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STRATIFIED RADIATIVE TRANSFER IN A FLUID AND NUMERICAL APPLICATIONS TO EARTH SCIENCE*

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Abstract. New mathematical results are given for the radiative transfer equations alone and coupled with the temperature equation of a fluid: existence, uniqueness, a maximum principle, and a convergent monotone iterative scheme. Numerical tests for Earth's atmosphere and the heating of a pool by the Sun are included.

Key words. radiative transfer, temperature equation, integral equation, numerical analysis, climate modelling

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1. Introduction. Radiative transfer (RT) is an important field of physics. It appears in astronomy, nuclear physics, and heat transfer in fluid mechanics. It is also a key ingredient of climate models.

Books on RT for the atmosphere are numerous, such as [22], [15], and [4], the numerically oriented [28], and the two mathematically oriented [6] and [9].

When Planck's theory of black bodies is used, radiation involves a continuum of frequencies governed by the temperature of the emitting bodies. Studies based on the interactions of the photons with the atoms of the medium, such as [3], are currently unusable numerically in large physical domains. A much simpler formulation was proposed a hundred years ago, known as the RT equations, which is based on the energy conservation principles of continuum mechanics.

Even when the interactions with the background fluid are neglected, the RT equations involves five "spatial" variables (three coordinates for the position of each photon, and the two components of its direction). Existence of solutions of the RT equations can be proved by a Schauder-type compactness argument (see [1]), with uniqueness under appropriate additional boundedness (see Proposition 2 in [23] and [27]), or monotonicity assumptions (see Corollary 2 in [23], together with [12]).

Given the intricacy of the RT equations, several simplifying assumptions have been studied in the literature. If the scattering and absorption coefficients do not depend on the frequencies of the radiation source, the RT equations can be averaged in the frequency variable, leading to a closed system of equations for the temperature and frequency-averaged radiative intensity, known as the "grey" model. However, the frequency dependence of the scattering and absorption coefficients is fundamental to understanding several important effects in the Earth's atmosphere. For instance, Rayleigh explained the blue color of the sky by the fact that the scattering coefficient is proportional to the fourth power of the radiation frequency. Likewise, the fact that some components of Earth's atmosphere are opaque to infrared radiations seems

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important to understanding the greenhouse effect. Another simplification, of a purely geometric nature, consists of assuming that the temperature and radiative intensity are uniform on a foliation of the space by parallel planes, and therefore depend on a single position variable. As a result, the radiative intensity depends only on the projection of the photon's direction on the orthogonal axis to these planes. This is known as the "slab symmetry" assumption, which appears in the "Milne problem" for planetary or stellar atmospheres (see [6] for a detailed physical discussion of the Milne problem, and [11] for the corresponding mathematical theory).

The term "radiative transfer" usually refers to the interaction of radiation with a fixed background material. But, of course, radiation obviously deposits energy in the background fluid, gas, or plasma, as well as momentum, through the radiation pressure and, conversely, high speed fluid motion obviously modifies such processes as Compton scattering (scattering of a photon by a free electron at rest) by Doppler effect. Therefore, in full generality, the equation for the radiation intensity is coupled with the fluid equations. This coupling is studied under the name of "radiation hydrodynamics" (see [26] for the coupling with ideal fluids, and [24]).

The most general studies of radiation hydrodynamics mentioned above involve high speed (possibly relativistic) fluid motion. In the present paper, we consider radiation passing through an incompressible fluid, or a compressible fluid at low Mach number. Thus our setting will be intermediate between radiation hydrodynamics as in [26], [24], and as in [10]. This last reference considers the coupling of the grey model of RT with a background material at rest. See also [27][1] for an existence result for the general system in three dimensions (3D), yet without the monotone properties used by the numerical algorithm, which is at the core of this study. The radiation energy is deposited in the background medium in the form of heat, and appears as a source term in the heat equation for the temperature, while the black body radiation of the background medium appears as a source term in the RT equation for the radiative intensity. Our model retains the fluid motion equation, as well as the frequency dependence of the radiation field, which is essential for applications to Earth's climate.

We shall, however, make another simplification, referred to as the "stratification or parallel plane assumption": while the radiation intensity and temperature depend on all three position coordinates, only one of these coordinates is retained in the computation of the streaming operator acting on the radiative intensity, while the two other coordinates appear only as parameters in the RT equation. The stratified approximation is used when the radiation source is far—as in the case of the Sun—and the radiative intensity deposited at the boundary of the computational domain is uniform or at least slowly varying in the tangential directions to this surface.

In 2005 Evans and Marshak wrote in Chapter 4 of [22] a review of the numerical methods available for radiative transfer alone. Today, judging from [5], the situation has not changed: SHDOM (Spherical Harmonic Discrete Ordinate Method) and Monte-Carlo are the two most popular methods. While reviewing the current situation for the RT equations in [2] we implemented a finite element version of SHDOM and found that the method was incapable, unless a huge number of degrees of freedom is used, of giving results with the accuracy needed to differentiate between small variations on the absorption coefficient.

On the other hand an integral formulation present in [6] turned out to be much more precise and also computationally much cheaper. A fixed-point iteration of this

[1]While this paper was being reviewed, [27] was brought to our attention.

nonlinear integral formulation, known in the RT community as "iterations on the sources," was shown to be monotone in [25], a property which seems to have escaped earlier studies. Finally, in [14] the method was extended to include the temperature equation of the fluid and also to handle Rayleigh scattering while retaining monotonicity. While [14] is more numerically oriented, the present article gives the convergence proofs as well.

The RT equations are presented in section 2. After this, a cascade of simplifications are discussed: the stratified approximation, the decoupling from the fluid, and Milne problem techniques originating from [11] (see also [23]).

In section 3, the stratified RT decoupled from the fluid is analyzed in the case of isotropic scattering. Existence of a solution is proved by using the convergent monotone iterative scheme proposed in [2]. A maximum principle in the line of [23], [11] is also presented.

Uniqueness issues are discussed in section 4. The proofs are far from straightforward, and heavily rely on ideas in [23]. It may be interesting to compare Mercier's monotonicity structure for the RT equation, which is quite involved, with the general observation [7] on order preserving maps in L^1 leaving the integral invariant.

In section 5, the above results are extended to the nonisotropic case of scattering with the Rayleigh phase function.

Finally, in section 6 existence, uniqueness, and monotone convergence of the fixed-point iterations are proved for the RT equation coupled with the temperature equation of a fluid whose velocity field is known.

Three numerical applications are presented in section 7. The first one is a numerical simulation of the RT in the atmosphere with real data for the frequency dependent absorption coefficient κ_ν. The numerical method is sufficiently accurate to study the effect of variations of κ_ν in part of the spectrum, much like changing the composition of the atmosphere by adding more $\mathtt{CO}_2$ or other greenhouse gases. The problem is one-dimensional in space. The second example is the study of the temperature in a pond heated by the Sun. For this problem RT is coupled with the Navier–Stokes equations. The geometry is academic, in two dimensions (2D); its object is to show the feasibility of the numerical method for such coupled problems. The third problem is also a feasibility study which shows that it is possible to make a three-dimensional computation of the wind in the atmosphere of a planet heated by the Sun and subject to thermal diffusion. The computing times show that the method could be used in real-life situations.

2. Fundamental equations and approximations. Finding the temperature T in a fluid heated by electromagnetic radiations is a complex problem because interactions of photons with atoms of the medium involve rather intricate quantum phenomena. A first simplifying assumption is that of local thermodynamic equilibrium (LTE): at each point in the fluid, there is a well-defined electronic temperature. In that case, one can write a kinetic equation for the radiative intensity $I_\nu(\boldsymbol{x}, \boldsymbol{\omega}, t)$ at time t, at position $\boldsymbol{x}$, and in the direction ω for photons of frequency ν, in terms of the temperature field $T(\boldsymbol{x}, t)$:

$$\begin{aligned}\frac{1}{c}\partial_t I_\nu + \boldsymbol{\omega}\cdot\nabla I_\nu + \rho\bar{\kappa}_\nu a_\nu \left[I_\nu - \tfrac{1}{4\pi}\int_{\mathbb{S}^2} p_\nu(\boldsymbol{\omega},\boldsymbol{\omega}')I_\nu(\boldsymbol{\omega}')\mathrm{d}\omega'\right] \\ = \rho\bar{\kappa}_\nu(1-a_\nu)[B_\nu(T)-I_\nu].\end{aligned} \tag{2.1}$$

In this equation, ∇ designates the gradient with respect to the position $\boldsymbol{x}$, while

$$B_\nu(T) = \frac{2h\nu^3}{c^2[\mathrm{e}^{\frac{h\nu}{kT}} - 1]} \tag{2.2}$$

is the Planck function at temperature T, with h the Planck constant, c the speed of light in the medium (assumed to be constant), and k the Boltzmann constant. Notice that

$$\int_0^\infty B_\nu(T)\mathrm{d}\nu = \bar{\sigma}T^4\,, \qquad \bar{\sigma} = \frac{2\pi^4 k^4}{15c^2h^3}\,, \tag{2.3}$$

where $\pi\bar{\sigma}$ is the Stefan–Boltzmann constant.

The intricacy of the interaction of photons with atoms of the medium is contained in three quantities: (1) the mass-absorption $\bar{\kappa}_\nu$, which is the fraction of radiative intensity at frequency ν that is absorbed per unit length; (2) the scattering albedo a_ν; and (3) a probability of scattering from directions $\boldsymbol{\omega}'$ to $\boldsymbol{\omega}$. Indeed, a photon of frequency ν travelling in a direction $\boldsymbol{\omega}'$ may be deflected by the atoms of the medium in a new direction $\boldsymbol{\omega}$. The proportion of deflected photons $a_\nu \in (0,1)$ is called the scattering albedo. Furthermore, if $p_\nu(\boldsymbol{\omega},\boldsymbol{\omega}') \geq 0$ is the probability density of scattering from $\boldsymbol{\omega}'$ to ω the scattered intensity is (see [9, p. 74]): $\frac{a_\nu\bar{\kappa}_\nu}{4\pi}\int_{\mathbb{S}^2} p_\nu(\boldsymbol{\omega},\boldsymbol{\omega}')I_\nu(\boldsymbol{\omega}')\mathrm{d}\omega'$. Probabilities sum up to 1, so $\frac{1}{4\pi}\int_{\mathbb{S}^2} p_\nu(\boldsymbol{\omega},\boldsymbol{\omega}')\mathrm{d}\omega' = \frac{1}{4\pi}\int_{\mathbb{S}^2} p_\nu(\boldsymbol{\omega},\boldsymbol{\omega}')\mathrm{d}\omega = 1$.

The kinetic equation (2.1) is coupled to the fluid equations solely by the local conservation of energy. When the fluid is incompressible, density ρ, pressure p, and velocity fields $\boldsymbol{u}$ satisfy the Navier–Stokes equations

$$\begin{cases} \partial_t\rho + \boldsymbol{u}\cdot\nabla\rho = 0, & \nabla\cdot\boldsymbol{u} = 0, \\ \partial_t\boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} - \dfrac{\mu_F}{\rho}\Delta\boldsymbol{u} + \dfrac{1}{\rho}\nabla p = \mathbf{g}\,, \end{cases} \tag{2.4}$$

where Δ is the Laplacian in the $\boldsymbol{x}$ variable. Here, $\mathbf{g}$ is the gravity, while μ_F is the fluid viscosity. For the applications discussed in section 7, namely the Earth's atmosphere below 12km and pools, air and water are incompressible to a very good precision (see the low Mach number limit theorem in [18]).

The total energy density is the sum of the kinetic energy density of the fluid, of the internal energy of the fluid, and of the radiative energy. Subtracting the kinetic energy balance equation from the local conservation of energy, neglecting the viscous heating term $\frac{1}{2}\mu_F|\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T|^2$ on the right-hand side of the equality above, which is legitimate assuming that the variations of $|\boldsymbol{u}|^2$ times μ_F are small, we arrive at

$$\begin{aligned} \rho c_V(\partial_t T + \boldsymbol{u}\cdot\nabla T) =& \nabla\cdot(\rho c_P\kappa_T\nabla T) \\ &+ \int_0^\infty \rho\bar{\kappa}_\nu(1-a_\nu)\left(\int_{\mathbb{S}^2} I_\nu(\boldsymbol{\omega})\mathrm{d}\boldsymbol{\omega} - 4\pi B_\nu(T)\right)\mathrm{d}\nu\,, \end{aligned} \tag{2.5}$$

where T is the temperature, while c_V, c_P are the specific heat capacity at constant volume and constant pressure, respectively, and κ_T is the thermal diffusivity.

Summarizing, the kinetic equation (2.1) for the radiative intensity is coupled to the incompressible Navier–Stokes equations (2.4) and to the drift diffusion equation

(2.5) for the temperature. The resulting system is

$$
\begin{cases}
\frac{1}{c}\partial_t I_\nu + \boldsymbol{\omega}\cdot\nabla I_\nu + \rho\bar{\kappa}_\nu a_\nu \left[I_\nu - \frac{1}{4\pi}\int_{\mathbb{S}^2} p_\nu(\boldsymbol{\omega},\boldsymbol{\omega}') I_\nu(\boldsymbol{\omega}')\mathrm{d}\omega' \right] \\
\qquad\qquad\qquad\qquad = \rho\bar{\kappa}_\nu(1-a_\nu)[B_\nu(T) - I_\nu]\,, \\
\rho c_V(\partial_t T + \boldsymbol{u}\cdot\nabla T) - \nabla\cdot(\rho c_P \kappa_T \nabla T) \\
\qquad = \int_0^\infty \rho\bar{\kappa}_\nu(1-a_\nu)\left(\int_{\mathbb{S}^2} I_\nu(\boldsymbol{\omega})\mathrm{d}\boldsymbol{\omega} - 4\pi B_\nu(T) \right)\mathrm{d}\nu\,, \\
\partial_t \boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} - \frac{\mu_F}{\rho}\Delta\boldsymbol{u} + \frac{1}{\rho}\nabla p = \mathbf{g}\,, \\
\partial_t\rho + \boldsymbol{u}\cdot\nabla\rho = 0, \qquad \nabla\cdot\boldsymbol{u} = 0\,.
\end{cases}
\tag{2.6}
$$

This system is supplemented with appropriate initial and boundary conditions. Assuming, for instance, that the spatial domain is an open subset Ω of $\mathbb{R}^3$ with C^1, or piecewise C^1 boundary $\partial\Omega$, and denoting by $\boldsymbol{n}$ the outward unit normal field on $\partial\Omega$, the following boundary conditions are natural:

$$
\begin{aligned}
&I_\nu(\boldsymbol{x},\boldsymbol{\omega},t) = Q_\nu(x,\omega,t)\,, \qquad x\in\partial\Omega\,,\ \boldsymbol{\omega}\cdot\boldsymbol{n}_{\boldsymbol{x}} < 0\,,\ \nu > 0\,, \\
&\boldsymbol{u}|_{\partial\Omega} = 0\,, \qquad \left.\frac{\partial T}{\partial n}\right|_{\partial\Omega} = 0\,.
\end{aligned}
\tag{2.7}
$$

The first boundary condition tells us that the radiative intensity of incoming photons ($\boldsymbol{\omega}\cdot\boldsymbol{n}_x < 0$) at the boundary of the spatial domain is known, which is a typical admissible boundary condition for kinetic models; the second boundary condition is the classical Dirichlet boundary condition for the velocity field, solution of the Navier–Stokes equations, while the last boundary condition, the Neuman condition for the temperature, corresponds to the absence of heat flux at the boundary of the spatial domain. (Of course, this is just one example of a boundary condition for the heat equation; other boundary conditions could also be considered—for instance, one could have mixed Dirichlet–Neuman, or even Robin conditions on the temperature.) Notice that there is no boundary condition for the density ρ, since the velocity field $\boldsymbol{u}$ is tangent (and even vanishes) at the boundary $\partial\Omega$.

Finally, one should specify initial conditions of the form

$$
\begin{aligned}
&I_\nu(\boldsymbol{x},\boldsymbol{\omega},0) = I_\nu^{in}(x,\omega)\,, \qquad x\in\Omega\,,\ \boldsymbol{\omega}\in\mathbb{S}^2\,,\ \nu>0\,, \\
&\rho|_{t=0} = \rho^{in}\,, \qquad \boldsymbol{u}|_{t=0} = \boldsymbol{u}^{in}\,, \qquad T|_{t=0} = T^{in}.
\end{aligned}
\tag{2.8}
$$

Neglecting the viscous heating term as explained above has an important consequence on the structure of this system, which can be thought of as "block triangular." In other words, one can first solve for $\rho, \boldsymbol{u}, p$ the Navier–Stokes equations (2.4), then the last three equations in the system (2.6) above. The mathematical theory of (2.4) has been discussed in great detail by Lions in [21]. Then, the density ρ and velocity field $\boldsymbol{u}$ are known and appear as coefficients in the coupled system of the RT equation (2.1) and of the heat drift-diffusion equation (2.5). This coupling must be studied in detail. In the next two sections, we discuss simplified model equations deduced from (2.6).

2.1. Stratified RT. Let (x, y, z) be the Cartesian coordinates of the point $\boldsymbol{x}\in\mathbb{R}^3$, with z denoting the altitude/depth.

Assume that the radiation source (henceforth referred to as "the Sun") is far away in the direction $z > 0$, and is independent of x and y. The radiation spectrum of this source is that of a black body at temperature T_S, that is, the Planck function $B_\nu(T_S)$. With such a radiation source, it is natural to assume that the temperature field T is slowly varying with x and y, so that $|\partial_x T|+|\partial_y T|\ll|\partial_z T|$ and that I_ν is also slowly varying in x and y so that $|\partial_x I_\nu|+|\partial_y I_\nu|\ll|\partial_z I_\nu|$.

Similarly, we further assume that $|\frac{1}{c}\partial_t I_\nu|\ll|\partial_z I_\nu|$, and forget the initial condition on I_ν, so that the time dependence of the radiative intensity is governed solely by the evolution of the temperature field through the RT equation (2.1).

With this assumption, the streaming term $\frac{1}{c}\partial_t I_\nu+\boldsymbol{\omega}\cdot\nabla I_\nu$ reduces to $\mu\partial_z I_\nu$, where μ is the cosine of the angle of $\boldsymbol{\omega}$ with the z axis. Henceforth, the spatial domain is $\Omega=\mathbb{O}\times(z_m,z_M)$, where $\mathbb{O}$ is an open subset of $\mathbb{R}^2$ with C^1 boundary.

Then (2.6) becomes (see [28])

$$\begin{cases}\mu\partial_z I_\nu+\rho\bar\kappa_\nu I_\nu=\rho\bar\kappa_\nu(1-a_\nu)B_\nu(T)+\frac12\rho\bar\kappa_\nu a_\nu\displaystyle\int_{-1}^1 p_\nu(\mu,\mu')I_\nu(z,\mu',t)\mathrm{d}\mu'\,,\\ \partial_t T+\boldsymbol{u}\cdot\nabla T-\frac{c_P}{c_V}\kappa_T\Delta T=\frac{4\pi}{c_V}\displaystyle\int_0^\infty\bar\kappa_\nu(1-a_\nu)\left(\frac12\int_{-1}^1 I_\nu\mathrm{d}\mu-B_\nu(T)\right)\mathrm{d}\nu\,,\\ I_\nu(x,y,z_M,\mu,t)|_{\mu<0}=Q^-(\mu)B_\nu(T_S)\,,\qquad I_\nu(x,y,z_m,\mu,t)|_{\mu>0}=Q_\nu^+(\mu)\,,\\ \left.\dfrac{\partial T}{\partial n}\right|_{\partial\Omega}=0\,,\qquad\qquad T|_{t=0}=T^{in}\,.\end{cases}\tag{2.9}$$

That $I_\nu(z_m,\mu,t)|_{\mu>0}=0$, i.e., $Q_\nu^+(\mu)=0$, is natural since no radiation comes from the bottom of the spatial domain. Yet, by the law of black bodies, radiation could also come from the bottom, but more general boundary conditions could be handled by the same analysis. In fact, in [9] and other references, it is assumed that most of the energy from the Sun is in the form of visible light and is essentially unaffected by crossing the atmosphere, so that it is equivalent to a source of energy located at $z=0$. Recall that it makes physical sense to take $Q^-(\mu)=\mu Q'\cos\theta$, where θ is the latitude on Earth, while μ is the cosine of the observation angle. The fluid velocity field $\boldsymbol{u}$ is given, assumed to be divergence-free, and regular enough for (2.9) to make sense. Note that by rescaling the time variable, $\boldsymbol{u}$, and κ_T appropriately, the factor $4\pi/\rho c_V$ can be replaced with 1.

2.2. RT decoupled from hydrodynamics. When $\kappa_T=0$, and the fluid is at rest, the left-hand side of temperature equation is zero, so that the fluid equations are decoupled from the RT equation (2.1). Let us first consider the case of isotropic scattering, namely $p_\nu(\mu,\mu')=1$ at all frequencies ν. Then the system becomes (see [2])

$$(\mu\partial_\tau+\kappa_\nu)I_\nu(\tau,\mu)=\kappa_\nu a_\nu J_\nu(\tau)+\kappa_\nu(1-a_\nu)B_\nu(T(\tau))\,,\tag{2.10}$$

$$I_\nu(0,\mu)=Q_\nu^+(\mu)\,,\quad I_\nu(Z,-\mu)=Q_\nu^-(\mu)\,,\qquad 0<\mu<1\,,\tag{2.11}$$

$$\int_0^\infty\kappa_\nu(1-a_\nu)B_\nu(T(\tau))\mathrm{d}\nu=\int_0^\infty\kappa_\nu(1-a_\nu)J_\nu(\tau)\mathrm{d}\nu,\tag{2.12}$$

with the notation $Q_\nu^-(\mu)=Q^-(-\mu)B_\nu(T_S)$ and

$$J_\nu(\tau):=\frac12\int_{-1}^1 I_\nu(\tau,\mu)\mathrm{d}\mu\,.\tag{2.13}$$

In these equations, we have replaced $\bar\kappa_\nu$ by κ_ν and the height $z\in(z_m,z_M)$ by τ, analogous to the "optical depth" (see, for instance, [9], or formula (51) in Chapter I of [6]), defined as follows.

Pick $\rho_0 > 0$, some "reference" density of the fluid. For instance, ρ_0 could be the average density in the fluid, or the density at some reference altitude z. Indeed, the following expressions for the atmospheric density ρ in terms of the altitude z are found in the literature: $\rho(z) = \rho_0 \mathrm{e}^{-z}$ or $\rho(z) = \rho_0 - \rho_1 z$. The new variable τ and the absorption coefficient κ_ν are defined as follows:

$$\tau := \int_{z_m}^{z} \frac{\rho(\zeta)}{\rho_0} \mathrm{d}\zeta \qquad \text{and } \kappa_\nu := \rho_0 \bar{\kappa}_\nu \,. \tag{2.14}$$

Equations (2.10) and (2.12) imply that

$$\partial_\tau \int_0^\infty \int_{-1}^1 \mu I_\nu(\tau,\mu) \mathrm{d}\mu \mathrm{d}\nu = 0. \tag{2.15}$$

We have ignored the dependence in x, y of T and I_ν, since x, y are mere parameters in these equations, which are anyway completely decoupled from the fluid equations.

Assuming that $0 < \kappa_\nu \le \kappa_M$ and $0 \le a_\nu < 1$ for all $\nu > 0$, we see that (2.12) and (2.13) define T as a functional of I, henceforth denoted $T[I]$. Equivalently, one can consider J_ν as a radiative intensity independent of μ, and observe that (2.12) and (2.13) imply that $T[I]$ is also a $T[J]$. Thus (2.10)–(2.12) can be recast as

$$\begin{cases} (\mu\partial_\tau + \kappa_\nu) I_\nu(\tau,\mu) = \kappa_\nu \mathcal{S}_\nu[J] := \kappa_\nu \left(a_\nu J_\nu(\tau) + \kappa_\nu(1-a_\nu) B_\nu(T[J](\tau))\right) , \\ I_\nu(0,\mu) = Q_\nu^+(\mu) \,, \qquad I_\nu(Z,-\mu) = Q_\nu^-(\mu) \,, \qquad 0 < \mu < 1 \,. \end{cases} \tag{2.16}$$

Throughout this article, we use the exponential integrals

$$E_p(X) := X^{1-p} \int_X^\infty \frac{e^{-z}}{z^p} \mathrm{d}z = \int_0^1 e^{-X/\mu} \mu^{p-2} \mathrm{d}\mu \,, \qquad X > 0 \,. \tag{2.17}$$

Lemma 2.1. *The following inequality holds:*

$$\tfrac{1}{2} \sup_{0 \le t \le Z} \int_0^Z E_1(\kappa|\tau - t|) \kappa \mathrm{d}\tau \le C_1(\kappa) \,,$$

where $\kappa \mapsto C_1(\kappa)$ is monotone increasing from $\mathbb{R}^+$ to $\mathbb{R}^+$, and less than 1.

Proof. With $s = \kappa t$, observe that

(2.18) ◻

$$\begin{aligned} \int_0^Z E_1(\kappa|\tau - t|) \kappa \mathrm{d}\tau &= \int_0^{\kappa Z} E_1(|\sigma - s|) \mathrm{d}\sigma = \int_{\mathbf{R}} E_1(|\sigma - s|) 1_{[0,\kappa Z]}(\sigma) \mathrm{d}\sigma \\ &= \int_{\mathbf{R}} E_1(|\theta|) 1_{[-s,\kappa Z - s]}(\theta) \mathrm{d}\theta \le \int_{\mathbf{R}} E_1(|\theta|) 1_{[-\kappa Z/2, \kappa Z/2]}(\theta) \mathrm{d}\theta \\ &= 2 \int_0^{\kappa Z/2} E_1(\theta) \mathrm{d}\theta \le 2 \int_0^{Z\kappa_M/2} E_1(\theta) \mathrm{d}\theta =: 2C_1(\kappa) \,. \end{aligned}$$

The first inequality above is the elementary rearrangement inequality (Theorem 3.4 in [20]). Now C_1 is obviously increasing since $E_1 > 0$, and

$$C_1(\kappa) = \int_0^{Z\kappa/2} E_1(\theta) \mathrm{d}\theta < \int_0^\infty E_1(\theta) \mathrm{d}\theta = \int_0^\infty \left(\int_1^\infty \frac{e^{-\theta y}}{y} \mathrm{d}y \right) \mathrm{d}y = \int_1^\infty \frac{\mathrm{d}y}{y^2} = 1 \,.$$

LEMMA 2.2. *Let*

$$S_\nu(\tau) = \tfrac{1}{2}\int_0^1 \left(e^{-\frac{\kappa_\nu\tau}{\mu}}Q_\nu^+(\mu) + e^{-\frac{\kappa_\nu(Z-\tau)}{\mu}}Q_\nu^-(\mu)\right)\mathrm{d}\mu\,. \tag{2.19}$$

Problem (2.10)–(2.13) *is equivalent to* (2.12), *plus the integral equation*

$$J_\nu(\tau) = S_\nu(\tau) + \tfrac{1}{2}\int_0^Z E_1(\kappa_\nu|\tau-t|)\kappa_\nu(a_\nu J_\nu(t) + (1-a_\nu)B_\nu(T(t)))\mathrm{d}t\,. \tag{2.20}$$

Proof. Applying the method of characteristics shows that

$$\begin{aligned}I_\nu(\tau,\mu) =& e^{-\frac{\kappa_\nu\tau}{\mu}}Q_\nu^+(\mu)\mathbf{1}_{\mu>0} + e^{-\frac{\kappa_\nu(Z-\tau)}{|\mu|}}Q_\nu^-(|\mu|)\mathbf{1}_{\mu<0}\\ &+ \mathbf{1}_{\mu>0}\int_0^\tau e^{-\frac{\kappa_\nu(\tau-t)}{\mu}}\tfrac{\kappa_\nu}{\mu}\mathcal{S}_\nu[J](t)\mathrm{d}t + \mathbf{1}_{\mu<0}\int_\tau^Z e^{-\frac{\kappa_\nu(t-\tau)}{|\mu|}}\tfrac{\kappa_\nu}{\mu}\mathcal{S}_\nu[J](t)\mathrm{d}t\,.\end{aligned} \tag{2.21}$$

One integrates both sides of this identity in μ, exchanges the order of integration by Tonelli's theorem, and changes variables in the inner integral, observing that

$$\int_0^1 e^{-\frac{X}{\mu}}\frac{\mathrm{d}\mu}{\mu} = \int_1^\infty \frac{e^{-Xy}}{y}\mathrm{d}y = \int_X^\infty \frac{e^{-z}}{z}\mathrm{d}z = E_1(X)\,.$$

Thus (2.20) holds ∎

3. Analysis of problem (2.10)–(2.12). In order to solve numerically (2.10)–(2.12), one uses the method of iteration on the sources. Starting from some appropriate (I_ν^0, T^0), one constructs a sequence (I_ν^n, T^n) by the following prescription:

$$\begin{cases}(\mu\partial_\tau + \kappa_\nu)I_\nu^{n+1}(\tau,\mu) = \kappa_\nu\mathcal{S}_\nu[J^n],\\ I_\nu^{n+1}(0,\mu) = Q_\nu^+(\mu)\,, \qquad I_\nu^{n+1}(Z,-\mu) = Q_\nu^-(\mu)\,, \qquad 0<\mu<1\,.\end{cases} \tag{3.1}$$

Note that $\mathcal{S}_\nu[J^n] := a_\nu J_\nu^n(t) + (1-a_\nu)B_\nu(T^n(t))$ does not depend on μ. Hence, it is

$$\begin{aligned}&J_\nu^{n+1}(\tau) = S_\nu(\tau) + \tfrac{1}{2}\int_0^Z E_1(\kappa_\nu|\tau-t|)\kappa_\nu(a_\nu J_\nu^n(t) + (1-a_\nu)B_\nu(T^n(t)))\mathrm{d}t\,,\\ &\int_0^\infty \kappa_\nu(1-a_\nu)B_\nu(T^{n+1}(\tau))\mathrm{d}\nu = \int_0^\infty \kappa_\nu(1-a_\nu)J_\nu^{n+1}(\tau)\mathrm{d}\nu\,.\end{aligned} \tag{3.2}$$

As in (2.21), the method of characteristics shows that

$$\begin{aligned}I_\nu^{n+1}(\tau,\mu) =& e^{-\frac{\kappa_\nu\tau}{\mu}}Q_\nu^+(\mu)\mathbf{1}_{\mu>0} + e^{-\frac{\kappa_\nu(Z-\tau)}{|\mu|}}Q_\nu^-(|\mu|)\mathbf{1}_{\mu<0}\\ &+ \mathbf{1}_{\mu>0}\int_0^\tau e^{-\frac{\kappa_\nu(\tau-t)}{\mu}}\tfrac{\kappa_\nu}{\mu}\mathcal{S}_\nu[J^n]\mathrm{d}t + \mathbf{1}_{\mu<0}\int_\tau^Z e^{-\frac{\kappa_\nu(t-\tau)}{|\mu|}}\tfrac{\kappa_\nu}{|\mu|}\mathcal{S}_\nu[J^n]\mathrm{d}t\,.\end{aligned} \tag{3.3}$$

Since $B_\nu \geq 0$, this formula shows, by a straightforward induction argument, that

$$I_\nu^0 \geq 0\,, \quad T^0 \geq 0\,, \quad Q_\nu^\pm \geq 0 \implies I_\nu^n \geq 0\,.$$

Moreover,

$$
\begin{aligned}
I_\nu^{n+1}(\tau,\mu)-I_\nu^n(\tau,\mu)=\mathbf{1}_{\mu>0}\int_0^\tau e^{-\frac{\kappa_\nu(\tau-t)}{\mu}}\tfrac{\kappa_\nu}{\mu}a_\nu(J_\nu^n(t)-J_\nu^{n-1}(t))\mathrm{d}t\\
+\mathbf{1}_{\mu>0}\int_0^\tau e^{-\frac{\kappa_\nu(\tau-t)}{\mu}}\tfrac{\kappa_\nu}{\mu}(1-a_\nu)(B_\nu(T^n(t))-B_\nu(T^{n-1}(t)))\mathrm{d}t\\
+\mathbf{1}_{\mu<0}\int_\tau^Z e^{-\frac{\kappa_\nu(t-\tau)}{|\mu|}}\tfrac{\kappa_\nu}{|\mu|}a_\nu(J_\nu^n(t)-J_\nu^{n-1}(t))\mathrm{d}t\\
+\mathbf{1}_{\mu<0}\int_\tau^Z e^{-\frac{\kappa_\nu(t-\tau)}{|\mu|}}\tfrac{\kappa_\nu}{|\mu|}(1-a_\nu)(B_\nu(T^n(t))-B_\nu(T^{n-1}(t)))\mathrm{d}t\,.
\end{aligned}
$$

Since B_ν is nondecreasing for each $\nu>0$, formula (2.12) shows that

$$J_\nu^n\ge J_\nu^{n-1}\implies T^n\ge T^{n-1}\,,$$

and we conclude from the equality above that

$$I_\nu^0=0\,,\quad T^0=0\,,\quad Q_\nu^\pm\ge 0\implies\begin{cases}0\le I_\nu^1\le I_\nu^2\le\cdots\le I_\nu^n\le\cdots\,,\\ 0\le T^1\le T^2\le\cdots\le T^n\le\cdots\,.\end{cases}$$

Integrating both sides of (3.2) over $[0,Z]$ in τ implies that

$$
\begin{aligned}
\int_0^Z J_\nu^{n+1}(\tau)\mathrm{d}\tau&=\int_0^Z S_\nu(\tau)\mathrm{d}\tau+\tfrac12\int_0^Z\left(\int_0^Z E_1(\kappa_\nu|\tau-t|)\kappa_\nu\mathrm{d}\tau\right)\mathcal{S}_\nu[J^n]\mathrm{d}t\\
&\le\int_0^Z S_\nu(\tau)\mathrm{d}\tau+\tfrac12\sup_{0\le t\le Z}\int_0^Z E_1(\kappa_\nu|\tau-t|)\kappa_\nu\mathrm{d}\tau\int_0^Z\mathcal{S}_\nu[J^n]\mathrm{d}t\,.
\end{aligned}
$$

Thus by Lemma 2.1

$$\int_0^Z J_\nu^{n+1}(\tau)\mathrm{d}\tau\le\int_0^Z S_\nu(\tau)\mathrm{d}\tau+C_1(\kappa_\nu)\int_0^Z\mathcal{S}_\nu[J^n]\mathrm{d}t\,.$$

By multiplying both sides of this inequality by κ_ν and integrating into ν, one finds that

$$\int_0^\infty\int_0^Z\kappa_\nu J_\nu^{n+1}(\tau)\mathrm{d}\tau\mathrm{d}\nu\le\int_0^\infty\int_0^Z\left(\kappa_\nu S_\nu(\tau)+C_1(\kappa_M)\kappa_\nu\mathcal{S}_\nu[J^n]\right)\mathrm{d}t\mathrm{d}\nu\,.$$

At this point, we recall that $T^n=T[J_\nu^n]$, so that

$$\int_0^\infty\kappa_\nu(1-a_\nu)B_\nu(T^n(t))\mathrm{d}\nu=\int_0^\infty\kappa_\nu(1-a_\nu)J_\nu^n(t)\mathrm{d}\nu\,,\tag{3.4}$$

and hence

$$\int_0^\infty\int_0^Z\kappa_\nu J_\nu^{n+1}(\tau)\mathrm{d}\tau\mathrm{d}\nu\le C_1(\kappa_M)\int_0^\infty\int_0^Z\kappa_\nu J_\nu^n(t)\mathrm{d}t\mathrm{d}\nu+\int_0^\infty\int_0^Z\kappa_\nu S_\nu(\tau)\mathrm{d}\tau\mathrm{d}\nu\,.$$

The expression of the source term can be slightly reduced, by integrating out the τ variable:

$$\int_0^Z\kappa_\nu e^{-\frac{\kappa_\nu\tau}{\mu}}\mathrm{d}\tau=\int_0^Z\kappa_\nu e^{-\frac{\kappa_\nu(Z-\tau)}{\mu}}\mathrm{d}\tau=\mu\left(1-e^{-\frac{\kappa_\nu Z}{\mu}}\right)\,,$$

so that

$$0\le\int_0^\infty\kappa_\nu\int_0^Z S_\nu(\tau)\mathrm{d}\tau\mathrm{d}\nu\le\tfrac12\int_0^\infty\kappa_\nu\int_0^1(Q_\nu^+(\mu)+Q_\nu^-(\mu))\mu\mathrm{d}\mu\mathrm{d}\nu=:\mathcal{Q}\,.$$
$$\implies\quad\int_0^\infty\int_0^Z\kappa_\nu J_\nu^{n+1}(\tau)\mathrm{d}\tau\mathrm{d}\nu\le C_1(\kappa_M)\int_0^\infty\int_0^Z\kappa_\nu J_\nu^n(t)\mathrm{d}t\mathrm{d}\nu+\mathcal{Q}\,.$$

Initializing the sequence I_ν^n with $I_\nu^0=0$ and $T^0=T[J_\nu^0]=0$, one finds that

$$\int_0^\infty\int_0^Z\kappa_\nu J_\nu^1(\tau)\mathrm{d}\tau\mathrm{d}\nu\le\mathcal{Q}\,,\qquad\int_0^\infty\int_0^Z\kappa_\nu J_\nu^2(\tau)\mathrm{d}\tau\mathrm{d}\nu\le C_1(\kappa_M)\mathcal{Q}+\mathcal{Q},$$

and by induction

$$\int_0^\infty\int_0^Z\kappa_\nu J_\nu^{n+1}(\tau)\mathrm{d}\tau\mathrm{d}\nu\le\mathcal{Q}\sum_{j=0}^n C_1(\kappa_M)^j\,.$$

Since $C_1(\kappa_M)<1$, the series above converges and one has the uniform bound

$$\int_0^\infty\int_0^Z\kappa_\nu J_\nu^{n+1}(\tau)\mathrm{d}\tau\mathrm{d}\nu\le\frac{\mathcal{Q}}{1-C_1(\kappa_M)}\,.$$

Furthermore, as

$$0\le I_\nu^1\le I_\nu^2\le\cdots\le I_\nu^n\le I_\nu^{n+1}\le\cdots\,,$$

the bound above and the monotone convergence theorem imply that the sequence $I_\nu^{n+1}(\tau,\mu)$ converges for a.e. $(\tau,\mu,\nu)\in(0,Z)\times(-1,1)\times(0,+\infty)$ to a limit denoted $I_\nu(\tau,\mu)$ as $n\to\infty$. Since

$$0\le T^1\le T^2\le\cdots\le T^n\le T^{n+1}\le\cdots\,,$$

we conclude from (2.15) and the monotone convergence theorem that $T^{n+1}(\tau)$ converges for a.e. $\tau\in(0,Z)$ to a limit denoted $T(\tau)$ as $n\to\infty$.

Then we can pass to the limit in (3.3) as $n\to\infty$ by monotone convergence, so that (2.21) holds for a.e. $(\tau,\mu,\nu)\in(0,Z)\times(-1,1)\times(0,+\infty)$. One recognizes in this equality the integral formulation of (2.10)–(2.12). Besides, we have seen that

$$\begin{aligned}0=I_\nu^0\le I_\nu^1\le I_\nu^2\le\cdots\le I_\nu^n\le I_\nu^{n+1}\le\cdots\le I_\nu\,,\\0=T^0\le T^1\le T^2\le\cdots\le T^n\le T^{n+1}\le\cdots\le T\,,\end{aligned}$$

so that

$$\begin{aligned}0\le\int_0^Z(J_\nu^{n+1}-J_\nu^n)(\tau)\mathrm{d}\tau&=\tfrac12\int_0^Z\left(\int_0^Z E_1(\kappa_\nu|\tau-t|)\kappa_\nu\mathrm{d}\tau\right)a_\nu(J_\nu^n-J_\nu^{n-1})(t)\mathrm{d}t\\&+\tfrac12\int_0^Z\left(\int_0^Z E_1(\kappa_\nu|\tau-t|)\kappa_\nu\mathrm{d}\tau\right)(1-a_\nu)(B_\nu(T^n(t))-B_\nu(T^{n-1}(t)))\mathrm{d}t\\&\le C_1(\kappa_M)\int_0^Z(a_\nu(J_\nu^n-J_\nu^{n-1})(t)+(1-a_\nu)(B_\nu(T^n(t))-B_\nu(T^{n-1}(t))))\mathrm{d}t\,.\end{aligned}$$

Using again (3.4), we conclude that

$$0\le\int_0^Z\int_0^\infty\kappa_\nu(J_\nu^{n+1}-J_\nu^n)(\tau)\mathrm{d}\nu\mathrm{d}\tau\le C_1(\kappa_M)\int_0^Z\int_0^\infty\kappa_\nu(J_\nu^n-J_\nu^{n-1})(t)\mathrm{d}t\,.$$

Hence

$$0 \le \int_0^Z \int_0^\infty \kappa_\nu (J_\nu^{n+1} - J_\nu^n)(\tau)\mathrm{d}\nu\mathrm{d}\tau \le C_1(\kappa_M)^n \int_0^\infty \kappa_\nu J_\nu^1(\tau)\mathrm{d}\nu\mathrm{d}\tau \le C_1(\kappa_M)^n \mathcal{Q}\,,$$

so that

$$0 \le \int_0^Z \int_0^\infty \kappa_\nu (J_\nu - J_\nu^n)(\tau)\mathrm{d}\nu\mathrm{d}\tau \le C_1(\kappa_M)^n \int_0^\infty \kappa_\nu J_\nu^1(\tau)\mathrm{d}\nu\mathrm{d}\tau \le \frac{C_1(\kappa_M)^n \mathcal{Q}}{1 - C_1(\kappa_M)}\,.$$

Summarizing, we have proved the following result.

THEOREM 3.1. *Assume that* $0 < \kappa_\nu \le \kappa_M$, *while* $0 \le a_\nu < 1$ *for all* $\nu > 0$. *Let* $Q_\nu^\pm(\mu)$ *satisfy*

$$\mathcal{Q} := \tfrac{1}{2} \int_0^\infty \kappa_\nu \int_0^1 (Q_\nu^+(\mu) + Q_\nu^-(\mu))\mu\mathrm{d}\mu < \infty\,.$$

Choose $I_\nu^0 = 0$ *and* $T^0 = 0$, *and let* I_ν^n *and* $T^n = T[J_\nu^n]$ *be the solution of* (3.1). *Then*

$$I_\nu^n(\tau,\mu) \to I_\nu(\tau,\mu) \quad \text{and} \quad T^n(\tau) \to T(\tau)$$

for $(\tau,\mu,\nu) \in (0,Z) \times (-1,1) \times (0,+\infty)$ *as* $n \to \infty$, *where* (I_ν, T) *is a solution of* (2.10)–(2.12). *This method converges exponentially fast, in the sense that*

$$0 \le \int_0^Z \int_0^\infty \kappa_\nu (J_\nu - J_\nu^n)(\tau)\mathrm{d}\nu\mathrm{d}\tau \le \frac{C_1(\kappa_M)^n \mathcal{Q}}{1 - C_1(\kappa_M)}\,,$$

and, if $0 \le a_\nu \le a_M < 1$ *while* $0 < \kappa_m \le \kappa_\nu$, *one has*

$$0 \le \int_0^Z \bar\sigma (T(t)^4 - T^n(t)^4)\mathrm{d}t \le \frac{C_1(\kappa_M)^n \mathcal{Q}}{\kappa_m(1 - a_M)(1 - C_1(\kappa_M))}\,.$$

The last bound comes from the defining equality for the temperature in terms of the radiative intensity

$$\begin{aligned} \kappa_m(1 - a_M)\bar\sigma(T^4 - (T^n)^4) &= \kappa_m(1 - a_M)\int_0^\infty (B_\nu(T) - B_\nu(T^n))\mathrm{d}\nu \\ \le \int_0^\infty \kappa_\nu(1 - a_\nu)(B_\nu(T) - B_\nu(T^n))\mathrm{d}\nu &= \int_0^\infty \kappa_\nu(1 - a_\nu)(J_\nu - J_\nu^n)\mathrm{d}\nu \\ &\le \int_0^\infty \kappa_\nu(J_\nu - J_\nu^n)\mathrm{d}\nu\,. \end{aligned}$$

4. Uniqueness, maximum principle for (2.10)–(2.12). This section follows computations in [11] (in the case $Z = +\infty$ and with $a_\nu = 0$) and the rather subtle monotonicity structure of the RT equations, a striking result[2] found by Mercier in [23]. The following theorem shows that two solutions of the problem (2.10)–(2.12) are ordered exactly as their boundary data. (This situation is analogous to the case of harmonic functions, except that the RT equations (2.10)–(2.12) are nonlinear, at variance with the Laplace equation.)

[2]In fact, Mercier's original argument is even more complex, because he assumes that the opacity $K_\nu := \kappa_\nu(1 - a_\nu)$ depends on the temperature T, and is a nonincreasing function of T for each $\nu > 0$ while $T \mapsto K_\nu(T)B_\nu(T)$ is nondecreasing.

THEOREM 4.1. *Assume that* $0 < \kappa_\nu \le \kappa_M$, *while* $0 \le a_\nu < 1$ *for all* $\nu > 0$. *Let* $Q^\pm, {Q'}^\pm \in L^1((0,1)\times(0,\infty))$ *satisfy*

$$0 \le Q^\pm_\nu(\mu) \le {Q'}^\pm_\nu(\mu) \quad \textit{for a.e. } (\mu,\nu) \in (0,1)\times(0,\infty)\,.$$

Then, the solutions $(I_\nu, T[I])$ *of* (2.10)–(2.12), *and* $({I'}_\nu, T[I'])$ *of* (2.10)–(2.12), *with boundary data* $Q^\pm_\nu(\mu)$ *replaced with* ${Q'}^\pm_\nu(\mu)$ *satisfy*

$$I_\nu(\tau,\mu) \le {I'}_\nu(\tau,\mu) \quad \textit{and} \quad T[I](\tau) \le T[I'](\tau) \quad \textit{for a.e. } (\tau,\mu) \in (-1,1)\times(0,\infty)\,.$$

In particular,

$$Q^\pm_\nu(\mu) = {Q'}^\pm_\nu(\mu) \ \textit{a.e. } \mu,\nu \implies I_\nu(\tau,\mu) = {I'}_\nu(\tau,\mu) \ \textit{and } T[I](\tau) = T[I'](\tau)$$
$$\textit{for a.e. } \tau,\mu \in (-1,1)\times(0,\infty)\,.$$

The proof of this result is deferred to the appendix at the very end of this paper.

One has also the following form of maximum principle for the RT equation. (If one keeps in mind the analogy with harmonic functions recalled before Theorem 4.1, the maximum principle below is a *consequence* of the monotonicity of the dependence of the solution of (2.10)–(2.12) in terms of its boundary data, whereas the analogous monotonicity in the case of harmonic functions is *deduced* from the maximum principle for the Laplace equation.)

Corollary 4.2. *Assume that* $0 < \kappa_\nu \le \kappa_M$, *while* $0 \le a_\nu < 1$ *for all* $\nu > 0$. *Let* $Q^\pm_\nu(\mu) \le B_\nu(T_M)$ *(resp.,* $Q^\pm_\nu(\mu) \ge B_\nu(T_m)$*) for a.e.* $(\mu,\nu) \in (0,1)\times(0,\infty)$. *Then*

$$I_\nu(\tau,\mu) \le B_\nu(T_M) \ \textit{and } T[I](\tau) \le T_M$$
$$(\textit{resp., } I_\nu(\tau,\mu) \ge B_\nu(T_m) \ \textit{and } T[I](\tau) \ge T_m\ )$$
$$\textit{for a.e.} (\tau,\mu) \in (-1,1)\times(0,\infty)\,.$$

Proof. Indeed, $I'_\nu = B_\nu(T_M)$ and $T[I'] = T_M$ (resp., $I'_\nu = B_\nu(T_m)$ and $T[I'] = T_m$) is the solution of (2.11) with boundary data ${Q'}^\pm_\nu(\mu) = B_\nu(T_M)$ (resp., ${Q'}^\pm_\nu(\mu) = B_\nu(T_m)$). The announced inequalities follow from the comparison of solutions obtained in Theorem 4.1. ∎

Remark 4.3. In Theorem 3.1, if one has the stronger condition

$$0 \le Q^\pm_\nu(\mu) \le B_\nu(T_M) \quad \text{for a.e. } (\mu,\nu) \in (0,1)\times(0,\infty)\,,$$

one obtains the following bound for the numerical and theoretical solutions:

$$0 \le I^1_\nu \le \cdots \le I^n_\nu \le \cdots I_\nu \le B_\nu(T_M) \text{ and } 0 \le T^1 \le \cdots \le T^n \le \cdots \le T \le T_M\,.$$

5. RT with Rayleigh phase function. In this section, we discuss the same problem as in the previous section, with the isotropic scattering kernel replaced by the Rayleigh phase function. In the case of slab symmetry, the Rayleigh phase function is

$$p(\mu,\mu') = \tfrac{3}{16}(3-\mu^2) + \tfrac{3}{16}(3\mu^2-1)\mu'^2$$

(see section 11.2 in Chapter I of [6]). Observe that

$$p(\mu,\mu') = \tfrac{3}{16}(3 + 3\mu^2\mu'^2 - \mu^2 - \mu'^2) \ge \tfrac{3}{16} > 0\,, \tag{5.1}$$

while

$$\tfrac{1}{2}\int_{-1}^{1} p(\mu,\mu')\mathrm{d}\mu = \tfrac{3}{16}(6 + 3\cdot\tfrac{2}{3}\mu'^2 - \tfrac{2}{3} - 2\mu'^2) = 1\,. \tag{5.2}$$

Keeping (2.12) as the defining equation for $T[I]$, the problem becomes

$$\begin{cases} (\mu\partial_\tau + \kappa_\nu)I_\nu(\tau,\mu) = \tfrac{3}{8}\kappa_\nu a_\nu((3-\mu^2)J_\nu(\tau) + (3\mu^2-1)K_\nu(\tau)) \\ \qquad\qquad\qquad + \kappa_\nu(1-a_\nu)B_\nu(T[J](\tau))\,, \\ I_\nu(0,\mu) = Q_\nu^+(\mu)\,, \qquad I_\nu(Z,-\mu) = Q_\nu^-(\mu)\,, \qquad 0<\mu<1\,, \end{cases} \tag{5.3}$$

with

$$J_\nu := \tfrac{1}{2}\int_{-1}^{1}\mu I_\nu \mathrm{d}\mu\,, \qquad K_\nu = \tfrac{1}{2}\int_{-1}^{1}\mu^2 I_\nu\mathrm{d}\mu \tag{5.4}$$

and (2.12). Starting from $I_\nu^0(\tau,\mu)=0$ and $T^0(\tau)=0$, one solves for I^{n+1}

$$\begin{cases} (\mu\partial_\tau + \kappa_\nu)I_\nu^{n+1}(\tau,\mu) = \tfrac{3}{8}\kappa_\nu a_\nu((3-\mu^2)J_\nu^n(\tau) + (3\mu^2-1)K_\nu^n(\tau)) \\ \qquad\qquad\qquad + \kappa_\nu(1-a_\nu)B_\nu(T^n(\tau))\,, \qquad T^n := T[I^n], \\ I_\nu^{n+1}(0,\mu) = Q_\nu^+(\mu)\,, \qquad I_\nu^{n+1}(Z,-\mu) = Q_\nu^-(\mu)\,, \qquad 0<\mu<1\,. \end{cases} \tag{5.5}$$

Since B_ν is nondecreasing for each $\nu>0$, one easily checks with (5.1) that

$$\begin{aligned} &0 = I_\nu^0 \le I_\nu^1 \le I_\nu^2 \le \cdots \le I_\nu^n \le I_\nu^{n+1} \le \cdots\,, \\ &0 = T^0 \le T^1 \le T^2 \le \cdots \le T^n \le T^{n+1} \le \cdots\,. \end{aligned}$$

The construction of these sequences is straightforward:

$$\begin{aligned} J_\nu^{n+1}(\tau) =\,& S_\nu(\tau) + \tfrac{3}{16}\int_0^Z E_1(\kappa_\nu|\tau-t|)\kappa_\nu a_\nu(3J_\nu^n(t) - K_\nu^n(t))\mathrm{d}t \\ &+ \tfrac{3}{16}\int_0^Z E_3(\kappa_\nu|\tau-t|)\kappa_\nu a_\nu(3K_\nu^n(t) - J_\nu^n(t))\mathrm{d}t \\ &+ \tfrac{1}{2}\int_0^Z E_1(\kappa_\nu|\tau-t|)\kappa_\nu(1-a_\nu)B_\nu(T^n(t))\mathrm{d}t\,, \\ K_\nu^{n+1}(\tau) =\,& \tfrac{1}{2}\int_0^1 \Big(e^{-\frac{\kappa_\nu\tau}{\mu}}Q_\nu^+(\mu)\mathbf{1}_{\mu>0} + e^{-\frac{\kappa_\nu(Z-\tau)}{|\mu|}}Q_\nu^-(|\mu|)\mathbf{1}_{\mu<0}\Big)\mu^2\mathrm{d}\mu \\ &+ \tfrac{3}{16}\int_0^Z E_3(\kappa_\nu|\tau-t|)\kappa_\nu a_\nu(3J_\nu^n(t) - K_\nu^n(t))\mathrm{d}t \\ &+ \tfrac{3}{16}\int_0^Z E_5(\kappa_\nu|\tau-t|)\kappa_\nu a_\nu(3K_\nu^n(t) - J_\nu^n(t))\mathrm{d}t \\ &+ \tfrac{1}{2}\int_0^Z E_3(\kappa_\nu|\tau-t|)\kappa_\nu(1-a_\nu)B_\nu(T^n(t))\mathrm{d}t\,, \\ \int_0^\infty \kappa_\nu(1-a_\nu)&B_\nu(T^{n+1})\mathrm{d}\nu = \int_0^\infty \kappa_\nu(1-a_\nu)J_\nu^{n+1}\mathrm{d}\nu\,. \end{aligned} \tag{5.6}$$

Notice that the radiative intensity is eliminated, but can be recovered by

$$
\begin{aligned}
I_\nu^{n+1}(\tau,\mu) = {}& e^{-\frac{\kappa_\nu\tau}{\mu}}Q_\nu^+(\mu)\mathbf{1}_{\mu>0}+e^{-\frac{\kappa_\nu(Z-\tau)}{|\mu|}}Q_\nu^-(|\mu|)\mathbf{1}_{\mu<0}\\
&+\mathbf{1}_{\mu>0}\int_0^\tau e^{-\frac{\kappa_\nu(\tau-t)}{\mu}}\tfrac{\kappa_\nu}{\mu}\tfrac{3}{8}a_\nu((3-\mu^2)J_\nu^n(t)+(3\mu^2-1)K_\nu^n(t))\mathrm{d}t\\
&+\mathbf{1}_{\mu>0}\int_0^\tau e^{-\frac{\kappa_\nu(\tau-t)}{\mu}}\tfrac{\kappa_\nu}{\mu}(1-a_\nu)B_\nu(T^n(t))\mathrm{d}t\\
&+\mathbf{1}_{\mu<0}\int_t^Z e^{-\frac{\kappa_\nu|t-\tau|}{|\mu|}}\tfrac{\kappa_\nu}{|\mu|}\tfrac{3}{8}a_\nu((3-\mu^2)J_\nu^n(t)+(3\mu^2-1)K_\nu^n(t))\mathrm{d}t\\
&+\mathbf{1}_{\mu<0}\int_0^Z e^{-\frac{\kappa_\nu|t-\tau|}{|\mu|}}\tfrac{\kappa_\nu}{|\mu|}(1-a_\nu)B_\nu(T^n(t))\mathrm{d}t\,.
\end{aligned}
\tag{5.7}
$$

Assume that $0\le Q_\nu^\pm\le B_\nu(T_M)\,,0\le I_\nu^n\le B_\nu(T_M)$, and $0\le T^n\le T_M$. Thus $0\le J_\nu^n\le B_\nu(T_M)$ and $0\le K_\nu^n\le\frac13 B_\nu(T_M)$, so that

$$
\begin{aligned}
I_\nu^{n+1}(\tau,\mu)\le{}&\left(e^{-\frac{\kappa_\nu\tau}{\mu}}\mathbf{1}_{\mu>0}+e^{-\frac{\kappa_\nu(Z-\tau)}{|\mu|}}\mathbf{1}_{\mu<0}\right)B_\nu(T_M)\\
&+\mathbf{1}_{\mu>0}\int_0^\tau e^{-\frac{\kappa_\nu(\tau-t)}{\mu}}\tfrac{\kappa_\nu}{\mu}\tfrac38 a_\nu((3-\mu^2)B_\nu(T_M)+(\mu^2-\tfrac13)B_\nu(T_M))\mathrm{d}t\\
&+\mathbf{1}_{\mu>0}\int_0^\tau e^{-\frac{\kappa_\nu(\tau-t)}{\mu}}\tfrac{\kappa_\nu}{\mu}(1-a_\nu)B_\nu(T_M)\mathrm{d}t\\
&+\mathbf{1}_{\mu<0}\int_\tau^Z e^{-\frac{\kappa_\nu(t-\tau)}{|\mu|}}\tfrac{\kappa_\nu}{|\mu|}\tfrac38 a_\nu((3-\mu^2)B_\nu(T_M)+(\mu^2-\tfrac13)B_\nu(T_M))\mathrm{d}t\\
&+\mathbf{1}_{\mu<0}\int_\tau^Z e^{-\frac{\kappa_\nu(t-\tau)}{|\mu|}}\tfrac{\kappa_\nu}{|\mu|}(1-a_\nu)B_\nu(T_M)\mathrm{d}t
\end{aligned}
\tag{5.8}
$$

$$
\begin{aligned}
={}&B_\nu(T_M)\mathbf{1}_{\mu>0}\left(e^{-\frac{\kappa_\nu\tau}{\mu}}+\int_0^\tau e^{-\frac{\kappa_\nu(\tau-t)}{\mu}}\tfrac{\kappa_\nu}{\mu}(\tfrac38 a_\nu(3-\tfrac13)+(1-a_\nu))\mathrm{d}t\right)\\
&+B_\nu(T_M)\mathbf{1}_{\mu<0}\left(e^{-\frac{\kappa_\nu(Z-\tau)}{|\mu|}}+\int_\tau^Z e^{-\frac{\kappa_\nu(t-\tau)}{|\mu|}}\tfrac{\kappa_\nu}{|\mu|}(\tfrac38 a_\nu(3-\tfrac13)+(1-a_\nu))\mathrm{d}t\right)=B_\nu(T_M).
\end{aligned}
$$

Besides, using again that $T\mapsto B_\nu(T)$ is increasing for each $\nu>0$ while $\kappa_\nu(1-a_\nu)>0$ for all $\nu>0$,

$$T^{n+1}=T[I^{n+1}]\le T[B_\nu(T_M)]=T_M.$$

Summarizing, we have proved the following result.

THEOREM 5.1. *Assume that $\kappa_\nu>0$ while $0\le a_\nu<1$ for all $\nu>0$. Let the boundary data $Q_\nu^\pm$ satisfy*

$$0\le Q_\nu^\pm(\mu)\le B_\nu(T_M)\qquad\text{for all }\mu\in(-1,1)\text{ and }\nu>0\,.$$

Equation (5.6) *defines an increasing sequence of radiative intensities I_ν^n and temperatures T^n converging pointwise to I_ν and $T=T[I]$, respectively, which is a solution of* (5.3).

The argument above is based on the monotonicity of the sequences I_ν^n and T^n, and does not give any information on the convergence rate.

Remark 5.2. One can easily check that the uniqueness theorem, Theorem 4.1, holds verbatim for the problem (5.3) with Rayleigh phase function. See Appendix A at the end of this paper for the proof.

6. RT in a fluid with thermal diffusion. For clarity we consider the case of a lake; we neglect the wind above the lake and we assume that the sunlight hits the surface of the lake with a given energy. The depth of the lake should vary slowly with x, y, but for the sake of simplicity, it is assumed to be uniform: $\Omega = \mathbb{O} \times (0, Z)$ for some open set $\mathbb{O} \subset \mathbb{R}^2$ with C^1 boundary, or piecewise C^1 boundary.

With $\boldsymbol{u} \in H^1(\Omega)$ satisfying $\nabla \cdot \boldsymbol{u} = 0$ and $\boldsymbol{u} \cdot \boldsymbol{n}|_{\partial\Omega} = 0$, consider again the system (2.9). Throughout this section, we assume isotropic scattering, with

$$0 \le a_\nu \le a_M < 1\,, \qquad 0 < \kappa_m \le \kappa_\nu \le \kappa_M\,, \qquad \nu > 0\,. \tag{6.1}$$

Here, ρ is assumed to be a constant, and we choose $\rho_0 = \rho$ in (2.14), so that $\kappa_\nu = \rho\bar{\kappa}_\nu$, and $\tau = z$.

We further assume that the fluid flow is steady, and consider the system

$$\mu\partial_z I_\nu + \kappa_\nu I_\nu = \kappa_\nu(1 - a_\nu)B_\nu(T) + \kappa_\nu a_\nu J_\nu, \quad J_\nu := \tfrac{1}{2}\int_{-1}^{1} I_\nu \mathrm{d}\mu, \tag{6.2}$$

$$\boldsymbol{u} \cdot \nabla T - \tfrac{c_P}{c_V}\kappa_T \Delta T = \tfrac{4\pi}{\rho c_V}\int_0^\infty \kappa_\nu(1 - a_\nu)(J_\nu - B_\nu(T))\mathrm{d}\nu, \tag{6.3}$$

$$I_\nu|_{z=Z,\mu<0} = Q_\nu^-(x, y, -\mu), \quad I_\nu|_{z=0,\mu>0} = Q_\nu^+(x, y, \mu), \quad \frac{\partial T}{\partial n}\Big|_{\partial\Omega} = 0. \tag{6.4}$$

The boundary sources $Q_\nu^\pm(x, y, \mu)$ are bounded, measurable, nonnegative functions defined a.e. on $\mathbb{O} \times (-1, 1) \times (0, \infty)$.

As a first reduction, we solve (6.2) for the radiative intensity I_ν in terms of the angle-averaged intensity J_ν and of the temperature T, and average the resulting expression in μ: proceeding as in Lemma 2.2, we arrive at the system

$$\begin{cases} J_\nu(x, y, z) = S_\nu(x, y, z) \\ \quad + \tfrac{1}{2}\displaystyle\int_0^Z \kappa_\nu E_1(\kappa_\nu|z - \zeta|)\left(a_\nu J_\nu(x, y, \zeta) + (1 - a_\nu)B_\nu(T(x, y, \zeta))\right)\mathrm{d}\zeta, \\ \boldsymbol{u}(\boldsymbol{x}) \cdot \nabla T(\boldsymbol{x}) - \tfrac{c_P}{c_V}\kappa_T \Delta T(\boldsymbol{x}) = \tfrac{4\pi}{\rho c_V}\displaystyle\int_0^\infty \kappa_\nu(1 - a_\nu)(J_\nu(\boldsymbol{x}) - B_\nu(T(\boldsymbol{x})))\mathrm{d}\nu, \\ \dfrac{\partial T}{\partial n}\Big|_{\partial\Omega} = 0\,, \end{cases} \tag{6.5}$$

where

$$S_\nu(x, y, z) := \tfrac{1}{2}\int_0^1 \left(e^{-\frac{\kappa_\nu z}{\mu}} Q_\nu^+(x, y, \mu) + e^{-\frac{\kappa_\nu(Z-z)}{\mu}} Q_\nu^-(x, y, \mu)\right)\mathrm{d}\mu. \tag{6.6}$$

Once the angle-averaged radiative intensity is known J_ν, the radiative intensity I_ν itself is easily obtained by solving the transfer equation (6.2) by the method of characteristics; see (2.21).

THEOREM 6.1. *Assume that the absorption coefficient κ_ν and the scattering albedo a_ν satisfy* (6.1)*. Let the boundary source terms $Q_\nu^\pm$ satisfy the following: for some T_M,*

$$0 \le Q_\nu^\pm(\mu) \le B_\nu(T_M)\,, \qquad 0 < \mu < 1\,, \quad \nu > 0\,.$$

Consider $\{J_\nu^n, T^n\}_{n\ge0}$ to be initiated by T^0 given and generated by

$$\begin{aligned} &J_\nu^{n+1}(x, y, z) = S_\nu(x, y, z) \\ &+ \tfrac{1}{2}\int_0^Z \kappa_\nu E_1(\kappa_\nu|z - \zeta|)\left(a_\nu J_\nu^n(x, y, \zeta) + (1 - a_\nu)B_\nu(T^n(x, y, \zeta))\right)\mathrm{d}\zeta, \end{aligned} \tag{6.7}$$

$$
(6.8) \qquad \begin{cases} \boldsymbol{u}\cdot\nabla T^{n+1} - \frac{c_P}{c_V}\kappa_T\Delta T^{n+1} + \frac{4\pi}{\rho c_V}\displaystyle\int_0^\infty \kappa_\nu(1-a_\nu)B_\nu(T_+^{n+1})\mathrm{d}\nu \\ \qquad\qquad = \frac{4\pi}{\rho c_V}\displaystyle\int_0^\infty \kappa_\nu(1-a_\nu)J_\nu^{n+1}\mathrm{d}\nu, \qquad \frac{\partial T}{\partial n}\Big|_{\partial\Omega} = 0\,. \end{cases}
$$

Then

$$
\begin{aligned} S_\nu(\boldsymbol{x}) = J_\nu^0(\boldsymbol{x}) \le J_\nu^1(\boldsymbol{x}) \le \cdots \le J_\nu^n(\boldsymbol{x}) \le J_\nu^{n+1}(\boldsymbol{x}) \le \cdots \le B_\nu(T_M)\,, \qquad & \nu > 0\,, \\ 0 = T^0 \le T^1(\boldsymbol{x}) \le \cdots \le T^n(\boldsymbol{x}) \le T^{n+1}(\boldsymbol{x}) \le \cdots \le T_M\,, \qquad & \boldsymbol{x}\in\Omega\,, \end{aligned}
$$

and convergence to a solution (J,T) *of the system* (6.5) *holds.*

Define

$$
\mathcal{B}(T) := \int_0^\infty \kappa_\nu(1-a_\nu)B_\nu(T_+)\mathrm{d}\nu\,.
$$

Observe that

$$
\kappa_m(1-a_M)\bar\sigma T_+^4 \le \mathcal{B}(T) \le \kappa_M\bar\sigma T_+^4\,,
$$

where $\pi\bar\sigma$ is the Stefan–Boltzmann constant (see (2.3)). Observe also that the function $\mathcal{B}:\mathbf{R}\to\mathbf{R}$ is nondecreasing, and increasing on $(0,+\infty)$ by construction, since B_ν is increasing on $[0,+\infty)$ for each $\nu>0$.

For the sake of notational simplicity, in order to keep the number of physical constants to a strict minimum, we assume henceforth that $\rho c_P\kappa_T/4\pi = 1$, and replace $\boldsymbol{u}$ with $\rho c_V\boldsymbol{u}/4\pi$.

The key argument in the proof of this theorem is the following lemma.

LEMMA 6.2. *Let* $R\in L^{6/5}(\Omega)$. *There exists at least one weak solution of*

$$
-\Delta T + \boldsymbol{u}\cdot\nabla T + \mathcal{B}(T) = R\,, \qquad \frac{\partial T}{\partial n}\Big|_{\partial\Omega} = 0\,.
$$

If $R\ge 0$ *a.e. and* $|\{x\in\Omega \text{ s.t. } R(x)>0\}|>0$, *the weak solution of the problem above is unique and satisfies* $T\ge 0$ *a.e. on* Ω.

Moreover, if $R'\in L^{6/5}(\Omega)$ *and* $R'\ge R$ *a.e. on* Ω, *the weak solution* T' *of the problem above with the right-hand side* R' *satisfies* $T\le T'$ *a.e. on* Ω.

Proof. For each $0<\varepsilon<1$, the problem

$$
\varepsilon T_\varepsilon - \Delta T_\varepsilon + \boldsymbol{u}\cdot\nabla T_\varepsilon + \mathcal{B}(T_\varepsilon) = R\,, \qquad \frac{\partial T}{\partial n}\Big|_{\partial\Omega} = 0
$$

has a weak solution in $H^1(\Omega)$.

To see this, apply Theorem 1 of [19] with $V = H^1(\Omega)$ to the nonlinear operator $\mathcal{A}_\varepsilon : V\mapsto V'$ defined by

$$
\langle\mathcal{A}_\varepsilon T,\phi\rangle_{V',V} = \int_\Omega (\varepsilon T\phi + \nabla T\cdot\nabla\phi + \phi\boldsymbol{u}\cdot\nabla T + \mathcal{B}(T)\phi)\mathrm{d}\boldsymbol{x}\,.
$$

That $\mathcal{A}_\varepsilon$ is continuous from V to V' easily follows from the Sobolev embedding $H^1(\Omega)\subset L^6(\Omega)$, which implies by duality the continuous inclusion $L^{6/5}(\Omega)\subset V'$. Since $\boldsymbol{u}\in H^1(\Omega)\subset L^6(\Omega)$, one has

$$
\boldsymbol{u}\cdot\nabla T \in L^{3/2}(\Omega)\subset L^{6/5}(\Omega)\subset V' \quad \text{ with } \|\boldsymbol{u}\cdot\nabla T\|_{L^{3/2}(\Omega)} \le \|\boldsymbol{u}\|_{L^6(\Omega)}\|T\|_{H^1(\Omega)}
$$

and

$$\mathcal{B}(T) \in L^{3/2}(\Omega) \subset L^{6/5}(\Omega) \subset V' \quad \text{with } \|\mathcal{B}(T)\|_{L^{3/2}(\Omega)} \le \kappa_M \bar{\sigma} \|T_+\|^4_{L^6(\Omega)}\,.$$

Since $\boldsymbol{u}$ is a divergence free vector in $H^1(\Omega)$ satisfying $\boldsymbol{u} \cdot n = 0$ on $\partial\Omega$, the bilinear functional

$$H^1(\Omega) \times H^1(\Omega) \ni (T, \phi) \mapsto \int_\Omega \phi\, \boldsymbol{u} \cdot \nabla T \mathrm{d}\boldsymbol{x} \in \mathbf{R}$$

is skew-symmetric, and $\mathcal{B}(T(x)) = 0$ if $T(x) \le 0$ by definition, so that

$$\langle \mathcal{A}_\varepsilon T, T \rangle_{V',V} = \varepsilon \|T\|^2_{L^2(\Omega)} + \|\nabla T\|^2_{L^2(\Omega)} + \int_\Omega \mathcal{B}(T) T \mathrm{d}\boldsymbol{x} \ge \varepsilon \|T\|^2_{H^1(\Omega)}\,.$$

Hence $\mathcal{A}_\varepsilon$ is coercive on V. Besides, for all $T_1, T_2 \in H^1(\Omega)$,

$$\begin{aligned}\langle \mathcal{A}_\varepsilon T_1 - \mathcal{A} T_2, T_1 - T_2 \rangle_{V',V} = & \varepsilon \|T_1 - T_2\|^2_{L^2(\Omega)} + \|\nabla(T_1 - T_2)\|^2_{L^2(\Omega)} \\ & + \int_\Omega (T_1 - T_2)(\mathcal{B}(T_1) - \mathcal{B}(T_2)) \mathrm{d}\boldsymbol{x} \ge 0\,.\end{aligned}$$

Theorem 1 in [19] implies the desired existence result for each $\varepsilon \in (0, 1)$.

Then, since $R \ge 0$ a.e. on Ω, one has $RT_\varepsilon \le RT_{\varepsilon+}$ a.e. on Ω, and therefore,

$$\begin{aligned}\varepsilon \|T_\varepsilon\|^2_{L^2(\Omega)} + \|\nabla T_\varepsilon\|^2_{L^2(\Omega)} + \bar{\sigma} \kappa_m (1 - a_M) \int_\Omega T_\varepsilon(\boldsymbol{x})^5_+ \mathrm{d}\boldsymbol{x} \le \langle \mathcal{A}_\varepsilon T, T \rangle_{V',V} \\ \le \int_\Omega R(\boldsymbol{x}) T_\varepsilon(\boldsymbol{x})_+ \mathrm{d}\boldsymbol{x} \le \|R\|_{L^{6/5}(\Omega)} \|T_{\varepsilon+}\|_{L^6(\Omega)} \le C_S \|R\|_{L^{6/5}(\Omega)} \|T_{\varepsilon+}\|_{H^1(\Omega)}\,.\end{aligned}$$

By Hölder's inequality

$$\int_\Omega T_\varepsilon(\boldsymbol{x})^5_+ \mathrm{d}\boldsymbol{x} \ge \tfrac{1}{|\Omega|^{3/2}} \|T_{\varepsilon+}\|^5_{L^2(\Omega)}\,,$$

and since $\|\nabla T_{\varepsilon+}\|_{L^2(\Omega)} \le \|\nabla T_\varepsilon\|_{L^2(\Omega)}$, we see that

$$\|\nabla T_\varepsilon\|^2_{L^2(\Omega)} + \tfrac{\bar{\sigma} \kappa_m (1 - a_M)}{|\Omega|^{3/2}} \|T_{\varepsilon+}\|^5_{L^2(\Omega)} \le C_S \|R\|_{L^{6/5}(\Omega)} \left(\|T_{\varepsilon+}\|^2_{L^2(\Omega)} + \|\nabla T_\varepsilon\|^2_{L^2(\Omega)} \right)^{\frac{1}{2}}$$

so that

$$\sup_{0<\varepsilon<1} \left(\|\nabla T_\varepsilon\|_{L^2(\Omega)} + \|T_{\varepsilon+}\|_{L^2(\Omega)} \right) < \infty\,.$$

By the Banach–Alaoglu and Rellich theorems, there exists a subsequence of T_ε (still denoted T_ε for simplicity) such that

$$T_{\varepsilon+} \to T_+ \quad \text{in } L^p(\Omega) \quad \text{and} \quad \nabla T_\varepsilon \to \nabla T \quad \text{weakly in } L^2(\Omega)$$

for all $p \in [1, 6)$ while $\varepsilon^{1/2} T_\varepsilon$ is bounded in $L^2(\Omega)$. Hence, for each $\phi \in H^1(\Omega)$, one has

$$\begin{aligned} 0 &= \int_\Omega (\varepsilon T_\varepsilon \phi + \nabla T_\varepsilon \cdot \nabla \phi + \phi\, \boldsymbol{u} \cdot \nabla T_\varepsilon + \mathcal{B}(T_\varepsilon) \phi) \mathrm{d}\boldsymbol{x} \\ &\to \int_\Omega (\nabla T \cdot \nabla \phi + \phi\, \boldsymbol{u} \cdot \nabla T + \mathcal{B}(T) \phi) \mathrm{d}\boldsymbol{x} =: \langle \mathcal{A} T, \phi \rangle_{V',V}\end{aligned}$$

in the limit as $\varepsilon \to 0$, so that T is a weak solution of

$$-\Delta T + \boldsymbol{u} \cdot \nabla T + \mathcal{B}(T) = R \,, \qquad \frac{\partial T}{\partial n}\Big|_{\partial\Omega} = 0 \,.$$

Observe that

$$\langle \mathcal{A}T - \mathcal{A}T', (T-T')_+\rangle_{V',V} = \|\nabla(T-T')_+\|^2_{L^2(\Omega)} + \int_\Omega (\mathcal{B}(T) - \mathcal{B}(T'))(T-T')_+ \mathrm{d}\boldsymbol{x} \geq 0 \,,$$

since

$$\int_\Omega (T-T')_+ \boldsymbol{u} \cdot \nabla (T-T') \mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{u} \cdot \nabla \tfrac{1}{2}(T-T')^2_+ \mathrm{d}\boldsymbol{x} = \int_{\partial\Omega} \tfrac{1}{2}(T-T')^2_+ \boldsymbol{u} \cdot n \mathrm{d}\sigma(\boldsymbol{x}) = 0 \,,$$

denoting by $\mathrm{d}\sigma(\boldsymbol{x})$ the surface element on $\partial\Omega$. Hence

$$R \leq R' \text{ a.e. on } \Omega \implies \langle (R-R'), (T-T')_+\rangle_{V',V} = \|\nabla(T-T')_+\|_{L^2(\Omega)} = 0 \,.$$

Since Ω is connected, $(T-T')_+ = c$ a.e. on Ω for some constant $c \geq 0$.

A first consequence of this remark is that if $R' \geq 0$ a.e. on Ω, weak solutions of

$$-\Delta T' + \boldsymbol{u} \cdot \nabla T' + \mathcal{B}(T') = R' \,, \qquad \frac{\partial T'}{\partial n}\Big|_{\partial\Omega} = 0$$

satisfy

$$T' \geq 0 \text{ a.e. on } \Omega, \quad \text{ unless } R' = 0 \text{ a.e. on } \Omega \,, \quad \text{ in which case } T' = \text{Const.} \leq 0 \,.$$

A second consequence is that if $R' \geq R \geq 0$, with $|\{x \in \Omega \text{ s.t. } R \geq 0\}| > 0$, the solutions T and T' of

$$-\Delta T + \boldsymbol{u} \cdot \nabla T + \mathcal{B}(T) = R \,, \qquad \frac{\partial T}{\partial n}\Big|_{\partial\Omega} = 0$$

satisfy $T \geq 0$ and $T' \geq 0$, and $(T-T')_+ = c$ a.e. on Ω for some constant $c \geq 0$. Besides

$$\begin{aligned} 0 = \langle R - R', (T-T')_+\rangle_{V',V} &= \langle \mathcal{A}T - \mathcal{A}T', (T-T')_+\rangle_{V',V} = \|\nabla(T-T')_+\|^2_{L^2(\Omega)} \\ &+ \int_\Omega (\mathcal{B}(T) - \mathcal{B}(T'))(T-T')_+ \mathrm{d}\boldsymbol{x} = c \int_\Omega (\mathcal{B}(T'+c) - \mathcal{B}(T'))\mathrm{d}\boldsymbol{x} \,. \end{aligned}$$

Since $T' \geq 0$ a.e. on Ω, and since $\mathcal{B}$ is increasing, this implies that $c = 0$. Therefore

$$R' \geq R \geq 0 \text{ with } |\{x \in \Omega \text{ s.t. } R \geq 0\}| > 0 \implies (T-T')_+ = 0 \,.$$

Hence $T \leq T'$ a.e. on Ω. ∎

Proof of Theorem 6.1. For the sake of clarity, we systematically omit the tangential variables x, y in the integral equations for the averaged radiative intensity J^n_ν (as well as for the radiative intensity I_ν itself), since these variables are only parameters in all these formulas. We start from

$$T^0 \equiv 0 \,, \qquad J^0_\nu(z) = S_\nu(z) > 0 \,.$$

Construct iteratively $(T^n, J^n_\nu)_{n\geq 0}$ by the following recursion formula: first, compute

$$J^{n+1}_\nu(z) = S_\nu(z) + \tfrac{1}{2}\int_0^Z \kappa_\nu E(\kappa_\nu|z-t|)(a_\nu J^n_\nu(t) + (1-a_\nu)B_\nu(T^n(t)))\mathrm{d}t \,.$$

Then let T^{n+1} be the solution of

$$(6.9)\quad -\Delta T^{n+1}+\boldsymbol{u}\cdot\nabla T^{n+1}+\mathcal{B}(T^{n+1})=\int_0^\infty \kappa_\nu(1-a_\nu)J_\nu^{n+1}\mathrm{d}\nu\,,\qquad \frac{\partial T^{n+1}}{\partial n}\Big|_{\partial\Omega}=0\,.$$

Obviously, $J_\nu^1 \geq J_\nu^0 > 0$, and applying Lemma 6.2 implies that $T^1 \geq T^0$ a.e. on Ω. Moreover,

$$T^n \geq T^{n-1} \quad \text{ and } \quad J_\nu^n \geq J_\nu^{n-1} > 0 \implies J_\nu^{n+1} \geq J_\nu^n > 0\,,$$

and applying Lemma 6.2 shows that $T^{n+1} \geq T^n$ a.e. on Ω.

Assume that $Q_\nu^\pm(\mu) \leq B_\nu(T_M)$. It will be more convenient to deal with radiative intensities I_ν instead of their angle-averaged variants J_ν. Therefore, we define I_ν^n to be the solution of

$$\begin{aligned}
&(\mu\partial_z+\kappa_\nu)I_\nu^{n+1}=\kappa_\nu(1-a_\nu)B_\nu(T^n)+\kappa_\nu a_\nu J_\nu^n\,, \qquad J_\nu^n=\tilde{I}_\nu^n\,,\\
&I_\nu^{n+1}(Z,-\mu)=Q_\nu^-(-\mu)\,,\qquad I_\nu^{n+1}(0,+\mu)=Q_\nu^+(+\mu),\quad 0<\mu<1\,.
\end{aligned}$$

Let us prove by induction that

$$\begin{aligned}
&I_\nu^n \leq B_\nu(T_M) \text{ a.e. on } \Omega\times(-1,1)\times(0,+\infty)\,,\\
&J_\nu^n \leq B_\nu(T_M) \text{ a.e. on } \Omega\times(0,+\infty)\,,\qquad T^n\leq T_M \text{ a.e. on } \Omega\,.
\end{aligned}$$

This is true for $n=0$ since $T^0\equiv 0$, while

$$\begin{aligned}
I_\nu^0(z,\mu)&=1_{0<\mu<1}e^{-\kappa_\nu z/\mu}Q_\nu^+(\mu)+\mathbf{1}_{0<-\mu<1}e^{-\kappa_\nu(Z-z)/|\mu|}Q_\nu^-(-\mu)\\
&\leq(\mathbf{1}_{0<\mu<1}+\mathbf{1}_{0<-\mu<1})B_\nu(T_M)\,,\qquad \text{so that } 0\leq J_\nu^0\leq B_\nu(T_M)\,.
\end{aligned}$$

If this is true for some $n\geq 0$, then

$$\begin{aligned}
&(\mu\partial_z+\kappa_\nu)I_\nu^{n+1}=\kappa_\nu\Sigma_\nu^n\,, && 0\leq\Sigma_\nu^n\leq B_\nu(T_M)\,,\\
&I_\nu^{n+1}(Z,-\mu)\Big|_{0<\mu<1}=Q_\nu^-(-\mu)\,, && I_\nu^{n+1}(0,+\mu)\Big|_{0<\mu<1}=Q_\nu^+(+\mu)\,.
\end{aligned}$$

Thus, proceeding as (5.8) shows that $I_\nu^{n+1}\leq B_\nu(T_M)$. Hence $J_\nu^{n+1}\leq B_\nu(T_M)$, and one solves (6.9) for T^{n+1}. Since $J_\nu^n\geq S_\nu>0$ and

$$\int_0^\infty \kappa_\nu(1-a_\nu)J_\nu^{n+1}\mathrm{d}\nu\leq\int_0^\infty \kappa_\nu(1-a_\nu)B_\nu(T_M)\mathrm{d}\nu=\mathcal{B}(T_M)\,,$$

we conclude from Lemma 6.2 that T^{n+1} is a.e. less than or equal to the solution of the problem

$$-\Delta T+\boldsymbol{u}\cdot\nabla T+\mathcal{B}(T)=\mathcal{B}(T_M)\,,\qquad \frac{\partial T}{\partial n}\Big|_{\partial\Omega}=0\,,$$

which is obviously the constant T_M. Hence $T^{n+1}\leq T_M$ a.e. on Ω, so that we have proved by induction the desired chain of inequalities.

From these inequalities, we conclude that the sequences J_ν^n and T^n converge a.e. pointwise on $\Omega\times(0,\infty)$ and on Ω, respectively, to limits denoted J_ν and T, and that this convergence also holds in $L^p(\Omega\times(0,\infty))$ and $L^p(\Omega)$ for all $p\in[1,\infty)$ by dominated convergence.

Passing to the limit in (6.7) immediately shows that J_ν, T satisfy the first equation in (6.5). As for the second equation, one can pass to the limit in the right-hand side and in the nonlinear term on the left-hand side of (6.8). Since T^{n+1} is a weak solution of (6.8), one has $T^{n+1}\in H^1(\Omega)$ and

$$\int_\Omega \nabla T^{n+1}(\boldsymbol{x})\cdot\nabla\phi(\boldsymbol{x})\mathrm{d}\boldsymbol{x}-\int_\Omega T^{n+1}(\boldsymbol{x})\boldsymbol{u}(\boldsymbol{x})\cdot\nabla\phi(\boldsymbol{x})\mathrm{d}\boldsymbol{x}=\int_\Omega h_{n+1}(\boldsymbol{x})\phi(\boldsymbol{x})\mathrm{d}\boldsymbol{x}\tag{6.10}$$

for all $\phi\in H^1(\Omega)$, with

$$h_{n+1}:=\int_0^\infty \kappa_\nu(1-a_\nu)(J_\nu^{n+1}-B_\nu(T^{n+1}))\mathrm{d}\nu$$

so that h_{n+1} is bounded in $L^p(\Omega)$ for all $p\in[1,\infty)$. Taking $\phi=T^{n+1}$, and observing that

$$\int_\Omega T^{n+1}(\boldsymbol{x})\boldsymbol{u}(\boldsymbol{x})\cdot\nabla T^{n+1}(\boldsymbol{x})\mathrm{d}\boldsymbol{x}=\int_{\partial\Omega}\tfrac12 T^{n+1}(\boldsymbol{x})^2\boldsymbol{u}(\boldsymbol{x})\cdot\boldsymbol{n_x}\mathrm{d}\sigma(\boldsymbol{x})=0$$

since $\boldsymbol{u}\cdot\boldsymbol{n}|_{\partial\Omega}=0$ shows that T^{n+1} is bounded, and therefore weakly relatively compact in $H^1(\Omega)$. Since we already know that $T^{n+1}\to T$ in $L^p(\Omega)$ for all $p\in[1,\infty)$ as $n\to\infty$, we conclude that $T^{n+1}\to T$ weakly in $H^1(\Omega)$. At this point, we can pass to the limit in the weak formulation of (6.10), and this shows that T satisfies the second equation in (6.5). ∎

Next, we discuss the convergence rate of (6.7). We shall use the monotonic structure of the RT equations. Consider the upper approximating sequence

$$\begin{aligned}
&\mu\partial_z H_\nu^n=\kappa_\nu(a_\nu K_\nu^{n-1}+(1-a_\nu)B_\nu(\Theta^{n-1})-H_\nu^n)\,,\quad K_\nu=\tfrac12\int_{-1}^1 H_\nu d\mu\,,\\
&\qquad\qquad\boldsymbol{u}\cdot\nabla\Theta^n-\Delta\Theta^n=\int_0^\infty\kappa_\nu(1-a_\nu)(K_\nu^n-B_\nu(\Theta^n))d\nu\,,\\
&H_\nu^n(0,\mu)=Q_\nu^+(\mu)\,,\quad H_\nu^n(Z,-\mu)=Q_\nu^-(\mu)\,,\quad 0<\mu<1\,,\qquad \frac{\partial\Theta^n}{\partial n}\Big|_{\partial\Omega}=0
\end{aligned}$$

for all $n\geq1$, initialized with $\Theta^0=T_M$ and $H_\nu^0=K_\nu^0=B_\nu(\Theta^0)$.

THEOREM 6.3. *Assume that the absorption coefficient κ_ν and the scattering albedo a_ν satisfy* (6.1). *Assume, moreover, that the constant C_1 defined in* (2.18) *satisfies*

$$0\leq\gamma:=\left(\sup_{\nu>0}(1-a_\nu)C_1(\kappa_\nu)+\sup_{\nu>0}a_\nu C_1(\kappa_\nu)\right)<1\,.\tag{6.11}$$

Let the boundary source terms $Q_\nu^\pm$ satisfy the bound

$$0\leq Q_\nu^\pm(\mu)\leq B_\nu(T_M)\,,\qquad 0<\mu<1\,,\quad \nu>0\,.$$

Then one has

$$\begin{aligned}
&0\leq T^0\leq\cdots\leq T^{n-1}\leq\Theta^n\leq\cdots\Theta^1\leq T_M\,,\\
&0\leq J_\nu^0\cdots\leq J_\nu^{n-1}\leq K_\nu^n\leq\cdots\leq K_\nu^1\leq B_\nu(T_M)\,;\\
&\|\mathcal{B}(T^{n+1})-\mathcal{B}(T^n)\|_{L^1(\Omega)}\leq\|\mathcal{B}(\Theta^{n+1})-\mathcal{B}(T^n)\|_{L^1(\Omega)}\leq\gamma^n|\Omega|\mathcal{B}(T_M)\,,\\
&\|J_\nu^{n+1}-J_\nu^n\|_{L^1(\Omega\times(0,+\infty))}\leq\|K_\nu^{n+1}-J_\nu^n\|_{L^1(\Omega\times(0,+\infty))}\leq\frac{\gamma^n|\Omega|\mathcal{B}(T_M)}{\kappa_m(1-a_M)}\,;\\
&\|\mathcal{B}(T)-\mathcal{B}(T^n)\|_{L^1(\Omega)}\leq\frac{\gamma^n}{1-\gamma}|\Omega|\mathcal{B}(T_M)\,,\\
&\|J_\nu-J_\nu^n\|_{L^1(\Omega\times(0,+\infty))}\leq\frac{\gamma^n|\Omega|\mathcal{B}(T_M)}{\kappa_m(1-a_M)(1-\gamma)}\,.
\end{aligned}\tag{6.12}$$

Proof. First, one has

$$
\begin{aligned}
&\mu\partial_z H_\nu^1 + \kappa_\nu H_\nu^1 = \kappa_\nu B_\nu(T_M) \geq 0\,, \quad 0 < z < Z\,,\\
&0 \leq H_\nu^1(0,+\mu) = Q_\nu^+(\mu) \leq B_\nu(T_M)\,, \quad 0 < \mu < 1\,,\\
&0 \leq H_\nu^1(Z,-\mu) = Q_\nu^-(\mu) \leq B_\nu(T_M)\,, \quad 0 < \mu < 1\,,\\
&\implies H_\nu^1(z,\mu) = 1_{0<\mu<1}\left(e^{-\kappa_\nu z/\mu}Q_\nu^+(\mu) + (1-e^{-\kappa_\nu z/\mu})B_\nu(T_M)\right)\\
&\qquad\qquad + 1_{0<-\mu<1}\left(e^{-\kappa_\nu (Z-z)/|\mu|}Q_\nu^-(-\mu) + (1-e^{-\kappa_\nu (Z-z)/\mu})B_\nu(T_M)\right)\\
&\qquad\qquad 0 \leq I_\nu^0 \leq H_\nu^1 \leq B_\nu(T_M)\,, \qquad 0 \leq J_\nu^0 \leq K_\nu^1 \leq B_\nu(T_M)\,.
\end{aligned}
$$

Hence

$$
\mathcal{B}(\Theta^1) + \boldsymbol{u}\cdot\nabla\Theta^1 - \Delta\Theta^1 = \int_0^\infty \kappa_\nu(1-a_\nu)K_\nu^1 \mathrm{d}\nu \leq \mathcal{B}(T_M)\,,
$$

so that $0 \leq T^0 \leq \Theta^1 \leq T_M$ by Lemma 6.2. The same induction argument as in the proof of Theorem 6.1 shows that

$$
\begin{aligned}
0 \leq \cdots \leq \Theta^n \leq \Theta^{n-1} \leq T_M\,,\\
0 \leq \cdots \leq H_\nu^n \leq H_\nu^{n-1} \leq B_\nu(T_M)\,, \quad 0 \leq \cdots \leq K_\nu^n \leq K_\nu^{n-1} \leq B_\nu(T_M)\,.
\end{aligned}
$$

Moreover, assume that we have proved that

$$
\begin{aligned}
0 \leq T^0 \leq \cdots \leq T^{n-1} \leq \Theta^n \leq \cdots \Theta^1 \leq T_M\,,\\
0 \leq I_\nu^0 \leq \cdots \leq I_\nu^{n-1} \leq H_\nu^n \leq \cdots H_\nu^1 \leq B_\nu(T_M)\,,\\
0 \leq J_\nu^0 \cdots \leq J_\nu^{n-1} \leq K_\nu^n \leq \cdots \leq K_\nu^0 \leq B_\nu(T_M)\,.
\end{aligned}
$$

Then

$$
\begin{aligned}
\mu\partial_z(H_\nu^{n+1} - I_\nu^n) + \kappa_\nu(H_\nu^{n+1} - I_\nu^n) &= \kappa_\nu a_\nu(K_\nu^n - J_\nu^{n-1})\\
&\quad + \kappa_\nu(1-a_\nu)(B_\nu(\Theta^n) - B_\nu(T^{n-1})) \geq 0\,,\\
(H_\nu^{n+1} - I_\nu^n)(0,+\mu) = (H_\nu^{n+1} - I_\nu^n)(Z,-\mu) &= 0\,, \quad 0 < \mu < 1\,,
\end{aligned}
$$

so that $I_\nu^n \leq H_\nu^{n+1}$ and $J_\nu^n \leq K_\nu^{n+1}$. Then $\frac{\partial\Theta^{n+1}}{\partial n}\Big|_{\partial\Omega} = \frac{\partial T^n}{\partial n}\Big|_{\partial\Omega} = 0$ and

$$
\begin{aligned}
\mathcal{B}(\Theta^{n+1}) + \boldsymbol{u}\cdot\nabla\Theta^{n+1} - \Delta\Theta^{n+1} &= \int_0^\infty \kappa_\nu(1-a_\nu)K_\nu^{n+1}\mathrm{d}\nu\,,\\
\mathcal{B}(T^n) + \boldsymbol{u}\cdot\nabla T^n - \Delta T^n &= \int_0^\infty \kappa_\nu(1-a_\nu)J_\nu^n\mathrm{d}\nu\,,
\end{aligned}
$$

and Lemma 6.2 implies that $T^n \leq \Theta^{n+1}$. Hence we have proved by induction that

$$
\begin{aligned}
0 \leq T^0 \leq \cdots \leq T^{n-1} \leq \Theta^n \leq \cdots \Theta^1 \leq T_M\,,\\
0 \leq I_\nu^0 \leq \cdots \leq I_\nu^{n-1} \leq H_\nu^n \leq \cdots H_\nu^1 \leq B_\nu(T_M)\,,\\
0 \leq J_\nu^0 \cdots \leq J_\nu^{n-1} \leq K_\nu^n \leq \cdots \leq K_\nu^1 \leq B_\nu(T_M) \text{ for all } n \geq 1,
\end{aligned}
$$

which implies the two first chains of inequalities in (6.12).

Then

$$
\begin{aligned}
&\mathcal{B}(\Theta^{n+1})-\mathcal{B}(T^n)+\boldsymbol{u}\cdot\nabla(\Theta^{n+1}-T^n)-\Delta(\Theta^{n+1}-T^n)\\
&=\int_0^\infty \kappa_\nu(1-a_\nu)(K_\nu^{n+1}-J_\nu^n)\mathrm{d}\nu\,,\quad \frac{\partial(\Theta^{n+1}-T^n)}{\partial n}\Big|_{\partial\Omega}=0\,,\\
\implies\quad &\int_\Omega(\mathcal{B}(\Theta^{n+1})-\mathcal{B}(T^n))\mathrm{d}\boldsymbol{x}=\int_\Omega\int_0^\infty \kappa_\nu(1-a_\nu)(K_\nu^{n+1}-J_\nu^n)\mathrm{d}\nu\mathrm{d}\boldsymbol{x}\,,
\end{aligned}
$$

because
$$\int_{\partial\Omega}\left((\Theta^{n+1}-T^n)\boldsymbol{u}\cdot\boldsymbol{n}_{\boldsymbol{x}}-\frac{\partial(\Theta^{n+1}-T^n)}{\partial n}\right)\mathrm{d}\sigma(\boldsymbol{x})=0\,.$$

Then

$$
\begin{aligned}
&K_\nu^{n+1}(\boldsymbol{x})-J_\nu^n(\boldsymbol{x})\\
&=\tfrac12\int_0^Z \kappa_\nu E_1(\kappa_\nu|z-\zeta|)(1-a_\nu)(B_\nu(\Theta^n)-B_\nu(T^{n-1}))(x,y,\zeta)\mathrm{d}\zeta\\
&\quad+\tfrac12\int_0^Z \kappa_\nu E_1(\kappa_\nu|z-\zeta|)a_\nu(K_\nu^n-J_\nu^{n-1})(x,y,\zeta)\mathrm{d}\zeta\,.\\
\implies\quad &\epsilon_n:=\int_\Omega\int_0^\infty \kappa_\nu(1-a_\nu)(K_\nu^{n+1}-J_\nu^n)\mathrm{d}\nu\mathrm{d}\boldsymbol{x}=\tfrac12\int_{\mathbb{O}}\mathrm{d}x\mathrm{d}y\int_0^\infty\mathrm{d}\nu\int_0^Z\mathrm{d}z\int_0^Z\\
&\qquad\kappa_\nu^2E_1(\kappa_\nu|z-\zeta|)\cdot(1-a_\nu)^2(B_\nu(\Theta^n)-B_\nu(T^{n-1}))(x,y,\zeta)\mathrm{d}\zeta\\
&+\tfrac12\int_{\mathbb{O}}\mathrm{d}x\mathrm{d}y\int_0^\infty\mathrm{d}\nu\int_0^Z\mathrm{d}z\int_0^Z\kappa_\nu^2E_1(\kappa_\nu|z-\zeta|)\cdot(1-a_\nu)a_\nu(K_\nu^n-J_\nu^{n-1})(x,y,\zeta)\mathrm{d}\zeta\,.
\end{aligned}
$$

At this point, we integrate first in z and use (2.18) to obtain

$$
\begin{aligned}
\epsilon_n&=\int_\Omega\int_0^\infty \kappa_\nu(1-a_\nu)(K_\nu^{n+1}-J_\nu^n)\mathrm{d}\nu\mathrm{d}\boldsymbol{x}\\
&\le\int_{\mathbb{O}}\mathrm{d}x\mathrm{d}y\int_0^\infty\mathrm{d}\nu\int_0^Z C_1(\kappa_\nu)\kappa_\nu(1-a_\nu)^2(B_\nu(\Theta^n)-B_\nu(T^{n-1}))(x,y,\zeta)\mathrm{d}\zeta\\
&\quad+\int_{\mathbb{O}}\mathrm{d}x\mathrm{d}y\int_0^\infty\mathrm{d}\nu\int_0^Z C_1(\kappa_\nu)\kappa_\nu(1-a_\nu)a_\nu(K_\nu^n-J_\nu^{n-1})(x,y,\zeta)\mathrm{d}\zeta\\
&\le\sup_{\nu>0}(1-a_\nu)C_1(\kappa_\nu)\int_\Omega\int_0^\infty\kappa_\nu(1-a_\nu)(B_\nu(\Theta^n)-B_\nu(T^{n-1}))(\boldsymbol{x})\mathrm{d}\nu\mathrm{d}\boldsymbol{x}\\
&\quad+\sup_{\nu>0}a_\nu C_1(\kappa_\nu)\int_\Omega\int_0^\infty\kappa_\nu(1-a_\nu)(K_\nu^n-J_\nu^{n-1})(\boldsymbol{x})\mathrm{d}\nu\mathrm{d}\boldsymbol{x}\\
&\le\sup_{\nu>0}(1-a_\nu)C_1(\kappa_\nu)\int_\Omega(\mathcal{B}(\Theta^n)-\mathcal{B}(T^{n-1}))(\boldsymbol{x})\mathrm{d}\boldsymbol{x}\\
&\quad+\sup_{\nu>0}a_\nu C_1(\kappa_\nu)\int_\Omega\int_0^\infty\kappa_\nu(1-a_\nu)(K_\nu^n-J_\nu^{n-1})(\boldsymbol{x})\mathrm{d}\nu\mathrm{d}\boldsymbol{x}\\
&=\epsilon_{n-1}\left(\sup_{\nu>0}(1-a_\nu)C_1(\kappa_\nu)+\sup_{\nu>0}a_\nu C_1(\kappa_\nu)\right)\,.
\end{aligned}
$$

Hence $\epsilon_n\le\epsilon_0\gamma^n$ with $\gamma:=(\sup_{\nu>0}(1-a_\nu)C_1(\kappa_\nu)+\sup_{\nu>0}a_\nu C_1(\kappa_\nu))\in[0,1)$, while $\epsilon_0\le|\Omega|\mathcal{B}(T_M)<\infty$. Hence the sequence $(K_\nu^n,\Theta^n)_{n\ge1}$ of upper approximations and the sequence (J_ν^n,T^n) of lower approximations provided by (6.7) are adjacent. In particular,

$$\|\mathcal{B}(T^{n+1}) - \mathcal{B}(T^n)\|_{L^1(\Omega)} = \int_\Omega (\mathcal{B}(T^{n+1}) - \mathcal{B}(T^n))\mathrm{d}\boldsymbol{x} \\ \leq \int_\Omega (\mathcal{B}(\Theta^{n+1}) - \mathcal{B}(T^n))\mathrm{d}\boldsymbol{x} \leq \epsilon_0\gamma^n$$

for all $n \geq 1$, so that $\|\mathcal{B}(T) - \mathcal{B}(T^n)\|_{L^1(\Omega)} \leq \frac{\epsilon_0\gamma^n}{1-\gamma}$. Similarly,

$$\int_\Omega \int_0^\infty \kappa_\nu(1-a_\nu)(J_\nu^{n+1} - J_\nu^n)\mathrm{d}\nu\mathrm{d}\boldsymbol{x} \\ \leq \int_\Omega \int_0^\infty \kappa_\nu(1-a_\nu)(K_\nu^{n+1} - J_\nu^n)\mathrm{d}\nu\mathrm{d}\boldsymbol{x} \leq \epsilon_0\gamma^n\,,$$

$$\kappa_m(1-a_M)\|J_\nu - J_\nu^n\|_{L^1(\Omega\times(0,\infty))} \leq \sum_{m\geq n}\int_\Omega \int_0^\infty \kappa_\nu(1-a_\nu)(J_\nu^{m+1} - J_\nu^m)\mathrm{d}\nu\mathrm{d}\boldsymbol{x} \\ \leq \sum_{m\geq n}\int_\Omega \int_0^\infty \kappa_\nu(1-a_\nu)(K_\nu^{m+1} - J_\nu^m)\mathrm{d}\nu\mathrm{d}\boldsymbol{x} \leq \frac{\epsilon_0\gamma^n}{1-\gamma}\,.$$

This concludes the proof of the convergence statements in (6.12). ∎

Remark 6.4. The condition $\sup_{\nu>0}(1-a_\nu)C_1(\kappa_\nu) < 1$ implies that the absorption-emission nonlinearity is a contraction, while $\sup_{\nu>0} a_\nu C_1(\kappa_\nu) < 1$ implies that the scattering term is also a contraction. The condition $\gamma < 1$ implies that these two terms are contractions separately, leading to the exponential rate in Theorem 6.3 (3). As $a_\nu \in [0,1]$ and $\kappa_\nu \mapsto C_1(\kappa_\nu)$ is monotone increasing from 0 to 1, for a given a_ν there is always a κ^* such that (6.11) holds for all $\kappa_\nu < \kappa^*$. Conversely, if it is known that $\kappa_\nu < \kappa^*$, for some κ^*, for all ν, there is a maximum a^* for which (6.11) for all $a_\nu < a^*$. By Lemma 2.1, $C_1 < 1$. Hence $\gamma < 1$ if a_ν is independent of ν, whatever the upper bound κ_M in (6.1) is. The more a_ν varies between 0 and 1, the lower κ_M must be to satisfy $\gamma < 1$.

With the monotonic structure of the RT equations, our argument will also provide the uniqueness of the solution of the system (6.2)–(6.4).

THEOREM 6.5. *Under the same assumptions as in Theorem* 6.3*, there exists at most one solution* (I_ν, T) *of the problem* (6.2)–(6.4) *such that* $T \in L^\infty(\Omega)$,

$$I_\nu \geq 0 \text{ a.e. on } \Omega \times (-1,1) \times (0,\infty) \quad \text{and} \quad T \geq 0 \text{ a.e. on } \Omega\,.$$

Proof. Let (I_ν, T) be a solution of (6.2)–(6.4), and assume that the upper approximating sequence $(H_\nu^n, \Theta^n)_{n\geq 1}$ satisfies $I_\nu \leq H_\nu^n$ and $J_\nu \leq K_\nu^n$, with $T \leq \Theta^n$. Then, one has

$$\mu\partial_z(H_\nu^{n+1} - I_\nu) + \kappa_\nu(H_\nu^{n+1} - I_\nu) = \kappa_\nu a_\nu(K_\nu^n - J_\nu) \\ + \kappa_\nu(1-a_\nu)(B_\nu(\Theta^n) - B_\nu(T)) \geq 0\,,$$
$$(H_\nu^{n+1} - I_\nu)(0, +\mu) = (H_\nu^{n+1} - I_\nu)(Z, -\mu) = 0\,, \qquad 0 < \mu < 1\,.$$

Solving this equation for $(H_\nu^{n+1} - I_\nu)$ by the method of characteristics shows that $I_\nu \leq H_\nu^{n+1}$, and therefore, $J_\nu \leq K_\nu^{n+1}$. Next, one has

$$\mathcal{B}(\Theta^{n+1}) - \mathcal{B}(T) + \boldsymbol{u}\cdot\nabla(\Theta^{n+1} - T) - \Delta(\Theta^{n+1} - T) \\ = \int_0^\infty \kappa_\nu(1-a_\nu)(K_\nu^{n+1} - J_\nu)\mathrm{d}\nu \geq 0\,, \quad \frac{\partial(\Theta^{n+1} - T)}{\partial n}\Big|_{\partial\Omega} = 0,$$

so that $T \leq \Theta^{n+1}$ according to Lemma 6.2.

It remains to check the initial step of this induction argument. Since $T \in L^\infty(\Omega)$, we pick $\Theta^0 = \max(T_M, \|T\|_{L^\infty(\Omega)})$ and $H^0_\nu = K^0_\nu = B_\nu(\Theta^0)$. Hence $T \leq \Theta^0$ by construction. Next, we prove that $I_\nu \leq B_\nu(\Theta^0)$. Multiplying both sides of (6.2) by $s_+(I_\nu - B_\nu(\Theta^0))$, we repeat the argument of the proof of Theorem 4.1:

$$\begin{aligned}&\partial_z\langle\mu(I_\nu - B_\nu(\Theta^0))_+\rangle\\ &= -\langle\kappa_\nu(1-a_\nu)(I_\nu - B_\nu(\Theta^0)) - (B_\nu(T) - B_\nu(\Theta^0))s_+(I_\nu - B_\nu(\Theta^0))\rangle\\ &-\langle\kappa_\nu a_\nu(I_\nu - B_\nu(\Theta^0)) - (J_\nu - B_\nu(\Theta^0))s_+(I_\nu - B_\nu(\Theta^0))\rangle = -D_1 - D_2\,.\end{aligned}$$

We have seen in the proof of Theorem 4.1 that

$$\begin{aligned}&D_2 = \langle\kappa_\nu a_\nu(I_\nu - B_\nu(\Theta^0)) - (J_\nu - B_\nu(\Theta^0))s_+(I_\nu - B_\nu(\Theta^0))\rangle\\ &= \langle\kappa_\nu a_\nu(I_\nu - B_\nu(\Theta^0)) - (J_\nu - B_\nu(\Theta^0))(s_+(I_\nu - B_\nu(\Theta^0))) - s_+(J_\nu - B_\nu(\Theta^0))\rangle \geq 0\,.\end{aligned}$$

As for D_1, observe that

$$D_1 = \langle\kappa_\nu(1-a_\nu)((I_\nu - B_\nu(\Theta^0)) - (B_\nu(T) - B_\nu(\Theta^0)))(s_+(I_\nu - B_\nu(\Theta^0)) - s_+(T-\Theta^0))\rangle,$$

which is positive by our assumption on T which implies that $s_+(T - \Theta^0) = 0$. Integrating on Ω, we conclude that

$$\int_{\mathbb{O}}\langle\mu_+(I_\nu - B_\nu(\Theta^0))_+\rangle(x,y,Z)\mathrm{d}x\mathrm{d}y = \int_{\mathbb{O}}\langle\mu_-(I_\nu - B_\nu(\Theta^0))_+\rangle(x,y,0)\mathrm{d}x\mathrm{d}y = 0$$

and that $D_1 = D_2 = 0$ a.e. on Ω. Now, since $\kappa_\nu(1 - a_\nu) \geq \kappa_m(1 - a_M) > 0$, the condition $D_1 = 0$ implies that

$$\begin{aligned}&((I_\nu - B_\nu(\Theta^0)) - (B_\nu(T) - B_\nu(\Theta^0)))(s_+(I_\nu - B_\nu(\Theta^0)) - s_+(T-\Theta^0)) = 0,\\ &\qquad\text{which implies in turn that } s_+(I_\nu - B_\nu(\Theta^0)) = s_+(T-\Theta^0) = 0.\end{aligned}$$

Hence $I_\nu \leq B_\nu(\Theta^0)$, which completes the proof of the initialization of our induction argument. Summarizing, we have proved that if one chooses $\Theta^0 = \max(T_M, \|T\|_{L^\infty(\Omega)})$, the solution (I_ν, T) of (6.2)–(6.4) considered satisfies

$$I_\nu \leq H^n_\nu \leq H^{n-1}_\nu \leq \cdots \leq H^0_\nu = B_\nu(\Theta^0)\,,\ \text{while } T \leq \Theta^n \leq \Theta^{n-1} \leq \cdots \leq \Theta^0\,,$$

where (H^n_ν, Θ^n) is the upper approximating sequence. A similar argument (with a slightly simpler initialization) shows that

$$I_\nu \geq I^n_\nu \geq I^{n-1}_\nu \geq \cdots \geq I^0_\nu = 0\,,\ \text{while } T \geq T^n \geq T^{n-1} \geq \cdots \geq T^0 = 0\,.$$

With this, we easily prove the uniqueness of the solution of (6.2)–(6.4). If (I_ν, T) and (I'_ν, T') are two solutions satisfying the assumptions of Theorem 6.5, we initialize the upper approximating sequence with $\Theta^0 = \max(T_M, \|T\|_{L^\infty(\Omega)}, \|T'\|_{L^\infty(\Omega)})$. The argument above shows that $I^n_\nu \leq I_\nu$, $I'_\nu \leq H^{n+1}_\nu$ while $T^n \leq T$, $T' \leq \Theta^{n+1}$. Hence

$$\begin{aligned}&\|J_\nu - J'_\nu\|_{L^1(\Omega\times(0,\infty))} \leq \|K^{n+1}_\nu - J^n_\nu\|_{L^1(\Omega\times(0,\infty))} \leq \frac{|\Omega|\gamma^n}{\kappa_m(1-a_M)}\mathcal{B}(\Theta^0)\,,\\ &\|\mathcal{B}(T) - \mathcal{B}(T')\|_{L^1(\Omega)} \leq \|\Theta^{n+1} - T^n\|_{L^1(\Omega)} \leq \gamma^n|\Omega|\mathcal{B}(\Theta^0)\,.\end{aligned}$$

When $n \to \infty$ it shows that $T = T'$ a.e. on Ω and $J_\nu = J'_\nu$ a.e. on $\Omega \times (0,\infty)$. Once it is known that $J_\nu = J'_\nu$ a.e. on $\Omega \times (0,\infty)$, solving (6.2) for I_ν and I'_ν by the method of characteristics shows that $I_\nu = I'_\nu$ a.e. on $\Omega \times (-1,1) \times (0,\infty)$. ∎

Several remarks regarding Theorems 6.1, 6.3, and 6.5 are in order.

Remarks.

(1) One can treat slightly more general situations with the same techniques. For instance, one could assume that the scattering rate a_ν depends on z, and is a slowly varying function of x, y. This may be useful to include a layer of clouds in our problem. Similarly, one can treat the case where ρ is not a constant, but for instance a function of z, by introducing an optical length defined as in (2.14). Typically, one could assume that $0 < \rho_m \le \rho(z) \le \rho_M < \infty$, and recast the RT equation in terms of the variable τ instead of z. Of course, this will modify the drift-diffusion operator in the left-hand side of (6.3), but in a way that should be tractable by the same methods.

(2) One could enrich the class of boundary conditions considered here by taking into account the albedo coefficients of the boundary at $z = 0$ and $z = Z$. This should lead to more serious modifications of the strategy discussed above, but we expect that some of our results can be modified to handle these more general boundary conditions.

(3) Until now, we have treated the case of an incompressible fluid with constant density. This is the reason for the factor c_P/c_V multiplying the heat diffusivity. One can treat in the same manner the case of low Mach number flows of a compressible fluid which could be useful for the stratosphere. (In the case of water at 20°C, one finds that $c_P/c_V = 1.007$, so that this ratio is very close to 1 for all practical purposes.)

(4) Including Boussinesq's approximation in our model in order to take into account the buoyancy created by the temperature dependence of the density is a more difficult problem in the first place because the motion equation of the fluid becomes coupled to the simple system considered here. We will address this problem in future work.

7. Numerical simulations. This section is meant to show that iterations (3.2), (5.6), and (6.7), proposed in the previous sections, are monotone, implementable, robust, and computationally fairly fast. Here, robustness means that there are no singular integrals and convergence is not subject to the adjustment of sensitive parameters; in other words, the mathematical properties derived above are observed numerically.

Two computer programs have been written: one in `C++` with (3.2) or (5.6) for the case $\kappa_T = 0$ and the other in the `FreeFEM` language [17] with (6.7) for the general case, either in Cartesian coordinates (2D) or in spherical ones (3D).

The programming is straightforward except at three places:

1. Writing a function to compute the exponential integrals is simple due to two formulas

$$\begin{aligned} &\mathrm{E}_1(x) = -\gamma - \ln x + \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^k}{k\, k!}, \qquad \gamma = 0.577215664901533, \\ &\mathrm{E}_{n+1}(x) = \frac{\mathrm{e}^{-x}}{n} - \frac{x}{n}\mathrm{E}_n(x), \end{aligned} \tag{7.1}$$

   but the tail of the series falls below machine precision if $x > 18$. For practical purposes, keeping $9 + (\mathrm{int}(x) - 1) \cdot 5$ terms in the series is more than enough.
2. When thermal diffusion is neglected, one must solve for T, with J_ν given,

$$\int_0^\infty \kappa_\nu (1 - a_\nu) B_\nu(T) \mathrm{d}\nu = \int_0^\infty \kappa_\nu (1 - a_\nu) J_\nu \mathrm{d}\nu.$$

   Newton iterations are used combined with dichotomy. The integrals are approximated with the trapezoidal rule on a mesh which is uniform in wavelength with up to 900 points, though 300 are usually more than enough.

3. When thermal diffusion is not neglected, the temperature equation has a similar nonlinearity which requires iterations. We use the time dependent problem, discretized by a method of characteristics, as follows, which is unconditionally stable:
(7.2)
$$\frac{1}{\delta t}(T^{m+1}(x) - T^m(x-\delta t u(x))) - \kappa_T \Delta T^{m+1} + \int_0^\infty \kappa_\nu (1-a_\nu) B_\nu(T^{m+1}) \mathrm{d}\nu$$
$$= \int_0^\infty \kappa_\nu (1-a_\nu) J_\nu \mathrm{d}\nu,$$
with Dirichlet or Neumann conditions on the boundaries. Then a standard P^1 finite element approximation of the temperature equation is applied for the discretization in a finite-dimensional space V_h on a triangular (2D) or tetrahedral (3D) mesh. Then the numerical approximation of T^{m+1} is also the solution of the minimization problem below, T^m and J_ν given, which can be solved by a Broyden–Fletcher–Goldfarb–Shanno (BFGS) method:
$$\begin{aligned}(7.3)\quad &\min_{T\in V_h} \int_\Omega \left[\frac{T^2}{2\delta t} + \frac{\kappa_T}{2}|\nabla T|^2 + \int_0^\infty \left(\kappa_\nu(1-a_\nu)\int_0^T B_\nu(T')\mathrm{d}T'\right)\mathrm{d}\nu\right]\mathrm{d}x \\ &- \int_\Omega T\left(\frac{1}{\delta T}T^m(x-\delta t u(x)) + \int_0^\infty \kappa_\nu(1-a_\nu)J_\nu \mathrm{d}\nu\right)\mathrm{d}x.\end{aligned}$$
Speed-up can be achieved by using for initial value in BFGS, the temperature computed by the Newton algorithm mentioned above with $\kappa_T = 0$.

The first set of tests is for the RT system decoupled from the temperature equation. The second set of test involves the complete system in 2D and the third is also with RT coupled with the temperature equation but in 3D.

7.1. RT in the troposphere without thermal diffusion. The troposphere is roughly 12km thick. When air density is $\rho(z) = \rho_0 e^{-z}$, with $\rho_0 = 1.225 \cdot 10^{-3}$, a change of vertical coordinate is made, $\tau = 1 - e^{-z}$ to remove the exponential from the equations; thus $\tau \in (0, Z)$ with $Z = 1 - e^{-12}$.

We wish to study the influence of κ_ν on T. As said earlier, $\bar\kappa_\nu$ is the mass-extinction coefficient, and $\kappa_\nu = \rho_0 \bar\kappa_\nu$ is the absorption coefficient, defined as a dimensionless parameter between 0 and 1 which measures the output to input ratio of ν-light crossing a horizontal unit length (here 1 km) of air layer. Note, however, that we are not restricted to $\kappa_\nu \in (0,1)$ because of the following observation.

Remark 7.1. When Z is large, $T(\tau)$ computed by (3.2) with κ_ν is equal to $T(\frac{\tau}{L})$ computed by (3.2) with $\kappa_\nu L$.

Incidently, this implies that if $\tau \mapsto T(\tau)$ is decreasing, increasing κ uniformly in ν will cause a uniform decrease of temperature.

The problem is as follows: find $I_\nu(\tau,\mu)$ and $T(\tau)$ verifying (2.10), (2.12), and the boundary conditions used in [9]:

$$I(0,\mu)|_{\mu>0} = Q_\nu \mu, \quad I(Z,\mu)|_{\mu<0} = 0. \tag{7.4}$$

The first one implies that the Earth receives sunlight on its surface and that the computation does not include the effect of the atmosphere on the sun rays during their downward travel ($\mu < 0$). It is generally assumed that visible light is unaffected by air.

Due to Planck's law for black bodies, Earth radiates ($\mu > 0$) infrared radiations upward; the second boundary condition says that these escape at $\tau = Z$ without back-scattering.

The frequency spectrum of interest is $\nu \in (0, 20 \cdot 10^{14})$. It is convenient to rescale some variables:

$$\nu' = 10^{-14}\nu, \quad T' = 10^{-14}\frac{k}{h}T = 10^{-14}\frac{1.381 \cdot 10^{-23}}{6.626 \cdot 10^{-34}}T = \frac{T}{4798},$$

so as to write

$$B_\nu(T) = B_0 \frac{\nu'^3}{e^{\frac{\nu'}{T'}} - 1}, \quad \text{with } B_0 = \frac{2h}{c^2}10^{42} = \frac{2 \times 6.626 \cdot 10^{-34}}{2.998^2 \cdot 10^{16}}10^{42} = 1.4744 \cdot 10^{-8}.$$

We may work with B_ν/B_0 and I_ν/B_0 so that, forgetting the primes, we have (2.10) with (2.12) and (7.4) with

$$B_\nu(T) = \frac{\nu^3}{e^{\frac{\nu}{T}} - 1}, \quad Q_\nu = Q_0 B_\nu(1.209), \quad Q_0 = 2 \cdot 10^{-5}, \tag{7.5}$$

because with T_{Sun} being $5800°K$, it is now $5800/4798 = 1.209$; Q_0 is found from the sunlight energy sent to Earth, $Q_{sun} = 1370\text{Watt}/m^2$:

$$Q_{sun} = \int_0^\infty Q_0 B_0 B_\nu(1.209) 10^{14} \mathrm{d}\nu = Q_0 1.4744 \cdot 10^6 \frac{(1.209\pi)^4}{15} = 1.023 \cdot 10^7 Q_0. \tag{7.6}$$

This leads to $Q_0 = 13.4 \cdot 10^{-5}$, but the Sun sees Earth as a disk of surface πR^2 while the Earth surface re-emitting radiations is $2\pi R^2$, so $6.7 \cdot 10^{-5}$ should be used instead. Yet this value is too high as it gives an Earth temperature of around 400°K. It comes down to 3.1 when it is corrected by the latitude, $\frac{1}{\sqrt{2}}$ at 45°, and by the Earth albedo: 35% of the Sun energy is reflected, i.e., not absorbed, by the Earth surface. Furthermore, due to the alternation of days and nights only a portion of the final value should be retained [9]. Thus Q_0 is in the range $(1.5, 3) \cdot 10^{-5}$. A reasonable value is $Q_0 = 2 \cdot 10^{-5}$, because, with a constant $\kappa = 0.5$, the temperature near the ground is found to be around 24°C; but it should not be taken for its face value because rains, clouds, etc. are not accounted for.

Scattering is the sum of an isotropic part and a Rayleigh part; both have their own a_ν, function of altitude (i.e., τ) and ν.

To simulate clouds, isotropic scattering is activated between altitude Z_1 and $Z_2 > Z_1$ and

$$a_\nu(z) = \alpha[4 \max(z - Z_1, 0) \max(Z_2 - z, 0.)/(Z_2 - Z_1)^2].$$

It is known that Rayleigh scattering is a function of ν^4 in the ultraviolet range at high altitude, so it is switched on above altitude Z_2 and is $O(\nu^4)$ for $\nu \in (0.8, 1.2)$:

$$a'_\nu(z) = \alpha[40 \max(\nu - 0.8, 0)^2 \max(1.2 - \nu, 0)^2 \max(z - Z_2, 0)/(Z - Z_2)].$$

The values of the physical and numerical parameters are as follows:

- $\alpha = \frac{1}{2}$ or zero; $Z_1 = 6$km, $Z_2 = 9$km.
- Absorption coefficient κ_ν digitalized from Gemini measurements.
- Discretization: 60 altitude stations, 485 frequencies corresponding to a uniform grid in wavelength in $(1,20)\mu m$.
- Number of iterations 20.

The Gemini measurements of the absorption are posted on the internet in
https://www.gemini.edu/observing/telescopes-and-sites/sites#Transmission.
Figure 1 shows κ_ν^0 versus wavelength c/ν. Recall that visible light is in the range $0.4 - 0.7\mu m$ (i.e., 450–750 THz) and relevant infrared radiations are in the range $0.8 - 20\mu m$ (i.e., 0.03–0.4 THz).

To assess the sensitivity of the temperature to gas like carbon dioxide opaque, for wavelengths in 7–9μm and 1–3μm, we constructed κ_ν^1 by increasing κ_ν^0 by a factor 3, and capped at 1, in the infrared range 7–8μm. Similarly, we constructed κ_ν^2 by increasing κ_ν^0 by a factor of 3, and capped at 1, in the range 1–3μm. These are displayed in Figure 1.

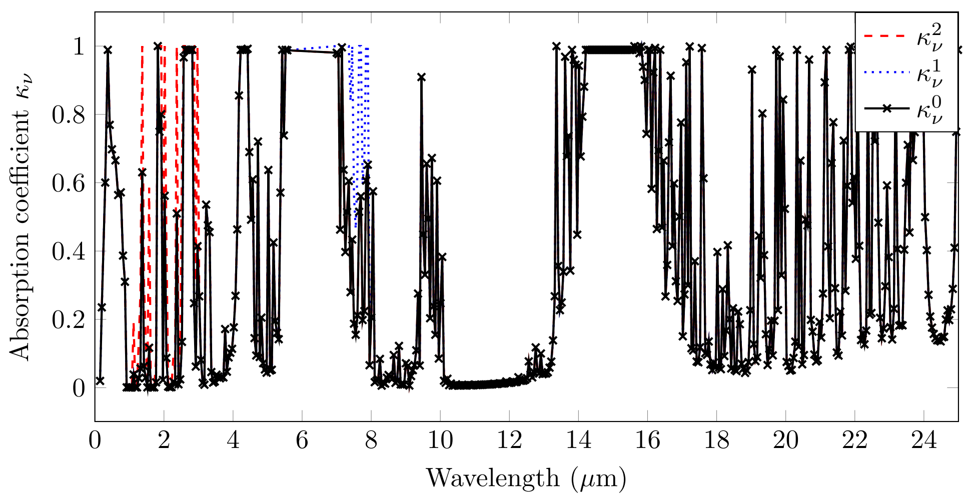

Fig. 1. *Absorption κ_ν^0 versus wavelength $(3/\nu)$ read from Gemini measurements; κ_ν^1 is κ_ν^0 increased in the infrared range 2–3μm and κ_ν^2 is κ_ν^0 increased in the range 8–14μm. The $\times$ marks show the 487 grid points for the integrals in ν.*

Convergence of the lower increasing and upper decreasing sequences is studied with and without Rayleigh scattering.

The convergence of the lower sequences is faster and is slightly slower in the presence of scattering. Yet, for both, 20 iterations seem appropriate for a three-digit precision.

Next, results are shown with κ_ν^0, κ_ν^1, and κ_ν^2, with and without scattering. Figures 3 and 4 show the mean radiation intensity J_ν versus wavelength at altitude 0 and 12km. Notice the dramatic changes when going from κ_ν^0 to κ_ν^1 and the smaller changes in the opposite direction when going from κ_ν^0 to κ_ν^2. Note too that scattering decreases J_ν. It is also interesting to note that in the frequency range where κ_ν^0 is very small such as wavelength 3–4μm and 10–14μm, J_ν is also small; it is because the Planck function with the Earth temperature (3.2) cannot create ν-waves in regions where κ_ν is small.

Figure 5 shows the scaled temperatures versus altitude computed with κ_ν^0, κ_ν^1, and κ_ν^2 with and without scattering. Note that going from κ_ν^0 to κ_ν^1 decreases the temperatures by 5%. On the other hand, going from κ_ν^0 to κ_ν^2 increases the temperatures by 2%.

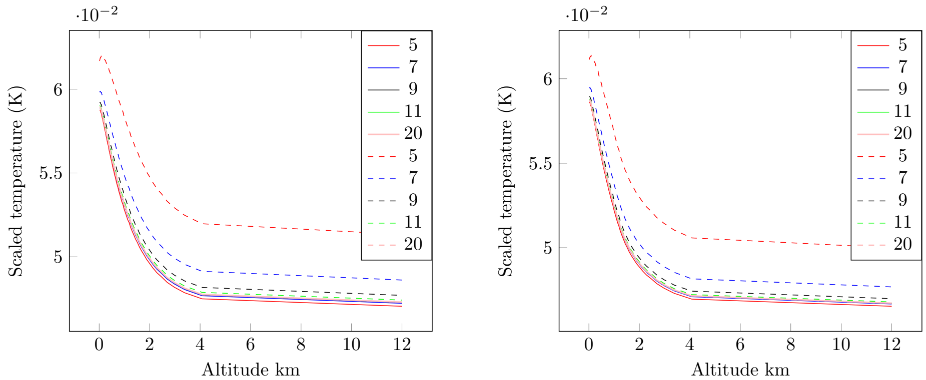

Fig. 2. *Temperatures scaled by* $4,798$ *without (left) and with (right) scattering: convergence history. The dashed curves are computed with an initial* $T^0 = T_{Sun}/10$ *and the solid curves with* $T^0 = 0$*. Notice the monotonic convergence towards a solution after* 20 *iterations. The iterations shown for the upper and lower solutions are* $(5, 7, 9, 11, 20)$*. This computation has used* $Q_0 = 3{\cdot}10^{-5}$.

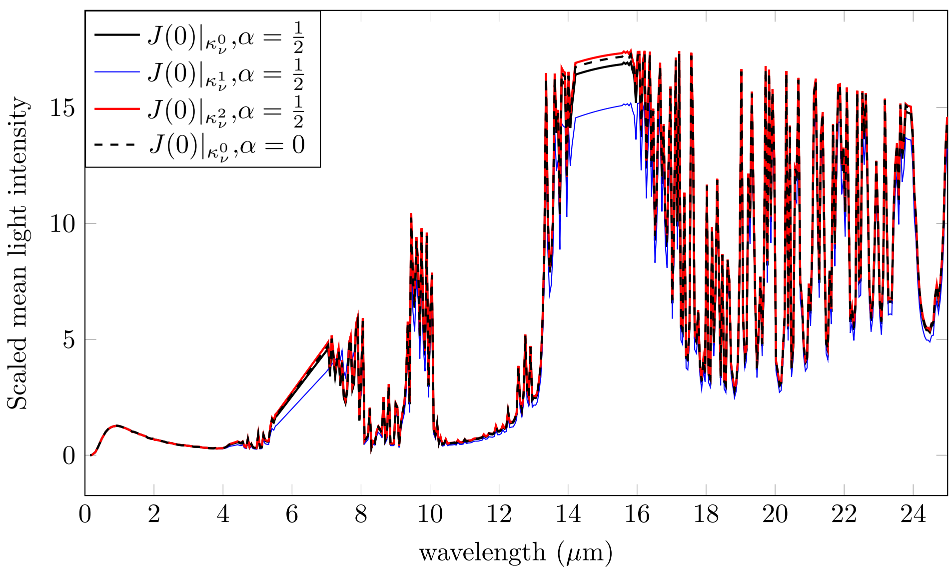

Fig. 3. *Computed mean radiation intensities* $10^5 \cdot J_\nu(0)$ *at the ground level for* κ_ν^0, κ_ν^1, κ_ν^2 *with scattering* ($\alpha = \frac{1}{2}$) *and for* κ_ν^0 *without scattering.*

Comments.

- CPU time is $20''$ on a Macbook air M1, but with a smoother κ_ν, 50 ν-integration points are sufficient, cutting the CPU time by 10 to $2''$.
- We observed that a highly oscillating κ_ν did not cause any programming or convergence problems. The total light intensities J plotted on Figures 3 and 4 show clearly that the method traces the small or large changes on κ_ν.
- Figure 2: Monotone convergence from below and from above is observed. The convergence from below, i.e., starting with $T^0 = 0$, is faster than the one from above, starting from $T = T_{sun}/10$, and it is slightly slower in the presence of scattering.

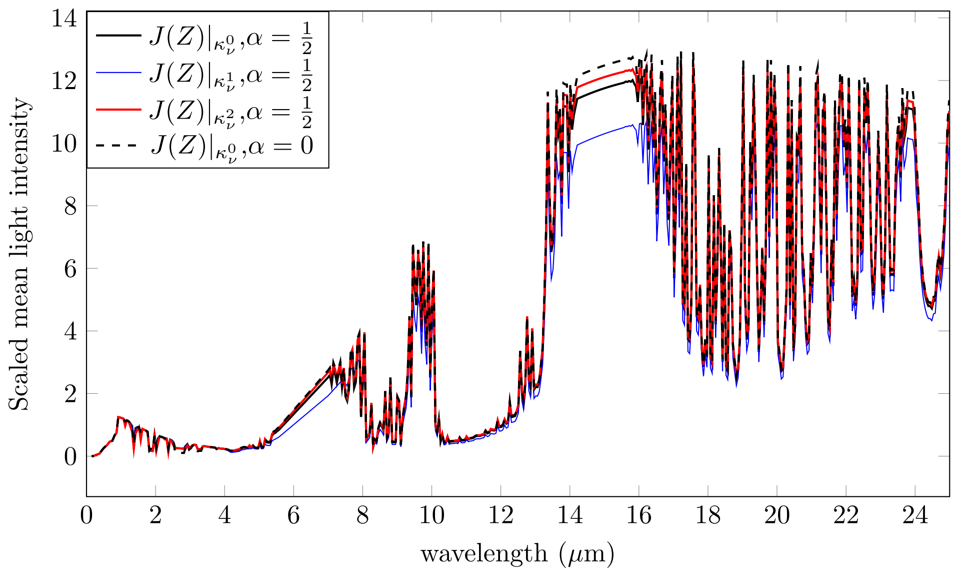

FIG. 4. *Computed mean radiation intensities $10^5 \cdot J_\nu(Z)$ at the top of the troposphere for κ_ν^0, κ_ν^1, κ_ν^2 with scattering ($\alpha = \frac{1}{2}$) and for κ_ν^0 without scattering.*

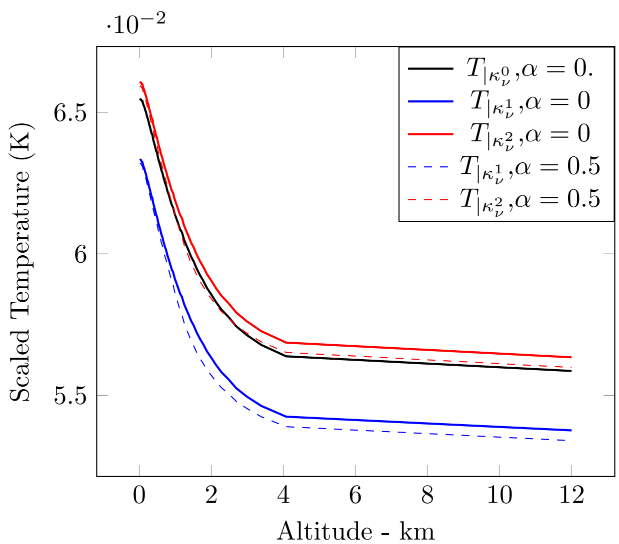

FIG. 5. *Temperatures in Kelvin divided by $4,798$ versus altitude, computed with κ_ν^0, κ_ν^1, and κ_ν^2 without scattering ($\alpha = 0$) and with a scattering $\alpha = \frac{1}{2}$.*

- Figure 5: Increasing κ_ν in the Earth infrared range can cause either an increase or a decrease of temperature, depending on the position of the change in the infrared spectrum.
- Isotropic and Rayleigh scattering did not change the above conclusion (see Figure 5).

Finally, note that the Earth albedo and the clouds seem to play an important role in the effect of the greenhouse gases on the temperature of the atmosphere [8]. If it is modeled by a Lambert condition of the type

$$I_\nu(0,\mu) - \beta I_\nu(0,-\mu) = \mu Q_0 B_\nu(T_{Sun}) \quad \forall\mu > 0;$$

then the present numerical method can handle it and our preliminary test shows an increase of temperature when β increases; while this is another story, it is yet another proof of the versatility of the present numerical formulation for climate modeling.

7.1.1. Relevance to global warming. The simulations made above indicate that an increase of opacity in the atmosphere may cause cooling or warming depending on the range of frequencies where the change of opacity occurs. It is known that $\mathtt{CO}_2$ is opaque to wavelengths around $\lambda_1 = 2\mu m$ and around $\lambda_2 = 6\mu m$. According to Figure 1 the λ_1 peak heats the atmosphere and the λ_2 peak cools it. Cooling does not go against the physical observations because it is known that $\mathtt{CO}_2$ cools the high atmosphere: see Figure 13 in [8] and this Belgian website, for instance:
www.aeronomie.be/en/news/2021/rising-co2-levels-also-cause-cooling-upper-layers-atmosphere.

What differentiates high and low altitudes? Clouds, for one thing, probably play a big role; also the absorption coefficient depends on the pressure, i.e., on altitude. The present formulation does not allow it, but it is not hard to see that by taking the greatest value for each frequency on the left-hand side of (3.2) and compensating for the difference on the right-hand side, the iterations on the source are still convergent. Thus there are many opportunities for future developments. We will show also, in [13], that the method is not confined to stratified atmospheres and that the full three-dimensional problem can be solved by iterations on the source in an integral formulation; it is much more expensive computationally but still a lot cheaper than SHDOM and Monte-Carlo.

One should be cautious not to draw early conclusions before the full problem is solved; the purpose of the present study is to show that here is a method which is mathematically well understood and numerically faster than others.

7.2. Radiative transfer with thermal diffusion in a pool. Consider the vertical cross-section of a pool, Ω, heated from above, possibly by the Sun, and subject to wind on its surface, but without evaporation. The bottom is elliptical with maximum length 3 and height 1.

The time dependent Navier–Stokes equations are solved with a kinematic viscosity $\nu_F = 0.05$. A no-slip condition $\boldsymbol{u} = (0,0)^T$ is applied on the bottom boundary and a Dirichlet condition on the horizontal boundary $\boldsymbol{u} = (10,0)^T$ to simulate the wind velocity.

The Taylor–Hood finite element method is used with the space V_h of continuous piecewise quadratic velocities on a triangulation and the space Q_h of piecewise linear pressures on the same triangulation. Galerkin-characteristic discretization in time is used: at each time step $n+1$, find $\boldsymbol{u}_h^{n+1} \in V_h$, satisfying the boundary conditions, and $p_h^{n+1} \in Q_h$, such that

$$\begin{aligned}(7.7)\quad &\int_{\Omega_h} \left(\tfrac{1}{\delta t}\boldsymbol{u}_h^{n+1}\cdot\hat{\boldsymbol{u}}_h + \nu_F \nabla\boldsymbol{u}_h^{n+1}\cdot\nabla\hat{\boldsymbol{u}}_h - p_h^{n+1}\nabla\cdot\hat{\boldsymbol{u}}_h + \hat{p}_h\nabla\cdot\boldsymbol{u}_h^{n+1}\right)\mathrm{d}x \\ &= \int_{\Omega_h} \tfrac{1}{\delta t}\boldsymbol{u}_h^n(\boldsymbol{x} - \boldsymbol{u}_h^n(\boldsymbol{x})\delta t)\cdot\hat{\boldsymbol{u}}_h \mathrm{d}x \quad \forall\hat{p}_h \in Q_h,\ \forall\hat{\boldsymbol{u}}_h \in V_h,\ \text{with } \hat{\boldsymbol{u}}_h|_{\partial\Omega} = 0.\end{aligned}$$

There are 764 vertices in the triangulation. Figure 6 displays the velocity vectors after 50 time steps of size 0.02; stationarity is reached. The computation takes 12″.

For the temperature, (6.5) is rescaled and discretized by (7.3). We chose $\kappa_T = 0.5$, $a_\nu = 0$, with vertical RT in the fluid, from its surface down into the liquid and Dirichlet conditions on the bottom boundary $T = 0.057$ which is approximately the reduced temperature found in the previous section.

The liquid water absorption parameter κ_ν can be found in https://en.wikipedia.org/wiki/Electromagnetic_absorption_by_water.

It turned out to be CPU prohibitive to solve the problem with such a detailed κ_ν; the bottleneck is in the computation of the integral in T of $B_\nu(T)$ required by the variational principle (7.3). Hence we approximated κ_ν by its regression line in the range $\nu \in (0.02, 7)10^{-14}$:

$$\kappa_\nu = \kappa_0 - \kappa_1 \nu \text{ with } \nu \in (0.02, \nu_{max}) \quad \nu_{max} = 7, \quad \kappa_0 = 0.7, \quad \kappa_1 = 0.5/\nu_{max}.$$

Then the integral of $\kappa_\nu B_\nu(T)$ can be computed analytically:

$$\int_0^\infty \kappa_\nu B_\nu(T) \mathrm{d}\nu = T^4 \kappa_0 \frac{\pi^4}{15} - 24 T^5 \kappa_1 \zeta(5),$$

where ζ is the Riemann function, $\zeta(5) = 1.03693$.

The time dependent temperature equation is solved until convergence to a stationary state with 50 time steps of size 0.1. The convection terms are treated explicitly so as to use (7.3) which is solved by the `BFGS` module in `FreeFEM++`. The computation takes 326″. The solution is shown in Figure 6. One sees the effect of the current in the fluid on the temperature distribution which has shifted to the right.

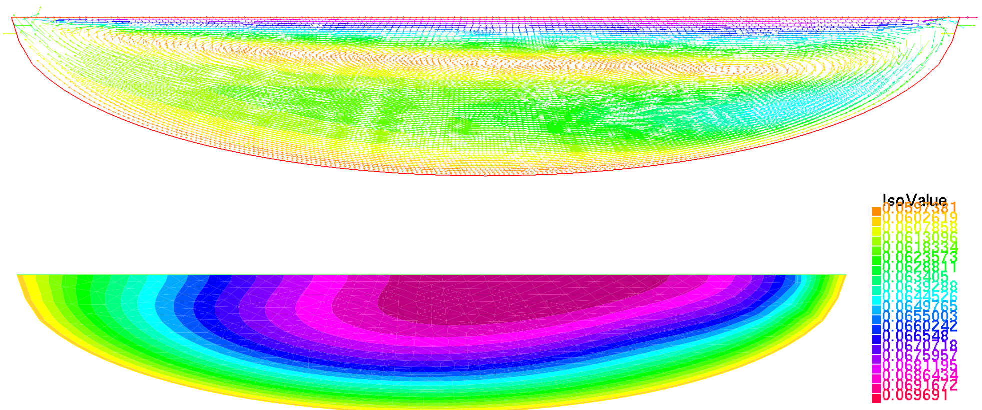

FIG. 6. *Velocity vectors and temperature in a pool subject to wind on its top boundary and given temperature on the bottom. The wind creates a large eddy rotating clockwise which, in turn, moves the hotter fluid region to the right.*

Note that with a Neumann condition on the bottom the temperature would keep rising with time and even with a Dirichlet condition on the bottom boundary there is a critical value for κ_T and/or Q_0 below which the temperature rises with time. Here $Q_0 = 0.02$, which is much bigger than the value for the sunlight, but with the latter the temperature is almost constant everywhere, equal to its bottom value 0.06.

7.3. RT with thermal diffusion in the atmosphere of a planet. Consider the atmosphere of a spherical planet, heated by the Sun, with a known ground temperature T_e. The computational domain is the space between a sphere of radius R_2 and a sphere of radius $R_1 < R_2$.

As before the sunrays travel unaffected and hit the ground; so the radiative part is governed by the first equation in (2.10) with (2.12) and (7.4), i.e., the second equation in (2.10) is replaced by (7.2). The density of the atmosphere is constant rather than decaying exponentially with altitude. The absorption parameter chosen for the computation is also constant $\kappa = 0.5$. The wind velocity is a rotating Poiseuille flow around an axis $(\sin\bar{\psi}, 0, \cos\bar{\psi})^T$ which is not aligned with the direction of the Sun. In spherical coordinates it is

$$u = r(H - r)[\cos\psi, \sin\psi, 0]^T, \quad r \text{ is the distance to the ground,}$$

where $H = R_2 - R_1$. The time dependent equation (7.2) is solved in spherical coordinates (details can be found in [16, Appendix A]). The computational domain becomes a solid rectangle with periodic conditions; it is discretized with a uniform distribution of vertices $16 \times 8 \times 8$ in the domain $(0, 2\pi) \times (0, \pi) \times (0, Z)$ with $Z = 1$.

The equations are discretized in time and space by a Galerkin-characteristic method and piecewise linear conforming finite elements on tetrahedra. The time step is $\delta t = 0.1$ and the thermal diffusion is $\kappa_T = 0.01$. The stratified approximation requires R_1 to be large and H small. For the visualizations, however, we map the solid rectangle onto the spherical domain with $R_1 = 1$ and $R_2' = 2$. As before $T_{Sun} = 1.209$ and $Q_0 = 2 \cdot 10^{-5}$. Initially $T_{t=0}$ is set to $T_e = T_{sun}\frac{\kappa}{2}\left(Q^0 E_3(\kappa z)\right)^{\frac{1}{4}}$. On the surface of the planet T is set to $0.95T_e(0)$.

Figure 7 shows the temperatures after 15 iterations without wind. The computing time is $357''$. The Sun is at infinity in the direction opposite to the blue region. Blue means cold; it corresponds to the night on this part of the planet. Yet with more time iterations we would see this zone heated by thermal diffusion due to the fixed temperature of the planet.

Figure 8 compares the temperatures with and without wind. The planar views correspond to cross-sections of the domain by the plane $z = 0$. Here, the Sun is in the horizontal direction on the right, but the wind transports its heat counterclockwise.

7.4. Conclusion. In this article a special case of radiative and heat transport has been studied, the so-called stratified approximation. The one-dimensional RT equations are coupled with the temperature equation. Existence and uniqueness have been established with almost no restriction on the absorption and scattering parameters. Furthermore, the proofs are based on a formulation of the problem which gives rise to an efficient numerical algorithm for RT coupled with the heat equation for a fluid. Upper and lower positive solutions can be computed and the convergence to the unique solution is polynomial.

The method has been implemented numerically and indeed arbitrary precision can be obtained, even with highly oscillating absorption or scattering coefficients. Furthermore, it is computationally very fast when the thermal diffusion is neglected and reasonably fast otherwise, at least with absorption coefficients which are polynomial functions of the frequencies.

It has been applied to the computation of the temperature in the Earth atmosphere, to that of a pool heated from above, and to the atmosphere of a planet with a large thermal diffusion. However, these are test cases rather than a full solution of physical problems and so one should be cautious not to draw early conclusions from these computations; the purpose of the present study is to show that here is a method which is mathematically well understood and numerically faster than others.

There are many other applications, especially for climate modelling and in nuclear engineering for which these new mathematical and numerical results should be useful.

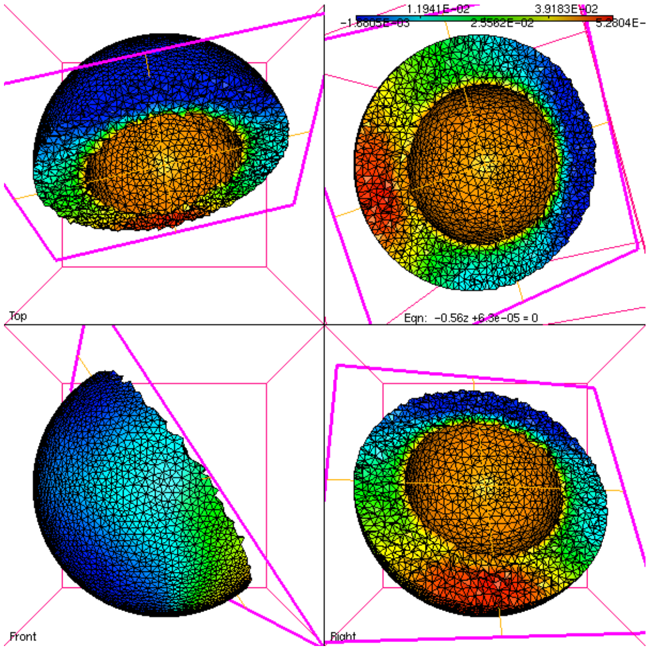

FIG. 7. *Temperature in the atmosphere of a planet heated by a sun, when thermal diffusion propagates heat in unlit regions and also in the presence of a counterclockwise rotating wind. Note that the thickness of the atmosphere has been expanded for readability.*

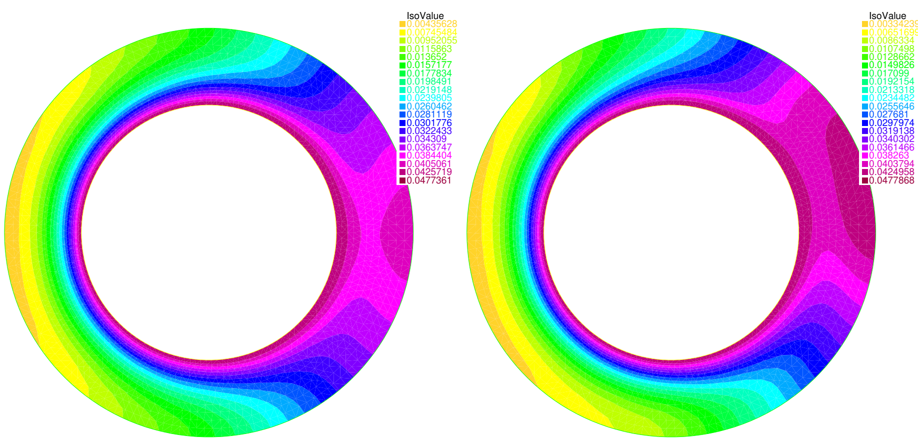

FIG. 8. *Temperature in the atmosphere of a planet heated by a sun on the right with wind (right) and without wind (left); it is a counterclockwise rotating wind around an axis almost (but not quite) perpendicular to the figure. Thermal diffusion propagates heat in unlit regions and the wind transports the heat counterclockwise. Note that the thickness of the atmosphere has been expanded for readability.*

Appendix A. Proof of Theorem 4.1.

Set $s_+(z) = 1_{z\geq 0}$. We recall that $z_+ = \max(z,0) = zs_+(z)$, while $z_- = \max(-z,0)$. We multiply both sides of the RT equation for two solutions I_ν and

I'_ν by $s_+(I_\nu - I'_\nu)$ and integrate in all variables with the notation

$$\langle \Phi \rangle := \int_0^\infty \int_{-1}^1 \Phi(\mu,\nu) \mathrm{d}\mu d\nu \,.$$

With $T = T[I]$ and $T' = T[I']$ defined by (2.16), let us compute

$$\begin{aligned} D &:= \langle \kappa_\nu((I_\nu - I'_\nu) - a_\nu(J_\nu - J'_\nu) - (1-a_\nu)(B_\nu(T) - B_\nu(T')))s_+(I_\nu - I'_\nu)\rangle \\ &= \langle \kappa_\nu(1-a_\nu)((I_\nu - I'_\nu) - (B_\nu(T) - B_\nu(T')))s_+(I_\nu - I'_\nu)\rangle \\ &\quad + \langle \kappa_\nu a_\nu((I_\nu - I'_\nu) - (J_\nu - J'_\nu))s_+(I_\nu - I'_\nu)\rangle =: D_1 + D_2 \,. \end{aligned}$$

Since

$$\int_{-1}^1 ((I_\nu - I'_\nu)(\tau,\mu) - (J_\nu - J'_\nu)(\tau))\mathrm{d}\mu = 0$$

and since $s_+(J_\nu - J'_\nu)$ is independent of μ, one has

$$D_2 = \langle \kappa_\nu a_\nu((I_\nu - I'_\nu) - (J_\nu - J'_\nu))(s_+(I_\nu - I'_\nu) - s_+(J_\nu - J'_\nu))\rangle \geq 0$$

since the function $z \mapsto s_+(z)$ is nondecreasing and $\kappa_\nu a_\nu \geq 0$. Similarly,

$$T = T[I] \text{ and } T' = T[I'] \implies \langle \kappa_\nu(1-a_\nu)((I_\nu - I'_\nu) - (B_\nu(T) - B_\nu(T')))\rangle = 0,$$

and since $s_+(T - T')$ is independent of μ and ν, one has

$$D_1 = \langle \kappa_\nu(1-a_\nu)((I_\nu - I'_\nu) - (B_\nu(T) - B_\nu(T')))(s_+(I_\nu - I'_\nu) - s_+(T-T'))\rangle \,.$$

Since B_ν is increasing for each $\nu > 0$, one has $s_+(T - T') = s_+(B_\nu(T) - B_\nu(T'))$. Hence

$$D_1 = \langle \kappa_\nu(1-a_\nu)((I_\nu - I'_\nu) - (B_\nu(T) - B_\nu(T')))(s_+(I_\nu - I'_\nu) - s_+(B_\nu(T) - B_\nu(T')))\rangle \geq 0$$

since $\kappa_\nu(1-a_\nu) \geq 0$ and $z \mapsto s_+(z)$ is nondecreasing.

Let I_ν and $I'{}_\nu$ be two solutions of (2.11) with boundary data

$$\begin{aligned} I_\nu(0,\mu) &= Q^+_\nu(\mu)\,, \quad I_\nu(Z,-\mu) = Q^-_\nu(\mu)\,, \qquad 0<\mu<1\,, \\ I'{}_\nu(0,\mu) &= Q'{}^+_\nu(\mu)\,, \quad I'{}_\nu(Z,-\mu) = Q'{}^-_\nu(\mu)\,, \qquad 0<\mu<1\,. \end{aligned}$$

Assume that

$$Q^\pm_\nu(\mu) \leq Q'{}^\pm_\nu(\mu) \quad \text{ for a.e. } (\mu,\nu) \in (0,1)\times(0,\infty)\,.$$

Then

$$\partial_\tau \langle \mu(I_\nu - I'{}_\nu)_+\rangle = -D_1 - D_2 \leq 0$$

so that $\tau \mapsto \langle \mu(I_\nu - I'{}_\nu)_+\rangle(\tau)$ is nonincreasing. Since

$$\begin{aligned} Q^-_\nu \leq Q'{}^-_\nu &\implies \langle \mu(I_\nu - I'{}_\nu)_+\rangle(Z) = \langle \mu_+(I_\nu - I'{}_\nu)_+\rangle(Z) \geq 0\,, \\ Q^+_\nu \leq Q'{}^+_\nu &\implies \langle \mu(I_\nu - I'{}_\nu)_+\rangle(0) = -\langle \mu_-(I_\nu - I'{}_\nu)_+\rangle(0) \leq 0\,, \end{aligned}$$

one has

$$\begin{aligned} 0 = \langle \mu(I_\nu - I'{}_\nu)_+\rangle = D_1 = D_2 &\quad \text{for a.e. } \tau \in (0,Z), \\ (I_\nu - I'{}_\nu)_+(0,-\mu) = (I_\nu - I'{}_\nu)_+(Z,\mu) = 0 &\quad \text{for a.e. } \mu \in (0,1)\,. \end{aligned}$$

Besides, since $\kappa_\nu(1-a_\nu)>0$ for all $\nu>0$,

$$D_1=0 \implies s_+(I_\nu(\tau,\mu)-I'_\nu(\tau,\mu))=s_+(T[I]-T[I']) \text{ for a.e. } (\tau,\mu,\nu)\,.$$

Next, we use the K-invariant (in the terminology of section 10 in Chapter I of Chandrasekhar [6]) for solutions of the RT equation with slab symmetry. We compute

$$\begin{aligned}\partial_\tau\left\langle\frac{\mu^2}{\kappa_\nu}(I_\nu-I'_\nu)_+\right\rangle&=-\langle a_\nu\mu((I_\nu-I'_\nu)-(I'_\nu-\tilde{I}'_\nu))s_+(T[I]-T[I'])\rangle\\&\quad-\langle(1-a_\nu)\mu((I_\nu-I'_\nu)-(B_\nu(T[I])-B_\nu(T[I'])))s_+(T[I]-T[I'])\rangle\\&=-\langle\mu(I_\nu-I'_\nu)s_+(T[I]-T[I'])\rangle=-\langle\mu(I_\nu-I'_\nu)_+\rangle=0\,,\end{aligned}$$

since

$$\int_{-1}^1\mu(I'_\nu(\tau)-\tilde{I}'_\nu(\tau))\mathrm{d}\mu=\int_{-1}^1\mu(B_\nu(T[I])-B_\nu(T[I']))\mathrm{d}\mu=0\,.$$

Next, we integrate in $\tau\in(0,Z)$, and observe that

$$\begin{aligned}&(I_\nu-I'_\nu)_+(0,-\mu)=0 \text{ and } Q^+_\nu(\mu)\le {Q'}^+_\nu(\mu) \qquad \text{for a.e. } \mu\in(0,1)\\&\implies\left\langle\frac{\mu^2}{\kappa_\nu}(I_\nu-I'_\nu)_+\right\rangle(\tau)=\left\langle\frac{\mu^2}{\kappa_\nu}(I_\nu-I'_\nu)_+\right\rangle(0)=0\,.\end{aligned}$$

Thus, we have proved that

$$\begin{aligned}&Q^\pm_\nu(\mu)\le {Q'_\nu}^\pm(\mu) \quad \text{for a.e. } (\mu,\nu)\in(0,1)\times(0,\infty)\\&\implies I_\nu(\tau,\mu)\le I'_\nu(\tau,\mu) \quad \text{for a.e. } (\tau,\mu,\nu)\in(0,Z)\times(-1,1)\times(0,\infty)\\&\implies T[I](\tau)\le T[I'](\tau) \quad \text{for a.e. } \tau\in(0,Z)\,.\end{aligned}$$

Exchanging $Q^\pm_\nu(\mu)$ and ${Q'_\nu}^\pm(\mu)$ above shows that $I_\nu=I'_\nu$ and $T[I]=T[I']$, which is the announced uniqueness.

Proof of Remark 5.2. Let $(I_\nu,T[I])$ and $(I'_\nu,T[I'])$ be the solutions of (5.3) corresponding to the boundary data $Q^\pm_\nu$ and ${Q'}^\pm_\nu$, respectively, such that $Q^\pm_\nu(\mu)\le {Q'}^\pm_\nu(\mu)$ for a.e. $(\mu,\nu)\in(0,1)\times(0,\infty)$. First, we slightly modify the treatment of D_2 as follows:

$$\begin{aligned}D_2&=\tfrac12\int_0^\infty\kappa_\nu a_\nu\int_{-1}^1(I_\nu-I'_\nu)_+(\mu)\mathrm{d}\mu\mathrm{d}\nu\\&-\tfrac12\int_0^\infty\kappa_\nu a_\nu\int_{-1}^1\int_{-1}^1 p(\mu,\mu')(I_\nu-I'_\nu)(\mu')s_+(I_\nu-I'_\nu)(\mu)\mathrm{d}\mu'\mathrm{d}\mu\mathrm{d}\nu\,.\end{aligned}$$

Since $p\ge0$ and $\frac12\int_{-1}^1 p(\mu,\mu')\mathrm{d}\mu=1$, one has

$$p(\mu,\mu')(I_\nu-I'_\nu)(\mu')s_+(I_\nu-I'_\nu)(\mu)\le p(\mu,\mu')(I_\nu-I'_\nu)_+(\mu')\,,$$

so that

$$\begin{aligned}D_2&\ge\tfrac12\int_0^\infty\kappa_\nu a_\nu\int_{-1}^1(I_\nu-I'_\nu)_+(\mu)\mathrm{d}\mu\mathrm{d}\nu\\&-\tfrac12\int_0^\infty\kappa_\nu a_\nu\int_{-1}^1\int_{-1}^1 p(\mu,\mu')(I_\nu-I'_\nu)_+(\mu')\mathrm{d}\mu'\mathrm{d}\mu\mathrm{d}\nu=0\,.\end{aligned}$$

As in the proof of Theorem 4.1, we see that

$$\langle \mu(I_\nu - I'_\nu)_+\rangle(\tau) = 0 \text{ for a.e. } \tau \in (0, Z)\,,$$

and

$$s_+(I_\nu(\tau,\mu) - I'_\nu(\tau,\mu)) = s_+(T[I](\tau) - T[I'](\tau))$$

for a.e. $(\tau,\mu,\nu) \in (0,Z)\times(-1,1)\times(0,\infty)$, while

$$(I_\nu - I'_\nu)_+(0,-\mu) = (I_\nu - I'_\nu)_+(Z,\mu) = 0 \qquad \text{for a.e. } \mu \in (0,1)\,.$$

Next, we compute

$$\begin{aligned}
\partial_\tau \left\langle \frac{\mu^2}{\kappa_\nu}(I_\nu - I'_\nu)_+\right\rangle &= -\tfrac12 \int_0^\infty a_\nu \int_{-1}^1 \mu(I_\nu - I'_\nu)_+(\tau,\mu)\mathrm{d}\mu\mathrm{d}\nu \\
+\tfrac12 \int_0^\infty a_\nu \int_{-1}^1 \mu \int_{-1}^1 & p(\mu,\mu')(I_\nu - I'_\nu)_+(\tau,\mu')\mathrm{d}\mu'\mathrm{d}\mu\mathrm{d}\nu\, s_+(T[I](\tau) - T[I'](\tau)) \\
&-\langle(1-a_\nu)\mu((I_\nu - I'_\nu) - (B_\nu(T[I]) - B_\nu(T[I'])))s_+(T[I]-T[I'])\rangle \\
= -\langle a_\nu\mu(I_\nu - I'_\nu)s_+(T[I]-T[I'])\rangle & - \langle(1-a_\nu)\mu(I_\nu - I'_\nu)s_+(T[I]-T[I'])\rangle \\
&= -\langle\mu(I_\nu - I'_\nu)s_+(T[I]-T[I'])\rangle = -\langle\mu(I_\nu - I'_\nu)_+\rangle = 0\,,
\end{aligned}$$

since

$$\int_{-1}^1 \mu p(\mu,\mu')\mathrm{d}\mu = \int_{-1}^1 \mu(B_\nu(T[I]) - B_\nu(T[I']))\mathrm{d}\mu = 0\,.$$

Finally, we integrate in $\tau \in (0,Z)$, and conclude as in the previous section that

$$\begin{aligned}
(I_\nu - I'_\nu)_+(0,-\mu) = 0 \text{ and } Q_\nu^+(\mu) \le {Q'}_\nu^+(\mu) \qquad &\text{for a.e. } \mu \in (0,1) \\
\Longrightarrow \left\langle \frac{\mu^2}{\kappa_\nu}(I_\nu - I'_\nu)_+\right\rangle(\tau) = \left\langle \frac{\mu^2}{\kappa_\nu}(I_\nu - I'_\nu)_+\right\rangle(0) = 0\,.
\end{aligned}$$

Hence $Q_\nu^\pm(\mu) \le {Q'}_\nu^\pm(\mu)$ for a.e. $(\mu,\nu) \in (0,1)\times(0,\infty)$ implies that $I_\nu(\tau,\mu) \le I'_\nu(\tau,\mu)$ for a.e. $(\tau,\mu,\nu) \in (0,Z)\times(-1,1)\times(0,\infty)$, and $T[I](\tau) \le T[I'](\tau)$ for a.e. $\tau \in (0,Z)$. This implies the uniqueness of the solution as explained in the proof of Theorem 4.1. ∎

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REFERENCES

[1] C. Bardos, F. Golse, B. Perthame, and R. Sentis, *The nonaccretive radiative transfer equations: Existence of solutions and Rosseland approximation*, J. Funct. Anal., 77 (1988), pp. 434–460.

[2] C. Bardos and O. Pironneau, *On the Greenhouse Effect*, https://hal.sorbonne-universite.fr/hal-03094855, submitted to SeMA J., 2022.

[3] V. Barlakasa, A. Macke, and M. Wendischa, *Sparta – solver for polarized atmospheric radiative transfer applications*, J. Quant. Spectrosc. Radiat. Transf., 178 (2016), pp. 77–92.

[4] C. F. Bohren and E. E. Clothiaux, *Fundamentals of Atmospheric Radiation*, Wiley-VCH Verlag, 2006.

[5] R. F. Cahalan, L. Oreopoulos, A. Marshak, K. F. Evans, A. B. Davis, R. Pincus, K. H. Yetzer, B. Mayer, R. Davies, T. P. Ackerman, H. W. Barker, E. E. Clothiaux, R. G. Ellingson, M. J. Garay, E. Kassianov, S. Kinne, A. Macke, W. O'hirok, P. T. Partain, S. M. Prigarin, A. N. Rublev, G. L. Stephens, F. Szczap, E. E. Takara, T. Várnai, G. Wen, and T. B. Zhuravleva, *THE I3RC: Bringing together the most advanced radiative transfer tools for cloudy atmospheres*, Bull. Am. Meteorol. Soc., 86 (2005), pp. 1275–1294.

[6] S. Chandrasekhar, *Radiative Transfer*, Clarendon Press, Oxford, 1950.

[7] M. G. Crandall and L. Tartar, *Some relations between nonexpansive and order preserving mappings*, Proc. Amer. Math. Soc., 78 (1980), pp. 385–390.

[8] J. Dufresne, V. Eymet, C. Crevoisier, and J. Grandpeix, *Greenhouse effect: The relative contributions of emission height and total absorption*, J. Clim., 33 (2020), pp. 3827–3844.

[9] A. Fowler, *Mathematical Geoscience*, Springer, Berlin, 2011.

[10] M. Ghattassi, X. Huo, and N. Masmoudi, *On the Diffusive Limits of Radiative Heat Transfer System* I*: Well Prepared Initial and Boundary Conditions*, preprint, https://arxiv.org/abs/2007.13209, 2020.

[11] F. Golse, *The Milne problem for the radiative transfer equations (with frequency dependence)*, Trans. Amer. Math. Soc., 303 (1987), pp. 125–143.

[12] F. Golse and B. Perthame, *Generalized solutions of the radiative transfer equations in a singular case*, Comm. Math. Phys., 106 (1986), pp. 211–239.

[13] F. Golse and O. Pironneau, *Radiative Transfer in a Fluid*, RACSAM, Springer, 2022.

[14] F. Golse and O. Pironneau, *Stratified radiative transfer for multidimensional fluids*, Compte-Rendus de l'Académie des Sciences (Mécamique), to appear, 2022.

[15] R. M. Goody and Y. L. Yung, *Atmospheric Radiation*, 2nd ed., Clarendon Press, Oxford, 1996.

[16] J. Happel and H. Brenner, *Low Reynolds Number Hydrodynamics with Special Applications to Particulate Media*, 2nd rev. ed., 4th printing, Kluwer Academic Publishers, Dordrecht, 1986.

[17] F. Hecht, *New developments in freefem++*, J. Numer. Math., 20 (2012), pp. 251–265.

[18] S. Klainerman and A. Majda, *Compressible and incompressible fluids*, Comm. Pure Appl. Math., 35 (1982), pp. 629–651.

[19] J. Leray and J.-L. Lions, *Quelques résultats de Visik sur les problèmes elliptiques non linéaires par les méthodes de Minty-Browder*, Bull. Soc. Math. Fr., 93 (1965), pp. 97–107.

[20] E. H. Lieb and M. Loss, *Analysis*, 2nd ed., American Mathematical Society, Providence, RI, 2001.

[21] P.-L. Lions, *Mathematical Topics in Fluid Mechanics. Vol.* 1*: Incompressible Models*, Clarendon Press, Oxford, 1996.

[22] A. Marshak and A. Davis, eds., 3*D Radiative Transfer in Cloudy Atmospheres*, Physics of Earth and Space Environments 5117, Springer, 2005.

[23] B. Mercier, *Application of accretive operators theory to the radiative transfer equations*, SIAM J. Math. Anal., 18 (1987), pp. 393–408, https://doi.org/10.1137/0518030.

[24] D. Mihalas and B. Weibel Mihalas, *Foundations of Radiation Hydrodynamics*, Oxford University Press, New York, Oxford, 1984.

[25] O. Pironneau, *A fast and accurate numerical method for radiative transfer in the atmosphere*, C. R. Math. Acad. Sci. Paris, 359 (2021), pp. 1179–1189.

[26] G. Pomraning, *The Equations of Radiation Hydrodynamics*, Dover, Mineola, NY, 1973.

[27] M. Porzio and O. Lopez-Pouso, *Application of accretive operators theory to evolutive combined conduction, convection and radiation*, Rev. Mat. Iberoamericana, 20 (2004), pp. 257–275.

[28] W. Zdunkowski and T. Trautmann, *Radiation in the Atmosphere. A Course in Theoretical Meteorology*, Cambridge University Press, 2007.