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Patrice Le Calvez

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HANDEL'S FIXED POINT THEOREM: A MORSE THEORETICAL POINT OF VIEW

PATRICE LE CALVEZ

Abstract. Michael Handel has proved in [Ha] a fixed point theorem for an orientation preserving homeomorphism of the open unit disk, that turned out to be an efficient tool in the study of the dynamics of surface homeomorphisms. The present article fits into a series of articles by the author [LeC2] and by Juliana Xavier [X1], [X2], where proofs were given, related to the classical Brouwer Theory, instead of the Homotopical Brouwer Theory used in the original article. Like in [LeC2], [X1] and [X2], we will use "free brick decompositions" but will present a more conceptual Morse theoretical argument. It is based on a new preliminary lemma, that gives a nice "condition at infinity" for our problem.

Keywords: Brick decomposition, Brouwer theory, translation arc

MSC 2020: 37B20, 37E30

À la mémoire du professeur Yu Jia Rong

1. Introduction

In this article, the unit circle $\mathbb{S} = \{z \in \mathbb{C} \,|\, |z| = 1\}$ of the complex plane is, as usual, oriented counterclockwise and this orientation induces an orientation on every interval of $\mathbb{S}$. If a and b are two distinct points of $\mathbb{S}$, we denote (a, b) the open interval of $\mathbb{S}$ that joins a to b for the induced orientation. We will denote $\mathbb{D} = \{z \in \mathbb{C} \,|\, |z| = 1\}$ the unit disk. Let f be a homeomorphism of $\mathbb{D}$ and $\Gamma \subset \mathbb{D}$ a simple loop that does not contain any fixed point of f. Recall that the *index* $i(f, \Gamma)$ is the degree of the map

$$s \mapsto \frac{f(\Gamma(s)) - \Gamma(s)}{|f(\Gamma(s)) - \Gamma(s)|},$$

where $s \mapsto \Gamma(s)$ is a parametrization of Γ defined on $\mathbb{S}$. It is well known that if $i(f, \Gamma) \neq 0$, the simply connected component of $\mathbb{D} \setminus \Gamma$ contains at least one fixed point of f.

Let us state now the main result of the article:

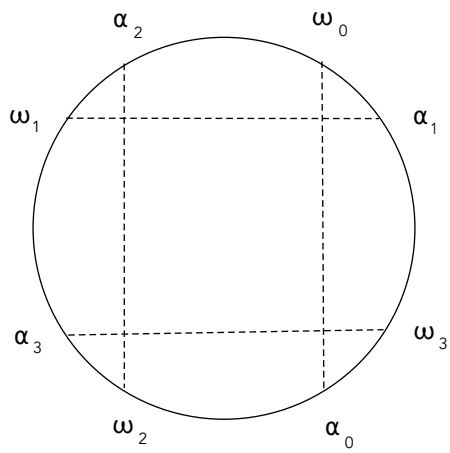
Theorem 1.1. *Let f be an orientation preserving homeomorphism of $\mathbb{D}$ and $p \geq 3$ an integer. We suppose that there exists a family $(z_i)_{i\in\mathbb{Z}/p\mathbb{Z}}$ in $\mathbb{D}$ and two families $(\alpha_i)_{i\in\mathbb{Z}/p\mathbb{Z}}$ and $(\omega_i)_{i\in\mathbb{Z}/p\mathbb{Z}}$ in $\mathbb{S}$, such that:*

- *the $2p$ points α_i, ω_i, $i \in \mathbb{Z}/p\mathbb{Z}$, are distinct;*
- *among these points, the unique one that belongs to (ω_{i-1}, ω_i) is α_{i+1};*
- *for every $i \in \mathbb{Z}/p\mathbb{Z}$, it holds that*

$$\lim_{k\to-\infty} f^k(z_i) = \alpha_i, \quad \lim_{k\to+\infty} f^k(z_i) = \omega_i;$$

- *f can be extended continuously to $\mathbb{D} \cup \left(\bigcup_{i\in\mathbb{Z}/p\mathbb{Z}}\{\alpha_i, \omega_i\}\right)$.*

Then, there exists a simple loop $\Gamma \subset \mathbb{D}$ that does not contain any fixed point of f, such that $i(f, \Gamma) = 1$. Consequently, f has at least one fixed point.



This theorem is usually called *Handel's fixed point theorem.* The original version (see [Ha]) is a little bit weaker: f is supposed to extend continuously to $\mathbb{S}$ and the conclusion is that f has at least one fixed point. It is a highly important result in the study of the dynamics of surface homeomorphisms. For very nice applications, see Franks [Fr2], [Fr3] or Matsumoto [M], for instance. The proof of Handel, which is judged very tough, is based on *Homotopical Brouwer Theory* which is the study of the mapping class group of the complement of finitely many orbits of a *Brouwer homeomorphism*, meaning a fixed point free orientation preserving homeomorphism of $\mathbb{D}$ (see [Ha], [LeR2] or [Ba]).

A proof of Theorem 1.1 was given in [LeC2], based on the classical *Brouwer Theory*, that used only topological arguments (and no geometrical argument). The fundamental fact is that the hypothesis of the theorem implies that f is *recurrent*, a property known to imply the conclusion. This property will be defined very soon in the text, like all the notions we will introduce in the following lines. The proof given in [LeC2] is a proof by contradiction. Assuming that the map is not recurrent, a preliminary lemma is proved in terms of *translation arcs*, giving some description of the dynamics in neighborhoods of the α_i and the ω_i. A *maximal free brick decomposition* can be constructed on the complement of the fixed point set, that "contains"

the dynamical elements described in the lemma. Then it is possible to show that this brick decomposition contains a *closed chain*, property that implies that the map is recurrent, which contradicts the hypothesis. A simplification of the proof was obtained by Xavier in [X1] and [X2]. Starting with the same preliminary lemma, a different argument was used to prove the existence of a closed chain in the brick decomposition, whose main interest is that it works for a wider class of brick decompositions that the ones considered in [LeC2]. It must be noticed that generalizations of Handel's fixed point theorem have been stated by Xavier, replacing the "elliptic" situation expressed in Theorem 1.1 by "hyperbolic" or "degenerate" situations.

In the present article, we display a new proof of Theorem 1.1 which follows the same outline as the proofs of [LeC2], [X1] and [X2]: first an introductory lemma, like in the cited articles, then a construction of a brick decomposition similar to the construction done by Xavier and then a proof of the existence of a closed chain. The introductory lemma used in the present article, slightly different from the one appearing in [LeC2], [X1] and [X2] is much easier to prove. It seems weaker than the old one but curiously turns out to be more efficient for our purpose. The arguments, relative to a brick decomposition, that appear later in the proof are more conceptual than those described in [LeC2], [X1] and [X2]: a Morse theoretical reasoning using a condition at infinity given by the introductory lemma. It must be pointed out that one of the nicest applications of Handel's fixed point theorem is the extension by Matsumoto of the Arnold conjecture to Hamiltonian surface homeomorphisms, and that the classical version for diffeomorphisms (see [Fl] or [Si]) is related to Morse theory and its extensions, like Floer homology, like so many results of symplectic topology.

The natural question being at the origin of [LeC2] is the following: can Theorem 1.1 be directly deduced from the *foliated version of Brouwer Translation Theorem* ? The preliminary lemma that is given here permits to answer positively to this question, at least to prove the existence of a fixed point.

Let us explain now the plan of the article. We will recall in the next section the principal results of Brouwer Theory and will define precisely the many objects introduced above. In a short Section 3 we will introduce definitions related to ordered sets that will be used throughout the text. In Section 4 we will recall classical results about brick decompositions on surfaces. The introductory lemma will be stated and proved in Section 5, and it will be used in Section 6 to construct a nice brick decomposition. The proof of Theorem 1.1 will be done in Section 7. Some supplementary comments will be done in Section 8. In particular the links with the foliated version of Brouwer Translation Theorem will be explained there. We have tried to make the article as self-contained as possible, adding precise references for the most technical details.

I would like to thank the referees for their careful reading and their relevant suggestions.

2. Recurrent homeomorphisms and Brouwer Theory

Usually, Brouwer Theory concerns with homeomorphisms of $\mathbb{C}$ but can be translated immediately to homeomorphisms of a topological set homeomorphic to $\mathbb{C}$. As we are interested in homeomorphisms of $\mathbb{D}$ in this article, we will state the results in $\mathbb{D}$.

An *open disk* of a topological surface S is a subset $V \subset S$ homeomorphic to $\mathbb{D}$. A *path* is a continuous map $\gamma : I \to S$, where I is a non trivial real interval. Suppose that γ is injective and proper: it is a *segment* if I is compact, a *line* if I is open, and a *half line* if $I = [a, +\infty)$ or $I = (-\infty, a]$. As usual, we will often identify a path and its image.

2.1. **Recurrent homeomorphisms.** Let S be a topological surface and f a homeomorphism of S. A subset $X \subset S$ is *f-free* if $f(X) \cap X = \emptyset$ (we will say simply *free* if there is no ambiguity). Following [LeC2], we say that f is *recurrent* if there exists a family of pairwise disjoint free open disks $(V_j)_{j\in\mathbb{Z}/r\mathbb{Z}}$, $r \geq 1$, and a family of positive integers $(k_j)_{i\in\mathbb{Z}/r\mathbb{Z}}$ such that $f^{k_i}(V_j) \cap V_{j+1} \neq \emptyset$, for every $j \in \mathbb{Z}/r\mathbb{Z}$ (such a family is usually called a *closed chain of free disks of length r*, see [Fr1]). This property is clearly invariant by conjugacy. A simple example of a recurrent homeomorphism of S is given by a homeomorphism that contains a non wandering free open disk, meaning a free open disk V such that there exists $k > 0$ such that $f^k(V) \cap V \neq \emptyset$. In particular, if the set of non wandering points is larger than the fixed point set, then f is recurrent. There exist more surprising examples of recurrent homeomorphisms, like the map $f : z \mapsto z/2$ defined on $\mathbb{C}$. Indeed the segment $\gamma : t \mapsto (8\pi - 6\pi t)e^{2i\pi t}$ defined on $[0, 1]$ is free but it holds that $f^2(\gamma) \cap \gamma \neq \emptyset$. So, one can find a free open disk V that contains γ and one knows that $f^2(V) \cap V \neq \emptyset$.

The fundamental property about recurrent homeomorphisms noticed by Franks in [Fr1] is the fact that a periodic orbit can be obtained by continuous deformation, without adding any fixed point. Indeed, suppose that f is recurrent and write $\mathrm{fix}(f)$ for the fixed point set. One can find an integer r such that there exists a free disks closed chain of length r but no free disks closed chain of length $< r$. Fix such a chain $(V_j)_{j\in\mathbb{Z}/r\mathbb{Z}}$. There exists a sequence of positive integers $(k_j)_{j\in\mathbb{Z}/r\mathbb{Z}}$, uniquely defined, such that:

- $f^{k_j}(V_j) \cap V_{j+1} \neq \emptyset$;
- $f^k(V_j) \cap V_{j+1} = \emptyset$, if $1 < k < k_j$.

Fix $z_j \in V_j \cap f^{-k_j}(V_{j+1})$ and then consider a homeomorphism h of S supported on $\bigcup_{j\in\mathbb{Z}/r\mathbb{Z}} V_j$ such that $h(f^{k_j}(z_i)) = z_{j+1}$, for every $j \in \mathbb{Z}/r\mathbb{Z}$. The minimality of r implies that the only disk V_l, $l \in \mathbb{Z}/r\mathbb{Z}$, met by $\bigcup_{k\geq 1} f^k(V_j)$ is V_{j+1}, and so $h \circ f$ admits a periodic point of period $\sum_{j\in\mathbb{Z}/r\mathbb{Z}} k_j > 1$. Moreover it holds that $\mathrm{fix}(h \circ f) = \mathrm{fix}(f)$. In fact, one can find an isotopy $(h_t)_{t\in[0,1]}$ joining Id to h such that every h_t is supported on $\bigcup_{j\in\mathbb{Z}/r\mathbb{Z}} V_j$ and so it holds that $\mathrm{fix}(h_t \circ f) = \mathrm{fix}(f)$, for every $t \in [0, 1]$.

2.2. **Brouwer Theory.** If λ is an oriented line of $\mathbb{D}$, denote $R(\lambda)$ the connected component of $\mathbb{D} \setminus \lambda$, lying on the right of λ, and $L(\lambda)$ the component lying on its left. If f is an orientation preserving homeomorphism of

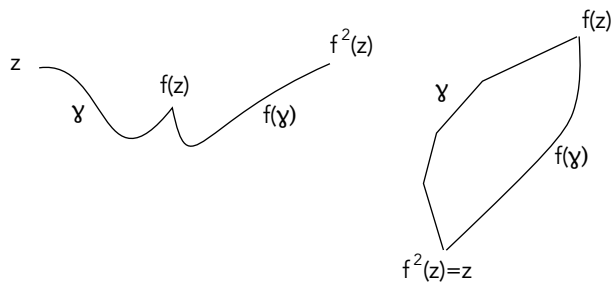
$\mathbb{D}$, say that λ is a *Brouwer line* of f if $f(\overline{L(\lambda)}) \subset L(\lambda)$ or equivalently if $f^{-1}(\overline{R(\lambda)}) \subset R(\lambda)$.

The Brouwer Translation Theorem (see [Br]), asserts that if f is a fixed point free orientation preserving homeomorphism of $\mathbb{D}$ (recall that f is called a Brouwer homeomorphism), then $\mathbb{D}$ can be covered with Brouwer lines: every point $z \in \mathbb{D}$ belongs to such a line. In particular the set

$$W = \bigcup_{k \geq 0} f^k(R(\lambda)) \cap \bigcup_{k \leq 0} f^{-k}(L(\lambda))$$

is an f-invariant open set homeomorphic to $\mathbb{C}$ and the map $f_{|W}$ is conjugate to a non trivial translation of $\mathbb{C}$. As a consequence, we know that every point z of $\mathbb{D}$ is wandering, meaning that there is a neighborhood U of z such that the $f^k(U)$, $k \in \mathbb{Z}$, are pairwise disjoint. The proof of Brouwer Translation Theorem uses a preliminary result, stating that if an orientation preserving homeomorphism f of $\mathbb{D}$ has a periodic point of period $p > 1$, then there exists a simple loop $\Gamma \subset \mathbb{D} \setminus \mathrm{fix}(f)$ such that $i(f, \Gamma) = 1$ and consequently f has a fixed point. In fact this preliminary result can be expressed in terms of translation arcs (see [Br], [Fa] or [G]). A segment $\gamma : [0, 1] \to \mathbb{D}$ is a *translation arc* of f if it joins a point $z \notin \mathrm{fix}(f)$ to $f(z)$ and if:

- either $\gamma \cap f(\gamma) = \{z\}$;
- or $\gamma \cap f(\gamma) = \{z, f(z)\}$ and $f^2(z) = z$.



Proposition 2.1. *Let f be an orientation preserving homeomorphism of $\mathbb{D}$ and γ a translation arc of f joining a point z to $f(z)$. If $\gamma \cap f(\gamma)$ is not reduced to $f(z)$ or if there exists $k > 1$ such that $\gamma \cap f^k(\gamma) \neq \emptyset$, then one can construct a simple loop $\Gamma \subset \mathbb{D} \setminus \mathrm{fix}(f)$ such that $i(f, \Gamma) = 1$.*

It is not difficult to construct a translation arc γ passing through a given point z (we will see the construction in Section 5). If z is periodic, then γ satisfies the hypothesis of Proposition 2.1 and so, there exists a simple loop $\Gamma \subset \mathbb{D} \setminus \mathrm{fix}(f)$ such that $i(f, \Gamma) = 1$.

Suppose that the hypothesis of Proposition 2.1 are satisfied. If z is periodic, then one can find a free open disk V containing z, and one knows that there exists $k > 0$ such that $f^k(V) \cap V \neq \emptyset$. If z is not periodic, there exists $k > 1$ such that $\gamma \cap f^k(\gamma) \neq \emptyset$. Fix $z' \in \gamma \cap f^{-k}(\gamma)$ and denote γ' the sub-segment of γ that joins z' to $f^k(z')$. It is different from γ because z is not periodic and consequently it is free. So, one can find a free open disk V that contains γ' and one knows that $f^k(V) \cap V \neq \emptyset$. In all cases, there exists a non wandering free open disk and so f is recurrent.

Let us recall now the following result of Franks [Fr1]:

Proposition 2.2. *If f is an orientation preserving recurrent homeomorphism of $\mathbb{D}$, then there exists a simple loop $\Gamma \subset \mathbb{D} \setminus \mathrm{fix}(f)$ such that $i(f, \Gamma) = 1$. Consequently, f has at least one fixed point.*

Indeed, as seen in Section 2.1, there exists a continuous family $(h_t)_{t\in[0,1]}$ of homeomorphisms such that:

- $h_t = \mathrm{Id}$;
- $\mathrm{fix}(h_t \circ f) = \mathrm{fix}(f)$ for every $t \in [0, 1]$;
- $h_1 \circ f$ has a periodic point of period $p > 1$.

By Proposition 2.1, there exists a simple loop $\Gamma \subset \mathbb{D} \setminus \mathrm{fix}(h_1 \circ f)$ that does not contain any fixed point of $h_1 \circ f$, such that $i(h_1 \circ f, \Gamma) = 1$. The integer $i(h_t \circ f, \Gamma)$ is well defined for every $t \in [0, 1]$, because $\mathrm{fix}(h_t \circ f) = \mathrm{fix}(f)$ for every $t \in [0, 1]$, and depends continuously on t. So it is independent of t, and we have $i(f, \Gamma) = 1$.

Let us conclude with a characterization of the recurrence due to Guillou and Le Roux (see [LeR1], page 39). If Σ is a submanifold of $\mathbb{D}$ (with or without boundary, of dimension 0, 1, or 2) we will denote $\partial\Sigma$ the boundary of Σ and define $\mathrm{int}(\Sigma) = \Sigma \setminus \partial\Sigma$.

Proposition 2.3. *Let f be an orientation preserving homeomorphism of $\mathbb{D}$. Suppose that there exists a family $(\Sigma_j)_{j\in\mathbb{Z}/r\mathbb{Z}}$ of free connected submanifolds of $\mathbb{D}$ such that the sets $\mathrm{int}(\Sigma_j)$, $j \in \mathbb{Z}/r\mathbb{Z}$, are pairwise disjoint, and a family of positive integers $(k_j)_{j\in\mathbb{Z}/r\mathbb{Z}}$ such that $f^{k_j}(\Sigma_j) \cap \Sigma_{j+1} \neq \emptyset$, for every $j \in \mathbb{Z}/r\mathbb{Z}$. Then f is recurrent.*

Here again, recall the idea of the proof. Suppose that $(\Sigma_j)_{j\in\mathbb{Z}/r\mathbb{Z}}$ is a family of submanifolds satisfying the hypothesis and that r is minimal, among all such families. In particular, the only manifold Σ_l, $l \in \mathbb{Z}/r\mathbb{Z}$, met by $\bigcup_{k\geq 1} f^k(\Sigma_j)$ is Σ_{j+1}. Let $(k_j)_{j\in\mathbb{Z}/r\mathbb{Z}}$ be the family of integers given by the hypothesis, chosen smallest as possible. Choose $z_j \in \Sigma_j \cap f^{-k_j}(\Sigma_{j+1})$. One can find a path $\gamma_j \subset \Sigma_j$ joining $f^{k_{j-1}}(z_{j-1})$ to z_j such that $\gamma_j \setminus \{f^{k_{j-1}}(z_{j-1}), z_j\} \subset \mathrm{int}(\Sigma_j)$. This implies that the γ_j are pairwise disjoint. Note that they are free and that $f^{k_j}(\gamma_j) \cap \gamma_{j+1} \neq \emptyset$, for every $j \in \mathbb{Z}/r\mathbb{Z}$. One can find a family of pairwise disjoint free open disks $(V_j)_{j\in\mathbb{Z}/r\mathbb{Z}}$ such that V_j contains γ_j. We have proved that f is recurrent because $f^{k_j}(V_j) \cap V_{j+1} \neq \emptyset$.

3. Ordered sets and cuts

We will consider different orders in the article, so let us introduce some definitions and notations that will be used later.

Let $(X, \leq)$ be an ordered set. For every $x \in B$, we denote

$$x_{\leq} = \{y \in X \mid y \leq x\},$$

and define similarly the sets $x_{\geq}$, $x_{<}$ and $x_{>}$.

A *cut* of $\leq$ is a partition $c = (X_c^-, X_c^+)$ of X in two sets X_c^- and X_c^+ (possibly empty), such that

$$x \in X_c^- \Rightarrow x_{\leq} \subset X_c^- \ , \ \ x \in X_c^+ \Rightarrow x_{\geq} \subset X_c^+ .$$

Denote $\mathcal{C}$ the set of cuts. The order $\leq$ admits a natural extension on $X \sqcup \mathcal{C}$ by the relations:

- $c \leq c'$ if $X_c^- \subset X_{c'}^-$;
- $x \leq c$ if $x \in X_c^-$;
- $x \geq c$ if $x \in X_c^+$.

Note that the cut $(\emptyset, X)$ is the smallest element of $X \sqcup \mathcal{C}$ and that $(X, \emptyset)$ is the largest. Note also that if $\leq$ is a total order, then its natural extension is a total order.

To conclude this section, recall that if $(X, \leq)$ is an ordered set, there exists a total order $\preceq$ on X, which is *weaker* than $\leq$, meaning that for every x, x' in X, we have:

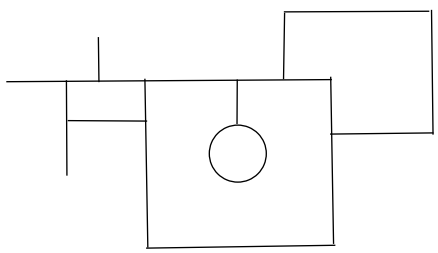
$$x \leq x' \Rightarrow x \preceq x'.$$

In particular every cut of $\preceq$ is a cut of $\leq$. Indeed, one can put an order $\ll$ on the set of orders $\mathcal{O}$, writing $\leq_1 \ll \leq_2$ if $\leq_2$ is weaker than $\leq_1$. If $(\leq_j)_{j \in J}$ is a totally ordered family in $\mathcal{O}$, then one gets an upper bound $\leq$ of this family, defining $x \leq y$ if there exists $j \in J$ such that $x \leq_j y$. By Zorn's lemma one can find an order $\preceq$, maximal for the order $\ll$, which is weaker than $\leq$. It remains to prove that $\preceq$ is a total order. Argue by contradiction and suppose that there exist x_0 and y_0 that are non comparable. One gets an order $\preceq'$ strictly weaker than $\preceq$ by defining $x \preceq' y$ if $x \preceq y$ or if $x \preceq x_0$ and $y_0 \preceq y$, in contradiction with the maximality of $\preceq$.

4. Brick decompositions

In this section, we will recall some facts about brick decompositions (see Sauzet [Sa] or [LeC1]). The results of the last sub-section are the only ones which are new. Nevertheless it seems useful to include some proofs of known results to make this article as self-contained as possible.

4.1. **Definitions.** A *brick decomposition* $\mathcal{D} = (V, E, B)$ on an orientable surface S is defined by a locally finite graph $\Sigma(\mathcal{D})$ of S, the *skeleton* of $\mathcal{D}$, such that any vertex is locally the extremity of exactly three edges. Here V is the set of vertices, E the set of edges (meaning the closures in S of the connected components of $\Sigma(\mathcal{D}) \setminus V$) and B the set of bricks (meaning the closures in S of the connected components of $S \setminus \Sigma(\mathcal{D})$). Note that every edge is the image of a proper topological embedding of $[0, 1]$, $[0, +\infty)$, $\mathbb{R}$ or $\mathbb{S}$ and that every brick is a surface (usually with boundary).



In fact, for every $X \in \mathcal{P}(B)$, the union of bricks $b \in X$ is a sub-surface of S with a boundary contained in $\Sigma(\mathcal{D})$. It is connected if and only if for every pair of bricks $b, b' \in X$, there exists a sequence $(b_j)_{0 \leq j \leq n}$ such that:

- $b_0 = b$ and $b_n = b'$;
- b_j and b_{j+1} are *adjacent* for $i \in \{0, \dots, n-1\}$ (meaning that they contain a common edge).

The maximal connected subsets of a set $X \in \mathcal{P}(B)$ will be called the *connected components* of X.[1] For simplicity we will denote by the same letter a set of bricks and the union of these bricks, and will write $X \subset B$ or $X \subset S$ if specification is needed. We will say that two disjoint sets X and Y of B are *adjacent* if $\partial X \cap \partial Y \neq \emptyset$ or equivalently if there is an edge $e \in E$ that is contained in both a brick of X and a brick of Y.

A brick decomposition $\mathcal{D}' = (V', E', B')$ is a *sub-decomposition* of $\mathcal{D}$ if $\Sigma(\mathcal{D}') \subset \Sigma(\mathcal{D})$. In that case V' is a subset of V, every edge in E' a union of edges in E and every brick in B' a union of bricks in B. Observe that one may have $B' = B$ even if $\mathcal{D}' \neq \mathcal{D}$.

4.2. **The induced maps.** Let f be a homeomorphism of S and $\mathcal{D} = (V, E, B)$ a brick decomposition of S. If $X \subset B$, then $f(X) \subset S$ is well defined. To define $\varphi_+(X) \subset B$, it is sufficient to set:

$$\begin{aligned}\varphi_+(X) &= \{b \in B \mid \text{there exists } b' \in X \text{ such that } b \cap f(b') \neq \emptyset\}\\ &= \{b \in B \mid b \cap f(X) \neq \emptyset\}.\end{aligned}$$

It is easy to see that the map $\varphi_+ : \mathcal{P}(B) \to \mathcal{P}(B)$ sends connected subsets into connected subsets and satisfies

$$\varphi_+\left(\bigcup_{j\in J} X_j\right) = \bigcup_{j\in J} \varphi_+(X_j), \quad \varphi_+\left(\bigcap_{j\in J} X_i\right) \subset \bigcap_{j\in J} \varphi_+(X_j),$$

for every family $(X_j)_{j\in J}$ of subsets of B.

Similarly, one defines $\varphi_- : \mathcal{P}(B) \to \mathcal{P}(B)$ writing :

$$\begin{aligned}\varphi_-(X) &= \{b \in B \mid \text{there exists } b' \in X \text{ such that } b \cap f^{-1}(b') \neq \emptyset\}\\ &= \{b \in B \mid b \cap f^{-1}(X) \neq \emptyset\}.\end{aligned}$$

The relations $b' \in \varphi_+^n(\{b\})$ and $b \in \varphi_-^n(\{b'\})$ are equivalent. Saying that $b \in \varphi_+^n(\{b\})$, $n \geq 1$, means that there exists a sequence $(b_j)_{0\leq j\leq n}$ in B such that:

- $b_0 = b_n = b$;
- $f(b_j) \cap b_{j+1} \neq \emptyset$ for every $i \in \{0, \dots, n-1\}$.

In that case we have a *cycle of bricks*.

4.3. **Free brick decompositions, the induced order.** Let f be a fixed point free homeomorphism of S. Say that $\mathcal{D} = (V, E, B)$ is a *free brick decomposition* of f if every brick $b \in B$ is free, meaning that $f(b) \cap b = \emptyset$. Such a decomposition always exists. More precisely, if $\mathcal{D}' = (V', E', B')$ is a brick decomposition of S, there exists a free brick decomposition $\mathcal{D} = (V, E, B)$ such that $\mathcal{D}'$ is a sub-decomposition of $\mathcal{D}$. Let $\mathcal{D} = (V, E, B)$ be a free brick decomposition. An easy application of Zorn's lemma tells us that there exists a partition $(X_j)_{j\in J}$ of B into free connected subsets, such that if

[1] Connected components will usually simply be called components in the text.

$j \neq j'$ then $X_j \cup X_{j'}$ is not free if connected. The set $\bigcup_{j\in J} \partial X_j$ is the skeleton of a sub-decomposition $\mathcal{D}'$ of $\mathcal{D}$ whose bricks are the X_j and such that every edge is contained in two bricks (and not one) of the decomposition. We obtain in that way a *maximal free brick decomposition*: it is free and every strict sub-decomposition of $\mathcal{D}'$ is not free.

Suppose now that f is a non recurrent fixed point free homeomorphism of S and that $\mathcal{D} = (V, E, B)$ is a free brick decomposition of f. Applying Proposition 2.3, one deduces that there is no cycle of bricks. Consequently, one gets an order $\leq$ on B, writing :

$$b \leq b' \iff b' \in \bigcup_{n\geq 0} \varphi_+^n(\{b\}).$$

In other words, $b \leq b'$ if and only if there exists a sequence $(b_j)_{0\leq j\leq n}$, $n \geq 0$, in B such that:

- $b_0 = b$ and $b'_n = b'$;
- $f(b_j) \cap b_{j+1} \neq \emptyset$ for every $j \in \{0, \dots, n-1\}$.

In the case where $\mathcal{D}$ is maximal, then two adjacent bricks b and b' are comparable because $b \cup b'$ is not free. More precisely, there is a natural orientation of $\Sigma(D)$ defined as follows: every edge $e \in E$ is oriented in such a way that $f(r(e)) \cap l(e) \neq \emptyset$, where $r(e)$ and $l(e)$ are the two bricks containing e, the first one on the right, the second one on the left. Note that a vertex is neither the final point of three edges nor the initial point of three edges (otherwise there would be a cycle of bricks of length 3).

Lemma 4.1. *Let f be a non recurrent fixed point free homeomorphism of S and $\mathcal{D} = (V, E, B)$ be a maximal free brick decomposition of f. The following conditions are equivalent, for a given brick $b \in B$:*

(1) *there exists b' adjacent to b such that $f(b) \cap b' \neq \emptyset$,*
(2) *$b_\geq$ is connected;*

and they imply that $b_>$ is connected. In the case where these conditions are satisfied, we will say that b is positively regular. *Otherwise we will say that b* is positively singular.

Proof. The inclusion (2) $\Rightarrow$ (1) is obvious. Indeed, $b_\geq$ contains b and is not reduced to b, so its contains a brick b' adjacent to b if it is connected. Of course, $f(b') \cap b = \emptyset$ because $b' \geq b$ and so we have $f(b) \cap b' \neq \emptyset$. To prove the inclusion (1) $\Rightarrow$ (2), note that $\varphi_+(\{b\})$ is connected for every $b \in B$ and so $\{b\} \cup \varphi_+(\{b\})$ is also connected if (1) is satisfied. Consequently, each set

$$\varphi_+^n(\{b\}) \cup \varphi_+^{n+1}(\{b\}) = \varphi_+^n(\{b\} \cup \varphi_+(\{b\})),$$

is connected. Noting that $\varphi_+^n(\{b\}) \cup \varphi_+^{n+1}(\{b\})$ and $\varphi_+^{n+1}(\{b\}) \cup \varphi_+^{n+2}(\{b\})$ intersect and writing

$$b_\geq = \bigcup_{n\geq 0} \varphi_+^n(\{b\}) = \bigcup_{n\geq 0} \varphi_+^n(\{b\}) \cup \varphi_+^{n+1}(\{b\}),$$

we deduce that $b_\geq$ is connected. We also deduce that $b_>$ is connected because

$$b_> = \bigcup_{n\geq 1} \varphi_+^n(\{b\}) = \bigcup_{n\geq 1} \varphi_+^n(\{b\}) \cup \varphi_+^{n+1}(\{b\}).$$

□

Similarly we will say that b is *negatively regular* if $b_{\leq}$ is connected, meaning that there is a brick b' adjacent to b such that $b' \leq b$. Otherwise it will be called *negatively singular*. We will say that b is *regular* if it is both positively and negatively regular and *singular* otherwise.

4.4. **Attractors and repellers.** Let f be a homeomorphism of S and $\mathcal{D} = (V, E, B)$ a brick decomposition of S. Let us call *attractor* a set $A \subset B$ such that $\varphi_+(A) \subset A$ or equivalently such that $f(A) \subset \mathrm{int}(A)$. In particular, if $(A_j)_{j\in J}$ is a family of attractors, then $\bigcup_{j\in J} A_j$ and $\bigcap_{j\in J} A_j$ are attractors.

Of course, if A is an attractor, φ_+ sends components of A into components. A component C satisfying $\varphi_+(C) \subset C$ is called *regular*, a component C satisfying $\varphi_+(C) \cap C = \emptyset$ is called *singular*.

In the case where f is a non recurrent fixed point free homeomorphism of S and $\mathcal{D} = (V, E, B)$ is a maximal free brick decomposition of S, then C is regular if and only if contains at least two bricks, and singular if and only if it is reduced to a positively singular brick. Indeed, let C be a component of A and denote C' the component of A containing $\varphi_+(C)$. If $\#C \geq 2$, then C contains two adjacent bricks and so $\varphi_+(C) \cap C \neq \emptyset$. Consequently one has $C' = C$. If C is reduced to a brick b, then $C' \neq C$ because b is free. Moreover, $C \cup C'$ is not connected, which implies that $\varphi_+(\{b\})$ does not contain a brick adjacent to b. So b is positively singular.

Similarly, a *repeller* is a set $R \subset B$ such that $\varphi_-(R) \subset R$ or equivalently such that $f^{-1}(R) \subset \mathrm{int}(R)$. We have a similar classification for its components. Note also that R is a repeller if and only if $B \setminus R$ is an attractor. In fact, a couple (R, A) such that $\{R, A\}$ is a partition of B that consists of a repeller R and an attractor A, is a cut of $\leq$. Observe that $\partial R = \partial A$.

4.5. **New and useful results.** We will finish this section by proving some new results on brick decompositions that will be useful to our purposes. We suppose that f is a non recurrent orientation preserving homeomorphism of the 2-sphere $\mathbb{S}^2$ and that $\mathcal{D} = (V, E, B)$ is a maximal free brick decomposition of $f_{|\mathbb{S}^2\setminus \mathrm{fix}(f)}$. We denote $\leq$ the induced order on B and φ_+, φ_- the induced maps on $\mathcal{P}(B)$[2].

We begin with the following lemma:

Lemma 4.2. *The boundary of a singular brick is a line of* $\mathbb{S}^2 \setminus \mathrm{fix}(f)$*, whose closure in* $\mathbb{S}^2$ *meets a unique connected component of* $\mathrm{fix}(f)$*.*

Proof. It is sufficient to prove the lemma for negatively singular bricks. Let b be such a brick. Its boundary is a boundaryless 1-submanifold of $\mathbb{S}^2 \setminus \mathrm{fix}(f)$ and each of its component is

- a simple loop,
- or a line of $\mathbb{S}^2 \setminus \mathrm{fix}(f)$ whose closure in $\mathbb{S}^2$ meets a unique connected component K of $\mathrm{fix}(f)$,
- or a line of $\mathbb{S}^2 \setminus \mathrm{fix}(f)$ whose closure in $\mathbb{S}^2$ meets exactly two connected component K_- and K_+ of $\mathrm{fix}(f)$.

[2]Of course, by considering its extension to the Alexandrov compactification of $\mathbb{D}$, everything will work for a homeomorphism f of $\mathbb{D}$ if we replace $\mathbb{S}^2 \setminus \mathrm{fix}(f)$ with $\mathbb{D} \setminus \mathrm{fix}(f)$.

Let us prove first that if there exists a component Γ of ∂b which is a line of $\mathbb{S}^2 \setminus \mathrm{fix}(f)$ whose closure in $\mathbb{S}^2$ meets exactly one connected component K of $\mathrm{fix}(f)$, then $\partial b = \Gamma$. The line Γ is contained in a component W of $\mathbb{S}^2 \setminus K$, this component is homeomorphic to $\mathbb{D}$ and Γ separates W. We denote U the connected component of $W \setminus \Gamma$ that contains $\mathrm{int}(b)$ and V the other one. The bricks $b' \neq b$ adjacent to Γ are included in $\overline{V}$. The brick b being negatively singular, $f(b)$ meets every brick adjacent to b. Moreover $f(b)$ is connected and disjoint from Γ because b is free. So every brick adjacent to b is in $\overline{V}$. This implies that ∂b has no component but Γ.

Let us prove now that no component of ∂b is a loop. We argue by contradiction and suppose that such a component Γ exists. It separates $\mathbb{S}^2$. We denote U the connected component of $\mathbb{S}^2 \setminus \Gamma$ that contains $\mathrm{int}(b)$ and V the other one. The same proof as above tells us that every brick adjacent to b is in $\overline{V}$. One deduces that $b = \overline{U}$. In this situation, the set X of bricks adjacent to b is finite. For every $b' \in X$, the set $f^{-1}(b')$ meets b but is not included in b (because the image by f^{-1} of an edge $e \subset b \cap b'$ is disjoint from b). Moreover $f^{-1}(b')$ is connected and so it meets another brick of X. We have proved that for every $b' \in X$ there exists $b'' \in X$ such that $b'' < b'$. This is not compatible with the finiteness of X.

Let us conclude by proving that the third case never occurs. We argue by contradiction and consider a component Γ of ∂b which is a line of $\mathbb{S}^2 \setminus \mathrm{fix}(f)$ whose closure in $\mathbb{S}^2$ meets exactly two connected component K_- and K_+ of $\mathrm{fix}(f)$. The line Γ is contained in a component W of $\mathbb{S}^2 \setminus (K_- \cup K_+)$, this component is homeomorphic to the annulus $\mathbb{S} \times \mathbb{R}$ and Γ does not separate W but joins its two ends. Consequently, Γ is not the unique component of ∂b. The brick b, being negatively singular, is positively regular and so Lemma 4.1 tells us that $b \geq$ and $b_> = b_\geq \setminus \{b\}$ are connected. The brick b being negatively singular, every brick adjacent to b belongs to $b_>$. The connectedness of b and $b_>$ implies that $b_\geq$ contains an essential simple loop Γ', meaning non homotopic to zero in W. Its image by f is contained in the interior of $b_>$ and consequently does not meet Γ. So it cannot be essential and we have got a contradiction. □

Corollary 4.3. *Let (R, A) be a cut of $\leq$. If a singular component R' of R is adjacent to a component A' of A, then A' is the only component of A that is adjacent to R'.*

Proof. Recall that R' is reduced to a single brick, which is negatively recurrent. The sets R and A have the same boundary and every component of this boundary is the boundary of a unique component of R and of a unique component of A. By Lemma 4.2, the boundary of R' is connected. It is a connected component of the boundary of R, and so is a connected component of the boundary of a component A' of A, moreover A' is the unique component of A that is adjacent to R'. □

Of course, one proves similarly that if a singular component A' of A is adjacent to a component R' of R, then R' is the only component of R that is adjacent to A'.

Proposition 4.4. *Let (R, A) be a cut of $\leq$. If R' is a regular component of R and A' a regular component of A and if there exists $n \geq 1$ such that $f^n(R') \cap A' \neq \emptyset$, then R' and A' are adjacent.*

Proof. Let X be the union of R' and of the components of A that are adjacent to R'. It is a connected set and we want to prove that it contains A'. Let e be an edge included in ∂X. Denote b the brick containing e and included in X and b' the other brick. Of course $b \notin R'$, otherwise b' would belong to a component of A adjacent to R' and so would belong to X. So, b belongs to a connected component A'' of A that is adjacent to R' and b' is contained in a connected component of R adjacent to A'' and different from R'. By Corollary 4.3, it implies that A'' is regular and we have $f(e) \subset f(A'') \subset \mathrm{int}(A'') \subset \mathrm{int}(X)$. So, X is a connected set such that $f(\partial X) \subset \mathrm{int}(X)$. Moreover, if A' is not adjacent to R', then $A' \cap X = \emptyset$ in B but also in $\mathbb{S}^2$. By hypothesis, the sequence $(f^{-k}(A'))_{k\geq 0}$ is increasing and for k large enough it holds that $f^{-k}(A') \cap R' \neq \emptyset$. Consequently, there exists $k \geq 0$ such that $f^{-k}(A') \cap X = \emptyset$ and $f^{-k-1}(A') \cap X \neq \emptyset$. But $f^{-k-1}(A')$ is connected and not included in X. It implies that $f^{-k-1}(A') \cap \partial X \neq \emptyset$, which implies that $f^{-k}(A') \cap \mathrm{int}(X) \neq \emptyset$. We have found a contradiction. □

5. A preliminary lemma

We begin from now on the proof of Theorem 1.1. We suppose in this section that $f : \mathbb{D} \to \mathbb{D}$ satisfies the hypothesis of the theorem and is not recurrent. The final goal is to prove that there is a contradiction. For every $i \in \mathbb{Z}/p\mathbb{Z}$ and every $k \in \mathbb{Z}$ we define $z_i^k = f^k(z_i)$.

Let $(X^k)_{k\geq 0}$ be a sequence of subsets of $\mathbb{D}$ and $\zeta \in \mathbb{S}$. We will write $\lim_{k\to+\infty} X^k = \zeta$ if for every neighborhood U of ζ in $\mathbb{D} \cup \{\zeta\}$, there exists $K \geq 0$ such that $X^k \subset U$, for every $k \geq K$.

Let us begin with a result whose detailed proof can be found in [LeC2]:

Lemma 5.1. *There exists a sequence $(V_i^k)_{k\in\mathbb{Z}}$ of open disks such that:*

(1) *V_i^k contains z_i^k and z_i^{k+1};*
(2) *$V_i^k \cap \mathrm{fix}(f) = \emptyset$;*
(3) *$\lim_{k\to-\infty} V_i^k = \alpha_i$;*
(4) *$\lim_{k\to+\infty} V_i^k = \omega_i$.*

Proof. By hypothesis, one can find a sequence $(U_i^k)_{k\in\mathbb{Z}}$ of open disks such that

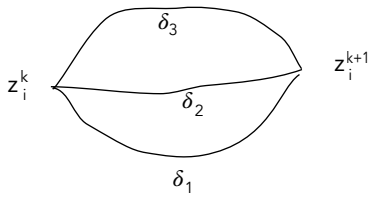
- U_i^k contains z_i^k and z_i^{k+1};
- $\lim_{k\to-\infty} U_i^k = \alpha_i$;
- $\lim_{k\to+\infty} U_i^k = \omega_i$.

Fix $i \in \mathbb{Z}/p\mathbb{Z}$ and $k \in \mathbb{Z}$ and choose a homeomorphism $h : \mathbb{C} \to U_i^k$ such that

$$h(-1) = z_i^k, \;\; h(1) = z_i^{k+1}.$$

Consider the following segments, defined on [0,1], that join z_i^k to z_i^{k+1}:

$$\delta_1 : t \mapsto h\left(e^{i\pi(1+t)}\right), \;\; \delta_2 : t \mapsto h(-1+2t), \;\; \delta_3 : t \mapsto h\left(e^{i\pi(1-t)}\right).$$



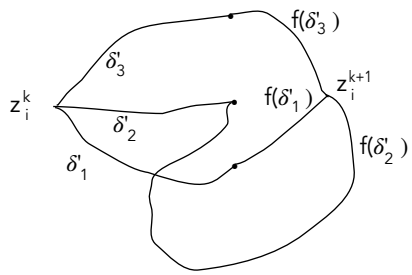
In the case where there exists $j \in \{1,2,3\}$ such that $\delta_j \cap \mathrm{fix}(f) = \emptyset$, one can find an open disk $V_j^k \subset U_j^k$, close enough to δ_j to satisfy $V_j^k \cap \mathrm{fix}(f) = \emptyset$. In the case where every δ_j meets $\mathrm{fix}(f)$, define $\delta'_j = \delta_{j|[0,t_j)}$, where

$$t_j = \min\{t \in [0,1] \,|\, \delta_j(t) \in \mathrm{fix}(f)\}.$$

The fact that f preserves the orientation implies that the cyclic order of the germs of δ_1, δ_2 and δ_3 at z_i^k coincides with the cyclic order of the germs of $f(\delta_1)$, $f(\delta_2)$ and $f(\delta_3)$ at z_i^{k+1}. This implies that there exist j_1 and j_2 in $\{1,2,3\}$ such that $f(\delta'_{j_1}) \cap \delta'_{j_2} \neq \emptyset$. Write

$$t'_{j_1} = \min\{t \in [0, t_{j_1}) \,|\, f(\delta_j(t)) \in \delta'_{j_2}\}$$

and define $t'_{j_2} \in [0, t_{j_2})$ such that $\delta_{j_2}(t'_{j_2}) = f(\delta_{j_1}(t'_{j_1}))$. One can find an open disk $V_j^k \subset U_j^k \cup f(U_j^k)$, close enough to the segment $\delta_{j_2}([0,t'_{j_2}]) \cup f(\delta_{j_1}([0,t'_{j_1}])$, to satisfy $V_j^k \cap \mathrm{fix}(f) = \emptyset$.



The fact that f extends to a homeomorphism of $\mathbb{D} \cup_{i \in \mathbb{Z}/p\mathbb{Z}} \{\alpha_i, \omega_i\}$ implies that

- $\lim_{k \to -\infty} f(U_i^k) = \alpha_i$,
- $\lim_{k \to +\infty} f(U_i^k) = \omega_i$;

and we deduce that

- $\lim_{k \to -\infty} V_i^k = \alpha_i$,
- $\lim_{k \to +\infty} V_i^k = \omega_i$.

□

We will now state the result which will be fundamental in our proof:

Lemma 5.2. *There exists a sequence $(\gamma_i^k)_{k \in \mathbb{Z}}$ of translation arcs such that:*

(1) *γ_i^k joins z_i^k to z_i^{k+1};*
(2) *$\gamma_i^{k-1} \cap \gamma_i^k$ coincide in a neighborhood of z_i^k;*
(3) *$\lim_{k \to -\infty} \gamma_i^k = \alpha_i$;*
(4) *$\lim_{k \to +\infty} \gamma_i^k = \omega_i$.*

Proof. Let $(V_i^k)_{k\in\mathbb{Z}}$ be a sequence of open disks given by Lemma 5.1. Fix $i \in \mathbb{Z}/p\mathbb{Z}$ and $k \in \mathbb{Z}$ and choose a homeomorphism $h : \mathbb{D} \to V_i^k$ such that $h(0) = z_i^k$.

For every $r < 1$, write $D_r = h(\{z \in \mathbb{D} \,|\, |z| \leq r\})$. Then define $D_* = D_{r_*}$, where

$$r_* = \sup\{r > 0 \,|\, f(D_r) \cap D_r = \emptyset\}.$$

In particular we have

$$f(\mathrm{int}(D_*)) \cap \mathrm{int}(D_*) = \emptyset \text{ and } f(\partial D_*) \cap \partial D_* \neq \emptyset.$$

Now, fix a point $y_i^k \in f(\partial D_*) \cap \partial D_*$. It is not fixed because D_* is included in V_i^k. So, one can construct a segment $\sigma_i^k = \beta_i^k \delta_i^k$ joining $f^{-1}(y_i^k)$ to y_i^k, where:

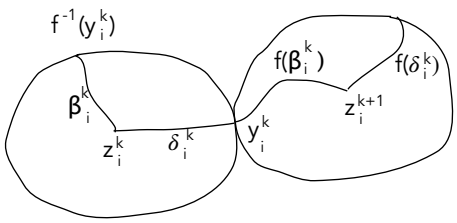
- β_i^k joins $f^{-1}(y_i^k)$ to z_i^k and δ_i^k joins z_i^k to y_i^k;
- $\beta_i^k \setminus \{f^{-1}(y_i^k)\}$ and $\delta_i^k \setminus \{y_i^k\}$ are included in $\mathrm{int}(D_*)$.

We have $f^{-1}(y_i^k) \neq f(y_i^k)$ because f is not recurrent and so $f(\sigma_i^k) \cap \sigma_i^k$ is reduced to y_i^k. Consequently, σ_i^k is a translation arc. As explained in the previous section, the segments $f^r(\sigma_i^k)$ and $f^{r'}(\sigma_i^k)$ are disjoint if $|r - r'| > 1$. In particular $\sigma_i^k\, f(\sigma_i^k)\, f^2(\sigma_i^k)$ is a segment. But this implies that $\gamma_i^k = \delta_i^k\, f(\beta_i^k)$ is a translation arc joining z_i^k to z_i^{k+1} that is contained in $D_* \cup f(D_*)$ and so is contained in $V_i^k \cup f(V_i^k)$. The fact that f extends to a homeomorphism of $\mathbb{D} \setminus \{\alpha_i, \omega_i\}$ implies that

- $\lim_{k\to-\infty} f(V_i^k) = \alpha_i$,
- $\lim_{k\to+\infty} f(V_i^k) = \omega_i$;

and we deduce that

- $\lim_{k\to-\infty} \gamma_i^k = \alpha_i$;
- $\lim_{k\to+\infty} \gamma_i^k = \omega_i$.



To finish the proof it remains to explain why the previous construction can be done in such a way that γ_i^{k-1} and γ_i^k coincide in a neighborhood of z_i^k, which means that $f(\beta_i^{k-1})$ and δ_i^k coincide in a neighborhood of z_i^k. The reason is that we have a lot of freedom when choosing the β_i^k and the δ_i^k. While choosing our paths β_i^k, $k \geq 0$, we can always suppose by an induction process that for every $k \geq 0$, the paths $f(\beta_i^k)$ and β_i^{k+1} intersect only at z_i^k in a neighborhood of this point. Then, again by induction, while constructing our paths β_i^k, $k < 0$, one can suppose that the previous property is true for every $k \in \mathbb{Z}$. Finally, while constructing our paths δ_i^k, $k \in \mathbb{Z}$, there is no longer any obstruction to supposing that $f(\beta_i^{k-1})$ and δ_i^k coincide in a neighborhood of z_i^k. $\square$

6. An adapted brick decomposition

Here again, we suppose that $f : \mathbb{D} \to \mathbb{D}$ satisfies the hypothesis of Theorem 1.1 and is not recurrent. We consider, for every $i \in \mathbb{Z}/p\mathbb{Z}$, a sequence $(\gamma_i^k)_{k\in\mathbb{Z}}$ of segments satisfying the conclusion of Lemma 5.2.

6.1. **Construction of critical graphs.** Each γ_i^k is a translation arc and so the path obtained by concatenation of all $f^{k'}(\gamma_i^k)$, $k' \in \mathbb{Z}$, is a simple path. In particular, $z_i^{k'} \notin \gamma_i^k$ if $k' \notin \{k, k+1\}$. We parametrize every γ_i^k on $[0,1]$. The fact that $\lim_{k\to-\infty} \gamma_i^k = \alpha_i$ and $\lim_{k\to+\infty} \gamma_i^k = \omega_i$ implies that one can define

$$k_0 = \min\{k' \leq 0 \,|\, \gamma_i^{k'} \cap \bigcup_{k>0} \gamma_i^k \neq \emptyset\}$$

and

$$k_1 = \max\{k > 0 \,|\, \gamma_i^k \cap \gamma_i^{k_0} \neq \emptyset\}.$$

We define

$$t_0 = \min\{t \in [0,1] \,|\, \gamma_i^{k_0}(t) \in \gamma_i^{k_1}\}$$

and then $t_1 \in [0,1]$ such that $\gamma_i^{k_0}(t_0) = \gamma_i^{k_1}(t_1)$, noting that

$$t_1 = \max\{t \in [0,1] \,|\, \gamma_i^{k_1}(t) \in {\gamma_i^{k_0}}_{|[0,t_0]}\}.$$

The fact that $z_i^{k'} \notin \bigcup_{k>0} \gamma_i^k$ if $k' \leq 0$ implies that $t_0 \neq 0$, and moreover that $t_0 \neq 1$ if $k_0 < 0$. For the same reasons, $t_0 \neq 1$, if $k_0 = 0$ and $k_1 > 1$. Finally $t_0 \neq 1$, if $k_0 = 0$ and $k_1 = 1$, because γ_i^0 and γ_i^1 coincide in a neighborhood of z_i^1. Consequently, in every situation, it holds that $t_0 \in (0,1)$. We deduce that $t_1 \in (0,1)$ because $\gamma_i^{k_0}(t_0)$ does not belong to the orbit of z_i, the path $\gamma_i^{k_0}$ being a translation arc.

We define inductively an increasing sequence $(k_l)_{l>0}$ of positive integers and a sequence $(t_l)_{l>0}$ by the relations (satisfied for $l > 0$):

$$k_{l+1} = \max\{k > k_l \,|\, \gamma_i^k \cap {\gamma_i^{k_l}}_{|[t_l,1]} \neq \emptyset\},$$

$$t_{l+1} = \max\{t \in [0,1] \,|\, \gamma_i^{k_{l+1}}(t) \in {\gamma_i^{k_l}}_{|[t_l,1]}\}.$$

Of course, $\gamma_i^{k_l}(t_l) \neq \gamma_i^{k_{l-1}}(t_{l-1})$ because $\gamma_i^{k_{l-1}}(t_{l-1}) \notin \gamma_i^{k_l}$. Note also that $t_l \neq 0$ and more generally that $\gamma_i^{k_l}(t_l)$ is not in the orbit of z_i because $z_i^{k_l}$ and $z_i^{k_l+1}$ are the only points of the orbit of z_i that are contained in $\gamma_i^{k_l}$ and because $\gamma_i^{k_l}$ and $\gamma_i^{k_l-1}$ coincide in a neighborhood of $z_i^{k_l}$

Now we define inductively an increasing sequence $(k_l)_{l<0}$ of negative integers and a sequences $(t_l)_{l\leq 0}$ by the relations (satisfied for $l \leq 0$):

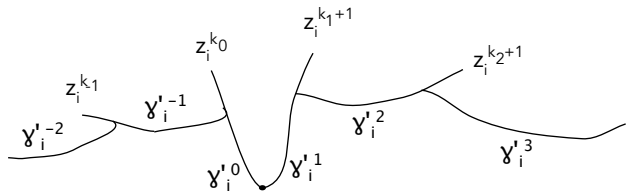
$$k_{l-1} = \min\{k < k_l \,|\, \gamma_i^k \cap {\gamma_i^{k_l}}_{|[0,t_l]} \neq \emptyset\}$$

$$t_{l-1} = \min\{t \in [0,1] \,|\, \gamma_i^{k_{l-1}}(t) \in {\gamma_i^{k_l}}_{|[0,t_l]}\}.$$

Similarly, we prove that $t_l \in (0,1)$ and that $\gamma_i^{k_l}(t_l)$ is not an end of ${\gamma_i^{k_{l+1}}}_{|[0,t_{l+1}]}$.

Now we define a family of segments $({\gamma'}_i^l)_{l\in\mathbb{Z}\setminus\{0\}}$ by the relations:

$${\gamma'}_i^l = \begin{cases} {\gamma_i^{k_l}}_{|[t_l,1]} & \text{if } l > 0 \\ {\gamma_i^{k_l}}_{|[0,t_l]} & \text{if } l \leq 0 \end{cases}$$



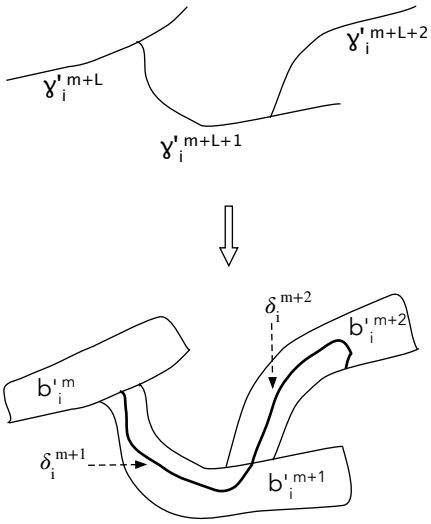
These segments are free and have disjoint interiors. Moreover, if $l' > l$, there exists $k > 0$ such that $f^k(\gamma'^l_i) \cap \gamma'^{l'}_i \neq \emptyset$ because γ'^l_i contains $z_i^{k_l+1}$ if $l > 0$, and contains $z_i^{k_l}$ if $l \leq 0$. Consequently, by Proposition 2.3, for every $k > 0$, one has $f^k(\gamma'^{l'}_i) \cap \gamma'^l_i = \emptyset$. Note also that

- $\lim_{l\to-\infty} \gamma'^l_i = \alpha_i$;
- $\lim_{l\to+\infty} \gamma'^l_i = \omega_i$.

In particular, there exists L such that:

- $\gamma'^l_i \cap \gamma'^{l'}_i = \emptyset$ if $l < -L$ and $l' > L$;
- $\gamma'^l_i \cap \gamma'^{\,l'}_{i'} = \emptyset$ if $i \neq i'$, $|l| > L$ and $|l'| > L$.

6.2. **Construction of an adapted brick decomposition.** We can find a free brick decomposition $\mathcal{D}' = (B', E', V')$ of f defined on $\mathbb{D} \setminus \mathrm{fix}(f)$ and a family of bricks $(b'^m_i)_{m\in\mathbb{Z}\setminus\{0\},\, i\in\mathbb{Z}/p\mathbb{Z}}$ in B' such that b'^m_i contains γ'^{m+L}_i if $m > 0$ and contains γ'^{m-L}_i if $m < 0$. To construct this decomposition we begin to enlarge the paths γ'^l_i, $|l| > L$, to construct the bricks b'^m_i and then we decompose the complement of the union of these bricks in free bricks. Then we consider a maximal free disk decomposition $\mathcal{D} = (B, E, V)$ which is a sub-decomposition of $\mathcal{D}'$ and we denote b^m_i the brick that contains b'^m_i. Note that for every $i \in \mathbb{Z}/p\mathbb{Z}$, and for every m, m' in $\mathbb{Z} \setminus \{0\}$ such that $m < m'$, there exists $k > 0$ such that $f^k(b^m_i) \cap b^{m'}_i \neq \emptyset$. In particular we have $b^m_i < b^{m'}_i$. Note also that for every $m > 0$, the bricks b^m_i and b^{m+1}_i are adjacent. Consequently every brick b^m_i, $m > 1$, is regular and b^1_i is positively regular. Shifting the indexation if necessary (and so starting the sequence with b^2_i instead of b^1_i) one can suppose that every b^m_i, $m > 0$, is regular. Similarly, one can suppose that every every b^m_i, $m < 0$, is regular.



To conclude, note that we can find a family of segments $(\delta_i^m)_{m\in\mathbb{Z}\setminus\{0\},\, i\in\mathbb{Z}/p\mathbb{Z}}$ satisfying:

- $\partial\delta_i^m \subset \partial b'^m_i$;
- $\mathrm{int}(\delta_i^m) \subset \mathrm{int}(b'^m_i)$;
- if $m > 0$, then δ_i^m and δ_i^{m+1} intersect at a single point;
- if $m < 0$, then δ_i^m and δ_i^{m-1} intersect at a single point.

Using the fact that

- $b'^m_i \subset b_i^m$;
- $\lim_{m\to+\infty} b'^m_i = \omega_i$;
- $\lim_{m\to-\infty} b'^m_i = \alpha_i$;

one obtains, for $m > 0$, a half line λ_i^m by concatenation of the $\delta_i^{m'}$, $m' \geq m$, such that

- $\partial\lambda_i^m \subset \partial\left(\bigcup_{m'\geq m} b_i^{m'}\right)$;
- $\mathrm{int}(\lambda_i^m) \subset \mathrm{int}\left(\bigcup_{m'\geq m} b_i^{m'}\right)$;
- $\lim_{t\to+\infty} \lambda_i^m(t) = \omega_i$ if λ_i^m is parametrized on $[0, +\infty)$;

and for $m < 0$, a half line λ_i^m by concatenation of the $\delta_i^{m'}$, $m' \leq m$, such that

- $\partial\lambda_i^m \subset \partial\left(\bigcup_{m'\leq m} b_i^{m'}\right)$;
- $\mathrm{int}(\lambda_i^m) \subset \mathrm{int}\left(\bigcup_{m'\leq m} b_i^{m'}\right)$;
- $\lim_{t\to-\infty} \lambda_i^m(t) = \alpha_i$ if λ_i^m is parametrized on $(-\infty, 0]$.

7. Proof of Theorem 1.1

We consider the brick decomposition $\mathcal{D} = (B, E, V)$ built in the previous section and keep the notations introduced there. In this section, when we talk about a component of a set $X \in \mathcal{P}(B)$, we will mean either a connected component of X or the empty set. We will write $\mathbb{Z}^* = \mathbb{Z} \setminus \{0\}$.

Lemma 7.1. *Let (R, A) be a cut of $\leq$. For every $i \in \mathbb{Z}/p\mathbb{Z}$, there is a cut (Z_i^-, Z_i^+) of $\mathbb{Z}^*$, a component R_i of R and a component A_i of A such that:*

- *R_i contains every b_i^m, $m \in Z_i^-$, and is empty if Z_i^- is empty;*
- *A_i contains every b_i^m, $m \in Z_i^+$, and is empty if Z_i^+ is empty;*
- *R_i and A_i are adjacent if Z_i^- and Z_i^+ are not empty.*

Proof. Let Z_i^- be the set of $m \in \mathbb{Z}^*$ such that $b_i^m \in R$ and Z_i^+ the set of $m \in \mathbb{Z}^*$ such that $b_i^m \in A$. The couple (Z_i^-, Z_i^+) is a cut of $\mathbb{Z}^*$ because $b_i^m \leq b_i^{m'}$ if $m \leq m'$. Every brick b_i^m, $m \in \mathbb{Z}^*$, is regular. So, for every $m \in Z_i^-$, the component of R that contains b_i^m is itself a repeller and contains every $b_i^{m'}$, $m' \leq m$. Similarly, if $m \in Z_i^+$, the component A' of A that contains b_i^m is an attractor that contains every $b_i^{m'}$, $m' \geq m$. So, if $Z_i^- \neq \emptyset$, the bricks b_i^m, $m \in Z_i^-$, belong to the same connected component R_i of R. Similarly, if $Z_i^+ \neq \emptyset$, the bricks b_i^m, $m \in Z_i^+$, belong to the same connected component A_i of A. Defining $R_i = \emptyset$ if $Z_i^- = \emptyset$ and $A_i = \emptyset$ if $Z_i^+ = \emptyset$, we get the two first assertions. To get the third one, use Proposition 4.4 and the fact that if $m < m'$, there exists $k > 0$ such that $f^k(b_i^m) \cap b_i^{m'} \neq \emptyset$. □

Say that $X \in \mathcal{P}(B)$ *contains* α_i if there exists $m < 0$ such that X contains every $b_i^{m'}$, $m' \leq m$ and that X *contains* ω_i if there exists $m > 0$ such that X contains every $b_i^{m'}$, $m' \geq m$.

Lemma 7.2. *Suppose that η, ζ, η', ζ', are four different points of*

$$\{\alpha_i \,|\, i \in \mathbb{Z}/p\mathbb{Z}\} \cup \{\omega_i \,|\, i \in \mathbb{Z}/p\mathbb{Z}\},$$

such that $\eta' \in (\eta, \zeta)$ and $\zeta' \in (\zeta, \eta)$. If $X \in \mathcal{P}(B)$ is connected and contains η and ζ and if $X' \in \mathcal{P}(B)$ is connected and contains η' and ζ', then X and X' have at least one common brick.

Proof. : Recall that we have defined half lines λ_i^m, $i \in \mathbb{Z}/p\mathbb{Z}$, $m \in \mathbb{Z}^*$, at the end of Section 6.2. The set $\mathrm{int}(X)$ is connected and contains two such half lines, one converging to η, the other one converging to ζ. So, there is a line $\lambda \subset \mathrm{int}(X)$ that joins η and ζ. Similarly, there is a line $\lambda' \subset \mathrm{int}(X')$ that joins η' and ζ'. Consequently, one has $\lambda \cap \lambda' \neq \emptyset$, which implies that X and X' intersect in B. □

Lemma 7.3. *If (R, A) is a cut of $\leq$, then exactly one of the following situations holds:*

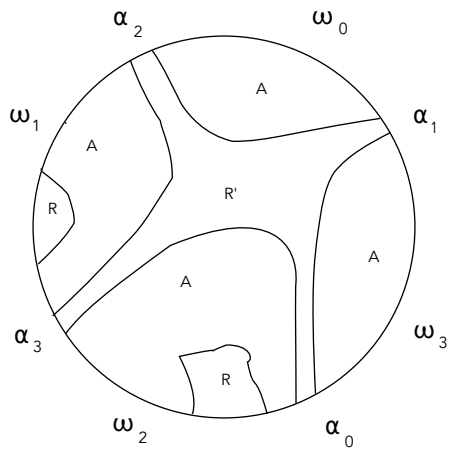
(1) *there is a component R' of R that contains every α_i, $i \in \mathbb{Z}/p\mathbb{Z}$;*
(2) *there is a component A' of A that contains every ω_i, $i \in \mathbb{Z}/p\mathbb{Z}$.*

Moreover, if (1) *holds then, every component of A contains at most one ω_i, $i \in \mathbb{Z}/p\mathbb{Z}$, and if* (2) *holds, every component of R contains at most one α_i, $i \in \mathbb{Z}/p\mathbb{Z}$.*

Proof. By Lemma 7.1, one knows that for every $i \in \mathbb{Z}/p\mathbb{Z}$, the set $R_i \cup A_i$ is connected and contains α_i and ω_i. Suppose first that there exists $i \in \mathbb{Z}/p\mathbb{Z}$ such that $R_i = R_{i+1}$. Then R_i contains α_i and α_{i+1}, while $R_{i-1} \cup A_{i-1}$ contains α_{i-1} and ω_{i-1}. By Lemma 7.2, it implies that R_i and $R_{i-1} \cup A_{i-1}$ intersect in B, which means that $R_{i-1} = R_i$. By induction one deduces that all R_i are equal and so (1) holds. Similarly, if $A_i = A_{i+1}$, then A_{i+1} and $R_{i+2} \cup A_{i+2}$ intersect in B and so $A_{i+1} = A_{i+2}$. One deduces that (2) holds. Applying Lemma 7.2 to $R_i \cup A_i$ and $R_{i+1} \cup A_{i+1}$, one deduces that these two

sets intersect in B and so, either $R_i = R_{i+1}$ or $A_i = A_{i+1}$. Consequently (1) or (2) holds.

Suppose now that (1) holds. Applying again Lemma 7.2, and the fact that $A \cap R' = \emptyset$ (in B), one deduces that every component of A contains at most one ω_i. Similarly, if (2) holds, every component of R contains at most one α_i. In particular (1) and (2) cannot occur simultaneously. □



Proof of Theorem 1.1 . As explained in section 3 there exists a total order $\preceq$ on B, which is weaker than $\leq$, meaning that for every b, b' in B, we have:

$$b \leq b' \Rightarrow b \preceq b'.$$

In particular every cut of $\preceq$ is a cut of $\leq$. Denote $\mathcal{C}$ the set of cuts $\Gamma = (R_\Gamma, A_\Gamma)$ of $\preceq$ and recall that $\preceq$ has a natural extension on $\mathcal{C} \sqcup B$. By Lemma 7.3, we have a partition $\mathcal{C} = \mathcal{C}_- \cup \mathcal{C}_+$, where:

- Γ belongs to $\mathcal{C}_-$ if there is a component of A_Γ containing every ω_i;
- Γ belongs to $\mathcal{C}_+$ if there is a component of R_Γ containing every α_i.

It is clear that $(\mathcal{C}_-, \mathcal{C}_+)$ is a cut of $\mathcal{C}$. Note also that $\mathcal{C}_-$ and $\mathcal{C}_+$ are not empty because $(\emptyset, B) \in \mathcal{C}_-$ and $(B, \emptyset) \in \mathcal{C}_+$.

Lemma 7.4. *The cut*

$$\Gamma_-^{\max} = \left(\bigcup_{\Gamma \in \mathcal{C}_-} R_\Gamma, \bigcap_{\Gamma \in \mathcal{C}_-} A_\Gamma \right)$$

is the greatest element of $\mathcal{C}_-$ *and the cut*

$$\Gamma_+^{\min} = \left(\bigcap_{\Gamma \in \mathcal{C}_+} R_\Gamma, \bigcup_{\Gamma \in \mathcal{C}_+} A_\Gamma \right)$$

the smallest element of $\mathcal{C}_+$.

Proof. It is sufficient to prove that $\Gamma_-^{\max}$ belongs to $\mathcal{C}_-$, the same proof will give us that $\Gamma_+^{\min}$ belongs to $\mathcal{C}_+$. We argue by contradiction and suppose that $\Gamma_-^{\max}$ belongs to $\mathcal{C}_+$. There is a component R' of $R_{\Gamma_-^{\max}}$ that contains every α_i and so for every $i \in \mathbb{Z}/p\mathbb{Z}$, there exists $m_i < 0$ such that $b_i^{m_i} \in R'$.

This implies that there exists a finite connected subset $X \subset R_{\Gamma_-^{\max}}$ that contains every $b_i^{m_i}$, $i \in \mathbb{Z}/p\mathbb{Z}$. Using the equality $R_{\Gamma_-^{\max}} = \bigcup_{\Gamma \in \mathcal{C}_-} R_\Gamma$, one knows that for every $b \in X$, there exists $\Gamma_b \in \mathcal{C}_-$ such that $b \in R_{\Gamma_b}$. Setting $\Gamma = \max_{b \in X} \Gamma_b$, we deduce that $X \subset R_\Gamma$.[3] This implies that the connected component of R_Γ that contains X is regular and contains all the α_i, $i \in \mathbb{Z}/p\mathbb{Z}$, in contradiction with the definition of $\mathcal{C}_-$. □

Lemma 7.4 tells us that the cut $(\mathcal{C}_-, \mathcal{C}_+)$ of $\mathcal{C}$ is a *gap*: $\mathcal{C}_-$ has a maximum and $\mathcal{C}_+$ has a minimum. In particular, there exists a brick b, uniquely defined, such that

$$R_{\Gamma_+^{\min}} = R_{\Gamma_-^{\max}} \cup \{b\}, \;\; A_{\Gamma_-^{\max}} = A_{\Gamma_+^{\min}} \cup \{b\}.$$

Moreover, it holds that

$$\mathcal{C}_- = \{\Gamma \in \mathcal{C} \,|\, \Gamma \prec b\}, \;\; \mathcal{C}_+ = \{\Gamma \in \mathcal{C} \,|\, b \prec \Gamma\}.$$

In other terms, a cut $(R, A) \in \mathcal{C}$ satisfies the condition (1) of Lemma 7.3 if $b \in R$ and the condition (2) if $b \in A$.

Indeed $R_{\Gamma_+^{\min}}$ is larger than $R_{\Gamma_-^{\max}}$, because $\Gamma_-^{\max} \prec \Gamma_+^{\min}$ and so $R_{\Gamma_+^{\min}} \setminus R_{\Gamma_-^{\max}}$ contains at least one brick. Suppose that $R_{\Gamma_+^{\min}} \setminus R_{\Gamma_-^{\max}}$ contains two bricks b, b'. One can assume that $b \prec b'$ because $\preceq$ is a total order. The cut $\Gamma = (b_\preceq, b_\succ)$ satisfies $\Gamma_-^{\max} \prec \Gamma \prec \Gamma_+^{\min}$ because $b \in R(\Gamma) \setminus R(\Gamma_-^{\max})$ and $b' \in R(\Gamma_+^{\min}) \setminus R(\Gamma)$, which contradicts the fact that $\Gamma \in \mathcal{C}_- \cup \mathcal{C}_+$.

Let us explain now where is the final contradiction that will imply Theorem 1.1. By definition of $\mathcal{C}_-$, there exists a component A' of $A_{\Gamma_-^{\max}}$ that contains every ω_i, $i \in \mathbb{Z}/p\mathbb{Z}$, which means that for every $i \in \mathbb{Z}/p\mathbb{Z}$, there exists $m_i > 0$ such that $b_i^{m_i} \in A'$. Moreover we know that A' is a connected attractor, which implies that $\varphi_+(A')$ is also a connected attractor. The set $\varphi_+(A')$ contains every $\varphi_+^n(\{b_i^{m_i}\}$, $n > 0$, because $b_i^{m_i} \in A'$. So $\varphi_+(A')$ contains $b_i^{m_i+1}$ because $b_i^{m_i+1} \in \bigcup_{n>0} \varphi_+^n(\{b_i^{m_i}\})$. We have proved that $\varphi_+(A')$ is a connected attractor that contains every ω_i, $i \in \mathbb{Z}$. To get a contradiction it remains to prove that $\varphi_+(A')$ is included in $A_{\Gamma_+^{\min}}$. Indeed, by definition of $\mathcal{C}_-$, no component of $A_{\Gamma_+^{\min}}$ contains all the ω_i.

The brick b belongs to $A_{\Gamma_-^{\max}}$ but is free. So, it holds that

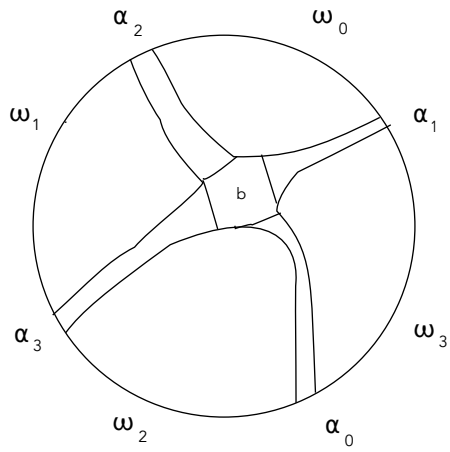
$$\varphi_+(\{b\}) \subset A_{\Gamma_-^{\max}} \setminus \{b\} = A_{\Gamma_+^{\min}},$$

which implies that

$$\varphi_+(A') \subset \varphi_+(A_{\Gamma_-^{\max}}) = \varphi_+(A_{\Gamma_+^{\min}} \cup \{b\}) = \varphi_+(A_{\Gamma_+^{\min}}) \cup \varphi_+(\{b\}) \subset A_{\Gamma_+^{\min}}.$$

□

[3] It is here that we need to work with a total order.



Remark. The underlying idea of Morse theory is to study a manifold by analyzing smooth functions defined on that manifold. If the manifold is compact and the critical points are not degenerate, the topology of the sublevels and superlevels change at every critical value and the change of topology can be understood relative to the Morse indices of the associated critical points. To prove such a thing, one needs to choose an adequate Riemannian metric on the manifold and to look at the dynamics of the associated gradient flow. In particular, if there is no critical value inside a segment $[a, b]$, the sublevels $\{x \in M \mid f(x) \leq a\}$ and $\{x \in M \mid f(x) \leq b\}$ are diffeomorphic. More precisely, after a reparametrization of the gradient vector field, the first set is the image of the second one by the time one map of the induced flow. There are many non compact situations where one can prove the existence of critical points of a given smooth function, or more generally of singular points of a given gradient-like vector field, but one needs conditions at infinity to do so (for instance imposing the function to be quadratic at infinity when M can be written $M = N \times \mathbb{R}^n$, where N is a compact manifold and $n \geq 1$.)

The principle of our given proof is very similar, but stated in a discrete setting, with an argument by contradiction. We suppose that our map is non recurrent and start by choosing an adequate free brick decomposition. Then we transform this decomposition into a maximal one (we do not have any control on this construction). The set of bricks of this new decomposition is furnished with a natural order $\leq$ and we extend this order into a maximal one (we still do not have any control on this extension). These objects (brick decomposition and total order) play the role of the Riemannian metric used in Morse theory. The cuts Γ play the role of level sets, with their sublevels A_Γ and their superlevels R_Γ. We have a "condition at infinity" satisfied by the original chosen brick decomposition, consequence of the fundamental lemma. Now, from one side, there is a change of "topology" (expressed in terms of the behaviour at infinity) among the sublevels, and on the other side, there is no change of topology because every brick b being free, it holds that $f(b_\preceq) \subset \varphi_+(b_\preceq) \subset b_\prec$ (the bricks play the role of flow-boxes in the classical setting).

8. Further comments about the proof

8.1. **Transverse foliations.** As recalled in Section 2, the Brouwer Translation Theorem asserts that if f is a Brouwer homeomorphism of $\mathbb{D}$, then $\mathbb{D}$ can be covered with Brouwer lines λ, meaning oriented lines such that $f(\overline{L(\lambda)}) \subset L(\lambda)$ or equivalently such that $f^{-1}(\overline{R(\lambda)}) \subset R(\lambda)$. Its foliated version (see [LeC1]) says the following: there exists a non singular C^0 foliation $\mathcal{F}$ on $\mathbb{D}$ such that every leaf is a Brouwer line.

Let $\gamma : I \to \mathbb{D}$ be a path defined on a non trivial real interval. Say that γ is *positively transverse* to $\mathcal{F}$ if it locally crosses leaves from the right to the left. Such a path must be injective because $\mathcal{F}$ is non singular. In fact, as explained in [HaeR], the space of leaves furnished with the quotient topology is an oriented one dimensional manifold, which is Hausdorff if and only if the foliation is trivial, meaning conjugate to the foliation of $\mathbb{C}$ by verticals. A positively transverse path projects onto a simple path in the manifold $\mathcal{F}$, compatible with the natural orientation. Note that there is a natural order $\leq$ on $\mathcal{F}$ defined as follows: $\lambda < \lambda'$ if there exists a transverse path $\gamma : [0,1] \to \mathbb{D}$ such that $\gamma(0) \in \lambda$ and $\gamma(1) \in \lambda'$. Let us state now an easy but important result proved in [LeC1]: for every $z \in \mathbb{D}$, there exists a positively transverse path that joins z to $f(z)$. Noting ϕ_z the leaf containing a point $z \in \mathbb{D}$, this result tells that the function $z \mapsto \phi_z$ is a *Lyapunov function* of f with values in $\mathcal{F}$: one has $\phi_{f(z)} < \phi_z$ for every $z \in \mathbb{D}$.

Let us explain now how to use the foliated version of Brouwer Translation Theorem to prove that a homeomorphism satisfying the hypothesis of Theorem 1.1 has at least one fixed point. Here again, it will be a proof by contradiction. The simplest way to construct a transverse foliation is to start with a maximal free brick decomposition $\mathcal{D} = (B, E, V)$, to extend the order $\leq$ naturally induced on B by a total order $\preceq$ and to consider the set $\mathcal{L}$ of lines $\lambda \subset \Sigma(\mathcal{D})$, where $\lambda \in \mathcal{L}$ if and only if there exists a cut $c = (R_c, A_c)$ of $\preceq$ such that λ is a connected component of $\partial R_c = \partial A_c$ (see [LeC3]). The orientation of $\Sigma(D)$ defines a natural orientation on each $\lambda \in \mathcal{L}$, and then $\lambda \in \mathcal{L}$ is a Brouwer line. There is a natural topology and a natural order on $\mathcal{L}$ (partial but locally total) that makes $\mathcal{L}$ similar to a lamination (every λ has a neighborhood that is homeomorphic to a totally disconnected compact subset of a real interval). By a process of desingularization, one can "blow up" the space $\mathcal{L}$ and transform it into a real lamination of $\mathbb{D}$ by Brouwer lines, homeomorphic to $\mathcal{L}$. To construct $\mathcal{F}$, it remains to fill the complement of this lamination.

The hypothesis of Theorem 1.1 tells us that, for every $i \in \mathbb{Z}/p\mathbb{Z}$, there exists a line Γ'_i that is transverse to $\mathcal{F}$ and that passes through every point $f^k(z_i)$, $k \in \mathbb{Z}$. Of course this line accumulates onto α_i and ω_i but can have other accumulation points on $\mathbb{S}$. If we start with the adapted brick decomposition defined in Section 6, we get an extra condition: there exists a line Γ_i that is transverse to $\mathcal{F}$,whose projection in the space of leaves is the same as the projection of Γ'_i and that accumulates only onto α_i and ω_i.

It is easy now to find a contradiction. The important fact being that $\Gamma_i \cap \Gamma_{i+1} \neq \emptyset$, for every $i \in \mathbb{Z}$. A first argument is to consider the function ν_i defined on the complement of Γ_i, equal to 0 on $R(\Gamma_i)$ and to 1 on $L(\Gamma_i)$, then to consider the fonction $\nu = \sum_{i \in \mathbb{Z}/p\mathbb{Z}} \nu_i$. In a neighborhood of $\mathbb{S}$, when it is

defined, the function ν takes the values 0 or 1. The fact that $\Gamma_i \cap \Gamma_{i+1} \neq \emptyset$ implies that ν takes at least 3 values on its domain of definition, and so either it holds that $\max \nu > 1$ or it holds that $\min \nu < 0$. Suppose that we are in the first case and consider a connected component U of the domain of definition of ν, where ν reaches its maximum. It is relatively compact. It is easy to show that U is the interior of a surface with boundary, whose boundary is transverse to $\mathcal{F}$, the leaves leaving U. This is incompatible with the fact that the leaves are lines of $\mathbb{D}$ (one can also say that there is no simple loop transverse to $\mathcal{F}$). In the case where $\min \nu < 0$ we have a similar situation with leaves entering into the domain.

Let us give another argument. For every $i \in \mathbb{Z}/p\mathbb{Z}$ and every $z \in \Gamma_i$ denote $\Gamma_i^-(z)$ and $\Gamma_i^+(z)$ the half lines contained in Γ_i arriving at z and starting at z respectively. Let us consider the first point $z \in \Gamma_i \cap \Gamma_{i+1}$ where Γ_i meets Γ_{i+1}. The union of $\Gamma_i^-(z)$ and $\Gamma_{i+1}^-(z)$ is a line. Moreover, the component of its complement that contains the half-lines $\Gamma_{i-1}^+(z')$, z' close to ω_i, is included in $R(\phi_z)$. Consequently $R(\phi_z)$ contains the whole line Γ_{i-1}. One deduces that $R(\phi_z)$ contains $\Gamma_{i-2}^+(z')$ if z' is close to ω_{i-2} and so it contains Γ_{i-2}. By iteration of this process we prove that $R(\phi_z)$ contains all $\Gamma_{i'}$, $i' \in \mathbb{Z}/p\mathbb{Z}$, which of course is absurd.

The original idea to get a proof of Handel's theorem via Brouwer Theory was to use the argument above. Unfortunately, Lemma 5.2 was missing, making impossible the use of the lines Γ_i, $i \in \mathbb{Z}/p\mathbb{Z}$, as above (the same argument does not work with the lines Γ_i'). What is proved and used in [LeC2] but also in [X1] and [X2] is the following:

Lemma 8.1. *There exists a sequence $(\gamma_i^k)_{k\in\mathbb{Z}}$ of translation arcs such that:*

(1) *γ_i^k joins $f^k(z_i)$ to $f^{k+1}(z_i)$;*
(2) *$f(\gamma_i^k) \cap \gamma_i^{k'} = \emptyset$ if $k' < k$;*
(3) *$\lim_{k\to-\infty} \gamma_i^k = \alpha_i$;*
(4) *$\lim_{k\to+\infty} \gamma_i^k = \omega_i$.*

Surprisingly, the proof of Lemma 5.2 is much simpler than the proof of Lemma 8.1. The second condition of Lemma 5.2, the fact that γ_i^k and γ_i^{k+1} coincide in a neighborhood of z_i^k, is fundamental in the construction of the adapted brick decomposition, but as it can be seen in the proof of Lemma 5.2, is very easy to get.

It is a natural question whether it is possible to construct a similar foliation in case f is not recurrent and get a complete proof of Theorem 1.1. The answer is probably yes. Nevertheless it is not clear if it will pertinent to try and do so. Let us give a first reason. To construct a transverse foliation for a Brouwer homeomorphism, it is necessary to prove first that there is no singular brick. The characterization that is given in Lemma 4.2 leads us to believe that singular bricks do not exist in the more general case of non recurrent map. But the proof will not be too easy and will certainly need to use topological arguments that cannot be expressed in terms of brick decompositions, like the majority of the arguments used here. In fact, singular bricks do not really cause problems (we will see very soon how to avoid them if necessary). A second reason not to use foliations is that the desingularisation and the filling processes necessary to get a foliation

is quite complicated, long and not easy to write. Moreover, Brouwer lines should be replaced by more complex manifolds (unions of lines) like what is done in the thesis of Tran Ngoc Diep [T], where a Brouwer type foliation is constructed for some orientation reversing plane homeomorphisms.

8.2. **As a conclusion.** We will conclude the article with some comments, that should permit us to understand why working with brick decompositions is not very different from working with transverse foliations. Let us first display a special case, where the proof of Theorem 1.1 is particularly simple. Let $\mathcal{D} = (B, E, V)$ be the adapted brick decomposition built in Section 6. Suppose that $\mathcal{D}$ satisfies the following condition: for every bricks b, b' such that $b \leq b'$, there exists a sequence $(b_j)_{0\leq j\leq n}$ in B such that:

- $b_0 = b$ and $b_n = b'$;
- $b_j < b_{j+1}$ for every $j \in \{0, \dots, n-1\}$;
- b_j and b_{j+1} are adjacent for every $j \in \{0, \dots, n-1\}$.

In that case, one easily constructs, for every $i \in \mathbb{Z}/p\mathbb{Z}$, a sequence $(\hat{b}_i^m)_{m\in\mathbb{Z}}$ and an integer $M_i \geq 0$ such that

- $\hat{b}_i^m = b_i^m$, for every $m < 0$;
- $\hat{b}_i^m = b_i^{m-M_i}$, for every $m > M_i$;
- $\hat{b}_i^m < \hat{b}_i^{m+1}$ for every $m \in \mathbb{Z}$;
- $\hat{b}_i^m$ and $\hat{b}_i^{m+1}$ are adjacent for every $m \in \mathbb{Z}$.

Note that the $\hat{b}_i^m$, $i \in \mathbb{Z}/p\mathbb{Z}$, $m \in \mathbb{Z}$, are all regular, note also that the set $\{\hat{b}_i^m\,,\, m \in \mathbb{Z}\}$ is connected and contains α_i and ω_i. So, for every $i \in \mathbb{Z}/p\mathbb{Z}$, there exists $m_i \in \mathbb{Z}$ and $m_i' \in \mathbb{Z}$ such that $\hat{b}_i^{m_i} = \hat{b}_{i+1}^{m_i'}$. Consider the cut $((\hat{b}_i^{m_i})_\leq, (\hat{b}_i^{m_i})_>)$. Both sets $(\hat{b}_i^{m_i})_\leq$ and $(\hat{b}_i^{m_i})_>$ are connected because $\hat{b}_i^{m_i}$ is regular. The first set contains α_i and α_{i+1} and the second one contains ω_i and ω_{i+1}. By Lemma 7.3 one deduces that $(\hat{b}_i^{m_i})_\leq$ contains every $\omega_{i'}$, $i' \in \mathbb{Z}/pZ$, and $(\hat{b}_i^{m_i})_>$ contains every $\alpha_{i'}$, $i' \in \mathbb{Z}/pZ$. This contradicts Lemma 7.3.

The last proposition, tells us that this "connectedness" property remains valid if one adds the cuts to the bricks. What we get is analogous to the fact that, for a Brouwer homeomorphism furnished with a transverse foliation, every point can be joined to its image by a transverse path.

Proposition 8.2. *We suppose that f is a non recurrent orientation preserving homeomorphism of the 2-sphere $\mathbb{S}^2$ and that $\mathcal{D} = (V, E, B)$ is a maximal free brick decomposition of $f_{|\mathbb{S}^2\setminus \mathrm{fix}(f)}$. We denote $\leq$ the induced order on B and consider a total order $\preceq$ weaker than $\leq$. Then, for every regular bricks b, b' such that $b \leq b'$, there exists a sequence $(b_j)_{0\leq j\leq n}$ of regular bricks, satisfying $b_0 = b$ and $b_n = b'$, and such that for every $j \in \{0, \dots, n-1\}$, it holds that:*

- *$b_j < b_{j+1}$;*
- *b_{j+1} is adjacent to the component of $(b_j)_\preceq$ that contains b_j;*
- *b_j is adjacent to the component of $(b_j)_\succ$ that contains b_{j+1}.*

Proof. The brick b' being regular, the set $b'_\leq$ is connected, and by hypothesis, it contains b. So, there exists a sequence $(\hat{b}_k)_{0\leq k\leq m}$ in $b'_\leq$, such that:

- $\hat{b}_0 = b$ and $\hat{b}_m = b'$;
- $\hat{b}_k$ and $\hat{b}_{k+1}$ are adjacent for every every $k \in \{0, \dots, m-1\}$.

Let us explain first why the bricks can be supposed to be positively regular. Indeed suppose that there exists $k \in \{0, \dots, m\}$ such that $\hat{b}_k$ is positively singular. By hypothesis, $k \notin \{0, m\}$ and the bricks $\hat{b}_{k-1}$ and $\hat{b}_{k+1}$ are adjacent to $\hat{b}_k$. We have proved in Lemma 4.2 than $\partial \hat{b}_k$ is reduced to a line of $\mathbb{D} \setminus \mathrm{fix}(f)$. This implies that there exists a sequence $(\check{b}_l)_{0 \le l \le q}$ in B such that:

- $\check{b}_0 = \hat{b}_{k-1}$ and $\check{b}_q = \hat{b}_{k+1}$;
- $\check{b}_l$ is adjacent to $\hat{b}_k$ for every every $l \in \{0, \dots, q\}$;
- $\check{b}_l$ and $\check{b}_{l+1}$ are adjacent for every $l \in \{0, \dots, q-1\}$.

Note that the bricks $\check{b}_l$, $l \in \{0, \dots, q\}$, are all positively regular and included in $b'_{\le}$ because we have $\check{b}_l \le \hat{b}_k \le b'$. So, this process permit to avoid positively singular bricks by extending our original sequence of bricks. Then we can shorten our sequence in such a way that a brick appears at most once.

Let us consider now the increasing sequence $(k_j)_{0 \le j \le n}$ uniquely defined by the following properties:

- $k_0 = 0$ and $k_n = m$;
- $\hat{b}_{k_j} \succ \hat{b}_{k_{j-1}}$ if $0 < j \le n$;
- $\hat{b}_k \preceq \hat{b}_{k_{j-1}}$ if $0 < j \le n$ and $0 \le k < k_j$.

Setting $b_j = \hat{b}_{k_j}$, we will prove that the sequence $(b_j)_{0 \le j \le n}$ satisfies the conclusions of Proposition 8.2.

We denote $\mathcal{C}$ the set of cuts of $\preceq$ and define $c_i = ((b_i)_{\preceq}, (b_i)_{\succ}) \in \mathcal{C}$, noting that $b_0 \prec c_0 \prec b_1 \prec \cdots \prec c_{n-1} \prec b_n$.

Note first that the bricks b_j, $0 \le j \le n$, are all regular. It is true if $j \in \{0, n\}$ by hypothesis. Suppose now that $0 < j < n$. The bricks $\hat{b}_{k_j - 1}$ and $\hat{b}_{k_j}$ are comparable for $\le$ because they are adjacent and we have $\hat{b}_{k_j-1} \preceq \hat{b}_{k_{j-1}} \prec \hat{b}_{k_j}$. So it holds that $\hat{b}_{k_j-1} < \hat{b}_{k_j}$. Consequently, $\hat{b}_{k_j}$ is regular, being positively regular by hypothesis.

Fix $j \in \{0, \dots, n-1\}$. The brick b_j being regular, the set $(b_j)_>$ is connected, and by hypothesis, it contains b'. So it is contained in the component A_j of $(b_j)_{\succ}$ that contains b'. Note that b_j is adjacent to A_j because, being positively regular, it is adjacent to $(b_j)_>$. Note R_j the component of $(b_j)_{\preceq}$ that contains b_j. It contains all b_k, $0 \le k < k_{j+1}$, because these bricks belong to $(b_j)_{\preceq}$ and their union is connected. Consequently $b_{j+1} = \hat{b}_{k_{j+1}}$ is adjacent to R_j. The fact that b_{j+1} is positively regular implies that the component of $(b_j)_{\succ}$ that contains b_{j+1} is regular and so is an attractor. One deduces that $(b_{j+1})_{\ge}$ is included in this component. By hypothesis $b' \ge b_{j+1}$ and so, b_{j+1} belongs to A_j. □

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Institut de Mathématiques de Jussieu-Paris Rive Gauche, IMJ-PRG, Sorbonne Université, Université Paris-Diderot, CNRS, F-75005, Paris, France & Institut Universitaire de France

E-mail address: patrice.le-calvez@imj-prg.fr