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► **To cite this version:**

| François Golse, Olivier Pironneau. Stratified Radiative Transfer in a Fluid *. 2021. hal-03419670v1

HAL Id: hal-03419670

<https://hal.sorbonne-universite.fr/hal-03419670v1>

Preprint submitted on 8 Nov 2021 (v1), last revised 1 Feb 2024 (v2)

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Stratified Radiative Transfer in a Fluid*

François Golse[†] Olivier Pironneau[‡]

November 8, 2021

Abstract

New mathematical results are given for the Radiative Transfer equations alone and coupled with the temperature equation of a fluid: existence, uniqueness, a maximum principle and a convergent monotone iterative scheme. Numerical tests for Earth’s atmosphere and the heating of a pool by the Sun are included.

Keywords Radiative transfer, Temperature equation, Integral equation, Numerical analysis

AMS3510, 35Q35, 35Q85, 80A21, 80M10

1 Introduction

Radiative transfer is an important field of physics including astronomy, nuclear physics and heat transfer in fluid mechanics. It is also a key ingredient of climate models.

Books on radiative transfer for the atmosphere are numerous, such as [14], [4], the numerically oriented [27] and the two mathematically oriented [5] and [9].

When Planck’s theory of black bodies is used, the radiation involves a continuum of frequencies governed by the temperature of the emitting bodies. Studies based on the interactions of the photons with the atoms of the medium, such as [8], are currently unusable numerically in large physical domains. A much simpler formulation has been proposed a hundred years ago, known as the radiative transfer equations, which is based on the energy conservation principles of continuum mechanics.

Even when the interactions with the background fluid are neglected, the radiative transfer equations involves 5 “spatial” variables (3 coordinates for the position of each photon, and the 2 components of its direction). Existence of

*Submitted to the editors DATE.

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solutions of the radiative transfer equations can be proved by a Schauder-type compactness argument (see [1]), with uniqueness under appropriate additional boundedness (see Proposition 2 in [20]), or monotonicity assumptions (see Corollary 2 in [20], together with [12]).

Given the intricacy of the radiative transfer equations, several simplifying assumptions have been studied in the literature. If the scattering and absorption coefficients do not depend on the frequencies of the radiation source, the radiative transfer equations can be averaged in the frequency variable, leading to a closed system of equations for the temperature and frequency-averaged radiative intensity, known as the “grey” model. However the frequency dependence of the scattering and absorption coefficients is fundamental to understand several important effects in Earth’s atmosphere (for instance, Rayleigh explained the blue color of the sky by the fact that the scattering coefficient is proportional to the fourth power of the radiation frequency; likewise, the fact that some components of Earth’s atmosphere are opaque to infrared radiations seems important to understand the greenhouse effect). Another simplification, of a purely geometric nature, consists in assuming that the temperature and radiative intensity are uniform on a foliation of the space by parallel planes, and therefore depend on a single position variable. As a result, the radiative intensity depends only on the projection of the photon’s direction on the orthogonal axis to these planes. This is known as the “slab symmetry” assumption, which appears in the “Milne problem” for planetary or stellar atmospheres (see [5] for a detailed physical discussion of the Milne problem, and [11] for the corresponding mathematical theory).

The term “radiative transfer” usually refers to the interaction of radiation with a fixed background material. But of course, radiation obviously deposits energy in the background fluid, gas or plasma, as well as momentum, through the radiation pressure, and conversely, high speed fluid motion obviously modifies such processes as Compton scattering (scattering of a photon by a free electron at rest) by Doppler effect. Therefore, in full generality, the equation for the radiation intensity are coupled with the fluid equations. This coupling is studied under the name of “radiation hydrodynamics” (see [23] for the coupling with ideal fluids, and [21]).

The most general studies of radiation hydrodynamics mentioned above involve high speed (possibly relativistic) fluid motion. In the present paper, we consider radiation passing through an incompressible fluid, or a compressible fluid at low Mach number, whose velocity field is uniformly small. Thus our setting will be intermediate between radiation hydrodynamics as [23],[21], and [10]. This last reference considers the coupling of the grey model of radiative transfer with a background material at rest. The radiation energy is deposited in the background medium in the form of heat, and appears as a source term in the heat equation for the temperature, while the black body radiation of the background medium appears as a source term in the radiative transfer equation for the radiative intensity. Our model is close to the one in [10], but retains the fluid motion equation, as well as the frequency dependence of the radiation field, which is essential for applications to Earth’s climate.

We shall however make another simplification, referred to as the “stratification assumption”: while the radiation intensity and temperature depend on all 3 position coordinates, only one of these coordinates is retained in the computation of the streaming operator acting on the radiative intensity, while the two other coordinates appear only as parameters in the radiative transfer equation. The stratified approximation is used when the radiation source is far — as in the case of the Sun — and the radiative intensity deposited at the boundary of the computational domain is uniform or at least slowly varying in the tangential directions to this surface.

While reviewing the current situation for the radiative transfer equations in [2] and updating the numerical possibilities on modern computers, it was found that brute force discretization of the equations by finite difference or finite element methods were incapable of giving results with the accuracy needed to differentiate between small variations on the absorption coefficient.

On the other hand an integral formulation present in [5] turned out to be much more precise and also computationally much cheaper. A fixed-point iteration of this nonlinear integral formulation was shown to be monotone in [22]. Finally in [13] the method was extended to include the temperature equation of the fluid and also to handle Rayleigh scattering while retaining monotonicity. While [13] is more numerically oriented, the present article is more focussed on the convergence proofs.

The radiative transfer equations are presented in Section 2. After this, a cascade of simplifications are discussed: the stratified approximation, the decoupling from the fluid, and Milne problem techniques originating from [11] (see also [20]).

In Section 3, the stratified radiative transfer decoupled from the fluid is analyzed in the case of isotropic scattering. Existence of a solution is proved by using the convergent monotone iterative scheme proposed in [2]. A maximum principle in the line of [20, 11] is also presented.

Uniqueness issues are discussed in Section 4. The proof is far from straightforward, and heavily relies on ideas in [20]. It may be interesting to compare Mercier’s monotonicity structure for the radiative transfer equation, which is quite involved, with the general observation [6] on order preserving maps in L^1 leaving the integral invariant.

In Section 5 the above results are extended to the non isotropic case of scattering with the Rayleigh phase function.

Finally in Section 6 existence, uniqueness and monotone convergence of the fixed-point iterations are proved for the radiative transfer equation coupled with the temperature equation of a fluid whose velocity field is known

2 Fundamental equations and approximations

Finding the temperature T in a fluid heated by electromagnetic radiations is a complex problem because interactions of photons with atoms of the medium involve rather intricate quantum phenomena. A first simplifying assumption is

that of local thermodynamic equilibrium (LTE): at each point in the fluid, there is a well-defined electronic temperature. In that case, one can write a kinetic equation for the radiative intensity $I_\nu(\mathbf{x}, \boldsymbol{\omega}, t)$ at time t , at position \mathbf{x} and in the direction $\boldsymbol{\omega}$ for photons of frequency ν , in terms of the temperature field $T(\mathbf{x}, t)$:

$$\begin{aligned} \frac{1}{c} \partial_t I_\nu + \boldsymbol{\omega} \cdot \nabla I_\nu + \rho \bar{\kappa}_\nu a_\nu \left[I_\nu - \frac{1}{4\pi} \int_{\mathbb{S}^2} p(\boldsymbol{\omega}, \boldsymbol{\omega}') I_\nu(\boldsymbol{\omega}') d\boldsymbol{\omega}' \right] \\ = \rho \bar{\kappa}_\nu (1 - a_\nu) [B_\nu(T) - I_\nu]. \end{aligned} \quad (1)$$

In this equation, ∇ designates the gradient with respect to the position \mathbf{x} , while

$$B_\nu(T) = \frac{2\hbar\nu^3}{c^2 [e^{\frac{\hbar\nu}{kT}} - 1]} \quad (2)$$

is the Planck function at temperature T , with \hbar the Planck constant, c the speed of light in the medium (assumed to be constant) and k the Boltzmann constant. Notice that

$$\int_0^\infty B_\nu(T) d\nu = \bar{\sigma} T^4, \quad \bar{\sigma} = \frac{2\pi^4 k^4}{15c^2 \hbar^3}, \quad (3)$$

where $\pi\bar{\sigma}$ is the Stefan-Boltzmann constant.

The intricacy of the interaction of photons with the atoms of the medium is contained in the mass-absorption $\bar{\kappa}_\nu$, the fraction of radiative intensity at frequency ν that is absorbed by fluid per unit length.

The coefficient $a_\nu \in (0, 1)$ is the scattering albedo, and $\frac{1}{4\pi} p(\boldsymbol{\omega}, \boldsymbol{\omega}') d\boldsymbol{\omega}$ is the probability that an incident ray of light with direction $\boldsymbol{\omega}'$ scatters in the infinitesimal element of solid angle $d\boldsymbol{\omega}$ centered at $\boldsymbol{\omega}$. This coefficient $\kappa_\nu(1 - a_\nu)$, also referred to as the ‘‘opacity’’, is obtained by rather complex quantum mechanics computations [24].

The kinetic equation (1) is coupled to the fluid equations solely by the local conservation of energy. The total energy density is the sum of the kinetic energy density of the fluid, of the internal energy of the fluid, and of the radiative energy:

$$\begin{aligned} \partial_t \left(\rho \left(\frac{1}{2} \mathbf{u}^2 + c_V T \right) + \frac{1}{c} \int_0^\infty \int_{\mathbb{S}^2} I_\nu d\boldsymbol{\omega} d\nu \right) \\ + \nabla \cdot \left(\mathbf{u} \rho \left(\frac{1}{2} \mathbf{u}^2 + c_V T \right) + (p - \mathbf{g} \cdot \mathbf{x}) \mathbf{u} + \int_0^\infty \int_{\mathbb{S}^2} \boldsymbol{\omega} I_\nu d\boldsymbol{\omega} d\nu \right) \\ = \nabla \cdot (\rho c_P \kappa_T \nabla T) + \nabla \cdot (\mu_F (\nabla \mathbf{u} + (\nabla \mathbf{u})^T) \mathbf{u}). \end{aligned}$$

Here, ρ is the fluid density, \mathbf{u} the velocity fluid, T the temperature, p the pressure and \mathbf{g} the gravity, while c_V , c_P are the specific heat capacity at constant volume and constant pressure respectively, μ_F is the fluid viscosity and κ_T is the thermal diffusivity. When the fluid is incompressible, density ρ , pressure p and velocity fields \mathbf{u} satisfy the Navier-Stokes equations

$$\begin{cases} \partial_t \rho + \mathbf{u} \cdot \nabla \rho = 0, & \nabla \cdot \mathbf{u} = 0, \\ \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} - \frac{\mu_F}{\rho} \Delta \mathbf{u} + \frac{1}{\rho} \nabla p = \mathbf{g}, \end{cases} \quad (4)$$

where Δ is the Laplacian in the \mathbf{x} variable. Subtracting the kinetic energy balance equation from the local conservation of energy leads to

$$\begin{aligned} \rho c_V (\partial_t T + \mathbf{u} \cdot \nabla T) &= \nabla \cdot (\rho c_P \kappa_T \nabla T) + \frac{1}{2} \mu_F (\nabla \mathbf{u} + (\nabla \mathbf{u})^T)^2 \\ &+ \int_0^\infty \rho \bar{\kappa}_\nu \left(1 - a_\nu \frac{1}{4\pi} \int_{\mathbb{S}^2} p(\boldsymbol{\omega}, \boldsymbol{\omega}') d\boldsymbol{\omega} \right) \int_{\mathbb{S}^2} I_\nu(\boldsymbol{\omega}') d\boldsymbol{\omega}' d\nu \\ &\quad - 4\pi \int_0^\infty \rho \bar{\kappa}_\nu (1 - a_\nu) B_\nu(T) d\nu. \end{aligned}$$

Since $\boldsymbol{\omega} \mapsto \frac{1}{4\pi} p(\boldsymbol{\omega}, \boldsymbol{\omega}')$ is a probability density on \mathbb{S}^2 for each $\boldsymbol{\omega}' \in \mathbb{S}^2$, neglecting the viscous heating term $\frac{1}{2} \mu_F (\nabla \mathbf{u} + (\nabla \mathbf{u})^T)^2$ on the right hand side of the equality above, which is legitimate under the assumption that the square fluid velocity $|\mathbf{u}|^2$ is small, we arrive at the equation

$$\begin{aligned} \rho c_V (\partial_t T + \mathbf{u} \cdot \nabla T) &= \nabla \cdot (\rho c_P \kappa_T \nabla T) \\ &+ \int_0^\infty \rho \bar{\kappa}_\nu (1 - a_\nu) \left(\int_{\mathbb{S}^2} I_\nu(\boldsymbol{\omega}) d\boldsymbol{\omega} - 4\pi B_\nu(T) \right) d\nu. \end{aligned} \quad (5)$$

Summarizing, the kinetic equation (1) for the radiative intensity is coupled to the incompressible Navier-Stokes equations (4) and to the energy balance equation in the form of the drift diffusion equation (5) for the temperature. The resulting system is

$$\begin{cases} \frac{1}{c} \partial_t I_\nu + \boldsymbol{\omega} \cdot \nabla I_\nu + \rho \bar{\kappa}_\nu a_\nu \left[I_\nu - \frac{1}{4\pi} \int_{\mathbb{S}^2} p(\boldsymbol{\omega}, \boldsymbol{\omega}') I_\nu(\boldsymbol{\omega}') d\boldsymbol{\omega}' \right] \\ \quad = \rho \bar{\kappa}_\nu (1 - a_\nu) [B_\nu(T) - I_\nu], \\ \rho c_V (\partial_t T + \mathbf{u} \cdot \nabla T) - \nabla \cdot (\rho c_P \kappa_T \nabla T) \\ \quad = \int_0^\infty \rho \bar{\kappa}_\nu (1 - a_\nu) \left(\int_{\mathbb{S}^2} I_\nu(\boldsymbol{\omega}) d\boldsymbol{\omega} - 4\pi B_\nu(T) \right) d\nu, \\ \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} - \frac{\mu_F}{\rho} \Delta \mathbf{u} + \frac{1}{\rho} \nabla p = \mathbf{g}, \\ \partial_t \rho + \mathbf{u} \cdot \nabla \rho = 0, \quad \nabla \cdot \mathbf{u} = 0. \end{cases} \quad (6)$$

This system is supplemented with appropriate initial and boundary conditions. Assuming for instance that the spatial domain is an open subset Ω of \mathbb{R}^3 with C^1 , or piecewise C^1 boundary $\partial\Omega$, and denoting by \mathbf{n} the outward unit normal field on $\partial\Omega$, the following boundary conditions are natural:

$$\begin{aligned} I_\nu(\mathbf{x}, \boldsymbol{\omega}, t) &= Q_\nu(x, \boldsymbol{\omega}, t), \quad x \in \partial\Omega, \quad \boldsymbol{\omega} \cdot \mathbf{n}_x < 0, \quad \nu > 0, \\ \mathbf{u}|_{\partial\Omega} &= 0, \quad \frac{\partial T}{\partial \mathbf{n}} \Big|_{\partial\Omega} = 0. \end{aligned} \quad (7)$$

The first boundary condition tells us that the radiative intensity of incoming photons ($\boldsymbol{\omega} \cdot \mathbf{n}_x < 0$) at the boundary of the spatial domain is known, which is a typical admissible boundary condition for kinetic models; the second boundary condition is the classical Dirichlet boundary condition for the velocity field,

solution of the Navier-Stokes equations, while the last boundary condition, the Neuman condition for the temperature, corresponds to the absence of heat flux at the boundary of the spatial domain. (Of course, this is just one example of boundary condition for the heat equation, other boundary conditions could also be considered — for instance, one could have mixed Dirichlet-Neuman, or even Robin conditions on the temperature.) Notice that there is no boundary condition for the density ρ , since the velocity field \mathbf{u} is tangent (and even vanishes) at the boundary $\partial\Omega$.

Finally, one should specify initial conditions of the form

$$\begin{aligned} I_\nu(\mathbf{x}, \boldsymbol{\omega}, 0) &= I_\nu^{in}(x, \boldsymbol{\omega}), & x \in \Omega, \boldsymbol{\omega} \in \mathbb{S}^2, \nu > 0, \\ \rho|_{t=0} &= \rho^{in}, & \mathbf{u}|_{t=0} &= \mathbf{u}^{in}, & T|_{t=0} &= T^{in}. \end{aligned} \tag{8}$$

Neglecting the viscous heating term as explained above has an important consequence on the structure of this system, which can be thought of as “block triangular”.

In other words, one can first solve for ρ, \mathbf{u}, p the Navier-Stokes equations (4), then the last three equations in the system (6) above. The mathematical theory of (4) has been discussed in great detail by P.-L. Lions in [19].

Once this is done, the density ρ and velocity field \mathbf{u} are known, and appear as coefficients in the coupled system of the radiative transfer equation (1) and of the heat drift-diffusion equation (5). This coupling has to be studied in detail, since there is no further triangular structure by which one can first solve for one of the remaining unknowns (in this case the radiative intensity I_ν and the temperature T).

The reader familiar with radiation hydrodynamics will observe another simplifying assumption in (6) leading to the “block triangular” structure described above. In general, the radiative intensity also appears in the momentum balance equation, through the time derivative of the radiation flux and the divergence of the radiation pressure tensor (see equations (1.16) and (1.19) in chapter I, section 2 of [23] for a precise definition of these notions). The radiative pressure tensor and the radiative flux are of order $1/c$ and $1/c^2$ respectively (see equation (9.83) in chapter IX, section 3 of [23]). In the physical context considered here, these terms are obviously negligible, and this is why the coupling between the radiation field and the hydrodynamics appears only in the local conservation of energy.

In the next two sections, we discuss simplified model equations deduced from (6).

2.1 Stratified radiative transfer

Let (x, y, z) be the cartesian coordinates of the point $\mathbf{x} \in \mathbb{R}^3$, with z denoting the altitude/depth.

Assume that the radiation source (henceforth referred to as “the Sun”) is far away in the direction $z > 0$, and is independent of x and y . The radiation spectrum of this source is that of a black body at temperature T_S , that is,

the Planck function $B_\nu(T_S)$. With such a radiation source, it is natural to assume that the temperature field T is slowly varying with x and y , so that $|\partial_x T| + |\partial_y T| \ll |\partial_z T|$ and that I_ν is also slowly varying in x and y so that $|\partial_x I_\nu| + |\partial_y I_\nu| \ll |\partial_z I_\nu|$.

Similarly, we further assume that $|\frac{1}{c}\partial_t I_\nu| \ll |\partial_z I_\nu|$, and forget the initial condition on I_ν , so that the time dependence of the radiative intensity is governed solely by the evolution of the temperature field through the radiative transfer equation (1).

With this assumption, the streaming term $\frac{1}{c}\partial_t I_\nu + \boldsymbol{\omega} \cdot \nabla I_\nu$ reduces to $\mu\partial_z I_\nu$, where μ is the cosine of the angle of $\boldsymbol{\omega}$ with the z axis. Henceforth, the spatial domain is $\Omega = \mathbb{O} \times (z_m, z_M)$, where \mathbb{O} is an open subset of \mathbb{R}^2 with C^1 boundary.

Then (6) becomes (see [27]):

$$\begin{cases} \mu\partial_z I_\nu + \rho\bar{\kappa}_\nu I_\nu = \rho\bar{\kappa}_\nu(1 - a_\nu)B_\nu(T) + \frac{1}{2}\rho\bar{\kappa}_\nu a_\nu \int_{-1}^1 p(\mu, \mu')I_\nu(z, \mu')d\mu', \\ \partial_t T + \mathbf{u} \cdot \nabla T - \frac{c_P}{c_V}\kappa_T \Delta T = \frac{4\pi}{c_V} \int_0^\infty \bar{\kappa}_\nu(1 - a_\nu) \left(\frac{1}{2} \int_{-1}^1 I_\nu d\mu - B_\nu(T) \right) d\nu, \\ I_\nu(x, y, z_M, \mu)|_{\mu < 0} = Q^-(\mu)B_\nu(T_S), \quad I_\nu(x, y, z_m, \mu)|_{\mu > 0} = Q_\nu^+(\mu), \\ \frac{\partial T}{\partial n} \Big|_{\partial\Omega} = 0, \quad T|_{t=0} = T^{in}. \end{cases} \quad (9)$$

That $I_\nu(z_m, \mu)|_{\mu > 0} = 0$, i.e. $Q_\nu^+(\mu) = 0$, is natural since no radiation comes from the bottom of the spatial domain. Yet, by the law of black bodies, radiations could also come from the bottom but more general boundary conditions could be handled by the same analysis. In fact in [9] and others, it is assumed that most of the energy from the Sun is in the form of visible light and is essentially unaffected by crossing the atmosphere, so that it is equivalent to a source of energy located at $z = 0$. Recall that it make physical sense to take $Q^-(\mu) = \mu Q' \cos \theta$, where θ is the latitude on Earth, while μ is the cosine of the observation angle. The fluid velocity field \mathbf{u} is given, assumed to be divergence free and regular enough for (9) to make sense. Note that by rescaling the time variable, \mathbf{u} and κ_T appropriately, the factor $4\pi/\rho c_V$ can be replaced with 1.

2.2 Radiative transfer decoupled from hydrodynamics

When $\kappa_T = 0$, and the fluid is at rest, the left-hand side of temperature equation is zero, so that the fluid equations are decoupled from the radiative transfer equation (1). Let us consider first the case of isotropic scattering, namely $p(\mu, \mu') = 1$. Then the system becomes (see [2])

$$(\mu\partial_\tau + \kappa_\nu)I_\nu(\tau, \mu) = \kappa_\nu a_\nu J_\nu(\tau) + \kappa_\nu(1 - a_\nu)B_\nu(T(\tau)), \quad (10)$$

$$I_\nu(0, \mu) = Q_\nu^+(\mu), \quad I_\nu(Z, -\mu) = Q_\nu^-(\mu), \quad 0 < \mu < 1, \quad (11)$$

$$\int_0^\infty \kappa_\nu(1 - a_\nu)B_\nu(T(\tau))d\nu = \int_0^\infty \kappa_\nu(1 - a_\nu)J_\nu(\tau)d\nu, \quad (12)$$

with the notation $Q_\nu^-(\mu) = Q^-(-\mu)B_\nu(T_S)$ and

$$J_\nu(\tau) := \frac{1}{2} \int_{-1}^1 I_\nu(\tau, \mu) d\mu. \quad (13)$$

In these equations, we have replaced the height $z \in (z_m, z_M)$ by τ , analogous to the ‘‘optical depth’’ (see for instance [9], or formula (51) in chapter I of [5]), defined as follows. Pick $\rho_0 > 0$, some ‘‘reference’’ density of the fluid. (For instance, ρ_0 could be the average density in the fluid, or the density at some reference altitude z . Indeed, the following expressions for the atmospheric density ρ in terms of the altitude z are found in the literature: $\rho(z) = \rho_0 e^{-z}$ or $\rho(z) = \rho_0 - \rho_1 z$.) The new variable τ , and the absorption coefficient κ_ν are defined as follows:

$$\tau := \int_{z_m}^z \frac{\rho(\zeta)}{\rho_0} d\zeta, \quad \text{and } \kappa_\nu := \rho_0 \bar{\kappa}_\nu. \quad (14)$$

Equations (10) and (12) imply that

$$\partial_\tau \int_0^\infty \int_{-1}^1 \mu I_\nu(\tau, \mu) d\mu d\nu = 0. \quad (15)$$

We have ignored the dependence in x, y of T and I_ν , since x, y are mere parameters in these equations, which are anyway completely decoupled from the fluid equations.

Assuming that $0 < \kappa_\nu \leq \kappa_M$ and $0 \leq a_\nu < 1$ for all $\nu > 0$, we see that (12) and (13) define T as a functional of I , henceforth denoted $T[I]$. Equivalently, one can consider J_ν as a radiative intensity independent of μ , and observe that (12) and (13) imply that $T[I] = T[J]$. Thus (10),(11),(12) can be recast as

$$\begin{cases} (\mu \partial_\tau + \kappa_\nu) I_\nu(\tau, \mu) = \kappa_\nu a_\nu J_\nu(\tau) + \kappa_\nu (1 - a_\nu) B_\nu(T[J](\tau)), \\ I_\nu(0, \mu) = Q_\nu^+(\mu), \quad I_\nu(Z, -\mu) = Q_\nu^-(\mu), \quad 0 < \mu < 1. \end{cases} \quad (16)$$

Throughout this article we use the exponential integrals

$$E_p(X) := \int_X^\infty \frac{e^{-z}}{z^p} dz = \int_X^\infty \frac{e^{-z}}{z^n} dz = \int_0^1 e^{-X/\mu} \mu^{p-2} d\mu, \quad X > 0. \quad (17)$$

Lemma 1 *The following inequality holds:*

$$\frac{1}{2} \sup_{0 \leq t \leq Z} \int_0^Z E_1(\kappa|\tau - t|) \kappa d\tau \leq C_1(\kappa),$$

where $\kappa \mapsto C_1(\kappa)$ is monotone increasing from \mathbb{R}^+ to \mathbb{R}^+ , and less than 1.

Proof Observe that

$$\begin{aligned}
\int_0^Z E_1(\kappa|\tau - t|)\kappa d\tau &= \int_0^{\kappa Z} E_1(|\sigma - s|)d\sigma \\
&= \int_{\mathbf{R}} E_1(|\sigma - s|)1_{[0, \kappa Z]}(\sigma)d\sigma \\
&= \int_{\mathbf{R}} E_1(|\theta|)1_{[-s, \kappa Z - s]}(\theta)d\theta \\
&\leq \int_{\mathbf{R}} E_1(|\theta|)1_{[-\kappa Z/2, \kappa Z/2]}(\theta)d\theta \\
&= 2 \int_0^{\kappa Z/2} E_1(\theta)d\theta \leq 2 \int_0^{Z\kappa_M/2} E_1(\theta)d\theta =: 2C_1(\kappa).
\end{aligned} \tag{18}$$

The first inequality above is the elementary rearrangement inequality (Theorem 3.4 in [18]). Now C_1 is obviously increasing since $E_1 > 0$, and

$$C_1(\kappa) = \int_0^{Z\kappa/2} E_1(\theta)d\theta < \int_0^\infty E_1(\theta)d\theta = \int_0^\infty \left(\int_1^\infty \frac{e^{-\theta y}}{y} dy \right) d\theta = \int_1^\infty \frac{dy}{y^2} = 1.$$

□

Lemma 2 *Let*

$$S_\nu(\tau) = \frac{1}{2} \int_0^1 \left(e^{-\frac{\kappa_\nu \tau}{\mu}} Q_\nu^+(\mu) + e^{-\frac{\kappa_\nu(Z-\tau)}{\mu}} Q_\nu^-(\mu) \right) d\mu. \tag{19}$$

Problem (10),(11),(12),(13) is equivalent to (12), plus the integral equation

$$J_\nu(\tau) = S_\nu(\tau) + \frac{1}{2} \int_0^Z E_1(\kappa_\nu|\tau - t|)\kappa_\nu(a_\nu J_\nu(t) + (1 - a_\nu)B_\nu(T(t)))dt. \tag{20}$$

Proof Applying the method of characteristics shows that

$$\begin{aligned}
I_\nu(\tau, \mu) &= e^{-\frac{\kappa_\nu \tau}{\mu}} Q_\nu^+(\mu) \mathbf{1}_{\mu > 0} + e^{-\frac{\kappa_\nu(Z-\tau)}{|\mu|}} Q_\nu^-(|\mu|) \mathbf{1}_{\mu < 0} \\
&\quad + \mathbf{1}_{\mu > 0} \int_0^\tau e^{-\frac{\kappa_\nu(\tau-t)}{\mu}} \frac{\kappa_\nu}{\mu} (a_\nu J_\nu(t) + (1 - a_\nu)B_\nu(T(t)))dt \\
&\quad + \mathbf{1}_{\mu < 0} \int_\tau^Z e^{-\frac{\kappa_\nu(t-\tau)}{|\mu|}} \frac{\kappa_\nu}{|\mu|} (a_\nu J_\nu(t) + (1 - a_\nu)B_\nu(T(t)))dt.
\end{aligned} \tag{21}$$

One integrates both sides of this identity in μ , exchange the order of integration by Tonelli's theorem, and change variables in the inner integral, observing that

$$\int_0^1 e^{-\frac{x}{\mu}} \frac{d\mu}{\mu} = \int_1^\infty \frac{e^{-Xy}}{y} dy = \int_X^\infty \frac{e^{-z}}{z} dz = E_1(X).$$

Thus

$$\begin{aligned}
J_\nu(\tau) = & \frac{1}{2} \int_0^1 \left(e^{-\frac{\kappa_\nu \tau}{\mu}} Q_\nu^+(\mu) + e^{-\frac{\kappa_\nu(Z-\tau)}{\mu}} Q_\nu^-(\mu) \right) d\mu \\
& + \frac{1}{2} \int_0^Z E_1(\kappa_\nu|\tau-t|) \kappa_\nu (a_\nu J_\nu(t) + (1-a_\nu) B_\nu(T(t))) dt.
\end{aligned} \tag{22}$$

□

3 Analysis of problem (10)-(12)

In order to solve numerically (10)-(12), one uses the method of iteration on the sources. Starting from some appropriate (I_ν^0, T^0) , one constructs a sequence (I_ν^n, T^n) by the following prescription

$$\begin{cases}
(\mu \partial_\tau + \kappa_\nu) I_\nu^{n+1}(\tau, \mu) = \kappa_\nu a_\nu J_\nu^n(\tau) + \kappa_\nu (1-a_\nu) B_\nu(T^n(\tau)), & T^n = T[J_\nu^n] \\
I_\nu^{n+1}(0, \mu) = Q_\nu^+(\mu), & I_\nu^{n+1}(Z, -\mu) = Q_\nu^-(\mu), & 0 < \mu < 1,
\end{cases} \tag{23}$$

Note that $a_\nu J_\nu^n(t) + (1-a_\nu) B_\nu(T^n(t))$ does not depend on μ . As in (21), the method of characteristics shows that

$$\begin{aligned}
I_\nu^{n+1}(\tau, \mu) = & e^{-\frac{\kappa_\nu \tau}{\mu}} Q_\nu^+(\mu) \mathbf{1}_{\mu > 0} + e^{-\frac{\kappa_\nu(Z-\tau)}{|\mu|}} Q_\nu^-(|\mu|) \mathbf{1}_{\mu < 0} \\
& + \mathbf{1}_{\mu > 0} \int_0^\tau e^{-\frac{\kappa_\nu(\tau-t)}{\mu}} \frac{\kappa_\nu}{\mu} (a_\nu J_\nu^n(t) + (1-a_\nu) B_\nu(T^n(t))) dt \\
& + \mathbf{1}_{\mu < 0} \int_\tau^Z e^{-\frac{\kappa_\nu(t-\tau)}{|\mu|}} \frac{\kappa_\nu}{|\mu|} (a_\nu J_\nu^n(t) + (1-a_\nu) B_\nu(T^n(t))) dt.
\end{aligned} \tag{24}$$

Since $B_\nu \geq 0$, this formula shows, by a straightforward induction argument, that

$$I_\nu^0 \geq 0, T^0 \geq 0, Q_\nu^\pm \geq 0 \implies I_\nu^n \geq 0.$$

Moreover

$$\begin{aligned}
& I_\nu^{n+1}(\tau, \mu) - I_\nu^n(\tau, \mu) = \\
& + \mathbf{1}_{\mu > 0} \int_0^\tau e^{-\frac{\kappa_\nu(\tau-t)}{\mu}} \frac{\kappa_\nu}{\mu} a_\nu (J_\nu^n(t) - J_\nu^{n-1}(t)) dt \\
& + \mathbf{1}_{\mu > 0} \int_0^\tau e^{-\frac{\kappa_\nu(\tau-t)}{\mu}} \frac{\kappa_\nu}{\mu} (1-a_\nu) (B_\nu(T^n(t)) - B_\nu(T^{n-1}(t))) dt \\
& + \mathbf{1}_{\mu < 0} \int_\tau^Z e^{-\frac{\kappa_\nu(t-\tau)}{|\mu|}} \frac{\kappa_\nu}{|\mu|} a_\nu (J_\nu^n(t) - J_\nu^{n-1}(t)) dt \\
& + \mathbf{1}_{\mu < 0} \int_\tau^Z e^{-\frac{\kappa_\nu(t-\tau)}{|\mu|}} \frac{\kappa_\nu}{|\mu|} (1-a_\nu) (B_\nu(T^n(t)) - B_\nu(T^{n-1}(t))) dt.
\end{aligned}$$

Since B_ν is nondecreasing for each $\nu > 0$, formula (12) shows that

$$J_\nu^n \geq J_\nu^{n-1} \implies T^n \geq T^{n-1},$$

and we conclude from the equality above that

$$I_\nu^0 = 0, T^0 = 0, Q_\nu^\pm \geq 0 \implies \begin{cases} 0 \leq I_\nu^1 \leq I_\nu^2 \leq \dots \leq I_\nu^n \leq \dots \\ 0 \leq T^1 \leq T^2 \leq \dots \leq T^n \leq \dots \end{cases}$$

By Lemma 2

$$J_\nu^{n+1}(\tau) = S_\nu(\tau) + \frac{1}{2} \int_0^Z E_1(\kappa_\nu|\tau - t|) \kappa_\nu (a_\nu J_\nu^n(t) + (1 - a_\nu) B_\nu(T^n(t))) dt.$$

Integrating over $[0, Z]$ in τ implies that

$$\begin{aligned} \int_0^Z J_\nu^{n+1}(\tau) d\tau &= \int_0^Z S_\nu(\tau) d\tau \\ &+ \frac{1}{2} \int_0^Z \left(\int_0^Z E_1(\kappa_\nu|\tau - t|) \kappa_\nu d\tau \right) (a_\nu J_\nu^n(t) + (1 - a_\nu) B_\nu(T^n(t))) dt \\ &\leq \int_0^Z S_\nu(\tau) d\tau \\ &+ \frac{1}{2} \left(\sup_{0 \leq t \leq Z} \int_0^Z E_1(\kappa_\nu|\tau - t|) \kappa_\nu d\tau \right) \int_0^Z (a_\nu J_\nu^n(t) + (1 - a_\nu) B_\nu(T^n(t))) dt. \end{aligned}$$

Thus by Lemma 1

$$\int_0^Z J_\nu^{n+1}(\tau) d\tau \leq \int_0^Z S_\nu(\tau) d\tau + C_1(\kappa_\nu) \int_0^Z (a_\nu J_\nu^n(t) + (1 - a_\nu) B_\nu(T^n(t))) dt.$$

Multiply both sides of this inequality by κ_ν and integrate in ν : one finds that

$$\begin{aligned} \int_0^\infty \int_0^Z \kappa_\nu J_\nu^{n+1}(\tau) d\tau d\nu &\leq \int_0^\infty \int_0^Z S_\nu(\tau) d\tau d\nu \\ &+ C_1(\kappa_M) \int_0^\infty \int_0^Z \kappa_\nu (a_\nu J_\nu^n(t) + (1 - a_\nu) B_\nu(T^n(t))) dt d\nu. \end{aligned}$$

At this point, we recall that $T^n = T[J_\nu^n]$, so that

$$\int_0^\infty \kappa_\nu (1 - a_\nu) B_\nu(T^n(t)) d\nu = \int_0^\infty \kappa_\nu (1 - a_\nu) J_\nu^n(t) d\nu, \quad (25)$$

and hence

$$\int_0^\infty \int_0^Z \kappa_\nu J_\nu^{n+1}(\tau) d\tau d\nu \leq C_1(\kappa_M) \int_0^\infty \int_0^Z \kappa_\nu J_\nu^n(t) dt d\nu + \int_0^\infty \int_0^Z S_\nu(\tau) d\tau d\nu.$$

The expression of the source term can be slightly reduced, by integrating out the τ variable:

$$\int_0^Z \kappa_\nu e^{-\frac{\kappa_\nu \tau}{\mu}} d\tau = \int_0^Z \kappa_\nu e^{-\frac{\kappa_\nu(Z-\tau)}{\mu}} d\tau = \mu \left(1 - e^{-\frac{\kappa_\nu Z}{\mu}}\right),$$

so that

$$\begin{aligned} 0 &\leq \frac{1}{2} \int_0^\infty \int_0^Z \int_0^1 \kappa_\nu \left(e^{-\frac{\kappa_\nu \tau}{\mu}} Q_\nu^+(\mu) + e^{-\frac{\kappa_\nu(Z-\tau)}{\mu}} Q_\nu^-(\mu) \right) d\mu d\tau d\nu \\ &\leq \frac{1}{2} \int_0^\infty \int_0^1 (Q_\nu^+(\mu) + Q_\nu^-(\mu)) \mu d\mu =: \mathcal{Q}. \end{aligned}$$

Thus

$$\int_0^\infty \int_0^Z \kappa_\nu J_\nu^{n+1}(\tau) d\tau d\nu \leq C_1(\kappa_M) \int_0^\infty \int_0^Z \kappa_\nu J_\nu^n(t) dt d\nu + \mathcal{Q}.$$

Initializing the sequence I_ν^n with $I_\nu^0 = 0$ and $T^0 = T[J_\nu^0] = 0$, one finds that

$$\begin{aligned} \int_0^\infty \int_0^Z \kappa_\nu J_\nu^1(\tau) d\tau d\nu &\leq \mathcal{Q}, \\ \int_0^\infty \int_0^Z \kappa_\nu J_\nu^2(\tau) d\tau d\nu &\leq C_1(\kappa_M) \mathcal{Q} + \mathcal{Q} \\ \int_0^\infty \int_0^Z \kappa_\nu J_\nu^3(\tau) d\tau d\nu &\leq C_1(\kappa_M)^2 \mathcal{Q} + C_1(\kappa_M) \mathcal{Q} + \mathcal{Q} \end{aligned}$$

and by induction

$$\int_0^\infty \int_0^Z \kappa_\nu J_\nu^{n+1}(\tau) d\tau d\nu \leq \mathcal{Q} \sum_{j=0}^n C_1(\kappa_M)^j.$$

Since $C_1(\kappa_M) < 1$, the series above converges and one has the uniform bound

$$\int_0^\infty \int_0^Z \kappa_\nu J_\nu^{n+1}(\tau) d\tau d\nu \leq \frac{\mathcal{Q}}{1 - C_1(\kappa_M)}.$$

Furthermore, as

$$0 \leq I_\nu^1 \leq I_\nu^2 \leq \dots \leq I_\nu^n \leq I_\nu^{n+1} \leq \dots$$

the bound above and the Monotone Convergence Theorem implies that the sequence $I_\nu^{n+1}(\tau, \mu)$ converges for a.e. $(\tau, \mu, \nu) \in (0, Z) \times (-1, 1) \times (0, +\infty)$ to a limit denoted $I_\nu(\tau, \mu)$ as $n \rightarrow \infty$. Since

$$0 \leq T^1 \leq T^2 \leq \dots \leq T^n \leq T^{n+1} \leq \dots$$

we conclude from (15) and the Monotone Convergence Theorem that $T^{n+1}(\tau)$ converges for a.e. $\tau \in (0, Z)$ to a limit denoted $T(\tau)$ as $n \rightarrow \infty$.

Then we can pass to the limit in (24) as $n \rightarrow \infty$ by monotone convergence, so that

$$\begin{aligned} I_\nu(\tau, \mu) &= e^{-\frac{\kappa_\nu \tau}{\mu}} Q_\nu^+(\mu) \mathbf{1}_{\mu > 0} + e^{-\frac{\kappa_\nu(Z-\tau)}{|\mu|}} Q_\nu^- (|\mu|) \mathbf{1}_{\mu < 0} \\ &\quad + \mathbf{1}_{\mu > 0} \int_0^\tau e^{-\frac{\kappa_\nu(\tau-t)}{\mu}} \frac{\kappa_\nu}{\mu} (a_\nu J_\nu(t) + (1-a_\nu) B_\nu(T(t))) dt \\ &\quad + \mathbf{1}_{\mu < 0} \int_\tau^Z e^{-\frac{\kappa_\nu(t-\tau)}{|\mu|}} \frac{\kappa_\nu}{|\mu|} (a_\nu J_\nu(t) + (1-a_\nu) B_\nu(T(t))) dt \end{aligned}$$

for a.e. $(\tau, \mu, \nu) \in (0, Z) \times (-1, 1) \times (0, +\infty)$. One recognizes in this equality the integral formulation of (10)-(12). Besides, we have seen that

$$\begin{aligned} 0 &= I_\nu^0 \leq I_\nu^1 \leq I_\nu^2 \leq \dots \leq I_\nu^n \leq I_\nu^{n+1} \leq \dots \leq I_\nu, \\ 0 &= T^0 \leq T^1 \leq T^2 \leq \dots \leq T^n \leq T^{n+1} \leq \dots \leq T, \end{aligned}$$

so that

$$\begin{aligned} 0 &\leq \int_0^Z (J_\nu^{n+1} - J_\nu^n)(\tau) d\tau \\ &= \frac{1}{2} \int_0^Z \left(\int_0^Z E_1(\kappa_\nu |\tau - t|) \kappa_\nu d\tau \right) a_\nu (J_\nu^n - J_\nu^{n-1})(t) dt \\ &\quad + \frac{1}{2} \int_0^Z \left(\int_0^Z E_1(\kappa_\nu |\tau - t|) \kappa_\nu d\tau \right) (1-a_\nu) (B_\nu(T^n(t)) - B_\nu(T^{n-1}(t))) dt \\ &\leq C_1(\kappa_M) \int_0^Z (a_\nu (J_\nu^n - J_\nu^{n-1})(t) + (1-a_\nu) (B_\nu(T^n(t)) - B_\nu(T^{n-1}(t)))) dt. \end{aligned}$$

Using again the equality (25), we conclude that

$$0 \leq \int_0^Z \int_0^\infty \kappa_\nu (J_\nu^{n+1} - J_\nu^n)(\tau) d\nu d\tau \leq C_1(\kappa_M) \int_0^Z \int_0^\infty \kappa_\nu (J_\nu^n - J_\nu^{n-1})(t) dt d\tau.$$

Hence

$$0 \leq \int_0^Z \int_0^\infty \kappa_\nu (J_\nu^{n+1} - J_\nu^n)(\tau) d\nu d\tau \leq C_1(\kappa_M)^n \int_0^\infty \kappa_\nu J_\nu^1(\tau) d\nu d\tau \leq C_1(\kappa_M)^n \mathcal{Q},$$

so that

$$0 \leq \int_0^Z \int_0^\infty \kappa_\nu (J_\nu - J_\nu^n)(\tau) d\nu d\tau \leq C_1(\kappa_M)^n \int_0^\infty \kappa_\nu J_\nu^1(\tau) d\nu d\tau \leq \frac{C_1(\kappa_M)^n \mathcal{Q}}{1 - C_1(\kappa_M)}.$$

Summarizing, we have proved the following result.

Theorem 1 *Assume that $0 < \kappa_\nu \leq \kappa_M$, while $0 \leq a_\nu < 1$ for all $\nu > 0$. Let $Q_\nu^\pm(\mu)$ satisfy*

$$\int_0^\infty \int_0^1 \mu Q_\nu^\pm(\mu) d\mu d\nu \leq \mathcal{Q} := \frac{1}{2} \int_0^\infty \int_0^1 (Q_\nu^+(\mu) + Q_\nu^-(\mu)) \mu d\mu.$$

Choose $I_\nu^0 = 0$ and $T^0 = 0$, and let I_ν^n and $T^n = T[J_\nu^n]$ be the solution of (23). Then

$$I_\nu^n(\tau, \mu) \rightarrow I_\nu(\tau, \mu) \quad \text{and} \quad T^n(\tau) \rightarrow T(\tau)$$

for $(\tau, \mu, \nu) \in (0, Z) \times (-1, 1) \times (0, +\infty)$ as $n \rightarrow \infty$, where (I_ν, T) is a solution of (10)-(12). This method converges exponentially fast, in the sense that

$$0 \leq \int_0^Z \int_0^\infty \kappa_\nu (J_\nu - J_\nu^n)(\tau) d\nu d\tau \leq \frac{C_1(\kappa_M)^n \mathcal{Q}}{1 - C_1(\kappa_M)},$$

and, if $0 \leq a_\nu \leq a_M < 1$ while $0 < \kappa_m \leq \kappa_\nu$, one has

$$0 \leq \int_0^Z \bar{\sigma} (T(t)^4 - T^n(t)^4) dt \leq \frac{C_1(\kappa_M)^n \mathcal{Q}}{\kappa_m(1 - a_M)(1 - C_1(\kappa_M))}.$$

The last bound comes from the defining equality for the temperature in terms of the radiative intensity

$$\begin{aligned} \kappa_m(1 - a_M)\bar{\sigma}(T^4 - (T^n)^4) &= \kappa_m(1 - a_M) \int_0^\infty (B_\nu(T) - B_\nu(T^n)) d\nu \\ &\leq \int_0^\infty \kappa_\nu(1 - a_\nu)(B_\nu(T) - B_\nu(T^n)) d\nu \\ &= \int_0^\infty \kappa_\nu(1 - a_\nu)(J_\nu - J_\nu^n) d\nu \\ &\leq \int_0^\infty \kappa_\nu(J_\nu - J_\nu^n) d\nu. \end{aligned}$$

4 Uniqueness, Maximum Principle for (10)-(12)

This section follows computations in [11] (in the case $Z = +\infty$ and with $a_\nu = 0$) and in [20].

The rather subtle monotonicity structure of the radiative transfer equations is a striking result, discovered by Mercier in [20]. In view of the complexity of his computations, it may be useful to keep in mind the following simple remarks, which should be viewed as a motivation.

Consider first the following steady radiative transfer equation (11) without scattering ($a_\nu = 0$) in the whole space with a source $0 \leq S_\nu \in L^1(\mathbf{R} \times (-1, 1) \times (0, \infty))$:

$$\lambda I_\nu(\tau, \mu) + \mu \partial_\tau I_\nu(\tau, \mu) + \kappa_\nu I_\nu(\tau, \mu) = \kappa_\nu B_\nu(T[I]) + \lambda S_\nu(\tau, \mu), \quad \tau \in \mathbf{R}, \quad |\mu| < 1,$$

where $\lambda > 0$. (One should think of this equation as the Laplace transform in time of the time-dependent radiative transfer equation, with initial data λS_ν , where the parameter λ is the Laplace variable, or as the implicit time-discretization of the same time-dependent radiative transfer equation, where $\lambda = 1/c\Delta t$ and S is the radiative intensity at the beginning of the time-step.)

By definition of $T[I]$, one easily checks that

$$\int_{-\infty}^{\infty} \int_{-1}^1 \int_0^{\infty} I_{\nu}(\tau, \mu) d\nu d\mu d\tau = \int_{-\infty}^{\infty} \int_{-1}^1 \int_0^{\infty} S_{\nu}(\tau, \mu) d\nu d\mu d\tau.$$

The radiative intensity is given in terms of the temperature $T[I]$ and the source S_{ν} by the explicit formula

$$\begin{aligned} I_{\nu}(\tau, \mu) = & \mathbf{1}_{\mu > 0} \int_{-\infty}^{\tau} e^{-\frac{(\lambda + \kappa_{\nu})(\tau - t)}{\mu}} \frac{\kappa_{\nu} B_{\nu}(T[I](t)) + \lambda S_{\nu}(t, \mu)}{\mu} dt \\ & + \mathbf{1}_{\mu < 0} \int_{\tau}^{\infty} e^{-\frac{(\lambda + \kappa_{\nu})(t - \tau)}{|\mu|}} \frac{\kappa_{\nu} B_{\nu}(T[I](t)) + \lambda S_{\nu}(t, \mu)}{|\mu|} dt. \end{aligned}$$

Now, if one replaces the source of radiation S_{ν} in the right hand side of this equation with a larger source $S'_{\nu} \geq S_{\nu}$, it is natural to expect that the resulting radiation intensity I'_{ν} will be such that $T[I'] \geq T[I]$. Observe now that the function $T \mapsto B_{\nu}(T)$ is increasing on $(0, +\infty)$ for each $\nu > 0$; the explicit formula for I_{ν} in terms of S_{ν} and $T[I]$ shows that $I'_{\nu}(\tau, \mu) \geq I_{\nu}(\tau, \mu)$.

Of course, this argument is by no means rigorous, since it rests on the assumption that $S'_{\nu} \geq S_{\nu} \implies T[I'] \geq T[I]$, which, although physically plausible, has not been proved yet. Notice however that

$$I'_{\nu} \geq I_{\nu} \implies T[I'] \geq T[I]$$

by (15), since the Planck function B_{ν} is increasing for each $\nu > 0$. Thus, the map $S_{\nu} \mapsto I_{\nu}$ preserves both the integral and the order between radiation intensities. Now there is a clever characterization of order preserving maps on L^1 leaving the integral invariant, which is due to Crandall and Tartar. Roughly speaking, a map from L^1 to itself that preserves the integral is order preserving if and only if it is nonexpansive in L^1 . This brings in the notion of L^1 -accretivity, which is at the heart of Mercier's remarkable discovery.

Indeed, the monotonicity argument above, together with Proposition 1 of [6] (with $C = L^1(\mathbf{R} \times (-1, 1) \times (0, \infty))^+$, which is the set of a.e. positive elements of $L^1(\mathbf{R} \times (-1, 1) \times (0, \infty))$), strongly suggest that it might be a good idea¹ to study

$$\int_{-\infty}^{\infty} \int_{-1}^1 \int_0^{\infty} (I_{\nu}^2 - I_{\nu}^1)_+ d\nu d\mu d\tau \text{ in terms of } \int_{-\infty}^{\infty} \int_{-1}^1 \int_0^{\infty} (S_{\nu}^2 - S_{\nu}^1)_+ d\nu d\mu d\tau,$$

where $S_{\nu}^1, S_{\nu}^2 \in C$ and I_{ν}^1, I_{ν}^2 are the solutions of the steady radiative transfer equation above with source terms S_{ν}^1 and S_{ν}^2 respectively².

¹This may be a reconstruction of Mercier's argument; he might have found the L^1 -accretivity structure of the radiative transfer equations by some different argument.

²In fact, Mercier's original argument is even more complex, because he assumes that the opacity $K_{\nu} := \kappa_{\nu}(1 - a_{\nu})$ depends on the temperature T , and is a nonincreasing function of T for each $\nu > 0$ while $T \mapsto K_{\nu}(T)B_{\nu}(T)$ is nondecreasing; the reader can easily verify that the intuitive argument above still applies, provided of course that our physically natural assumption that $S'_{\nu} \geq S_{\nu} \implies T[I'] \geq T[I]$ remains valid in this case as well.

After these preliminary remarks, we return to the problem (10)-(12). The following theorem shows that two solutions of this problem are ordered exactly as their boundary data. (This situation is analogous to the case of harmonic functions, except that the radiative transfer equations (10)-(12) are nonlinear, at variance with the Laplace equation.)

Theorem 2 *Assume that $0 < \kappa_\nu \leq \kappa_M$, while $0 \leq a_\nu < 1$ for all $\nu > 0$. Let $Q^\pm, Q'^\pm \in L^1((0, 1) \times (0, \infty))$ satisfy*

$$0 \leq Q_\nu^\pm(\mu) \leq Q'_\nu^\pm(\mu) \quad \text{for a.e. } (\mu, \nu) \in (0, 1) \times (0, \infty).$$

Then, the solutions $(I_\nu, T[I])$ of (10)-(12), and $(I'_\nu, T[I'])$ of (10)-(12), with boundary data $Q_\nu^\pm(\mu)$ replaced with $Q'^\pm_\nu(\mu)$ satisfy

$$I_\nu(\tau, \mu) \leq I'_\nu(\tau, \mu) \text{ and } T[I](\tau) \leq T[I'](\tau) \quad \text{for a.e. } (\tau, \mu) \in (-1, 1) \times (0, \infty).$$

In particular,

$$Q_\nu^\pm(\mu) = Q'^\pm_\nu(\mu) \text{ a.e. } \mu, \nu \implies I_\nu(\tau, \mu) = I'_\nu(\tau, \mu) \text{ and } T[I](\tau) = T[I'](\tau) \\ \text{for a.e. } \tau, \mu \in (-1, 1) \times (0, \infty).$$

Proof Define $s_+(z) = 1_{z \geq 0}$, and $z_+ = \max(z, 0)$ while $z_- = \max(-z, 0)$. Thus

$$z = z_+ - z_-, \quad |z| = z_+ + z_-, \quad z_+ = z s_+(z).$$

In accordance with the discussion above, as in [20], we multiply both sides of the radiative transfer equation for two solutions I_ν and I'_ν by $s_+(I_\nu - I'_\nu)$ and integrate in all variables.

Denote

$$\langle \Phi \rangle := \int_0^\infty \int_{-1}^1 \Phi(\mu, \nu) d\mu d\nu.$$

With $T = T[I]$ and $T' = T[I']$ defined by (16), let us compute

$$D := \langle \kappa_\nu ((I_\nu - I'_\nu) - a_\nu (J_\nu - J'_\nu) - (1 - a_\nu)(B_\nu(T) - B_\nu(T'))) s_+(I_\nu - I'_\nu) \rangle \\ = \langle \kappa_\nu (1 - a_\nu) ((I_\nu - I'_\nu) - (B_\nu(T) - B_\nu(T'))) s_+(I_\nu - I'_\nu) \rangle \\ + \langle \kappa_\nu a_\nu ((I_\nu - I'_\nu) - (J_\nu - J'_\nu)) s_+(I_\nu - I'_\nu) \rangle =: D_1 + D_2.$$

Observe that

$$(J_\nu - J'_\nu) s_+(I_\nu(\mu) - I'_\nu(\mu)) = \frac{1}{2} \int_{-1}^1 (I_\nu - I'_\nu)(\mu') s_+(I_\nu - I'_\nu)(\mu) d\mu' \\ \leq \frac{1}{2} \int_{-1}^1 (I_\nu - I'_\nu)_+(\mu') d\mu',$$

so that $D_2 \geq 0$. Next

$$D_1 = \langle \kappa_\nu (1 - a_\nu) ((I_\nu - I'_\nu) - (B_\nu(T) - B_\nu(T'))) (s_+(I_\nu - I'_\nu) - s_+(T - T')) \rangle$$

because

$$T = T[I] \text{ and } T' = T[I'] \implies \langle \kappa_\nu(1 - a_\nu)((I_\nu - I'_\nu) - (B_\nu(T) - B_\nu(T'))) \rangle = 0.$$

Since B_ν is increasing for each $\nu > 0$, one has

$$s_+(T - T') = s_+(B_\nu(T) - B_\nu(T')),$$

so that

$$D_1 = \langle \kappa_\nu(1 - a_\nu)((I_\nu - I'_\nu) - (B_\nu(T) - B_\nu(T'))) (s_+(I_\nu - I'_\nu) - s_+(B_\nu(T) - B_\nu(T'))) \rangle$$

and

$$s_+ \text{ nondecreasing} \implies D_1 \geq 0.$$

Let I_ν and I'_ν be two solutions of (11) with boundary data

$$\begin{aligned} I_\nu(0, \mu) &= Q_\nu^+(\mu), & I_\nu(Z, -\mu) &= Q_\nu^-(\mu), & 0 < \mu < 1, \\ I'_\nu(0, \mu) &= Q'^+_\nu(\mu), & I'_\nu(Z, -\mu) &= Q'^-_\nu(\mu), & 0 < \mu < 1. \end{aligned}$$

Assume that

$$Q_\nu^\pm(\mu) \leq Q'^\pm_\nu(\mu) \quad \text{for a.e. } (\mu, \nu) \in (0, 1) \times (0, \infty).$$

Then

$$\begin{aligned} & \partial_\tau \langle \mu(I_\nu - I'_\nu)_+ \rangle \\ & \leq -\langle \kappa_\nu(1 - a_\nu)((I_\nu - I'_\nu) - (B_\nu(T[I]) - B_\nu(T[J]))) s_+(I_\nu - I'_\nu) \rangle \\ & \quad - \langle \kappa_\nu a_\nu((I_\nu - I'_\nu) - (I'_\nu - \tilde{I}'_\nu)) s_+(I_\nu - I'_\nu) \rangle \leq 0, \end{aligned}$$

so that $\tau \mapsto \langle \mu(I_\nu - I'_\nu)_+ \rangle(\tau)$ is nonincreasing. Since

$$\begin{aligned} Q_\nu^- \leq Q'^-_\nu & \implies \langle \mu(I_\nu - I'_\nu)_+ \rangle(Z) = \langle \mu_+(I_\nu - I'_\nu)_+ \rangle(Z) \geq 0, \\ Q_\nu^+ \leq Q'^+_\nu & \implies \langle \mu(I_\nu - I'_\nu)_+ \rangle(0) = -\langle \mu_-(I_\nu - I'_\nu)_+ \rangle(0) \leq 0, \end{aligned}$$

one has

$$\begin{aligned} & \text{for a.e. } \tau \in (0, Z) \quad 0 = \langle \mu(I_\nu - I'_\nu)_+ \rangle \\ & = \langle \kappa_\nu a_\nu((I_\nu - I'_\nu) - (I'_\nu - \tilde{I}'_\nu)) s_+(I_\nu - I'_\nu) \rangle \\ & = \langle \kappa_\nu(1 - a_\nu)((I_\nu - I'_\nu) - (B_\nu(T[I]) - B_\nu(T[J]))) s_+(I_\nu - I'_\nu) \rangle, \end{aligned}$$

and

$$(I_\nu - I'_\nu)_+(0, -\mu) = (I_\nu - I'_\nu)_+(Z, \mu) = 0 \quad \text{for a.e. } \mu \in (0, 1).$$

Besides, since $\kappa_\nu(1 - a_\nu) > 0$ for all $\nu > 0$

$$\begin{aligned} 0 &= \langle \kappa_\nu(1 - a_\nu)((I_\nu - I'_\nu) - (B_\nu(T[I]) - B_\nu(T[J]))) s_+(I_\nu - I'_\nu) \rangle \\ &= \langle \kappa_\nu(1 - a_\nu)((I_\nu - I'_\nu) - (B_\nu(T[I]) - B_\nu(T[J]))) \\ & \quad (s_+(I_\nu - I'_\nu) - s_+(T[I] - T[I'])) \rangle \\ &\implies s_+(I_\nu(\tau, \mu) - I'_\nu(\tau, \mu)) = s_+(T[I] - T[I']) \text{ for a.e. } (\tau, \mu, \nu). \end{aligned}$$

At this point, we must resort to an additional idea, which is not present in [20]. Since we are dealing with solutions of the radiative transfer equation having the slab symmetry, it is natural to use the K -invariant (in the terminology of section 10 in chapter I of Chandrasekhar [5]). This idea³ is at the heart of the exponential decay estimate for the Milne problem obtained in [11], and will be used here for a different purpose.

We compute

$$\begin{aligned} \partial_\tau \left\langle \frac{\mu^2}{\kappa_\nu} (I_\nu - I'_\nu)_+ \right\rangle &= -\langle a_\nu \mu ((I_\nu - I'_\nu) - (I'_\nu - \tilde{I}'_\nu)) s_+(T[I] - T[I']) \rangle \\ &\quad - \langle (1 - a_\nu) \mu ((I_\nu - I'_\nu) - (B_\nu(T[I]) - B_\nu(T[I']))) s_+(T[I] - T[I']) \rangle \\ &= -\langle a_\nu \mu (I_\nu - I'_\nu) s_+(T[I] - T[I']) \rangle - \langle (1 - a_\nu) \mu (I_\nu - I'_\nu) s_+(T[I] - T[I']) \rangle \\ &= -\langle \mu (I_\nu - I'_\nu) s_+(T[I] - T[I']) \rangle = -\langle \mu (I_\nu - I'_\nu)_+ \rangle = 0, \end{aligned}$$

since

$$\int_{-1}^1 \mu (I'_\nu(\tau) - \tilde{I}'_\nu(\tau)) d\mu = \int_{-1}^1 \mu (B_\nu(T[I]) - B_\nu(T[I'])) d\mu = 0.$$

Next we integrate in $\tau \in (0, Z)$, and observe that

$$\begin{aligned} (I_\nu - I'_\nu)_+(0, -\mu) &= 0 \text{ and } Q_\nu^+(\mu) \leq Q'^+(\mu) \quad \text{for a.e. } \mu \in (0, 1) \\ \implies \left\langle \frac{\mu^2}{\kappa_\nu} (I_\nu - I'_\nu)_+ \right\rangle(\tau) &= \left\langle \frac{\mu^2}{\kappa_\nu} (I_\nu - I'_\nu)_+ \right\rangle(0) = 0. \end{aligned}$$

Thus, we have proved that

$$I_\nu(\tau, \mu) \leq I'_\nu(\tau, \mu) \quad \text{for a.e. } (\tau, \mu, \nu) \in (0, Z) \times (-1, 1) \times (0, \infty),$$

so that

$$T[I](\tau) \leq T[I'](\tau) \quad \text{for a.e. } \tau \in (0, Z),$$

under the assumption that

$$Q_\nu^\pm(\mu) \leq Q'_\nu^\pm(\mu) \quad \text{for a.e. } (\mu, \nu) \in (0, 1) \times (0, \infty).$$

Exchanging $Q_\nu^\pm(\mu)$ and $Q'_\nu^\pm(\mu)$ in the above argument shows that

$$I_\nu(\tau, \mu) = I'_\nu(\tau, \mu) \quad \text{for a.e. } (\tau, \mu, \nu) \in (0, Z) \times (-1, 1) \times (0, \infty),$$

so that

$$T[I](\tau) = T[I'](\tau) \quad \text{for a.e. } \tau \in (0, Z),$$

under the assumption that

$$Q_\nu^\pm(\mu) = Q'_\nu^\pm(\mu) \quad \text{for a.e. } (\mu, \nu) \in (0, 1) \times (0, \infty),$$

³A somewhat similar idea, unfortunately unpublished, had been used by R. Sentis to simplify the uniqueness proof for the linear Milne problem studied in [3].

which is precisely the announced uniqueness argument. \square

One has also the following form of Maximum Principle for the radiative transfer equation. (If one keeps in mind the analogy with harmonic functions recalled before (2), the Maximum Principle below is a *consequence* of the monotonicity of the dependence of the solution of (10)-(12) in terms of its boundary data, whereas the analogous monotonicity in the case of harmonic functions is *deduced* from the Maximum Principle for the Laplace equation.)

Corollary 1 *Assume that $0 < \kappa_\nu \leq \kappa_M$, while $0 \leq a_\nu < 1$ for all $\nu > 0$. Let $Q_\nu^\pm(\mu) \leq B_\nu(T_M)$ (resp. $Q_\nu^\pm(\mu) \geq B_\nu(T_m)$) for a.e. $(\mu, \nu) \in (0, 1) \times (0, \infty)$. Then*

$$\begin{aligned} I_\nu(\tau, \mu) &\leq B_\nu(T_M) \text{ and } T[I](\tau) \leq T_M \\ (\text{resp. } I_\nu(\tau, \mu) &\geq B_\nu(T_m) \text{ and } T[I](\tau) \geq T_m) \\ &\text{for a.e. } (\tau, \mu) \in (-1, 1) \times (0, \infty). \end{aligned}$$

Proof Indeed, $I'_\nu = B_\nu(T_M)$ and $T[I'] = T_M$ (resp. $I'_\nu = B_\nu(T_m)$ and $T[I'] = T_m$) is the solution of (11) with boundary data $Q'^\pm_\nu(\mu) = B_\nu(T_M)$ (resp. $Q'^\pm_\nu(\mu) = B_\nu(T_m)$). The announced inequalities follow from the comparison of solutions obtained in Theorem 2. \square

Remark 1 *In (1), if one has the stronger condition*

$$0 \leq Q_\nu^\pm(\mu) \leq B_\nu(T_M) \quad \text{for a.e. } (\mu, \nu) \in (0, 1) \times (0, \infty),$$

one obtains the following bound for the numerical and theoretical solutions

$$0 \leq I_\nu^1 \leq \dots \leq I_\nu^n \leq \dots \leq I_\nu \leq B_\nu(T_M), \quad \text{and } 0 \leq T^1 \leq \dots \leq T^n \leq \dots \leq T \leq T_M.$$

5 Radiative Transfer with Rayleigh Phase Function

In this section, we discuss the same problem as in the previous section, with the isotropic scattering kernel replaced by the Rayleigh phase function. In the case of slab symmetry, the Rayleigh phase function is

$$p(\mu, \mu') = \frac{3}{16}(3 - \mu^2) + \frac{3}{16}(3\mu^2 - 1)\mu'^2$$

(see section 11.2 in chapter I of [5]). Observe that

$$p(\mu, \mu') = \frac{3}{16}(3 + 3\mu^2\mu'^2 - \mu^2 - \mu'^2) \geq \frac{3}{16} > 0, \quad (26)$$

while

$$\frac{1}{2} \int_{-1}^1 p(\mu, \mu') d\mu = \frac{3}{16}(6 + 3 \cdot \frac{2}{3}\mu'^2 - \frac{2}{3} - 2\mu'^2) = 1. \quad (27)$$

Keeping (12) as the defining equation for $T[I]$, the problem becomes

$$\begin{cases} (\mu\partial_\tau + \kappa_\nu)I_\nu(\tau, \mu) = \frac{3}{8}\kappa_\nu a_\nu((3 - \mu^2)J_\nu(\tau) + (3\mu^2 - 1)K_\nu(\tau)) \\ \quad + \kappa_\nu(1 - a_\nu)B_\nu(T[J](\tau)), \\ I_\nu(0, \mu) = Q_\nu^+(\mu), \quad I_\nu(Z, -\mu) = Q_\nu^-(\mu), \quad 0 < \mu < 1, \end{cases} \quad (28)$$

with

$$J_\nu := \frac{1}{2} \int_{-1}^1 \mu I_\nu d\mu, \quad K_\nu = \frac{1}{2} \int_{-1}^1 \mu^2 I_\nu d\mu \quad (29)$$

and (12). Starting from $I_\nu^0(\tau, \mu) = 0$ and $T^0(\tau) = 0$, one solves for I_ν^{n+1}

$$\begin{cases} (\mu\partial_\tau + \kappa_\nu)I_\nu^{n+1}(\tau, \mu) = \frac{3}{8}\kappa_\nu a_\nu((3 - \mu^2)J_\nu^n(\tau) + (3\mu^2 - 1)K_\nu^n(\tau)) \\ \quad + \kappa_\nu(1 - a_\nu)B_\nu(T^n(\tau)), \quad T^n := T[I^n] \\ I_\nu^{n+1}(0, \mu) = Q_\nu^+(\mu), \quad I_\nu^{n+1}(Z, -\mu) = Q_\nu^-(\mu), \quad 0 < \mu < 1. \end{cases} \quad (30)$$

Since B_ν is nondecreasing for each $\nu > 0$, one easily checks with (26) that

$$\begin{aligned} 0 &= I_\nu^0 \leq I_\nu^1 \leq I_\nu^2 \leq \dots \leq I_\nu^n \leq I_\nu^{n+1} \leq \dots \\ 0 &= T^0 \leq T^1 \leq T^2 \leq \dots \leq T^n \leq T^{n+1} \leq \dots \end{aligned}$$

The construction of these sequences is referred to as (1).

Notice that the radiative intensity is eliminated from (1), but can be recovered by the explicit formula

$$\begin{aligned} I_\nu^{n+1}(\tau, \mu) &= e^{-\frac{\kappa_\nu \tau}{\mu}} Q_\nu^+(\mu) \mathbf{1}_{\mu > 0} + e^{-\frac{\kappa_\nu (Z - \tau)}{|\mu|}} Q_\nu^-(|\mu|) \mathbf{1}_{\mu < 0} \\ &+ \int_0^Z e^{-\frac{\kappa_\nu |t - \tau|}{|\mu|}} \frac{\kappa_\nu}{|\mu|} \frac{3}{8} a_\nu((3 - \mu^2)J_\nu^n(t) + (3\mu^2 - 1)K_\nu^n(t)) dt \\ &+ \mathbf{1}_{\mu > 0} \int_0^\tau e^{-\frac{\kappa_\nu (\tau - t)}{\mu}} \frac{\kappa_\nu}{\mu} (1 - a_\nu) B_\nu(T^n(t)) dt \\ &+ \int_0^Z e^{-\frac{\kappa_\nu |t - \tau|}{|\mu|}} \frac{\kappa_\nu}{|\mu|} \frac{3}{8} a_\nu((3 - \mu^2)J_\nu^n(t) + (3\mu^2 - 1)K_\nu^n(t)) dt \\ &+ \int_0^Z e^{-\frac{\kappa_\nu |t - \tau|}{|\mu|}} \frac{\kappa_\nu}{|\mu|} (1 - a_\nu) B_\nu(T^n(t)) dt. \end{aligned} \quad (34)$$

Returning to (34), assume that

$$0 \leq Q_\nu^\pm \leq B_\nu(T_M), \quad 0 \leq I_\nu^n \leq B_\nu(T_M) \text{ and } 0 \leq T^n \leq T_M.$$

Algorithm 1 to solve (28),(15).

- 1: Data: $Q_\nu^\pm, S_\nu(\tau)$ by ((19), κ_ν and a_ν).
- 2: For all $\tau \in (0, Z)$ and all $\nu \in (0, \infty)$, choose $J_\nu^0(\tau)$ and $K_\nu^0(\tau)$.
- 3: **for** $n = 0, 1, \dots, N - 1$ **do**
- 4: For all $\nu \in (0, \infty)$ and all $\tau \in (0, Z)$, compute $J_\nu^{n+1}(\tau)$ by

$$\begin{aligned}
J_\nu^{n+1}(\tau) = & S_\nu(\tau) + \frac{3}{16} \int_0^Z E_1(\kappa_\nu|\tau - t|)\kappa_\nu a_\nu(3J_\nu^n(t) - K_\nu^n(t))dt \\
& + \frac{3}{16} \int_0^Z E_3(\kappa_\nu|\tau - t|)\kappa_\nu a_\nu(3K_\nu^n(t) - J_\nu^n(t))dt \quad (31) \\
& + \frac{1}{2} \int_0^Z E_1(\kappa_\nu|\tau - t|)\kappa_\nu(1 - a_\nu)B_\nu(T^n(t))dt,
\end{aligned}$$

- 5: For all $\nu \in (0, \infty)$ and all $\tau \in (0, Z)$, compute $K_\nu^{n+1}(\tau)$ by

$$\begin{aligned}
K_\nu^{n+1}(\tau) = & \frac{1}{2} \int_0^1 \left(e^{-\frac{\kappa_\nu \tau}{\mu}} Q_\nu^+(\mu) \mathbf{1}_{\mu > 0} + e^{-\frac{\kappa_\nu(Z-\tau)}{|\mu|}} Q_\nu^- (|\mu|) \mathbf{1}_{\mu < 0} \right) \mu^2 d\mu \\
& + \frac{3}{16} \int_0^Z E_3(\kappa_\nu|\tau - t|)\kappa_\nu a_\nu(3J_\nu^n(t) - K_\nu^n(t))dt \\
& + \frac{3}{16} \int_0^Z E_5(\kappa_\nu|\tau - t|)\kappa_\nu a_\nu(3K_\nu^n(t) - J_\nu^n(t))dt \\
& + \frac{1}{2} \int_0^Z E_3(\kappa_\nu|\tau - t|)\kappa_\nu(1 - a_\nu)B_\nu(T^n(t))dt, \quad (32)
\end{aligned}$$

- 6: For each $\tau \in (0, Z)$, find $T^{n+1}(z)$ by solving with a Newton algorithm

$$\int_0^\infty \kappa_\nu(1 - a_\nu)B_\nu(T^{n+1})d\nu = \int_0^\infty \kappa_\nu(1 - a_\nu)J_\nu^{n+1}d\nu. \quad (33)$$

- 7: **end for**
 - 8: **return** $\{z \mapsto T^N(z)\}$
-

Then

$$\begin{aligned}
I_\nu^{n+1}(\tau, \mu) &\leq \left(e^{-\frac{\kappa_\nu \tau}{\mu}} \mathbf{1}_{\mu>0} + e^{-\frac{\kappa_\nu(Z-\tau)}{|\mu|}} \mathbf{1}_{\mu<0} \right) B_\nu(T_M) \\
&\quad + \mathbf{1}_{\mu>0} \int_0^\tau e^{-\frac{\kappa_\nu(\tau-t)}{\mu}} \frac{\kappa_\nu}{\mu} \frac{3}{8} a_\nu ((3-\mu^2)B_\nu(T_M) + (\mu^2 - \frac{1}{3})B_\nu(T_M)) dt \\
&\quad + \mathbf{1}_{\mu>0} \int_0^\tau e^{-\frac{\kappa_\nu(\tau-t)}{\mu}} \frac{\kappa_\nu}{\mu} (1-a_\nu) B_\nu(T_M) dt \\
&\quad + \mathbf{1}_{\mu<0} \int_\tau^Z e^{-\frac{\kappa_\nu(t-\tau)}{|\mu|}} \frac{\kappa_\nu}{|\mu|} \frac{3}{8} a_\nu ((3-\mu^2)B_\nu(T_M) + (\mu^2 - \frac{1}{3})B_\nu(T_M)) dt \\
&\quad + \mathbf{1}_{\mu<0} \int_\tau^Z e^{-\frac{\kappa_\nu(t-\tau)}{|\mu|}} \frac{\kappa_\nu}{|\mu|} (1-a_\nu) B_\nu(T_M) dt \\
&= \left(e^{-\frac{\kappa_\nu \tau}{\mu}} \mathbf{1}_{\mu>0} + e^{-\frac{\kappa_\nu(Z-\tau)}{|\mu|}} \mathbf{1}_{\mu<0} \right) B_\nu(T_M) \\
&\quad + \mathbf{1}_{\mu>0} \int_0^\tau e^{-\frac{\kappa_\nu(\tau-t)}{\mu}} \frac{\kappa_\nu}{\mu} \left(\frac{3}{8} a_\nu (3 - \frac{1}{3}) + (1 - a_\nu) \right) B_\nu(T_M) dt \\
&\quad + \mathbf{1}_{\mu<0} \int_\tau^Z e^{-\frac{\kappa_\nu(t-\tau)}{|\mu|}} \frac{\kappa_\nu}{|\mu|} \left(\frac{3}{8} a_\nu (3 - \frac{1}{3}) + (1 - a_\nu) \right) B_\nu(T_M) dt \\
&= \left(e^{-\frac{\kappa_\nu \tau}{\mu}} \mathbf{1}_{\mu>0} + e^{-\frac{\kappa_\nu(Z-\tau)}{|\mu|}} \mathbf{1}_{\mu<0} \right) B_\nu(T_M) \\
&\quad + B_\nu(T_M) \left(\mathbf{1}_{\mu>0} \int_0^\tau e^{-\frac{\kappa_\nu(\tau-t)}{\mu}} \frac{\kappa_\nu}{\mu} dt + \mathbf{1}_{\mu<0} \int_\tau^Z e^{-\frac{\kappa_\nu(t-\tau)}{|\mu|}} \frac{\kappa_\nu}{|\mu|} dt \right) \\
&= \left(e^{-\frac{\kappa_\nu \tau}{\mu}} \mathbf{1}_{\mu>0} + e^{-\frac{\kappa_\nu(Z-\tau)}{|\mu|}} \mathbf{1}_{\mu<0} \right) B_\nu(T_M) \\
&\quad + B_\nu(T_M) \left(\mathbf{1}_{\mu>0} \left(1 - e^{-\frac{\kappa_\nu \tau}{\mu}} \right) + \mathbf{1}_{\mu<0} \left(1 - e^{-\frac{\kappa_\nu(Z-\tau)}{|\mu|}} \right) \right) = B_\nu(T_M).
\end{aligned}$$

Besides, using again that $T \mapsto B_\nu(T)$ is increasing for each $\nu > 0$ while $\kappa_\nu(1 - a_\nu) > 0$ for all $\nu > 0$,

$$T^{n+1} = T[I^{n+1}] \leq T[B_\nu(T_M)] = T_M.$$

Summarizing, we have proved the following result.

Theorem 3 *Assume that $\kappa_\nu > 0$ while $0 \leq a_\nu < 1$ for all $\nu > 0$. Let the boundary data Q_ν^\pm satisfy*

$$0 \leq Q_\nu^\pm(\mu) \leq B_\nu(T_M) \quad \text{for all } \mu \in (-1, 1) \text{ and } \nu > 0.$$

(1) *defines an increasing sequence of radiative intensities I_ν^n and temperatures T^n converging pointwise to I_ν and $T = T[I]$ respectively, which is a solution of (28).*

The argument above is based on the monotonicity of the sequences I_ν^n and T^n , and does not give any information on the convergence rate.

Finally, (2) holds verbatim for the problem (28). Here are the (slight) modifications to the proof of the comparison argument due to the Rayleigh phase function. As in Theorem 2, consider two boundary data Q_ν^\pm and Q'_ν^\pm such that $Q_\nu^\pm(\mu) \leq Q'_\nu^\pm(\mu)$ for a.e. $(\mu, \nu) \in (0, 1) \times (0, \infty)$. Let $(I_\nu, T[I])$ and $(I'_\nu, T[I'])$ the solutions of (28) corresponding to the boundary data Q_ν^\pm and Q'_ν^\pm respectively.

First, we slightly modify the argument concerning the term D_2 as follows. In the case of the Rayleigh phase function

$$D_2 = \frac{1}{2} \int_0^\infty \kappa_\nu a_\nu \int_{-1}^1 (I_\nu - I'_\nu)_+(\mu) d\mu d\nu \\ - \frac{1}{2} \int_0^\infty \kappa_\nu a_\nu \int_{-1}^1 \int_{-1}^1 p(\mu, \mu') (I_\nu - I'_\nu)(\mu') s_+(I_\nu - I'_\nu)(\mu) d\mu' d\mu d\nu.$$

Since $p \geq 0$, one has

$$p(\mu, \mu') (I_\nu - I'_\nu)(\mu') s_+(I_\nu - I'_\nu)(\mu) \leq p(\mu, \mu') (I_\nu - I'_\nu)_+(\mu'),$$

so that

$$D_2 \geq \frac{1}{2} \int_0^\infty \kappa_\nu a_\nu \int_{-1}^1 (I_\nu - I'_\nu)_+(\mu) d\mu d\nu \\ - \frac{1}{2} \int_0^\infty \kappa_\nu a_\nu \int_{-1}^1 \int_{-1}^1 p(\mu, \mu') (I_\nu - I'_\nu)_+(\mu') d\mu' d\mu d\nu = 0,$$

since

$$\frac{1}{2} \int_{-1}^1 p(\mu, \mu') d\mu = 1.$$

Therefore, following the proof of (2), in the same manner, we obtain the following conclusions

$$\langle \mu (I_\nu - I'_\nu)_+ \rangle(\tau) = 0 \text{ for a.e. } \tau \in (0, Z),$$

and

$$s_+(I_\nu(\tau, \mu) - I'_\nu(\tau, \mu)) = s_+(T[I](\tau) - T[I'](\tau)) \\ \text{for a.e. } (\tau, \mu, \nu) \in (0, Z) \times (-1, 1) \times (0, \infty),$$

while

$$(I_\nu - I'_\nu)_+(0, -\mu) = (I_\nu - I'_\nu)_+(Z, \mu) = 0 \quad \text{for a.e. } \mu \in (0, 1).$$

Next we compute

$$\partial_\tau \left\langle \frac{\mu^2}{\kappa_\nu} (I_\nu - I'_\nu)_+ \right\rangle = -\frac{1}{2} \int_0^\infty a_\nu \int_{-1}^1 \mu (I_\nu - I'_\nu)_+(\tau, \mu) d\mu d\nu \\ + \frac{1}{2} \int_0^\infty a_\nu \int_{-1}^1 \mu \int_{-1}^1 p(\mu, \mu') (I_\nu - I'_\nu)_+(\tau, \mu') d\mu' d\mu d\nu s_+(T[I](\tau) - T[I'](\tau)) \\ - \langle (1 - a_\nu) \mu ((I_\nu - I'_\nu) - (B_\nu(T[I]) - B_\nu(T[I']))) s_+(T[I] - T[I']) \rangle \\ = -\langle a_\nu \mu (I_\nu - I'_\nu) s_+(T[I] - T[I']) \rangle - \langle (1 - a_\nu) \mu (I_\nu - I'_\nu) s_+(T[I] - T[I']) \rangle \\ = -\langle \mu (I_\nu - I'_\nu) s_+(T[I] - T[I']) \rangle = -\langle \mu (I_\nu - I'_\nu)_+ \rangle = 0,$$

since

$$\int_{-1}^1 \mu p(\mu, \mu') d\mu = \int_{-1}^1 \mu (B_\nu(T[I]) - B_\nu(T[I'])) d\mu = 0.$$

Finally we integrate in $\tau \in (0, Z)$, and conclude as in the previous section that

$$\begin{aligned} (I_\nu - I'_\nu)_+(0, -\mu) = 0 \text{ and } Q_\nu^\pm(\mu) \leq Q'^\pm_\nu(\mu) \quad \text{for a.e. } \mu \in (0, 1) \\ \implies \left\langle \frac{\mu^2}{\kappa_\nu} (I_\nu - I'_\nu)_+ \right\rangle(\tau) = \left\langle \frac{\mu^2}{\kappa_\nu} (I_\nu - I'_\nu)_+ \right\rangle(0) = 0. \end{aligned}$$

Hence $Q_\nu^\pm(\mu) \leq Q'^\pm_\nu(\mu)$ for a.e. $(\mu, \nu) \in (0, 1) \times (0, \infty)$ implies that $I_\nu(\tau, \mu) \leq I'_\nu(\tau, \mu)$ for a.e. $(\tau, \mu, \nu) \in (0, Z) \times (-1, 1) \times (0, \infty)$, and $T[I](\tau) \leq T[I'](\tau)$ for a.e. $\tau \in (0, Z)$. This comparison result implies the uniqueness of the solution as explained in the proof of Theorem 2.

6 Radiative transfer in a fluid with thermal diffusion

For clarity we consider the case of a lake; we neglect the wind above the lake and we assume that the sunlight hits the surface of the lake with a given energy. The depth of the lake should vary slowly with x, y , but for the sake of simplicity, it is assumed to be uniform: $\Omega = \mathbb{O} \times (0, Z)$, for some open set $\mathbb{O} \subset \mathbb{R}^2$ with C^1 boundary, or piecewise C^1 boundary.

With $\mathbf{u} \in H^1(\Omega)$ satisfying $\nabla \cdot \mathbf{u} = 0$ and $\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0$, consider again the system (9). Throughout this section, we assume isotropic scattering, with

$$0 \leq a_\nu \leq a_M < 1, \quad 0 < \kappa_m \leq \kappa_\nu \leq \kappa_M, \quad \nu > 0. \quad (35)$$

Here, ρ is assumed to be a constant, and we choose $\rho_0 = \rho$ in (14), so that $\kappa_\nu = \rho \bar{\kappa}_\nu$, and $\tau = z$.

We further assume that the fluid flow is steady, and consider the system

$$\mu \partial_z I_\nu + \kappa_\nu I_\nu = \kappa_\nu (1 - a_\nu) B_\nu(T) + \kappa_\nu a_\nu J_\nu, \quad J_\nu := \frac{1}{2} \int_{-1}^1 I_\nu d\mu, \quad (36)$$

$$\mathbf{u} \cdot \nabla T - \frac{c_P}{c_V} \kappa_T \Delta T = \frac{4\pi}{\rho c_V} \int_0^\infty \kappa_\nu (1 - a_\nu) (J_\nu - B_\nu(T)) d\nu, \quad (37)$$

$$I_\nu|_{z=Z, \mu < 0} = Q_\nu^-(x, y, -\mu), \quad I_\nu|_{z=0, \mu > 0} = Q_\nu^+(x, y, \mu), \quad \frac{\partial T}{\partial n} \Big|_{\partial\Omega} = 0. \quad (38)$$

The boundary sources $Q_\nu^\pm(x, y, \mu)$ are bounded, measurable, nonnegative functions defined a.e. on $\mathbb{O} \times (-1, 1) \times (0, \infty)$.

As a first reduction, we solve (36) for the radiative intensity I_ν in terms of the angle-averaged intensity J_ν and of the temperature T , and average the

resulting expression in μ : proceeding as in (2), we arrive at the system

$$\begin{cases} J_\nu(x, y, z) = S_\nu(x, y, z) \\ + \frac{1}{2} \int_0^Z \kappa_\nu E_1(\kappa_\nu |z - \zeta|) (a_\nu J_\nu(x, y, \zeta) + (1 - a_\nu) B_\nu(T(x, y, \zeta))) d\zeta, \\ \mathbf{u}(\mathbf{x}) \cdot \nabla T(\mathbf{x}) - \frac{c_E}{c_V} \kappa_T \Delta T(\mathbf{x}) = \frac{4\pi}{\rho c_V} \int_0^\infty \kappa_\nu (1 - a_\nu) (J_\nu(\mathbf{x}) - B_\nu(T(\mathbf{x}))) d\nu, \\ \left. \frac{\partial T}{\partial n} \right|_{\partial\Omega} = 0, \end{cases} \quad (39)$$

where

$$S_\nu(x, y, z) := \frac{1}{2} \int_0^1 \left(e^{-\frac{\kappa_\nu z}{\mu}} Q_\nu^+(x, y, \mu) + e^{-\frac{\kappa_\nu(Z-z)}{\mu}} Q_\nu^-(x, y, \mu) \right) d\mu \quad (40)$$

Once the angle-averaged radiative intensity is known J_ν , the radiative intensity I_ν itself is easily obtained by solving the transfer equation (36) by the method of characteristics: see (21).

Starting from $T^0 = 0$ and $J_\nu^0 = S_\nu$, consider the following iterative algorithm. In the present section, we shall study the convergence of (2).

Algorithm 2 to solve (36),(37),(38)

- 1: Data: Q_ν^\pm , $S_\nu(x, y, z)$ by (40), κ_ν , a_ν , κ_T , \mathbf{u} .
- 2: Choose $T^0(\mathbf{x})$, $\forall \mathbf{x} \in \Omega$.
- 3: **for** all $(x, y) \in \mathbb{O}$, **do**
- 4: For all $\nu \in (0, \infty)$ and all $z \in (0, Z)$ compute $J_\nu^{n+1}(z)$ by

$$\begin{aligned} J_\nu^{n+1}(x, y, z) &= S_\nu(x, y, z) + \\ &\frac{1}{2} \int_0^Z \kappa_\nu E_1(\kappa_\nu |z - \zeta|) (a_\nu J_\nu^n(x, y, \zeta) + (1 - a_\nu) B_\nu(T^n(x, y, \zeta))) d\zeta. \end{aligned} \quad (41)$$

- 5: Compute T^{n+1} by solving

$$\begin{cases} \mathbf{u} \cdot \nabla T^{n+1} - \frac{c_E}{c_V} \kappa_T \Delta T^{n+1} + \frac{4\pi}{\rho c_V} \int_0^\infty \kappa_\nu (1 - a_\nu) B_\nu(T^{n+1}) d\nu \\ = \frac{4\pi}{\rho c_V} \int_0^\infty \kappa_\nu (1 - a_\nu) J_\nu^{n+1} d\nu, \quad \left. \frac{\partial T}{\partial n} \right|_{\partial\Omega} = 0. \end{cases} \quad (42)$$

- 6: **end for**
 - 7: **return** T
-

Theorem 4 Assume that the absorption coefficient κ_ν and the scattering albedo a_ν satisfy (35). Let the boundary source terms Q_ν^\pm satisfy the bound: for some T_M ,

$$0 \leq Q_\nu^\pm(\mu) \leq B_\nu(T_M), \quad 0 < \mu < 1, \quad \nu > 0.$$

Then the sequence $\{J_\nu^n, T^n\}_{n \geq 0}$ generated by (2) satisfies

$$\begin{aligned} S_\nu(\mathbf{x}) = J_\nu^0(\mathbf{x}) \leq J_\nu^1(\mathbf{x}) \leq \dots \leq J_\nu^n(\mathbf{x}) \leq J_\nu^{n+1}(\mathbf{x}) \leq \dots \leq B_\nu(T_M), \quad \nu > 0, \\ 0 = T^0 \leq T^1(\mathbf{x}) \leq \dots \leq T^n(\mathbf{x}) \leq T^{n+1}(\mathbf{x}) \leq \dots \leq T_M, \quad \mathbf{x} \in \Omega, \end{aligned}$$

and converges to a solution (J, T) of the system (39).

Define

$$\mathcal{B}(T) := \int_0^\infty \kappa_\nu (1 - a_\nu) B_\nu(T_+) d\nu.$$

Observe that

$$\kappa_m (1 - a_M) \bar{\sigma} T_+^4 \leq \mathcal{B}(T) \leq \kappa_M \bar{\sigma} T_+^4,$$

where $\pi \bar{\sigma}$ is the Stefan-Boltzmann constant (see (3)). Observe also that the function $\mathcal{B} : \mathbf{R} \rightarrow \mathbf{R}$ is increasing by construction, since B_ν is increasing on $[0, +\infty)$ for each $\nu > 0$.

For the sake of notational simplicity, in order to keep the number of physical constants to a strict minimum, we assume henceforth that $\rho_{CP} \kappa_T / 4\pi = 1$, and replace \mathbf{u} with $\rho_{CV} \mathbf{u} / 4\pi$.

The key argument in the proof of this theorem is the following lemma.

Lemma 3 *Let $R \in L^{6/5}(\Omega)$. There exists at least one weak solution of*

$$-\Delta T + \mathbf{u} \cdot \nabla T + \mathcal{B}(T) = R, \quad \frac{\partial T}{\partial n} \Big|_{\partial\Omega} = 0.$$

If $R \geq 0$ a.e. and $|\{x \in \Omega \text{ s.t. } R(x) > 0\}| > 0$, the weak solution of the problem above is unique and satisfies $T \geq 0$ a.e. on Ω .

Moreover, if $R' \in L^{6/5}(\Omega)$ and $R' \geq R$ a.e. on Ω , the weak solution T' of the problem above with right hand side R' satisfies $T \leq T'$ a.e. on Ω .

Proof [Proof of (3)] For each $0 < \varepsilon < 1$, the problem

$$\varepsilon T_\varepsilon - \Delta T_\varepsilon + \mathbf{u} \cdot \nabla T_\varepsilon + \mathcal{B}(T_\varepsilon) = R, \quad \frac{\partial T}{\partial n} \Big|_{\partial\Omega} = 0$$

has a weak solution in $H^1(\Omega)$.

To see this, apply Theorem 1 of [17] with $V = H^1(\Omega)$ to the nonlinear operator $\mathcal{A}_\varepsilon : V \mapsto V'$ defined by

$$\langle \mathcal{A}_\varepsilon T, \phi \rangle_{V', V} = \int_\Omega (\varepsilon T \phi + \nabla T \cdot \nabla \phi + \phi \mathbf{u} \cdot \nabla T + \mathcal{B}(T) \phi) d\mathbf{x}.$$

That \mathcal{A}_ε is continuous from V to V' easily follows from the Sobolev embedding $H^1(\Omega) \subset L^6(\Omega)$, which implies the continuous inclusion $L^{6/5}(\Omega) \subset V'$. Since $\mathbf{u} \in H^1(\Omega) \subset L^6(\Omega)$, one has

$$\mathbf{u} \cdot \nabla T \in L^{3/2}(\Omega) \subset L^{6/5}(\Omega) \subset V' \quad \text{with } \|\mathbf{u} \cdot \nabla T\|_{L^{3/2}(\Omega)} \leq \|\mathbf{u}\|_{L^6(\Omega)} \|T\|_{H^1(\Omega)},$$

and

$$\mathcal{B}(T) \in L^{3/2}(\Omega) \subset L^{6/5}(\Omega) \subset V' \quad \text{with } \|\mathcal{B}(T)\|_{L^{3/2}(\Omega)} \leq \kappa_M \bar{\sigma} \|T_+\|_{L^6(\Omega)}^4.$$

Since \mathbf{u} is a divergence free vector in $H^1(\Omega)$ satisfying $\mathbf{u} \cdot \mathbf{n} = 0$ on $\partial\Omega$, the bilinear functional

$$H^1(\Omega) \times H^1(\Omega) \ni (T, \phi) \mapsto \int_{\Omega} \phi \mathbf{u} \cdot \nabla T \, d\mathbf{x} \in \mathbf{R}$$

is skew-symmetric, and $\mathcal{B}(T(x)) = 0$ if $T(x) \leq 0$ by definition, so that

$$\langle \mathcal{A}_\varepsilon T, T \rangle_{V', V} = \varepsilon \|T\|_{L^2(\Omega)}^2 + \|\nabla T\|_{L^2(\Omega)}^2 + \int_{\Omega} \mathcal{B}(T) T \, d\mathbf{x} \geq \varepsilon \|T\|_{H^1(\Omega)}^2.$$

Hence \mathcal{A} is coercive on V . Besides, for all $T_1, T_2 \in H^1(\Omega)$

$$\begin{aligned} \langle \mathcal{A}_\varepsilon T_1 - \mathcal{A} T_2, T_1 - T_2 \rangle_{V', V} &= \varepsilon \|T_1 - T_2\|_{L^2(\Omega)}^2 + \|\nabla(T_1 - T_2)\|_{L^2(\Omega)}^2 \\ &\quad + \int_{\Omega} (T_1 - T_2)(\mathcal{B}(T_1) - \mathcal{B}(T_2)) \, d\mathbf{x} \geq 0. \end{aligned}$$

Theorem 1 in [17], implies the desired existence result for each $\varepsilon \in (0, 1)$.

Then, since $R \geq 0$ a.e. on Ω , one has $RT_\varepsilon \leq RT_{\varepsilon+}$ a.e. on Ω , and therefore

$$\begin{aligned} \varepsilon \|T_\varepsilon\|_{L^2(\Omega)}^2 + \|\nabla T_\varepsilon\|_{L^2(\Omega)}^2 + \bar{\sigma} \kappa_m (1 - a_M) \int_{\Omega} T_\varepsilon(\mathbf{x})_+^5 \, d\mathbf{x} &\leq \langle \mathcal{A}_\varepsilon T, T \rangle_{V', V} \\ &\leq \int_{\Omega} R(\mathbf{x}) T_\varepsilon(\mathbf{x})_+ \, d\mathbf{x} \leq \|R\|_{L^{6/5}(\Omega)} \|T_{\varepsilon+}\|_{L^6(\Omega)} \leq C_S \|R\|_{L^{6/5}(\Omega)} \|T_{\varepsilon+}\|_{H^1(\Omega)}. \end{aligned}$$

By Hölder's inequality

$$\int_{\Omega} T_\varepsilon(\mathbf{x})_+^5 \, d\mathbf{x} \geq \frac{1}{|\Omega|^3} \|T_{\varepsilon+}\|_{L^2(\Omega)}^5,$$

and since $\|\nabla T_{\varepsilon+}\|_{L^2(\Omega)} \leq \|\nabla T_\varepsilon\|_{L^2(\Omega)}$, we see that

$$\|\nabla T_\varepsilon\|_{L^2(\Omega)}^2 + \frac{\bar{\sigma} \kappa_m (1 - a_M)}{|\Omega|^3} \|T_{\varepsilon+}\|_{L^2(\Omega)}^5 \leq C_S \|R\|_{L^{6/5}(\Omega)} \left(\|T_{\varepsilon+}\|_{L^2(\Omega)}^2 + \|\nabla T_\varepsilon\|_{L^2(\Omega)}^2 \right)^{\frac{1}{2}}$$

so that

$$\sup_{0 < \varepsilon < 1} \left(\|\nabla T_\varepsilon\|_{L^2(\Omega)} + \|T_{\varepsilon+}\|_{L^2(\Omega)} \right) < \infty.$$

By the Banach-Alaoglu and the Rellich theorems, there exists a subsequence of T_ε (still denoted T_ε for simplicity) such that

$$T_{\varepsilon+} \rightarrow T_+ \quad \text{in } L^p(\Omega) \quad \text{and} \quad \nabla T_\varepsilon \rightarrow \nabla T \quad \text{weakly in } L^2(\Omega)$$

for all $p \in [1, 6)$ while $\varepsilon^{1/2} T_\varepsilon$ is bounded in $L^2(\Omega)$. Hence, for each $\phi \in H^1(\Omega)$, one has

$$\begin{aligned} 0 &= \int_{\Omega} (\varepsilon T_\varepsilon \phi + \nabla T_\varepsilon \cdot \nabla \phi + \phi \mathbf{u} \cdot \nabla T_\varepsilon + \mathcal{B}(T_\varepsilon) \phi) \, d\mathbf{x} \\ &\rightarrow \int_{\Omega} (\nabla T \cdot \nabla \phi + \phi \mathbf{u} \cdot \nabla T + \mathcal{B}(T) \phi) \, d\mathbf{x} =: \langle \mathcal{A} T, \phi \rangle_{V', V} \end{aligned}$$

in the limit as $\varepsilon \rightarrow 0$, so that T is a weak solution of

$$-\Delta T + \mathbf{u} \cdot \nabla T + \mathcal{B}(T) = R, \quad \frac{\partial T}{\partial n} \Big|_{\partial\Omega} = 0.$$

Observe that

$$\langle \mathcal{A}T - \mathcal{A}T', (T - T')_+ \rangle_{V', V} = \|\nabla(T - T')_+\|_{L^2(\Omega)}^2 + \int_{\Omega} (\mathcal{B}(T) - \mathcal{B}(T'))(T - T')_+ \, d\mathbf{x} \geq 0,$$

since

$$\int_{\Omega} (T - T')_+ \mathbf{u} \cdot \nabla(T - T') \, d\mathbf{x} = \int_{\Omega} \mathbf{u} \cdot \nabla \frac{1}{2}(T - T')_+^2 \, d\mathbf{x} = \int_{\partial\Omega} \frac{1}{2}(T - T')_+^2 \mathbf{u} \cdot \mathbf{n} \, d\sigma(\mathbf{x}) = 0,$$

denoting by $d\sigma(\mathbf{x})$ the surface element on $\partial\Omega$. Hence

$$R \leq R' \text{ a.e. on } \Omega \implies \langle (R - R'), (T - T')_+ \rangle_{V', V} = \|\nabla(T - T')_+\|_{L^2(\Omega)} = 0.$$

Since Ω is connected, $(T - T')_+ = c$ a.e. on Ω for some constant $c \geq 0$.

A first consequence of this remark is that, if $R' \geq 0$ a.e. on Ω , weak solutions of

$$-\Delta T' + \mathbf{u} \cdot \nabla T' + \mathcal{B}(T') = R', \quad \frac{\partial T'}{\partial n} \Big|_{\partial\Omega} = 0$$

satisfy

$$T' \geq 0 \text{ a.e. on } \Omega, \quad \text{unless } R' = 0 \text{ a.e. on } \Omega, \quad \text{in which case } T' = \text{Const.} \leq 0.$$

A second consequence is that, if $R' \geq R \geq 0$, with $|\{x \in \Omega \text{ s.t. } R \geq 0\}| > 0$, the solutions T and T' of

$$\begin{aligned} -\Delta T + \mathbf{u} \cdot \nabla T + \mathcal{B}(T) &= R, & \frac{\partial T}{\partial n} \Big|_{\partial\Omega} &= 0, \\ -\Delta T' + \mathbf{u} \cdot \nabla T' + \mathcal{B}(T') &= R', & \frac{\partial T'}{\partial n} \Big|_{\partial\Omega} &= 0, \end{aligned}$$

satisfy, for some constant $c \geq 0$,

$$T \geq 0 \text{ and } T' \geq 0 \text{ a.e. on } \Omega, \quad \text{and } (T - T')_+ = c \text{ a.e. on } \Omega.$$

Besides

$$\begin{aligned} 0 &= \langle R - R', (T - T')_+ \rangle_{V', V} = \langle \mathcal{A}T - \mathcal{A}T', (T - T')_+ \rangle_{V', V} \\ &= \|\nabla(T - T')_+\|_{L^2(\Omega)}^2 + \int_{\Omega} (\mathcal{B}(T) - \mathcal{B}(T'))(T - T')_+ \, d\mathbf{x} \\ &= c \int_{\Omega} (\mathcal{B}(T' + c) - \mathcal{B}(T')) \, d\mathbf{x}. \end{aligned}$$

Since $T' \geq 0$ a.e. on Ω , and since \mathcal{B} is increasing, this implies that $c = 0$. Therefore

$$R' \geq R \geq 0 \text{ with } |\{x \in \Omega \text{ s.t. } R \geq 0\}| > 0 \implies (T - T')_+ = 0.$$

Hence $T \leq T'$ a.e. on Ω . □

Proof[Proof of (4)] For the sake of clarity, we systematically omit the tangential variables x, y in the integral equations for the averaged radiative intensity J_ν^n (as well as for the radiative intensity I_ν itself), since these variables are only parameters in all these formulas. Start from

$$T^0 \equiv 0, \quad J_\nu^0(z) = S_\nu(z) > 0.$$

Construct iteratively $(T^n, J_\nu^n)_{n \geq 0}$ by the following recursion formula: first, compute

$$J_\nu^{n+1}(z) = S_\nu(z) + \frac{1}{2} \int_0^Z \kappa_\nu E(\kappa_\nu |z - t|) (a_\nu J_\nu^n(t) + (1 - a_\nu) B_\nu(T^n(t))) dt;$$

and then let T^{n+1} be the solution of

$$-\Delta T^{n+1} + \mathbf{u} \cdot \nabla T^{n+1} + \mathcal{B}(T^{n+1}) = \int_0^\infty \kappa_\nu (1 - a_\nu) J_\nu^{n+1} d\nu, \quad \left. \frac{\partial T^{n+1}}{\partial n} \right|_{\partial\Omega} = 0. \quad (43)$$

Obviously $J_\nu^1 \geq J_\nu^0 > 0$, and applying Lemma 3 implies that $T^1 \geq T^0$ a.e. on Ω . Moreover

$$T^n \geq T^{n-1} \quad \text{and} \quad J_\nu^n \geq J_\nu^{n-1} > 0 \implies J_\nu^{n+1} \geq J_\nu^n > 0,$$

and applying the Lemma 3 shows that

$$T^{n+1} \geq T^n \text{ a.e. on } \Omega.$$

Assume that $Q_\nu^\pm(\mu) \leq B_\nu(T_M)$. It will be more convenient to deal with radiative intensities I_ν instead of their angle-averaged variants J_ν . Therefore, we define I_ν^n to be the solution of

$$\begin{aligned} (\mu \partial_z + \kappa_\nu) I_\nu^{n+1} &= \kappa_\nu (1 - a_\nu) B_\nu(T^n) + \kappa_\nu a_\nu J_\nu^n, & J_\nu^n &= \tilde{I}_\nu^n, \\ I_\nu^{n+1}(Z, -\mu) &= Q_\nu^-(-\mu), & I_\nu^{n+1}(0, +\mu) &= Q_\nu^+(+\mu), \quad 0 < \mu < 1. \end{aligned}$$

Let us prove by induction that

$$\begin{aligned} I_\nu^n &\leq B_\nu(T_M) \text{ a.e. on } \Omega \times (-1, 1) \times (0, +\infty), \\ J_\nu^n &\leq B_\nu(T_M) \text{ a.e. on } \Omega \times (0, +\infty), \\ T^n &\leq T_M \text{ a.e. on } \Omega. \end{aligned}$$

This is true for $n = 0$ since $T^0 \equiv 0$, while

$$\begin{aligned} I_\nu^0(z, \mu) &= \mathbf{1}_{0 < \mu < 1} e^{-\kappa_\nu z / \mu} Q_\nu^+(\mu) + \mathbf{1}_{0 < -\mu < 1} e^{-\kappa_\nu (Z - z) / |\mu|} Q_\nu^-(-\mu) \\ &\leq (\mathbf{1}_{0 < \mu < 1} + \mathbf{1}_{0 < -\mu < 1}) B_\nu(T_M), \quad \text{so that } 0 \leq J_\nu^0 \leq B_\nu(T_M). \end{aligned}$$

If this is true for some $n \geq 0$, then

$$\begin{aligned} (\mu \partial_z + \kappa_\nu) I_\nu^{n+1} &= \kappa_\nu \Sigma_\nu^n, & 0 \leq \Sigma_\nu^n &\leq B_\nu(T_M), \\ I_\nu^{n+1}(Z, -\mu) \Big|_{0 < \mu < 1} &= Q_\nu^-(-\mu), & I_\nu^{n+1}(0, +\mu) \Big|_{0 < \mu < 1} &= Q_\nu^+(+\mu). \end{aligned}$$

Thus

$$\begin{aligned} I_\nu^{n+1}(z) &= \mathbf{1}_{0 < \mu < 1} e^{-\kappa_\nu z / \mu} Q_\nu^+(\mu) + \mathbf{1}_{0 < \mu < 1} \int_0^z \frac{\kappa_\nu}{\mu} e^{-\frac{\kappa_\nu(z-t)}{\mu}} \Sigma_\nu^n(t) dt \\ &\quad + \mathbf{1}_{0 < -\mu < 1} e^{-\kappa_\nu(Z-z)/|\mu|} Q_\nu^-(-\mu) + \mathbf{1}_{0 < -\mu < 1} \int_z^Z \frac{\kappa_\nu}{|\mu|} e^{-\frac{\kappa_\nu(t-z)}{|\mu|}} \Sigma_\nu^n(t) dt \\ &\leq \left(\mathbf{1}_{0 < \mu < 1} e^{-\kappa_\nu z / \mu} + \mathbf{1}_{0 < \mu < 1} \int_0^z \frac{\kappa_\nu}{\mu} e^{-\frac{\kappa_\nu(z-t)}{\mu}} dt \right) B_\nu(T_M) \\ &\quad + \left(\mathbf{1}_{0 < -\mu < 1} e^{-\kappa_\nu(Z-z)/|\mu|} + \mathbf{1}_{0 < -\mu < 1} \int_z^Z \frac{\kappa_\nu}{|\mu|} e^{-\frac{\kappa_\nu(t-z)}{|\mu|}} dt \right) B_\nu(T_M) \\ &\leq \left(\mathbf{1}_{0 < \mu < 1} (e^{-\kappa_\nu z / \mu} + 1 - e^{-\kappa_\nu z / \mu}) \right) B_\nu(T_M) \\ &\quad + \left(\mathbf{1}_{0 < -\mu < 1} (e^{-\kappa_\nu(Z-z)/|\mu|} + 1 - e^{-\kappa_\nu(Z-z)/|\mu|}) \right) B_\nu(T_M) \leq B_\nu(T_M). \end{aligned}$$

Hence $J_\nu^{n+1} \leq B_\nu(T_M)$, and one solves (43) for T^{n+1} . Since $J_\nu^n \geq S_\nu > 0$ and

$$\int_0^\infty \kappa_\nu (1 - a_\nu) J_\nu^{n+1} d\nu \leq \int_0^\infty \kappa_\nu (1 - a_\nu) B_\nu(T_M) d\nu = \mathcal{B}(T_M),$$

we conclude from Lemma 3 that T^{n+1} is a.e. less than or equal to the solution of the problem

$$-\Delta T + \mathbf{u} \cdot \nabla T + \mathcal{B}(T) = \mathcal{B}(T_M), \quad \frac{\partial T}{\partial n} \Big|_{\partial \Omega} = 0,$$

which is obviously the constant T_M . Hence $T^{n+1} \leq T_M$ a.e. on Ω , so that we have proved by induction the desired chain of inequalities.

From these inequalities, we conclude that the sequences J_ν^n and T^n converge a.e. pointwise on $\Omega \times (0, \infty)$ and on Ω respectively to limits denoted J_ν and T , and that this convergence also holds in $L^p(\Omega \times (0, \infty))$ and $L^p(\Omega)$ for all $p \in [1, \infty)$ by dominated convergence.

Passing to the limit in (41) immediately shows that J_ν, T satisfy the first equation in (39). As for the second equation, one can pass to the limit in the right hand side and in the nonlinear term on the left hand side of (42). Since T^{n+1} is a weak solution of (42), one has $T^{n+1} \in H^1(\Omega)$ and

$$\int_\Omega \nabla T^{n+1}(\mathbf{x}) \cdot \nabla \phi(\mathbf{x}) d\mathbf{x} - \int_\Omega T^{n+1}(\mathbf{x}) \mathbf{u}(\mathbf{x}) \cdot \nabla \phi(\mathbf{x}) d\mathbf{x} = \int_\Omega h_{n+1}(\mathbf{x}) \phi(\mathbf{x}) d\mathbf{x} \quad (44)$$

for all $\phi \in H^1(\Omega)$, with

$$h_{n+1} := \int_0^\infty \kappa_\nu(1 - a_\nu)(J_\nu^{n+1} - B_\nu(T^{n+1}))d\nu$$

so that h_{n+1} is bounded in $L^p(\Omega)$ for all $p \in [1, \infty)$. Taking $\phi = T^{n+1}$, and observing that

$$\int_\Omega T^{n+1}(\mathbf{x})\mathbf{u}(\mathbf{x}) \cdot \nabla T^{n+1}(\mathbf{x})d\mathbf{x} = \int_{\partial\Omega} \frac{1}{2}T^{n+1}(\mathbf{x})^2\mathbf{u}(\mathbf{x}) \cdot \mathbf{n}_\mathbf{x}d\sigma(\mathbf{x}) = 0$$

since $\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0$ shows that T^{n+1} is bounded, and therefore weakly relatively compact in $H^1(\Omega)$. Since we already know that $T^{n+1} \rightarrow T$ in $L^p(\Omega)$ for all $p \in [1, \infty)$ as $n \rightarrow \infty$, we conclude that $T^{n+1} \rightarrow T$ weakly in $H^1(\Omega)$. At this point, we can pass to the limit in the weak formulation of (44), and this shows that T satisfies the second equation in (39). \square

Next we discuss the convergence rate of (2). We shall use the monotonic structure of the radiative transfer equations. Consider the upper approximating sequence

$$\begin{aligned} \mu\partial_z H_\nu^n &= \kappa_\nu(a_\nu K_\nu^{n-1} + (1 - a_\nu)B_\nu(\Theta^{n-1}) - H_\nu^n), \quad K_\nu = \frac{1}{2} \int_{-1}^1 H_\nu d\mu, \\ \mathbf{u} \cdot \nabla \Theta^n - \Delta \Theta^n &= \int_0^\infty \kappa_\nu(1 - a_\nu)(K_\nu^n - B_\nu(\Theta^n))d\nu, \\ H_\nu^n(0, \mu) &= Q_\nu^+(\mu), \quad H_\nu^n(Z, -\mu) = Q_\nu^-(\mu), \quad 0 < \mu < 1, \quad \frac{\partial \Theta^n}{\partial n} \Big|_{\partial\Omega} = 0, \end{aligned}$$

for all $n \geq 1$, initialized with $\Theta^0 = T_M$ and $H_\nu^0 = K_\nu^0 = B_\nu(\Theta^0)$.

Theorem 5 *Assume that the absorption coefficient κ_ν and the scattering albedo a_ν satisfy (35). Assume moreover that the constant C_1 defined in (18) satisfies*

$$0 \leq \gamma := \left(\sup_{\nu>0} (1 - a_\nu)C_1(\kappa_\nu) + \sup_{\nu>0} a_\nu C_1(\kappa_\nu) \right) < 1.$$

Let the boundary source terms Q_ν^\pm satisfy the bound

$$0 \leq Q_\nu^\pm(\mu) \leq B_\nu(T_M), \quad 0 < \mu < 1, \quad \nu > 0.$$

Then

(1) one has

$$\begin{aligned} 0 \leq T^0 \leq \dots \leq T^{n-1} \leq \Theta^n \leq \dots \leq \Theta^1 \leq T_M, \\ 0 \leq J_\nu^0 \dots \leq J_\nu^{n-1} \leq K_\nu^n \leq \dots \leq K_\nu^1 \leq B_\nu(T_M); \end{aligned} \tag{45}$$

(2) one has

$$\|\mathcal{B}(T^{n+1}) - \mathcal{B}(T^n)\|_{L^1(\Omega)} \leq \|\mathcal{B}(\Theta^{n+1}) - \mathcal{B}(T^n)\|_{L^1(\Omega)} \leq \gamma^n |\Omega| \mathcal{B}(T_M),$$

and

$$\|J_\nu^{n+1} - J_\nu^n\|_{L^1(\Omega \times (0, +\infty))} \leq \|K_\nu^{n+1} - J_\nu^n\|_{L^1(\Omega \times (0, +\infty))} \leq \frac{\gamma^n |\Omega| \mathcal{B}(T_M)}{\kappa_m (1 - a_M)};$$

(3) finally

$$\|\mathcal{B}(T) - \mathcal{B}(T^n)\|_{L^1(\Omega)} \leq \frac{\gamma^n}{1 - \gamma} |\Omega| \mathcal{B}(T_M),$$

and

$$\|J_\nu - J_\nu^n\|_{L^1(\Omega \times (0, +\infty))} \leq \frac{\gamma^n |\Omega| \mathcal{B}(T_M)}{\kappa_m (1 - a_M) (1 - \gamma)}.$$

Proof First, one has

$$\begin{aligned} \mu \partial_z H_\nu^1 + \kappa_\nu H_\nu^1 &= \kappa_\nu B_\nu(T_M) \geq 0, & 0 < z < Z, \\ 0 \leq H_\nu^1(0, +\mu) &= Q_\nu^+(\mu) \leq B_\nu(T_M), & 0 < \mu < 1, \\ 0 \leq H_\nu^1(Z, -\mu) &= Q_\nu^-(\mu) \leq B_\nu(T_M), & 0 < \mu < 1, \end{aligned}$$

so that

$$\begin{aligned} H_\nu^1(z, \mu) &= 1_{0 < \mu < 1} \left(e^{-\kappa_\nu z / \mu} Q_\nu^+(\mu) + (1 - e^{-\kappa_\nu z / \mu}) B_\nu(T_M) \right) \\ &\quad + 1_{0 < -\mu < 1} \left(e^{-\kappa_\nu (Z-z) / |\mu|} Q_\nu^-(\mu) + (1 - e^{-\kappa_\nu (Z-z) / \mu}) B_\nu(T_M) \right) \\ 0 \leq I_\nu^0 &\leq H_\nu^1 \leq B_\nu(T_M), & 0 \leq J_\nu^0 &\leq K_\nu^1 \leq B_\nu(T_M). \end{aligned}$$

Hence

$$\mathcal{B}(\Theta^1) + \mathbf{u} \cdot \nabla \Theta^1 - \Delta \Theta^1 = \int_0^\infty \kappa_\nu (1 - a_\nu) K_\nu^1 d\nu \leq \mathcal{B}(T_M),$$

so that

$$0 \leq T^0 \leq \Theta^1 \leq T_M$$

by (3). The same induction argument as in the proof of (4) shows that

$$\begin{aligned} 0 &\leq \dots \leq \Theta^n \leq \Theta^{n-1} \leq T_M, \\ 0 &\leq \dots \leq H_\nu^n \leq H_\nu^{n-1} \leq B_\nu(T_M), \\ 0 &\leq \dots \leq K_\nu^n \leq K_\nu^{n-1} \leq B_\nu(T_M). \end{aligned}$$

Moreover, assume that we have proved that

$$\begin{aligned} 0 &\leq T^0 \leq \dots \leq T^{n-1} \leq \Theta^n \leq \dots \leq \Theta^1 \leq T_M, \\ 0 &\leq I_\nu^0 \leq \dots \leq I_\nu^{n-1} \leq H_\nu^n \leq \dots \leq H_\nu^1 \leq B_\nu(T_M), \\ 0 &\leq J_\nu^0 \leq \dots \leq J_\nu^{n-1} \leq K_\nu^n \leq \dots \leq K_\nu^0 \leq B_\nu(T_M). \end{aligned}$$

Then

$$\begin{aligned} \mu \partial_z (H_\nu^{n+1} - I_\nu^n) + \kappa_\nu (H_\nu^{n+1} - I_\nu^n) &= \kappa_\nu a_\nu (K_\nu^n - J_\nu^{n-1}) \\ &\quad + \kappa_\nu (1 - a_\nu) (B_\nu(\Theta^n) - B_\nu(T^{n-1})) \geq 0, \\ (H_\nu^{n+1} - I_\nu^n)(0, +\mu) &= (H_\nu^{n+1} - I_\nu^n)(Z, -\mu) = 0, \quad 0 < \mu < 1, \end{aligned}$$

so that

$$I_\nu^n \leq H_\nu^{n+1}, \quad \text{and} \quad J_\nu^n \leq K_\nu^{n+1}.$$

Then

$$\begin{aligned} \mathcal{B}(\Theta^{n+1}) + \mathbf{u} \cdot \nabla \Theta^{n+1} - \Delta \Theta^{n+1} &= \int_0^\infty \kappa_\nu (1 - a_\nu) K_\nu^{n+1} d\nu, \\ \mathcal{B}(T^n) + \mathbf{u} \cdot \nabla T^n - \Delta T^n &= \int_0^\infty \kappa_\nu (1 - a_\nu) J_\nu^n d\nu, \\ \frac{\partial \Theta^{n+1}}{\partial n} \Big|_{\partial \Omega} &= \frac{\partial T^n}{\partial n} \Big|_{\partial \Omega} = 0, \end{aligned}$$

and (3) implies that $T^n \leq \Theta^{n+1}$. Hence we have proved by induction that, for all $n \geq 1$,

$$\begin{aligned} 0 \leq T^0 \leq \dots \leq T^{n-1} \leq \Theta^n \leq \dots \leq \Theta^1 \leq T_M, \\ 0 \leq I_\nu^0 \leq \dots \leq I_\nu^{n-1} \leq H_\nu^n \leq \dots \leq H_\nu^1 \leq B_\nu(T_M), \\ 0 \leq J_\nu^0 \leq \dots \leq J_\nu^{n-1} \leq K_\nu^n \leq \dots \leq K_\nu^1 \leq B_\nu(T_M), \end{aligned}$$

which implies (45).

Then

$$\begin{aligned} \mathcal{B}(\Theta^{n+1}) - \mathcal{B}(T^n) + \mathbf{u} \cdot \nabla (\Theta^{n+1} - T^n) - \Delta (\Theta^{n+1} - T^n) \\ = \int_0^\infty \kappa_\nu (1 - a_\nu) (K_\nu^{n+1} - J_\nu^n) d\nu, \quad \frac{\partial (\Theta^{n+1} - T^n)}{\partial n} \Big|_{\partial \Omega} = 0, \end{aligned}$$

so that

$$\int_\Omega (\mathcal{B}(\Theta^{n+1}) - \mathcal{B}(T^n)) d\mathbf{x} = \int_\Omega \int_0^\infty \kappa_\nu (1 - a_\nu) (K_\nu^{n+1} - J_\nu^n) d\nu d\mathbf{x},$$

since

$$\int_{\partial \Omega} \left((\Theta^{n+1} - T^n) \mathbf{u} \cdot \mathbf{n}_x - \frac{\partial (\Theta^{n+1} - T^n)}{\partial n} \right) d\sigma(\mathbf{x}) = 0.$$

Then

$$\begin{aligned} &K_\nu^{n+1}(\mathbf{x}) - J_\nu^n(\mathbf{x}) \\ &= \frac{1}{2} \int_0^Z \kappa_\nu E_1(\kappa_\nu |z - \zeta|) (1 - a_\nu) (B_\nu(\Theta^n) - B_\nu(T^{n-1}))(x, y, \zeta) d\zeta \\ &\quad + \frac{1}{2} \int_0^Z \kappa_\nu E_1(\kappa_\nu |z - \zeta|) a_\nu (K_\nu^n - J_\nu^{n-1})(x, y, \zeta) d\zeta. \end{aligned}$$

Thus

$$\begin{aligned}
\epsilon_n &:= \int_{\Omega} \int_0^{\infty} \kappa_{\nu}(1 - a_{\nu})(K_{\nu}^{n+1} - J_{\nu}^n) d\nu d\mathbf{x} \\
&= \frac{1}{2} \int_{\mathbb{O}} dx dy \int_0^{\infty} d\nu \int_0^Z dz \int_0^Z \kappa_{\nu}^2 E_1(\kappa_{\nu}|z - \zeta|) \\
&\quad \times (1 - a_{\nu})^2 (B_{\nu}(\Theta^n) - B_{\nu}(T^{n-1}))(x, y, \zeta) d\zeta \\
&+ \frac{1}{2} \int_{\mathbb{O}} dx dy \int_0^{\infty} d\nu \int_0^Z dz \int_0^Z \kappa_{\nu}^2 E_1(\kappa_{\nu}|z - \zeta|) \\
&\quad \times (1 - a_{\nu}) a_{\nu} (K_{\nu}^n - J_{\nu}^{n-1})(x, y, \zeta) d\zeta.
\end{aligned}$$

At this point, we integrate first in z and use (18), to obtain

$$\begin{aligned}
\epsilon_n &= \int_{\Omega} \int_0^{\infty} \kappa_{\nu}(1 - a_{\nu})(K_{\nu}^{n+1} - J_{\nu}^n) d\nu d\mathbf{x} \\
&\leq \int_{\mathbb{O}} dx dy \int_0^{\infty} d\nu \int_0^Z C_1(\kappa_{\nu}) \kappa_{\nu} (1 - a_{\nu})^2 (B_{\nu}(\Theta^n) - B_{\nu}(T^{n-1}))(x, y, \zeta) d\zeta \\
&\quad + \int_{\mathbb{O}} dx dy \int_0^{\infty} d\nu \int_0^Z C_1(\kappa_{\nu}) \kappa_{\nu} (1 - a_{\nu}) a_{\nu} (K_{\nu}^n - J_{\nu}^{n-1})(x, y, \zeta) d\zeta \\
&\leq \sup_{\nu > 0} (1 - a_{\nu}) C_1(\kappa_{\nu}) \int_{\Omega} \int_0^{\infty} \kappa_{\nu} (1 - a_{\nu}) (B_{\nu}(\Theta^n) - B_{\nu}(T^{n-1}))(\mathbf{x}) d\nu d\mathbf{x} \\
&\quad + \sup_{\nu > 0} a_{\nu} C_1(\kappa_{\nu}) \int_{\Omega} \int_0^{\infty} \kappa_{\nu} (1 - a_{\nu}) (K_{\nu}^n - J_{\nu}^{n-1})(\mathbf{x}) d\nu d\mathbf{x} \\
&\leq \sup_{\nu > 0} (1 - a_{\nu}) C_1(\kappa_{\nu}) \int_{\Omega} (\mathcal{B}(\Theta^n) - \mathcal{B}(T^{n-1}))(\mathbf{x}) d\mathbf{x} \\
&\quad + \sup_{\nu > 0} a_{\nu} C_1(\kappa_{\nu}) \int_{\Omega} \int_0^{\infty} \kappa_{\nu} (1 - a_{\nu}) (K_{\nu}^n - J_{\nu}^{n-1})(\mathbf{x}) d\nu d\mathbf{x} \\
&= \epsilon_{n-1} \left(\sup_{\nu > 0} (1 - a_{\nu}) C_1(\kappa_{\nu}) + \sup_{\nu > 0} a_{\nu} C_1(\kappa_{\nu}) \right).
\end{aligned}$$

Hence $\epsilon_n \leq \epsilon_0 \gamma^n$ with

$$\gamma := \left(\sup_{\nu > 0} (1 - a_{\nu}) C_1(\kappa_{\nu}) + \sup_{\nu > 0} a_{\nu} C_1(\kappa_{\nu}) \right) \in [0, 1),$$

while $\epsilon_0 \leq |\Omega| \mathcal{B}(T_M) < \infty$. Hence the sequence $(K_{\nu}^n, \Theta^n)_{n \geq 1}$ of upper approximations and the sequence (J_{ν}^n, T^n) of lower approximations provided by (2) are adjacent. In particular

$$\begin{aligned}
\|\mathcal{B}(T^{n+1}) - \mathcal{B}(T^n)\|_{L^1(\Omega)} &= \int_{\Omega} (\mathcal{B}(T^{n+1}) - \mathcal{B}(T^n)) d\mathbf{x} \\
&\leq \int_{\Omega} (\mathcal{B}(\Theta^{n+1}) - \mathcal{B}(T^n)) d\mathbf{x} \leq \epsilon_0 \gamma^n
\end{aligned}$$

for all $n \geq 1$, so that

$$\|\mathcal{B}(T) - \mathcal{B}(T^n)\|_{L^1(\Omega)} \leq \frac{\epsilon_0 \gamma^n}{1 - \gamma}.$$

Similarly

$$\begin{aligned} & \int_{\Omega} \int_0^{\infty} \kappa_{\nu}(1 - a_{\nu})(J_{\nu}^{n+1} - J_{\nu}^n) d\nu d\mathbf{x} \\ & \leq \int_{\Omega} \int_0^{\infty} \kappa_{\nu}(1 - a_{\nu})(K_{\nu}^{n+1} - J_{\nu}^n) d\nu d\mathbf{x} \leq \epsilon_0 \gamma^n, \end{aligned}$$

and

$$\begin{aligned} & \kappa_m(1 - a_M) \|J_{\nu} - J_{\nu}^n\|_{L^1(\Omega \times (0, \infty))} \\ & \leq \sum_{m \geq n} \int_{\Omega} \int_0^{\infty} \kappa_{\nu}(1 - a_{\nu})(J_{\nu}^{m+1} - J_{\nu}^m) d\nu d\mathbf{x} \\ & \leq \sum_{m \geq n} \int_{\Omega} \int_0^{\infty} \kappa_{\nu}(1 - a_{\nu})(K_{\nu}^{m+1} - J_{\nu}^m) d\nu d\mathbf{x} \leq \frac{\epsilon_0 \gamma^n}{1 - \gamma}. \end{aligned}$$

This concludes the proof of statements (2) and (3). \square

With the monotonic structure of the radiative transfer equations, our argument will also provide the uniqueness of the solution of the system (36)-(37)-(38).

Theorem 6 *Under the same assumptions as in (5), there exists at most one solution (I_{ν}, T) of the problem (36)-(37)-(38) such that*

$$T \in L^{\infty}(\Omega), \quad \text{and} \quad T \geq 0 \text{ a.e. on } \Omega,$$

and

$$I_{\nu} \geq 0 \text{ a.e. on } \Omega \times (-1, 1) \times (0, \infty).$$

Proof Let (I_{ν}, T) be a solution of (36)-(37)-(38), and assume that the upper approximating sequence $(H_{\nu}^n, \Theta^n)_{n \geq 1}$ satisfies

$$I_{\nu} \leq H_{\nu}^n, \quad J_{\nu} \leq K_{\nu}^n, \quad \text{and} \quad T \leq \Theta^n.$$

Then, one has

$$\begin{aligned} \mu \partial_z (H_{\nu}^{n+1} - I_{\nu}) + \kappa_{\nu} (H_{\nu}^{n+1} - I_{\nu}) &= \kappa_{\nu} a_{\nu} (K_{\nu}^n - J_{\nu}), \\ &+ \kappa_{\nu} (1 - a_{\nu}) (B_{\nu}(\Theta^n) - B_{\nu}(T)) \geq 0 \\ (H_{\nu}^{n+1} - I_{\nu})(0, +\mu) &= (H_{\nu}^{n+1} - I_{\nu})(Z, -\mu) = 0, \quad 0 < \mu < 1. \end{aligned}$$

Solving this equation for $(H_{\nu}^{n+1} - I_{\nu})$ by the method of characteristics shows that

$$I_{\nu} \leq H_{\nu}^{n+1}, \quad \text{and therefore} \quad J_{\nu} \leq K_{\nu}^{n+1}.$$

Next, one has

$$\begin{aligned} & \mathcal{B}(\Theta^{n+1}) - \mathcal{B}(T) + \mathbf{u} \cdot \nabla(\Theta^{n+1} - T) - \Delta(\Theta^{n+1} - T) \\ &= \int_0^{\infty} \kappa_{\nu}(1 - a_{\nu})(K_{\nu}^{n+1} - J_{\nu}) d\nu \geq 0, \quad \frac{\partial(\Theta^{n+1} - T)}{\partial n} \Big|_{\partial\Omega} = 0, \end{aligned}$$

so that, according to (3), $T \leq \Theta^{n+1}$.

It remains to check the initial step of this induction argument. Since $T \in L^\infty(\Omega)$, we pick

$$\Theta^0 = \max(T_M, \|T\|_{L^\infty(\Omega)}), \quad \text{and } H_\nu^0 = K_\nu^0 = B_\nu(\Theta^0).$$

Hence $T \leq \Theta^0$ by construction. Next we prove that $I_\nu \leq B_\nu(\Theta^0)$. Multiplying both sides of (36) by $s_+(I_\nu - B_\nu(\Theta^0))$, we repeat the argument of the proof of (2):

$$\begin{aligned} & \partial_z \langle \mu(I_\nu - B_\nu(\Theta^0))_+ \rangle \\ = & -\langle \kappa_\nu(1 - a_\nu)(I_\nu - B_\nu(\Theta^0)) - (B_\nu(T) - B_\nu(\Theta^0)) \rangle s_+(I_\nu - B_\nu(\Theta^0)) \\ & - \langle \kappa_\nu a_\nu(I_\nu - B_\nu(\Theta^0)) - (J_\nu - B_\nu(\Theta^0)) \rangle s_+(I_\nu - B_\nu(\Theta^0)) \\ = & -D_1 - D_2. \end{aligned}$$

We have seen in the proof of (2) that

$$\begin{aligned} D_2 &= \langle \kappa_\nu a_\nu(I_\nu - B_\nu(\Theta^0)) - (J_\nu - B_\nu(\Theta^0)) \rangle s_+(I_\nu - B_\nu(\Theta^0)) \\ &= \langle \kappa_\nu a_\nu(I_\nu - B_\nu(\Theta^0)) - (J_\nu - B_\nu(\Theta^0)) \rangle (s_+(I_\nu - B_\nu(\Theta^0)) - s_+(J_\nu - B_\nu(\Theta^0))) \geq 0. \end{aligned}$$

As for D_1 , observe that

$$\begin{aligned} D_1 &= \langle \kappa_\nu(1 - a_\nu)((I_\nu - B_\nu(\Theta^0)) - (B_\nu(T) - B_\nu(\Theta^0))) \rangle (s_+(I_\nu - B_\nu(\Theta^0)) - s_+(T - \Theta^0)) \\ &\geq 0 \end{aligned}$$

since our assumption on T implies that $s_+(T - \Theta^0) = 0$. Integrating on Ω , we conclude that

$$\int_{\mathbb{O}} \langle \mu_+(I_\nu - B_\nu(\Theta^0))_+ \rangle(x, y, Z) dx dy = \int_{\mathbb{O}} \langle \mu_-(I_\nu - B_\nu(\Theta^0))_+ \rangle(x, y, 0) dx dy = 0$$

and that

$$D_1 = D_2 = 0 \quad \text{a.e. on } \Omega.$$

Since $\kappa_\nu(1 - a_\nu) \geq \kappa_m(1 - a_M) > 0$, the condition $D_1 = 0$ implies that

$$\begin{aligned} & ((I_\nu - B_\nu(\Theta^0)) - (B_\nu(T) - B_\nu(\Theta^0))) (s_+(I_\nu - B_\nu(\Theta^0)) - s_+(T - \Theta^0)) = 0 \\ & \text{which implies in turn that } s_+(I_\nu - B_\nu(\Theta^0)) = s_+(T - \Theta^0) = 0 \end{aligned}$$

Hence $I_\nu \leq B_\nu(\Theta^0)$, which completes the proof of the initialization of our induction argument.

Summarizing, we have proved that, if one chooses $\Theta^0 = \max(T_M, \|T\|_{L^\infty(\Omega)})$, the solution (I_ν, T) of (36)-(37)-(38) considered satisfies

$$I_\nu \leq H_\nu^n \leq H_\nu^{n-1} \leq \dots \leq H_\nu^0 = B_\nu(\Theta^0),$$

while

$$T \leq \Theta^n \leq \Theta^{n-1} \leq \dots \leq \Theta^0,$$

where (H_ν^n, Θ^n) is the upper approximating sequence. A similar argument (with a slightly simpler initialization) shows that

$$I_\nu \geq I_\nu^n \geq I_\nu^{n-1} \geq \dots \geq I_\nu^0 = 0,$$

while

$$T \geq T^n \geq T^{n-1} \geq \dots \geq T^0 = 0.$$

With this, we easily prove the uniqueness of the solution of (36)-(37)-(38). If (I_ν, T) and (I'_ν, T') are two solutions satisfying the assumptions of Theorem 6, we initialize the upper approximating sequence with

$$\Theta^0 = \max(T_M, \|T\|_{L^\infty(\Omega)}, \|T'\|_{L^\infty(\Omega)}).$$

The argument above shows that

$$I_\nu^n \leq I_\nu, \quad I'_\nu \leq H_\nu^{n+1}, \quad T^n \leq T, \quad T' \leq \Theta^{n+1}.$$

Hence

$$\|J_\nu - J'_\nu\|_{L^1(\Omega \times (0, \infty))} \leq \|K_\nu^{n+1} - J_\nu^n\|_{L^1(\Omega \times (0, \infty))} \leq \frac{|\Omega| \gamma^n}{\kappa_m (1 - a_M)} \mathcal{B}(\Theta^0),$$

and

$$\|\mathcal{B}(T) - \mathcal{B}(T')\|_{L^1(\Omega)} \leq \|\Theta^{n+1} - T^n\|_{L^1(\Omega)} \leq \gamma^n |\Omega| \mathcal{B}(\Theta^0).$$

Passing to the limit as $n \rightarrow \infty$ shows that

$$T = T' \text{ a.e. on } \Omega, \quad \text{and } J_\nu = J'_\nu \text{ a.e. on } \Omega \times (0, \infty).$$

Once it is known that $J_\nu = J'_\nu$ a.e. on $\Omega \times (0, \infty)$, solving (36) for I_ν and I'_ν by the method of characteristics shows that $I_\nu = I'_\nu$ a.e. on $\Omega \times (-1, 1) \times (0, \infty)$. \square

Several remarks regarding Theorems (4), (5) and (6) are in order.

Remarks.

(1) One can treat slightly more general situations with the same techniques. For instance, one could assume that the scattering rate a_ν depends on z , and is a slowly varying function of x, y . This may be useful to include a layer of clouds in our problem. Similarly, one can treat the case where ρ is not a constant, but for instance a function of z , by introducing an optical length defined as in (14). Typically, one could assume that $0 < \rho_m \leq \rho(z) \leq \rho_M < \infty$, and recast the radiative transfer equation in terms of the variable τ instead of z . Of course, this will modify the drift-diffusion operator in the left hand side of (37), but in a way that should be tractable by the same methods.

(2) One could enrich the class of boundary conditions considered here by taking into account the albedo coefficients of the boundary at $z = 0$ and $z = Z$. This should lead to more serious modifications of the strategy discussed above, but

we expect that some of our results can be modified to handle these more general boundary conditions.

(3) Until now, we have treated the case of an incompressible fluid with constant density. This is the reason for the factor c_P/c_V multiplying the heat diffusivity. One can treat in the same manner the case of low Mach number flows of a compressible fluid, typically a gas, such as Earth's atmosphere, in which case the prefactor c_P/c_V multiplying the heat diffusivity should be replaced with 1. The reason for this difference is due to the work of the pressure in the case of a compressible fluid at low Mach number, which is not identically zero, at variance with the case of an incompressible fluid with constant density: see footnotes 46 and 47 on p. 107 of [26], or footnote 6 on p. 93 in [25] for a detailed explanation. (In the case of water at 20°C, one finds that $c_P/c_V = 1.007$, so that this ratio is very close to 1 for all practical purposes.)

(4) Including Boussinesq's approximation in our model in order to take into account the buoyancy created by the temperature dependence of the density is a more difficult problem — in the first place because the motion equation of the fluid becomes coupled to the simple system considered here. We keep this problem for future work.

7 Numerical Simulations

This section is meant to show that (1) and (2), proposed in the previous sections, is implementable, robust and computationally fairly fast. Here, robustness means that there are no singular integrals and convergence is not subject to the adjustment of sensitive parameters; in other words, the mathematical properties derived above are observed numerically.

Two computer programs have been written: one in **C++** for the case $\kappa_T = 0$ and the other in the **FreeFEM** language[16] for the general case, either in cartesian coordinates or in spherical ones.

The programming is straightforward except at two places:

1. Writing a function to compute the exponential integrals is simple due to two formulas

$$\begin{aligned} E_1(x) &= -\gamma - \ln x + \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^k}{k k!}, & \gamma &= 0.577215664901533, \\ E_{n+1}(x) &= \frac{e^{-x}}{n} - \frac{x}{n} E_n(x), \end{aligned} \tag{46}$$

but the series is destroyed by machine precision if $x > 18$. From practical purpose keeping $9 + (x - 1) * 5$ terms in the series is more than enough.

2. When thermal diffusion is neglected, one must solve for T , with given J_ν

$$\int_0^\infty \kappa_\nu(1 - a_\nu) B_\nu(T) d\nu = \int_0^\infty \kappa_\nu(1 - a_\nu) J_\nu d\nu.$$

Newton iterations are used combined with dichotomy.

3. When thermal diffusion is not neglected, the temperature equation has a similar nonlinearity which requires iterations. We use the time dependent problem, discretized by a method of characteristics, as follows, which is unconditionally stable:

$$\begin{aligned} \frac{1}{\delta t}(T^{m+1}(x) - T^m(x - \delta t u(x))) - \kappa_T \Delta T^{m+1} + \int_0^\infty \kappa_\nu(1 - a_\nu) B_\nu(T^{m+1}) d\nu \\ = \int_0^\infty \kappa_\nu(1 - a_\nu) J_\nu d\nu, \end{aligned} \quad (47)$$

with Dirichlet or Neumann conditions on the boundaries. Then a standard P^1 Finite Element approximation of the temperature equation is applied for the discretization in a finite dimensional space V_h . Here J_ν and T^m are known. Then the numerical approximation of T^{m+1} is also the solution of the minimization problem below, which can be solved by a BFGS method:

$$\begin{aligned} \min_{T \in V_h} \int_\Omega \left[\frac{T^2}{2\delta t} + \frac{\kappa_T}{2} |\nabla T|^2 + \int_0^\infty \left(\kappa_\nu(1 - a_\nu) \int_0^T B_\nu(T') dT' \right) d\nu \right] dx \\ - \int_\Omega T \left(\frac{1}{\delta T} T^m(x - \delta t u(x)) + \int_0^\infty \kappa_\nu(1 - a_\nu) J_\nu d\nu \right) dx \end{aligned} \quad (48)$$

The first test is for the radiative transfer system decoupled from the temperature equation. The second test involves the complete system in 2D and the third is also with radiative transfer coupled with the temperature equation but in 3D.

7.1 Radiative Transfer in the Troposphere without Thermal Diffusion

The troposphere is roughly 12km thick. When air density is $\rho(z) = \rho_0 e^{-z}$, with $\rho_0 = 1.225 \cdot 10^{-3}$, a change of vertical coordinate is made, $\tau = 1 - e^{-z}$ to remove the exponential from the equations; thus $\tau \in (0, Z)$, $Z = 1 - e^{-12}$.

If $\bar{\kappa}_\nu$ is the mass-extinction coefficient, $\kappa_\nu = \rho_0 \bar{\kappa}_\nu$, is the absorption coefficient, for which one may forget its atomic origin and define it as a dimensionless parameter between 0 and 1 which measures the output to input ratio of ν -light crossing an horizontal kilometer of air layer.

The problem is: find $I_\nu(\tau, \mu)$ and $T(\tau)$ such that

$$\begin{aligned} \mu \partial_\tau I_\nu + \kappa_\nu I_\nu = \frac{\kappa_\nu a_\nu}{2} \int_{-1}^1 p(\mu, \mu') I_\nu(\tau, \mu') d\mu' + \kappa_\nu(1 - a_\nu) B_\nu(T) \quad \forall \mu \in (-1, 1), \\ \int_0^\infty \kappa_\nu(1 - a_\nu) \left(B_\nu(T) - \frac{1}{2} \int_{-1}^1 I_\nu d\mu \right) d\nu = 0, \quad \forall \tau, \nu \in (0, Z) \times \mathbb{R}^+. \end{aligned} \quad (49)$$

The boundary conditions used by [9] are considered:

$$I(0, \mu)|_{\mu > 0} = Q_\nu \mu, \quad I(Z, \mu)|_{\mu < 0} = 0. \quad (50)$$

They imply that Earth's surface receives sunlight energy $\nu \mapsto Q_\nu = Q_{Sun} B_\nu(T_{Sun})$, $T_{Sun} = 5800K$, and no light rays come back ($\mu < 0$) from the top of the troposphere. Due to Planck's law for black bodies, Earth radiates (infrared) light up ($\mu > 0$) which escapes at $\tau = Z$ without back-scattering.

The frequency spectrum of interest is $\nu \in (0, 20 \cdot 10^{14})$. It is convenient to rescale some variables:

$$\nu' = 10^{-14}\nu, \quad T' = 10^{-14} \frac{k}{\hbar} T = 10^{-14} \frac{1.381 \cdot 10^{-23}}{6.626 \cdot 10^{-34}} T = \frac{T}{4798},$$

so as to write

$$B_\nu(T) = B_0 \frac{\nu'^3}{e^{\frac{\nu'}{T'}} - 1}, \quad \text{with } B_0 = \frac{2\hbar}{c^2} 10^{42} = \frac{2 \times 6.626 \cdot 10^{-34}}{2.998^2 \cdot 10^{16}} 10^{42} = 1.4744 \cdot 10^{-8}.$$

We may work with B_ν/B_0 and I_ν/B_0 so that, forgetting the primes, we have (49) and (50) with

$$B_\nu(T) = \frac{\nu^3}{e^{\frac{\nu}{T}} - 1}, \quad Q_\nu = Q_0 B_\nu(1.209), \quad Q_0 = 2.03 \cdot 10^{-5}, \quad (51)$$

because T_{Sun} is now $5800/4798 = 1.209$; Q_0 is derived from the knowledge of the energy sent by the Sun to the troposphere, $Q_{Sun} = 1370 \text{Watt}/m^2$, while from (51) it is

$$\int_0^1 \mu \int_0^\infty Q_0 B_0 B_\nu(1.209) 10^{14} d\nu = \frac{1}{2} Q_0 1.4744 \cdot 10^6 \frac{(1.209\pi)^4}{15} = 1.023 \cdot 10^7 Q_0.$$

This leads to $Q_0 = 9.03 \cdot 10^{-5}$; but this value is too high as it gives an Earth temperature around 430K. So it is corrected by the latitude, $\frac{1}{\sqrt{2}}$ at 45° , and by the Earth albedo: 36% of the Sun energy is reflected, i.e. not absorbed, by the Earth surface. Furthermore due to the alternation of days and nights only half the final value is retained [9]. With such a value, i.e. $Q_0 = 2.03 \cdot 10^{-5}$, and a constant $\kappa = 0.5$, the temperature near the ground is found to be around 24°C ; note that it cannot be taken for its face value because rains, clouds etc, are not taken into account.

Scattering is the sum of an isotropic part and a Rayleigh part; both have their own a_ν , function of altitude (i.e. on τ) and ν .

To simulate clouds, isotropic scattering is activated between altitude Z_1 and $Z_2 > Z_1$ and

$$a_\nu(z) = \alpha \max(z - Z_1, 0) \max(Z_2 - z, 0) * 4/(Z_2 - Z_1)^2.$$

It is known that Rayleigh is function of ν^4 in the ultraviolet range at high altitude, so this scattering is switched on beyond altitude Z_2 and is $O(\nu^4)$ for $\nu \in (0.8, 1.2)$:

$$a_\nu(z) = \alpha (\max(\nu - 0.8, 0))^2 \max(1.2 - \nu, 0)^2 \max(z - Z_2, 0)/(Z - Z_2) \cdot 40$$

. The values of the physical and numerical parameters are

- $\alpha = \frac{1}{2}$ or zero; , $Z_1 = 6\text{km}$, $Z_2 = 9\text{km}$
- Absorption coefficient κ_ν digitalized from Gemini measurements.
- Discretization: 60 altitude stations, 485 frequencies corresponding to a uniform grid in wavelength in $(1,20)\mu\text{m}$.
- Number of iterations 20.

The Gemini measurements of the absorption are posted on wikipedia in <https://www.gemini.edu/observing/telescopes-and-sites/sites#Transmission> Figure 1 shows κ_ν^0 versus wavelength c/ν . Recall that visible light is in the range $0.4 - 0.7\mu\text{m}$ (i.e. 450-750 THz) and relevant infrared radiations are in the range $0.8 - 20\mu\text{m}$ (i.e. 0.03 - 0.4 THz).

To assess the sensitivity of the temperature to opaque gas like carbon dioxide and methane we constructed κ_ν^1 by increasing κ_ν^0 by a 1.5 factor in the infrared range $2 - 3\mu\text{m}$. Similarly we construct κ_ν^2 by increasing κ_ν^0 by a 1.5 factor in the range $8 - 14\mu\text{m}$. These are displayed in Figure 1.

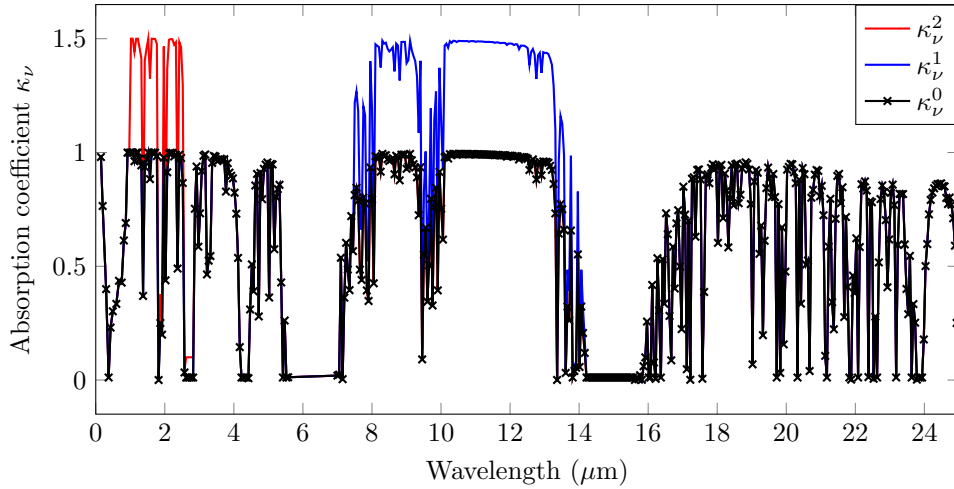


Figure 1: Absorption κ_ν^0 read from Gemini measurements; κ_ν^1 , is κ_ν^0 increased in the infrared range $2 - 3\mu\text{m}$ and κ_ν^2 is κ_ν^0 increased in the range $8 - 14\mu\text{m}$. The \times marks show the 487 grid points for the integrals in ν . Enhanced high values are truncated at $\kappa = 1.5$.

Convergence of the lower increasing and upper decreasing sequences is studied with and without Rayleigh scattering.

The convergence of the lower sequences is faster and it is slightly slower in the presence of scattering. Yet for both 20 iterations seem appropriate for a 3 digits precision.

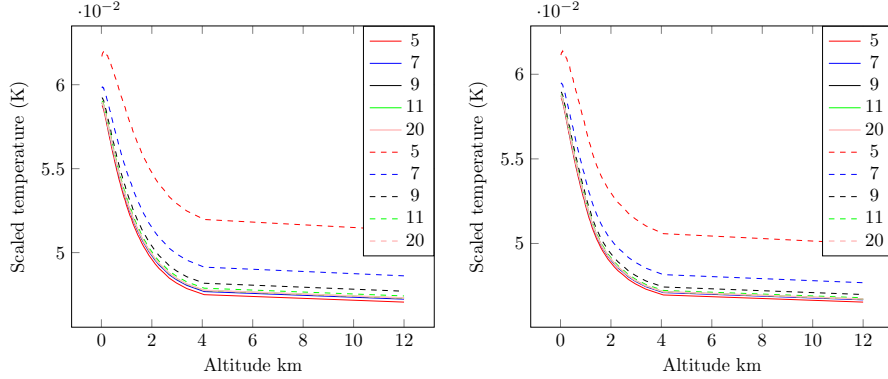


Figure 2: Temperatures scaled by 4798 without (left) and with (right) scattering: convergence history. The dashed curves are computed with an initial $T^0 = T_{Sun}/10$ and the solid curves with $T^0 = 0$. Notice the monotonic convergence towards a solution after 20 iterations. The iterations shown for the upper and lower solutions are (5,7,9,11,20). This computation has used $Q_0 = 3.03 \cdot 10^{-5}$.

Next, results are shown with κ_ν^0 , κ_ν^1 and κ_ν^2 , with and without scattering. Figures 3 and 4 shows the mean radiation intensity J_ν versus wavelength at altitude 0 and 12km. Notice the dramatic changes when going from κ_ν^0 to κ_ν^1 and the smaller changes in the opposite direction when going from κ_ν^0 to κ_ν^2 . Note too that scattering increases only slightly J_ν .

Figure 5 shows the scaled temperatures versus altitude for κ_ν^0 , κ_ν^1 and κ_ν^2 with and without scattering. Note that going from κ_ν^0 to κ_ν^1 decreases the temperatures by 5%. On the other hand going from κ_ν^0 to κ_ν^2 increases the temperatures by 2%.

Comments

- CPU time is 20" on an Macbook air M1, but with a smoother κ_ν , 50 ν -integration points are sufficient, cutting the CPU time by 10 to 2".
- We observed that a highly oscillating κ_ν did not cause any programming or convergence problems. The total light intensities J plotted on Figures 3 and 4 show clearly that the method tracts the small or wide changes on κ_ν .
- Figure 2: Monotone convergence from below and from above is observed. The convergence from below, i.e. starting with $T^0 = 0$, is faster than the

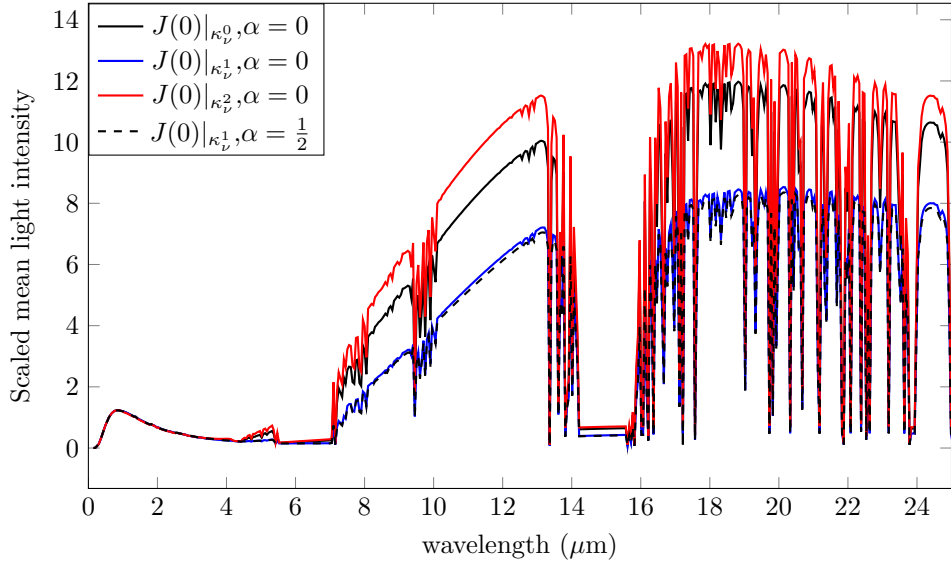


Figure 3: Computed mean radiation intensities $J_\nu(0)$ at the ground level for κ_ν^0 , κ_ν^1 , κ_ν^2 without scattering and for κ_ν^0 with scattering.

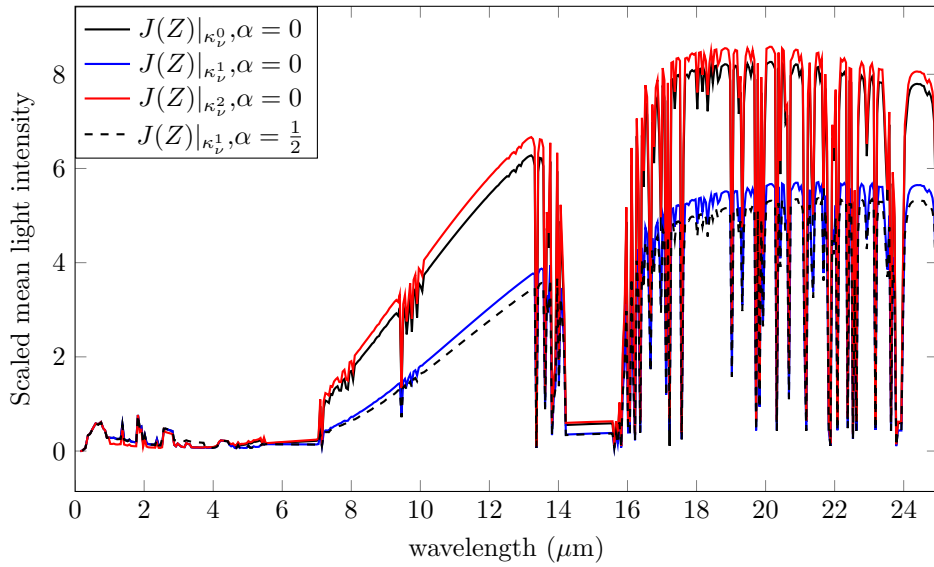


Figure 4: Computed mean radiation intensities $J_\nu(Z)$ at the top of the troposphere for κ_ν^0 , κ_ν^1 , κ_ν^2 without scattering and for κ_ν^0 without and with scattering.

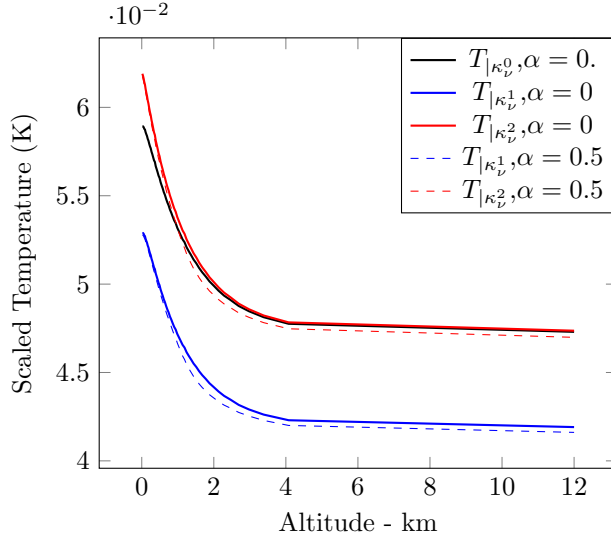


Figure 5: Temperatures in Kelvin divided by 4798 $z \rightarrow T(z)$ computed with κ_ν^0 , κ_ν^1 and κ_ν^2 without scattering ($\alpha = 0$) and with a scattering $\alpha = \frac{1}{2}$.

one from above, starting from $T = T_{sun}/10$, and it is slightly slower in the presence of scattering.

- . Figure 5: Increasing κ_ν in the Earth infrared range can cause either an increase or a decrease of temperature, depending on the position of the change in the infrared spectrum.
- Isotropic and Rayleigh scattering did not change the above conclusion (see Figure 5).

Finally, note that the Earth albedo seems to play an important role on the effect of the green house gases on the temperature of the atmosphere [7]. If it is modeled by a Lambert condition of the type

$$I_\nu(0, \mu) - \beta I_\nu(0, -\mu) = \mu Q_0 B_\nu(T_{Sun}), \quad \forall \mu > 0,$$

then the present numerical method can handle it and our preliminary test show an increase of temperature when β increases; while this is another story, it is yet another proof of the versatility of the present numerical formulation for climate modeling.

7.2 Radiative Transfer with Thermal Diffusion in a Pool

The liquid water absorption spectrum can be found in

https://en.wikipedia.org/wiki/Electromagnetic_absorption_by_water

Consider a pool, heated by the Sun and subject to wind on its surface. The maximum length and height are 3 and 1. The numerical viscosity is 0.05. Rescaled to be in (0,1) it is approximately

$$\kappa_\nu = \min\{1, 0.6 \cdot \left(\frac{3.2}{\nu} - 0.4\right)^+\}^{\frac{1}{4}}.$$

The solution of the time dependent Navier-Stokes equations in a vertical cut of the pool is shown on Figure 6 after 100 time steps of size 0.02; stationarity is reached. Dirichlet conditions are applied due to the wind velocity, $(10, 0)^T$ on the horizontal boundary and $(0, 0)^T$ elsewhere. The Hood-Taylor finite element method is used with Galerkin-characteristics discretization in time.

For the temperature equation, $\kappa_T = 0.5$, $\kappa_\nu = 1$ and $Q_0 = 2 \cdot 10^{-5}$, with vertical radiative transfer in the fluid, from its surface and Dirichlet conditions on the bottom boundary $T = 0.0572$ which is the temperature given by the radiative transfer equations without thermal diffusion.

The time dependent temperature equation is solved until convergence to a stationary state with 50 time steps of size 0.1. The convection terms are treated explicitly so as to use (48). Note that with a Neumann condition on the bottom the temperature would keep rising with time and even with a Dirichlet condition on the bottom boundary there is a critical value for κ_T below which the temperature rises with time.

The solution is shown on Figure 6. One sees the effect of the current in the fluid on the temperature distribution which has shifted to the right. There are

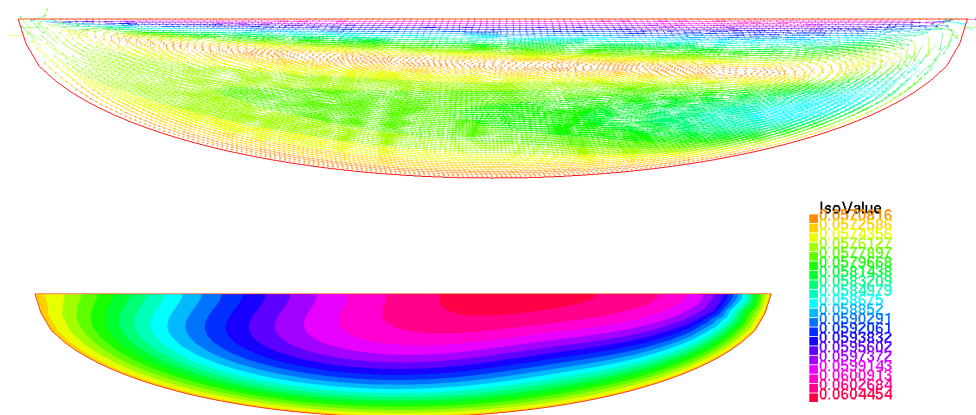


Figure 6: Velocity vectors and Temperature in a pool subject to wind on its top boundary and given temperature on the bottom.

1157 vertices in the triangulation; the computation of the flow takes 12". The computation of the temperature takes 126".

7.3 Radiative Transfer with Thermal Diffusion in the Atmosphere of a planet

Consider the atmosphere of a spherical planet with a known ground temperature T_e , heated by the Sun. The computational domain is the space between a sphere of radius R_2 and a sphere of radius $R_1 < R_2$.

As before the sunrays travel unaffected and hit the ground; so the radiative part is governed by the first equation in (49) and (50). The second equation in (49) is replaced by (47), solved with spherical coordinates. The density of the atmosphere is constant and the absorption parameter is constant $\kappa = 1$. The wind velocity is a rotating Poiseuille flow around an axis $(\sin \bar{\psi}, 0, \cos \bar{\psi})^T$ which is not aligned with the direction of the Sun. In spherical coordinates it is

$$u = r(H - r)[\cos \psi, \sin \psi, 0]^T, \quad r \text{ is the distance to the ground}$$

In spherical coordinates the domain becomes a solid rectange with periodic conditions; it is discretized with a uniform distribution of vertices $24 \times 12 \times 12$ in the domain $(0, 2\pi) \times (0, \pi) \times (0, Z)$.

The time dependent temperature equation, are solved in spherical coordinates (see [15]-appendix A), discretized in time and space by a Galerkin-Characteristic method and piecewise linear conforming finite elements on tetraedras. The time step is $\delta t = 0.1$, the thermal diffusion is $\kappa_T = 0.01$. The stratified approximation requires R_1 to be large and $R_2 - R_1$ small. But for the visualizations we map the solid rectangle onto the spherical domain with $R_1 = 1$ and $R_2 = 2$.

As before $\kappa = 1$, $T_{Sun} = 1.209$ and $Q_0 = 2 \cdot 10^{-5}$. Initially $T_{t=0}$ is the temperature when $\kappa_T = 0$. Figure 7 shows the temperatures after 15 iterations without wind. The computing time was 108". The Sun is on the right. Blue means cold on the left side because it does not receive the light. Yet with more time iterations we would see this zone heated by thermal diffusion due to the fixed temperature of the planet.

Figure 8 compares the temperatures with and without wind. The planar views correspond to cross sections of the domain by the plane $z = 0$. Here the Sun in the horizontal direction on the right but the wind transports its heat counterclockwise.

7.4 Conclusion

In this article a special case of radiative and heat transport has been studied, the so called stratified approximation. Existence and uniqueness has been established with almost no restriction on the absorption and scattering parameters. Furthermore the proofs are based on a formulation of the problem which gives rise to an efficient numerical algorithm for radiative transfer coupled with the heat equation for a fluid. Upper and lower positive solutions can be computed and the convergence to the unique solution is polynomial.

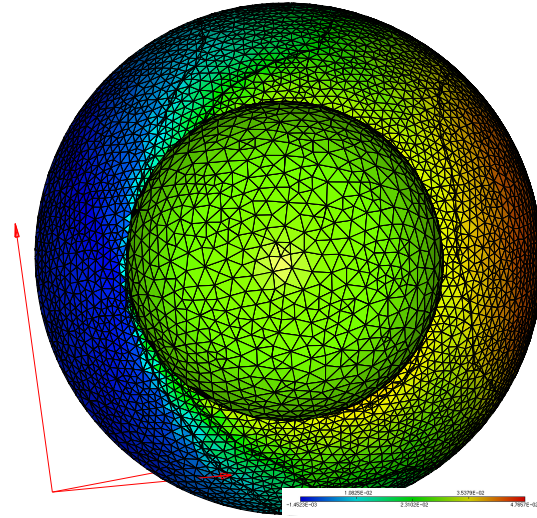


Figure 7: Temperature in the atmosphere of a planet heated by a Sun, when thermal diffusion propagates heat in unlit regions and also in the presence of a counter clockwise rotating wind.

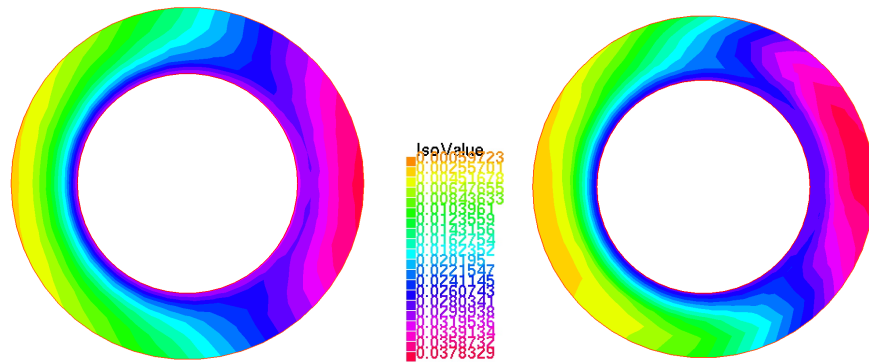


Figure 8: Temperature in the atmosphere of a planet heated by a Sun on the right with (right) and without (left) almost counterclockwise rotating wind (the axis of rotation is not perpendicular to the figure). Thermal diffusion propagates heat in unlit regions and the wind transports the heat counterclockwise.

The method has been implemented numerically and indeed arbitrary precision can be obtained, even with highly oscillating absorption or scattering coefficients. Furthermore it is computationally very fast when the thermal diffusion is neglected and reasonably fast otherwise.

It has been applied to the computation of the temperature in the Earth atmosphere, to that of a pool heated by the Sun and to a the atmosphere of a planet with a large thermal diffusion. However these are test cases rather than full solution of physical problems.

There are many other applications, especially for climate modelling and in nuclear engineering for which these new mathematical and numerical results should be useful.

Acknowledgments

The authors would like to thank Prof. Claude Bardos for the numerous discussions and references given.

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