Linear stability of thick sprays equations

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C. Buet[*], B. Després[†], L. Desvillettes [‡]

Abstract

The coupling through both drag force and volume fraction (of gas) of a kinetic equation of Vlasov type and a system of Euler or Navier-Stokes type (in which the volume fraction explicity appears) leads to the so-called thick sprays equations. Those equations are used to describe sprays (droplets or dust specks in a surrounding gas) in which the volume fraction of the disperse phase is non negligible. As for other multiphase flows systems, the issues related to the linear stability around homogeneous solutions is important for the applications. We show in this paper that this stability indeed holds for thick sprays equations, under physically reasonable assumptions. The analysis which is performed makes use of Lyapunov functionals for the linearized equations.

1 Introduction

We present new stability properties of solutions to a family of multiphase models for sprays. In this work, we will denote by spray a disperse liquid or solid phase evolving in a surrounding gas. The models are based on a coupling between a kinetic equation of Vlasov type (for the droplets or dust specks constituting the disperse phase) and a system of (compressible) fluid equations for the gas, so that they belong to the class of coupled kinetic-(compressible) fluid models. Before presenting our results, we emphasize that despite a rich history which can be traced back to the seminal publications [15, 20, 27, 25, 28], which were motivated by application needs, and which were followed by [1, 2, 6], the mathematical analysis of coupled kinetic (compressible) fluid models is not yet fully developed. We refer to [22, 8] for very recent results. The mathematical theory of continuum physics and hyperbolic balance laws is developed in [9, 17], where the fundamental elements of the fluid part of our model is developed. For this fluid part of our model, local existence and stability of strong solutions is established for short time before some singularities (shocks, contact discontinuities)

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are created. For the kind of coupled kinetic-fluid models discussed in this work, one can expect similar behavior. However we will focus on more basic issues such as the establishment of an entropy and the stability of linearized equations around self-similar profiles.

Before describing our findings, we present the specificity of the family of models that we use, originated for example in the publication [6]. The system is made of a kinetic equation and a system of (compressible) fluid equations which are coupled not only through a drag force, but also through the volume fraction $\alpha := \alpha(t, \mathbf{x})$ of gas at time t and point $\mathbf{x} \in \mathbb{R}^{\mathbf{3}}$. Sprays modeled in such a way are sometimes called "thick sprays".

We assume here that the gas is described by the compressible Euler (or Navier-Stokes) equations. The variables are the (mass) density $\rho := \rho(t, \mathbf{x}) \geq 0$, the velocity $\mathbf{u} := \mathbf{u}(t, \mathbf{x}) \in \mathbb{R}^{\mathbf{3}}$ and the internal energy $e := e(t, \mathbf{x}) \geq 0$ at time t and point $\mathbf{x}$. The pressure $p \geq 0$ is a function of ρ and e, that is $p := p(\rho, e)$. For the sake of simplicity, we assume a perfect gas pressure law

$$p = (\gamma - 1)\rho e, \qquad \gamma > 1, \tag{1}$$

and an energy law (in terms of temperature)

$$e = C_v\, T, \tag{2}$$

where $T > 0$ is the temperature, and $C_v > 0$ is a constant.

The disperse phase is described by a phase space density $f := f(t, \mathbf{x}, \mathbf{v}) \geq 0$ of particles (droplets or dust specks) which at time t and point $\mathbf{x}$ move with velocity $\mathbf{v} \in \mathbb{R}^{\mathbf{3}}$. The force acting on the droplets is $m_\star \mathbf{\Gamma}$, with $m_\star := \frac{4}{3}\,\pi\, r^3$, where $r > 0$ is the radius of one droplet, and $m_\star$ is the mass of one droplet, in a system of units where the density of the material constituting the droplets is equal to 1. The spray is assumed to be mono-disperse, which means that all droplets have the same radius. This force is usually decomposed in two parts, one related to the pressure gradient, and the other one related to the drag or friction between the two phases:

$$m_\star \mathbf{\Gamma} = -m_\star \nabla p - D_\star(\mathbf{v} - \mathbf{u}), \tag{3}$$

where $D_* > 0$. On physical grounds, the drag coefficient $D_\star$ may depend also on the volume fraction of the gas defined by

$$\alpha := 1 - m_\star \int f dv, \tag{4}$$

on the density of the gas ρ, and on the modulus of difference of velocities $|\mathbf{v} - \mathbf{u}|$. However, still for the sake of simplicity, we consider in this paper that it is a constant (all quantities indexed with a $\star$ are also constants in the sequel). Taking into account the retroaction of the droplets on the fluid, one obtains the model system of partial differential equations with the identities (1) – (4),

$$\begin{cases} \partial_t(\alpha\rho) + \nabla\cdot(\alpha\rho\mathbf{u}) = 0, \\ \partial_t(\alpha\rho\mathbf{u}) + \nabla\cdot(\alpha\rho\mathbf{u}\otimes\mathbf{u}) + \nabla p = -m_\star \int \mathbf{\Gamma} f\,dv, \\ \partial_t(\alpha\rho e) + \nabla\cdot(\alpha\rho e\mathbf{u}) + p\,(\partial_t\alpha + \nabla\cdot(\alpha\mathbf{u})) = D_\star \int |\mathbf{v}-\mathbf{u}|^2 f\,dv, \\ \partial_t f + \mathbf{v}\cdot\nabla_x f + \nabla_v\cdot(\mathbf{\Gamma} f) = 0. \end{cases} \tag{5}$$

In the last equation one can substitute

$$\nabla_v\cdot(\mathbf{\Gamma} f) = -\nabla_x p\cdot\nabla_v f - \frac{D_\star}{m_\star}\nabla_v\cdot[(\mathbf{v}-\mathbf{u})f].$$

Such systems are sometimes called "Eulerian-Lagrangian" or "gas-particles" for thick sprays. We shall also investigate a simplified model of the same type, without energy equation (belonging to the "barotropic" family), and more complex models (at the end of this work). In those more complex models, extra parameters can enter the description of additional physical effects for the particles.

When $\alpha \approx 1$, it is possible to simplify eq. (5) by enforcing $\alpha = 1$ in the equations, and by discarding the constraint (4). The spray is then said to be thin. One of the simplest thin spray model is the so-called (compressible, barotropic) Euler-Vlasov system, which writes

$$\begin{cases} \partial_t\rho + \nabla\cdot(\rho\mathbf{u}) = 0, \\ \partial_t(\rho\mathbf{u}) + \nabla\cdot(\rho\mathbf{u}\otimes\mathbf{u}) + \nabla p = D_\star \int (\mathbf{v}-\mathbf{u})\,f\,dv, \\ \partial_t f + \mathbf{v}\cdot\nabla_x f + \nabla_v\cdot(\frac{D_\star}{m_\star}(\mathbf{u}-\mathbf{v})f) = 0. \end{cases} \tag{6}$$

Existence of smooth local in time solutions for a system of this kind was obtained in [2], and the derivation of related systems from fully kinetic equations at the formal level was described in [11]. The most recent works that we are aware of (for example [22, 8] and the references therein) address the well-posedness of thin spray systems corresponding to slightly different physical situations (viscous flows, incompressible flows). We refer to [4, 5] for the derivation of those systems. We emphasize that thanks to those works (and the works cited therein), it is well established that thin spray systems typically do not suffer from linear instability, and are locally well-posed.

In our case, we deal with thick sprays which model physical situations in which α can be significantly smaller than 1. That is why we will keep in the equations the most general domain of validity for the volume fraction of the gas

$$\alpha \in (0, 1]. \tag{7}$$

One of our goals is to investigate whether or not a fluid-kinetic model for thick sprays such as (5) suffers from the same kind of instability phenomena as the one described in [19, 3]. Note that fluid-kinetic models for thick sprays (with collision kernels) are linked to multiphase (fluid-fluid) models by an identified asymptotics (cf. [12]), so that it is indeed a natural question to ask. Notice also that such an investigation is unavoidably more complex for the system (5) than in the case of a coupling between fluid equations only. Indeed the linear operators associated to equations (5) are infinite-dimensional, whereas the linearized equations are 4 or 6-dimensional for the coupling between fluid equations only (cf. [26] for example). If ever some instability phenomenon is present in the solutions of (5), then it would rule out the possibility to extend the theoretical results from [22, 8] to models like (5), and it would raise issues on the modeling of real phenomena by those models when α is not very close to 1.

- Our first result in Proposition 2.1 below is that the volume fraction stays strictly positive under mild smoothness and boundedness conditions on velocities only: if $\alpha(0, \mathbf{x})$ is bounded below by a strictly positive constant, then $\alpha(t, \mathbf{x}) > 0$ for further times $t > 0$ and for all $\mathbf{x}$. The proof combines nonlinear entropy estimates and simple bounds.
- Our second family of results investigates the linear stability of solutions around space-homogeneous profiles which are adapted from classical solutions in plasma physics [24, 18, 10]. We linearize the equations of the problem around a reference solution and we analyze the stability of perturbations using weighted quadratic norms. The major difficulty is that the reference solution is non constant in time due to the drag force, so that some of the weights display a dependence with respect to the time variable also. More precisely, the concentration of droplets in velocity space is such that the reference solution behaves like a Dirac mass in velocity at the limit $t \to \infty$, so that the estimates must be robust enough with respect to this behavior. Linear stability is proved in Proposition 3.4 for the barotropic system which is a simplification of (5), and in Proposition 4.2 for the full system (5).
- Some hints about the possible extension of these results to more general models where the particles are submitted to additional physical effects (such as collisions, or temperature exchanges, or viscosity effects) are collected in Section 5. We first explain how the introduction of elastic collisions preserves the Lyapunov functional for specific reference solutions. We then show how to get the entropy consistency of a system with thermal effects for the particles, and then how to find explicit solutions to this new system, with a technique of separation of variables. Finally we explain how the presence of viscosity preserves the Lyapunov functional (of the barotropic case).

The organization of the paper is as follows. Preliminary remarks and the nonlinear positivity principle for volume fractions is established in Section 2. Linear stability is shown for the simpler barotropic system in Section 3. The analysis is generalized to the full system (5) in Section 4. Finally in Section 5, we show how to extend the results to more challenging systems involving extra physical effects.

2 Preliminary remarks and positivity of the volume fraction

2.1 Conservation identities and entropy equation

Considerations on moment equations, thermodynamics and entropy are classical in the mathematical treatment of continuum physics [9, 16, 17]. For the self consistency of this work, these properties are established below for our model. We begin with natural remarks on conservation identities and entropy inequality for system (5) (with (1) – (4)).

One observes that the mass of each phase is preserved since $\partial_t(\alpha\rho) + \nabla \cdot (\alpha\rho\mathbf{u}) = 0$, and

$$\partial_t\left(m_\star \int f\, dv\right) + \nabla \cdot \left(m_* \int \mathbf{v}\, f\, dv\right) = 0. \tag{8}$$

The momentum equation for the gas in (5) can be recast as $\partial_t\mathbf{u} + (\mathbf{u}\cdot\nabla)\boldsymbol{u} + (\alpha\rho)^{-1}\nabla p = \frac{1}{\alpha\rho}\left(-m_\star \int \mathbf{\Gamma} f dv\right)$. Multiplying the kinetic equation by $m_\star\mathbf{v}$ and integrating with respect to the velocity variable, one obtains the momentum equation for the particles

$$\partial_t\left(m_\star \int \mathbf{v} f dv\right) + \nabla \cdot \left(m_\star \int \mathbf{v}\otimes\mathbf{v} f dv\right) = m_\star \int \mathbf{\Gamma} f dv.$$

Adding this identity with the momentum equation for the gas in (5), one obtains the conservation law for the total momentum equation

$$\partial_t\left(\alpha\rho\mathbf{u} + m_\star \int f\mathbf{v} dv\right) + \nabla\cdot(\alpha\rho\mathbf{u}\otimes\mathbf{u}) + \nabla p + \nabla\cdot\left(m_* \int \mathbf{v}\otimes\mathbf{v} f dv\right) = 0. \tag{9}$$

In view of the definition (3) of the force $m_\star\mathbf{\Gamma}$ and of identity (4), the momentum equation of the fluid can be rewritten as

$$\partial_t(\alpha\rho\mathbf{u}) + \nabla\cdot(\alpha\rho\mathbf{u}\otimes\mathbf{u}) + \alpha\nabla p = D_\star \int (\mathbf{v}-\mathbf{u}) f dv. \tag{10}$$

One obtains then a balance law for the kinetic energy of the fluid

$$\partial_t\left(\alpha\rho\frac{|\mathbf{u}|^2}{2}\right) + \nabla\cdot\left(\alpha\rho\frac{|\mathbf{u}|^2}{2}\mathbf{u}\right) + \alpha\mathbf{u}\cdot\nabla p = D_\star\mathbf{u}\cdot\int(\mathbf{v}-\mathbf{u}) f dv. \tag{11}$$

The same balance law for the particles is obtained from the kinetic equation $\partial_t\left(m_\star \int \frac{|\mathbf{v}|^2}{2} f dv\right) + \nabla_x\cdot\left(m_\star \int \frac{|\mathbf{v}|^2}{2}\mathbf{v} f dv\right) = \left(\int m_\star\mathbf{v}\cdot\mathbf{\Gamma} f dv\right)$, that is

$$\partial_t\left(m_\star \int \frac{|\mathbf{v}|^2}{2} f dv\right) + \nabla_x\cdot\left(m_\star \int \frac{|\mathbf{v}|^2}{2}\mathbf{v} f dv\right) = -m_\star\nabla p\cdot\int \mathbf{v} f dv - D_\star \int (\mathbf{v}-\mathbf{u})\cdot\mathbf{v} f dv. \tag{12}$$

Making use of (4) and (8), the fluid internal energy equation (5) can be expanded as

$$\partial_t(\alpha\rho e) + \nabla\cdot(\alpha\rho e\mathbf{u}) + m_\star p\nabla\cdot\int \mathbf{v} f dv + p\nabla\cdot(\alpha\mathbf{u}) = D_\star\int|\mathbf{v}-\mathbf{u}|^2 f dv. \quad (13)$$

The summation of (11) – (13) yields the total energy equation in conservation form

$$\partial_t\left(\alpha\rho E + m_\star\int f\frac{|\mathbf{v}|^{\mathbf{2}}}{2}dv\right) \quad (14)$$
$$+\nabla\cdot\left(\alpha\rho E\mathbf{u} + m_\star\int\frac{|\mathbf{v}|^{\mathbf{2}}}{2}\mathbf{v} f dv + \alpha p\mathbf{u} + p m_\star\int f\mathbf{v} dv\right) = 0.$$

Here $E = e + \frac{|\mathbf{u}|^2}{2}$ is the fluid total energy, and the viscous tensor in the right hand side is defined by

$$(u\nabla\mathbf{u})_j = \mathbf{u}\cdot\partial_{x_j}\mathbf{u}.$$

In order to obtain the fluid entropy equation, we observe that the density equation yields

$$\rho\left(\partial_t\alpha + \nabla\cdot(\alpha\mathbf{u})\right) + \alpha D_t\rho = 0 \Longleftrightarrow \partial_t\alpha + \nabla\cdot(\alpha\mathbf{u}) = \alpha\rho D_t\tau,$$

where $\tau = 1/\rho > 0$ is the specific volume, and $D_t := \partial_t + \mathbf{u}\cdot\nabla$. The internal energy equation in (5) can be rewritten as

$$\alpha\rho\left(D_t e + pD_t\tau\right) = D_\star\int|\mathbf{v}-\mathbf{u}|^2 f dv.$$

Since we take the perfect gas pressure law (1) and the energy law (2), the entropy is defined by

$$S = C_v\log\left(e\rho^{1-\gamma}\right). \quad (15)$$

The second principle of thermodynamics writes

$$TdS = de + pd\tau, \quad T > 0,$$

and as a consequence,

$$\alpha\rho D_t S = \frac{D_\star}{T}\int|\mathbf{v}-\mathbf{u}|^2 f dv. \quad (16)$$

which can be rewritten

$$\partial_t\left(\alpha\rho S\right) + \nabla\cdot\left(\alpha\rho S\mathbf{u}\right) = \frac{D_\star}{T}\int|\mathbf{v}-\mathbf{u}|^2 f dv. \quad (17)$$

This entropy inequality shows the thermodynamical consistency of the model.

2.2 Positivity of the volume fraction for smooth solutions

The nonlinear stability of the model is related to the possibility of showing that $\alpha > 0$ for all times, provided that this property initially holds. This means that the concentration of particles cannot exceed the critical value $1/m_\star$. In what follows, we prove such a property for smooth flows defined in the time-space domain $\Omega = [0, T_{\text{end}}) \times \mathbb{R}^3$ for some $T_{\text{end}} \in (0, \infty]$.

We make standard assumptions about the positivity of the initial density of particles, that is

$$f(0, \mathbf{x}, \mathbf{v}) > 0, \qquad \text{for} \qquad (t, x) \in \Omega, \tag{18}$$

and about the positivity and boundedness at initial time of the fluid initial density

$$0 < \rho_- = \inf_{\mathbf{x} \in \mathbb{R}^3} \rho(0, \mathbf{x}) \leq \rho_+ = \sup_{\mathbf{x} \in \mathbb{R}^3} \rho(0, \mathbf{x}) < \infty. \tag{19}$$

The fluid entropy is also assumed to be initially lower bounded,

$$-\infty < S_- \leq \inf_{\mathbf{x} \in \mathbb{R}^3} S(0, \mathbf{x}), \tag{20}$$

and the volume fraction of the fluid to be positive at initial time

$$0 < \alpha_- = \inf_{\mathbf{x} \in \mathbb{R}^3} \alpha(0, \mathbf{x}) \leq 1. \tag{21}$$

This assumption implies that

$$\sup_{\mathbf{x} \in \mathbb{R}^3} \int f(0, \mathbf{x}, \mathbf{v}) dv \leq \frac{1 - \alpha_-}{m_\star} < \frac{1}{m_\star},$$

which is an upper bound for the initial density of particles.

We will then make the following regularity hypothesis on the velocity variables:

$$\mathbf{u} \in W^{1,\infty}(\Omega), \qquad \frac{\int f \mathbf{v} dv}{\int f dv} \in L^\infty(\Omega). \tag{22}$$

For the model problem (5), it is natural to consider that the pressure vanishes at infinity. We will therefore use the following assumption (in which there is uniformity in time of this limit), that is for all $\epsilon > 0$, there exists $A > 0$ such that

$$0 < p(t, \mathbf{x}) = (\gamma - 1)\rho(t, \mathbf{x})e(t, \mathbf{x})) < \epsilon, \quad \text{for } 0 \leq t < T_{\text{end}} \text{ and } |\mathbf{x}| > A. \tag{23}$$

This assumption is very natural for smooth flows defined on the whole space $\mathbb{R}^3$ (note that for a flow in a bounded domain, the boundary condition sometimes induces the boundedness of the pressure under the form $0 < p(t, \mathbf{y}) = (\gamma - 1)\rho(t, \mathbf{y})e(t, \mathbf{y}) < B$ for some particular $\mathbf{y} \in \mathbb{R}^3$).

We now write down a result of positivity of the volume fraction under the above assumptions.

Proposition 2.1. *Assume that a solution of (5) defined on the whole space $\mathbb{R}^3$ (and $\mathbb{R}^3 \times \mathbb{R}^3$ for f) is smooth on $[0, T_{\rm end})$ for some $T_{\rm end} \in]0, \infty[$. We also suppose that the assumptions on the initial data (18) – (21) hold, that $\alpha \in [0,1]$ on $[0, T_{\rm end}) \times \mathbb{R}^3$, and that the boundedness assumptions (22) – (23) hold. Then for some $C > 0$ depending only on $T_{\rm end}$, ρ_+, ρ_-, α_-, S_-, A (corresponding to $\varepsilon = 1$) and $||u||_{W^{1,\infty}(\Omega)}$, $\left|\left|\frac{\int f \mathbf{v} dv}{\int f dv}\right|\right|_{L^\infty(\Omega)}$, the following estimate holds:*

$$C \leq \alpha(t, \mathbf{x}) \leq 1, \qquad t \in [0, T_{\rm end}),\ \mathbf{x} \in \mathbb{R}^3. \tag{24}$$

Proof. The regularity $\mathbf{u} \in W^{1,\infty}(\Omega)$ and a classical treatment of the characteristic curves of the first eq. in (5) yields that for some $C_-, C_+ > 0$,

$$C_- = \inf_{\mathbf{x} \in \mathbb{R}^3} (\alpha\rho)(t, \mathbf{x}) \leq \sup_{\mathbf{x} \in \mathbb{R}^3} (\alpha\rho)(t, \mathbf{x}) = C_+, \qquad 0 \leq t < T_{\rm end}. \tag{25}$$

Therefore, if one manages to obtain an upper bound on the density ρ, it will yield a positive lower bound on α, which is the claim.

The momentum equation (10) yields

$$\alpha \nabla p = -\alpha\rho D_t \mathbf{u} + D_\star \frac{1-\alpha}{m_\star} \times \left(\frac{\int f \mathbf{v} dv}{\int f dv} - \mathbf{u} \right),$$

so that one can write for some $C > 0$ (in the rest of the proof, we use $C > 0$ for various constants depending on the parameters cited in Proposition 2.1)

$$\|\alpha \nabla p\|_{L^\infty(\Omega)} \leq C \left(1 + \|1 - \alpha\|_{L^\infty(\Omega)} \right). \tag{26}$$

For the perfect gas pressure and energy laws (2), one has

$$p = (\gamma - 1)\, \rho^\gamma\, e^{S/C_v}. \tag{27}$$

Therefore, the following identity holds:

$$\alpha \nabla p = \left[(\gamma - 1)^{\frac{1}{\gamma}} e^{S/(C_v \gamma)} \alpha\rho \right] \frac{1}{p^{\frac{1}{\gamma}}} \nabla p = \left[\frac{(\gamma - 1)^{\frac{1}{\gamma}} e^{S/(C_v \gamma)}}{1 - 1/\gamma} \alpha\rho \right] \nabla(p^{1 - 1/\gamma}).$$

Since $\alpha\rho$ is lower bounded by a positive constant thanks to (25) and S is lower bounded by means of (16), we see that thanks to estimate (26),

$$\left\| \nabla(p^{1-1/\gamma}) \right\|_{L^\infty(\Omega)} \leq C \left(1 + \|1 - \alpha\|_{L^\infty(\Omega)} \right).$$

Thanks to the hypothesis (23), one gets after integration wit respect to $\mathbf{x}$,

$$\left\| p^{1-1/\gamma} \right\|_{L^\infty(\Omega)} \leq C \left(1 + \|1 - \alpha\|_{L^\infty(\Omega)} \right).$$

By means of (27) and the lower boundedness of the entropy S, the density ρ is bounded

$$\left\| \rho^{\gamma - 1} \right\|_{L^\infty(\Omega)} \leq C \left(1 + \|1 - \alpha\|_{L^\infty(\Omega)} \right).$$

Then thanks to (25), one has $\|1/\alpha\|_{L^\infty(\Omega)} \leq C \left(1 + \|1 - \alpha\|_{L^\infty(\Omega)}^{1/(\gamma - 1)} \right) \leq C$.

□

3 Linear stability explained for the barotropic system

3.1 Presentation of a simplified system

What we mean by linear stability is stability of the linearized equations around a specific family of interesting exact solution, for some weighted L^2 norm. The method that we present below is quite explicit, however the algebra is somewhat cumbersome because the linear integro-differential system has coefficients which are time dependent. This is why it is useful to present the method for a model simpler than the initial one (5). We consider therefore a simpler model, with a barotropic type hypothesis, where the pressure law depends only on the density, that is

$$p = p(\rho), \qquad p'(\rho) > 0, \tag{28}$$

so that the equation for the energy is not needed.

It writes

$$\begin{cases} \partial_t(\alpha\rho) + \nabla \cdot (\alpha\rho\mathbf{u}) = 0, \\ \partial_t(\alpha\rho\mathbf{u}) + \nabla \cdot (\alpha\rho\mathbf{u} \otimes \mathbf{u}) + \nabla p = -m_\star \int \mathbf{\Gamma} f dv, \\ \partial_t f + \mathbf{v} \cdot \nabla_x f + \nabla_v \cdot (\mathbf{\Gamma} f) = 0, \end{cases} \tag{29}$$

with the closure relations (3) and (4).

3.2 An explicit solution of the system

We compute an explicit solution to system (29). We consider initial data for the gas which are homogeneous (independent on $\mathbf{x}$). Using Galilean invariance, we can moreover impose that the initial velocity is equal to 0. Those assumptions write

$$\begin{aligned} &\rho(0, \mathbf{x}) := \rho_0 > 0, \\ &\mathbf{u}(0, \mathbf{x}) := 0, \\ &\alpha(0, \mathbf{x}) := \alpha_0 > 0. \end{aligned}$$

Then we assume that the particles are also initially distributed in an homogeneous way w.r.t. $\mathbf{x}$, and in an isotropic way with respect to $\mathbf{v}$, that is

$$f(0, \mathbf{x}, \mathbf{v}) = \frac{n_0}{(K\,T_\star)^{\frac{3}{2}}} F\left(\frac{|\mathbf{v}|^2}{2T_\star}\right),$$

where $n_0 > 0$ does not depend on x, $T_\star > 0$ also does not depend on x (the introduction of this last constant is not really mandatory, but is useful if one wishes to fix some moment of the profile F), and F is a smooth function from $\mathbb{R}_+$ to $\mathbb{R}_+$. The case of a Maxwellian distribution function corresponds to $F(w) = e^{-w}$ for $w \geq 0$. In the above formula, $K > 0$ is defined by

$$K^{3/2} := 4\pi \int_0^\infty F(w)\,\sqrt{2w}\,dw,$$

so that $\int_{\mathbb{R}^3} f(0,\mathbf{x},\mathbf{v})\,dv = n_0$ and $\int_{\mathbb{R}^3} f(0,\mathbf{x},\mathbf{v})\,\mathbf{v}\,dv = 0$.

Lemma 3.1. *A solution to the system (29) with the closure relations (3), (4), (28) is*

$$\begin{array}{l} \rho(t,\mathbf{x}) = \rho_0, \\ \mathbf{u}(t,\mathbf{x}) = 0, \\ \alpha(t,\mathbf{x}) = \alpha_0 = 1 - m_\star\, n_0, \\ \mathbf{\Gamma} = -d_\star \mathbf{v}, \\ f(t,\mathbf{x},\mathbf{v}) = e^{3d_\star t} f(0,\mathbf{x},e^{d_\star t}\mathbf{v}) = e^{3d_\star t}\, \frac{n_0}{(K\,T_\star)^{\frac{3}{2}}}\, F\left(\frac{e^{2d_\star t}\,|\mathbf{v}|^2}{2T_\star}\right), \end{array} \tag{30}$$

where $d_* = \frac{D_\star}{m_\star} \geq 0$. If the drag/friction coefficient is non zero, then $d_* > 0$.

Remark 3.2. *The distribution function of the particles is homogeneous in space. It is equivalent to write*

$$f(t,\mathbf{x},v) = f_0(t,v) := \frac{n_0}{(K\,T_k(t))^{\frac{3}{2}}}\, F\left(\frac{|v|^2}{2T_k(t)}\right) \tag{31}$$

where $v = |\mathbf{v}|$ and the kinetic temperature $T_k(t)$ of the particules is

$$T_k(t) = T_\star e^{-2d_\star t}. \tag{32}$$

Note that $\int_{\mathbb{R}^3} f(t,\mathbf{x},v)\,dv = n_0$ and $\int_{\mathbb{R}^3} f(t,\mathbf{x},v)\,\mathbf{v}\,dv = 0$ for all time $t \geq 0$.

Remark 3.3. *The originality of the family of profiles (30) with respect to what is usually considered in stability analysis, is that it is time dependent (for non zero drag/friction $d_\star > 0$). Note that the density of particles tends to a Dirac mass in v as $t \to \infty$.*

Proof. In the first equation of (29), both terms vanish. In the second equation of (29), the only term which might be non zero is proportional to $\int \mathbf{v} f dv$. As already noticed in the remark, the function f is radially symmetric with respect to $\mathbf{v}$, so $\int \mathbf{v} f dv = 0$ and the second equation is verified. Finally, concerning the last equation of (29), it is easy to check that

$$\begin{aligned} &\partial_t f + \nabla_v \cdot (\mathbf{\Gamma} f) \\ &= \partial_t f + \mathbf{\Gamma}\cdot\nabla_v f + f\nabla_v\cdot\mathbf{\Gamma} \\ &= \partial_t f - d_\star \mathbf{v}\cdot\nabla_v f - 3d_\star f \\ &= e^{3d_\star t}\left[\partial_t g - d_\star \mathbf{v}\cdot\nabla_v g\right] \qquad (\text{where } g = e^{-3d_\star t} f) \\ &= 0, \end{aligned} \tag{33}$$

which ends the verification. □

3.3 Linearization

We linearize the system (29), (3 − 4) around the state $\rho_0 > 0$, $u_0 = 0$, $\alpha_0 = 1 - m_* n_0 > 0$ and f_0 (defined in (31)), known to be a solution thanks to the results of Lemma 3.1.

We consider therefore

$$\begin{cases} \rho(t,\mathbf{x}) &= \rho_0 + \varepsilon\rho_1(t,\mathbf{x}) + O(\varepsilon^2), \\ \mathbf{u}(t,\mathbf{x}) &= \varepsilon\mathbf{u}_1(t,\mathbf{x}) + O(\varepsilon^2), \\ \alpha(t,\mathbf{x}) &= \alpha_0 + \varepsilon\alpha_1(t,\mathbf{x}) + O(\varepsilon^2), \\ f(t,\mathbf{x},\mathbf{v}) &= f_0(t,v) + \varepsilon\sqrt{f_0(t,v)}e^{d_\star t} g_1(t,\mathbf{x},\mathbf{v}) + O(\varepsilon^2). \end{cases}$$

The linearization of the density equation writes

$$\alpha_0 \partial_t \rho_1 + \rho_0 \partial_t \alpha_1 + \alpha_0 \rho_0 \nabla\cdot\mathbf{u}_1 = 0,$$

which can be rewritten, using the linearized specific volume $\tau_1 = -\frac{\rho_1}{\rho_0^2}$, under the form

$$\begin{aligned} \alpha_0\rho_0\partial_t\tau_1 &= \alpha_0\nabla\cdot\mathbf{u}_1 + \partial_t\alpha_1 \\ &= \alpha_0\nabla\cdot\mathbf{u}_1 - m_*\,\partial_t\Big(\int\sqrt{f_0}\,e^{d_\star t}\,g_1 dv\Big) \\ &= \alpha_0\nabla\cdot\mathbf{u}_1 + m_\star\nabla\cdot\Big(\int\sqrt{f_0}\,e^{d_\star t}\,g_1\,\mathbf{v}\,dv\Big), \end{aligned}$$

thanks to the linearization of the mass conservation of the disperse phase

$$\partial_t\Big(m_*\int\sqrt{f_0}\,e^{d_\star t}\,g_1\,dv\Big) + \nabla\cdot\Big(m_*\int\sqrt{f_0}\,e^{d_\star t}\,g_1\,\mathbf{v}\,dv\Big) = 0.$$

Defining the speed of sound $c_0 := \sqrt{p'(\rho_0)}$, one can see that $\rho_1\,p'(\rho_0) = -\rho_0^2\,c_0^2\,\tau_1$, so that the linearization of the momentum equation can be written under the form

$$\alpha_0\rho_0\partial_t\mathbf{u}_1 = \alpha_0\rho_0^2c_0^2\nabla\tau_1 + m_\star d_\star\int\mathbf{v}\sqrt{f_0}e^{d_\star t}g_1 dv - m_\star d_\star\mathbf{u}_1\int f_0 dv.$$

The equation for g_1 is more tricky to get and we keep the notation $f_0(t,v)$ to make some terms clearer. One has

$$\begin{aligned} &\sqrt{f_0(t,v)}e^{d_\star t}\partial_t g_1 + g_1\partial_t(\sqrt{f_0(t,v)}e^{d_\star t}) + \sqrt{f_0(t,v)}e^{d_\star t}\mathbf{v}\cdot\nabla_x g_1 \\ &\qquad + \nabla_v\cdot\Big(\mathbf{\Gamma}_0\sqrt{f_0(t,v)}e^{d_\star t}g_1 + \mathbf{\Gamma}_1 f_0(t,v)\Big) = 0, \end{aligned}$$

where $\mathbf{\Gamma}_0 = -d_\star\mathbf{v}$ and $\mathbf{\Gamma}_1 = -\nabla(p'(\rho_0)\,\rho_1) + d_\star\mathbf{u}_1 = \rho_0^2c_0^2\nabla\tau_1 + d_\star\mathbf{u}_1$. So

$$\partial_t g_1 + \mathbf{v}\cdot\nabla_x g_1 + (\sqrt{f_0(t,v)}e^{d_\star t})^{-1}\left(\rho_0^2c_0^2\nabla\tau_1\cdot\nabla_v f_0\right)$$

$$= (\sqrt{f_0(t,v)}e^{d_\star t})^{-1}\Big[- g_1\partial_t(\sqrt{f_0(t,v)}e^{d_\star t}) - \nabla_v\cdot\Big(\mathbf{\Gamma}_0\sqrt{f_0(t,v)}e^{d_\star t}g_1\Big)$$

$$-d_\star \mathbf{u}_1 \cdot \nabla_v f_0(t,v)\Big].$$

One has the following formula for f_0

$$\nabla_v f_0(t,|\mathbf{v}|) = -\frac{\mathbf{v}}{T_\star} e^{2d_\star t}\left(-\frac{F'}{F}\right)\left(\frac{|v|^2}{2T_k(t)}\right) f_0(t,v),$$

which yields the identity

$$(\sqrt{f_0(t,v)}e^{d_\star t})^{-1}\nabla_v f_0(t,v) = -\frac{\mathbf{v}}{T_\star}\sqrt{f_0(t,v)}e^{d_\star t}\left(-\frac{F'}{F}\right)\left(\frac{|v|^2}{2T_k(t)}\right). \quad (34)$$

One gets

$$\partial_t g_1 + \mathbf{v}\cdot\nabla_x g_1 - \frac{\rho_0^2 c_0^2}{T_\star}\sqrt{f_0(t,v)}e^{d_\star t}\mathbf{v}\cdot\nabla\tau_1\left(-\frac{F'}{F}\right)\left(\frac{|v|^2}{2T_k(t)}\right)$$

$$= \frac{d_\star}{T_\star}\sqrt{f_0(t,v)}e^{d_\star t}\mathbf{v}\cdot\mathbf{u}_1\left(-\frac{F'}{F}\right)\left(\frac{|v|^2}{2T_k(t)}\right)$$

$$-(\sqrt{f_0(t,v)}e^{d_\star t})^{-1}\left[g_1\partial_t(\sqrt{f_0(t,v)}e^{d_\star t}) + \nabla_v\cdot\left(\mathbf{\Gamma}_0\sqrt{f_0(t,v)}e^{d_\star t}g_1\right)\right].$$

The opposite of the last term in the right hand side is

$$(\sqrt{f_0(t,v)}e^{d_\star t})^{-1}\left[g_1\partial_t(\sqrt{f_0(t,v)}e^{d_\star t}) + \nabla_v\cdot\left(\mathbf{\Gamma}_0\sqrt{f_0(t,v)}e^{d_\star t}g_1\right)\right]$$

$$= (\sqrt{f_0(t,v)}e^{d_\star t})^{-1}\left[\partial_t(\sqrt{f_0(t,v)}e^{d_\star t}) + \nabla_v\cdot\left(\mathbf{\Gamma}_0\sqrt{f_0(t,v)}e^{d_\star t}\right)\right]g_1 + \mathbf{\Gamma}_0\cdot\nabla_v g_1$$

$$= (\sqrt{f_0(t,v)}e^{d_\star t})^{-1}\Big[\partial_t(\sqrt{f_0(t,v)}e^{d_\star t}) + \nabla_v\cdot\left(\mathbf{\Gamma}_0\sqrt{f_0(t,v)}e^{d_\star t}\right)$$

$$-\frac{1}{2}[\nabla_v\cdot\mathbf{\Gamma}_0]\,(\sqrt{f_0(t,v)}e^{d_\star t})\Big]g_1 + \frac{1}{g_1}\nabla_v\cdot\left(\frac{1}{2}\mathbf{\Gamma}_0 g_1^2\right).$$

The term in front of g_1 is

$$(\sqrt{f_0(t,v)}e^{d_\star t})^{-1}$$

$$\times\left[\partial_t(\sqrt{f_0(t,v)}e^{d_\star t}) + \nabla_v\cdot\left(\mathbf{\Gamma}_0\sqrt{f_0(t,v)}e^{d_\star t}\right) - \frac{1}{2}[\nabla_v\cdot\mathbf{\Gamma}_0]\,(\sqrt{f_0(t,v)}e^{d_\star t})\right]$$

$$= (\sqrt{f_0(t,v)}e^{d_\star t})^{-1}\Big[\sqrt{f_0(t,v)}\partial_t e^{d_\star t} + e^{d_\star t}\partial_t\sqrt{f_0(t,v)}$$

$$+\mathbf{\Gamma}_0\cdot\nabla_v(\sqrt{f_0(t,v)}e^{d_\star t}) + \frac{1}{2}\nabla_v\cdot\mathbf{\Gamma}_0(\sqrt{f_0(t,v)}e^{d_\star t})\Big]$$

$$= (\sqrt{f_0(t,v)})^{-1}\left[\partial_t\sqrt{f_0(t,v)} + \mathbf{\Gamma}_0\cdot\nabla_v\sqrt{f_0(t,v)}\right] + \left(d_\star + \frac{1}{2}\nabla_v\cdot\mathbf{\Gamma}_0\right)$$

$$= (2f_0(t,v))^{-1}\left[\partial_t f_0(t,v) + \mathbf{\Gamma}_0\cdot\nabla_v f_0(t,v)\right] + \left(d_\star + \frac{1}{2}\nabla_v\cdot\mathbf{\Gamma}_0\right)$$

$$= -\frac{1}{2}\nabla_v \cdot \Gamma_0 + \left(d_\star + \frac{1}{2}\nabla_v \cdot \mathbf{\Gamma}_0\right) = d_*,$$

because f_0 which is defined by (31) satisfies (cf. eq. (33))

$$\partial_t f_0(t,v) + \mathbf{\Gamma}_0 \cdot \nabla_v f_0(t,v) + f_0(t,v)\nabla_v \cdot \mathbf{\Gamma}_0 = 0.$$

One gets

$$\partial_t g_1 + \mathbf{v}\cdot\nabla_x g_1 - \frac{\rho_0^2 c_0^2}{T_\star}\sqrt{f_0}\, e^{d_\star t}\mathbf{v}\cdot\nabla\tau_1 \left(-\frac{F'}{F}\right)\left(\frac{|v|^2}{2T_k(t)}\right) \tag{35}$$

$$= \frac{d_\star}{T_\star}\sqrt{f_0}\, e^{d_\star t}\mathbf{v}\cdot\mathbf{u}_1 \left(-\frac{F'}{F}\right)\left(\frac{|v|^2}{2T_k(t)}\right) - d_\star g_1 - \frac{1}{g_1}\nabla_v \cdot \left(\frac{1}{2}\mathbf{\Gamma}_0 g_1^2\right).$$

Regrouping the linearized equations for τ_1, u_1 and g_1, we end up with the system

$$\begin{cases} \alpha_0\rho_0\,\partial_t\tau_1 = \alpha_0\,\nabla\cdot\mathbf{u}_1 + m_\star\nabla\cdot\int\sqrt{f_0}\,e^{d_\star t}\,g_1\,\mathbf{v}\,dv, \\ \alpha_0\rho_0\,\partial_t\mathbf{u}_1 = \alpha_0\rho_0^2c_0^2\,\nabla\tau_1 + m_\star d_\star\int\mathbf{v}\sqrt{f_0}\,e^{d_\star t}\,g_1\,dv - m_\star d_\star\mathbf{u}_1\int f_0\,dv, \\ \partial_t g_1 + \mathbf{v}\cdot\nabla_x g_1 - \frac{\rho_0^2c_0^2}{T_\star}\sqrt{f_0}\,e^{d_\star t}\,\mathbf{v}\cdot\nabla\tau_1\left(-\frac{F'}{F}\right)\left(\frac{|v|^2}{2T_k(t)}\right) \\ = \frac{d_\star}{T_\star}\sqrt{f_0}\,e^{d_\star t}\,\mathbf{v}\cdot\mathbf{u}_1\left(-\frac{F'}{F}\right)\left(\frac{|v|^2}{2T_k(t)}\right) - d_\star g_1 + \frac{d_\star}{g_1}\nabla_v\cdot\left(\frac{1}{2}\mathbf{v}g_1^2\right). \end{cases} \tag{36}$$

This is a linear integro-differential system of equations, with coefficients which are homogeneous in space, but with a dependency in time. More precisely, if $d_\star = 0$ (no friction), then the coefficients become constant in space and time, however for non zero friction $d_\star > 0$, then terms like $e^{d_\star t}$ display exponential increase in time. Such coefficients are not a surprise, because the system is derived from the linearization around kinetic profiles which tend to a Dirac mass as $t \to \infty$.

3.4 Lyapunov functional

Taking our inspiration from stability analysis in plasma physics, we introduce a quadratic Lyapunov functional for the linearized system (36). This approach has been introduced in plasma physics in 58', we refer to [21, 14]. We point out its recent use for the analysis of the asymptotic stability around Maxwellian profiles in the context of the Vlasov Poisson equation, cf. [10].

For a given smooth function $F \geq 0$, we introduce the function

$$R(t,\mathbf{v}) = -\frac{F}{F'}\left(\frac{|\mathbf{v}|^2}{2T_k(t)}\right). \tag{37}$$

If we assume that F is strictly decreasing (more precisely, if $F' < 0$), then R is positive and well defined. In plasma physics, the strict monotony of F is a classical way to satisfy the Penrose stability criterion around radial profiles [24][page 45 and remark 2.2] and [18]. In the rest of this work, we will therefore make the assumption $R > 0$, which corresponds to $F' < 0$. A Maxwellian reference profile corresponds to $F(w) = e^{-\beta w}$, so that $R = \beta > 0$ is a constant in this case.

Proposition 3.4. *We consider constants $m_\star, T_\star > 0$ and $d_\star \geq 0$. We also consider constants $\rho_0 > 0$, $\alpha_0 = 1 - m_\star n_0 \in (0,1)$, $c_0 > 0$, and f_0 defined by (32), (31), where F is a smooth function such that $F' < 0$.*

Then, for any $T_{end} \in]0, \infty]$, all smooth quickly decaying when $|x| \to \infty$ (and $|v| \to \infty$ for g_1) solution $(\tau_1, \mathbf{u}_1, g_1)$ to the system (36) on $[0, T_{end}) \times \mathbb{R}^3$ ($\times \mathbb{R}^3$ for g_1) satisfy on $[0, T_{end})$ the differential inequality

$$\frac{d}{dt}\int \left[\alpha_0\rho_0\left(\frac{\rho_0^2 c_0^2}{2}\tau_1^2 + \frac{1}{2}|\mathbf{u}_1|^2\right) + m_\star T_\star \int \frac{1}{2} g_1^2\, R\, dv\right] dx \leq 0. \tag{38}$$

Proof. We compute

$$\partial_t\left(\alpha_0\rho_0\left(\frac{\rho_0^2 c_0^2}{2}\tau_1^2 + \frac{1}{2}|\mathbf{u}_1|^2\right) + m_\star T_\star \int \frac{1}{2} g_1^2\, R\, dv\right)$$

$$= \rho_0^2 c_0^2 \tau_1\, \partial_t(\alpha_0\rho_0\tau_1) + \mathbf{u_1}\cdot\partial_t(\alpha_0\rho_0\mathbf{u_1}) + m_\star\, T_\star \int g_1\, \partial_t g_1\, R\, dv + \frac{1}{2}\, m_\star\, T_\star \int g_1^2\, \partial_t R\, dv$$

$$= \rho_0^2 c_0^2 \tau_1\, \alpha_0\, \nabla\cdot\mathbf{u}_1 + \rho_0^2 c_0^2 \tau_1\, m_\star \nabla\cdot \int \sqrt{f_0}\, e^{d_\star t}\, g_1\, \mathbf{v}\, dv$$

$$+\mathbf{u_1}\cdot\alpha_0\rho_0^2 c_0^2\, \nabla\tau_1 + \mathbf{u_1}\cdot m_\star d_\star \int \mathbf{v}\sqrt{f_0}\, e^{d_\star t}\, g_1\, dv - m_\star d_\star\, |\mathbf{u_1}|^2 \int f_0\, dv$$

$$+ m_\star\, T_\star \int g_1\, [-\mathbf{v}\cdot\nabla_x g_1]\, R\, dv + m_\star \int g_1\, \rho_0^2 c_0^2\, \sqrt{f_0}\, e^{d_\star t}\, \mathbf{v}\cdot\nabla\tau_1 \left(-\frac{F'}{F}\right)\left(\frac{|v|^2}{2T_k(t)}\right) R\, dv$$

$$+ m_\star\, d_\star \int g_1\, \sqrt{f_0}\, e^{d_\star t}\, \mathbf{v}\cdot\mathbf{u}_1 \left(-\frac{F'}{F}\right)\left(\frac{|v|^2}{2T_k(t)}\right) R\, dv - m_\star\, T_\star\, d_\star \int g_1^2\, R\, dv$$

$$+ m_\star\, T_\star\, d_\star \int \nabla_v\cdot\left(\frac{1}{2}\mathbf{v} g_1^2\right) R\, dv + \frac{1}{2}\, m_\star\, T_\star \int g_1^2\, \partial_t R\, dv.$$

We see that

$$\partial_t\left(\alpha_0\rho_0\left(\frac{\rho_0^2 c_0^2}{2}\tau_1^2 + \frac{1}{2}|\mathbf{u}_1|^2\right) + m_\star T_\star \int \frac{1}{2} g_1^2\, R\, dv\right)$$

$$= \nabla\cdot\left(\alpha_0\rho_0^2 c_0^2\, \tau_1\, \mathbf{u_1} - m_\star\, T_\star \int \frac{g_1^2}{2}\, \mathbf{v}\, R\, dv + \rho_0^2 c_0^2\, m_\star \tau_1 \int \sqrt{f_0}\, e^{d_\star t}\, g_1\, \mathbf{v}\, dv\right)$$

$$+\mathbf{u_1}\cdot D_\star \int \mathbf{v}\sqrt{f_0}\, e^{d_\star t}\, g_1\, dv - D_\star\, |\mathbf{u_1}|^2 \int f_0\, dv$$

$$+m_\star\, d_\star \int g_1\, \sqrt{f_0}\, e^{d_\star t}\, \mathbf{v}\, dv\cdot\mathbf{u}_1 - m_\star\, T_\star\, d_\star \int g_1^2\, R\, dv$$

$$+\frac{1}{2}\, m_\star\, T_\star \int g_1^2\, (\partial_t R - d_*\, \mathbf{v}\cdot\nabla_v R)\, dv$$

$$= \nabla \cdot G - D_\star\, Q + \frac{1}{2}\, m_\star\, T_\star \int g_1^2 \left(- \frac{F}{F'} \right)' \left(\frac{|v|^2}{2T_k(t)} \right) \left(- \frac{\partial_t T_k}{T_k^2} \frac{|v|^2}{2} - d_* \frac{|v|^2}{T_k} \right) dv$$

$$= \nabla \cdot G - D_\star\, Q,$$

where

$$G := \alpha_0 \rho_0^2 c_0^2\, \tau_1\, \mathbf{u_1} - m_\star\, T_\star \int \frac{g_1^2}{2}\, \mathbf{v}\, R\, dv + \rho_0^2 c_0^2\, m_\star \tau_1 \int \sqrt{f_0}\, e^{d_\star t}\, g_1\, \mathbf{v}\, dv,$$

and $Q := |\mathbf{u_1}|^2 \int f_0\, dv - 2 \int g_1 \sqrt{f_0}\, e^{d_\star t}\, \mathbf{v}\, dv \cdot \mathbf{u}_1 + T_\star \int g_1^2\, R\, dv$. Lemma 3.5 below shows that Q can be recast as

$$Q = T_\star \int |h_1|^2\, R\, dv \geq 0, \qquad h_1 = \frac{1}{T_\star} \mathbf{u}_1 \cdot \mathbf{v} \sqrt{f_0}\, e^{d_\star t}\, R^{-1} - g_1. \tag{39}$$

Therefore one has by integration in space

$$\frac{d}{dt} \int \left(\alpha_0 \rho_0 \left(\frac{\rho_0^2 c_0^2}{2} \tau_1^2 + \frac{1}{2} |\mathbf{u_1}|^2 \right) + m_\star T_\star \int \frac{1}{2} g_1^2\, R\, dv \right) dx$$

$$= -D_\star \int Q\, dx \leq 0,$$

which is the claim. □

Lemma 3.5. *The following formula holds:*

$$T_\star \int \left| \frac{1}{T_\star} \mathbf{u}_1 \cdot \mathbf{v} \sqrt{f_0}\, e^{d_\star t}\, R^{-1} \right|^2 R\, dv = |\mathbf{u_1}|^2 \int f_0\, dv.$$

Proof. We denote $\mathbf{u}_1 = (\alpha, \beta, \gamma)$ and $\mathbf{v} = (v_1, v_2, v_3)$ and use classical symmetry arguments. One has

$$T_\star \int_{\mathbf{v} \in \mathbb{R}^3} \left| \frac{1}{T_\star} \mathbf{u}_1 \cdot \mathbf{v} \sqrt{f_0}\, e^{d_\star t}\, R^{-1} \right|^2 R\, dv = \frac{1}{T_\star} \int_{\mathbf{v} \in \mathbb{R}^3} |\mathbf{u}_1 \cdot \mathbf{v}|^2 f_0\, e^{2 d_\star t}\, R^{-1}\, dv$$

$$= \frac{1}{T_k(t)} \int_{\mathbf{v} \in \mathbb{R}^3} |\mathbf{u}_1 \cdot \mathbf{v}|^2 \frac{n_0}{(K\, T_k(t))^{\frac{3}{2}}}\, F\left(\frac{|v|^2}{2T_k(t)} \right) \left(- \frac{F'}{F} \right) \left(\frac{|v|^2}{2T_k(t)} \right) dv$$

$$= - \frac{1}{T_k(t)} \int |\alpha v_1 + \beta v_2 + \gamma v_3|^2\, \frac{n_0}{(K\, T_k(t))^{\frac{3}{2}}}\, F'\left(\frac{|v|^2}{2T_k(t)} \right) dv$$

$$= - \frac{1}{T_k(t)} \int \left(\alpha^2 v_1^2 + \beta^2 v_2^2 + \gamma^2 v_3^2 \right) \frac{n_0}{(K\, T_k(t))^{\frac{3}{2}}}\, F'\left(\frac{|v|^2}{2T_k(t)} \right) dv$$

$$= - \frac{1}{T_k(t)} \int \left(\alpha^2 + \beta^2 + \gamma^2 \right) v_1^2\, \frac{n_0}{(K\, T_k(t))^{\frac{3}{2}}}\, F'\left(\frac{|v|^2}{2T_k(t)} \right) dv$$

$$= - \frac{|\mathbf{u_1}|^{\mathbf{2}}}{T_k(t)} \int \frac{|v|^2}{3}\, \frac{n_0}{(K\, T_k(t))^{\frac{3}{2}}}\, F'\left(\frac{|v|^2}{2T_k(t)} \right) dv$$

$$= -\frac{2}{3}\,\frac{n_0\,|\mathbf{u_1}|^2}{K^{3/2}}\,4\pi\,\sqrt{2}\int u^{3/2}\,F'(u)\,du$$

$$= n_0\,|\mathbf{u}_1|^2\,K^{-3/2}\,4\pi\,\sqrt{2}\int u^{1/2}\,F(u)\,du$$

$$= n_0\,|\mathbf{u}_1|^2 = |\mathbf{u_1}|^2\int f_0\,dv.$$

□

4 Linear stability for the general case

The extension to the full Euler equations (including the equation for the energy) coupled with particles (5) is needed for applications [6]. We show how to linearize around the new reference exact solution and how to complement to Lyapunov functional with a Gronwall technique.

4.1 Exact solution

The exact solution of Section 3.2 can be generalized to the system (5) by giving the value of the internal energy $e_0(t)$.

Lemma 4.1. *The functions (30) – (32), complemented by*

$$e_0(t) = e_0(0) + \left(1 - e^{-2d_\star t}\right)\frac{m_\star}{\alpha_0\rho_0}K_\star, \tag{40}$$

with $K_* = \frac{1}{2}\int v^2\,f_0(0,\mathbf{v})\,dv$*, is a solution to the system (5). Under the physical condition* $e_0(0) > 0$*, the following estimate holds:*

$$0 < e_0(0) \le e_0(t) \le e_0(0) + \frac{m_\star}{\alpha_0\rho_0}\,K_\star < \infty \qquad 0 \le t < \infty. \tag{41}$$

Proof. Plugging (30) – (32) in the internal energy equation (5) or in the total energy equation (14) yields the equations

$$\alpha_0\rho_0 e_0'(t) = D_\star\int |\mathbf{v}|^2 f_0(t,\mathbf{v})dv.$$

The kinetic energy of particles is

$$\mathrm{Kin}(t) := \frac{1}{2}\int |\mathbf{v}|^2 f_0(t,\mathbf{v})dv = \int \frac{|\mathbf{v}|^2}{2}\,\frac{n_0}{(KT_k(t))^{\frac{3}{2}}}\,F\left(\frac{|\mathbf{v}|^2}{2T_k(t)}\right)\,dv = e^{-2d_\star t}\mathrm{Kin}(0).$$

Using this to calculate $e_0(t)$, one obtains (40) which is the first part of the claim. The last part is obtained directly from (40). □

Contrary to the density ρ_0 which is constant in time, the internal energy e_0 is monotone increasing with respect to time. It illustrates a transfer of the kinetic energy of the particles to the internal energy of the fluid. Note that the pressure $p_0(t) = p(\rho_0, e_0(t))$ is also monotone increasing because of eq. (1).

4.2 Linearization

We introduce the ansatz

$$e(t,x) = e_0(t) + \varepsilon\, e_1(t,\mathbf{x}) + O(\varepsilon^2),$$

$$p(t,x) = (\gamma_-1)\,\rho_0\, e_0(t) + \varepsilon\, p_1(t,\mathbf{x}) + O(\varepsilon^2),$$

and

$$S(t,x) = S_0(t) + \varepsilon\, S_1(t,\mathbf{x}) + O(\varepsilon^2),$$

where S is the entropy of the gas.

The linearized system associated with the full system (5) contains the following equations, which are close to those of (36):

$$\begin{cases} \alpha_0\rho_0\,\partial_t\tau_1 = \alpha_0\,\nabla\cdot\mathbf{u}_1 + m_\star\nabla\cdot\int\sqrt{f_0}\,e^{d_\star t}\,g_1\,\mathbf{v}\,dv, \\ \alpha_0\rho_0\,\partial_t\mathbf{u}_1 = -\alpha_0\nabla p_1 + D_\star\int\mathbf{v}\sqrt{f_0}\,e^{d_\star t}\,g_1\,dv - D_\star\mathbf{u}_1\int f_0\,dv, \\ \partial_t g_1 + \mathbf{v}\cdot\nabla_x g_1 + \frac{1}{T_\star}\sqrt{f_0}\,e^{d_\star t}\,\mathbf{v}\cdot\nabla p_1\left(-\frac{F'}{F}\right)\left(\frac{|v|^2}{2T_k(t)}\right) \\ = \frac{d_\star}{T_\star}\sqrt{f_0}\,e^{d_\star t}\,\mathbf{v}\cdot\mathbf{u}_1\left(-\frac{F'}{F}\right)\left(\frac{|v|^2}{2T_k(t)}\right) - d_\star g_1 + \frac{d_\star}{g_1}\nabla_v\cdot\left(\frac{1}{2}\mathbf{v}g_1^2\right). \end{cases} \tag{42}$$

The only difference between (36) (that is, the linearized equations in the barotropic case) and (42) is the first term in the right hand side of the velocity equation. Next we eliminate τ_1 in (42) since it can be written in terms of p_1 and S_1. For the perfect gas pressure law (1), (2), one has $dp = -(\rho^2c^2)d\tau + p/C_v dS$, where

$$c^2 := \gamma\,\frac{p}{\rho} = \gamma\,(\gamma-1)\,e$$

is the square of the speed of sound in the gas.

It yields the identity

$$\tau_1 = -\frac{1}{\rho_0^2c_0^2}p_1 + \frac{1}{\gamma C_v\rho_0}S_1.$$

One must take care that $c_0(t)^2 = \gamma(\gamma-1)e_0(t)$ is non constant with respect to the time variable (note however that it is bounded below and above by a strictly positive constant). This sole fact explains some of the unavoidable technicalities when treating the full model with internal energy. After differentiation with respect to the time variable, one gets

$$\partial_t\tau_1 = -\frac{1}{\rho_0c_0}\partial_t\left(\frac{p_1}{\rho_0c_0}\right) - \frac{p_1}{\rho_0c_0}\partial_t\left(\frac{1}{\rho_0c_0}\right) + \frac{1}{\gamma C_v\rho_0}\partial_t S_1.$$

Therefore it is possible to rewrite (42) as

$$\begin{cases} \alpha_0\rho_0\,\partial_t\left(\frac{p_1}{\rho_0 c_0}\right) = -\alpha_0\rho_0 c_0\nabla\cdot\mathbf{u}_1 - m_\star\rho_0 c_0\nabla\cdot\int\sqrt{f_0}\,e^{d_\star t}\,g_1\,\mathbf{v}\,dv + W, \\ \alpha_0\rho_0\,\partial_t\mathbf{u}_1 = -\alpha_0\nabla p_1 + D_\star\int\mathbf{v}\sqrt{f_0}\,e^{d_\star t}\,g_1\,dv - D_\star\mathbf{u}_1\int f_0\,dv, \\ \partial_t g_1 + \mathbf{v}\cdot\nabla_x g_1 + \frac{1}{T_\star}\sqrt{f_0}\,e^{d_\star t}\,\mathbf{v}\cdot\nabla p_1\left(-\frac{F'}{F}\right)\left(\frac{|v|^2}{2T_k(t)}\right) \\ = \frac{d_\star}{T_\star}\sqrt{f_0}\,e^{d_\star t}\,\mathbf{v}\cdot\mathbf{u}_1\left(-\frac{F'}{F}\right)\left(\frac{|v|^2}{2T_k(t)}\right) - d_\star g_1 + \frac{d_\star}{g_1}\nabla_v\cdot\left(\frac{1}{2}\mathbf{v}g_1^2\right). \end{cases} \tag{43}$$

The additional term in the first equation is

$$W = \alpha_0\rho_0^2 c_0\left(\frac{1}{\gamma C_v\rho_0}\partial_t S_1\right) - \alpha_0\rho_0 p_1\partial_t\left(\frac{1}{\rho_0 c_0}\right), \tag{44}$$

it is not present in the barotropic case.

In order to get a closed linearized system, one must complement the system (43) with one more equation for one of the unknowns e_1, p_1 or S_1. If one starts from the entropy identity (16), the calculations can be handled swiftly. One first writes thanks to (16)

$$\alpha\rho T D_t S = D_\star\int|\mathbf{v}-\mathbf{u}|^2 f dv.$$

Using the perfect gas pressure law (1), (15), one obtains

$$\alpha p D_t S = (\gamma-1)C_v D_\star\int|\mathbf{v}-\mathbf{u}|^2 f dv. \tag{45}$$

The linearization of this identity around the solution defined by (30), (31), (40) yields (using the radial symmetry of f_0 and $\mathbf{u}_0 = 0$ to simplify the right hand side)

$$\alpha_0 p_0\partial_t S_1 + \alpha_0 p_1\partial_t S_0 + \alpha_1 p_0\partial_t S_0 = (\gamma-1)C_v D_\star\int|\mathbf{v}|^2\sqrt{f_0}e^{d_\star t}g_1 dv. \tag{46}$$

Using the notation (39), it is rewritten as

$$\alpha_0 p_0\partial_t S_1 = -\alpha_0 p_1\partial_t S_0 - \alpha_1 p_0\partial_t S_0 + (\gamma-1)C_v D_\star\int|\mathbf{v}|^2\sqrt{f_0}e^{d_\star t}h_1 dv. \tag{47}$$

Two comments can be made about (47). The first one is that from the definition of the perfect gas equation of state (1), (2) and the identity (40), the following estimate holds:

$$\partial_t S_0 = C_v\frac{e_0'}{e_0} = O(e^{-2d_\star t}). \tag{48}$$

The second comment is that in the second term in the right hand side of eq. (47), the term α_1 can be rewritten as

$$\begin{aligned} \alpha_1 &= -e^{d_\star t}m_\star\int\sqrt{f_0}g_1 dv \\ &= -e^{d_\star t}m_\star\int\sqrt{f_0}h_1 dv + e^{d_\star t}m_\star\int\sqrt{f_0}(h_1-g_1)dv \\ &= -e^{d_\star t}m_\star\int\sqrt{f_0}h_1 dv. \end{aligned} \tag{49}$$

Indeed the difference $h_1 - g_1$ is an odd function with respect to any component of the velocity variable $\mathbf{v}$ because of (39), so the corresponding integral vanishes.

4.3 Linear stability

Linear stability is shown thanks to an adaptation of the Lyapunov functional argument of Section 3.4 to the system (43), (47). Let us define

$$a = \left\| \frac{p_1}{\rho_0\, c_0} \right\|^2_{L^2(\mathbb{R}^3)} + \|\mathbf{u}_1\|^2_{L^2(\mathbb{R}^3)} + \|S_1\|^2_{L^2(\mathbb{R}^3)} + \frac{m_\star\, T_\star}{\alpha_0\, \rho_0} \left\| g_1 \sqrt{R} \right\|^2_{L^2(\mathbb{R}^6)}, \quad (50)$$

where R is defined by (37).

Proposition 4.2. *We consider constants $m_\star, T_\star > 0$, $d_\star \geq 0$, $C_v > 0$, $\gamma > 1$. We also consider constants $\rho_0 > 0$, $\alpha_0 = 1 - m_\star\, n_0 \in (0,1)$, f_0 defined by (32), (31), where F is a smooth strictly decreasing function such that (remembering definition (37))*

$$r_- \leq R(t, \mathbf{v}) \leq r_+ \qquad \textit{for all } t, \mathbf{v}, \quad (51)$$

for some $0 < r_- \leq r_+$, and e_0 is defined by (40) (with $e_0(0) > 0$).

Then, for any $T_{end} \in]0, \infty]$, all smooth quickly decaying when $|x| \to \infty$ (and $|v| \to \infty$ for g_1) solution $(p_1, \mathbf{u}_1, S_1, g_1)$ to the system (43), (46) on $[0, T_{end}) \times \mathbb{R}^3$ ($\times \mathbb{R}^3$ for g_1) satisfy on $[0, T_{end})$ the inequality

$$a(t) \leq C\, a(0),$$

where a is defined in (50) and $C > 0$ is a constant which depends only on the parameters introduced in the proposition. Note that p_0, c_0 and S_0 used in the definition of the system (43), (46) are assumed to be related to ρ_0 and e_0 (and C_v, γ) by the thermodynamical identities coming out of (1), (2).

Proof. Multiplication of the first (resp. second, resp. third) equation of (43) by $\frac{p_1}{\rho_0 c_0}$ (resp. $\mathbf{u}_1$, resp. g_1) and integration yields (using notations (39) and (44))

$$\frac{d}{dt}\left[\alpha_0 \rho_0 \int \left(\frac{1}{2}\left(\frac{p_1}{\rho_0 c_0}\right)^2 + \frac{1}{2}|\mathbf{u}_1|^2 \right) dx + m_\star T_\star \int\int \frac{1}{2} g_1^2\, R\, dx dv \right]$$
$$= -D_\star T_\star \int\int \frac{1}{2} h_1^2\, R\, dx dv + \int W \frac{p_1}{\rho_0 c_0} dx.$$

It can be simplified as ($C > 0$ being here and in the sequel a generic constant depending only on the parameters introduced in the statement of the proposition)

$$\frac{d}{dt}\left(\left\| \frac{p_1}{\rho_0\, c_0} \right\|^2_x + \|\mathbf{u}_1\|^2_x + \frac{m_\star\, T_\star}{\alpha_0\, \rho_0} \left\| g_1 \sqrt{R} \right\|^2_{xv} \right) \leq -C \left\| h_1 \sqrt{R} \right\|^2_{xv} + C\, \|p_1\|_x \|W\|_x, \quad (52)$$

where we use the shorthands $|| \, ||_x$ and $|| \, ||_{xv}$ for the L^2 norms on $\mathbb{R}^3$ and $\mathbb{R}^3 \times \mathbb{R}^3$.

From (47) – (49), one gets

$$\|\partial_t S_1\|_x \le C\, e^{-2d_\star t}\, \|p_1\|_x + C\, e^{-d_\star t}\, \left\|\int \sqrt{f_0} h_1 dv\right\|_x + C\, e^{d_\star t}\, \left\|\int |\mathbf{v}|^2 \sqrt{f_0} h_1 dv\right\|_x . \tag{53}$$

We bound the second term in the right hand side of (53) by

$$\left\|\int \sqrt{f_0} h_1 dv\right\|_x \le \left(\int f_0 dv\right)^{\frac{1}{2}} \|h_1\|_{xv} \le C \left\|h_1 \sqrt{R}\right\|_{xv} .$$

Concerning the third term in the right hand side, one has $\left\|\int |\mathbf{v}|^2 \sqrt{f_0} h_1 \, dv\right\|_x \le \left(\int |\mathbf{v}|^4 f_0 dv\right)^{\frac{1}{2}} \|h_1\|_{xv}$. By virtue of the definition (31) of f_0, one can bound $\int |\mathbf{v}|^4 f_0 dv \le C e^{-4d_\star t}$. It yields

$$\left\|\int |\mathbf{v}|^2 \sqrt{f_0} h_1 dv\right\|_x \le C\, e^{-2d_\star t} \|h_1\|_{xv} \le C\, e^{-2d_\star t} \|h_1 \sqrt{R}\|_{xv}.$$

From (53), one obtains

$$\|\partial_t S_1\|_x \le C e^{-2d_\star t} \|p_1\|_x + C e^{-d_\star t} \|h_1 \sqrt{R}\|_{xv}. \tag{54}$$

Identity (44) yields (remembering (40) and $c_0^2 = \gamma\, (\gamma - 1)\, e_0$)

$$\|W\|_x \le C e^{-2d_\star t} \|p_1\|_x + \|\partial_t S_1\|_x,$$

which turns, thanks to (54), into

$$\|W\|_x \le C\, e^{-2d_\star t} \|p_1\|_x + C\, e^{-d_\star t} \|h_1 \sqrt{R}\|_{xv}.$$

Plugging this estimate in (52), one obtains

$$\begin{aligned} \frac{d}{dt}\left(\|\tfrac{p_1}{\rho_0\, c_0}\|_x^2 + \|\mathbf{u}_1\|_x^2 + \tfrac{m_\star\, T_\star}{\alpha_0\, \rho_0} \left\|g_1 \sqrt{R}\right\|_{xv}^2\right) \quad \le \quad & -C\, \|h_1 \sqrt{R}\|_{xv}^2 + C\, e^{-2d_\star t} \|p_1\|_x^2 \\ & + C\, e^{-d_\star t} \|p_1\|_x \|h_1 \sqrt{R}\|_{xv}. \end{aligned} \tag{55}$$

Inequality (54) also yields

$$\frac{d}{dt} \|S_1\|_x^2 \le C\, e^{-2d_\star t} \|p_1\|_x \|S_1\|_x + C\, e^{-d_\star t} \|h_1 \sqrt{R}\|_{xv} \|S_1\|_x. \tag{56}$$

The summation of (55) and (56) and a Cauchy-Schwarz inequality yield a control

$$\frac{d}{dt} a(t) \le -\, C\, \|h_1 \sqrt{R}\|_{xv}^2 + C\, e^{-2d_\star t} a(t) + C\, e^{-d_\star t} a(t)^{\frac{1}{2}} \|h_1 \sqrt{R}\|_{xv}.$$

Thanks to Young's inequality, it turns into

$$\frac{d}{dt} a(t) \le C\, e^{-2d_\star t} a(t).$$

A Gronwall lemma yields the final bound for all $t \ge 0$

$$a(t) \le a(0) e^{C\, \left(1 - e^{-2d_\star t}\right)}.$$

□

5 Models including more physics

In this section, we consider some natural physical extensions for which the tools developed in this work can be applied. In a first subsection, we consider colliding particles. In the second subsection, we consider particles with their own internal energy. We restrict in this last subsection the discussion to the presentation of the new terms in the model.

5.1 Extension to colliding particles

We introduce here the elastic Boltzmann collision kernel for droplets. We assume that the fluid is barotropic as in section 3.

We recall that in the Boltzmann theory for hard spheres, the kernel $Q := Q(f)$ acts on functions of v only (that is, t and x are parameters), and writes (up to a constant)

$$Q(f)(\mathbf{v}) = \int_{\mathbf{v}_* \in \mathbb{R}^3} \int_{\omega \in S^2} \Big[f(\mathbf{v}'_*)\, f(\mathbf{v}') - f(\mathbf{v}_*)\, f(\mathbf{v}) \Big] \, |(\mathbf{v} - \mathbf{v}_*) \cdot \omega| \, d\omega dv_*,$$

where $\mathbf{v}_*$ represents the velocity of a droplet which has just collided with the droplet under study (of velocity $\mathbf{v}$), and $\mathbf{v}'$, $\mathbf{v}'_*$ represent the velocities of two droplets which after collision will have velocities $\mathbf{v}$ and $\mathbf{v}_*$. The quantities $\mathbf{v}'$ and $\mathbf{v}'_*$ are given by the following parametrization (involving the unit vector $\boldsymbol{\omega}$):

$$\mathbf{v}' = \mathbf{v} - (\boldsymbol{\omega} \cdot (\mathbf{v} - \mathbf{v}_*))\, \boldsymbol{\omega},$$

$$\mathbf{v}'_* = \mathbf{v}_* + (\boldsymbol{\omega} \cdot (\mathbf{v} - \mathbf{v}_*))\, \boldsymbol{\omega}.$$

The factor $|(\mathbf{v} - \mathbf{v}_*) \cdot \boldsymbol{\omega}|$ is that of hard spheres, which is natural for macroscopic particles. We refer to [23] for the use of Boltzmann operator in the context of (thin) sprays.

In this subsection, the equation for f in the barotropic system (29) is modified by adding the collision kernel in the right hand side. The resulting system is called in the rest of this subsection the modified system, it writes

$$\left\{ \begin{array}{l} \partial_t(\alpha\rho) + \nabla \cdot (\alpha\rho\mathbf{u}) = 0, \\ \partial_t(\alpha\rho\mathbf{u}) + \nabla \cdot (\alpha\rho\mathbf{u} \otimes \mathbf{u}) + \nabla p = -m_\star \int \mathbf{\Gamma} f dv, \\ \partial_t f + \mathbf{v} \cdot \nabla_x f + \nabla_v \cdot (\mathbf{\Gamma} f) = Q(f), \end{array} \right. \tag{57}$$

with the closure relations (3) and (4).

Then one linearizes Q around a function $f_0 \geq 0$ assumed to be a time-dependant Maxwellian (thus $F(u) = \exp(-\beta(t)\, u)$ with the notation of section 3), that is $\ln f_0(t, \mathbf{v}) = a(t) + \mathbf{b}(t) \cdot \mathbf{v} - c(t)|\mathbf{v}|^2$, with $a(t) \geq 0$, $\mathbf{b}(t) \in \mathbb{R}^3$ and $c(t) > 0$. Using the expansion $f = f_0 + \varepsilon \sqrt{f_0}\, g_1$, one gets thanks to a classical computation (cf. [7] for example) that $Q(f_0) = 0$, and that

$$Q(f_0 + \varepsilon \sqrt{f_0}\, g_1)(\mathbf{v}) = \varepsilon\, (Lg_1)(v) + O(\varepsilon^2),$$

where L is the linearized Boltzmann operator (around f_0):

$$(Lg_1)(\mathbf{v}) := (\sqrt{f_0}(\mathbf{v}))^{-1} \int_{\mathbf{v}_*\in\mathbb{R}^3} \int_{\omega\in S^2} f_0(\mathbf{v})\, f_0(\mathbf{v}_*)$$

$$\times \left[\frac{g_1(\mathbf{v}'_*)}{\sqrt{f}_0(\mathbf{v}'_*)} + \frac{g_1(\mathbf{v}')}{\sqrt{f}_0(\mathbf{v}')} - \frac{g_1(\mathbf{v}_*)}{\sqrt{f}_0(\mathbf{v}_*)} - \frac{g_1(\mathbf{v})}{\sqrt{f}_0(\mathbf{v})} \right] |(\mathbf{v}-\mathbf{v}_*)\cdot\boldsymbol{\omega}|\, d\omega dv_*.$$

Moreover, it is well known (this is the linearized version of Boltzmann equation) that

$$\int_{\mathbf{v}\in\mathbb{R}^3} g_1(\mathbf{v})\,(Lg_1)(\mathbf{v})\, dv \le 0. \tag{58}$$

Thus the linearization of the modified system (29) around the time dependent Maxwellian $f_0(t,\mathbf{v})$ is just the system (36) where the equation of g_1 is modified by adding Lg_1 in the right hand side, that is

$$\begin{cases} \alpha_0\rho_0\, \partial_t\tau_1 = \alpha_0\, \nabla\cdot\mathbf{u}_1 + m_\star\nabla\cdot\int\sqrt{f_0}\, e^{d_\star t}\, g_1\,\mathbf{v}\, dv, \\ \alpha_0\rho_0\, \partial_t\mathbf{u}_1 = \alpha_0\rho_0^2 c_0^2\,\nabla\tau_1 + m_\star d_\star\int\mathbf{v}\sqrt{f_0}\, e^{d_\star t}\, g_1\, dv - m_\star d_\star\mathbf{u}_1\int f_0\, dv, \\ \partial_t g_1 + \mathbf{v}\cdot\nabla_x g_1 - \frac{\rho_0^2c_0^2}{T_\star}\sqrt{f_0}\, e^{d_\star t}\,\mathbf{v}\cdot\nabla\tau_1 \left(-\frac{F'}{F}\right)\left(\frac{|v|^2}{2T_k(t)}\right) \\ = \frac{d_\star}{T_\star}\sqrt{f_0}\, e^{d_\star t}\,\mathbf{v}\cdot\mathbf{u}_1\left(-\frac{F'}{F}\right)\left(\frac{|v|^2}{2T_k(t)}\right) - d_\star g_1 + \frac{d_\star}{g_1}\nabla_v\cdot\left(\frac{1}{2}\mathbf{v}g_1^2\right) + Lg_1. \end{cases} \tag{59}$$

Our observation is that

$$\mathcal{E} := \int \left[\alpha_0\rho_0 \left(\frac{\rho_0^2 c_0^2}{2}\tau_1^2 + \frac{1}{2}|\mathbf{u}_1|^2 \right) + m_\star T_\star \int \frac{1}{2} g_1^2\, R\, dv \right] dx$$

is still a Lyapunov functional (for the case $F(u) = \exp(-\beta(t)\,u)$, in other cases, this is not necessarily true).

Indeed, due to (37), $R = \frac{1}{\beta}$, and the proof is similar to the proof of Proposition 3.4. The only change is that one needs to use inequality (58) for the linearized operator L.

5.2 Extension to particles with internal energy

We consider in this subsection droplets which have their own temperature T_p related to an internal energy $e_p \ge 0$ (satisfying $e_p = C_{vp}\, T_p$ for the sake of simplicity). Their phase space density is then $f := f(t,x,v,e_p) \ge 0$, cf. [6].

We assume that internal energy is exchanged between the gas and a given droplet, at a rate proportional to $T - T_p$. The coefficient $N_\star$ of proportionality is assumed to be an absolute constant (related to the Nusselt number) for the sake of simplicity. The model (5) becomes, with these assumptions,

$$\begin{cases} \partial_t(\alpha\rho) + \nabla\cdot(\alpha\rho\mathbf{u}) = 0, \\ \partial_t(\alpha\rho\mathbf{u}) + \nabla\cdot(\alpha\rho\mathbf{u}\otimes\mathbf{u}) + \nabla p = -m_\star \int\int \mathbf{\Gamma} f dv de_p, \\ \partial_t(\alpha\rho e) + \nabla\cdot(\alpha\rho e\mathbf{u}) + p\left(\partial_t\alpha + \nabla\cdot(\alpha\mathbf{u})\right) = D_\star \int\int |\mathbf{v}-\mathbf{u}|^2 f dv de_p \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad + N_\star \int\int (T_p - T) f dv de_p, \\ \alpha = 1 - m_\star \int\int f dv de_p, \\ \partial_t f + \mathbf{v}\cdot\nabla_x f + \nabla_v\cdot(\mathbf{\Gamma} f) + \partial_{e_p}(\Phi f) = 0, \\ m_\star \Phi = N_\star (T - T_p), \\ m_\star \mathbf{\Gamma} = -m_\star \nabla p - D_\star(\mathbf{v}-\mathbf{u}). \end{cases} \tag{60}$$

We still consider a linear relation between the energy of the fluid e and the temperature of the fluid T, that is $e = C_v T$, as in (2).

5.2.1 Entropy property

Before constructing an exact solution which is a generalization of (31), we show that this model is thermodynamically consistant. We define the entropy of the particles $s_p = C_{vp} \log e_p$ by the differential relation

$$T_p ds_p := de_p.$$

For the simplicity of the notations, we shall take a system of units where $C_{vp} = 1$. The total entropy of the particules is then

$$\mathcal{S}_p = \int\int m_\star s_p f dv de_p.$$

Lemma 5.1. *The model (60) is endowed with the following entropy law*

$$\frac{d}{dt}\int (\alpha\rho S + \mathcal{S}_p)\, dx = \int\int\int \left(\frac{D_\star}{T}|\mathbf{v}-\mathbf{u}|^2 + N_\star \frac{(T-T_p)^2}{TT_p}\right) f dv de_p dx \geq 0. \tag{61}$$

Proof. We first proceed as in (17), and obtain the entropy law of the fluid

$$\partial_t(\alpha\rho S) + \nabla\cdot(\alpha\rho S\mathbf{u}) = \frac{D_\star}{T}\int\int |\mathbf{v}-\mathbf{u}|^2 f dv de_p - N_\star \int\int \frac{1}{T}(T-T_p) f dv de_p. \tag{62}$$

We then derive an entropy law for the particles

$$\partial_t(m_\star s_p f) + \nabla\cdot(\mathbf{v} m_\star s_p f) + \nabla_v\cdot(\mathbf{\Gamma} m_\star s_p f) + m_\star s_p \partial_{e_p}(\Phi f) = 0,$$

which yields

$$\partial_t \mathcal{S}_p + \nabla\cdot\left(\int\int \mathbf{v} m_\star s_p f dv de_p\right) + \int\int m_\star s_p \partial_{e_p}(\Phi f) dv de_p = 0.$$

An integration by parts of the last term shows that

$$\partial_t \mathcal{S}_p + \nabla\cdot\left(\int\int \mathbf{v} m_\star s_p f dv de_p\right) = N_\star \int\int \frac{1}{T_p}(T-T_p) f dv de_p. \tag{63}$$

□

5.2.2 Exact solutions

We show how to generalize (30), (31) in the case of system (60). We use another notation for the right hand side of (31), that is, we define

$$G(t, \mathbf{v}) := \frac{n_0}{(K\, T_k(t))^{\frac{3}{2}}}\, F\left(\frac{|\mathbf{v}|^2}{2T_k(t)}\right), \tag{64}$$

where $T_k(t)$ is still defined by (32). By construction

$$\partial_t G + \nabla_v \cdot (\mathbf{\Gamma} G) = 0, \qquad \mathbf{\Gamma} = -d_\star \mathbf{v}. \tag{65}$$

Let us now consider a nonnegative function H which depends on t and e_p as follows

$$H(t, e_p) := e^{\lambda t} H_0\left(e^{\lambda t}(e_p - U(t))\right), \tag{66}$$

where

$$U'(t) + \lambda U(t) = \lambda T(t),$$

and

$$\lambda = \frac{N_\star}{m_\star}.$$

We take $U(0) = 0$ so that $H(0, e_p) = H_0(e_p)$. With the natural assumption that $T(t) \geq 0$, one has $U(t) \geq 0$ for all time.
We observe that

$$\partial_t H + \partial_{e_p}(\Phi H) = 0, \tag{67}$$

where Φ is defined as in (60), that is

$$\Phi = \lambda(T(t) - T_p) = \lambda(T(t) - e_p), \tag{68}$$

thanks to the following computation:

$$\begin{aligned}
\partial_t H + \partial_{e_p}(\Phi H) &= \lambda e^{\lambda t} H_0 + \lambda e^{2\lambda t}(e_p - U(t))H_0' - e^{2\lambda t} U'(t) H_0' \\
&\quad + (\partial_{e_p}\Phi) H + \Phi \partial_{e_p} H \\
&= \lambda e^{\lambda t} H_0 + \lambda e^{2\lambda t}(e_p - U(t))H_0' - e^{2\lambda t} U'(t) H_0' \\
&\quad - \lambda H + \lambda(T(t) - e_p) e^{2\lambda t} H_0' \\
&= \left(-U'(t) + \lambda T(t) - \lambda U(t)\right) e^{2\lambda t} H_0' \\
&= 0.
\end{aligned}$$

It is convenient to assume that H_0 is continuous on $\mathbb{R}$, and has a compact support included in $\mathbb{R}_+$. The consequence is that H is endowed with an homogeneous Dirichlet condition at $e_p = 0$, that is

$$H(t, 0) = e^{\lambda t} H_0(-e^{\lambda t} U(t)) = 0 \text{ for all } t \geq 0. \tag{69}$$

Let us finally consider the function

$$f(t, \mathbf{v}, e_p) = G(t, \mathbf{v}) H(t, e_p), \tag{70}$$

which exhibits concentration with respect to both the velocity variable $\mathbf{v}$ and the energy variable e_p.

We observe that the function f is a spatially homogeneous solution to the kinetic equation

$$\partial_t f + \mathbf{v} \cdot \nabla_x f + \nabla_v \cdot (\mathbf{\Gamma} f) + \partial_{e_p}(\Phi f) = 0.$$

Indeed, since $\mathbf{v}$ and e_p are separate variables,

$$\begin{array}{rcl} \partial_t f + \nabla_v \cdot (\mathbf{\Gamma} f) + \partial_{e_p}(\Phi f) & = & \partial_t(GH) + \nabla_v \cdot (\mathbf{\Gamma} GH) + \partial_{e_p}(\Phi GH) \\ & = & H\left[\partial_t G + \nabla_v \cdot (\mathbf{\Gamma} G)\right] + G\left[\partial_t H + \nabla_v \cdot (\mathbf{\Phi} H)\right] \\ & = & \qquad 0 \qquad\qquad + \qquad\qquad 0. \end{array}$$

We now consider

$$\rho(t,\mathbf{x}) := \rho_0, \qquad \mathbf{u}(t,\mathbf{x}) := 0, \qquad \alpha(t,\mathbf{x}) := \alpha_0 = 1 - m_\star n_0, \tag{71}$$

and $e := e(t, \mathbf{x})$ depending only on t and satisfying

$$\alpha_0 \rho_0 e'(t) = D_\star \int_{\mathbf{v}\in\mathbb{R}^3} \int_{e_p>0} |\mathbf{v}|^2 G(t,\mathbf{v}) H(t,e_p) d\mathbf{v} de_p \tag{72}$$
$$- \int_{\mathbf{v}\in\mathbb{R}^3} \int_{e_p>0} N_\star (T - T_p) G(t,\mathbf{v}) H(t,e_p) d\mathbf{v} de_p,$$

which can be rewritten

$$\alpha_0 \rho_0 e'(t) = c_1 T_\star e^{-2d_\star t} - c_2 (T(t) + U(t)) + c_3 e^{-\lambda t},$$

where the coefficients c_1, c_2, c_3 are defined by

$$\begin{cases} c_1 = D_\star \frac{n_0}{K^{\frac{3}{2}}} \int_{\mathbb{R}^3} |\mathbf{w}|^2 G\left(\frac{|\mathbf{w}|^2}{2}\right) dw \int_{\mathbb{R}_+} H_0(e_p) de_p, \\ c_2 = N_\star \frac{n_0}{K^{\frac{3}{2}}} \int_{\mathbb{R}^3} G\left(\frac{|\mathbf{w}|^2}{2}\right) dw \int_{\mathbb{R}_+} H_0(e_p) de_p, \\ c_3 = N_\star \frac{n_0}{K^{\frac{3}{2}}} \int_{\mathbb{R}^3} G\left(\frac{|\mathbf{w}|^2}{2}\right) dw \int_{\mathbb{R}_+} e_p H_0(e_p) de_p. \end{cases}$$

We see that the quantities defined by (71), (70), (64), (32), (66) constitute a spatially homogeneous solution of system (60) provided that $T := T(t)$ and $U := U(t)$ are solutions to the following linear differential system (with right hand side):

$$\begin{cases} \alpha_0 \rho_0 T'(t) = -c_2 (T(t) + U(t)) \quad + c_1 T_\star e^{-2d_\star t} + c_3 e^{-\lambda t}, \\ U'(t) = -\lambda (T(t) + U(t)). \end{cases} \tag{73}$$

This system can be quickly solved by observing that thanks to a linear combination,

$$(T+U)' = -\left[\lambda + \frac{c_2}{\alpha_0\,\rho_0}\right](T+U) + \frac{c_1\,T_\star}{\alpha_0\,\rho_0}\, e^{-2d_\star t} + \frac{c_3}{\alpha_0\,\rho_0}\, e^{-\lambda t},$$

so that

$$(T+U)(t) = (T+U)(0)\, e^{-\left(\lambda + \frac{c_2}{\alpha_0\,\rho_0}\right)\,t} + \frac{c_1\,T_\star}{\alpha_0\,\rho_0}\;\frac{e^{-2d_\star t} - e^{-\left(\lambda+\frac{c_2}{\alpha_0\,\rho_0}\right)\,t}}{\lambda + \frac{c_2}{\alpha_0\,\rho_0} - 2d_\star}$$

$$+\frac{c_3}{c_2}\,\left(e^{-\lambda t} - e^{-\left(\lambda+\frac{c_2}{\alpha_0\,\rho_0}\right)\,t}\right),$$

and T and U are obtained by a direct integration of the two equations of system (73).

The obtained explicit homogeneous solution to system (60) can hopefully be used to investigate the linear stability. We leave this issue to future works.

5.3 Models with viscosity

Usually, thick spray models are inviscid (though sometimes some turbulent diffusion terms are introduced in those models). We just notice here that the presence of viscosity terms in the equation for $\alpha\rho\mathbf{u}$ does not affect the Lyapunov structure obtained in Prop. 3.4, it only changes the dissipation of the functional.

References

[1] Baranger C, Boudin L, Jabin P-E and Mancini S 2005 A modelling of biospray for the upper airways, CEMRACS 2004-mathematics and applications to biology and medicine ESAIM Proc. 14 41-7.

[2] C. Baranger, L. Desvillettes Coupling Euler and Vlasov Equations in the Context of Sprays : Local Smooth Solutions. Journal of Hyperbolic Differential Equations, vol. 3, n.1, (2006), pp. 1-26.

[3] C. Bardos and A. Nouri, A Vlasov equation with Dirac potential used in fusion plasmas. J. Math. Phys. 53 (2012), no. 11, 115621.

[4] E. Bernard, L. Desvillettes, F. Golse, V. Ricci: A Derivation of the Vlasov-Navier-Stokes Model for Aerosol Flows from Kinetic Theory, Commun. Math. Sciences , vol. 15, n.5 (2017), 1703-1741.

[5] E. Bernard, L. Desvillettes, F. Golse, V. Ricci: A Derivation of the Vlasov-Stokes System for Aerosol Flows from the Kinetic Theory of Binary Gas Mixtures, Kinetic Related Models, vol. 11, n.1 (2018), 43-69.

[6] L. Boudin, L. Desvillettes and R. Motte. A modeling of compressible droplets in a fluid. Commun. Math. Sci. 1 (2003), no. 4, 657-669.

[7] C. Cercignani *The Boltzmann Equation and its Applications*. Springer, **1988**.

[8] Y.P. Choi and S.B. Yun, Global existence of weak solutions for Navier-Stokes-BGK system. Nonlinearity 33 (2020), no. 4, 1925-1955.

[9] Constantin M. Dafermos, Hyperbolic Conservation Laws in Continuum Physics, Springer, 2010.

[10] B. Després, Symmetrization of Vlasov-Poisson equations. SIAM J. Math. Anal. 46 (2014), no. 4, 2554-2580.

[11] L. Desvillettes, F. Golse, V. Ricci: A formal passage from a system of Boltzmann equations for mixtures towards a Vlasov-Euler system of compressible fluids, Acta Math. Appl. Sinica vol. 35, n.1 (2019), 158-173, special issue.

[12] L. Desvillettes, J. Mathiaud. Some aspects of the asymptotics leading from gas-particles equations towards multiphase flows equations. Journal of Statistical Physics, vol. 141, n.1, (2010), pp. 120-141.

[13] R. Danchin, Fourier Analysis Methods for the Compressible Navier-Stokes Equations, in Handbook of Mathematical Analysis in Mechanics of Viscous Fluids, Y. Giga and A. Novotny editors, Springer, 2016.

[14] C. De Witt and J. F. Detoeuf, La théorie des gaz neutres et ionisés : le problème des n corps à température non nulle (The theory of neutral and ionized gases) Ecole d'été de physique théorique, Les Houches, 1959.

[15] J.K. Dukowicz, A particle-fluid numerical model for liquid sprays, J. Comp. Phys., 35:229-253, 1980.

[16] E. Godlewski and P.-A. Raviart, Hyperbolic Systems of Conservation Laws. Ellipses, 1991.

[17] E. Godlewski and P.-A. Raviart, Numerical Approximation of Hyperbolic Systems of Conservation Laws, Springer, 1996.

[18] E. Grenier, T. T. Nguyen and I. Rodnianski, Landau damping for analytic and Gevrey data, arxiv:2004.05979v2, 2020.

[19] Y. Guo and W.A. Strauss. Nonlinear instability of double-humped equilibria. Ann. Inst. H. Poincaré Anal. Non Linéaire, 12(3):339-352, 1995.

[20] M. Ishii and T. Hibiki. Thermo-fluid dynamics of two-phase flow. Second edition. Springer, New York, 2011.

[21] M. D. Kruskal and C. R. Oberman, On the Stability of Plasma in Static Equilibrium The Physics of Fluids 1, 275, 1958.

[22] H.-L. Li and L.-Y.Shou, Global well-posedness of one-dimensional compressible Navier-Stokes-Vlasov system. J. Differential Equations 280 (2021), 841-890.

[23] J. Mathiaud, Local smooth solutions of a thin spray model with collisions. Mathematical Models and Methods in Applied Sciences 02/2010; 20(02).

[24] C. Mouhot and C. Villani, On Landau damping, Acta Math. 207(1): 29-201, 2011.

[25] P. O'Rourke, Collective drop effects on vaporising liquid sprays PhD, Thesis, 1981, Princeton University, Princeton, NJ.

[26] D. Ramos, Quelques résultats mathématiques et simulations numériques d'écoulements régis par des modèles bifluides, PhD Thesis, 2000 (in French), ENS Cachan.

[27] W. Ranz and W. Marshall, Evaporization from drops Chem. Eng. Prog. 48, 141-80, 1952.

[28] F.A. Williams, Spray combustion and atomization Phys. Fluids 1 541-55, 1958.