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Research Article

Properties of Total Transformation Graphs for General Sum-Connectivity Index

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The study of networks and graphs through structural properties is a massive area of research with developing significance. One of the methods used in studying structural properties is obtaining quantitative measures that encode structural data of the whole network by the real number. A large collection of numerical descriptors and associated graphs have been used to examine the whole structure of networks. In these analyses, degree-related topological indices have a significant position in theoretical chemistry and nanotechnology. Thus, the computation of degree-related indices is one of the successful topics of research. The general sum-connectivity (GSC) index of graph $\mathcal{Q}$ is described as $\chi_\rho(\mathcal{Q}) = \sum_{\mathfrak{q}\mathfrak{q}\prime\in E(\mathcal{Q})}(d^{(\mathfrak{q})} + d^{(\mathfrak{q}')})^\rho$, where $d^{(\mathfrak{q})}$ presents the degree of the vertex $\mathfrak{q}$ in $\mathcal{Q}$ and ρ is a real number. The total graph $T(\mathcal{Q})$ is a graph whose vertex set is $V(\mathcal{Q}) \cup E(\mathcal{Q})$, and two vertices are linked in $T(\mathcal{Q})$ if and only if they are either adjacent or incident in $\mathcal{Q}$. In this article, we study the general sum-connectivity index $\chi_\rho(\mathcal{Q})$ of total graphs for different values of ρ by using Jensen's inequality.

1. Introduction

Chemical graph theory [1] has given a massive area of research to chemists in which they apply graph theory to mathematical modeling of chemical phenomena. This area of research necessitates characteristics of structures, in which researchers deal with their grouping, properties, and ordering. The molecular graph furnishes a graph-theoretical depiction of the molecule and lays out gainful knowledge into the chemical phenomena. The atoms of a molecule are described as vertices, and links signify the covalent bonds between the atoms. A topological index is represented by a single number corresponding with a graph. There are a lot of categories of topological indices, some of them being distance-, counting-, eccentricity-, and degree-related indices and polynomials of graphs. Among the abovementioned categories of indices, the degree-dependent indices show a beneficial character in the chemical graph theory.

Here, we examine only simple and connected graphs. Let $\mathcal{Q}$ be the graph with vertex and edge sets, $V(\mathcal{Q})$ and $E(\mathcal{Q})$, respectively. For a vertex $\mathfrak{q}$ of $\mathcal{Q}$, $d^{(\mathfrak{q})}$ denotes the degree of $\mathfrak{q}$ which is described as the number of its incident edges. The Randić index was described in [2] as

$$\mathcal{R}(\mathcal{Q}) = \sum_{\mathfrak{q}\mathfrak{q}\prime\in E(\mathcal{Q})} \frac{1}{\sqrt{d^{(\mathfrak{q})} d^{(\mathfrak{q}')}}}. \tag{1}$$

It is one of the most popular molecular invariants in the study of structure-property and activity relationships [3, 4]. The general Randić index is described as follows:

$$\mathcal{R}_\rho(\mathcal{Q}) = \sum_{\mathfrak{q}\mathfrak{q}\prime\in E(\mathcal{Q})} \left(d^{(\mathfrak{q})} d^{(\mathfrak{q}')}\right)^\rho, \tag{2}$$

where ρ is a real number. The sum-connectivity index $\chi_{-1/2}(\mathcal{Q}) = \sum_{\mathfrak{q}\mathfrak{q}'\in E(\mathcal{Q})}(d^{(\mathfrak{q})} + d^{(\mathfrak{q}')})^{-1/2}$ is introduced by Zhou and Trinajstić [5]. Zhou and Trinajstić [6] investigated the idea of the general sum-connectivity index with the help of sum-connectivity index and defined it in following way:

$$\chi_\rho(\mathcal{Q}) = \sum_{\mathbf{qq}' \in E(\mathcal{Q})} \left(d^{(\mathbf{q})} + d^{(\mathbf{q}')}\right)^\rho, \tag{3}$$

where ρ is a real number. The Randić and sum-connectivity indices relate with the π-electron energy of benzenoid hydrocarbons [7]. A lot of extremal characteristics of the general sum-connectivity index for certain graph families were given [8–16]. For additional studies on extremal graphs with topological indices, we refer [17–23].

The total $T(\mathcal{Q})$ graph is the graph with $V(T(\mathcal{Q})) = V(\mathcal{Q}) \cup E(\mathcal{Q})$ and $\mathbf{qq}' \in E(V(\mathcal{Q}) \cup E(\mathcal{Q}))$ if and only if $\mathbf{q}$ and $\mathbf{q}'$ are either incident or linked in $\mathcal{Q}$. The total graphs of paths P_n and cycle C_6 are shown in Figure 1. Some characteristics of total graphs are described in the literature [24–29].

A graph whose vertex set is converted into two linked vertex sets $\mathcal{Q}_1$ and $\mathcal{Q}_2$, such that every vertex in $\mathcal{Q}_1$ is attached to every vertex in $\mathcal{Q}_2$, is recognized as a complete bipartite $K_{a,b}$ graph. A star graph S_n is a complete bipartite $K_{1,n}$ graph. The n-hypercube graph Q_l can be defined as a graph with vertices being a set of ordered l-tuples of $0's$ and $1's$ and whose vertices are joined if and only if ordered l-tuples differ in exactly one partition.

2. Parametric Inequalities

This section provides certain important lemmas that will be beneficial in Section 3 to derive the smallest and largest general sum-connectivity index for different graphs corresponding to their total transformations.

Lemma 1. *Let $v \geq 3$ be a positive integer; then, the function defined by*

$$f(v) = 2^{\rho-1} v (v-1)^{\rho+1} \left[1 - 2^\rho (v+1)\right], \tag{4}$$

is a strictly decreasing and a strictly increasing function for $\rho > -3$ and $\rho < -3$, respectively.

Proof. However, we have

$$\begin{aligned} f'(v) &= 2^{\rho-1}(v-1)^\rho \left[2v + \rho v - 1 - 2^\rho\left(3v^2 + \rho v^2 + \rho v - 1\right)\right] \\ &= 2^{\rho-1}(v-1)^\rho g(v), \end{aligned} \tag{5}$$

where the function $g(v)$ is

$$g(v) = 2v + \rho v - 1 - 2^\rho\left(3v^2 + \rho v^2 + \rho v - 1\right). \tag{6}$$

Thus, we obtain

$$g''(v) = -2^{\rho+1}(3+\rho) < 0, \tag{7}$$

for $\rho > -3$, and

$$g''(v) = -2^{\rho+1}(3+\rho) > 0, \tag{8}$$

for $\rho < -3$. Consequently, we have $f'(v) < 0$ and $f'(v) > 0$ when $\rho > -3$ and $\rho < -3$, respectively. Therefore $f(v)$ is a strictly decreasing function when $\rho > -3$ and is a strictly increasing function when $\rho < -3$, which finishes the proof. □

Lemma 2. *Let o, $t \geq 2$ be positive integers; then, the function defined by*

$$g(o,t) = t(o+t)^{\rho-1}(o+t+\rho o). \tag{9}$$

is a strictly increasing function for $\rho > 0$.

Proof. We obtain

$$\frac{\partial g(o,t)}{\partial o} = \rho t (o+t)^{\rho-2}(o+2t+\rho o) > 0, \tag{10}$$

for every $\rho > 0$, and

$$\frac{\partial g(o,t)}{\partial t} = (o+t)^{\rho-2}\left[(1+\rho)o^2 + \rho t^2 + (1+\rho)ot + \rho^2 o\right] > 0, \tag{11}$$

if and only if we have $(1+\rho)o^2 + \rho t^2 + (1+\rho)ot + \rho^2 o > 0$ for $o, t \geq 2$ and $\rho > 0$. Hence, $g(o,t)$ is a strictly increasing function when $\rho > 0$. □

Lemma 3. *Let $o, t \geq 2$ be positive integers; then, the function defined as*

$$\theta(o,t) = ot2^\rho(o+t)^\rho, \tag{12}$$

is a strictly increasing function for $\rho > 0$.

Proof. By taking derivative of $\theta(o,t)$, we acquire

$$\frac{\partial \theta(o,t)}{\partial o} = 2^\rho t (o+t)^{\rho-1}(o+t+\rho o) = 2^\rho t (o+t)^{\rho-1}\psi(o,t), \tag{13}$$

where $\psi(o,t) = o + t + \rho o$. The function $\psi(o,t)$ is increasing if and only if $\rho > 0$. Thus, we obtain $(\partial\theta(o,t)/\partial o) > 0$ and, similarly, we obtain $(\partial\theta(o,t)/\partial t) > 0$ when $\rho > 0$. Therefore, the function $\theta(o,t)$ is strictly increasing function when $\rho > 0$. □

Lemma 4. *Let the function be defined by*

$$\phi(o,t) = ot(o+t)^\rho\left[1 - 2^\rho(o+2)\right], \tag{14}$$

for $o, t \geq 2$, where o and t are the integers. Then, $\phi(o,t)$ is a decreasing function when $\rho > 0$.

Proof. It is necessary to show that $(\partial\phi(o,t)/\partial o) < 0$ and $(\partial\phi(o,t)/\partial t) < 0$ for each $o, t \geq 2$. We acquire

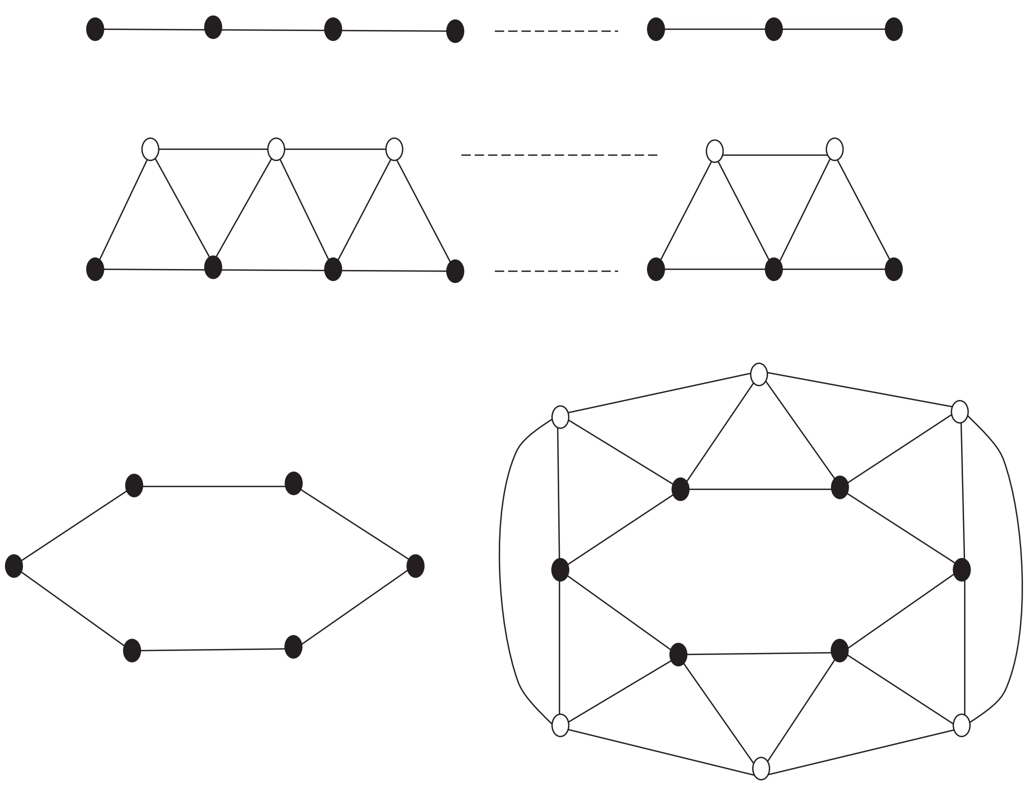

Figure 1: P_n, $Ts(P_n)$, C_6, and $T(C_6)$.

$$\frac{\partial\phi(o,t)}{\partial o} = t(o+t)^{\rho-1}(o+t+\rho o)\left[1-2^{\rho}(o+2)\right] - ot(o+t)^{\rho}2^{\rho} = g(o,t)h(o) - \theta(o,t), \tag{15}$$

where $g(o,t) = t(o+t)^{\rho-1}(o+t+\rho o)$, $h(o) = 1 - 2^{\rho}(o+2)$, and $t(o,t) = ot(o+t)^{\rho}2^{\rho}$.

By Lemma 2, $g(o,t)$ is a strictly increasing function when $\rho > 0$. Also, $h'(o) = 1 - 2^{\rho} < 0$ for each $\rho > 0$; therefore, $h(o)$ is also a strictly decreasing function when $\rho > 0$.

From Lemma 3, $\theta(o,t)$ is an increasing function for $\rho > 0$, which implies that $(\partial\phi(o,t)/\partial o) < 0$ and, similarly, we have $(\partial\phi(o,t)/\partial t) < 0$ where $\rho > 0$ and $o, t \geq 2$. Hence, the function $\phi(o,t)$ is a decreasing function for $\rho > 0$. □

Lemma 5. *Let $u \geq 2$ be a positive integer; then,*

$$f_1(u) = 2^{u+\rho-1}u^{\rho+1}\left[1 - 2^{\rho}(u+2)\right], \tag{16}$$

is the strictly decreasing function for $\rho > -3$.

Proof. It is easily seen that

$$f_1'(u) = 2^{u+\rho-1}u^{\rho}\left[(u\ln 2 + \rho + 1)\left(1 - 2^{\rho}(u+2)\right) - 2^{\rho}u\right] = 2^{u+\rho-1}u^{\rho}g(u), \tag{17}$$

where $g(u) = (u\ln 2 + \rho + 1)(1 - 2^{\rho}(u+2)) - 2^{\rho}u$. Then, $g''(u) = -2^{\rho+1}\ln 2 < 0$ for $\rho > -3$. Consequently, we obtain that $f_1'(u) < 0$; thus, $f_1(u)$ is the strictly decreasing function for $\rho > -3$. □

Lemma 6 (see [30]). *Let $w > 0$ be a real number. If $\rho < 0$ or $\rho > 1$, then $(1+w)^{\rho} > 1 + \rho w$; but for $0 < \rho < 1$, $(1+w)^{\rho} < 1 + \rho w$.*

Lemma 7. *Let $u \geq 4$ be a positive integer; then, the function defined as*

$$f(u) = (u-1)\left[\left(1 - 2^{\rho-1}u\right)u^{\rho} - (u+2)^{\rho} - (3u-2)^{\rho}\right], \tag{18}$$

is a strictly decreasing function for $2 \leq \rho < 3$.

Proof. From $f(u)$, we obtain

$$f'(u) = (u-1)\Big[u^{\rho-1}\big\{(\rho+1)u - \rho - 2^{\rho-1}u^2(\rho+1) + 2^{\rho-1}u(\rho-1)\big\} - (u+2)^{\rho-1}(u+2+\rho) - (3u-2)^{\rho-1}(3u-2+\rho)\Big]. \tag{19}$$

Since $u^{\rho-1}$ is strictly convex, by using Jensen's inequality, we acquire that

$$(3u-2)^{\rho-1} > 4u^{\rho-1} - (u+2)^{\rho-1}, \tag{20}$$

which yields that

$$f\prime(u) < (u-1)\left[u^{\rho-1}\left\{(\rho-11)u - 5\rho + 8 - 2^{\rho-1}u^2(\rho+1) + 2^{\rho-1}u(\rho-1)\right\} - 2(u+2)^{\rho-1}(2-u)\right]$$
$$= (u-1)u^{\rho-1}\left[(\rho-11)u - 5\rho + 8 - 2^{\rho-1}u^2(\rho+1) + 2^{\rho-1}u(\rho-1) - 2(2-u)\left(1+\frac{2}{u}\right)^{\rho-1}\right]. \quad (21)$$

Since $1 \le \rho - 1 < 2$, by Lemma 6, we obtain that $(1+(2/u))^{\rho-1} > 1 + (2(\rho-1)/u)$. This implies that $f\prime(u) < (u-1)u^{\rho-1}g(u,\rho)$, where we obtain the function

$$g(u,\rho) = \frac{1}{u}\left[a(\rho)u^3 + b(\rho)u^2 + c(\rho)u + d(\rho)\right]. \quad (22)$$

In this result, an easy routine computation yields that $a(\rho) = \rho + 2^{\rho-1}(\rho+1) < 0$, $b(\rho) = 2^{\rho-1}(\rho-1) + \rho - 9 < 0$, $c(\rho) = -\rho < 0$, and $d(\rho) = -8(\rho-1) < 0$. Consequently, the function $f(u)$ is a strictly decreasing function for $2 \le \rho < 3$. □

3. Main results

The present section focuses on finding the general sum-connectivity χ_ρ of total graphs of path P_n and cycle C_n, complete K_n graph, complete bipartite $K_{a,b}$ graph, hypercube graph Q_n, star graph S_n, and unicyclic graph with a fixed pendant vertex for different values of ρ.

In the next theorem, we determine the graphs having the smallest and largest general sum-connectivity (GSC) index in the case of total transformation of path P_n.

Theorem 1. *Let $T(P_n)$ be the total graph of path P_n, where $n \ge 4$. Then, P_n has the smallest and largest general sum-connectivity (GSC) index when $\rho > -2$ and $\rho < -2$, respectively.*

Proof. It can be easily seen that

$$\chi_\rho(P_n) - \chi_\rho(T(P_n)) = (n-3)4^\rho - (4n-13)8^\rho + 2\cdot 3^\rho - 2\cdot 5^\rho - 2\cdot 6^\rho - 4\cdot 7^\rho. \quad (23)$$

For $u \ge 4$, the function defined as $\theta(u) = (u-3)4^\rho - (4u-13)8^\rho$ is strictly decreasing when $\rho > -2$. Therefore,

$$\chi_\rho(P_n) - \chi_\rho(T(P_n)) \le \theta(u) \le \theta(4) \quad (24)$$

and

$$\begin{aligned}\theta(4) &= 4^\rho - 3\cdot 8^\rho + 2\cdot 3^\rho - 2\cdot 5^\rho - 2\cdot 6^\rho - 4\cdot 7^\rho \\ &= 2\left(3^\rho - 5^\rho - 6^\rho\right) - \left(3\cdot 8^\rho + 4\cdot 7^\rho - 4^\rho\right) < 0.\end{aligned} \quad (25)$$

This implies that $\chi_\rho(P_n) < \chi_\rho(T(P_n))$. Similarly, we obtain $\chi_\rho(P_n) > \chi_\rho(T(P_n))$ for $\rho < -2$, which finishes the proof.

In Theorem 2, we investigate the graphs with the smallest and largest general sum-connectivity (GSC) index in the case of total transformation of C_n. □

Theorem 2. *For every $n \ge 3$, let $T(C_n)$ be the total graph of cycle C_n. Then, the cycle C_n has a smallest and largest general sum-connectivity (GSC) index when $\rho > -2$ and $\rho < -2$, respectively. Moreover,*

$$\chi_\rho(C_n) = \chi_\rho(T(C_n)), \quad (26)$$

if and only if $\rho = -2$.

Proof. We have

$$\chi_\rho(C_n) - \chi_\rho(T(C_n)) = n\cdot 4^\rho - 4n\cdot 8^\rho. \quad (27)$$

For every $u \ge 3$, the function $\psi(u) = (4^\rho - 4\cdot 8^\rho)u$. Then, we acquire that $\psi\prime(u) = 4^\rho - 4\cdot 8^\rho < 0$, which shows that $\psi(u)$ is a decreasing function. Thus,

$$\chi_\rho(C_n) - \chi_\rho(T(C_n)) \le \psi(u) \le \psi(3). \quad (28)$$

Also, $\psi(3) = 3\cdot 4^\rho - 12\cdot 8^\rho < 0$ if and only if $(1/2)^\rho < 4$ which is proved for $\rho > -2$. Then, we obtain $\chi_\rho(C_n) < \chi_\rho(T(C_n))$ finally.

Moreover, it can be easily seen that $\psi(3) = 3\cdot 4^\rho - 12\cdot 8^\rho > 0$ if and only if $(1/2)^\rho > 4$, which is obeyed for $\rho < -2$. Then, we obtain in this case that $\chi_\rho(C_n) > \chi_\rho(T(C_n))$. For $\rho = -2$, we have $\chi_\rho(C_n) = \chi_\rho(T(C_n))$, which finishes the proof.

In the upcoming theorem, we determine graphs with the smallest and largest *GSC* index for the total transformation of the complete graph K_n. □

Theorem 3. *For $n \ge 3$, let $T(K_n)$ be the total transformation of K_n. Then, K_n has the smallest and largest GSC index for $\rho > -3$ and $\rho < -3$, respectively. Moreover,*

$$\chi_\rho(K_n) = \chi_\rho(T(K_n)), \quad (29)$$

if and only if we have $\rho = -3$ and $n = 3$.

Proof. As we have

$$\chi_\rho(K_n) - \chi_\rho(T(K_n)) = 2^{\rho-1}n(n-1)^{\rho+1}\left[1 - 2^\rho(n+1)\right], \quad (30)$$

for every $u \ge 3$, the function $\varphi(u) = 2^{\rho-1}u(u-1)^{\rho+1}[1 - 2^\rho(u+1)]$ is a decreasing function when $\rho > -3$ and increasing function for $\rho < -3$ by Lemma 1. Then, we acquire

$$\chi_\rho(K_n) - \chi_\rho(T(K_n)) \le \varphi(u) \le \varphi(3). \quad (31)$$

For all ρ, we have $\varphi(3) = 3(4^\rho - 4\cdot 8^\rho) < 0$ if and only if $(1/2)^\rho < 4$, which is proved for $\rho > -3$. Then, $\chi_\rho(K_n) < \chi_\rho(T(K_n))$. Similarly, $\varphi(3) = 3(4^\rho - 4\cdot 8^\rho) > 0$ if and only if $(1/2)^\rho > 4$, which is obeyed for $\rho < -3$. Thus, we finally obtain that $\chi_\rho(K_n) > \chi_\rho(T(K_n))$ with equality if and only if we have $\rho = -3$ and $n = 3$.

In the upcoming theorem, we derive graphs with the smallest *GSC* index for the total transformation of $K_{a,b}$. □

Theorem 4. *For $a, b \ge 2$, let $T(K_{a,b})$ be the total transformation of $K_{a,b}$. Then, the complete bipartite graph $K_{a,b}$ has the* *smallest GSC index for $\rho > 0$ with equality if and only if $\rho = -2$ and $a, b = 2$.*

Proof. We have

$$\chi_\rho\left(K_{a,b}\right) - \chi_\rho\left(T\left(K_{a,b}\right)\right) = ab(a+b)^\rho\left[1 - 2^\rho(a+2)\right]. \tag{32}$$

Let the function $\phi(u,v) = uv(u+v)^\rho\left[1 - 2^\rho(u+2)\right]$ for $u, v \geq 2$. By Lemma 4, we obtain that $\phi(u,v)$ is a decreasing function for $\rho > 0$. Thus, we have

$$\chi_\rho\left(K_{a,b}\right) - \chi_\rho\left(T\left(K_{a,b}\right)\right) \leq \phi(x,y) \leq \phi(2,2). \tag{33}$$

This is because for all ρ, we obtain $\phi(2,2) = 4(4^\rho - 4 \cdot 8^\rho) < 0$ if and only if $(1/2)^\rho < 4$, which is satisfied for $\rho > 0$. Thus, we obtain that

$$\chi_\rho\left(K_{a,b}\right) < \chi_\rho\left(T\left(K_{a,b}\right)\right), \tag{34}$$

for $x, y \geq 2$ and $\rho > 0$ with equality if and only if $\rho = -2$ and $a = b = 2$.

In the next theorem, we derive graphs with the smallest general sum-connectivity (GSC) index for the total transformation of the hypercubes Q_n. □

Theorem 5. *Let Q_n be the hypercube and $T(Q_n)$ be the total transformation of the hypercube. For every $n \geq 2$ and $\rho > -3$, the hypercube Q_n has the smallest general sum-connectivity (GSC) index and also*

$$\chi_\rho(Q_n) = \chi_\rho(T(Q_n)), \tag{35}$$

if and only if $n = 2$ and $\rho = -2$.

Proof. As we know,

$$\chi_\rho(Q_n) - \chi_\rho(T(Q_n) =)2^{n+\rho-1}n^{\rho+1}\left[1 - 2^\rho(n+2)\right]. \tag{36}$$

As the function defined by $f_1(u) = 2^{u+\rho-1}u^{\rho+1}\left[1 - 2^\rho(u+2)\right]$ is the strictly decreasing function by Lemma 5, $f_1(2) = 4(4^\rho - 4 \cdot 8^\rho) < 0$ if and only if $(1/2)^\rho < 4$, where $u \geq 2$ and $\rho > -3$. Thus, we acquire

$$\chi_\rho(Q_n) - \chi_\rho(T(Q_n)) \leq f_1(u) \leq f_1(2), \tag{37}$$

For all ρ, we have $f_1(2) = 4(4^\rho - 4 \cdot 8^\rho) < 0$ if and only if $(1/2)^\rho < 4$, which is satisfied for $\rho > -3$. Therefore,

$$\chi_\rho(Q_n) < \chi_\rho(T(Q_n)), \tag{38}$$

with equality if and only if we have $n = 2$ and $\rho = -2$, which finishes the proof.

Next, we present graphs with the smallest GSC index for the total transformation of the star graph S_n. □

Theorem 6. *For every $n \geq 4$, let $T(S_n)$ be the total transformation of star graph S_n. Then, the star graph S_n has the smallest GSC index where $2 \leq \rho < 3$.*

Proof. We easily acquire that

$$\chi_\rho(S_n) - \chi_\rho(T(S_n)) = (n-1)\left[\left(1 - 2^{\rho-1}n\right)n^\rho - (n+2)^\rho - (3n-2)^\rho\right]. \tag{39}$$

For every $u \geq 4$, let $f(u) = (u-1)\left[(1 - 2^{\rho-1}u)u^\rho - (u+2)^\rho - (3u-2)^\rho\right]$. Then, by using Lemma 7, the function $f(u)$ is a strictly decreasing function. Hence,

$$\chi_\rho(S_n) - \chi_\rho(T(S_n)) \leq f(u) \leq f(4) < 0. \tag{40}$$

This is because we have $f(4) = 3 \cdot 4^\rho - 3 \cdot 6^\rho - 6 \cdot 8^\rho - 3 \cdot 10^\rho < 0$ if and only if $((4^\rho - 6^\rho)/(2 \cdot 8^\rho + 10^\rho)) < 1$, which is satisfied by $2 \leq \rho < 3$. This implies that the star S_n has the smallest general sum-connectivity (GSC) index when $2 \leq \rho < 3$.

In the next theorem, we obtain graphs with the smallest and largest general sum-connectivity (*GSC*) index for the total transformation of the $\mathbb{U}_n^1$, where $\mathbb{U}_n^1$ is the unicyclic graph acquired by connecting a pendant vertex to a unique vertex of C_n. □

Theorem 7. *Let $T(\mathbb{U}_n^1)$ be the total transformation of $\mathbb{U}_n^1$. Then, for every $n \geq 3$, $\mathbb{U}_n^1$ has the smallest and largest general sum-connectivity (GSC) index when $\rho > -2$ and $\rho < -2$, respectively.*

Proof. We acquire

$$\chi_\rho\left(\mathbb{U}_n^1\right) - \chi_\rho\left(T\left(\mathbb{U}_n^1\right)\right) = (n-1)4^\rho - (4n-8)8^\rho + 2 \cdot 5^\rho - 6^\rho - 6 \cdot 9^\rho - 4 \cdot 10^\rho - 2 \cdot 11^\rho. \tag{41}$$

For every $u \geq 3$, let $f(u) = (u-1)4^\rho - (4u-8)8^\rho$ be a strictly decreasing function for $\rho > -2$. Then,

$$\chi_\rho\left(\mathbb{U}_n^1\right) - \chi_\rho\left(T\left(\mathbb{U}_n^1\right)\right) \leq f(u) \leq f(3). \tag{42}$$

For $\rho > -2$, we acquire

$$\begin{aligned} f(3) &= 2 \cdot 4^\rho - 8 \cdot 8^\rho + 2 \cdot 5^\rho - 6^\rho - 6 \cdot 9^\rho - 4 \cdot 10^\rho - 2 \cdot 11^\rho \\ &= 2\left(4^\rho + 5^\rho - 11^\rho\right) - 2\left(4 \cdot 8^\rho + 3 \cdot 9^\rho + 2 \cdot 10^\rho\right) - 6^\rho < 0. \end{aligned} \tag{43}$$

It follows that $\chi_\rho(\mathbb{U}_n^1) < \chi_\rho(T(\mathbb{U}_n^1))$. Similarly, one can also prove that

$$\chi_\rho\left(\mathbb{U}_n^1\right) > \chi_\rho\left(T\left(\mathbb{U}_n^1\right)\right), \tag{44}$$

for $\rho < -2$, which completes the proof. □

4. Conclusion

In this manuscript, we focus on finding the general sum-connectivity index in case of total graphs for a path P_n and cycle C_n, complete graph, complete bipartite graph $K_{a,b}$, hypercube graph Q_n, star graphS_n, and unicyclic graph with a fixed pendant vertex for different values of ρ. It would be most appealing to compute beneficial results for the extremal general sum-connectivity index of different graphs with fixed parameters and also to work on the Mostar, edge

Mostar and weighted Mostar indices for total graphs and certain transformations of graphs.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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