Rapid decay and polynomial growth for bicrossed products

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Abstract

We study the rapid decay property and polynomial growth for duals of bicrossed products coming from a matched pair of a discrete group and a compact group.

1 Introduction

In the breakthrough paper paper [Ha78], Haagerup showed that the norm of the reduced C*-algebra $C_r^*(\mathbb{F}_N)$ of the free group on N-generators $\mathbb{F}_N$, can be controlled by the Sobolev l^2-norms associated to the word length function on $\mathbb{F}_N$. This is a striking phenomenon which actually occurs in many more cases. Jolissaint recognized this phenomenon, called Rapid Decay (or property (RD)), and studied it in a systematic way in [Jo90]. Property (RD) has now many applications. Let us mention the remarkable one concerning K-theory. Property (RD) allowed Jolissaint [Jo89] to show that the K-theory and $C_r^*(\Gamma)$ equals the K-theory of subalgebras of rapidly decreasing functions on Γ (Jolissaint did attribute this result to Connes). This result was then used by V. Lafforgue in his approach to the Baum-Connes conjecture via Banach KK-theory [La00, La02].

In this paper, we view discrete quantum groups as duals of compact quantum groups. The theory of compact quantum groups has been developed by Woronowicz [Wo87, Wo88, Wo98]. Property (RD) for discrete quantum groups has been introduced and studied by Vergnioux [Ve07]. Property (RD) has been refined later [BVZ14] in order to fit in the context of non-unimodular discrete quantum groups.

In this paper, we study the permanence of property (RD) under the bicrossed product construction. This construction was initiated by Kac [Ka68] in the context of finite quantum groups and was extensively studied later by many authors in different settings. The general construction, for locally compact quantum groups, was developed by Vaes-Vainerman [VV03]. In the context of compact quantum groups given by matched pairs of classical groups, an easier approach, that we will follow, was given by Fima-Mukherjee-Patri [FMP17].

Following [FMP17], the bicrossed product construction associates to a matched pair (Γ, G) of a discrete group Γ and a compact group G (see Section 2.2) a compact quantum group $\mathbb{G}$, called the bicrossed product. Given a length function l on the set of equivalence classes $\mathrm{Irr}(\mathbb{G})$ of irreducible unitary representations of $\mathbb{G}$ one can associate in a canonical way, as explained in Proposition 4.2, a pair of length functions (l_Γ, l_G) on Γ and $\mathrm{Irr}(G)$ respectively. Such a pair satisfies some compatibility relations and every pair of length functions (l_Γ, l_G) on $(\Gamma, \mathrm{Irr}(G))$ satisfying those compatibility relations will be called matched (see Definition 4.1). Any matched pair (l_Γ, l_G) on $(\Gamma, \mathrm{Irr}(G))$ allows one to reconstruct a canonical length function on $\mathrm{Irr}(\mathbb{G})$. The main result of the present paper is the following.

Theorem A. *Let (Γ, G) be a matched pair of a discrete group Γ and a compact group G. Denote by $\mathbb{G}$ the bicrossed product. The following are equivalent.*

1. *$\widehat{\mathbb{G}}$ has property (RD).*
2. *There exists a matched pair of length function (l_Γ, l_G) on $(\Gamma, \mathrm{Irr}(G))$ such that both (Γ, l_Γ) and $(\widehat{G}, l_G)$ have (RD).*

For amenable discrete groups, property (RD) is equivalent to polynomial growth [Jo90] and the same occurs for discrete quantum groups [Ve07]. Hence, for the compact classical group G one has that $(\widehat{G}, l_G)$ has (RD) if and only if it has polynomial growth. Note that a bicrossed product of a matched pair (Γ, G) is co-amenable if and only if Γ is amenable [FMP17]. The following theorem shows the permanence of polynomial growth under the bicrossed product construction.

Theorem B. *Let (Γ, G) be a matched pair of a discrete group Γ and a compact group G. Denote by $\mathbb{G}$ the bicrossed product. The following are equivalent.*

1. *$\widehat{\mathbb{G}}$ has polynomial growth.*
2. *There exists a matched pair of length function (l_Γ, l_G) on $(\Gamma, \mathrm{Irr}(G))$ such that both (Γ, l_Γ) and $(\widehat{G}, l_G)$ have polynomial growth.*

The main ingredient to prove Theorem A and B is the classification of the irreducible unitary representation of a bicrossed product and the fusion rules.

The paper is organized as follows. Section 2 is a preliminary section in which we introduce our notations. In section 3 we classify the irreducible unitary representation of a bicrossed product and describe their fusion rules. Finally, in section 4, we prove Theorem A and Theorem B.

2 Preliminaries

2.1 Notations

For a Hilbert space H, we denote by $\mathcal{U}(H)$ its unitary group and by $\mathcal{B}(H)$ the C*-algebra of bounded linear operators on H. When H is finite dimensional, we denote by Tr the unique trace on $\mathcal{B}(H)$ such that $\mathrm{Tr}(1) = \dim(H)$. We use the same symbol $\otimes$ for the tensor product of Hilbert spaces, unitary representations of compact quantum groups, minimal tensor product of C*-algebras. For a compact quantum group G, we denote by $\mathrm{Irr}(G)$ the set of equivalence classes of irreducible unitary representations and $\mathrm{Rep}(G)$ the collection of finite dimensional unitary representations. We will often denote by $[u]$ the equivalence class of an irreducible unitary representation u. For $u \in \mathrm{Rep}(G)$, we denote by $\chi(u)$ its character, i.e., viewing $u \in \mathcal{B}(H) \otimes C(G)$ for some finite dimensional Hilbert space H, one has $\chi(u) := (\mathrm{Tr} \otimes \mathrm{id})(u) \in C(G)$. We denote by $\mathrm{Pol}(G)$ the unital C*-algebra obtained by taking the Span of the coefficients of irreducible unitary representation, by $C_m(G)$ the enveloping C*-algebra of $\mathrm{Pol}(G)$ and by $C(G)$ the C*-algebra generated by the GNS construction of the Haar state on $C_m(G)$. We also denote by $\varepsilon : C_m(G) \to \mathbb{C}$ the counit and we use the same symbol $\varepsilon \in \mathrm{Irr}(G)$ to denote the trivial representation and its class in $\mathrm{Irr}(G)$. In the entire paper, the word representation means a unitary and finite dimensional representation.

2.2 Compact bicrossed products

In this section, we follow the approach and the notations of [FMP17].

Let (Γ, G) be a pair of a countable discrete group Γ and a second countable compact group G with a left action $\alpha : \Gamma \to \mathrm{Homeo}(G)$ of Γ on the compact space G by homeomorphisms and a right action $\beta : G \to S(\Gamma)$ of G on the discrete space Γ, where $S(\Gamma)$ is the Polish group of bijections of Γ, the topology being the one of pointwise convergence i.e., the smallest one for which the evaluation maps $S(\Gamma) \to \Gamma$, $\sigma \mapsto \sigma(\gamma)$ are continuous, for all $\gamma \in \Gamma$, where Γ has the discrete topology. Here, α is a group homomorphism and β is an antihomomorphism. The pair (Γ, G) is called a matched pair if $\Gamma \cap G = \{e\}$ with e being the common unit for both G and Γ, and if the actions α and β satisfy the following matched pair relations:

$$\forall g, h \in G,\ \gamma, \mu \in \Gamma, \quad \alpha_\gamma(gh) = \alpha_\gamma(g)\alpha_{\beta_g(\gamma)}(h),\ \beta_g(\gamma\mu) = \beta_{\alpha_\mu(g)}(\gamma)\beta_g(\mu) \quad \text{and} \quad \alpha_\gamma(e) = \beta_g(e) = e. \tag{2.1}$$

We also write $\gamma \cdot g := \beta_g(\gamma)$. From now on, we assume (Γ, G) is matched. It is shown in [FMP17, Proposition 3.2] that β is automatically continuous. By continuity of β and compactness of G, every β orbit is finite. Moreover, the sets $G_{r,s} := \{g \in G : r \cdot g = s\}$ are clopen (see [FMP17, Section 2.1]). Let $v_{rs} = 1_{G_{r,s}} \in C(G)$ be the characteristic function of $G_{r,s}$. It is shown in [FMP17, Section 2.1] that, for all β-orbits $\gamma \cdot G \in \Gamma/G$, the unitary $v_{\gamma \cdot G} := \sum_{r,s \in \gamma \cdot G} e_{rs} \otimes v_{rs} \in \mathcal{B}(l^2(\gamma \cdot G)) \otimes C(G)$ is a unitary representation of G as well as a magic unitary, where $e_{rs} \in \mathcal{B}(l^2(\gamma \cdot G))$ are the canonical matrix units and the Haar probability measure ν on G is α-invariant.

It is shown in [FMP17, Theorem 3.4] that there exists a unique compact quantum group $\mathbb{G}$, called the bicrossed product of the matched pair (Γ, G), such that $C(\mathbb{G}) = \Gamma_\alpha \ltimes C(G)$ is the reduced C*-algebraic crossed product, generated by a copy of $C(G)$ and the unitaries u_γ, $\gamma \in \Gamma$ and $\Delta : C(\mathbb{G}) \to C(\mathbb{G}) \otimes C(\mathbb{G})$ is the unique unital $*$-homomorphism satisfying $\Delta|_{C(G)} = \Delta_G$ (the comultiplication on $C(G)$) and $\Delta(u_\gamma) = \sum_{r\in\gamma\cdot G} u_\gamma v_{\gamma r} \otimes u_r$ for all $\gamma \in \Gamma$. It is also shown that the Haar state on $\mathbb{G}$ is a trace and is given by the formula $h(u_\gamma F) = \delta_{\gamma,1} \int_G F d\nu$ for all $\gamma \in \Gamma$ and $F \in C(G)$.

3 Representation theory of bicrossed products

3.1 Classification of irreducible representations

In this section we classify the irreducible representations of a bicrossed product. Let (Γ, G) be a matched pair of a discrete countable group Γ and a second countable compact group G with actions α, β.

For $\gamma \in \Gamma$ we denote by $G_\gamma := G_{\gamma,\gamma}$ the stabilizer of γ for the action $\beta : \Gamma \curvearrowleft G$. Note that G_γ is an open (hence closed) subgroup of G, hence of finite index: its index is $|\gamma\cdot G|$. We view $C(G_\gamma) = v_{\gamma\gamma} C(G) \subset C(G)$ as a non-unital C*-subalgebra. Let us denote by ν the Haar probability measure on G and note that $\nu(G_\gamma) = \frac{1}{|\gamma\cdot G|}$ so that the Haar probability measure ν_γ on G_γ is given by $\nu_\gamma(A) = |\gamma \cdot G|\, \nu(A)$ for all Borel subset A of G_γ.

For $\gamma \in \Gamma$ we fix a section, still denoted γ, $\gamma : \gamma \cdot G \to G$ of the canonical surjection $G \to \gamma \cdot G : g \mapsto \gamma \cdot g$. This means that $\gamma : \gamma\cdot G \to G$ is an injective map such that $\gamma\cdot\gamma(r) = r$ for all $r \in \gamma\cdot G$. We choose the section γ such that $\gamma(\gamma) = 1$, for all $\gamma \in \Gamma$. For $r, s \in \gamma \cdot G$, we denote by $\psi^\gamma_{r,s}$ the ν-preserving homeomorphism of G defined by $\psi^\gamma_{r,s}(g) = \gamma(r) g \gamma(s)^{-1}$. It follows from our choices that $\psi^\gamma_{\gamma,\gamma} = \mathrm{id}$ for all $\gamma \in \Gamma$. Moreover, for all $g \in G$, one has $\psi^\gamma_{r,s}(g) \in G_\gamma$ if and only if $g \in G_{r,s}$. It follows that $\psi^\gamma_{r,r}$ is an isomorphism and an homeomorphism from G_r to G_γ intertwining the Haar probability measures.

Let $u : G_\gamma \to \mathcal{U}(H)$ be a unitary representation of G_γ and view u as a continuous function $G \to \mathcal{B}(H)$ which is zero outside G_γ i.e. a partial isometry in $\mathcal{B}(H) \otimes C(G)$ such that $uu^* = u^*u = \mathrm{id}_H \otimes v_{\gamma\gamma}$. Define, for $r, s \in \gamma \cdot G$, the partial isometry $u_{r,s} := u \circ \psi^\gamma_{r,s} := (g \mapsto u(\psi^\gamma_{r,s}(g))) \in \mathcal{B}(H) \otimes C(G)$ and note that $u^*_{r,s} u_{r,s} = u_{r,s} u^*_{r,s} = \mathrm{id}_H \otimes 1_{G_{r,s}}$. In the sequel we view $u_{r,s} \in \mathcal{B}(H) \otimes C(G) \subset \mathcal{B}(H) \otimes C(\mathbb{G})$ and we define:

$$\gamma(u) := \sum_{r,s\in\gamma\cdot G} e_{rs} \otimes (1 \otimes u_r v_{rs}) u_{r,s} \in \mathcal{B}(l^2(\gamma \cdot G)) \otimes \mathcal{B}(H) \otimes C(\mathbb{G}),$$

where we recall that e_{rs}, for $r, s \in \gamma \cdot G$, are the matrix units associated to the canonical orthonormal basis of $l^2(\gamma \cdot G)$.

The irreducible unitary representations of $\mathbb{G}$ are described as follows.

Theorem 3.1. *The following holds.*

1. *For all $\gamma \in \Gamma$ and $u \in \mathrm{Rep}(G_\gamma)$ one has $\gamma(u) \in \mathrm{Rep}(\mathbb{G})$.*
2. *The character of $\gamma(u)$ is $\chi(\gamma(u)) = \sum_{r\in\gamma\cdot G} u_r v_{rr} \chi(u) \circ \psi^\gamma_{r,r}$.*
3. *For all $\gamma, \mu \in \Gamma$, $u \in \mathrm{Rep}(G_\gamma)$ and $w \in \mathrm{Rep}(G_\mu)$ one has*
$$\dim(\mathrm{Mor}_{\mathbb{G}}(\gamma(u), \mu(w))) = \delta_{\gamma\cdot G,\mu\cdot G} \dim(\mathrm{Mor}_{G_\gamma}(u, w \circ \psi^\mu_{\gamma,\gamma})).$$
4. *For all $\gamma \in \Gamma$ and $u \in \mathrm{Rep}(G_\gamma)$ one has $\overline{\gamma(u)} \simeq \gamma^{-1}(\overline{u} \circ \alpha_{\gamma^{-1}})$ (which makes sense since $\alpha_{\gamma^{-1}} : G_{\gamma^{-1}} \to G_\gamma$ is a group isomorphism and an homeomorphism).*
5. *$\gamma(u)$ is irreducible if and only if u is irreducible. Moreover, for any irreducible unitary representation u of $\mathbb{G}$ there exists $\gamma \in \Gamma$ and v an irreducible representation of G_γ such that $u \simeq \gamma(v)$.*

Proof. (1). Writing $\gamma(u)=\sum_{r,s}e_{r,s}\otimes V_{r,s}$, where $V_{r,s}:=(1\otimes u_r v_{rs})u_{r,s}\in\mathcal{B}(H)\otimes C(\mathbb{G})$, it suffices to check that, for all $r,s\in\gamma\cdot G$ one has $(\mathrm{id}\otimes\Delta)(V_{r,s})=\sum_{t\in\gamma\cdot G}(V_{r,t})_{12}(V_{t,s})_{13}$. We first claim that, for all $r,s\in\gamma\cdot G$, $(\mathrm{id}\otimes\Delta)(u_{r,s})=\sum_{t\in\gamma\cdot G}(u_{r,t})_{12}(u_{t,s})_{13}$. To check our claim, first recall that, for all $r,s\in\gamma\cdot G$ one has $\psi^{\gamma}_{r,s}(g)\in G_\gamma$ if and only if $r\cdot g=s$. Let $r,s\in\gamma\cdot G$ and $g,h\in G$. For $t=r\cdot g\in\gamma\cdot G$ one has :

$$u_{r,s}(gh)=u(\gamma(r)g\gamma(t)^{-1}\gamma(t)h\gamma(s)^{-1})=u(\psi^{\gamma}_{r,t}(g)\psi^{\gamma}_{t,s}(h))=\begin{cases}u_{r,t}(g)u_{t,s}(h) & \text{if } r\cdot gh=s,\\ 0 & \text{otherwise.}\end{cases}$$

Since we also have $u_{t,s}(h)=0$ whenever $r\cdot gh\neq s$ we find, in both cases, that $u_{r,s}(gh)=u_{r,t}(g)u_{t,s}(h)$. Now, for $t\neq r\cdot g$ we have $u_{r,t}(g)=0$ so the following formulae holds for any $r,s\in\gamma\cdot G$ and any $g,h\in G$:

$$v_{r,t}(g)u_{r,s}(gh)=u_{r,t}(g)u_{t,s}(h).$$

Hence, for all $r,s,t\in\gamma\cdot G$, $(1\otimes v_{r,t}\otimes 1)(\mathrm{id}\otimes\Delta)(u_{r,s})=(u_{r,t})_{12}(u_{t,s})_{13}$. Using this we find:

$$\begin{aligned}\sum_{t\in\gamma\cdot G}(V_{r,t})_{12}(V_{t,s})_{13} &= \sum_t(1\otimes u_r v_{rt}\otimes 1)(u_{r,t})_{12}(1\otimes 1\otimes u_t v_{ts})(u_{t,s})_{13}\\ &= \sum_t(1\otimes u_r v_{rt}\otimes u_t v_{ts})(u_{r,t})_{12}(u_{t,s})_{13}=\left(1\otimes(\sum_t u_r v_{rt}\otimes u_t v_{ts})\right)(\mathrm{id}\otimes\Delta)(u_{r,s}).\end{aligned}$$

Since v_γ is a unitary representation of G and a magic unitary we also have:

$$\Delta(u_r v_{rs})=\sum_{t,t'}(u_r v_{rt}\otimes u_t)(v_{rt'}\otimes v_{t's})=\sum_t u_r v_{rt}\otimes u_t v_{ts}.$$

This shows that $\gamma(u)$ is a representation of $\mathbb{G}$. We now check that $\gamma(u)$ is unitary. As before, since for all $r,s\in\gamma\cdot G$ one has $\psi^{\gamma}_{r,s}(g)\in G_\gamma$ if and only if $r\cdot g=s$ and because u is a unitary representation of G_γ, we have, for all $r,t\in\gamma\cdot G$, $(1\otimes v_{rt})u_{r,t}u^*_{r,t}=1\otimes v_{rt}$. Hence,

$$\begin{aligned}\sum_{t\in\gamma\cdot G}V_{r,t}V^*_{s,t} &= \sum_t(1\otimes u_r)(1\otimes v_{rt})u_{r,t}u^*_{s,t}(1\otimes v_{st})(1\otimes u^*_s)\\ &= \delta_{r,s}(1\otimes u_r)\left(\sum_t(1\otimes v_{rt})u_{r,t}u^*_{r,t}\right)(1\otimes u^*_r)=\delta_{r,s}(1\otimes u_r)\left(\sum_t(1\otimes v_{rt})\right)(1\otimes u^*_r)\\ &= \delta_{r,s}.\end{aligned}$$

A similar computations shows that $\sum_{t\in\gamma\cdot G}V^*_{t,r}V_{t,s}=\delta_{r,s}$.
(2). The character of $\gamma(u)$ is given by

$$\chi(\gamma(u))=\sum_{r\in\gamma\cdot G}(\mathrm{Tr}\otimes\mathrm{id})(V_{r,r})=\sum_r u_r v_{rr}(\mathrm{Tr}\otimes\mathrm{id})(u_{r,r})=\sum_r u_r v_{rr}\chi(u)\circ\psi^{\gamma}_{r,r}.$$

(3). Let $\gamma,\mu\in\Gamma$ and u,w be representations of G_γ and G_μ respectively. Since the Haar measure on G is invariant under the action α and the homeomorphisms $\psi^{\gamma}_{r,r}$ and $\psi^{\mu}_{r,r}$, we find, by the character formulae in 2 and the crossed-product relations,

$$\begin{aligned}\dim(\mathrm{Mor}(\gamma(u),\mu(w))) &= h(\chi(\gamma(u))\chi(\mu(w))^*)=\sum_{r\in\gamma\cdot G,\,s\in\mu\cdot G}h(u_{rs^{-1}}\alpha_s(v_{rr}v_{ss}\chi(u)\circ\psi^{\gamma}_{r,r}(\chi(w)\circ\psi^{\mu}_{s,s})^*))\\ &= \delta_{\gamma\cdot G,\mu\cdot G}\sum_{r\in\gamma\cdot G}\int_G\alpha_r(v_{rr}(\chi(u)\circ\psi^{\gamma}_{r,r})(\overline{\chi(w)}\circ\psi^{\mu}_{r,r}))d\nu\\ &= \delta_{\gamma\cdot G,\mu\cdot G}\sum_{r\in\gamma\cdot G}\int_{G_r}(\chi(u)\circ\psi^{\gamma}_{r,r})(\chi(\overline{w})\circ\psi^{\mu}_{r,r})d\nu\\ &= \delta_{\gamma\cdot G,\mu\cdot G}\sum_{r\in\gamma\cdot G}\int_{G_\mu}\chi(u)\circ(\psi^{\mu}_{\gamma,\gamma})^{-1}(\chi(\overline{w})\circ\psi^{\mu}_{r,r}\circ(\psi^{\gamma}_{r,r})^{-1}\circ(\psi^{\mu}_{\gamma,\gamma})^{-1})d\nu\end{aligned}$$

Now, note that $\psi^{\mu}_{r,r} \circ (\psi^{\gamma}_{r,r})^{-1} \circ (\psi^{\mu}_{\gamma,\gamma})^{-1} = \mathrm{Ad}(h)$, where $h = \mu(r)\gamma(r)^{-1}\mu(\gamma)^{-1}$. Moreover, $\mu \cdot h = \mu$ since:

$$\mu \cdot \mu(r)\gamma(r)^{-1}\mu(\gamma)^{-1} = r \cdot \gamma(r)^{-1}\mu(\gamma)^{-1} = \gamma \cdot \mu(\gamma)^{-1} = \mu.$$

Hence, $h \in G_{\mu}$. Since the characters of finite dimensional unitary representation of a group Λ are central functions i.e. invariant under $\mathrm{Ad}(\lambda)$ for all $\lambda \in \Lambda$, we have $\chi(\overline{w}) \circ \psi^{\mu}_{r,r} \circ (\psi^{\gamma}_{r,r})^{-1} \circ (\psi^{\mu}_{\gamma,\gamma})^{-1} = \chi(\overline{w}) \circ \mathrm{Ad}(h) = \chi(\overline{w})$. Hence:

$$\begin{aligned}\dim(\mathrm{Mor}(\gamma(u),\mu(w))) &= \delta_{\gamma\cdot G,\mu\cdot G}\sum_{r\in\gamma\cdot G}\int_{G_\mu}\chi(u)\circ(\psi^{\mu}_{\gamma,\gamma})^{-1}\chi(\overline{w})d\nu = \delta_{\gamma\cdot G,\mu\cdot G}\int_{G_\mu}\chi(u)\circ(\psi^{\mu}_{\gamma,\gamma})^{-1}\chi(\overline{w})d\nu_\mu \\ &= \delta_{\gamma\cdot G,\mu\cdot G}\dim(\mathrm{Mor}_{G_\mu}(u\circ(\psi^{\mu}_{\gamma,\gamma})^{-1},w)) = \delta_{\gamma\cdot G,\mu\cdot G}\int_{G_\gamma}\chi(u)\chi(\overline{w}\circ\psi^{\mu}_{\gamma,\gamma})d\nu_\mu \\ &= \delta_{\gamma\cdot G,\mu\cdot G}\dim(\mathrm{Mor}_{G_\gamma}(u,w\circ\psi^{\mu}_{\gamma,\gamma}).\end{aligned}$$

(4). Note that, by the bicrossed product relations, we have, for all $\gamma \in \Gamma$ and $g \in G$, $(\gamma \cdot g)^{-1} = \gamma^{-1} \cdot \alpha_\gamma(g)$. Hence $v_{\gamma^{-1}\gamma^{-1}} \circ \alpha_\gamma = v_{\gamma\gamma}$ and $(\gamma \cdot G)^{-1} = \gamma^{-1} \cdot G$. In particular, $\alpha_\gamma : G_\gamma \to G_{\gamma^{-1}}$ is an homeomorphism and, by the bicrossed product relations, one has, for all $g \in G_\gamma$ and $h \in G$, $\alpha_\gamma(gh) = \alpha_\gamma(g)\alpha_{\gamma\cdot g}(h) = \alpha_\gamma(g)\alpha_\gamma(h)$ so that $\alpha_\gamma : G_\gamma \to G_{\gamma^{-1}}$ is also a group homomorphism.

For $r \in \gamma \cdot G$ one has $\gamma^{-1} \cdot \alpha_\gamma(\gamma(r)) = (\gamma \cdot \gamma(r))^{-1} = r^{-1} = \gamma^{-1} \cdot \gamma^{-1}(r^{-1})$. This implies that, for all $\gamma \in \Gamma$, there exists a map $\eta_\gamma : \gamma \cdot G \to G_{\gamma^{-1}}$ such that, for all $r \in \gamma \cdot G$, one has $\alpha_\gamma(\gamma(r)) = \eta_\gamma(r)\gamma^{-1}(r^{-1})$.

Let now $r \in \gamma \cdot G$ and $g \in G_r$. One has, using the bicrossed product relations, that $e = \alpha_r(\gamma(r)\gamma(r)^{-1}) = \alpha_\gamma(\gamma(r))\alpha_r(\gamma(r)^{-1})$, hence

$$(\alpha_\gamma \circ \psi^{\gamma}_{r,r})(g) = \alpha_\gamma(\gamma(r))\alpha_r(g)\alpha_r(\gamma(r)^{-1}) = \alpha_\gamma(\gamma(r))\alpha_r(g)\big(\alpha_\gamma(\gamma(r))\big)^{-1} = \eta_\gamma(r)(\psi^{\gamma^{-1}}_{r^{-1},r^{-1}} \circ \alpha_r)(g)(\eta_\gamma(r))^{-1}.$$

Hence, for all $\gamma \in \Gamma$, if $w \in \mathrm{Rep}(G_{\gamma^{-1}})$, since $\chi(w) \in C(G_{\gamma^{-1}})$ is central we have

$$\chi(w) \circ \alpha_\gamma \circ \psi^{\gamma}_{r,r}(g) = \chi(w) \circ \psi^{\gamma^{-1}}_{r^{-1},r^{-1}} \circ \alpha_r(g) \quad \text{for all } r \in \gamma \cdot G,\ g \in G_r.$$

Since, as we seen above, $\gamma^{-1} \cdot G = (\gamma \cdot G)^{-1}$ and because $\chi(\overline{u} \circ \alpha_{\gamma^{-1}}) = \chi(\overline{u}) \circ \alpha_{\gamma^{-1}}$ we find, by the character formulae in 2, $\chi(\gamma^{-1}(\overline{u} \circ \alpha_{\gamma^{-1}})) = \sum_{r\in\gamma\cdot G} u_{r^{-1}}v_{r^{-1}r^{-1}}\chi(\overline{u}) \circ \alpha_{\gamma^{-1}} \circ \psi^{\gamma^{-1}}_{r^{-1},r^{-1}}$. It then follows from the crossed-product relations and the discussion above :

$$\begin{aligned}\chi(\gamma^{-1}(\overline{u}\circ\alpha_{\gamma^{-1}})) &= \sum_{r\in\gamma\cdot G} u_{r^{-1}}v_{r^{-1}r^{-1}}\chi(\overline{u})\circ\alpha_{\gamma^{-1}}\circ\psi^{\gamma^{-1}}_{r^{-1},r^{-1}} \\ &= \sum_{r\in\gamma\cdot G}(\chi(\overline{u})\circ\alpha_{\gamma^{-1}}\circ\psi^{\gamma^{-1}}_{r^{-1},r^{-1}}\circ\alpha_r)(v_{r^{-1}r^{-1}}\circ\alpha_r)u_{r^{-1}} \\ &= \sum_{r\in\gamma\cdot G}\chi(\overline{u})\circ\psi^{\gamma}_{r,r}v_{rr}u_r^* = \sum_{r\in\gamma\cdot G}(\chi(u)\circ\psi^{\gamma}_{r,r}v_{rr})^*u_r^* \\ &= \chi(\gamma(u))^*\end{aligned}$$

(5). The statement on irreducibility following from 3, it suffices, by the general theory, to show that the linear span X of coefficients of representations of the form $\gamma(u)$, for $\gamma \in \Gamma$ and u an irreducible unitary representation of G_γ, is a dense subset of $C(\mathbb{G})$. Note that, for all $\gamma \in \Gamma$, the relation $1 = \sum_{r\in\gamma\cdot G} v_{\gamma r}$ implies that any function in $C(G)$ is a sum of continuous functions with support in $G_{\gamma,r} := \{g \in G : \gamma \cdot g = r\}$, for $r \in \gamma \cdot G$. Moreover, since $G_{\gamma,r} = (\psi^{\gamma}_{\gamma,r})^{-1}(G_\gamma)$, any continuous function on G with support in $G_{\gamma,r}$ is of the form $F \circ \psi^{\gamma}_{\gamma,r}$, where $F \in C(G_\gamma)$. Since the linear span of coefficients of irreducible unitary representation of G_γ is dense in $C(G_\gamma)$, it suffices to show that, for any $\gamma \in \Gamma$, for any irreducible unitary representation of G_γ, $u : G_\gamma \to \mathcal{U}(H)$, any coefficient $u_{ij} \in C(G_\gamma) = v_{\gamma\gamma}C(G) \subset C(G)$ satisfies $u_\gamma u_{ij} \in X$. But this is obvious since one has

$$u_\gamma u_{ij} = u_\gamma v_{\gamma\gamma}u_{i,j} = u_\gamma v_{\gamma\gamma}u_{i,j} \circ \psi^{\gamma}_{\gamma,\gamma} = \gamma(u)_{\gamma,\gamma,i,j} \in X. \qquad \square$$

Finally, the fusion rules are described as follows.

Let $\gamma,\mu\in\Gamma$, $u:G_\gamma\to\mathcal{U}(H_u)$, $v:G_\mu\to\mathcal{U}(H_v)$ by unitary representations of G_γ and G_μ respectively. For any $r\in(\gamma\cdot G)(\mu\cdot G)$, we define the r-twisted tensor product of u and v, denoted $u\underset{r}{\otimes}v$ as a unitary representation of G_r on $K_r\otimes H_u\otimes H_v$, where

$$K_r:=\mathrm{Span}(\{e_s\otimes e_t\,:\,s\in\gamma\cdot G\text{ and }t\in\mu\cdot G\text{ such that }st=r\})\subset l^2(\gamma\cdot G)\otimes l^2(\mu\cdot G).$$

For $g\in G$, we define:

$$(u\underset{r}{\otimes}v)(g)=\sum_{\substack{s,s'\in\gamma\cdot G\\ t,t'\in\mu\cdot G\\ st=r=s't'}}e_{ss'}\otimes e_{tt'}\otimes v_{ss'}(\alpha_t(g))v_{tt'}(g)u(\psi^\gamma_{s,s'}(\alpha_t(g)))\otimes v(\psi^\mu_{t,t'}(g))\in\mathcal{U}(K_r\otimes H_u\otimes H_v).$$

Theorem 3.2. *The following holds.*

1. *For all $\gamma,\mu\in\Gamma$, all $r\in(\gamma\cdot G)(\mu\cdot G)$ and all u,v finite dimensional unitary representations of G_γ, G_μ respectively the element $u\underset{r}{\otimes}v$ is a unitary representation of G_r.*
2. *The character of $u\underset{r}{\otimes}w$ is $\chi(u\underset{r}{\otimes}v)=\sum_{s\in\gamma\cdot G,t\in\mu\cdot G,\,st=r}(v_{ss}\circ\alpha_t)v_{tt}(\chi(u)\circ\psi^\gamma_{s,s}\circ\alpha_t)(\chi(v)\circ\psi^\mu_{t,t})$.*
3. *For all $\gamma_1,\gamma_2,\gamma_3\in\Gamma$ and all u,v,w unitary representations of G_{γ_1}, G_{γ_2} and G_{γ_3} respectively, the number $\dim(\mathrm{Mor}_{\mathbb{G}}(\gamma_1(u),\gamma_2(v)\otimes\gamma_3(w)))$ is equal to:*

$$\begin{cases}\frac{1}{|\gamma_1\cdot G|}\sum_{r\in\gamma_1\cdot G\cap(\gamma_2\cdot G)(\gamma_3\cdot G)}\dim(\mathrm{Mor}_{G_r}(u\circ\psi^{\gamma_1}_{r,r},v\underset{r}{\otimes}w)) & \text{if }\gamma_1\cdot G\cap(\gamma_2\cdot G)(\gamma_3\cdot G)\neq\emptyset,\\ 0 & \text{otherwise.}\end{cases}$$

Let us observe that, by the bicrossed product relations, we have, for all $\gamma_1,\gamma_2,\gamma_3\in\Gamma$,

$$\gamma_1\cdot G\cap(\gamma_2\cdot G)(\gamma_3\cdot G)\neq\emptyset\Leftrightarrow\gamma_1\cdot G\subset(\gamma_2\cdot G)(\gamma_3\cdot G).$$

Proof. (1). Put $w=u\underset{r}{\otimes}v$ and let $g,h\in G_r$. Then, $w(gh)$ is equal to:

$$\sum_{s,s'\in\gamma\cdot G,t,t'\in\mu\cdot G,st=s't'=r}e_{ss'}\otimes e_{tt'}\otimes v_{ss'}(\alpha_t(gh))v_{tt'}(gh)u(\psi^\gamma_{s,s'}(\alpha_t(gh)))\otimes v(\psi^\mu_{t,t'}(gh)).$$

Since $v_{ty}(g)\neq0$ precisely when $t\cdot g=y$, the factor $v_{ss'}(\alpha_t(gh))v_{tt'}(gh)u(\psi^\gamma_{s,s'}(\alpha_t(gh)))\otimes v(\psi^\mu_{t,t'}(gh))$ is equal to:

$$\sum_{x\in\gamma\cdot G,y\in\mu\cdot G}v_{sx}(\alpha_t(g))v_{xs'}(\alpha_{t\cdot g}(h))v_{ty}(g)v_{yt'}(h)u(\psi^\gamma_{s,x}(\alpha_t(g))u(\psi^\gamma_{x,s'}(\alpha_{t\cdot g}(h)))\otimes v(\psi^\mu_{t,y}(g))v(\psi^\mu_{y,t'}(h))$$

$$=\sum_{x\in\gamma\cdot G,y\in\mu\cdot G}v_{sx}(\alpha_t(g))v_{xs'}(\alpha_y(h))v_{ty}(g)v_{yt'}(h)u(\psi^\gamma_{s,x}(\alpha_t(g))u(\psi^\gamma_{x,s'}(\alpha_y(h)))\otimes v(\psi^\mu_{t,y}(g))v(\psi^\mu_{y,t'}(h)).$$

Moreover, since for all $g\in G_r$ and all s,t such that $st=r$, one has, whenever $t\cdot g=y$ and $s\cdot\alpha_t(g)=x$, that $xy=(s\cdot\alpha_t(g))(t\cdot g)=(st)\cdot g=r\cdot g=r$, it follows that the only non-zero terms in the last sum are for $x\in\gamma\cdot G$ and $y\in\mu\cdot G$ such that $xy=r$. By the properties of the matrix units we see immediately that $w(gh)=w(g)w(h)$. To end the proof of (1), it suffices to check that $w(1)=1$, which is clear, and that $w(g)^*=w(g^{-1})$ for all $g\in G_r$. So let $g\in G_r$. One has:

$$w(g)^*=\sum_{s,s'\in\gamma\cdot G,\,t,t'\in\mu\cdot G,\,st=r=s't'}e_{ss'}\otimes e_{tt'}\otimes v_{s's}(\alpha_{t'}(g))v_{t't}(g)u((\psi^\gamma_{s',s}(\alpha_{t'}(g)))^{-1})\otimes v((\psi^\mu_{t',t}(g))^{-1}).$$

Note that for all $t,t'\in\Gamma$ and all $g\in G$, one has $v_{s's}(g)=v_{ss'}(g^{-1})$. Also, using the bicrossed product relations one finds that $\alpha_r(g)^{-1}=\alpha_{r\cdot g}(g^{-1})$ for all $r\in\Gamma$ and $g\in G$. In particular, $v_{s's}(\alpha_{t'}(g))v_{t't}(g)=$

$v_{ss'}(\alpha_t(g^{-1}))v_{tt'}(g^{-1})$ and, when $t'\cdot g = t$, one has $\psi^{\gamma}_{s',s}(\alpha_{t'}(g)))^{-1} = \psi^{\gamma}_{s,s'}(\alpha_t(g^{-1}))$. It follows immediately that $w(g)^* = w(g^{-1})$.

(2). Is a direct computation.

(3). One has $\dim(\mathrm{Mor}_{\mathbb{G}}(\gamma_1(u), \gamma_2(v)\otimes\gamma_3(w))) = h(\chi(\gamma_1(u))^*\chi(\gamma_2(v))\chi(\gamma_3(w)))$ which is equal to:

$$\begin{aligned}
&\sum_{r\in\gamma_1\cdot G, s\in\gamma_2\cdot G, t\in\gamma_3\cdot G} h(\chi(\overline{u})\circ\psi^{\gamma_1}_{r,r}v_{rr}u_r^*u_sv_{ss}\chi(v)\circ\psi^{\gamma_2}_{s,s}u_tv_{tt}\chi(w)\circ\psi^{\gamma_3}_{t,t})\\
&= \sum_{r,s,t} h(u_{r^{-1}st}\alpha_{t^{-1}s^{-1}r}(\chi(\overline{u})\circ\psi^{\gamma_1}_{r,r}v_{rr})\alpha_{t^{-1}}(v_{ss}\chi(v)\circ\psi^{\gamma_2}_{s,s})v_{tt}\chi(w)\circ\psi^{\gamma_3}_{t,t})\\
&= \sum_{r\in\gamma_1\cdot G}\ \sum_{s\in\gamma_2\cdot G, t\in\gamma_3\cdot G, st=r} \int_G \chi(\overline{u})\circ\psi^{\gamma_1}_{r,r}v_{rr}\alpha_{t^{-1}}(v_{ss}\chi(v)\circ\psi^{\gamma_2}_{s,s})v_{tt}\chi(w)\circ\psi^{\gamma_3}_{t,t}d\nu\\
&= \sum_{r\in\gamma_1\cdot G\cap(\gamma_2\cdot G)(\gamma_3\cdot G)} \frac{1}{|r\cdot G|}\int_{G_r}\chi(\overline{u})\circ\psi^{\gamma_1}_{r,r}\chi(v\underset{r}{\otimes} w)d\nu_r\\
&= \frac{1}{|\gamma_1\cdot G|}\sum_{r\in\gamma_1\cdot G\cap(\gamma_2\cdot G)(\gamma_3\cdot G)} \dim(\mathrm{Mor}_{G_r}(u\circ\psi^{\gamma_1}_{r,r}, v\underset{r}{\otimes} w)).
\end{aligned}$$

Note that, whenever $\gamma_1\cdot G\cap((\gamma_2\cdot G)(\gamma_3\cdot G)) = \emptyset$, there is no non-zero terms in the sum above. □

3.2 The induced representation

In this section, we explain how the induced representation maybe viewed as a particular twisted tensor product.

For $\gamma\in\Gamma$ and $u : G_\gamma\to\mathcal{U}(H)$ is a unitary representation of G_γ we define the induced representation:

$$\mathrm{Ind}^G_\gamma(u) := \varepsilon_{G_{\gamma^{-1}}}\underset{1}{\otimes} u\ :\ G\to\mathcal{U}(l^2(\gamma\cdot G)\otimes H);\ g\mapsto\sum_{r,s\in\gamma\cdot G} e_{rs}\otimes v_{rs}(g)u(\psi^{\gamma}_{rs}(g)).$$

It follows from Theorem 3.2 that $\mathrm{Ind}^G_\gamma(u)$ is indeed a unitary representation of G. We collect some elementary and well known facts about this representation in the following Proposition. Note that, in property 3, we use the symbol $\mathrm{Res}^G_{G_\gamma}(u)$ for $u\in\mathrm{Rep}(G)$ to denote the restriction of u to a representation of G_γ. Hence, property 3 motivates the name induced representation for the representation $\mathrm{Ind}^G_\gamma(u)$.

Proposition 3.3. *The following holds.*

1. *For all $\gamma\in\Gamma$ and all $u\in\mathrm{Rep}(G_\gamma)$ one has $\chi(\mathrm{Ind}^G_\gamma(u))(g) = \sum_{r\in\gamma\cdot G} v_{rr}(g)\chi(u)(\psi^\gamma_{rr}(g))$ for all $g\in G$.*
2. *For all $\gamma\in\Gamma$ and all $u,v\in\mathrm{Rep}(G_\gamma)$ one has $u\simeq v \implies \mathrm{Ind}^G_\gamma(u)\simeq\mathrm{Ind}^G_\gamma(v)$.*
3. *For all $\gamma\in\Gamma$, $u\in\mathrm{Rep}(G)$ and $v\in\mathrm{Rep}(G_\gamma)$ one has $\dim(\mathrm{Mor}_G(u,\mathrm{Ind}^G_\gamma(v))) = \dim(\mathrm{Mor}_{G_\gamma}(\mathrm{Res}^G_{G_\gamma}(u), v)$.*

Proof. (1). It is obvious, by definition of $\mathrm{Ind}^G_\gamma(u)$.

(2). If $u\simeq v$ then $\chi(u)=\chi(v)$. Hence, $\chi(\mathrm{Ind}^G_\gamma(u)) = \chi(\mathrm{Ind}^G_\gamma(v))$ by (1). So $\mathrm{Ind}^G_\gamma(u)\simeq\mathrm{Ind}^G_\gamma(v)$.

(3). Let $\gamma\in\Gamma$, $u\in\mathrm{Rep}(G)$ and $v\in\mathrm{Rep}(G_\gamma)$. One has,

$$\dim(\mathrm{Mor}_G(u,\mathrm{Ind}^G_\gamma(v))) = \sum_{r\in\gamma\cdot G}\int_G\chi(\overline{u})v_{rr}\chi(v)\circ\psi^\gamma_{rr}d\nu = \frac{1}{|\gamma\cdot G|}\sum_{r\in\gamma\cdot G}\int_{G_r}\chi(\overline{u})\chi(v)\circ\psi^\gamma_{rr}d\nu_\gamma.$$

Since $\psi^{\gamma}_{rr} : G_r \to G_\gamma$ is a Haar probability preserving homeomorphism we obtain

$$\dim(\mathrm{Mor}_G(u, \mathrm{Ind}^G_\gamma(v))) = \frac{1}{|\gamma \cdot G|} \sum_{r \in \gamma \cdot G} \int_{G_\gamma} \chi(\overline{u}) \circ (\psi^{\gamma}_{rr})^{-1} \chi(v) d\nu_\gamma.$$

Finally, since, for all $g \in G$, $\chi(\overline{u}) \circ (\psi^{\gamma}_{rr})^{-1}(g) = \chi(\overline{u})(g)$ (because $\chi(\overline{u})$ is a central function on G) it follows that:

$$\dim(\mathrm{Mor}_G(u, \mathrm{Ind}^G_\gamma(v))) = \frac{1}{|\gamma \cdot G|} \sum_{r \in \gamma \cdot G} \int_{G_\gamma} \chi(\overline{u}) \chi(v) d\nu_\gamma = \dim(\mathrm{Mor}_{G_\gamma}(\mathrm{Res}^G_{G_\gamma}(u), v).$$

□

4 Length functions

Recall that given a compact quantum group $\mathbb{H}$, a function $l : \mathrm{Irr}(\mathbb{H}) \to [0, \infty)$ is called a *length function on* $\mathrm{Irr}(\mathbb{H})$ if $l([\epsilon]) = 0$, $l(\overline{x}) = l(x)$ and that $l(x) \leq l(y) + l(z)$ whenever $x \subset y \otimes z$. A length function on a discrete group Λ is a function $l : \Lambda \to [0, \infty)$ such that $l(1) = 0$, $l(r) = l(r^{-1})$ and $l(rs) \leq l(r) + l(s)$ for all $r, s \in \Lambda$.

Let (Γ, G) be a matched pair with bicrossed product $\mathbb{G}$. In view of the description of the irreducible representations of $\mathbb{G}$, the fusion rules and the contragredient representation, it is clear that to get a length function on $\mathrm{Irr}(\mathbb{G})$, we need a family of maps $l_\gamma : \mathrm{Irr}(G_\gamma) \to [0, +\infty[$, for $\gamma \in \Gamma$, satisfying the hypothesis of the following definition.

Definition 4.1. Let (Γ, G) be a matched pair, $l : \mathrm{Irr}(G) \to [0, +\infty[$ and $l_\Gamma : \Gamma \to [0, +\infty[$ be length functions. The pair (l, l_Γ) is *matched* if, for all $\gamma \in \Gamma$, there exists a function $l_\gamma : \mathrm{Irr}(G_\gamma) \to [0, +\infty[$ such that

(i) $l_1 = l$ and $l_\gamma(\varepsilon_{G_\gamma}) = l_\Gamma(\gamma)$.

(ii) For any $\gamma \in \Gamma$, $r \in \gamma \cdot G$, and $x \in \mathrm{Irr}(G_\gamma)$, we have $l_\gamma(x) = l_r([u^x \circ \psi^{\gamma}_{r,r}])$.

(iii) For any $\gamma \in \Gamma$, $x \in \mathrm{Irr}(G_\gamma)$, we have $l_\gamma(x) = l_{\gamma^{-1}}([\overline{u^x} \circ \alpha_{\gamma^{-1}}])$.

(iv) For any $\gamma_1, \gamma_2, \gamma_3 \in \Gamma$, $x \in \mathrm{Irr}(G_{\gamma_1})$, $y \in \mathrm{Irr}(G_{\gamma_2})$, $z \in \mathrm{Irr}(G_{\gamma_3})$, if $\gamma_3 \in (\gamma_1 \cdot G)(\gamma_2 \cdot G)$, and

$$\dim \mathrm{Mor}_{G_r}(u^z \circ \psi^{\gamma_3}_{r,r}, u^x \otimes_r u^y) \neq 0 \tag{4.1}$$

for some $r \in \gamma_3 \cdot G$, then

$$l_{\gamma_3}(z) \leq l_{\gamma_1}(x) + l_{\gamma_2}(y). \tag{4.2}$$

The next Proposition shows that our notion of matched pair for length functions is the good one, as expected.

Proposition 4.2. *Let (Γ, G) be a matched pair with bicrossed product $\mathbb{G}$.*

1. *If l is a length function on $\mathrm{Irr}(\mathbb{G})$ then the maps $l_G : \mathrm{Irr}(G) = \mathrm{Irr}(G_1) \to [0, +\infty[$, $x \mapsto l([1(x)])$ and $l_\Gamma : \Gamma \to [0, +\infty[$, $\gamma \mapsto l([\gamma(\varepsilon_{G_\gamma})])$ are length functions and the pair (l_Γ, l_G) is matched.*
2. *If l_Γ is any β-invariant length function on Γ then the map $l' : \mathrm{Irr}(\mathbb{G}) \mapsto [0, +\infty[$, $[\gamma(u^x)] \mapsto l_\Gamma(\gamma)$ is a well defined length function on $\mathrm{Irr}(\mathbb{G})$.*
3. *If (l_Γ, l_G) is a matched pair of length functions on $(\Gamma, \mathrm{Irr}(G))$ then l_Γ is β-invariant and the maps $l, \widetilde{l} : \mathrm{Irr}(\mathbb{G}) \to [0, +\infty[$, $l([\gamma(u^x)]) := l_\gamma(x)$ and $\widetilde{l}([\gamma(u^x)]) := l_\gamma(x) + l_\Gamma(\gamma)$ are well-defined length functions.*

Proof. (1). Since $1(\varepsilon_G)$ is the trivial representation of $\mathbb{G}$ one has $l_\Gamma(1) = 0$. Let $\gamma, \mu \in \Gamma$ and note that $\gamma\mu \in (\gamma \cdot G)(\mu \cdot G)$. Moreover,

$$\dim(\mathrm{Mor}(\varepsilon_{G_{\gamma\mu}}, \varepsilon_{G_\gamma} \underset{\gamma\mu}{\otimes} \varepsilon_{G_\mu})) = \int_{G_{\gamma\mu}} \chi(\varepsilon_{G_\gamma} \underset{\gamma\mu}{\otimes} \varepsilon_{G_\mu}) d\nu_{G_{\gamma\mu}} = |\gamma\mu \cdot G| \sum_{s \in \gamma \cdot G, t \in \mu \cdot G, st = \gamma\mu} \int_{G_{\gamma\mu}} (v_{ss} \circ \alpha_t) v_{tt} d\nu$$

$$\begin{aligned} &= |\gamma\mu\cdot G|\sum_{s\in\gamma\cdot G,t\in\mu\cdot G,st=\gamma\mu}\nu(\alpha_{t^{-1}}(G_s)\cap G_t\cap G_{\gamma\mu})\\ &\geq \nu(\alpha_{\mu^{-1}}(G_\gamma)\cap G_\mu\cap G_{\gamma\mu}).\end{aligned}$$

Hence, since $\alpha_{\mu^{-1}}(G_\gamma)\cap G_\mu\cap G_{\gamma\mu}$ is open and non empty (it contains 1) we deduce that

$$\dim(\mathrm{Mor}(\varepsilon_{G_{\gamma\mu}},\varepsilon_{G_\gamma}\underset{\gamma\mu}{\otimes}\varepsilon_{G_\mu}))>0.$$

So $\varepsilon_{G_{\gamma\mu}}\subset\varepsilon_{G_\gamma}\underset{\gamma\mu}{\otimes}\varepsilon_{G_\mu}$ and, by the fusion rules of $\mathbb{G}$ in Theorem 3.2, $(\gamma\mu)(\varepsilon_{G_{\gamma\mu}})\subset\gamma(\varepsilon_{G_\gamma})\otimes\mu(\varepsilon_{G_\mu})$. Hence, since l is a length function, $l_\Gamma(\gamma\mu)=l([\gamma\mu(\varepsilon_{G_{\gamma\mu}})])\leq l([\gamma(\varepsilon_{G_\gamma})])+l([\mu(\varepsilon_{G_\mu})])=l_\Gamma(\gamma)+l_\Gamma(\mu)$. Finally, note that, by point 4 of Theorem 3.1, for all $\gamma\in\Gamma$, one has $\gamma^{-1}(\varepsilon_{G_{\gamma^{-1}}})\simeq\overline{\gamma(\varepsilon_G)}$. Hence,

$$l_\Gamma(\gamma^{-1})=l([\gamma^{-1}(\varepsilon_{G_{\gamma^{-1}})]}=l([\overline{\gamma(\varepsilon_G)}])=l([\gamma(\varepsilon_G)])=l_\Gamma(\gamma).$$

So l_Γ is a length function on Γ. It is obvious that l_G is a length function on $\mathrm{Irr}(G)$. Let us prove that the pair (l_Γ,l_G) is matched. Indeed, defining $l_\gamma:\mathrm{Irr}(G_\gamma)\to[0,+\infty[$ by $l_\gamma(x)=l([\gamma(u^x)])$, point (i) of Definition 4.1 is clear while point (ii) follows from point 3 of Theorem 3.1, since it implies $[\gamma(u^x)]=[r(u^x\circ\psi^r_{\gamma,\gamma})]$, thus

$$l_\gamma(x)=l([\gamma(u^x)])=l([r(u^x\circ\psi^r_{\gamma,\gamma})])=l_r([u^x\circ\psi^r_{\gamma,\gamma}]).$$

Next, by point 4 of Theorem 3.1, we have $\overline{[\gamma(u^x)]}=[\gamma^{-1}(\overline{u^x})\circ\alpha_{\gamma^{-1}}]$ thus,

$$l_\gamma(x)=l(\overline{[\gamma(u^x)]})=l([\gamma^{-1}(\overline{u^x})\circ\alpha^{-1}])=l_{\gamma^{-1}}([\overline{u^x}\circ\alpha^{-1}]),$$

which proves point (ii) of Definition 4.1. Finally, for point (iv), the fusion rules in Theorem 3.2 imply

$$\dim\mathrm{Mor}(\gamma_3(u^z),\gamma_1(u^x)\otimes\gamma_2(u^y))=\frac{1}{|\gamma\cdot G|}\sum_{r\in\gamma_3\cdot G}\dim\mathrm{Mor}_{G_r}(u^z\circ\psi^{\gamma_3}_{r,r},u^x\otimes_r u^y).\tag{4.3}$$

If $\dim\mathrm{Mor}_{G_r}(u^z\circ\psi^{\gamma_3}_{r,r},u^x\otimes_r u^y)\neq 0$ for some $r\in\gamma_3\cdot G$, then (4.3) is also nonzero, which means, by irreducibility of $\gamma_3(u^z)$ that $[\gamma_3(u^z)]\subseteq[\gamma_1(u^x)]\otimes[\gamma_2(u^y)]$. Hence, since l is a length function on $\mathrm{Irr}(\mathbb{G})$,

$$l_{\gamma_3}(z)=l([\gamma_3(u^z)])\leq l([\gamma_1(u^x)])+l([\gamma_2(u^y)])=l_{\gamma_1}(x)+l_{\gamma_2}(y).$$

(2). Since l_Γ is β-invariant, the map l' is well defined by Theorem 3.1. It is clear that $l'(\varepsilon_{\mathbb{G}})=0$ and, by point 4 (and 5) of Theorem 3.1 and since l' is a length function we also have that $l'(z)=l'(\overline{z})$ for all $z\in\mathrm{Irr}(\mathbb{G})$. Let now $\gamma_1,\gamma_2,\gamma_3\in\Gamma$, $x\in\mathrm{Irr}(G_{\gamma_1})$, $y\in\mathrm{Irr}(G_{\gamma_2})$ and $z\in\mathrm{Irr}(G_{\gamma_3})$ be such that $\gamma_1(u^x)\subset\gamma_2(u^y)\otimes\gamma_3(u^z)$ then, by point 3 in Theorem 3.2, there exists $r\in\gamma_1\cdot G$, $s\in\gamma_2\cdot G$ and $t\in\gamma_3\cdot G$ such that $r=st$ (and $u^x\circ\psi^{\gamma_1}_{r,r}\subset u^y\underset{r}{\otimes}u^z$). Then,

$$l'([\gamma_1(u^x)])=l_\Gamma(\gamma_1)=l_\Gamma(r)\leq l_\Gamma(s)+l_\Gamma(t)=l_\Gamma(\gamma_2)+l_\Gamma(\gamma_3)=l'([\gamma_2(u^y)])+l'([\gamma_3(u^z)]).$$

(3). Let (l_Γ,l_G) be a matched pair of length functions. By points 1 and 2 of Definition 4.1 we have, for all $\gamma\in\Gamma$ and all $r\in\gamma\cdot G$, $l_\Gamma(\gamma)=l_\gamma(\varepsilon_{G_\gamma})=l_r([\varepsilon_{G_\gamma}\circ\psi^\gamma_{r,r}])=l_r(\varepsilon_{G_r})=l_\Gamma(r)$. Hence, l_Γ is β-invariant. By assertion (2) we just proved above, we get a length function l' on $\mathrm{Irr}(\mathbb{G})$. Now, it is clear from Definition 4.1, the fusion rules and the adjoint representation of a bicrossed product (point 3 of Theorem 3.2 and point 4 of Theorem 3.1) that $l:[\gamma(u^x)]\mapsto l_\gamma(x)$ is a length function on $\mathrm{Irr}(\mathbb{G})$. Since $\widetilde{l}=l+l'$, $\widetilde{l}$ is also a length function on $\mathrm{Irr}(\mathbb{G})$. □

5 Rapid decay and polynomial growth

In this section we study property (RD) and polynomial growth for bicrossed-products.

5.1 Generalities

We use the notion of property (RD) developed by Vergnioux in [Ve07] (see also [BVZ14]) and recall the definition below. Since we are only dealing with Kac algebras, we recall the definition of the Fourier transform and rapid decay only for Kac algebras.

Let $\mathbb{H}$ be a compact quantum group. We use the notation $l^\infty(\widehat{\mathbb{H}}) := \bigoplus_{x\in\mathrm{Irr}(\mathbb{H})}\mathcal{B}(H_x)$ to denote the l^∞ direct sum. The c_0 direct sum is denoted by $c_0(\widehat{\mathbb{H}}) \subset l^\infty(\widehat{\mathbb{H}})$ and the algebraic direct sum is denoted by $c_c(\widehat{\mathbb{H}}) \subset c_0(\widehat{\mathbb{H}})$. An element $a \in c_c(\widehat{\mathbb{H}})$ is said to have finite support and its finite support is denoted by $\mathrm{Supp}(a) := \{x \in \mathrm{Irr}(\mathbb{H}) : ap_x \neq 0\}$, where p_x, for $x \in \mathrm{Irr}(\mathbb{H})$ denotes the central minimal projection of $l^\infty(\widehat{\mathbb{H}})$ corresponding to the block $\mathcal{B}(H_x)$.

For a compact quantum group $\mathbb{H}$ which is always supposed to be of Kac type, and $a \in C_c(\widehat{\mathbb{H}})$ we define its Fourier transform as:

$$\mathcal{F}_{\mathbb{H}}(a) = \sum_{x\in\mathrm{Irr}(\mathbb{H})} \dim(x)(\mathrm{Tr}_x \otimes \mathrm{id})(u^x(ap_x \otimes 1)) \in \mathrm{Pol}(\mathbb{H}),$$

and its "Sobolev 0-norm" by $\|a\|^2_{\mathbb{H},0} = \sum_{x\in\mathrm{Irr}(\mathbb{H})}\dim(x)\mathrm{Tr}_x((a^*a)p_x)$.

Given a length function $l : \mathrm{Irr}(\mathbb{H}) \to [0,\infty)$, consider the element $L = \sum_{x\in\mathrm{Irr}(\mathbb{H})} l(x)p_x$ which is affilated to $c_0(\widehat{\mathbb{H}})$. Let q_n denote the spectral projections of L associated to the interval $[n, n+1)$.

The pair $(\widehat{\mathbb{H}}, l)$ is said to have:

- *Polynomial growth* if there exists a polynomial $P \in \mathbb{R}[X]$ such that for every $k \in \mathbb{N}$ one has

$$\sum_{x\in\mathrm{Irr}(\mathbb{H}),\, k\leq l(x)<k+1} \dim(x)^2 \leq P(k)$$

- *Property* (RD) if there exists a polynomial $P \in \mathbb{R}[X]$ such that for every $k \in \mathbb{N}$ and $a \in q_k c_c(\widehat{\mathbb{H}})$, we have $\|\mathcal{F}(a)\|_{C(\mathbb{H})} \leq P(k)\|a\|_{\mathbb{H},0}$.

Finally, $\widehat{\mathbb{H}}$ *is said to have polynomial growth (resp. property* (RD) if there exists a length function l on $\mathrm{Irr}(\mathbb{H})$ such that $(\widehat{\mathbb{H}}, l)$ has polynomial growth (resp. property (RD)).

It is known from [Ve07] that if $(\widehat{\mathbb{H}}, l)$ has polynomial growth then $(\widehat{\mathbb{H}}, l)$ has rapid decay and the converse also holds when we assume $\mathbb{H}$ to be co-amenable. Moreover, it is shown also shown in [Ve07] that duals of compact connected real Lie groups have polynomial growth. The fact that polynomial growth implies (RD) can easily be deduced from the following lemma.

Lemma 5.1. *Let $\mathbb{H}$ be a CQG, $F \subset \mathrm{Irr}(\mathbb{H})$ a finite subset and $a \in l^\infty(\widehat{\mathbb{H}})$ with $ap_x = 0$ for all $x \notin F$. Then,*

$$\|\mathcal{F}_{\mathbb{H}}(a)\| \leq 2\sqrt{\sum_{x\in F}\dim(x)^2}\,\|a\|_{\mathbb{H},0}.$$

Proof. One can copy the proof of Proposition 4.2, assertion (a), in [BVZ14] or the proof of Proposition 4.4, assertion (ii), in [Ve07]. ☐

5.2 Technicalities

Let (Γ, G) be a matched pair with actions (α, β) and denote by $\mathbb{G}$ the bicrossed product.

Recall that $\mathrm{Irr}(\mathbb{G}) = \sqcup_{\gamma\in I}\mathrm{Irr}(G_\gamma)$, where $I \subset \Gamma$ is a complete set of representatives for Γ/G. For $\gamma \in I$ and $x \in \mathrm{Irr}(G_\gamma)$, we denote by $\gamma(x)$ the corresponding element in $\mathrm{Irr}(\mathbb{G})$. If a complete set of representatives of $\mathrm{Irr}(G_\gamma)$, $x \in \mathrm{Irr}(G_\gamma)$ is given by $u^x \in \mathcal{B}(H_x) \otimes C(G_\gamma)$ then a representative for $\gamma(x)$ is given by

$$u^{\gamma(x)} := \sum_{r,s\in\gamma\cdot G} e_{rs} \otimes (1 \otimes u_r v_{rs})u^x \circ \psi_{r,s} \in \mathcal{B}(l^2(\gamma \cdot G)) \otimes C(\mathbb{G}).$$

The lemma below gives a way of obtaining an element $\widetilde{a} \in c_c(\widehat{G})$ from an $a \in c_c(\widehat{G_\gamma})$ in a suitable way so that they are compatible with the Fourier transforms.

Lemma 5.2. *Let $\gamma \in \Gamma$ and $a \in c_c(\widehat{G}_\gamma)$. Define $\widetilde{a} \in c_c(\widehat{G})$ by:*

$$\widetilde{a}p_y = \sum_{x\in\mathrm{supp}(a)\ and\ y\subset\mathrm{Ind}_\gamma^G(x)} \frac{\dim(x)}{\dim(y)} \sum_{i=1}^{\dim(\mathrm{Mor}_G(y,\mathrm{Ind}_\gamma^G(x)))} (S_i^y)^*(e_{\gamma\gamma}\otimes ap_x)S_i^y,$$

where $S_i^y \in \mathrm{Mor}(y, \mathrm{Ind}_\gamma^G(x))$ is a basis of isometries with pairwise orthogonal images. The following holds.

1. *If (l_Γ, l) is a matched pair of length functions on $(\Gamma, \mathrm{Irr}(G))$ then, for all $y \in \mathrm{supp}(\widetilde{a})$ one has*
$$l(y) \le \max(\{l_\gamma(x) : x \in \mathrm{supp}(a)\}) + l_\Gamma(\gamma),$$
where $(l_\gamma)_{\gamma\in\Gamma}$ is any family of maps realizing the compatibility relations of Definition 4.1.
2. *$\mathcal{F}_{G_\gamma}(a) = v_{\gamma\gamma}\mathcal{F}_G(\widetilde{a})$.*
3. *$\|\widetilde{a}\|_{G,0} \le \|a\|_{G_\gamma,0}$.*

Proof. (1). Since any $y \in \mathrm{supp}(\widetilde{a})$ is such that $y \subset \mathrm{Ind}_\gamma^G(x) = \varepsilon_{G_{\gamma^{-1}}} \underset{1}{\otimes} x$ for some $x \in \mathrm{supp}(a)$, it follows that any $y \in \mathrm{supp}(\widetilde{a})$ satisfies $l(y) = l_1(y) \le l_{\gamma^{-1}}(\varepsilon_{G_{\gamma^{-1}}}) + l_\gamma(x) = l_\Gamma(\gamma^{-1}) + l_\gamma(x) = l_\Gamma(\gamma) + l_\gamma(x)$ for some $x \in \mathrm{supp}(a)$.
(2). One has:

$$\begin{aligned}
v_{\gamma\gamma}\mathcal{F}_G(\widetilde{a}) &= v_{\gamma\gamma}\sum_y \dim(y)(\mathrm{Tr}_y\otimes\mathrm{id})(u^y\widetilde{a}p_y\otimes 1)\\
&= v_{\gamma\gamma}\sum_{x\in\mathrm{supp}(a),\, y\subset\mathrm{Ind}_\gamma^G(x)}\ \sum_{i=1}^{\dim(\mathrm{Mor}(y,\mathrm{Ind}_\gamma^G(x)))} \dim(x)(\mathrm{Tr}_y\otimes\mathrm{id})(u^y((S_i^y)^*(e_{\gamma\gamma}\otimes ap_x)S_i^y)\otimes 1)\\
&= v_{\gamma\gamma}\sum_{x,y,i}\dim(x)(\mathrm{Tr}_y\otimes\mathrm{id})(((S_i^y)^*\otimes 1)\mathrm{Ind}_\gamma^G(u^x)(e_{\gamma\gamma}\otimes ap_x\otimes 1)(S_i^y\otimes 1))\\
&= v_{\gamma\gamma}\sum_{x,y,i}\dim(x)(\mathrm{Tr}_{l^2(\gamma\cdot G)\otimes H_x}\otimes\mathrm{id})(\mathrm{Ind}_\gamma^G(u^x)(e_{\gamma\gamma}\otimes ap_x\otimes 1)(S_i^y(S_i^y)^*\otimes 1))\\
&= v_{\gamma\gamma}\sum_{x\in\mathrm{supp}(a)}\dim(x)(\mathrm{Tr}_{l^2(\gamma\cdot G)\otimes H_x}\otimes\mathrm{id})(\mathrm{Ind}_\gamma^G(u^x)(e_{\gamma\gamma}\otimes ap_x\otimes 1))\\
&= v_{\gamma\gamma}\sum_{x\in\mathrm{supp}(a)}\dim(x)(\mathrm{Tr}_x\otimes\mathrm{id})(u^x ap_x\otimes 1)) = \mathcal{F}_{G_\gamma}(a),
\end{aligned}$$

where, in the 3rd equation we use the fact that $(S_i^y)^* \in \mathrm{Mor}(\mathrm{Ind}_\gamma^G(x), y)$ and, in the last equation we use the definition of the representation $\mathrm{Ind}_\gamma^G(u^x)$.
(3). One has:

$$\begin{aligned}
\|\widetilde{a}\|_{G,0}^2 &= \sum_y \dim(y)\mathrm{Tr}_y(\widetilde{a}^*\widetilde{a}p_y)\\
&= \sum_{x\in\mathrm{supp}(a),\, y\subset\mathrm{Ind}_\gamma^G(x)}\ \sum_{i,j=1}^{\dim(\mathrm{Mor}(y,\mathrm{Ind}_\gamma^G(x)))} \dim(y)\frac{\dim(x)^2}{\dim(y)^2}\mathrm{Tr}_y((S_i^y)^*(e_{\gamma\gamma}\otimes a^*p_x)S_i^y(S_j^y)^*(e_{\gamma\gamma}\otimes ap_x)S_j^y)\\
&= \sum_{x,y,i}\dim(x)\left(\frac{\dim(x)}{\dim(y)}\right)\mathrm{Tr}_y((S_i^y)^*(e_{\gamma\gamma}\otimes a^*p_x)S_i^y(S_i^y)^*(e_{\gamma\gamma}\otimes ap_x)S_i^y)
\end{aligned}$$

Since, for all y,i, $S_i^y(S_i^y)^*$ is a projection, one has $S_i^y(S_i^y)^* \leq 1$ hence,

$$\mathrm{Tr}_y((S_i^y)^*(e_{\gamma\gamma}\otimes a^*p_x)S_i^y(S_i^y)^*(e_{\gamma\gamma}\otimes ap_x)S_i^y) \leq \mathrm{Tr}_y((S_i^y)^*(e_{\gamma\gamma}\otimes a^*ap_x)S_i^y).$$

Moreover, by Proposition 3.3, one has $y\subset \mathrm{Ind}_\gamma^G(x)$ if and only if

$$\dim(\mathrm{Mor}_{G_\gamma}(\mathrm{Res}^G_{G_\gamma}(y),x)) = \dim(\mathrm{Mor}_G(y,\mathrm{Ind}_\gamma^G(x)))\neq 0.$$

Since x is irreducible, we find that $y\subset \mathrm{Ind}_\gamma^G(x)\Leftrightarrow x\subset \mathrm{Res}^G_{G_\gamma}(y)$. In particular, for any $y\subset\mathrm{Ind}_\gamma^G(x)$ one has $\dim(x)\leq\dim(y)$. Hence,

$$\begin{aligned}\|\widetilde{a}\|^2_{G,0} &\leq \sum_{x,y,i}\dim(x)\mathrm{Tr}_y((S_i^y)^*(e_{\gamma\gamma}\otimes a^*ap_x)S_i^y) = \sum_{x,y,i}\dim(x)\mathrm{Tr}_{l^2(\gamma\cdot G)\otimes H_x}(e_{\gamma\gamma}\otimes a^*ap_x(S_i^y)^*S_i^y)\\ &= \sum_{x\in\mathrm{supp}(a)}\dim(x)\mathrm{Tr}_{l^2(\gamma\cdot G)\otimes H_x}(e_{\gamma\gamma}\otimes a^*ap_x) = \sum_{x\in\mathrm{supp}(a)}\dim(x)\mathrm{Tr}_x(a^*ap_x) = \|a\|^2_{G_\gamma,0}.\end{aligned}$$ □

Lemma 5.3. *Let (l_Γ,l) be a matched pair of length functions on $(\Gamma,\mathrm{Irr}(G))$. If $(\widehat{G},l)$ has polynomial growth then, there exists $C>0$ and $N\in\mathbb{N}$ such that:*

- $\|\mathcal{F}_G(a)\|\leq C(k+1)^N\|a\|_{G,0}$ *for all* $a\in c_c(\widehat{G})$ *with* $\mathrm{supp}(a)\subset\{x\in\mathrm{Irr}(G)\,:\,l(x)<k+1\}$.
- $|\gamma\cdot G|\dim(x)\leq C(l_\Gamma(\gamma)+l_\gamma(x)+1)^N$ *for all* $\gamma\in\Gamma$, $x\in\mathrm{Irr}(G_\gamma)$.
- *For all* $\gamma\in\Gamma$, $\sum_{x\in\mathrm{Irr}(G_\gamma),\,l_\gamma(x)<k+1}\dim(x)^2\leq C^2(k+l_\Gamma(\gamma)+1)^{2N}$.

Proof. Let $P\in\mathbb{R}[X]$ be such that $\sum_{x\in\mathrm{Irr}(G),\,k\leq l(x)<k+1}\dim(x)^2\leq P(k)$ for all $k\in\mathbb{N}$ and let $C_1>0$ and $N_1\in\mathbb{N}$ be such that $P(k)\leq C_1(k+1)^{N_1}$ for all $k\in\mathbb{N}$. By Lemma 5.1 one has, for all $a\in c_c(\widehat{G})$, with $\mathrm{supp}(a)\subset\{x\in\mathrm{Irr}(G)\,:\,k\leq l(x)<k+1\}$, $\|\mathcal{F}_G(a)\|\leq 2\sqrt{P(k)}\|a\|_{G,0}\leq\sqrt{C_1}(k+1)^{\frac{N_1}{2}}\|a\|_{G,0}$. Now, suppose that $\mathrm{supp}(a)\subset\{x\in\mathrm{Irr}(G)\,:\,l(x)<k+1\}$ so that $a\in q_kc_c(\widehat{G})$, where $q_k=\sum_{j=0}^k p_j$ and $p_j=\sum_{x\in\mathrm{Irr}(G),\,k\leq l(x)<k+1}$. One has,

$$\|\mathcal{F}_G(a)\| = \sum_{j=0}^k\|\mathcal{F}_G(ap_j)\|\leq\sum_{j=0}^k\sqrt{C_1}(j+1)^{\frac{N_1}{2}}\|a\|_{G,0}\leq\sqrt{C_1}(k+1)^{\frac{N_1}{2}+1}\|a\|_{G,0}. \tag{5.1}$$

Now, let $\gamma\in\Gamma$ and $x\in\mathrm{Irr}(G_\gamma)$. By Proposition 3.3 one has:

$$\begin{aligned}|\gamma\cdot G|\dim(x) &= \dim(\mathrm{Ind}_\gamma^G(x)) = \sum_{y\in\mathrm{Irr}(G)}\dim(\mathrm{Mor}_G(y,\mathrm{Ind}_\gamma^G(x)))\dim(y)\\ &= \sum_{y\in\mathrm{Irr}(G),\,y\subset\mathrm{Ind}_\gamma^G(x)}\dim(\mathrm{Mor}_{G_\gamma}(\mathrm{Res}^G_{G_\gamma}(y),x))\dim(y).\end{aligned}$$

Note that $\dim(\mathrm{Mor}_{G_\gamma}(\mathrm{Res}^G_{G_\gamma}(y),x))\leq\dim(y)$ for all x,y. Moreover, since $\mathrm{Ind}_\gamma^G(x)\simeq\varepsilon_{G_{\gamma^{-1}}}\underset{1}{\otimes}x$ and the pair (l_Γ,l) is matched, one has $\{y\in\mathrm{Irr}(G),\,y\subset\mathrm{Ind}_\gamma^G(x)\}\subset\{y\in\mathrm{Irr}(G)\,:\,l(y)\leq l_\Gamma(\gamma)+l_\gamma(x)\}$. Hence,

$$\begin{aligned}|\gamma\cdot G|\dim(x) &\leq \sum_{y\in\mathrm{Irr}(G),\,l(y)<l_\Gamma(\gamma)+l_\gamma(x)+1}\dim(y)^2 = \sum_{j=0}^{l_\Gamma(\gamma)+l_\gamma(x)}\sum_{y\in\mathrm{Irr}(G),\,j\leq l(y)<j+1}\dim(y)^2\\ &\leq \sum_{j=0}^{l_\Gamma(\gamma)+l_\gamma(x)}P(j)\leq C_1\sum_{j=0}^{l_\Gamma(\gamma)+l_\gamma(x)}(j+1)^{N_1}\leq C_1(l_\Gamma(\gamma)+l_\gamma(x)+1)^{N_1+1}.\end{aligned} \tag{5.2}$$

It follows from Equations (5.1) and (5.2) that $C:=\mathrm{Max}(C_1,\sqrt{C_1})$ and $N:=N_1+1$ do the job.

Let us show the last point. Fix $\gamma \in \Gamma$ and let $F \subset \mathrm{Irr}(G_\gamma)$ a finite subset. Define $p_F \in c_c(\widehat{G}_\gamma)$ by $p_F = \sum_{x\in F} p_x$ and note that $\mathcal{F}_{G_\gamma}(p_F) = \sum_{x\in F}\dim(x)\chi(x)$ and $\|a\|^2_{G_\gamma,0} = \sum_{x\in F}\dim(x)^2$. Suppose that $F \subset \{x \in \mathrm{Irr}(G_\gamma) : l_\gamma(x) < k+1\}$. Using Lemma 5.2 and the first part of the proof we find, since $\widetilde{p_F} \in c_c(\widehat{G})$ with $\mathrm{supp}(\widetilde{p_F}) \subset \{x \in \mathrm{Irr}(G) : l(x) < l_\Gamma(\gamma)+k+1\}$,

$$\begin{aligned}\left\|\sum_{x\in F}\dim(x)\chi(x)\right\|^2 &= \|\mathcal{F}_{G_\gamma}(p_F)\|^2 = \|v_{\gamma\gamma}\mathcal{F}_G(\widetilde{p_F})\|^2 \leq \|\mathcal{F}_G(\widetilde{p_F})\|^2 \leq C^2(k+l_\Gamma(\gamma)+1)^{2N}\|\widetilde{p_F}\|^2_{G,0}\\ &\leq C^2(k+l_\Gamma(\gamma)+1)^{2N}\|p_F\|^2_{G_\gamma,0} = C^2(k+l_\Gamma(\gamma)+1)^{2N}\sum_{x\in F}\dim(x)^2.\end{aligned}$$

It follows that:

$$\left(\sum_{x\in F}\dim(x)^2\right)^2 = \left(\sum_{x\in F}\dim(x)\chi(x)(1)\right)^2 \leq \left\|\sum_{x\in F}\dim(x)\chi(x)\right\|^2_{C(G)} \leq C^2(k+l_\Gamma(\gamma)+1)^{2N}\sum_{x\in F}\dim(x)^2.$$

Hence, for all non empty finite subsets $F \subset \{x \in \mathrm{Irr}(G_\gamma) : l_\gamma(x) < k+1\}$ one has $\sum_{x\in F}\dim(x)^2 \leq C^2(k+l_\Gamma(\gamma)+1)^{2N}$. The last assertion follows. □

5.3 Polynomial growth for bicrossed product

We start with the following result.

Theorem 5.4. *Suppose that that (l_G, l_Γ) is a matched pair of length functions on (Γ, G). If both (Γ, l_Γ) and $(\widehat{G}, l_G)$ has polynomial growth then $(\widehat{\mathbb{G}}, \widetilde{l})$ have polynomial growth.*

Proof. Let $I \subset \Gamma$ be a complete set of representatives for Γ/G so that $\mathrm{Irr}(\mathbb{G}) = \sqcup_{\gamma\in I}\mathrm{Irr}(G_\gamma)$. Let $k \geq 1$ and define

$$F_k := \{z \in \mathrm{Irr}(\mathbb{G}) : \widetilde{l}(z) < k\} \subset \sqcup_{\gamma\in I_k} T_{\gamma,k},$$

where $I_k := \{\gamma \in \Gamma : l_\Gamma(\gamma) < k+1\} \cap I$ and $T_{\gamma,k} := \{x \in \mathrm{Irr}(G_\gamma) : l_\gamma(x) < k+1\}$. Since (Γ, l_Γ) has polynomial growth, there exists a polynomial P_1 such that, for all $k \in \mathbb{N}$, $|I_k| \leq P_1(k)$. Moreover, since $(\widehat{G}, l_G)$ has polynomial growth, we can apply Lemma 5.3 to get $C > 0$ and $N \in N$ such that, for all $k \in \mathbb{N}$ and all $\gamma \in I_k$, one has $\sum_{x\in T_{\gamma,k}}\dim(x)^2 \leq C^2(2k+2)^{2N}$ and, $|\gamma\cdot G| = |\gamma\cdot G|\dim(\varepsilon_G) \leq C(2k+3)^N$. Hence, for all $k \geq 1$,

$$\begin{aligned}\sum_{z\in F_k}\dim(z)^2 &= \sum_{\gamma\in I_k}|\gamma\cdot G|^2\sum_{x\in T_{\gamma,k}}\dim(x)^2 \leq C^2(2k+2)^{2N}\sum_{\gamma\in I_k}|\gamma\cdot G|^2 \leq C^4(2k+2)^{2N}(2k+3)^{2N}|I_k|\\ &\leq C^4(2k+2)^{2N}(2k+3)^{2N}P_1(k).\end{aligned}$$
□

To complete the proof of Theorem B, we need the following Proposition.

Proposition 5.5. *Assume that there exists a length function l on $\mathrm{Irr}(\mathbb{G})$ such that $(\widehat{\mathbb{G}}, l)$ has polynomial growth and consider the matched pair of length functions (l_Γ, l_G) associated to l given in Proposition 4.2. Then (Γ, l_Γ) and $(\widehat{G}, l_G)$ both have polynomial growth.*

Proof. Assume that $(\widehat{\mathbb{G}}, l)$ has polynomial growth. Since the map $\mathrm{Irr}(G) \to \mathrm{Irr}(\mathbb{G})$, $x \mapsto 1(x)$ is injective, dimension preserving and length preserving (by definition of l_G), it is clear that $(\widehat{G}, l_G)$ has polynomial growth. Let us show that (Γ, l_Γ) also has polynomial growth. Let P be a polynomial witnessing (RD) for $(\widehat{\mathbb{G}}, l)$ and $k \in \mathbb{N}$. Define $F_k := \{\gamma \in \Gamma : k \leq l_\Gamma(\gamma) < k+1\}$. One has, for all $k \in \mathbb{N}$,

$$\begin{aligned}|F_k| &= \sum_{k\leq l([\gamma(\varepsilon_G)])<k+1} 1 \leq \sum_{k\leq l([\gamma(\varepsilon_G)])<k+1}|\gamma\cdot G|^2 = \sum_{k\leq l([\gamma(\varepsilon_G)])<k+1}\dim([\gamma(\varepsilon_G)])^2\\ &\leq \sum_{z\in\mathrm{Irr}(\mathbb{G}),\, k\leq l(z)<k+1}\dim(z)^2 \leq P(k).\end{aligned}$$
□

5.4 Rapid decay for bicrossed product

Recall that $l^\infty(\widehat{\mathbb{G}}) = \bigoplus_{\gamma\cdot G\in\Gamma/G}\bigoplus_{x\in\mathrm{Irr}(G_\gamma)}\mathcal{B}(l^2(\gamma\cdot G)\otimes H_x)$. Let us denote by $p_{\gamma(x)}$ the central projection of $l^\infty(\widehat{\mathbb{G}})$ corresponding to the block $\mathcal{B}(l^2(\gamma\cdot G)\otimes H_x)$ and define, for $\gamma\cdot G\in\Gamma/G$, the central projection :

$$p_\gamma := \sum_{x\in\mathrm{Irr}(G_\gamma)} p_{\gamma(x)} \in l^\infty(\widehat{\mathbb{G}}).$$

Note that $p_\gamma l^\infty(\widehat{\mathbb{G}}) = \bigoplus_{x\in\mathrm{Irr}(G_\gamma)}\mathcal{B}(l^2(\gamma\cdot G)\otimes H_x)\simeq\mathcal{B}(l^2(\gamma\cdot G))\otimes L(G_\gamma)$, where $L(G_\gamma)=\bigoplus_{x\in\mathrm{Irr}(G_\gamma)}\mathcal{B}(H_x)$ is the group von-Neumann algebra of G_γ (which is also the multiplier C*-algebra of $C_r^*(G_\gamma)=\bigoplus^{c_0}_{x\in\mathrm{Irr}(G_\gamma)}\mathcal{B}(H_x)$). Using this identification, we define $\pi_\gamma : c_0(\widehat{\mathbb{G}})\to\mathcal{B}(l^2(\gamma\cdot G))\otimes C_r^*(G_\gamma)\subset c_0(\widehat{\mathbb{G}})$ to be the $*$-homomorphism given by $\pi_\gamma(a)=ap_\gamma$, for all $a\in c_0(\widehat{\mathbb{G}})$. We also write, for $a\in c_0(\widehat{\mathbb{G}})$, $\pi_\gamma(a)=\sum_{r,s\in\gamma\cdot G}e_{rs}\otimes\pi^\gamma_{r,s}(a)$, where we recall that (e_{rs}) are the matrix units associated to the canonical orthonormal basis $(e_r)_{r\in\gamma\cdot G}$ of $l^2(\gamma\cdot G)$ and $\pi^\gamma_{r,s} : c_0(\widehat{\mathbb{G}})\to C_r^*(G_\gamma)$ is the completely bounded map defined by $\pi^\gamma_{r,s} := (\omega_{e_s,e_r}\otimes\mathrm{id})\circ\pi_\gamma$ and $\omega_{e_s,e_r}\in\mathcal{B}(l^2(\gamma\cdot G))$, $\omega_{e_s,e_r}(T)=\langle Te_s,e_r\rangle$.
We start with the following result.

Theorem 5.6. *Let (l_Γ,l_G) be a matched pair of length functions on $(\Gamma,\mathrm{Irr}(G))$. Suppose that $(\widehat{G},l_G)$ has polynomial growth and (Γ,l_Γ) has (RD). Then $(\widehat{\mathbb{G}},\widetilde{l})$ has (RD).*

Proof. Let $a\in c_c(\widehat{\mathbb{G}})$ and write $a=\sum_{\gamma\in S}\sum_{x\in T_\gamma}ap_{\gamma(x)}$, where $S\subset I$ and $T_\gamma\subset\mathrm{Irr}(G_\gamma)$ are finite subsets.
Claim. *The following holds.*

1. $\mathcal{F}_{\mathbb{G}}(a)=\sum_{\gamma\in S}|\gamma\cdot G|\left(\sum_{r,s\in\gamma\cdot G}u_rv_{rs}\mathcal{F}_{G_\gamma}(\pi^\gamma_{s,r}(a))\circ\psi^\gamma_{r,s}\right)$.
2. $\|a\|^2_{\mathbb{G},0}=\sum_{\gamma\in S}|\gamma\cdot G|\left(\sum_{r,s\in\gamma\cdot G}\|\pi^\gamma_{r,s}(a)\|^2_{G_\gamma,0}\right)$.

Proof of the Claim.(1). A direct computation gives:

$$\begin{aligned}\mathcal{F}_{\mathbb{G}}(a) &= \sum_{\gamma\in S,\,x\in T_\gamma}|\gamma\cdot G|\dim(x)(\mathrm{Tr}_{l^2(\gamma\cdot G)\otimes H_x}(\gamma(u^x)ap_{\gamma(x)}\otimes 1)\\ &= \sum_{\gamma\in S,x\in T_\gamma}|\gamma\cdot G|\dim(x)\sum_{r,s\in\gamma\cdot G}u_rv_{rs}(\mathrm{Tr}_x\otimes\mathrm{id})(u^x\circ\psi^\gamma_{r,s}\pi^\gamma_{s,r}(a)p_x\otimes 1)\\ &= \sum_{\gamma\in S}|\gamma\cdot G|\sum_{r,s\in\gamma\cdot G}u_rv_{rs}\mathcal{F}_{G_\gamma}(\pi^\gamma_{s,r}(a))\circ\psi^\gamma_{r,s}.\end{aligned}$$

(2). Since π_γ is a $*$-homomorphism, we have $\pi^\gamma_{r,s}(a^*a)=\sum_{t\in\gamma\cdot G}\pi^\gamma_{t,r}(a)^*\pi^\gamma_{t,s}(a)$ hence,

$$\begin{aligned}\|a\|^2_{\mathbb{G},0} &= \sum_{\gamma\in S,x\in T_\gamma}|\gamma\cdot G|\dim(x)\sum_{r,s\in\gamma\cdot G}(\mathrm{Tr}_x\otimes\mathrm{id})(\pi^\gamma_{s,r}(a)^*\pi^\gamma_{r,s}(a))\\ &= \sum_{\gamma\in S}|\gamma\cdot G|\sum_{r,s\in\gamma\cdot G}\|\pi^\gamma_{r,s}(a)\|^2_{G_\gamma,0}.\end{aligned}$$

Let us now prove the theorem. Let $b=\sum_{\gamma\in S'}\sum_{t,t'\in\gamma\cdot G}u_tv_{tt'}F_\gamma\circ\psi^\gamma_{t,t'}\in C(\mathbb{G})$, where $F_\gamma\in C(G_\gamma)$ and $S'\subset I$ is a finite subset. For all $r\in\Gamma$, we denote by γ_r the unique element in I such that $\gamma_r\cdot G=r\cdot G$. We may re-order the sums and write:

$$\mathcal{F}_{\mathbb{G}}(a)=\sum_{r\in\Gamma}1_{S\cdot G}(r)|r\cdot G|\left(\sum_{s\in r\cdot G}u_rv_{rs}\mathcal{F}_{G_{\gamma_r}}(\pi^{\gamma_r}_{s,r}(a))\circ\psi^{\gamma_r}_{r,s}\right)\text{ and }b=\sum_{t\in\Gamma}u_t1_{S'\cdot G}(t)\left(\sum_{t'\in t\cdot G}v_{tt'}F_{\gamma_t}\circ\psi^{\gamma_t}_{t,t'}\right).$$

Also, $\|a\|^2_{\mathbb{G},0}=\sum_{r\in\Gamma}1_{S\cdot G}(r)|r\cdot G|\left(\sum_{s\in r\cdot G}\|\pi^{\gamma_r}_{r,s}(a)\|^2_{G_{\gamma_r},0}\right)$. Then, $\|\mathcal{F}_{\mathbb{G}}(a)b\|^2_{2,h_{\mathbb{G}}}$ is equal to :

$$
\begin{aligned}
&\left\| \sum_{r,t\in\Gamma} u_{rt} 1_{S\cdot G}(r) 1_{S'\cdot G}(t) |r\cdot G| \left(\sum_{s\in r\cdot G, t'\in t\cdot G} v_{rs}\circ\alpha_t \mathcal{F}_{G_{\gamma_r}}(\pi^{\gamma_r}_{s,r}(a))\circ\psi^{\gamma_r}_{r,s}\circ\alpha_t v_{tt'} F_{\gamma_t}\circ\psi^{\gamma_t}_{t,t'} \right) \right\|^2_{2,h_{\mathbb{G}}} \\
&= \sum_{x\in\Gamma} \left\| \sum_{\substack{r,t\in\Gamma\\ rt=x}} 1_{S\cdot G}(r) 1_{S'\cdot G}(t) |r\cdot G| \left(\sum_{s\in r\cdot G, t'\in t\cdot G} v_{rs}\circ\alpha_t \mathcal{F}_{G_{\gamma_r}}(\pi^{\gamma_r}_{s,r}(a))\circ\psi^{\gamma_r}_{r,s}\circ\alpha_t v_{tt'} F_{\gamma_t}\circ\psi^{\gamma_t}_{t,t'} \right) \right\|^2_{2} \\
&= \sum_{x\in\Gamma} \left\| \sum_{\substack{r,t\in\Gamma\\ rt=x}} 1_{S\cdot G}(r) 1_{S'\cdot G}(t) |r\cdot G| \left(\sum_{s\in r\cdot G} v_{rs}\circ\alpha_t \mathcal{F}_{G_{\gamma_r}}(\pi^{\gamma_r}_{s,r}(a))\circ\psi^{\gamma_r}_{r,s}\circ\alpha_t \right) \left(\sum_{t'\in t\cdot G} v_{tt'} F_{\gamma_t}\circ\psi^{\gamma_t}_{t,t'} \right) \right\|^2_{2} \\
&\leq \sum_{x} \left(\sum_{\substack{r,t\in\Gamma\\ rt=x}} 1_{S\cdot G}(r) 1_{S'\cdot G}(t) |r\cdot G| \left\| \sum_{s\in r\cdot G} v_{rs}\circ\alpha_t \mathcal{F}_{G_{\gamma_r}}(\pi^{\gamma_r}_{s,r}(a))\circ\psi^{\gamma_r}_{r,s}\circ\alpha_t \right\|_\infty \left\| \sum_{t'\in t\cdot G} v_{tt'} F_{\gamma_t}\circ\psi^{\gamma_t}_{t,t'} \right\|_2 \right)^2 \\
&= \sum_{x} \left(\sum_{\substack{r,t\in\Gamma\\ rt=x}} \left(1_{S\cdot G}(r) |r\cdot G| \left\| \sum_{s\in r\cdot G} v_{rs} \mathcal{F}_{G_{\gamma_r}}(\pi^{\gamma_r}_{s,r}(a))\circ\psi^{\gamma_r}_{r,s} \right\|_\infty \right) \left(1_{S'\cdot G}(t) \left\| \sum_{t'\in t\cdot G} v_{tt'} F_{\gamma_t}\circ\psi^{\gamma_t}_{t,t'} \right\|_2 \right) \right)^2 \\
&= \|\psi*\phi\|^2_{l^2(\Gamma)},
\end{aligned}
$$

where $\|\cdot\|_2$ and $\|\cdot\|_\infty$ denote respectively the L^2-norm and the supremum norm on $C(G)$ and $\psi,\phi : \Gamma\to\mathbb{R}_+$ are finitely supported functions defined by :

$$
\psi(r) := 1_{S\cdot G}(r)|r\cdot G| \left\| \sum_{s\in r\cdot G} v_{rs}\mathcal{F}_{G_{\gamma_r}}(\pi^{\gamma_r}_{s,r}(a))\circ\psi^{\gamma_r}_{r,s} \right\|_\infty \text{ and } \phi(t) := 1_{S'\cdot G}(t) \left\| \sum_{t'\in t\cdot G} v_{tt'} F_{\gamma_t}\circ\psi^{\gamma_t}_{t,t'} \right\|_2,
$$

Note that $\|\phi\|^2_{l^2(\Gamma)} = \|b\|^2_{2,h_{\mathbb{G}}}$. Moreover, one has, since $\psi^\gamma_{r,s} : G_{r,s}\to G_\gamma$ is an homeomorphism,

$$
\begin{aligned}
\|\psi\|^2_{l^2(\Gamma)} &= \sum_{r\in\Gamma} 1_{S\cdot G}(r)|r\cdot G|^2 \left\| \sum_{s\in r\cdot G} v_{rs}\mathcal{F}_{G_{\gamma_r}}(\pi^{\gamma_r}_{s,r}(a))\circ\psi^{\gamma_r}_{r,s} \right\|^2_\infty \\
&\leq \sum_{r\in\Gamma} 1_{S\cdot G}(r)|r\cdot G|^3 \sum_{s\in r\cdot G} \left\| v_{rs}\mathcal{F}_{G_{\gamma_r}}(\pi^{\gamma_r}_{s,r}(a))\circ\psi^{\gamma_r}_{r,s} \right\|^2_\infty \\
&= \sum_{r\in\Gamma} 1_{S\cdot G}(r)|r\cdot G|^3 \sum_{s\in r\cdot G} \left\| \mathcal{F}_{G_{\gamma_r}}(\pi^{\gamma_r}_{s,r}(a)) \right\|^2_{C(G_{\gamma_r})}.
\end{aligned}
$$

For $k\in\mathbb{N}$ let $p_k = \sum_{\gamma\in I, x\in\mathrm{Irr}(G_\gamma) : k\leq l(\gamma(x))<k+1} p_{\gamma(x)} \in l^\infty(\widehat{\mathbb{G}})$, $p_k^{G_\gamma} = \sum_{x\in\mathrm{Irr}(G_\gamma) : k\leq l_{G_\gamma}(x)<k+1} p_x \in l^\infty(\widehat{G}_\gamma)$ and suppose from now on that $a\in p_k c_c(\widehat{\mathbb{G}})$. Hence, we must have $S\subset\{\gamma\in\Gamma : l_\Gamma(\gamma)<k+1\}$ and, for all $\gamma\in S$, $T_\gamma\subset\{x\in\mathrm{Irr}(G_\gamma) : l_{G_\gamma}(x)<k+1\}$. Hence, for all $\gamma\in S$ and all $r,s\in\gamma\cdot G$ one has $\pi^\gamma_{r,s}(a)\in q^\gamma_k c_c(\widehat{G}_\gamma)$, where $q^\gamma_k = \sum_{j=0}^k p_j^{G_\gamma}$.

Since $(\widehat{G}, l_G)$ has polynomial growth, there exists $C>0$ and $N\in\mathbb{N}$ satisfying the properties of Lemma 5.3. In particular, one has, for all $\gamma\in\Gamma$, $|\gamma\cdot G|\leq C(2l_\Gamma(\gamma)+1)^N$. Moreover, since $S\subset\{g\in\Gamma : l_\Gamma(g)<k+1\}$ and l_Γ is β-invariant, it follows that $S\cdot G\subset\{g\in\Gamma : l_\Gamma(g)<k+1\}$. By Lemma 5.2 (and Lemma 5.3) we deduce that:

$$
\begin{aligned}
\|\psi\|^2_{l^2(\Gamma)} &\leq \sum_{r\in\Gamma} 1_{S\cdot G}(r)|r\cdot G|^3 \sum_{s\in r\cdot G} \left\| v_{\gamma_r\gamma_r}\mathcal{F}_G(\widetilde{\pi^{\gamma_r}_{s,r}(a)}) \right\|^2 \leq \sum_{r\in\Gamma} 1_{S\cdot G}(r)|r\cdot G|^3 \sum_{s\in r\cdot G} \left\| \mathcal{F}_G(\widetilde{\pi^{\gamma_r}_{s,r}(a)}) \right\|^2 \\
&\leq \sum_{r\in\Gamma} 1_{S\cdot G}(r)|r\cdot G|^3 \sum_{s\in r\cdot G} C^2(k+l_\Gamma(\gamma_r)+1)^{2N} \left\| \widetilde{\pi^{\gamma_r}_{s,r}(a)} \right\|^2_{G,0}
\end{aligned}
$$

$$\leq \quad C^2(2k+2)^{2N}\sum_{r\in\Gamma}1_{S\cdot G}(r)|r\cdot G|^3\sum_{s\in r\cdot G}\left\|\pi^{\gamma_r}_{s,r}(a)\right\|^2_{G_{\gamma_r},0}$$

$$\leq \quad C^4(2k+3)^{4N}\sum_{r\in\Gamma}1_{S\cdot G}(r)|r\cdot G|\sum_{s\in r\cdot G}\left\|\pi^{\gamma_r}_{s,r}(a)\right\|^2_{G_{\gamma_r},0}=C^4(2k+3)^{4N}\|a\|^2_{\mathbb{G},0}.$$

Since (Γ,l_Γ) has (RD), let $C_2>0$ and $N_2\in\mathbb{N}$ such that for all $k\in\mathbb{N}$, for all function ξ on Γ supported on $\{g\in\Gamma\,:\,l_\Gamma(g)<k+1\}$, we have $\|\xi*\eta\|_{l^2(\Gamma)}\leq C_2(k+1)^{N_2}\|\xi\|_{l^2(\Gamma)}\|\eta\|_{l^2(\Gamma)}$. Note that ψ is supported on $S\cdot G$ and $S\cdot G\subset\{g\in\Gamma\,:\,l_\Gamma(g)<k+1\}$. Hence, it follows from the preceding computations that:

$$\begin{aligned}\|\mathcal{F}_{\mathbb{G}}(a)b\|^2_{2,h_{\mathbb{G}}} &\leq \|\psi*\phi\|^2_{l^2(\Gamma)}\leq C_2^2(k+1)^{2N_2}\|\psi\|_{l^2(\Gamma)}\|\phi\|_{l^2(\Gamma)}\leq C^4(2k+3)^{4N}C_2^2(k+1)^{2N_2}\|a\|^2_{\mathbb{G},0}\|b\|^2_{2,h_{\mathbb{G}}}\\ &= (P(k)\|a\|^2_{\mathbb{G},0}\|b\|_{2,h_{\mathbb{G}}})^2.\end{aligned}$$

where $P(X)=C^2C_2^2(2X+3)^{2N}(X+1)^{N_2}$. It concludes the proof. ☐

To complete the proof of Theorem A, we need the following Proposition.

Proposition 5.7. *Assume that there exists a length function l on* $\mathrm{Irr}(\mathbb{G})$ *such that $(\widehat{\mathbb{G}},l)$ has (RD) and consider the matched pair of length functions (l_Γ,l_G) associated to l given in Proposition 4.2. Then (Γ,l_Γ) has (RD) and $(\widehat{G},l_G)$ has polynomial growth.*

Proof. Suppose that $(\widehat{\mathbb{G}},l)$ has (RD). The fact that $(\widehat{G},l_G)$ has (RD) follows from the general theory (since $C(G)\subset C(\mathbb{G})$ intertwines the comultiplication and the associated injection $\mathrm{Irr}(G)\to\mathrm{Irr}(\mathbb{G})$, actually given by $(x\mapsto 1(x))$, preserves the length functions). Let us show that (Γ,l_Γ) has (RD). Let $k\in\mathbb{N}$ and $\xi\,:\,\Gamma\to\mathbb{C}$ be a finitely supported function with support in $\{\gamma\in\Gamma\,:\,k\leq l_\Gamma(\gamma)<k+1\}$. Define $\widetilde{\xi}\in c_c(\widehat{\mathbb{G}})$ by $\widetilde{\xi}=\sum_{\gamma\in I}\frac{1}{|\gamma\cdot G|}\left(\sum_{r\in\gamma\cdot G}\xi(r)e_{rr}\right)p_{\gamma(1)}$, where we recall $e_{rs}\in\mathcal{B}(l^2(\gamma\cdot G))$ for $r,s\in\gamma\cdot G$ are the matrix units associated to the canonical orthonormal basis. Then,

$$\mathcal{F}_{\mathbb{G}}(\widetilde{\xi})=\sum_{\gamma\in I}\sum_{r\in\gamma\cdot G}\xi(r)(\mathrm{Tr}_{l^2(\gamma\cdot G)}\otimes\mathrm{id})(u^{\gamma(1)}(e_{rr}\otimes 1))=\sum_{\gamma\in I}\sum_{r\in\gamma\cdot G}\xi(r)u_rv_{rr}\quad\text{also,}$$

$$\|\widetilde{\xi}\|^2_{\mathbb{G},0}=\sum_{\gamma\in I}|\gamma\cdot G|\mathrm{Tr}_{l^2(\gamma\cdot G)}(\sum_{r\in\gamma\cdot G}\frac{|\xi(r)|^2}{|\gamma\cdot G|^2}e_{rr})=\sum_{\gamma\in I}\frac{1}{|\gamma\cdot G|}\sum_{r\in\gamma\cdot G}|\xi(r)|^2\leq\sum_{\gamma\in I}\sum_{r\in\gamma\cdot G}|\xi(r)|^2=\|\xi\|_2^2.$$

Since ξ is supported in $\{\gamma\in\Gamma\,:\,k\leq l_\Gamma(\gamma)<k+1\}$ and l_Γ is β-invariant, it follows that $\mathrm{supp}(\widetilde{\xi})\subset\{z\in\mathrm{Irr}(\mathbb{G})\,:\,k\leq l(z)<k+1\}$. Hence, denoting by P a polynomial witnessing (RD) for $(\widehat{\mathbb{G}},l)$, we have:

$$\left\|\sum_{\gamma\in I}\sum_{r\in\gamma\cdot G}\xi(r)u_rv_{rr}\right\|\leq P(k)\|\xi\|_2.$$

Denote by Ψ the unital $*$-morphism $\Psi\,:\,C(\mathbb{G})=\Gamma\ltimes C(G)\to C^*_r(\Gamma)$ such that $\Psi(u_\gamma F)=\lambda_\gamma F(1)$ for all $\gamma\in\Gamma$ and $F\in C(G)$. Since Ψ has norm one, denoting by $\lambda(\xi)\in C^*_r(\Gamma)$ the convolution operator by ξ, we have

$$\|\lambda(\xi)\|=\left\|\sum_{\gamma\in I}\sum_{r\in\gamma\cdot G}\xi(r)\lambda_r\right\|=\left\|\Psi\left(\sum_{\gamma\in I}\sum_{r\in\gamma\cdot G}\xi(r)u_rv_{rr}\right)\right\|\leq\left\|\sum_{\gamma\in I}\sum_{r\in\gamma\cdot G}\xi(r)u_rv_{rr}\right\|\leq P(k)\|\xi\|_2.$$

This concludes the proof. ☐

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