and (32) we obtain that

$$\varepsilon(\widetilde{\psi}_{\underline{\lambda}}, f) = \varepsilon(\psi_{\underline{\lambda}'}, f) \cdot \varepsilon(\widetilde{\psi}_{\underline{\lambda}''}, f) = \varepsilon(\psi_{\lambda'}, f) \cdot \varepsilon(\psi_{\lambda''}, f). \tag{33}$$

The result follows from (30) and (33). □

4. Alternating groups: The global case

Let $\lambda = \lambda^*$. Denote by $\mathcal{C}_\lambda$ the conjugacy classes of $\mathfrak{S}_n$ of type $\mathfrak{D}(\lambda)$, that is, the lengths of the elements of $\mathfrak{D}(\lambda)$ are the cycle lengths of any element $x \in \mathcal{C}_\lambda$. Recall that the classes $\mathcal{C}_\lambda$ of $\mathfrak{S}_n$ split into two classes $\mathcal{C}_\lambda^+$ and $\mathcal{C}_\lambda^-$ of $\mathfrak{A}_n$, and that the restriction to $\mathfrak{A}_n$ of the irreducible character χ_λ splits into two constituents χ_λ^+ and χ_λ^- that take the same (integer) value on every class except on $\mathcal{C}_\lambda^{\pm}$, and by [8, 2.5.13] the labeling can be chosen such that for all η, $\nu \in \{-1, 1\}$

$$\chi_\lambda^{\eta}(x_\lambda^{\nu}) = \frac{1}{2}\left((-1)^{(n-d_\lambda)/2} + \eta\nu i^{(n-d_\lambda)/2}\sqrt{\prod_{h\in\mathfrak{D}(\lambda)} h}\right), \tag{34}$$

where x_λ^{ν} is a representative of $\mathcal{C}_\lambda^{\nu}$ and $d_\lambda = |\mathfrak{D}(\lambda)|$.

For any field automorphism f, if α is a root of $x^2 - q \in \mathbb{Q}[x]$, then $f(\alpha)$ is also a root of $x^2 - q$. We denote by $\varepsilon(\alpha, f) \in \{-1, 1\}$ the sign such that

$$f(\alpha) = \varepsilon(\alpha, f)\alpha. \tag{35}$$

Note that when $\lambda = \lambda^*$ and $f \in \mathcal{H}_{n!/2}$,

$$\varepsilon(\chi_\lambda, f) = \varepsilon\left(i^{(n-d_\lambda)/2}\sqrt{\prod_{h\in\mathfrak{D}(\lambda)} h}, f\right). \tag{36}$$

4.1. **Action of Galois automorphisms on square roots.** Let m be an odd number. For any integer r, we write $\left(\frac{r}{m}\right)$ for the Jacobi symbol.

Proposition 4.1. *Let m be an odd number, and f be a Galois automorphism. Denote by r an integer prime to m such that $f(\omega_m) = \omega_m^r$. Then*

$$f(\sqrt{m}) = \varepsilon(i, f)^{\frac{m-1}{2}}\left(\frac{r}{m}\right)\sqrt{m}.$$

Proof. Write $m = p_1^{a_1}\cdots p_s^{a_s}$ for the prime factorisation of m. Define by E and F respectively the set of indices $1 \leq j \leq s$ such that $p_j \equiv 1$ or 3 modulo 4.

Suppose $m \equiv 1 \mod 4$. Then $\sum_{j\in F} a_j$ is even, and

$$\sqrt{m} = \prod_{j\in E}\sqrt{p_j}^{a_j}\cdot\left(\eta\prod_{j\in F}(i\sqrt{p_j})^{a_j}\right), \tag{37}$$

where $\eta = -1$ if $\sum_{j\in F} a_j \equiv 2 \mod 4$ and $\eta = 1$ otherwise. Since f is a field automorphism fixing η, we deduce

$$\varepsilon(\sqrt{m}, f) = \prod_{j\in E}\varepsilon(\sqrt{p}_j, f)^{a_j}\cdot\prod_{j\in F}\varepsilon(i\sqrt{p}_j, f)^{a_j}. \tag{38}$$

Now, if we set $q_j = \sqrt{p_j}$ if $j \in E$ and $q_j = i\sqrt{p_j}$ if $j \in F$, then [4, Thm. 1] gives

$$\sum_{t=1}^{p_j-1}\left(\frac{t}{p_j}\right)\omega_{p_j}^t = q_j$$

Furthermore, one has $\omega_{p_j} = \omega_m^{m/p_j}$, so $f(\omega_{p_j}) = \omega_{p_j}^r$, and

$$f(q_j) = \sum_{t=1}^{p_j-1}\left(\frac{t}{p_j}\right)\omega_{p_j}^{rt} = \left(\frac{r}{p_j}\right)q_j$$

by [4, Prop. 6.3.1]. Hence, $\varepsilon(q_j, f) = \left(\frac{r}{p_j}\right)$ and the result follows from (38) and the definition of the Jacobi symbol.

Suppose that $m \equiv 3 \mod 4$. Then $\sum_{j\in F} a_j$ is odd, and in the formula (37), η is now equal to i up to a sign. When the formula (38) is multiplied by $\varepsilon(i, f)$, the result follows. $\square$

4.2. **Combinatorics of symmetric partitions.** Recall a partition λ is completely determined by the *rim* of its Young diagram $Y(\lambda)$, a path constituted of vertical and horizontal dashes of length one. Then λ can, by the association of 0 (resp. 1) to a vertical (resp. horizontal) dash of length one, be expressed by its *partition sequence* Λ. This is an infinite sequence taking its values in $\{0, 1\}$ and of the form $\overline{0}\cdots\overline{1}$, where $\overline{0}$ and $\overline{1}$ mean an infinite sequence of left-trailing and right-trailing 0s and of 1s, respectively. We refer the reader to Example 4.2.

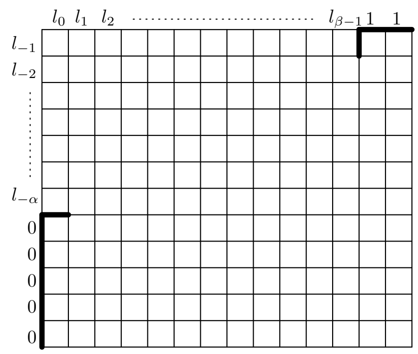


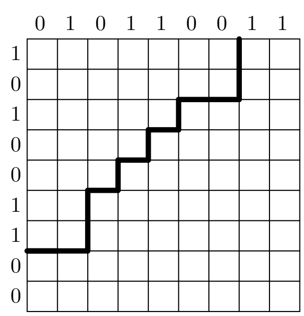
FIGURE 1. Construction of the rim from the sequence

Let Λ be the partition sequence associated to λ. Denote by α and β the numbers of zeroes and ones between the leftmost 1 and the rightmost 0 coming after it when we read the sequence from the left-to-right. Then there are $\alpha+\beta$ elements in the sequence between $\overline{0}$ and $\overline{1}$. We write

$$\Lambda = \overline{0} l_{-\alpha} l_{-\alpha+1} \cdots l_{-1} l_0 \cdots l_{\beta-1} \overline{1} = (l_u)_{u\in\mathbb{Z}}. \tag{39}$$

In particular $l_{-\alpha}=1$ and $l_\beta = 0$. If there is no 0 after the first 1, then $\alpha=\beta=0$ and the sequence is $\overline{0}\,\overline{1}$ and corresponds to the empty partition. The bijection between this labeling of partition sequences and partitions can be represented graphically as in Figure 1.

Example 4.2. *Consider the partition* $\lambda = (7^2, 5, 4, 3, 2^2)$.



The partition sequence of λ *is* $\Lambda = \overline{0}11001010101100\overline{1}$. *We have* $\alpha=\beta=7$, *and following the preceding convention,* l_0 *and* l_{-1} *are the numbers directly at the right and the left of the dash* 1100101|0101100. *Note that in the accompanying figure the partition sequence has been projected to the left-and-top border of the Young diagram.*

Furthermore, by [15, Lemma 2.2], the partition sequence of λ^*, denoted by Λ^*, is obtained from Λ by reading Λ from the right to the left with 0s and 1s interchanged. In other words

$$\Lambda^* = \overline{0}(1-l_{\beta-1})(1-l_{\alpha-2})\cdots(1-l_{-\alpha})\overline{1} = (1-l_{-u-1})_{u\in\mathbb{Z}}. \tag{40}$$

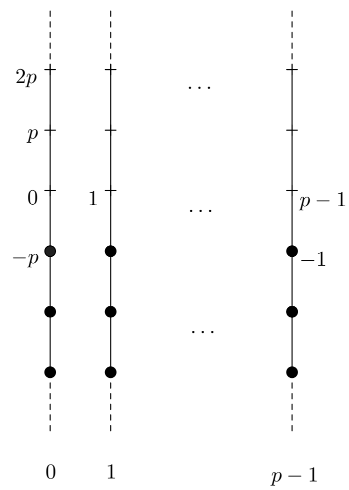


FIGURE 2. p-abacus of the empty partition

We now describe $\mathfrak{D}(\lambda)$ the diagonal hooks of λ using Λ. For $\delta \in \{0, 1\}$, write

$$H_\delta = \{0 \leq j \leq \beta - 1 \mid l_j = \delta\} \quad \text{and} \quad K_\delta = \{-\alpha \leq j \leq -1 \mid l_j = \delta\}.$$

Note that if $h = |H_0|$, then $|H_1| = \beta - h$ and $|K_1| = \beta - |H_1| = \beta - (\beta - h) = h$. Hence, $|H_0| = |K_1|$. On the other hand, by [15, p. 9] each hook of λ corresponds to a pair (i, j) such that $-\alpha \leq i < j \leq \beta - 1$ with $l_i = 1$ and $l_j = 0$. Such a hook $h_{(i,j)}$ has length $|j - i|$. In particular, the longest hook of λ is $h_{-\alpha,\beta-1}$ and it has to be the first diagonal hook of λ. When we remove it from λ, we obtain a new partition with the same sequence as λ except that $l_{-\alpha} = 0$ and $l_{\beta-1} = 1$. Since $|H_0| = |K_1|$, when we iterate this process $|H_0|$ times, we obtain the empty partition. In fact, we have removed from λ all diagonal hooks one by one. Thus, the diagonal hooks of λ are labeled by H_0 (and K_1).

Example 4.3. *In Example 4.2, we see that there are four* 0*s on the horizontal and four 1s on the vertical axe, corresponding to the four diagonal hooks of* λ.

Let p be an odd prime. We now consider a p-abacus with p runners, labeled from 0 to $p - 1$ from left-to-right. We choose a position on the first runner and we label it by 0. Then we label positions by integers moving left-to-right to the runner $p - 1$, then wrapping around to runner 0 one row above. In particular, the positions on the runner 0 are labeled by $\cdots, -3p, -2p, -p, 0, p, 2p \cdots$. Now, we fill the abacus so that there is a bead at the position labeled by j if and only if $l_j = 0$. For example, Figure 2 is the p-abacus of the empty partition.

We can also read the diagonal hooks $\mathfrak{D}(\lambda)$ directly off of the p-abacus: they are parametrized by the beads labeled by a non-negative integer. More precisely, if we set

$$l_{\gamma,j} = l_{jp+\gamma} \tag{41}$$

for all $j \in \mathbb{Z}$, then the beads on runner γ can be interpreted as the partition sequence $(l_{\gamma,j})_{j\in\mathbb{Z}}$ of a partition λ_γ.

Remark 4.4. In general, this labeling of the sequence is not compatible with that of (39). Indeed, there is no reason that there should be exactly the same number of 1s below $l_{\gamma,0}$ as the number of 0s above it.

We define λ_γ as the partition whose partition sequence can be read off the beads on runner γ. That is, the abacus position $\gamma + mp$ corresponds to a so-called γ-position m; that is, if λ has a bead in abacus position $\gamma + mp$ then λ_γ has a bead in position m on runner γ. Then $\mathcal{Q}uo_p(\lambda)$ is the *p-quotient* of λ, that is, the sequence $(\lambda_0, \ldots, \lambda_{p-1})$.

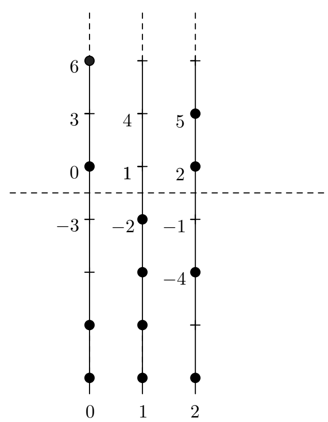
Now, for $0 \le \gamma \le p-1$, define

$$\mathcal{X}_\gamma = \{j \in \mathbb{Z} \mid pj + \gamma \ge 0 \text{ and } l_{\gamma,j} = 0\}.$$

Therefore, each $j \in \mathcal{X}_\gamma$ labels a diagonal hook of λ. Such hooks will be called *diagonal hooks of* λ *arising from runner* γ. Let $\mathcal{C}or_p(\lambda)$ be the *p-core* of λ, that is, the partition one obtains by removing all the p-hooks of λ. Such a partition is well-defined [8, p. 79]. Then, λ is uniquely determined by $\mathcal{C}or_p(\lambda)$ and $\mathcal{Q}uo_p(\lambda)$.

Let $\mathcal{C}or_p^{(0)}(\lambda) = \mathcal{C}or_p(\lambda)$. Now consider the p-tuple of p-abaci, one for each of the $\lambda_\gamma \in \mathcal{Q}uo_p(\lambda)$ above. Then $\mathcal{C}or_p^{(1)}(\lambda)$ will be a p-tuple defined to be the sequence $(\mathcal{C}or_p(\lambda_\gamma))$ for $0 \le \gamma \le p-1$. This naturally induces a p^2-tuple $(\mathcal{Q}uo_p(\lambda_0), \cdots, \mathcal{Q}uo_p(\lambda_{p-1}))$, that defines $\mathcal{C}or_p^{(2)}(\lambda)$. Iterating this process we define $\mathcal{C}or_p^{(k)}(\lambda)$ for any non-negative integer k, and obtain at the end the p-core tower $\mathcal{CT}(\lambda)$ of λ as in (2).

Example 4.5. *We continue with Example 4.2. Consider $p = 3$. Then the p-abacus of λ is*



Then λ has four diagonal hooks corresponding to the beads in positions 0, 2, 5 and 6. We have

$$\mathcal{X}_0 = \{0, 2\}, \quad \mathcal{X}_1 = \emptyset \quad \text{and} \quad \mathcal{X}_2 = \{0, 1\}.$$

By the discussion after Example 4.2, the diagonal hooks arising from the 0-runner have length 1 and 13. The ones arising from the 2-runner have length 5 and 11. The partition sequences of λ_0, λ_1 and λ_2 are respectively $\overline{0}11010\overline{1}$, $\overline{0}\,\overline{1}$ and $\overline{0}10100\overline{1}$. Thus,

$$\lambda_0 = (3, 2), \quad \lambda_1 = \emptyset \quad \text{and} \quad \lambda_2 = (2^2, 1).$$

Suppose $\lambda = \lambda^*$. Then $\Lambda^* = \Lambda$, and $l_{-\alpha} = 1 - l_{\alpha-1}$. Since, by definition, α is the number of zeroes before the leftmost 1, and β is the number of ones after the leftmost

0, this switch between 0 and 1 in each position implies that $\alpha = \beta$. Moreover, for $0 \leq u \leq \alpha - 1$ and $\delta \in \{0, 1\}$, we have $l_u = \delta$ if and only if $l_{-u-1} = 1 - \delta$. Denote by $\phi : \mathbb{Z} \mapsto \mathbb{Z}, u \to -u - 1$. We define

$$\mathcal{Y}_\gamma = \{j \in \mathbb{Z} \mid pj + \gamma \leq -1 \text{ and } l_{\gamma,j} = 1\}.$$

Lemma 4.6. *Suppose ϕ is as above. Then the following hold.*
(1) ϕ is a bijection from $\mathbb{Z}$ to $\mathbb{Z}$.
(2) ϕ induces a bijection $\phi|_{H_0} : H_0 \to H_1$ with inverse map $\phi|_{H_1} : H_1 \to H_0$.
(3) $\phi^2 = id$.
(4) ϕ induces a bijection from $\mathcal{X}_\gamma$ to $\mathcal{Y}_{p-\gamma-1}$.

Proof. (1) and (3) are immediate. For (2), note, in particular, ϕ and the diagonal hooks of λ are the $h_{u,\phi(u)}$ for $u \in H_0$ of length $2u + 1$. For $u \in H_0$, we denote the corresponding diagonal hook-length by

$$d_u = 2u + 1. \tag{42}$$

To see (4), suppose that $u = jp + \gamma$ for $j \in \mathbb{Z}$ and $0 \leq \gamma \leq p - 1$. Then $-u - 1 = -jp - \gamma - 1 = -(j + 1)p + p - 1 - \gamma$ with $0 \leq p - 1 - \gamma \leq p - 1$. Since $l_{\phi(u)} = 1$ if and only if $l_u = 0$, we have

$$l_{p-1-\gamma,j} = 1 - l_{\gamma,-(j+1)} \tag{43}$$

which is the partition sequence of the conjugate partition of λ_γ. ☐

Assume that $\gamma \neq (p-1)/2$. Since $\mathcal{X}_{p-1-\gamma}$ labels the diagonal hooks of λ arising from runner $(p - 1 - \gamma)$, $\mathcal{Y}_\gamma$ does too. Hence, the diagonal hooks of λ arising from the runners γ and $(p - 1 - \gamma)$ are parametrized by $\mathcal{X}_\gamma \cup \mathcal{Y}_\gamma$. By (42), for $x \in \mathcal{X}_\gamma$ and $x' \in \mathcal{Y}_\gamma$, the corresponding diagonal hook-lengths of λ are

$$d_x = 2(xp + \gamma) + 1 \quad \text{and} \quad d_{x'} = 2((-x' - 1)p + p - 1 - \gamma) + 1. \tag{44}$$

Denote by Γ a set of representatives of $\{\gamma, p-1-\gamma\}$ for $\{0, \ldots, j, \ldots, p-1\} \backslash \{(p-1)/2\}$. By the discussion above, we have the following.

Corollary 4.7. *The diagonal hooks of λ are parametrized by the elements of*

$$\mathcal{X}_{(p-1)/2} \cup \bigcup_{\gamma \in \Gamma} (X_\gamma \cup \mathcal{Y}_\gamma).$$

Assume now that $\lambda = \lambda^*$ with $\mathcal{C}or_p(\lambda) = \emptyset$. Furthermore, assume that $\lambda_{(p-1)/2} = \emptyset$ where $\lambda_{\frac{(p-1)}{2}} \in \mathcal{Q}uo_p(\lambda)$. Let $0 \leq \gamma \leq p - 1$. Consider the partition sequence $(l_{\gamma,j})_{j \in \mathbb{Z}}$ as in (41). Since the p-abacus of Figure 2 is the one that we obtain after removing all the p-hooks of λ (because $\mathcal{C}or_p(\lambda)$ is empty), it follows from the construction of the p-quotient that the number of beads above $j = 0$ is the same as the number of empty positions under and strictly below $j = 0$. In particular, the sequence $(l_{\gamma,j})_{j \in \mathbb{Z}}$ is compatible with the labeling of (39), and the beads over $j = 0$ correspond to the diagonal hooks of λ_γ and are in bijection with the diagonal hooks of λ arising from runner γ.

Since $\lambda^*_{p-1-\gamma} = \lambda_\gamma$, they have the same number of diagonal hooks. If d is the length of the jth-diagonal hook of λ_γ, then we denote by d^* the length of the jth-diagonal hook of $\lambda_{p-1-\gamma}$. Write $x \in \mathcal{X}_\gamma$ and $x^* \in \mathcal{Y}_\gamma$ such that $d = d_x$ and $d^* = d_{\phi(x^*)}$. Then (42) gives

$$d_x = 2(xp + \gamma) + 1 \quad \text{and} \quad d_{x^*} = 2(\phi(x^*)p + (p - 1) - \gamma) + 1. \tag{45}$$

Hence, if we set $w_{x,x^*} = x - x^*$, then

$$d_x + d_{x^*} = 2pw_{x,x^*}. \tag{46}$$

Moreover, by (4.7)

$$\mathfrak{d}(\lambda) = \bigcup_{\gamma\in\Gamma} \{d_x,\, d_{x^*} \mid x \in \mathcal{X}_\gamma\},$$

where $\mathfrak{d}(\lambda)$ is defined in (1)

Example 4.8. *Consider the partition $\lambda = (7^2, 5, 4, 3, 2^2)$ in Example 4.5. We see from the 3-abacus that $\mathcal{C}or_3(\lambda)$ is empty. We also see that*

$$\mathcal{Y}_0 = \{-2, -1\} \quad \textit{and} \quad \mathcal{Y}_2 = \{-1, -3\}.$$

The bijection between $\mathcal{Y}_0$ and $\mathcal{X}_2$ is

$$-2 \mapsto \phi(-2) = 2 - 1 = 1 \quad \textit{and} \quad -1 \mapsto \phi(-1) = 1 - 1 = 0.$$

Then $\mathfrak{d}(\lambda)$ is given by (44)

$$\{d_x \mid x \in \mathcal{X}_0\} = \{d_0,\, d_2\} = \{1, 13\} \quad \textit{and} \quad \{d_x \mid x \in \mathcal{Y}_0\} = \{d_{-2},\, d_{-1}\} = \{5, 11\}.$$

In particular, the diagonal hooks of length 1 of λ_0 and λ_2 are associated with $1 \in \mathcal{X}_0$ and $-1 \in \mathcal{Y}_0$. Similarly, the ones of length 4 correspond to $2 \in \mathcal{X}_0$ and $-3 \in \mathcal{Y}_0$. It follows that

$$1^* = -1 \quad \textit{and} \quad 2^* = -3.$$

4.3. **Diagonal hooks of regular partitions.** Let p be an odd prime, n an integer divisible by p, and $\lambda = \lambda^*$ be a partition of n. Let $n = n_1p + n_2p^2 + \cdots + n_sp^s$ be its p-adic expansion. Write $I = \{0, \dots, p-1\}$ as above, and the p-core tower $\mathcal{CT}(\lambda)$ of λ as in (2). We assume that the $\mathcal{C}or_p(\lambda) = \emptyset$. We say that λ is a *regular partition* when $c_k(\lambda) = n_k$ and $\lambda_{\underline{p}^*} = \emptyset$ where $\underline{p}^* \in I^k$ for any $1 \le k \le s$. On the other hand, λ is called *singular* whenever $\lambda_{\underline{j}} = \emptyset$, except possibly for $\underline{j} = \underline{p}^*(k) \in I^k$, where $\underline{p}^*(k)$ is defined in Equation (14).

For λ as above, we also define $\mathfrak{r}(\lambda)$ and $\mathfrak{s}(\lambda)$ the *regular* and *singular* parts (respectively) by giving their p-core towers as follows. For $k \ge 0$ and $\underline{j} \in I^k$, if $\underline{j} \ne \underline{p}^*(k)$, then we set $\lambda'_{\underline{j}} = \lambda_{\underline{j}}$ and $\lambda''_{\underline{j}} = \emptyset$. Otherwise, if $\underline{j} = \underline{p}^*(k)$, then write $\lambda'_{\underline{p}^*(k)} = \emptyset$ and $\lambda''_{\underline{p}^*(k)} = \lambda_{\underline{p}^*(k)}$.

Therefore, the p-core towers of $\mathfrak{r}(\lambda)$ and $\mathfrak{s}(\lambda)$ are given by

$$\mathcal{C}or_p^{(k)}(\mathfrak{r}(\lambda)) = \{\lambda'_{\underline{j}} \mid \underline{j} \in I^k\} \quad \text{and} \quad \mathcal{C}or_p^{(k)}(\mathfrak{s}(\lambda)) = \{\lambda''_{\underline{j}} \mid \underline{j} \in I^k\} \quad \text{for } k \ge 0. \tag{47}$$

Recall $\underline{p}^*(k) \in I^k$. Then that $c_k(\mathfrak{s}(\lambda)) = |\lambda_{\underline{p}^*(k)}|$ and $c_k(\mathfrak{r}(\lambda)) = c_k(\lambda) - c_k(\mathfrak{s}(\lambda))$ by construction. Hence, if we set $n' = \sum c_k(\mathfrak{r}(\lambda))p^k$ and $n'' = \sum c_k(\mathfrak{s}(\lambda))p^k$, then $n = n' + n''$ and $\mathfrak{r}(\lambda)$ and $\mathfrak{s}(\lambda)$ are respectively regular and singular partitions of n' and n'' in the previous sense.

Proposition 4.9. *Let n be an integer with p-adic expansion $n = n_1p + n_2p^2 + \cdots + n_sp^s$, where p is an odd prime. Let λ be a regular partition with p-core tower $\mathcal{C}or_p^{(k)}(\lambda) = \{\lambda_{\underline{j}} \mid \underline{j} \in I^k\}$ for $k \ge 0$. For any integer $0 \le i \le s-1$, write $\mathcal{H}_i$ for the set of diagonal hooks lengths of λ which are divisible by p^i but not by p^{i+1}. Then the elements of $\mathcal{H}_i$ are of the form $t_{u,i} = p^iu$ and $t^*_{u,i} = p^i(w_{u,i}p - u)$, where $u \in U_i$ is an odd integer relatively prime to p, and $w_{u,i} \in W_j$ is an even integer.*

Proof. We proceed by induction on $s \geq 1$. Suppose that $s = 1$. Then $n = n_1 p$. Note that $\mathcal{C}or_p(\lambda) = \emptyset$ by assumption, thus we are in the situation described above. By (45) we set $t_{u,0} = d_x$ and $t^*_{u,0} = d_{x^*}$, and (46) gives that $t^*_{u,0} = pw_u - t_{u,0}$ with $w_{u,0} = 2w_{x,x^*}$. In particular, $t_{u,0}$ and $t^*_{u,0}$ are odd and prime to p and $w_{u,0}$ is even. The result is true for $s = 1$.

Let $s \geq 1$. Suppose that the result holds for s. Let $n = n_1 p + n_2 p^2 + \cdots + n_s p^s + n_{s+1} p^{s+1}$, and λ be a partition of n that satisfies the assumption. Consider $\lambda' = \lambda_{(p-1)/2} \in \mathcal{Q}uo_p(\lambda)$ and $n' = |\lambda'|$. One has $n = p(n' + \sum_{j \neq (p-1)/2} |\lambda_j|)$ because $\mathcal{C}or_p(\lambda) = \emptyset$, and n' is divisible by p because $\mathcal{C}or_p(\lambda') = \emptyset$, since λ is regular. Thus, the p-adic expansion of n' is then of the form $n'_1 p + \cdots + n'_h p^h$ with $h \leq s$. By induction, the diagonal hooks of λ' are as required. Now, there is a bijection f between the diagonal hooks of λ divisible by p and the diagonal hooks of $\lambda_{(p-1)/2}$ such that $|f(h_{mm})| = p|h_{mm}|$, where h_{mm} is a diagonal hook of $\lambda_{(p-1)/2}$. In particular, for $1 \leq i \leq s$, we have $H_i = f(H'_{i-1})$ where H'_{i-1} is the set of diagonal hooks of λ' divisible by p^{i-1} but not by p^i. On the other hand, since $\mathcal{C}or_p(\lambda) = \emptyset$, H_0 is the set of diagonal hooks arising from $\mathcal{Q}uo_p(\lambda) = (\lambda_0, \ldots, \lambda_{(p-3)/2}, \emptyset, \lambda_{(p+1)/2}, \ldots, \lambda_{p-1})$, and (45) and (46) give the result. □

Proposition 4.10. *Let λ be a regular partition of n. If $f \in \mathcal{K}_{n!/2}$ then $\varepsilon(\chi_\lambda, f) = 1$. Moreover,*

$$\varepsilon(\chi_\lambda, \sigma_{n!/2}) = (-1)^{\frac{(p-1)n}{4}}.$$

Proof. First, we remark that if the p-adic expansion of n is $n_1 p + \cdots + n_s p^s$ then each n_i is even since $n_i = 2\sum_{\underline{j}} |\lambda_{\underline{j}}|$, where the sum runs over $\underline{j} \neq \underline{p}^*(k)$ and $\underline{j}$ is a representative of $\{\underline{j}, \underline{j}^*\}$. Here we use that λ is a symmetric partition and that $|\lambda_{\underline{j}}| = |\lambda^*_{\underline{j}}| = |\lambda_{\underline{j}^*}|$. Now, by Proposition 4.9, we have

$$\begin{aligned}\prod_{h \in \mathfrak{d}(\lambda)} h &= \prod_{i=0}^{s-1} \prod_{u \in U_i} t_{u,i} t^*_{u,i} \\ &= \prod_{i=0}^{s-1} \prod_{u \in U_i} p^{2i} u(w_{u,i} p - u).\end{aligned}$$

Let f be in $\mathcal{H}_{n!/2}$. With the notation (35), we have

$$\begin{aligned}\varepsilon\left(\sqrt{\prod_{h \in \mathfrak{D}(\lambda)} h}, f\right) &= \varepsilon\left(\prod_{i=0}^{s-1} \prod_{u \in U_i} \sqrt{u(w_{u,i} p - u)}, f\right) \qquad (48) \\ &= \prod_{i=0}^{s-1} \varepsilon\left(\prod_{u \in U_i} \sqrt{u(w_{u,i} p - u)}, f\right).\end{aligned}$$

Note that u and $(w_{u,i} p - u)$ are odd. Furthermore,

$$u(w_{u,i} p - u) = \begin{cases} 2 - u^2 \equiv 1 \mod 4 & \text{if } w_{u,i} \equiv 2 \mod 4, \\ -u^2 \equiv -1 \mod 4 & \text{if } w_{u,i} \equiv 0 \mod 4. \end{cases} \qquad (49)$$

We also have

$$n = \sum_{h \in D(\lambda)} h = \sum_{i=0}^{s-1} p^{i+1} \sum_{u \in U_i} w_{u,i}.$$

Since $w_{u,i}$ is even, there is an integer $w'_{u,i}$ such that $w_{u,i} = 2w'_{u,i}$, and

$$\begin{aligned}\frac{n}{2} &= \sum_{i=0}^{s-1} p^{i+1} \sum_{u \in U_i} w'_{u,i} \\ &\equiv \sum_{i=0}^{s-1} \sum_{u \in U_i} w'_{u,i} \mod 2,\end{aligned} \tag{50}$$

because p is odd. Now, write $A = \{w'_{u,i} \mid 0 \le i \le s-1,\, u \in U_i\}$, and A_{even} and A_{odd} for the subsets of even and odd elements of A, respectively. Then $|A| = \frac{d_\lambda}{2}$ and (50) gives

$$\frac{n}{2} \equiv \sum_{w \in A_{\text{odd}}} w \equiv \sum_{w \in A_{\text{odd}}} 1 \equiv |A_{\text{odd}}| \mod 2.$$

Since $|A| = |A_{\text{odd}}| + |A_{\text{even}}|$, we deduce from (49) that

$$\prod_{i=0}^{s-1} \prod_{u \in U_i} u(w_{u,i}p - u) \equiv (-1)^{|A_{\text{even}}|} \equiv (-1)^{|A|-|A_{\text{odd}}|} \equiv (-1)^{\frac{n-d_\lambda}{2}} \mod 4. \tag{51}$$

Thus, by (48) and Proposition 4.1 we obtain

$$\varepsilon\left(\sqrt{\prod_{h \in \mathfrak{d}(\lambda)} h}, f\right) = \varepsilon(i, f)^{\frac{(p-1)(n-d_\lambda)}{4}} \prod_{i=0}^{s-1} \prod_{u \in U_i} \left(\frac{r}{u}\right)\left(\frac{r}{w_{u,i}p - u}\right), \tag{52}$$

where r is such that $f(\omega_m) = \omega_m^r$ for $m = \prod_{i,u} u(w_{u,i}p-u)$. Note that if $f \in \mathcal{K}_{n!/2}$, then f acts trivially on i and on ω_m, that is $r = 1$, and (36) implies that $\varepsilon(\chi_\lambda, f) = 1$. Assume that $f = \sigma_{n!/2}$, that is $r = p$. On the other hand, by quadratic reciprocity, one has

$$\begin{aligned}\left(\frac{p}{u}\right)\left(\frac{p}{w_{u,i}p - u}\right) &= (-1)^{\frac{p-1}{2}\left(\frac{u-1}{2} + \frac{w_{u,i}p-u-1}{2}\right)}\left(\frac{-1}{p}\right) \\ &= (-1)^{\frac{p-1}{2}\left(\frac{u-1}{2} + \frac{w_{u,i}p-u-1}{2} + 1\right)} \\ &= (-1)^{\frac{(p-1)w_{u,i}}{4}} \\ &= (-1)^{\frac{(p-1)w'_{u,i}}{2}} \\ &= \begin{cases} 1 & \text{if } w'_{u,i} \equiv 0 \mod 2, \\ -1 & \text{if } w'_{u,i} \equiv 1 \mod 2. \end{cases}\end{aligned} \tag{53}$$

Using (36), it follows that

$$\begin{aligned}\varepsilon(\chi_\lambda, \sigma_{n!/2}) &= (-1)^{\frac{(p-1)(n-d_\lambda)}{4}} \varepsilon\left(\sqrt{\prod_{h \in \mathfrak{d}(\lambda)} h}, f\right) \\ &= (-1)^{\frac{(p-1)(n-d_\lambda)}{4}} \cdot (-1)^{\frac{(p-1)(n-d_\lambda)}{4}} \cdot (-1)^{\frac{(p-1)|A_{\text{odd}}|}{2}} \\ &= (-1)^{\frac{(p-1)n}{4}},\end{aligned}$$

as required. □

4.4. **Diagonal hooks of partitions with non-empty p-core.** For any partition λ, we denote by $\mathcal{Q}_p(\lambda)$ the partition with the same p-quotient as λ but with empty p-core. That is, $\mathcal{Q}uo_p(\mathcal{Q}_p(\lambda)) = \mathcal{Q}uo_p(\lambda)$ but $\mathcal{C}or_p(\mathcal{Q}_p(\lambda)) = \emptyset$.

Let $\lambda = \lambda^*$. Write $\mathcal{M} = (m_u)_{u\in\mathbb{Z}}$, $\mathcal{M}' = (m'_u)_{u\in\mathbb{Z}}$ and $\Lambda = (l_u)_{u\in\mathbb{Z}}$ for the partition sequences with the labeling as in (39) associated to λ, $\mathcal{C}or_p(\lambda)$ and $\mathcal{Q}_p(\lambda)$ respectively.

Since $\lambda = \lambda^*$ we have $\mathcal{C}or_p(\lambda) = \mathcal{C}or_p^*(\lambda)$ by [15, Prop. 3.5]. Let $0 \leq \gamma \leq p-1$. By definition of a p-core, if $m'_\gamma = 0$, then there is an integer $\delta_\gamma > 0$ such that $m'_{pj+\gamma} = 0$ if and only if $j \leq \delta_\gamma - 1$. Since $\mathcal{C}or_p(\lambda) = \mathcal{C}or_p^*(\lambda)$ it follows from §4.2 that $m'_{pj+(p-1)-\gamma} = 0$ if and only if $j < -\delta_\gamma$. If $m'_\gamma = 1$ and $m'_{-p-\gamma} = 0$ then $m'_{(p-1)-\gamma} = 1$ and $m'_{-p+(p-1)-\gamma} = 0$. In this last case, we set $\delta_\gamma = 0$. Let γ be such that $\delta_\gamma > 0$. Define $\Delta_\gamma = \{0 \leq j \leq \delta_\gamma - 1\}$. Then elements of $\mathfrak{D}(\mathcal{C}or_p(\lambda))$ are labeled by the elements of $\cup_{\delta_\gamma>0}\Delta_\gamma$. In particular $\mathcal{C}or_p(\lambda)$ has $\sum_{\delta_\gamma>0}\delta_\gamma$ diagonal hooks.

We construct the p-abacus of λ from that of $\mathcal{Q}_p(\lambda)$ as follows. If $\delta_\gamma = 0$ then the runners γ and $(p-1-\gamma)$ of $\mathcal{Q}_p(\lambda)$ and λ are identical. If $\delta_\gamma > 0$, then runner γ of λ (resp. the runner $p-1-\gamma$ of λ) is obtained by shifting up (resp. down) the corresponding runner of $\mathcal{Q}_p(\lambda)$ δ_γ positions. It follows that, for all $0 \leq \gamma \leq p-1$ such that $\delta_\gamma \geq 0$, one has

$$m_{(j+\delta_\gamma)p+\gamma} = l_{jp+\gamma} \quad \text{and} \quad m_{(j-\delta_\gamma)p+p-1-\gamma} = l_{jp+p-1-\gamma} \quad \text{for all } j \in \mathbb{Z}. \tag{54}$$

We will now describe how to obtain $\mathfrak{D}(\lambda)$ from $\mathfrak{D}(\mathcal{Q}_p(\lambda))$. For $\gamma \in \Gamma\cup\{(p-1)/2\}$, we denote by $\mathcal{X}_\gamma$ and $\mathcal{Y}_\gamma$ (respectively $\mathcal{X}'_\gamma$ and $\mathcal{Y}'_\gamma$) the sets as in (4.7) that label the diagonal hooks of $\mathcal{Q}_p(\lambda)$ (respectively, of λ).

We remark that if $\delta_\gamma = 0$, then $\mathcal{X}_\gamma = \mathcal{X}'_\gamma$ and $\mathcal{Y}_\gamma = \mathcal{Y}'_\gamma$, that is the hooks of λ and $\mathcal{Q}_p(\lambda)$ arising from runner γ are the same. Note that $\delta_{(p-1)/2} = 0$, since $\lambda = \lambda^*$.

Suppose $\delta_\gamma > 0$. We introduce four possibilities in passing from the diagonal hooks of $\mathcal{Q}_p(\lambda)$ to those of λ.

(i) Any $x \in \mathcal{X}_\gamma$ corresponds to a hook labeled by $x+\delta_\gamma \in \mathcal{X}'_\gamma$ of λ on the γ-runner. More precisely, by (54) we can associate to the hook of length d_x of $\mathcal{Q}_p(\lambda)$ labeled by x given in (45), a hook of λ of length

$$c(d_x) = 2((x+\delta_\gamma)p+\gamma)+1. \tag{55}$$

We will call this *an increase of the length of an existing hook with respect to γ.*

(ii) Similarly, for $x \in \mathcal{Y}_\gamma$ such that $x < -\delta_\gamma$, we have $\delta_\gamma + x < 0$, and $\delta_\gamma + x \in \mathcal{Y}'_\gamma$ by (54). By (45), we associate to d_x a hook of λ of length

$$c(d_x) = 2(\phi(\delta_\gamma + x)p + (p-1) - \gamma) + 1. \tag{56}$$

We will refer to this as *an increase of the length of an existing hook with respect to $\gamma^* = p - \gamma - 1$.*

(iii) Let $-\delta_\gamma \leq x \leq -1$ be such that $x \notin \mathcal{Y}_\gamma$, that is $l_{xp+\gamma} = 0$. Then $x + \delta_\gamma \geq 0$ and by (54), $x + \delta_\gamma \in \mathcal{X}'_\gamma$. Hence, a new diagonal hook of length

$$c_x = 2((\delta_\gamma + x)p + \gamma) + 1$$

appears in λ. This is also a diagonal hook of $\mathcal{C}or_p(\lambda)$. We will call this *the appearance of a new hook with respect to γ.*

(iv) Finally, let $-\delta_\gamma \le x \le -1$ be such that $x \in \mathcal{Y}_\gamma$, that is $l_{xp+\gamma} = 1$. Then $x+\delta_\gamma \notin \mathcal{X}'_\gamma$. Then the hook of $\mathcal{Q}_p(\lambda)$ labeled by x gives no hook of λ. We will call this *the disappearance of an existing hook with respect to* $\gamma^* = p-\gamma-1$.

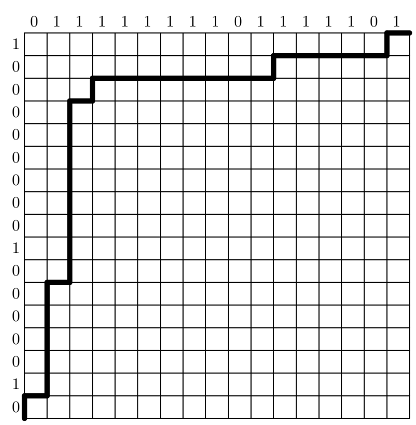
Remark 4.11. Let $\mathcal{A}_\gamma$ and $\mathcal{B}_\gamma$ be the set of $-\delta_\gamma \le x \le -1$ such that $l_{px+\gamma} = 0$ and $l_{px+\gamma} = 1$, respectively. Then $\mathcal{A}_\gamma \sqcup \mathcal{B}_\gamma$ labels the diagonal hooks of $\mathcal{C}or_p(\lambda))$ as follows: associate the set of diagonal hooks of $\mathcal{C}or_p(\lambda)$ of length

$$c_x = 2((\delta_\gamma + x)p + \gamma) + 1 \tag{57}$$

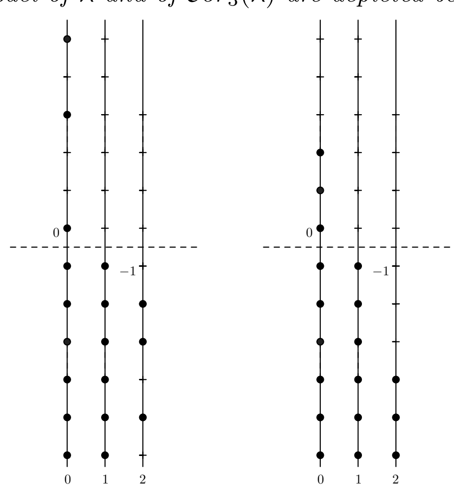
to $\mathcal{A}_\gamma \sqcup \mathcal{B}_\gamma$.

In the next example we use the fact that the p-abacus of $\mathcal{C}or_p(\lambda)$ is obtained from the p-abacus of λ by placing beads in empty positions one position below them on each runner until this is no longer possible, and then reading off the resulting partition from the new p-abacus configuration. by [8, p. 79].

Example 4.12. *Let* $\lambda = (16, 11, 3, 2^8, 1^5)$. *We find* $\mathfrak{D}(\lambda)$ *using the* 3*-abaci of* $\mathcal{C}or_3(\lambda)$ *and* $\mathcal{Q}_3(\lambda)$.

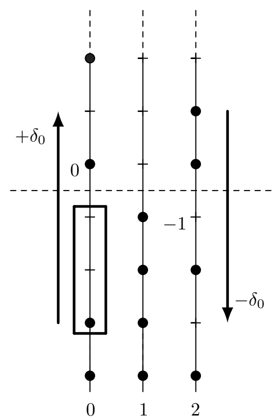


In particular, the 3*-abaci of* λ *and of* $\mathcal{C}or_3(\lambda)$ *are depicted below:*



λ $\qquad$ $\mathcal{C}or_3(\lambda)$

We can obtain $\mathcal{C}or_3(\lambda) = (7,5,3,2^2,1^2)$ from λ by pushing down beads and reading off the resulting bead positions. We have $\delta_0 = 3$ and $\delta_1 = 0$, and $\lambda_{(3)}$ has three diagonal hooks. Now consider the partition $\mathcal{Q}_3(\lambda)$ of Example 4.2. More precisely, by the previous discussion, $\mathfrak{D}(\lambda)$ can be obtained from the 3-abacus of $\mathcal{Q}_3(\lambda)$ and the Δ_γ. The 3-abacus of λ is obtained by shifting up the runner 0 of $\mathcal{Q}_3(\lambda)$ δ_0 positions and by shifting down the runner 2 $-\delta_0$ positions.



Consider runner 0 of $\mathcal{Q}_3(\lambda)$. Since $\delta_0 = 3$, one shifts it up three positions to obtain the 0-runner of λ. However (here we abuse notation) this causes $\mathcal{X}_0 \cup \mathcal{Y}_0(\lambda)$, to be altered from $\mathcal{X}_0' \cup \mathcal{Y}_0'(\lambda)$, and hence the number of diagonal hooks of λ arising from runner 0 is different the number of diagonal hooks of $\mathcal{Q}_3(\lambda)$ arising from its runner 0. In particular, the diagonal hooks in $\mathcal{Q}_3(\lambda)$ corresponding to positions 2 and 5 on runner 2 "disappear" for λ as they shift to new positions -1 and -4, while the bead in position -9 on the 3-abacus of $\mathcal{Q}_p(\lambda)$ introduces a new diagonal for λ as it shifts up to position 0.

Recall that the Durfee square of λ is the largest square that can be accommodated inside the Young diagram of λ (see for example [1, §2.3]). Let $\lambda^\square$ be the size of the Durfee square of λ, otherwise known as the Durfee number of λ. Let $\mathcal{Y}_\gamma^1 = \{-\delta_\gamma \geq x \geq -1 \mid x \notin \mathcal{Y}_\gamma\}$ and $\mathcal{Y}_\gamma^0 = \{-\delta_\gamma \geq x \geq -1 \mid x \in \mathcal{Y}_\gamma\}$, and $\mathcal{Y}_\gamma^\square = |\mathcal{Y}_\gamma^1| - |\mathcal{Y}_\gamma^0|$. Then steps (i) through (iv) in this section describe how to calculate the size of the Durfee square of a symmetric partition from the Durfee squares of its p-quotient and its p-core.

Lemma 4.13. *With the above notation, we have*

$$\lambda^\square = \sum_{\lambda_\gamma \in \mathcal{Q}uo_p(\lambda)} \lambda_\gamma^\square + \sum_{\delta_\gamma > 0} \mathcal{Y}_\gamma^\square.$$

Proof. We can rewrite the equation in the statement of the theorem as follows:

$$\lambda^\square = \sum_{\lambda_\gamma, \delta_\gamma > 0} (\lambda_\gamma^\square + \mathcal{Y}_\gamma^\square) + \sum_{\lambda_\gamma, \delta_\gamma = 0} \lambda_\gamma^\square.$$

The second of the two sums counts the contribution to the Durfee number from the runners that are not affected by the introduction of a core. The first of the two sums calculates the original contribution to the Durfee number from the runners on which the core appears, and then corrects it using $\mathcal{Y}_\gamma^\square$ for each $\delta_\gamma > 0$. In particular,

$\mathcal{Y}_\gamma^\square$ subtracts the disappearances of existing hooks with respect to $\gamma^* = p - \gamma - 1$ from the appearances of a new hooks with respect to γ. $\square$

The following two corollaries are immediate.

Corollary 4.14. *If λ is a p-core, that is $\lambda = \mathcal{C}or_p(\lambda)$, then $\lambda^\square = \sum_{\delta_\gamma > 0} \delta_\gamma$.*

Proof. In this case the Durfee number is calculated directly from the the p-core. $\square$

Corollary 4.15. *If λ has empty p-core, that is, $\lambda = \mathcal{Q}_p(\lambda)$, then*

$$\lambda^\square = \sum_{\lambda_\gamma \in \mathcal{Q}uo_p(\lambda)} \lambda_\gamma^\square.$$

Proof. In the case the p-core contributes nothing, no diagonal hooks appear, non disappear, and the Durfee number of λ is the the sum of the Durfee numbers of the quotient. $\square$

4.5. **The sign of the product of the diagonal hooks.**

Theorem 4.16. *Let w and r be non-negative integers, and set $n = pw + r$. Let $\lambda = \lambda^*$ be a partition of n such that $|\mathcal{C}or_p(\lambda)| = r$ and $\mathcal{Q}uo_p(\lambda) \in \mathcal{MP}(p, w)$, where $\mathcal{MP}(p, w)$ is the set of p-multipartitions of w. Assume that $\lambda_{(p-1)/2} = \emptyset$. Set*

$$d = \prod_{h \in \mathfrak{d}(\lambda)} h\,, \quad q = \prod_{h \in \mathfrak{d}(\mathcal{Q}_p(\lambda))} h \quad \textit{and} \quad c = \prod_{h \in \mathfrak{d}(\mathcal{C}or_p(\lambda))} h.$$

Then

$$\left(\frac{p}{d}\right) = \left(\frac{p}{q}\right)\left(\frac{p}{c}\right).$$

Furthermore, if $b = \sum_\gamma |\mathcal{B}_\gamma|$, then

$$d \equiv qc(-1)^b \mod 4,$$

where $\mathcal{B}_\gamma$ is the set defined in Remark 4.11.

Proof. Recall from §4.4 that $\mathfrak{D}(\lambda)$ is labeled by $\mathcal{X}'_\gamma$ and $\mathcal{Y}'_\gamma$ where $\gamma \in \Gamma$. We choose the representative $\gamma \in \Gamma$ such that $\delta_\gamma \geq 0$. We also recall that $\mathfrak{D}(\mathcal{Q}uo_p(\lambda))$ is labeled by $\mathcal{X}_\gamma \cup Y_\gamma$ and $\mathfrak{D}(\mathcal{C}or_p(\lambda))$ by Δ_γ for $\gamma \in \Gamma$. Furthermore, for $\gamma \in \Gamma$, if $\delta_\gamma = 0$, then $\mathcal{X}'_\gamma = \mathcal{X}_\gamma$, $\mathcal{Y}'_\gamma = \mathcal{Y}'_\gamma$ and $\Delta_\gamma = \emptyset$. Otherwise, if $\delta_\gamma > 0$, then with the notation (55), (56) and (57)

$\mathcal{X}'_\gamma = \{c(d_x) \mid x \in \mathcal{X}_\gamma\} \cup \{c_x \mid x \in \mathcal{A}_\gamma\}$ and $\mathcal{Y}'_\gamma = \{c(d_x) \mid x \in \mathcal{Y}_\gamma$ such that $x < -\delta_\gamma\}$.

Write

$$M = \prod_{\delta_\gamma > 0} \prod_{x \in \mathcal{X}'_\gamma \cup \mathcal{Y}'_\gamma} \left(\frac{p}{d'_x}\right),$$

where d'_x is the diagonal hook-length of λ corresponding to x. We remark that

$$\left(\frac{p}{d}\right) = M \prod_{\delta_\gamma = 0} \prod_{x \in \mathcal{X}_\gamma \cup Y_\gamma} \left(\frac{p}{d_x}\right).$$

But for any $c(d_x) \in \mathcal{Y}'_\gamma$, there is $c(d_{x^*}) \in \mathcal{X}_\gamma$, where d_x and d_{x^*} are diagonal hook lengths of λ as in (45). Furthermore, by (46), (55) and (56) we have

$$c(d_x) + c(d_x^*) = 2pw_{x,x^*}. \tag{58}$$

It follows that

$$c(d_x)c(d_{x^*}) \equiv 2c(d_x)pw_{x,x^*} - 1 \equiv 2pw_{x,x^*} - 1 \equiv d_x d_{x^*} \mod 4. \tag{59}$$

Hence,

$$\begin{aligned} \left(\frac{p}{c(d_x)c(d_{x^*})}\right) &= (-1)^{\frac{(p-1)(c(d_x)c(d_{x^*})-1)}{4}} \left(\frac{c(d_x)c(d_{x^*})}{p}\right) \\ &= (-1)^{\frac{(p-1)(d_x d_{x^*}-1)}{4}} \left(\frac{-1}{p}\right) \\ &= \left(\frac{p}{d_x d_{x^*}}\right). \end{aligned} \tag{60}$$

On the other hand, if $x \in \mathcal{Y}_\gamma$ is such that $-\delta_\gamma \leq x \leq -1$, that is $x \in \mathcal{B}_\gamma$, then there is a diagonal hook of μ of length $c(d_{x^*})$ with $x^* \in \mathcal{X}_\gamma$. So

$$\begin{aligned} M &= \prod_{\delta_\gamma > 0} \left(\prod_{x \in \mathcal{Y}_\gamma} \left(\frac{p}{c(d_x)c(d_{x^*})}\right) \prod_{x \in \mathcal{B}_\gamma} \left(\frac{p}{c(d_{x^*})}\right) \prod_{x \in \mathcal{A}_\gamma} \left(\frac{p}{c_x}\right) \right) \\ &= \prod_{\delta_\gamma > 0} \left(\prod_{x \in \mathcal{Y}_\gamma} \left(\frac{p}{d_x d_{x^*}}\right) \prod_{x \in \mathcal{B}_\gamma} \left(\frac{p}{c(d_{x^*})}\right) \prod_{x \in \mathcal{A}_\gamma} \left(\frac{p}{c_x}\right) \right). \end{aligned}$$

By Remark 4.11, recall that $\mathfrak{d}(\mathcal{C}or_p(\lambda)) = \{c_x \mid x \in \mathcal{A}_\gamma \cup \mathcal{B}_\gamma\}$, where c_x is given in (57). Then

$$\left(\frac{p}{d}\right) = \left(\frac{p}{q}\right) \left(\frac{p}{c}\right) \prod_{\delta_\gamma > 0} \prod_{x \in \mathcal{B}_\gamma} \left(\frac{p}{d_x d_{x^*} c(d_{x^*}) c_x}\right).$$

Let γ be such that $\delta_\gamma > 0$ and $x \in \mathcal{B}_\gamma$. By (55) and (57), we have $c_x \equiv c(d_{x^*}) \mod p$. Moreover

$$\begin{aligned} c_x c(d_{x^*}) &\equiv 1 + 2((\delta_\gamma + x)p + 1) + 2((\delta_\gamma + x^*)p + 1) \mod 4 \\ &\equiv 1 + 2x + 2x^* \mod 4 \\ &\equiv 1 + 2(x + x^*) \mod 4 \\ &= 1 + 2(x - x^*) \mod 4 \\ &= 1 + 2w_{x,x^*} \mod 4. \end{aligned} \tag{61}$$

Hence

$$\frac{c_x c(d_{x^*}) - 1}{2} \equiv w_{x,x^*} \mod 2,$$

and we obtain that

$$\left(\frac{p}{c(d_{x^*})c_x}\right) = (-1)^{\frac{(p-1)w_{x,x^*}}{2}} \left(\frac{c(d_{x^*})c_x}{p}\right) = (-1)^{\frac{(p-1)w_{x,x^*}}{2}}.$$

However, the computation (53) shows that $\left(\frac{p}{d_x d_{x^*}}\right) = (-1)^{\frac{(p-1)w_{x,x^*}}{2}}$, and

$$\left(\frac{p}{d_x d_{x^*} c(d_{x^*}) c_x}\right) = (-1)^{\frac{(p-1)w_{x,x^*}}{2}} \cdot (-1)^{\frac{(p-1)w_{x,x^*}}{2}} = 1.$$

The result follows.

We now prove the second part of the statement. Since an odd number is its own inverse modulo 4, we do the same computation as above and obtain that

$$d \equiv qc \prod_{\delta_\gamma > 0} \prod_{x \in \mathcal{B}_\gamma} d_x d_{x^*} c(d_{x^*}) c_x \mod 4.$$

But by (59) and (61), we have

$$d_x d_{x^*} c(d_{x^*}) c_x \equiv (2w_{x,x'} - 1)(2w_{x,x'} + 1) \equiv 4w_{x,x^*}^2 - 1 \equiv -1 \mod 4.$$

Thus, $d \equiv qc(-1)^b \mod 4$. □

Theorem 4.17. *Let* $\lambda = \lambda^*$*.Then for any* $f \in \mathcal{H}_{n!/2}$*,*

$$\varepsilon(\chi_\lambda, f) = \varepsilon(\chi_{\mathcal{Q}_p(\lambda)}, f)\varepsilon(\chi_{\mathcal{C}or_p(\lambda)}, f).$$

Proof. Write $n = pw + r$, where $r = |\mathcal{C}or_p(\lambda)|$.

First, we assume that $\lambda_{(p-1)/2} = \emptyset$. We define λ, d, q and c as in Theorem 4.16. Since $\lambda_{(p-1)/2} = \emptyset$, we have $\varepsilon(\chi_\lambda, f) = \varepsilon(\chi_{\mathcal{Q}_p(\lambda)}, f) = \varepsilon(\chi_{\mathcal{C}or_p(\lambda)}, f) = 1$ for all $f \in \mathcal{K}_{n!/2}$. We consider the case $f = \sigma_{n!/2}$. In the proof of 4.10, we see that $\varepsilon(\chi_\lambda, \sigma_{n!/2}) = \left(\frac{p}{q}\right)$. To simplify the notation, set $m = d_{\mathcal{C}or_p(\lambda)}$. By Theorem 4.16, $d \equiv (-1)^b qc \mod 4$. Furthermore,

$$\begin{aligned} d_\mu + d_\lambda &= \sum_{\gamma \in \Gamma} (|\mathcal{X}_\gamma| + |\mathcal{X}'_\gamma| + |\mathcal{Y}_\gamma| + |\mathcal{Y}'_\gamma|) \\ &= \sum_{\gamma \in \Gamma} (2|\mathcal{X}_\gamma| + |\mathcal{A}_\gamma| + 2|\mathcal{Y}_\gamma| - |\mathcal{B}_\gamma|) \\ &= 2\sum_{\gamma \in \Gamma} (\underbrace{|\mathcal{X}_\gamma)| + |\mathcal{Y}_\gamma|}_{\text{even}}) + m - 2b \\ &\equiv m + 2b \mod 4. \end{aligned}$$

Now, we derive from the proof of Proposition 4.10 $(-1)^{\frac{n-r+d_\lambda}{2}} = (-1)^{\frac{n-r-d_\lambda}{2}} = (-1)^{\frac{pw-d_\lambda}{2}} = (-1)^{\frac{q-1}{2}}$. In particular, $n - r + d_\lambda \equiv q - 1 \mod 4$. Thus,

$$\begin{aligned} n - d_\mu + d - 1 - r + m - c + 1 &= n - r + m - d_\mu + d - c \\ &\equiv n - r + d_\lambda + 2b + qc(-1)^b - c \mod 4 \\ &\equiv q - 1 + 2b + qc(-1)^b - c \mod 4. \end{aligned}$$

If b is even, then

$$n - d_\mu + d - 1 - r + m - c + 1 \equiv q - 1 + qc - c \equiv (q-1)(c+1) \equiv 0 \mod 4,$$

because q and c are odd. If b is odd, then

$$\begin{aligned} n - d_\mu + d - 1 - r + m - c + 1 &\equiv q - 1 + 2 - qc - c \mod 4 \\ &\equiv (q+1)(1-c) \mod 4 \\ &\equiv 0 \mod 4. \end{aligned}$$

Finally, using Propositions 4.10 and 4.1, and (36), we obtain

$$\begin{aligned}\varepsilon(\chi_\lambda,\sigma_{n!/2}) &= (-1)^{\frac{p-1}{4}(n-d_\mu+d-1)}\left(\frac{p}{d}\right)\\ &= (-1)^{\frac{p-1}{4}(n-d_\mu+d-1)}\left(\frac{p}{q}\right)\left(\frac{p}{c}\right)\\ &= (-1)^{\frac{p-1}{4}(n-d_\mu+d-1-r+m-c+1)}\varepsilon(\chi_{\mathcal{Q}_p(\lambda)},\sigma_{n!/2})\varepsilon(\chi_{\mathcal{C}or_p(\lambda)},\sigma_{n!/2})\\ &= \varepsilon(\chi_{\mathcal{Q}_p(\lambda)},\sigma_{n!/2})\varepsilon(\chi_{\mathcal{C}or_p(\lambda)},\sigma_{n!/2}).\end{aligned}$$

Assume now that $\lambda_{(p-1)/2}$ is non-empty. Since $\delta_{(p-1)/2}=\emptyset$, we have $\mathcal{X}'_{(p-1)/2}=\mathcal{X}_{(p-1)/2}$, that is, the diagonal hooks arising from the $(p-1)/2$-runner of λ and $\mathcal{Q}_p(\lambda)$ are the same. Denote by $\lambda^\vee$ the partition with same p-core and p-quotient as λ except $\lambda^\vee_{(p-1)/2}=\emptyset$. Then

$$\begin{aligned}\varepsilon(\chi_\lambda,f) &= \varepsilon(i,f)^{\frac{p|\lambda_{(p-1)/2}|-|\mathcal{X}_{(p-1)/2}|+d_{(p-1)/2}-1}{2}}\varepsilon(\sqrt{d_{(p-1)/2}},f)\varepsilon(\chi_{\lambda^\vee},f)\\ &= \varepsilon(i,f)^{\frac{p|\lambda_{(p-1)/2}|-|\mathcal{X}_{(p-1)/2}|+d_{(p-1)/2}-1}{2}}\varepsilon(\sqrt{d_{(p-1)/2}},f)\varepsilon(\chi_{\mathcal{Q}_p(\lambda)^\vee},f)\varepsilon(\chi_{\mathcal{C}or_p(\lambda)},f)\\ &= \varepsilon(\chi_{\mathcal{Q}_p(\lambda)},f)\varepsilon(\chi_{(\mathcal{C}or_p(\lambda)},f),\end{aligned}$$

where $d_{(p-1)/2}$ is the product of the diagonal hook lengths arising from the runner $(p-1)/2$. □

5. Verification of Navarro's conjecture for the alternating groups

We will now prove Theorem 1.1. Let n be a positive integer with p-adic expansion $n=n_0+pn_1+\cdots+n_sp^s$. Let λ be a partition of n with p-core tower $\mathcal{C}or_p^{(k)}(\lambda)=\{\lambda_{\underline{j}}\mid \underline{j}\in I^k\}$ for $k\geq 0$ such that $c_k(\lambda)=\sum_{\underline{j}\in I^k}|\lambda_{\underline{j}}|$. We then associate to λ the irreducible character of $\mathrm{N}_{\mathfrak{S}_n}(P)$

$$\psi_\lambda=\prod_{k\geq 0}\psi_{\underline{\lambda},k},$$

as above, where $\psi_{\underline{\lambda},k}\in\mathrm{Irr}(N_k)$ as in (7). If λ is not symmetric, then χ_λ and $\psi_{\underline{\lambda}}$ restrict irreducibly to $\mathfrak{A}_n$ and $\mathrm{N}_{\mathfrak{A}_n}(P)$. As above, we denote the restriction by the same symbol. If λ is symmetric, then the restriction of χ_λ to $\mathfrak{A}_n$ has two irreducible constituents χ_λ^+ and χ_λ^-. Similarly for ψ_λ. More precisely, for any $k\geq 0$ and $\underline{\lambda}\in\mathcal{MP}(p^k,n_k)$, we have $\underline{\lambda}^*=\underline{\lambda}$ and the restriction of $\psi_{\underline{\lambda},k}$ to $(Y_k\wr\mathfrak{S}_{n_k})^+$ splits into two irreducible characters $\psi^+_{\underline{\lambda},k}$ and $\psi^-_{\underline{\lambda},k}$. Then following §3.1, we label ψ_λ^+ such that $\prod_k\psi^+_{\underline{\lambda},k}$ is a constituent of $\mathrm{Res}_{\prod(Y^k\wr\mathfrak{S}_{n_k})^+}(\psi^+_{\underline{\lambda}})$. In particular, $\prod_k\psi^-_{\underline{\lambda},k}$ is a constituent of $\mathrm{Res}_{\prod(Y^k\wr\mathfrak{S}_{n_k})^+}(\psi^-_{\underline{\lambda}})$. Now, define $\Phi:\mathrm{Irr}_{p'}(\mathfrak{A}_n)\to\mathrm{Irr}_{p'}(\mathrm{N}_{\mathfrak{A}_n}(P))$ by setting

$$\Phi(\chi_\lambda)=\psi_\lambda \text{ if } \lambda\neq\lambda^*,\quad\text{and}\quad \Phi(\chi_\lambda^\pm)=\psi_\lambda^\pm \text{ otherwise.}\tag{62}$$

We need the following two lemmas.

Lemma 5.1. *If λ is a regular partition of n and $f\in\mathcal{H}_{n!/2}$, then*

$$\varepsilon(\chi_\lambda,f)=\varepsilon(\psi_\lambda,f).$$

Proof. Since λ is regular, $n=n_1p+n_2p^2+\cdots+n_sp^s$ with n_i even for all $1\leq i\leq s$. By Proposition 3.2

$$\varepsilon(\psi_\lambda,f)=\prod_{k=1}^s\varepsilon(\psi_{\underline{\lambda},k},f).$$

Hence, for $f \in \mathcal{K}_{n!/2}$, one has $\varepsilon(\psi_\lambda, f) = 1$ by Proposition 3.6, and

$$\begin{aligned}\varepsilon(\psi_\lambda, \sigma_{n!/2}) &= \prod_{k=1}^{s} \varepsilon(\psi_{\underline{\lambda},k}, \sigma_{n!/2}) \\ &= \prod_{k=1}^{s} (-1)^{\frac{(p-1)n_k}{2}} \\ &= (-1)^{\frac{p-1}{4}\sum_{k=1}^{s} n_k}.\end{aligned}$$

For $1 \leq k \leq s$, let $n'_k \in \mathbb{Z}$ such that $n_k = 2n'_k$. We have

$$\frac{n}{2} = \sum_{k=1}^{s} n'_k p^k \equiv \sum_{k=1}^{s} n'_k \mod 2,$$

because p is odd. Thus $\sum_{k=1}^{s} n_k \equiv n \mod 4$, and $\varepsilon(\psi_\lambda, \sigma_{n!/2}) = (-1)^{\frac{(p-1)n}{4}}$. The result now follows from Proposition 4.10. □

Lemma 5.2. *If λ is a singular partition of n with empty p-core, and $f \in \mathcal{H}_{n!/2}$, then*

$$\varepsilon(\chi_\lambda, f) = \varepsilon(\psi_\lambda, f).$$

Proof. By construction of λ from its p-core tower and §4.4, for all $k \geq 1$, we have

$$\mathfrak{d}_{p^k}(\lambda) = \{p^k h \mid h \in \mathfrak{d}_{\underline{\lambda},k}\},$$

where $\mathfrak{d}_{p^k}(\lambda)$ is the set of diagonal hooklengths of χ_λ divisible by p^k but not by p^{k+1} and $\mathfrak{d}_{\underline{\lambda},k}$ is the set of diagonal hooklengths of $\chi_{\underline{p}^*(k)}$ with $\underline{p}^*(k) \in I^k$. In the following, we write $\chi_k = \chi_{\underline{p}^*(k)}$. In particular, if d_λ and d_k are the number of diagonal hooks of λ and the partition with empty p-core tower except the position $\underline{p}^*(k)$ in the level k, that is equal to $\lambda_{\underline{p}^*(k)}$, then $d_\lambda = \sum_{k=1}^{s} d_k$.

Let $f \in \mathcal{H}_{n!/2}$. By (36), we obtain

$$\begin{aligned}\varepsilon(\chi_\lambda, f) &= \varepsilon(i,f)^{(n-d_\lambda)/2}\varepsilon\left(\sqrt{\prod_{h\in\mathfrak{d}(\lambda)} h}, f\right) \\ &= \prod_{k=1}^{s} \varepsilon(i,f)^{\frac{n_k p^k - d_k}{2}}\varepsilon\left(\sqrt{\prod_{h\in\mathfrak{d}_{p^k}(\lambda)} h}, f\right) \\ &= \prod_{k=1}^{s} \varepsilon(i,f)^{\frac{n_k(-1)^k - d_k}{2}}\varepsilon\left(\sqrt{p}^{kd_k}\sqrt{\prod_{h\in\mathfrak{d}_{\underline{\lambda},k}(\lambda)} h}, f\right) \\ &= \prod_{k=1}^{s} \varepsilon(i,f)^{\frac{n_k(-1)^k - d_k + n_k - d_k}{2}}\varepsilon(\sqrt{p}^{kd_k}, f)\varepsilon(\chi_k, f) \\ &= \prod_{k=1}^{s} \varepsilon(i,f)^{\frac{n_k(-1)^k - d_k + n_k - d_k + 2d_k k}{2}}\varepsilon(\psi_{\underline{\lambda},k}, f) \quad \text{by Prop. 3.7} \\ &= \varepsilon(i,f)^{\frac{1}{2}\sum_{k=1}^{s}(n_k(-1)^k - d_k + n_k - d_k + 2d_k k)}\varepsilon(\psi_\lambda, f),\end{aligned}$$

where the last equality comes from Proposition 3.2. However, if k is even, then

$$n_k(-1)^k - d_k + n_k - d_k + 2d_k k \equiv 2(n_k - d_k) \equiv 0 \mod 4$$

because n_k and d_k have the same parity. If k is odd, then

$$n_k(-1)^k - d_k + n_k - d_k + 2d_k k \equiv -n_k + 2d_k + n_k - 2d_k \equiv 0 \mod 4.$$

The result follows. □

Lemma 5.3. *Let λ be a symmetric partition with empty p-core. Then for $f \in \mathcal{H}_{n!/2}$*

$$\varepsilon(\chi_\lambda, f) = \varepsilon(\chi_{\mathfrak{r}(\lambda)}, f)\varepsilon(\chi_{\mathfrak{s}(\lambda)}, f),$$

where $\mathfrak{r}(\lambda)$ and $\mathfrak{s}(\lambda)$ are the regular and the singular parts of λ as in (47).

Proof. By assumption, the p-core of λ is empty. In particular,

$$\mathcal{D}(\lambda) = \mathcal{D}(\mathfrak{r}(\lambda)) \sqcup \mathcal{D}(\mathfrak{s}(\lambda)).$$

Write $d_\lambda = |\mathcal{D}(\lambda)|$, $d_{\mathfrak{r}(\lambda)} = |\mathcal{D}(\mathfrak{r}(\lambda))|$ and $d_{\mathfrak{s}(\lambda)} = |\mathcal{D}(\mathfrak{s}(\lambda))|$. We have

$$d_\lambda = d_{\mathfrak{r}(\lambda)} + d_{\mathfrak{s}(\lambda)}.$$

Furthermore, we have $|\lambda| = |\mathfrak{r}(\lambda)| + |\mathfrak{s}(\lambda)|$ by construction. Hence, for all $f \in \mathcal{H}_{n!}$, Equation (36) gives

$$\begin{aligned}
\varepsilon(\chi_\lambda, f) &= \varepsilon\left(i^{(|\lambda|-d_\lambda)/2}\sqrt{\prod_{h\in\mathcal{D}(\lambda)} h}, f\right)\\
&= \varepsilon\left(i^{(|\mathfrak{r}(\lambda)|-d_{\mathfrak{r}(\lambda)})/2}\sqrt{\prod_{h\in\mathcal{D}(\mathfrak{r}(\lambda))} h}, f\right)\varepsilon\left(i^{(|\mathfrak{s}(\lambda)|-d_{\mathfrak{s}(\lambda)})/2}\sqrt{\prod_{h\in\mathcal{D}(\mathfrak{s}(\lambda))} h}, f\right)\\
&= \varepsilon(\chi_{\mathfrak{r}(\lambda)}, f)\varepsilon(\chi_{\mathfrak{s}(\lambda)}, f),
\end{aligned}$$

as required. □

Assume that $\lambda = \lambda^*$. Recall $\mathcal{Q}_p(\lambda)$ is the partition with the same p-quotient as λ and with empty p-core. Proposition 4.17 and Lemma 5.3 give

$$\varepsilon(\chi_\lambda, f) = \varepsilon(\chi_{\mathcal{C}or_p(\lambda)}, f)\varepsilon(\chi_{\mathfrak{r}(\mathcal{Q}_p(\lambda))}, f)\varepsilon(\chi_{\mathfrak{s}\mathcal{Q}_p(\lambda))}, f). \tag{63}$$

Now, by Theorem 3.8 and Proposition 3.2, we have

$$\begin{aligned}
\varepsilon(\psi_\lambda, f) &= \varepsilon(\chi_{\mathcal{C}or_p(\lambda)}, f)\prod_{k=1}^{s}\varepsilon(\psi_{\underline{\mathfrak{r}(\mathcal{Q}_p(\lambda))},k}, f)\prod_{k=1}^{s}\varepsilon(\psi_{\underline{\mathfrak{s}(\mathcal{Q}_p(\lambda))},k}, f) \qquad (64)\\
&= \varepsilon(\chi_{\mathcal{C}or_p(\lambda)}, f)\varepsilon(\psi_{\mathfrak{r}(\mathcal{Q}_p(\lambda))}, f)\varepsilon(\psi_{\mathfrak{s}(\mathcal{Q}_p(\lambda))}, f).
\end{aligned}$$

However, by Lemmas 5.1 and 5.2, we have

$$\varepsilon(\chi_{\mathfrak{r}(\mathcal{Q}_p(\lambda))}, f) = \varepsilon(\psi_{\mathfrak{r}(\mathcal{Q}_p(\lambda))}, f) \quad \text{and} \quad \varepsilon(\chi_{\mathfrak{s}(\mathcal{Q}_p(\lambda))}, f) = \varepsilon(\psi_{\mathfrak{s}(\mathcal{Q}_p(\lambda))}, f).$$

Finally (63) and (64) give that

$$\varepsilon(\chi_\lambda, f) = \varepsilon(\psi_\lambda, f).$$

Hence, Φ is an $\mathcal{H}_{n!/2}$-equivariant bijection, as required.

6. Blockwise Navarro's conjecture for alternating groups

For any finite group G and any prime number p dividing $|G|$, recall that $\mathrm{Irr}(G)$ decomposes into families, the so-called p-blocks of G. Write $\mathrm{Bl}(G)$ for the set of p-blocks of G. Furthermore, we attach to any $B \in \mathrm{Bl}(G)$ its *p-defect group* D. This is a p-subgroup of G which is well-defined up to conjugation. Now, by Brauer's first main theorem [7, (15.45)], we can associate to any p-block B of G its Brauer correspondent $B' \in \mathrm{Bl}(\mathrm{N}_G(D))$. Then the *blockwise Navarro's conjecture* asserts that the number of height zero characters in B and B' fixed by $\sigma \in \mathcal{H}_n$ is the same.

6.1. **Case of p odd.** In order to discuss blockwise Navarro's conjecture for alternating groups, we will first recall some some facts about the p-blocks of symmetric and alternating groups.

It is well-know by the *Nakayama Conjecture* that for any prime p, the p-blocks of $\mathfrak{S}_n$ are labeled by the p-cores of partitions of n. More precisely, two irreducible characters of $\mathfrak{S}_n$ lie in the same p-block if and only if the partitions labeling them have the same p-core; see for example [15, Theorem 11.1]. In the following, such a p-core will be called a *p-core of n*. Note that there is here an abuse of terminology since a p-core of n is not in general a partition of n. For a p-core γ of n, we denote by B_γ the corresponding p-block of $\mathfrak{S}_n$, and we define the p-weight of B_γ by setting $w = \frac{n-|\gamma|}{p}$.

We can describe the height zero characters of B_γ in term of the p-core tower of partitions labeling characters of the block as follows. By [15, Proposition 11.5], an irreducible character χ_λ lying in the block B_γ has height zero if and only if $0 \leq c_k(\lambda) \leq p-1$ for all $k \geq 1$ with $c_k(\lambda) = \sum_{\underline{j} \in I^k} |\lambda_{\underline{j}}|$, where the Notation is as in (2).

Furthermore, without loss of generality, we can assume by [15, Proposition 11.3] that any Sylow p-subgroup D_γ of $\mathfrak{S}_{pw} \subseteq \mathfrak{S}_n$ is a defect group of B_γ. Let $pw = w_1 p + w_2 p^2 + \cdots$ denote the p-adic expansion of pw. Then by [2, page 159], we have

$$\mathrm{N}_{\mathfrak{S}_n}(D_\gamma)/D'_\gamma \simeq \mathfrak{S}_{|\gamma|} \times \prod_{k \geq 1} Y^k \wr \mathfrak{S}_{w_k}.$$

Moreover, by [2, page 158 and 159], the set $\mathrm{Irr}_0(B'_\gamma)$ of height zero characters of the Brauer correspondent $B'_\gamma \in \mathrm{N}_{\mathfrak{S}_n}(D_\gamma)$ of B_γ is

$$\mathrm{Irr}_0(B'_\gamma) = \left\{ \chi_\gamma \otimes \prod_{k \geq 1} \psi_{\underline{\lambda},k} \mid \chi_\gamma \in \mathrm{Irr}(\mathfrak{S}_{|\gamma|});\ \psi_{\underline{\lambda},k} \in \mathrm{Irr}(Y^k \wr \mathfrak{S}_{w_k}) \right\}.$$

From now on, assume p is odd. Note that $B_{\gamma^*} = \{\chi_{\lambda^*} \in \mathrm{Irr}(\mathfrak{S}_n) \mid \mathcal{C}or_p(\lambda) = \gamma\} = B^*_\gamma$. In particular, if $\gamma \neq \gamma^*$, then $B_\gamma \cap B_{\gamma^*} = \emptyset$ and B_γ contains no self-conjugate character. Then [12, (9.2)] implies that the two p-blocks B_γ and B_{γ^*} cover a unique p-block b_γ of $\mathfrak{A}_n$ (Note that $b_\gamma = b_{\gamma^*}$). Furthermore, if $\gamma = \gamma^*$ and B_γ has non-zero defect, then there is an irreducible character $\chi_\lambda \in B_\gamma$ with $\lambda \neq \lambda^*$ and [12, (9.2)] implies that B_γ again covers a unique p-block b_γ of $\mathfrak{A}_n$. Finally, for $n \geq 3$, if B_γ has defect zero and $\gamma = \gamma^*$, then $\{\chi^+_\gamma\}$ and $\{\chi^-_\gamma\}$ are two p-blocks of $\mathfrak{A}_n$ of defect zero. These two blocks are equal to their Brauer correspondent, and the blockwise Navarro's conjecture is then trivial in this case.

We remark that D_γ is a defect group of B_γ since p is odd, and $\mathrm{N}_{\mathfrak{A}_n}(D_\gamma) = \mathrm{N}_{\mathfrak{S}_n}(D_\gamma)^+$. Assume that B_γ has a non-zero defect. Then B'_γ covers a unique p-block of $\mathrm{N}_{\mathfrak{S}_n}(D_\gamma)^+$. Indeed, if $\gamma \neq \gamma^*$ then the restrictions to $\mathrm{N}_{\mathfrak{S}_n}(D_\gamma)^+$ of the characters of B'_γ form a p-block $b'_\gamma (= b'_{\gamma^*})$ of $\mathrm{N}_{\mathfrak{S}_n}(D_\gamma)^+$ covered by B'_γ and B'_{γ^*} by [12, (9.2)], and if $\gamma = \gamma^*$, then B'_γ has a self-conjugate character (since the block has a non-zero defect) and B'_γ covers a unique p-block b'_γ of $\mathrm{N}_{\mathfrak{S}_n}(D_\gamma)^+$ by [12, (9.2)]. Furthermore, by unicity of the covered block, b'_γ is the Brauer correspondent of b_γ by [12, (9.28)]. Therefore, the height zero characters of this block are identified (by lifting) with the set of irreductible characters of

$$\mathrm{N}_{\mathfrak{A}_n}(D_\gamma)/D_\gamma \simeq (\mathfrak{S}_\gamma \times \prod_{k\geq 1} Y^k \wr \mathfrak{S}_{w_k})^+.$$

Let λ be a partition of n with p-core γ and with height zero. Write $\mathcal{CT}(\lambda)$ for the p-core tower of λ with the Notation as in (2). In particular, $\lambda_\emptyset = \gamma$. Write

$$\psi_\lambda = \chi_\gamma \otimes \prod_{k\geq 1} \psi_{\underline{\lambda},k}$$

for the irreducible character of B'_γ labeled by λ (which is well-defined since $\chi_\lambda \in B_\gamma$ is of height zero). Then ψ_λ splits into one or two constituents of b'_γ whenever $\lambda \neq \lambda^*$ or $\lambda = \lambda^*$. We again write ψ_λ for the irreducible restriction in the first case, and we write $\psi_\lambda^\pm$ for the two irreducible constituents otherwise.

Theorem 6.1. *Let p be an odd prime. Let γ be a p-core of n. We assume $w > 0$. For a partition λ of n with p-core γ, define $\Phi : \mathrm{Irr}_0(b_\gamma) \to \mathrm{Irr}_0(b'_\gamma)$ by setting*

$$\Phi(\chi_\lambda) = \psi_\lambda \quad \text{if } \lambda \neq \lambda^* \quad \text{and} \quad \Phi(\chi_\lambda^\pm) = \psi_\lambda^\pm \quad \text{if } \lambda = \lambda^*.$$

Then Φ is a $\mathcal{H}_{n!/2}$-equivariant bijection. In particular, blockwise Navarro's conjecture holds for the p-blocks of alternating groups.

Proof. First, we remark that the map is well-defined. We only have to consider the case of an irreducible character $\chi_\lambda^\pm \in b_\gamma$ for $\lambda = \lambda^*$. In particular, $\mathcal{C}or_p(\lambda) = \gamma$, and by Equation (63) we have for any $f \in \mathcal{H}_{n!/2}$

$$\varepsilon(\chi_\lambda, f) = \varepsilon(\chi_\gamma, f)\varepsilon(\chi_{\mathfrak{r}(\mathcal{Q}_p(\lambda)),f})\varepsilon(\chi_{\mathfrak{s}(\mathcal{Q}_p(\lambda)),f}).$$

Now, applying the resuts of Section 3 to pw with the group $(\prod_{k\geq 1} Y^k \wr \mathfrak{S}_{w_k})^+$, Proposition 3.2 and Theorem 3.8 give

$$\varepsilon(\prod_{k\geq 1} \psi_{\lambda,k}, f) = \varepsilon(\psi_{\mathfrak{r}(\mathcal{Q}_p(\lambda))}, f)\varepsilon(\psi_{\mathfrak{r}(\mathcal{Q}_p(\lambda))}, f).$$

Again using Proposition 3.2, we obtain

$$\varepsilon(\psi_\lambda, f) = \varepsilon(\chi_\gamma, f)\varepsilon(\psi_{\mathfrak{r}(\mathcal{Q}_p(\lambda))}, f)\varepsilon(\psi_{\mathfrak{r}(\mathcal{Q}_p(\lambda))}, f),$$

and we conclude by Lemmas 5.1 and 5.2. □

6.2. **Case of $p = 2$.** First, we will prove that an analogue of Theorem 4.16 holds for $p = 2$.

Theorem 6.2. *Assume $p = 2$. We have*

$$\left(\frac{2}{d}\right) = \left(\frac{2}{q}\right)\left(\frac{2}{c}\right),$$

where c, d and q are as in Theorem 4.16.

Proof. Let $\gamma \in \{0,1\}$ be such that $\delta_\gamma \geq 0$. We write $\mathcal{X}'_\gamma$, $\mathcal{Y}'_\gamma$, $\mathcal{X}_\gamma$, $\mathcal{Y}_\gamma$ and Δ_γ for the sets labeling $\mathfrak{D}(\lambda)$, $\mathfrak{D}(\mathcal{Q}uo_p(\lambda))$, and $\mathfrak{D}(\mathcal{C}or_p(\lambda))$, respectively. If $\delta_\gamma = 0$, then the statement is trivial. Assume $\delta_\gamma > 0$. We also consider the set $\mathcal{B}_\gamma$ as in Remark 4.11. Let $x \in \mathcal{X}_\gamma$ and $\varepsilon \in \{1,3\}$ be such that $d_x = 4x+\varepsilon$. Hence, $d_{x^*} = 4\phi(x^*)+\varepsilon'$, where $\varepsilon' = 4-\varepsilon$. Furthermore, with the notation of (55),

$$c(d_x) = 4(x+\delta_\gamma)+\varepsilon \quad \text{and} \quad c(d_{x^*}) = 4(\phi(x^*)-\delta_\gamma)+\varepsilon',$$

where $c(d_{x^*})$ "exists" if and only if $\phi(x^*) \geq \delta_\gamma$. Assume $\phi(x^*) \geq \delta_\gamma$. Then

$$c(d_x)c(d_{x^*}) = d_x d_{x^*} + 4\delta_\gamma(d_{x^*}-d_x) - 16d^2 \equiv d_x d_{x^*} + 4\delta_\gamma(d_{x^*}-d_x) \mod 16.$$

Since $d_{x^*} - d_x$ is even, we obtain $(c(d_x)c(d_{x^*})^2 \equiv (d_x d_{x^*})^2 \mod 16$. Hence, $(c(d_x)c(d_{x^*})^2-1)/8 - ((d_x d_{x^*})^2-1)/8$ is even, whence

$$\left(\frac{2}{c(d_x)c(d_{x^*})}\right) = (-1)^{\frac{(c(d_x)c(d_{x^*}))^2-1}{8}} = (-1)^{\frac{(d_x d_{x^*})^2-1}{8}} = \left(\frac{2}{d_x d_{x^*}}\right). \tag{65}$$

Assume now that $0 \leq \phi(x^*) \leq \delta_\gamma - 1$. In particular, $x^* \in \mathcal{B}_\gamma$, and

$$c_{x^*} = 4(\delta_\gamma - 1 - \phi(x^*)) + \varepsilon = 4\delta_\gamma - 4 - d_{x^*} + \underbrace{\varepsilon+\varepsilon'}_{=4} = 4d - d_{x^*},$$

and we again have $(c(d_x)c_{x^*})^2 \equiv (d_x d_{x^*})^2 \mod 16$. Hence,

$$\left(\frac{2}{c(d_x)c_{x^*}}\right) = \left(\frac{2}{d_x d_{x^*}}\right). \tag{66}$$

Now, using Equations (65) and (66), like in the proof of Theorem 4.16, we obtain

$$\left(\frac{2}{d}\right) = \left(\frac{2}{q}\right)\left(\frac{2}{c}\right)\prod_{x\in\mathcal{B}_\gamma}\left(\frac{2}{d_x d_{x^*}}\right)\left(\frac{2}{c(d_{x^*})c_x}\right) = \left(\frac{2}{q}\right)\left(\frac{2}{c}\right),$$

□

Theorem 6.3. *The blockwise Navarro's conjecture holds for alternating groups at $p=2$.*

Proof. Let b_γ be a 2-block of $\mathfrak{A}_n$ covered by a 2-block B_γ of $\mathfrak{S}_n$ labeled by the 2-core γ. Write $r = |\gamma|$ and w for the 2-weight of B_γ. As above, we denote by χ_λ the irreductible character of $\mathfrak{S}_n$ labeled by λ. We also denote the irreducible characters of $\mathfrak{A}_n$ by ϑ_λ^+ for $\lambda \neq \lambda^*$ and $\vartheta_\lambda^\pm$ for $\lambda = \lambda^*$. We only have to consider the case that γ is self-conjugate and $w > 0$. Write $\mathcal{P}_\gamma$ for the set of partitions μ of $2w$ such that χ_μ has height zero or χ_μ is of height 1 and μ is self-conjugate. By [15, Proposition 12.5], we have

$$\mathrm{Irr}_0(b_\gamma) = \{\vartheta_\lambda^\pm \mid \mathcal{C}or_2(\lambda) = \gamma,\ \mathcal{Q}_2(\lambda) \in \mathcal{P}_\gamma\}.$$

By [15, (12.2)], B_γ covers only the block b_γ. Hence, b_γ is $\mathfrak{S}_n$-invariant, and by [12, Theorem 9.17] the defect group of b_γ is $D = \mathfrak{A}_n \cap \widetilde{D}$, where $\widetilde{D}$ is the defect group of B_γ. Since $\widetilde{D}$ is isomorphic to the Sylow 2-subgroup of a $\mathfrak{S}_{2w}$, it follows that D is isomorphic to the Sylow 2-subgroup of $\mathfrak{A}_{2w}$ and

$$\mathrm{N}_{\mathfrak{A}_n}(D) \simeq (\mathfrak{S}_r \times \mathrm{N}_{\mathfrak{S}_{2w}}(D))^+.$$

We remark that $(\mathrm{N}_{\mathfrak{S}_{2w}}(D))^+ = \mathrm{N}_{\mathfrak{A}_{2w}}(D)$. By [10, Theorem 5.6] applied to the principal 2-block of $\mathfrak{A}_{2w}$, the number of $2'$-characters of the principal blocks of $\mathfrak{A}_{2w}$ and of $\mathrm{N}_{\mathfrak{A}_{2w}}(D)$ is the same. By [15, Proposition 12.5] $\mathcal{P}_\gamma$ labels the set of

$2'$-characters of the principal block of $\mathfrak{A}_{2w}$. We choose a bijection θ between these two sets, and for $\mu \in \mathcal{P}_\gamma$, we set

$$\psi_\mu^{\pm} = \theta(\vartheta_\mu^{\pm}).$$

Now, the second author proved in [11] that if $w > 3$, then for all $\mu \in \mathcal{P}_\gamma$, $\vartheta_\mu^{\pm}$ and $\theta(\vartheta_\mu^{\pm})$ are fixed by all $f \in \mathcal{H}_{n!/2}$. Furthermore, for $w = 1$ and $w = 2$, the normalizer of the Sylow 2-subgroup of $\mathfrak{A}_{2w}$ is $\mathfrak{A}_{2w}$ itself. We can take θ to be the identity, that is automatically $\mathcal{H}_{n!/2}$-equivariant.

Write b'_γ for the Brauer correspondent of b_γ and B'_γ for the unique block of $\mathfrak{S}_r \times \mathrm{N}_{\mathfrak{S}_{2w}}(D)$ that covers b'_γ by [12, Corollary 9.6]. By Clifford Theory, for any $\mu \in \mathcal{P}_\gamma$, there is $\widetilde{\psi}_\mu \in \mathrm{Irr}(\mathrm{N}_{\mathfrak{S}_{2w}}(D))$ such that $\psi_\mu^{\pm}$ appears in its restriction to $(\mathrm{N}_{\mathfrak{S}_{2w}}(D))^+$ with multiplicity one. Hence, for any $\mu \in \mathcal{P}_\gamma$, we have $(\chi_\gamma \otimes \widetilde{\psi}_\mu)^{\pm} \in \mathrm{Irr}_0(b'_\gamma)$. On the other hand, by cardinality [10, Theorem 5.6], we deduce that

$$\mathrm{Irr}_0(b'_\gamma) = \{(\chi_\gamma \otimes \widetilde{\psi}_\mu)^{\pm} \mid \mu \in \mathcal{P}_\gamma\}.$$

Now, we define

$$\Phi : \mathrm{Irr}_0(b_\gamma) \to \mathrm{Irr}_0(b'_\gamma), \quad \vartheta_\lambda^{\pm} \mapsto (\chi_\gamma \otimes \psi_{\mathcal{Q}_2(\lambda)})^{\pm}.$$

We remark that Φ is a bijection by construction. If $\lambda \neq \lambda^*$, then ϑ_λ^+ and $\Phi(\vartheta_\lambda)^+$ are fixed by all $f \in \mathcal{H}_{n!/2}$. Assume that $\lambda = \lambda^*$. Write d_λ, $d_{\mathcal{Q}_2(\lambda)}$ and d_γ for the product of diagonal hooks of λ, $\mathcal{Q}_2(\lambda)$ and γ. Then by [11, Theorem 2.2] and Theorem 6.2, we obtain for any λ labeling a character of b_γ

$$\varepsilon(\chi_\lambda, f) = \left(\frac{2}{d_\lambda}\right) = \left(\frac{2}{d_\gamma}\right)\left(\frac{2}{d_{\mathcal{Q}_2(\lambda)}}\right) = \varepsilon(\chi_\gamma, f)\varepsilon(\chi_{\mathcal{Q}_2(\lambda)}, f).$$

On the other hand, by Proposition 3.2, for any $f \in \mathcal{H}_{n!/2}$

$$\varepsilon(\chi_\gamma \otimes \widetilde{\psi}_{\mathcal{Q}_2(\lambda)}, f) = \varepsilon(\chi_\gamma, f)\varepsilon(\widetilde{\psi}_{\mathcal{Q}_2(\lambda)}, f).$$

However, $\varepsilon(\widetilde{\psi}_{\mathcal{Q}_2(\lambda)}, f) = \varepsilon(\chi_{\mathcal{Q}_2(\lambda)}, f)$ because θ is $\mathcal{H}_{n!/2}$-equivariant. Thus,

$$\varepsilon(\chi_\lambda, f) = \varepsilon(\chi_\gamma \otimes \widetilde{\psi}_{\mathcal{Q}_2(\lambda)}, f),$$

as required.

□

Acknowledgements. The first author acknowledges the support of the ANR grant GeRepMod ANR-16-CE40-0010-01. The second author acknowledges the support of PSC-CUNY TRADA-47-785. The authors would like to thank the Banff International Research Station (BIRS) where conversations on this project began during the 2014 Global/Local Conjectures in Representation Theory of Finite Groups workshop. Part of this work was done at the Centre Interfacultaire Bernoulli (CIB) in the Ecole Polytechnique Federale de Lausanne (Switzerland), during the 2016 Semester Local Representation Theory and Simple Groups. The authors thank the CIB for the financial and logistical support. The authors would also like to thank the IMJ-PRG at the University of Paris Diderot and the Department of Mathematics at the Graduate Center, City University of New York for the financial and logistical support which allowed the completion of this project. The authors sincerely thank Gunter Malle for his precise reading of the paper and his helpful comments. Finally, the authors wish to thank the referee for their careful reading of the manuscript and for their suggestions that improves the reading of the paper.

References

[1] G. E. Andrews. *The theory of partitions*. Cambridge Mathematical Library. Cambridge University Press, Cambridge, 1998. Reprint of the 1976 original.
[2] P. Fong. The Isaacs-Navarro conjecture for symmetric groups. *J. Algebra.*, 260:154–161, 2003.
[3] F. Garvan, D. Kim, and D. Stanton. Cranks and t-cores. *Invent. Math.*, 101(1):1–17, 1990.
[4] K. Ireland and M. Rosen. *A Classical Introduction to Modern Number Theory*, volume 84 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1990.
[5] I. Isaacs, G. Malle, and G. Navarro. A reduction theorem for McKay conjecture. *Invent. Math.*, 170:33–101, 2007.
[6] I. M. Isaacs. Characters of solvable and symplectic groups. *Amer. J. Math.*, 95:594–635, 1973.
[7] I. M. Isaacs. *Character Theory of Finite Groups*. Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1976. Pure and Applied Mathematics, No. 69.
[8] G. James and A. Kerber. *The Representation Theory of the Symmetric Group*, volume 16 of *Encyclopedia of Mathematics and its Applications*. Addison-Wesley Publishing Co., Reading, Mass., 1981.
[9] G. Malle and B. Späth. Characters of odd degree. *Ann. of Math. (2)*, 184(3):869–908, 2016.
[10] G. O. Michler and J. B. Olsson. The Alperin-McKay conjecture holds in the covering groups of symmetric and alternating groups, $p \neq 2$. *J. Reine Angew. Math.*, 405:78–111, 1990.
[11] R. Nath. The Navarro conjecture for the alternating groups, $p = 2$. *J. Algebra Appl.*, 8(6):837–844, 2009.
[12] G. Navarro. *Characters and blocks of finite groups*, volume 250 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1998.
[13] G. Navarro. The McKay conjecture and Galois automorphisms. *Ann. of Math.*, 160:1129–1140, 2004.
[14] G. Navarro, B. Spaeth, and C. Vallejo. A reduction theorem for the Galois-McKay conjecture. *Trans. Amer. Math. Soc.*, To appear, 2020.
[15] J. B. Olsson. *Combinatorics and Representations of Finite Groups*. Vorlesungen aus dem Fachbereich Mathematik der Universität GH Essen, Heff 20, 1993.
[16] L. Ruhstorfer. The Navarro refinement of the McKay conjecture for finite groups of Lie type in defining characteristic. *Preprint, arXiv:1703.09006*, 2017.
[17] A. Turull. Above the Glauberman correspondence. *Adv. Math.*, 217(5):2170–2205, 2008.
[18] A. Turull. Strengthening the McKay conjecture to include local fields and local Schur indices. *J. Algebra*, 319(12):4853–4868, 2008.

Université Paris-Diderot Paris 7, Institut de mathématiques de Jussieu – Paris Rive Gauche, UFR de mathématiques, Case 7012, 75205 Paris Cedex 13, France.
Email address: olivier.brunat@imj-prg.fr

York College, City University of New York, 94–20 Guy R. Brewer Blvd., Jamaica, NY 11435, USA
Email address: rnath@york.cuny.edu